TEACHING ARITHMETIC

A HANDBOOK FOR TEACHERS

AND

A TEXTBOOK FOR NORMAL AND TRAINING SCHOOLS

BY

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PREFACE

THERE is a science of teaching arithmetic. The feeling of needs and the finding of means for their satisfaction is the law of advancement; logical division is the law of classifications and definitions; induction, deduction, and the complete method are the laws for the discovery of principles; experimentation and reasoning are the laws for the solution of simple problems; and the analysis of complex problems into simple problems is the law for the solution of problems in general.

In *Part I* the author has endeavored to develop and state these laws. In *Part II* he has endeavored to illustrate their application; much has been omitted but enough has been given to suggest the rest. In *Part III* he has endeavored to give an idea of the advance in complexity from grade to grade in elementary schools and of what is required to secure primary and higher licenses for teaching arithmetic.

As a handbook, this work should help the superintendent to secure uniformity in his schools; the principal to harmonize the work of the grades; and the teacher to get a clear view of fundamental laws.

As a textbook, this work should help the teacher in normal and training schools to supplement his lectures; and prospective teachers in school and out of school to prepare for their chosen profession.

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TEACHING ARITHMETIC

PART I. INTRODUCTION

1. Object and Scope. The office of this book is to discuss means of assisting pupils six years of age to pass in eight years from complete ignorance of mathematics to the knowledge and efficiency outlined in the course of study. The keynote is, *Every step in response to a need and as a means to an end.*

The race has advanced to its present stage and will advance to other stages in response to needs. The satisfaction of one need gives rise to others, their satisfaction to still others, and so on. Pupils must follow the same path but must accomplish in a few years what the race has accomplished in many centuries.

2. The Plan. The subject matter of mathematics is mental products — number and the operations upon number do not exist ready formed in nature. Assuming that the pupil, like the race, should be an inventor, we shall discuss the principal means of helping him to create subject matter for himself, and shall apply these means to the divisions of arithmetic.

LESSON 1. Needs

3. Discovering Needs. Pupils are to take every step in response to a need.

Place them in the actual situations that give rise to a need or cause them to image these situations, and suggest the need by a question. No labored effort should be made; the suggestion should be natural and simple.

Ill. *Number.* T. "On your desk there is a handful of sticks. How many are there?"

Ill. *Interest.* T. "A man borrowed $1000. With this money he bought goods which he sold for $1200. What should he be willing to do for the money borrowed?"

4. Discovering Species. Most terms have varieties. Thus, numbers are abstract or concrete, concrete numbers are denominate or not denominate, denominate numbers are simple or compound. We do not wish pupils to study these varieties ready formed and to memorize set definitions, but we wish them to find the varieties for themselves and to make definitions in terms of the development. The process of discovering the species of a genus is **logical division.**

Fix upon a basis of classification, find the differences of this basis, unite the genus and the differences, name the resulting species, define the species.

Basis. The basis of classification is determined by some need. Thus, we may wish to find what kinds of triangles there may be with reference to the relative lengths of the sides.

Differences. The differences of a basis of classification may be found by experiment or by the law, a thing must be or must not be. Thus, to find the differences in relative lengths of three lines we may draw several sets of three lines and examine all possible cases. Or we may reason, Three are equal or not three equal; of three not equal, two are equal or not two equal. This gives rise to three equal, two equal, or no two equal.

Names. Names are common words selected for their appropriateness, or unusual words from foreign languages. The pupil is helped to fix a name in mind by an explanation of its origin provided the explanation is within his comprehension. Thus, when no two sides of a triangle are equal, the sides if placed parallel are like the rungs of a ladder. The triangle is called **scalene** from the Latin for ladder.

Definitions. To define a term logically is to state its genus and differences. There are as many differences as there have been bases of classification. The proximate genus should be used when its meaning is understood because there is then only one difference. Thus, from the proximate genus, A scalene triangle is a *triangle* having no two of its sides equal; from a remote genus, A scalene triangle is a *polygon* having three sides and having no two of its sides equal.

Teaching. It would never do to speak to children about bases of classification, differences of the basis, genus, and species. The formal steps are to be in the mind of the teacher as a guide, the thought being that the teacher with such a guide will do a better piece of work than without it.

ILL. *Triangles.* T. "Let us find the different kinds of triangles with reference to the relative lengths of the sides.

"Draw a triangle with all its sides equal. Draw some other kind. John has a triangle with two sides equal, and James has a triangle with no two sides equal. Is any other kind possible? Draw a triangle with three sides equal and name it **equilateral** triangle, draw a triangle with two sides equal and name it **isosceles** triangle, draw a triangle with no two sides equal and name it **scalene** triangle. Define each kind."

Triangle
- Equilateral
- Isosceles
- Scalene

ILL. *Fractions.* T. "Let us find the different kinds of fractions with reference to the relative value of numerator and denominator.

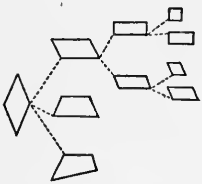
"Write a fraction. Mary, how does the numerator of your fraction compare with its denominator? The numerator is less than the denominator. This is true of every fraction that has been written. Write some other kind. John has a fraction whose numerator is equal to its denominator, and Henry has a fraction whose numerator is greater than its denominator. See if you can find some other kind. No one succeeds. We will call such fractions as $\frac{2}{3}$ **proper.** The name is appropriate because we can separate a unit into 3 equal parts and take 2 of them. Define a proper fraction. We will call such fractions as $\frac{4}{3}$ **improper.** After we have separated a unit into 3 equal parts we cannot take 4 of them. Hence, $\frac{4}{3}$ is not properly a fraction. We will call such fractions as $\frac{3}{3}$ **improper.** After we have separated a unit into 3 equal parts we can take the 3 parts, but 'fraction' means a part and '$\frac{3}{3}$' is a whole. Hence $\frac{3}{3}$ is appropriately called an improper fraction. Define an improper fraction. An improper fraction is a fraction whose numerator is equal to or greater than its denominator."

Fraction
- Nu. less than den.
- Nu. equal to den.
- Nu. greater than den.

Fraction
- Proper
- Improper
- Improper

5. Diagrams. At the end of a classification, a diagram should be made to relate the genus and the species. The diagram may emphasize the differences, the finished products, or the names. Following are diagrams illustrating the classification of quadrilaterals:

- Quad.
  - Both pair sides ∥
    - All angles right
      - Adjacent sides equal
      - Adj. sides not equal
    - Not all right
      - Adjacent sides equal
      - Adj. sides not equal
  - One pair sides ∥
  - Neither pair sides ∥



- Quad.
  - Parallelogram
    - Rectangle
      - Square
      - Oblong
    - Rhomboid
      - Rhombus
      - No name
  - Trapezoid
  - Trapezium

6. Exercises. **1.** Cause pupils to feel the need of addition. **2.** Cause pupils to feel the need of bills. **3.** Help pupils to classify integers with reference to divisibility by 2. **4.** Help pupils to classify number with reference to the expression of the unit. **5.** Explain the appropriateness of *equilateral* as the name of a triangle whose sides are all equal. **6.** Criticise this definition: An equilateral triangle is a three-sided triangle having its sides all equal. **7.** Criticise this definition: A triangle is one which has three sides. **8.** Teach the classification of quadrilaterals. **9.** Define square, using as genus: (*a*) rectangle; (*b*) quadrilateral.

LESSON 2. CASES — LOGICAL DIVISION

7. Discovering Cases. Different types of problems may involve the same terms. Thus in interest, the principal, time, and rate may be given to find the interest; the principal, time, and interest may be given to find the rate; and so on. Pupils should discover the different cases for themselves. The cases are not usually named.

ILL. *Cost of 1, En. Cost, Number.* T. "At 3¢ each the cost of 5 apples is 15¢. Let us find the different problems that can be formed by the omission of each term in succession.

"State the problem which arises from the omission of 15¢. At 3¢ each what is the cost of 5 apples? State the problem which arises from the omission of 3¢. If the cost of 5 apples is 15¢, what is the cost of 1 apple? State the problem which arises from the omission of 5 apples. At 3¢ each how many apples can be bought for 15¢?"

ILL. *Whole, Fraction of Whole, Part.* T. "Let us find the different problems which arise from $\frac{2}{5}$ of $20 = 8$ by the omission of each term in succession.

"State the problem which results from the omission of 8. *What is $\frac{2}{5}$ of 20?* State the problem which results from the omission of 20. *8 is $\frac{2}{5}$ of what number?* State the problem which results from the omission of $\frac{2}{5}$. *8 is what part of 20?* State these problems in general form."

1. To find a fractional part of a number.
2. To find a number when a fractional part is given.
3. To find what fractional part one number is of another.

The teacher should carry such developments farther than he proposes to carry them with pupils, for the sake of widening his own horizon, for he should know more than he attempts to teach.

Ill. *Cases in Percentage.* Let us find all the different cases in percentage from the formulæ, $P = B \times R$, $A = B + P$, and $D = B - P$.

There are three equations with five quantities. To solve these equations two of the quantities must be known, because it is impossible to solve three equations with more than three unknown quantities. Hence, there will be three cases in percentage for every two known terms.

The combinations of twos in the terms A, B, D, P, R are: AB, AD, AP, AR; BD, BP, BR; DP, DR; PR. That is, there are 10 times 3, or 30, cases in percentage. They are:

Given	*To Find*	*Given*	*To Find*
A and B;	D, P, R	B and P;	A, D, R
A and D;	B, P, R	B and R;	A, D, P
A and P;	B, D, R	D and P;	A, B, R
A and R;	B, D, P	D and R;	A, B, P
B and D;	A, P, R	P and R;	A, B, D

The teacher should form a concrete problem for each case grouped around some industry or activity.

Ill. *Sheep.* Let us take the activity of buying and selling sheep. A represents the number of sheep after a purchase or 53; B, the original number or 50; D, the number after a sale or 47; P, the number bought or sold or 3; R, the per cent of the original number or 6%.

Case 1. Given A and B to find D. A man had 50 sheep; after purchasing a certain number he had 53. If he had sold as many as he bought, how many would he have had left?

Case 6. Given A and D to find R. After purchasing a number of sheep a man had 53; if he had sold as many as he purchased, he would have had 47 left. What per cent of the original number did he purchase?

Case 10. Given A and R to find B. After purchasing a number of sheep a man had 53, which was 6% more than the original number. What was the original number?

8. Progressive Difficulty. The teacher must classify the examples under each subject in the order of their difficulty that the pupils may advance in the line of least resistance. He may need several bases of classification.

Ill. *Subtraction of Mixed Numbers.* The difficulty depends upon the sameness of the denominators, and upon the relative value of the fractions in the minuend and subtrahend. Classifying according to these bases, we obtain: denominators the same and fraction in the minuend the greater, denominators the same and fraction in the minuend the smaller; denominators different and fraction in the minuend the greater, denominators different and fraction in the minuend the smaller.

Thus: $8\frac{3}{4} - 5\frac{1}{4}$, $8\frac{1}{4} - 5\frac{3}{4}$; $8\frac{3}{4} - 5\frac{2}{3}$, $8\frac{2}{3} - 5\frac{3}{4}$.

9. Measurements. The teacher should have in mind the steps in all measurement as a guide to skilful presentation.

To measure an object, assert that it possesses as much of a quality as a well-known standard. Abbreviate the concept by a name. Thus, the act requires as much time as the rotation of the earth about its axis, or a **day**.

If the object possesses as much of the quality as the standard, as much more, as much more, and so on, introduce number. Thus, the act requires as much time as **two** rotations, or two days.

If the measurement requires a large number of repetitions of the standard, select a larger standard. Thus, the act requires as much time as the revolution of the earth about the sun, or a **year**.

If the object possesses less of the quality than the standard, select a smaller standard. Thus, the act requires as much time as the 24th of a day, or an **hour**.

If necessary, use two or more standards. Thus, the act requires 5 years 2 days and 3 hours.

Ill. *Length.* T. "How long is that line on the board? About so long (the hands are held apart). How far from your desk to the door? 16 steps (the pupil counts his steps).

"It is impossible to hold the hands the exact distance apart; steps are not all of the same length. Here is a rule which is as long as a certain king's foot; it is called a **foot**. Mary may measure the line; it is 2 feet long.

"Here is a rule 3 feet long; it is called a **yard**; it is more convenient for measuring long distances than a foot rule; it is used in measuring cloth. John, measure the long line; it is 2 yards long.

"With the foot rule Henry may measure the short line; it is 1 foot and a little more. To measure this little more, what must we have? Yes, a still shorter rule. This foot rule has been divided into 12 equal parts, and each part is called an **inch**. Measure the line again, Henry; it is 1 foot 4 inches. Who will give me the table? 12 inches make 1 foot, 3 feet make 1 yard."

10. Logical Steps. Preparatory to many exercises, the teacher must discover the steps which are taken in common practice to reach the desired end.

Perform the exercise and analyze the steps.

Ill. *Multiplication.* Let us discover the logical steps in the multiplication of an integer by a number of two orders.

In multiplying 264 by 24, we multiplied 264 by 4, we multiplied 264 by 20, and added the results. The logical steps are to multiply by the number in units' order, to multiply by the number in tens' order and the result by 10, and to add the products.

11. Exercises. 1. Teach pupils to find the problems which grow out of the statement, The interest of $200 for 2 years at 6% is $24. **2.** From the simple interest formulæ, $I = P \times T \times R$ *and* $A = P + I$, find the 20 cases. **3.** Why are the cases API to find T and R, impossible? **4.** The following divisors are to be used in long division, 71 and 17. (*a*) Arrange them in order of difficulty; (*b*) give reasons for your arrangement. **5.** Teach quart, pint, gallon, in liquid measure. **6.** State the logical steps in the addition of fractions.

LESSON 3. One-Line Diagrams

12. One-Line Diagrams. To get clear notions of the various operations upon fractions pupils should be taught to make and use diagrams for themselves. Diagrams made by teachers do little good to pupils. One-line diagrams are based upon separating a line into a number of equal parts and then into another number of equal parts in such a way as to show a common measure.

Ill. T. "We wish to separate a line into 8 equal parts and then into 6 equal parts so as to show a common measure.

"How shall we proceed? Separate the line by short vertical lines into 8 equal parts.

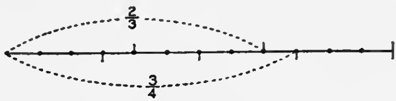
"Then what? The least number which exactly contains 8 and 6 is 24. Separate the line by dots into 24 equal parts. How can we do this? The line is already separated into 8 equal parts and we can separate each small portion into 24 ÷ 8 or 3 equal parts. Do so.

"Now what? We want to separate the line into 6 equal parts. How many dots to a part? 24 ÷ 6 or 4. Draw a short vertical line above at every 4th dot.

"What does each vertical line below show? $\frac{1}{8}$. What does each dot show? $\frac{1}{24}$. What does each vertical line above show? $\frac{1}{6}$. What is a common measure of $\frac{1}{8}$ and $\frac{1}{6}$? $\frac{1}{24}$. How many 24ths make 1 8th? 3. How many 24ths make 1 6th? 4."

Designating Parts. Designate a required portion by placing its value in the center of a dotted line which connects the extremities of the portion.

Ill. T. "Let us draw a diagram to represent $\frac{3}{4}$ and $\frac{2}{3}$. How shall we proceed? Divide a line into 4 equal parts and then into 3 equal parts. Mark off $\frac{3}{4}$ and $\frac{2}{3}$ as I have done."



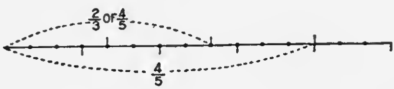
Uses. Pupils should solve examples in addition, subtraction, multiplication, and division of fractions as a preparation for solutions by rules.

Ill. 1. T. "You have just made a diagram to represent $\frac{3}{4}$ and $\frac{2}{3}$. Find the value of $\frac{3}{4} + \frac{2}{3}$. The value is $\frac{9}{12} + \frac{8}{12}$ or $1\frac{5}{12}$.

"Find the value of $\frac{3}{4} - \frac{2}{3}$. The value is $\frac{9}{12} - \frac{8}{12}$ or $\frac{1}{12}$.

"Find the value of $\frac{2}{3} \div \frac{3}{4}$. The value is 8 12ths ÷ 9 12ths or $\frac{8}{9}$."

Ill. 2. T. "Let us find $\frac{2}{3}$ of $\frac{4}{5}$. What must we do? Represent $\frac{4}{5}$, divide $\frac{4}{5}$ into 3 equal parts, and take 2 parts.



"$\frac{2}{3}$ of $\frac{4}{5}$ is 8 of the 15 equal parts into which the unit is divided or $\frac{2}{3}$ of $\frac{4}{5}$ is $\frac{8}{15}$."

13. Mechanical Aids. Pupils should be taught to use mechanical devices especially in mensuration.

Paper Folding and Cutting. Let us agree that the pupil shall do the work.

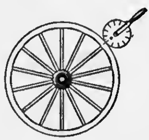
Ill. *Triangles.* T. "What is the sum of the angles of a triangle?

"How shall we proceed? Let us cut a triangle from paper and fold in such a way as to bring the vertices together at the same point. Fold the upper vertex over upon the base so as to make the folded edge parallel to the base. Fold the other parts. What seems to be the sum of the angles of a triangle? The sum of the angles of a triangle seems to be the sum of the angles about a point on one side of a straight line or 180°."

Ill. *Parallelograms.* T. "We want a rule for finding the area of a parallelogram. We know how to find the area of a rectangle. Let us find how the area of a parallelogram compares with the area of a rectangle. How shall we make this comparison? Cut a parallelogram from paper. Starting from one of the vertices cut off a right-angle triangle. Put the triangle first at one end of the second portion and then at the other. What seems to be true? The area of a parallelogram seems to be the same as the area of a rectangle which has the same base and altitude."

Crude Measurements. Some things are hard to measure, as the circumference of a circle or the surface of a sphere. Pupils should be asked to exercise their ingenuity on such measurements.

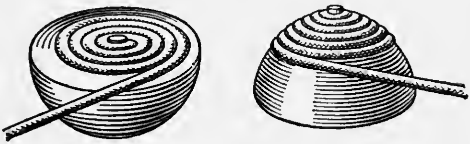
Ill. *Circumference of a Circle.* T. "Here is a circle on the board. How shall we find its circumference? Sometimes the blacksmith wants to cut off a strip of steel long enough to make the tire of a wagon wheel. Do you know how he finds the proper length? He has a rule made in the form of a circle, with a handle. He finds how many times this small wheel turns around in moving about the circumference of the wagon wheel.



"Can we use such a wheel to measure the circumference of this circle on the board? What shall we use? I have a piece of electric wire here. How can we use it? John may lay the wire on the circumference and then measure the wire. What is the length of the circumference? 44 in.

"Let us see if we can find an easier way. Measure the diameter of the circle. It is 14 in. Divide 44 in. by 14 in. The answer is $3\frac{1}{7}$. What seems to be the circumference of a circle? The product of its diameter by $3\frac{1}{7}$."

Ill. *Surface of a Sphere.* T. "How shall we measure the surface of this sphere?



"I have here a piece of waxed cord and the half of a croquet ball (a hemisphere) into which I have driven two tacks. I propose to wrap the cord about one tack until the string covers the curved surface, and then about the other tack until the cord covers the plane surface, and then to compare the lengths. Henry may do this. The cord about the curved surface is twice the length of that about the plane surface. What does this seem to show? The surface of a hemisphere seems to be twice the surface of a circle which has the same diameter, or the surface of a sphere seems to be 4 times the surface of a circle which has the same diameter."

14. Exercises. **1.** Divide a line into 4 equal parts and then into 8 equal parts. **2.** Represent $\frac{4}{5}$ and $\frac{3}{4}$. **3.** By diagram find: (*a*) $\frac{4}{5}+\frac{3}{4}$; (*b*) $\frac{4}{5}-\frac{3}{4}$; (*c*) $\frac{4}{5}\div\frac{3}{4}$. **4.** By diagram find $\frac{4}{5}$ of $\frac{2}{3}$. **5.** By paper cutting find the relation of a triangle to a parallelogram which has the same base and altitude. **6.** Cut a triangle from paper and fold as in finding the sum of its angles. Show that the area of a triangle seems to be the area of a rectangle which has the same base and half the same altitude.

LESSON 4. PRINCIPLES — INDUCTION

15. Inductive Method. There are many principles in arithmetic which must be established as guides to methods of procedure. It is better for the pupil to develop these for himself than to study them ready formulated by another. The inductive method is the process of establishing principles by experiment. Five canons of induction are discussed in logic, but only the canon of agreement will be considered in this treatise. *See logic* (J. & H., p. 215).

16. Canon of Agreement. Whatever is true of several individuals of a class is probably true of all the individuals of that class. Thus, by experiment it is found of several examples in subtraction that adding the same number to both minuend and subtrahend does not affect the remainder; this principle is probably true of all examples in subtraction. When a proposition is established in regard to individuals by experiment alone, the inference in regard to the class can never be more than probable unless every individual in the class has been examined. The greater the number of individuals the greater the probability.

Take several instances in which the phenomenon occurs, examine these instances for a common circumstance, and make the common circumstance the basis of a generalization.

ILL. T. "We wish to discover a rule for the divisibility of a number by 9.

"Let us examine several numbers which are exactly divisible by 9. To get such numbers those who sit in the first row may multiply a number of three figures by 9; those in the second row a number of 4 figures; all others, a number of 5 figures. Read some of the products: 2034, 3861, 23895, 808884. Do you find anything which these numbers have in common? Get the sums of the digits — 9, 18, 27, 36. Now do you find anything in common? The sum of the digits is exactly divisible by 9. What seems to be the rule for the divisibility of a number by 9? A number seems to be divisible by 9 if the sum of its digits is divisible by 9. Memorize this rule."

Weakened Form. An inference from a single instance can have little weight of itself, but is of value in a few cases where its establishment by other methods is too difficult for the grade. For examples, see § 13. It plays an important part also in the complete method (§ 20). Its use elsewhere is not recommended. Observe the weakness of the following:

Ill. T. "Find by diagram $\frac{2}{3}$ of $\frac{4}{5}$. The value is $\frac{8}{15}$ (§ 12). What seems to be a rule for multiplying fractions? To multiply fractions multiply the numerators for a new numerator, and the denominators for a new denominator."

Use in Arithmetic. Except as a part of the complete method, the inductive method should be rarely used in arithmetic. An examination of enough instances to warrant a conclusion is long and laborious; no appeal is made to the intelligence; an experimental method is not suited to an exact science; and at the best, the conclusion can never be more than probable. Used by itself, it is valuable for finding rules for divisibility in the lower grades, and for finding a few rules for mensuration in the upper grades, because in both cases the pupils are not sufficiently mature to use more satisfactory methods.

17. Deductive Method. The deductive method is the method of establishing principles by giving reasons for the steps. Several forms of deduction are discussed in logic but only the form, *A is B*, *B is C*, ∴ *A is C*, will be considered in this treatise. *See logic* (J. & H., p. 145).

18. A is B, etc. Whatever is true of a term is true of what is included within that term or of what is identical with that term. The argument may take the form *A is B*, *B is C*, ∴ *A is C*, or the form of a chain of such arguments, *A is B*, *B is C*, *C is D*, *D is E*, ∴ *A is E*. Each premise must be established by a definition, an axiom, or a proposition previously proved.

Ill. Multiplying both numerator and denominator of a fraction by the same number first multiplies the fraction by a number and then divides the result by the same number, because multiplying the numerator multiplies a fraction and multiplying the denominator divides a fraction (*A is B*).

Multiplying a fraction by a number and dividing the result by the same number does not change the value of a fraction by axiom (*B is C*).

Therefore, multiplying both numerator and denominator of a fraction by the same number does not change the value of a fraction (*A is C*).

Make some predication about the subject of the required proposition based upon a definition, an axiom, or a proposition already proved. This gives the form, *A is B*.

Make some predication about the predicate of the last proposition based upon a definition, an axiom, or a proposition already proved. This gives the form, *B is C*.

Continue as before until a serviceable predicate is found, and unite it with the subject of the first proposition. This may give the form, *A is E*.

Individual Subject. In teaching, it is better to use an individual than a general term because the process is then more vivid. Thus, 'multiplying the numerator and denominator of $\frac{3}{4}$ by 2' is more vivid than 'multiplying both numerator and denominator of a fraction by the same number.' Whatever is proved of the individual in this way is proved of the whole class which includes that individual because the principle could be proved of every other individual of the class in the same way.

ILL. *Multiplication of Decimals.* T. "We are going to learn how to multiply a decimal by a decimal. We know how to multiply a decimal by an integer and how to multiply by .1, .01, and so on.

$$\begin{array}{r} .24 \\ .6 \\ \hline .144 \end{array}$$

"Take .24 × .6. What is to multiply by .6? To multiply by 6 and to multiply the result by .1 because .6 is 6 × .1.

"You may all multiply by 6; the answer is 1.44; to multiply a decimal by an integer multiply as in integers and point off as many decimal places in the product as there are decimal places in the multiplicand. Multiply the result by .1; the answer is .1$_\times$44; to multiply by .1, .01, .001, and so on, move the decimal point as many places to the left as there are decimal places in the multiplier.

"What is to multiply .24 by .6 or to multiply a decimal by a decimal? Multiply as in integers and point off as many decimal places in the product as there are decimal places in both multiplicand and multiplier."

19. Exercises. **1.** Using the method of agreement, help pupils to find a rule for the divisibility of a number by 4. **2.** Determine whether the formula, $x^2 + x + 41 = a\ prime$, is true for all integral values of x. *Suggestion.* For $x = 0, 1, 2, 3, 4, 5, 6$, the values are 41, 43, 47, 53, 61, 71, 83, respectively. **3.** Using the form, *A is B, B is C*, and so on, assist pupils to find how to multiply integers by a number of two orders. **4.** Why is the deductive method better than the inductive method for establishing the principles of arithmetic?

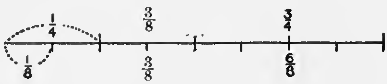
LESSON 5. Principles — Complete Method

20. The Complete Method. The race has established many important principles in mathematics by inferring some property from an examination of one or more individuals and by discovering why the property must be true of all the individuals of the class. Thus, the race found by measurement that of a right-angle triangle whose legs are 3 in. and 4 in. the hypotenuse is 5 in. From this and other measurements they inferred that the square of the hypotenuse of every right-angle triangle must be the sum of the squares of the other two sides. By the deductive methods of geometry they proved that their surmise was correct. The complete method consists in finding a principle by experiment (induction) and in proving it by reasons (deduction). It is the method of the discoverer. *See logic* (J. & H.. p. 249).

Find by experiment a proposition which is true of one or more individuals of a class and discover from definitions, axioms, or propositions already proved, why it must be true of the whole class.

Ill. *Mult. Terms of Fraction.* T. "We wish to discover the effect on a fraction of multiplying its numerator, multiplying its denominator, and multiplying both numerator and denominator by the same number.

"You may draw a diagram like mine showing $\frac{3}{8}$, $\frac{6}{8}$, and $\frac{3}{4}$.



"Let us discover the effect of *multiplying the numerator.* Take $\frac{3}{8}$ and multiply the numerator by 2; the result is $\frac{6}{8}$. From the diagram, tell me what has been done to the fraction; it has been multiplied by 2. What seems to be the effect of multiplying the numerator? To multiply the fraction. Who can tell me why? Multiplying the numerator multiplies the number of equal parts taken without affecting the size of the parts. Write the rule, multiplying the numerator multiplies the fraction.

"Let us discover the effect of *multiplying the denominator.* Take $\frac{3}{4}$ and multiply the denominator by 2; the result is $\frac{3}{8}$. From the diagram, tell me what has been done to the fraction; it has been divided by 2. What seems to be the effect of multiplying the denominator? To divide the fraction. Who can tell me why? What was the size of one of the equal parts before we multiplied the denominator? A fourth. After we multiplied the denominator? An eighth. What did we do to the size of one of the equal parts? Divided it by 2. Now who can tell why? Multiplying the denominator divides the size of the equal parts without affecting the number of parts taken. Write the rule, multiplying the denominator divides the fraction.

"Let us discover the effect of *multiplying both terms* by the same number. Take $\frac{3}{4}$ and multiply both terms by 2; the result is $\frac{6}{8}$. From the diagram, tell me what has been done to the fraction; its value has not been changed. What seems to be the effect on a fraction of multiplying both terms by the same number? It does not change the value of the fraction. Who can tell me why? When we multiplied the numerator by 2 what did we do to the fraction? We multiplied the fraction. When we multiplied the denominator of the result by 2 what did we do to the result? We divided the result by 2. If we multiply a fraction by 2 and divide the result by 2 what do we do to the fraction? Write the rule, multiplying both numerator and denominator of a fraction by the same number does not change the value of the fraction."

Brief Rules. After several rules of kindred nature have been established it is sometimes possible to make a brief rule which comprehends them all. A valuable illustration is found in the principles of multiplying just developed, together with the principles of division that may be developed in a similar manner.

ILL. T. "Multiplying the numerator multiplies the fraction, dividing the numerator divides the fraction. Who can express the two rules by one? In the case of multiplying and dividing, doing either thing to the numerator does the same thing to the fraction. Let us put it shorter. *Doing a thing to the numerator does the same thing to the fraction.*

"Multiplying the denominator divides the fraction, dividing the denominator multiplies the fraction. Give me a short rule for these two. *Doing a thing to the denominator does the opposite thing to the fraction.*

"Multiplying both numerator and denominator by the same number does not change the value of the fraction, dividing both numerator and denominator by the same number does not change the value of the fraction. Give me a short rule for these two. *Doing the same thing to both numerator and denominator does not change the value of the fraction.*"

Use in Arithmetic. The complete method should be used in developing nearly all the principles of arithmetic because it is the method of the discoverer. It discovers something by trial that *may be* true and proves by reasons that it *is* true.

ILL. *Multiplying Fractions.* T. "To-day we are going to learn how to multiply a fraction by a fraction. Take $\frac{2}{3}$ of $\frac{4}{5}$ and find the result by diagram; $\frac{8}{15}$. (*See diagram, p. 11.*)

"How can 8 be obtained from the numerators 2 and 4? How can 15 be obtained from the denominators 3 and 5? What seems to be the rule for multiplying fractions? Multiply the numerators for a new numerator and the denominators for a new denominator.

"Let us find why this rule is true. What does $\frac{2}{3}$ of $\frac{4}{5}$ mean? That $\frac{4}{5}$ is to be divided by 3 and the result multiplied by 2 because the denominator shows into how many equal parts a thing is to be divided, and the numerator shows how many equal parts are taken.

"Divide $\frac{4}{5}$ by 3; $\frac{4}{5 \times 3}$, because multiplying the denominator divides the fraction. Multiply the result by 2; $\frac{4 \times 2}{5 \times 3}$, because multiplying the numerator multiplies the fraction.

"Write, $\frac{2}{3}$ of $\frac{4}{5} = \frac{4 \times 2}{5 \times 3}$. Now we see why the rule is true.

"To multiply fractions, multiply the numerators for a new numerator, and the denominators for a new denominator. Write the rule."

ILL. *Dividing Fractions.* T. "We want a rule for dividing a fraction by a fraction.

"By diagram, divide $\frac{2}{3}$ by $\frac{3}{4}$; the result is $\frac{8}{9}$. (*See diagram, p. 11.*)

"How could we get $\frac{8}{9}$? By inverting the divisor and proceeding as in multiplication; $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$. What may possibly be a rule for dividing a fraction by a fraction? Invert the divisor and proceed as in multiplication.

"Let us see if we can find reasons for this rule. The other day we learned how to divide 1 by a fraction; invert the divisor. What is $1 \div \frac{3}{4}$? $\frac{4}{3}$. If '1 divided by $\frac{3}{4}$' is $\frac{4}{3}$, what is $\frac{2}{3}$ of '1 divided by $\frac{3}{4}$'? $\frac{2}{3}$ of $\frac{4}{3}$. The rule must be true. To divide a fraction by a fraction invert the divisor and proceed as in multiplication. Write the rule."

21. Exercises. **1.** Using the diagram, p. 18, teach pupils by the complete method to discover the effect on a fraction: (*a*) of dividing the numerator; (*b*) of dividing the denominator; (*c*) of dividing both numerator and denominator by the same number. **2.** Teach pupils by the complete method how to divide 1 by a fraction. **3.** Why is the complete method better for No. 1; (*a*) than the inductive method alone? (*b*) than the deductive method alone? **4.** What is the advantage of the brief rules suggested for the principles in fractions?

LESSON 6. Simple Problems — By Experiment

22. Kinds of Problems. An exercise involving number must state the operations directly or indirectly; this gives rise to examples and problems. A problem must involve one operation or more than one; this gives rise to simple problems and complex problems. A simple problem must involve addition, subtraction, multiplication, the first case in division known as quotition, or the second case in division known as partition.

Ill. *Example.* Multiply 3 ¢ by 5.

Ill. *Simple Problem.* At 3 ¢ each what is the cost of 5 apples?

Ill. *Complex Problem.* If 2 apples cost 6 ¢, how much do 5 apples cost?

23. Importance of Simple Problems. The solution of simple problems is the basis of the solution of all problems, because a complex problem can be separated into a chain of simple problems, and can be solved by solving each simple problem in succession.

24. Stages of Progress. In the advancement of the race there have been three stages; the stage of obtaining results by counting, the stage of obtaining results chiefly by addition, the stage of obtaining results by the most fitting operation. These three stages should be observed in the advancement of the child.

Ill. At 3 ¢ each what is the cost of 5 apples?

Counting Stage. Here are 5 apples, and 3 cents for each apple. They cost 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 cents.

Addition Stage. They cost 3, 6, 9, 12, 15 cents.

Advance Stage. They cost 5 times 3 cents, or 15 cents.

25. Counting Stage. As soon as pupils can count they have at command a means for solving the problems of their daily experience. They can represent the terms by objects and find the results by counting. This work exercises their ingenuity, gives them a feeling of power, and lays the foundation for all subsequent work.

Ill. *Simple Prob. in Add.* On each desk there is a bundle of sticks.

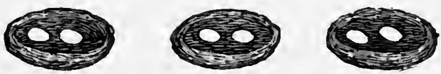
T. "If John has 3 apples and Joseph has 2 apples, how many apples have both? Use the sticks for apples and find out. Mary may explain."

M. "Here are John's apples, 1, 2, 3; here are Joseph's, 1, 2; both have 1, 2, 3, 4, 5 apples."

Ill. *Simple Prob. in Sub.* T. "Susan had 6 ¢ and spent 2 ¢. How many cents did she have left? Use the sticks for cents and find out. Henry may explain."

H. "Here are the 6 ¢ she had, 1, 2, 3, 4, 5, 6; she spent 1, 2 cents: she had left 1, 2, 3, 4 cents."

Ill. *Simple Prob. in Mult.* On each desk there are sticks and pieces of paper. T. "A boy found 3 nests; there were 2 eggs in each nest. How many eggs did he find? Use the pieces of paper for nests and the sticks for eggs. Susan may explain."

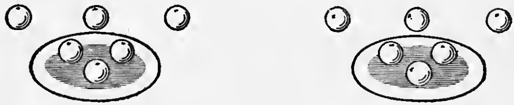


S. "Here are the three nests, 1, 2, 3; here are the 2 eggs in each nest. He found 1, 2, 3, 4, 5, 6 eggs."

Ill. *Simple Prob. in Quotition.* T. "A boy found 8 eggs; there were 2 eggs in each nest. How many nests were there? Use the pieces of paper for nests and the sticks for eggs. Joseph may explain."

J. "Here are the 8 eggs, 1, 2, 3, 4, 5, 6, 7, 8; these 2 eggs need a nest; these 2 need a nest; these 2 need a nest; these 2 need a nest; the eggs are all used; there are 1, 2, 3, 4 nests."

Ill. *Simple Prob. in Partition.* *T.* "A girl had 6 oranges and 2 plates; she put the same number of oranges on each plate. How many oranges did she put on each plate? Peter may explain."



P. "Here are the 6 oranges and the 2 plates; I put an orange on each plate; I put another orange on each plate; I put another orange on each plate; the oranges are all placed; there are 1, 2, 3 oranges on each plate."

26. Addition Stage. *Full Form.* As soon as pupils can add they have a still better means for solving the problems of their daily experience. They can represent the terms by objects and find the results by addition.

Ill. *S. P. in Add.* See § 25. Both have 3, 5 apples.

Ill. *S. P. in Mult.* See § 25. He found 2, 4, 6 eggs.

Ill. *S. P. in Quot.* See § 25. Here are 2, 4, 6, 8 eggs; they need 1, 2, 3, 4 nests.

Short Form. In solving such problems as are suggested on fruit stands, pupils may represent groups by objects instead of individuals.

Ill. T. "At 4 for 5¢ what is the cost of 12 oranges? Make a mark for each group of 4 oranges. What else will each mark represent? A group of 5 cents. Point to each mark and add by 5's. Ann may explain."

A. "There are 4, 8, 12 oranges; they cost 5, 10, 15 cents."

ILL. T. "At 3 for 5¢ how many oranges can be bought for 25¢? Shall we make a mark for each group of 3 oranges or for each group of 5¢? For each group of 5¢, because the number of cents in all is given. William may explain."

| | | | |

W. "There are 5, 10, 15, 20, 25 cents; they buy 3, 6, 9, 12, 15 oranges."

27. Advance Stage. In the advance stage, pupils select the operation best suited to the problem, dispensing with objects except in cases of perplexity. The impulse to use an operation comes as soon as the situation is grasped. If the situation can be imaged without objects no objects are necessary. If the situation cannot be imaged objects are imperative.

ILL. *Problems of Paragraph 25.* They both have 3 apples plus 2 apples or 5 apples. She had left 6¢ minus 2¢ or 4¢. The boy found 3 times 2 eggs or 6 eggs. The number of nests is the number of times 2 eggs is contained in 8 eggs or 4. On 1 plate the girl put $\frac{1}{2}$ of 6 oranges or 3 oranges.

28. Exercises. 1. State an example involving more than one operation. **2.** State a complex problem involving three operations. **3.** Separate the complex problem into its simple problems. **4.** State a simple problem in multiplication and solve it experimentally by counting objects. **5.** Solve the last problem experimentally by addition. **6.** Solve the last problem experimentally by the operation involved. **7.** State a fruit stand problem and solve it experimentally by representing the groups by objects.

LESSON 7. Simple Problems — By Reasons

29. Nature of Reasons. Reasons occur in pairs — the one to state a law and the other to state that the law applies. Thus, Smith is mortal because all men are mortal, and because he is a man. In the last lesson, simple problems are solved without statement of reasons. The situation is grasped either with or without the use of objects and the necessary operation is applied. Thus, Mary has 2¢ and wants to spend 6¢, how many cents does she lack? She lacks 6¢ minus 2¢ or 4¢.

Let us find the reasons. We subtract 2¢ from 6¢. Why do we subtract? Because what she lacks is what she must spend minus what she has. Why do we perform the operation on 6¢ and 2¢? Because what she must spend is 6¢ and what she has is 2¢.

30. Complete Analysis. The complete analysis states both reasons, why the operation is selected (the law), and why the terms are selected (why the law applies).

Ill. *S. P. Add.* If the cost is 6¢ and the gain is 2¢, what is the selling price?

Since the selling price is the cost plus the gain, and since the cost is 6¢ and the gain is 2¢, the selling price is 6¢ plus 2¢ or 8¢.

Ill. *S. P. Sub.* If the selling price is 10¢ and the gain is 3¢, what is the cost?

Since the cost is the selling price minus the gain, and since the selling price is 10¢ and the gain is 3¢, the cost is 10¢ minus 3¢ or 7¢.

Ill. *S. P. Mult.* If the cost of 1 apple is 3¢, what is the cost of 5 apples?

Since the cost of 5 apples is 5 times the cost of 1 apple, and since the cost of 1 apple is 3 ¢, the cost of 5 apples is 5 times 3 ¢ or 15 ¢.

Ill. *S. P. Quot.* If the cost of 1 apple is 3 ¢, how many apples can be bought for 15 ¢?

Since the number of apples for 15 ¢ is the number of times the cost of 1 apple is contained in 15 ¢, and since the cost of 1 apple is 3 ¢, the number of apples for 15 ¢ is the number of times 3 ¢ is contained in 15 ¢ or 5.

Ill. *S. P. Part.* If the cost of 5 apples is 15 ¢, what is the cost of 1 apple?

Since the cost of 1 apple is $\frac{1}{5}$ the cost of 5 apples, and since the cost of 5 apples is 15 ¢, the cost of 1 apple is $\frac{1}{5}$ of 15 ¢ or 3 ¢.

31. Major Analysis. Usually only one of a pair of reasons is stated. Thus, Smith is mortal because all men are mortal, or Smith is mortal because he is a man. The major analysis retains the reason for the selection of the operation.

Ill. *Add.* Since the selling price is the cost plus the gain, the selling price is 6 ¢ plus 2 ¢ or 8 ¢.

Ill. *Sub.* Since the cost is the selling price minus the gain, the cost is 10 ¢ minus 3 ¢ or 7 ¢.

Ill. *Mult.* Since the cost of 5 apples is 5 times the cost of 1 apple, the cost of 5 apples is 5 times 3 ¢ or 15 ¢.

Ill. *Quot.* Since the number of apples for 15 ¢ is the number of times the cost of 1 apple is contained in 15 ¢, the number of apples for 15 ¢ is the number of times 3 ¢ is contained in 15 ¢ or 5.

Ill. *Part.* Since the cost of 1 apple is $\frac{1}{5}$ the cost of 5 apples, the cost of 1 apple is $\frac{1}{5}$ of 15 ¢ or 3 ¢.

32. Minor Analysis. The minor analysis, often called the model analysis, retains the reason for the selection of the terms.

Ill. *Add.* Since the cost is 6 ¢ and the gain is 2 ¢, the selling price is 6 ¢ plus 2 ¢ or 8 ¢.

Ill. *Sub.* Since the selling price is 10 ¢ and the gain is 3 ¢, the cost is 10 ¢ minus 3 ¢ or 7 ¢.

Ill. *Mult.* Since the cost of 1 apple is 3 ¢, the cost of 5 apples is 5 times 3 ¢ or 15 ¢.

Ill. *Quot.* Since the cost of 1 apple is 3 ¢, the number of apples for 15 ¢ is the number of times 3 ¢ is contained in 15 ¢ or 5.

Ill. *Part.* Since the cost of 5 apples is 15 ¢, the cost of 1 apple is $\frac{1}{5}$ of 15 ¢ or 3 ¢.

33. Use in Arithmetic. The solution of a simple problem consists in grasping the situation and in applying the necessary operation. The office of analysis is to provide forms of expression after the plan has been formed. Thus, we discover that Mary lacks 6 ¢ minus 2 ¢ as soon as we realize that she has 2 ¢ and wants to spend 6 ¢. Some form of the reason may be in the mind subconsciously, but its statement comes after the discovery of how to proceed. Hence, it is a mistake to teach forms of analysis as aids to the solution of simple problems. The true use of analysis is to help pupils to give clear forms of explanation after the solution is understood.

The model analysis seems to be positively harmful. Since it gives the reason for the selection of the terms without indicating the operation, it draws attention away from the situation and fixes it upon a form of words.

Ill. *Form of Mult.* Since 1 apple costs 6 ¢, 2 apples cost 2 times 6 ¢.

Misled by similarity of statement, the pupil who looks for aid to model analysis instead of to situations, may be expected to say, "Since 1 man requires 6 days for a work, 2 men require 2 times 6 days or 12 days. Since a dog standing on 1 leg weighs 6 lb., standing on 2 legs he weighs 2 times 6 lb. or 12 lb. Since 1 boy gets up at 6 o'clock, 2 boys get up at 2 times 6 o'clock or 12 o'clock."

34. Teaching. After pupils reach the reason stage, it is recommended that they be required usually to give answers without any form of explanation, that they be asked occasionally for an explanation, and that in case of failure they be directed to study the situation.

The Usual. The pupil speaks or writes the answer only; all thought processes are unspoken and unwritten.

ILL. T. "If the selling price is 8¢ and the loss is 2¢, what is the cost?" P. "10¢."

T. "If a boy requires 1 hr. to walk 2 mi., how many hours will he require to walk 6 mi.?" P. "3 hr."

The Occasional. The pupil states the answer, and, when asked for an explanation, gives the conclusion of the complete analysis. If required, he states also the reason for the selection of the operation.

ILL. T. "If a boy earns $3 a week, in how many weeks will he earn $12?" P. "4." T. "Explain." P. "He will earn $12 in as many weeks as the times $3 is contained in $12 or 4." T. "Why?" P. "Because for each time he earns $3 there will be 1 week."

In Case of Failure. The pupil is directed to study the situation either with or without objects.

ILL. T. "How many barrels are required for 12 bu. of apples if 1 bbl. holds 3 bu.?" P. "I cannot tell." T. "Take bits of paper for barrels and sticks for bushels and find out." P. "4 bbl." T. "Why?" P. "For each 3 bu. there must be 1 bbl. There must be as many barrels as the times 3 bu. is contained in 12 bu. or 4."

T. "If a sum amounts to $1.06 in 1 yr. at simple interest, how much will it amount to in 2 yr.?" P. "$2.12." T. "State the parts which make up $1.06." P. "The principal and the interest for 1 yr." T. "Should both these parts be multiplied by 2? Try again." P. "The amount for 2 yr. cannot be found."

35. Exercises. **1**. State a simple problem not stated in the lesson of each of the five types and give the complete analysis. **2**. State the major analysis of each. **3**. State the minor analysis of each. **4**. Help a pupil who fails on, If 2 men require 6 da. for a work, how many days does 1 man require? **5**. Why should pupils be required *usually* to give answers to simple problems without explanations?

LESSON 8. Complex Problems — by Arithmetic

36. Into Simple Problems. A complex problem is a problem that involves more than one operation (§ 22). It can be separated into a chain of simple problems such that the answer to one becomes a given term in a second, the answer to a second becomes a given term in a third, and so on.

To solve a complex problem arithmetically, state a simple problem and its answer, state a second simple problem and its answer, and so on until the answer to the last simple problem is the answer to the complex problem.

Ill. A boy bought 10 lb. of pop corn at 12 ¢ a pound and sold it at 25 ¢ a pound. What was his entire gain?

If the cost of 1 lb. was 12 ¢ and the selling price 25 ¢, what was the gain on 1 lb.? 13 ¢. If the gain on 1 lb. was 13 ¢, what was the gain on 10 lb.? $1.30.

37. Finding Component Problems. The chief work in the solution of a complex problem is to find the component problems. It must be done by a study of the situation. There are two principal aids.

Analytic Aid. The value of the term required in the complex problem can be found from one or more terms by a single operation. Thus, the entire gain can be found from the gain on 1 lb. and the number of pounds. The value of each unknown term thus used can be found from one or more terms by a single operation, and so on. Thus, the gain on 1 lb. can be found from the cost of 1 lb. and the selling price of 1 lb. As soon as the values of all the terms are known the simple problems can be formed.

Consider from what terms the required term can be found by a single operation, consider from what terms each unknown term thus used can be found by a single operation, and so on. State the simple problems.

Make a diagram by joining each term to its dependent terms.

En. gain < Gain on 1 lb. < Cost 1 lb., 12 ¢ ; S.P. 1 lb., 25 ¢
En. gain < No. lbs., 10

The above may be read, The entire gain can be found from the gain on 1 lb. and the no. of pounds; the gain on 1 lb. can be found from the cost of 1 lb. and the selling price of 1 lb.

Ill. 1. T. "If the cost of 3 apples is 6 ¢, what is the cost of 5 apples?

"From what can we find the cost of 5 apples? From the cost of 1 apple. From what can we find the cost of 1 apple? From the cost of 3 apples.

Cost 5 ap. < Cost of 1 ap. < Cost of 3 ap., 6 ¢.

"Solve. If the cost of 3 apples is 6 ¢, what is the cost of 1 apple? 2 ¢. If the cost of 1 apple is 2 ¢, what is the cost of 5 apples? 10 ¢."

Ill. 2. T. "If A requires 2 da. for a work and B requires 3 da., how many days do both require?

"From what can we find the no. of days both require? From the part both can do in 1 da. From what can we find the part both can do in 1 da.? From the part A can do in 1 da. and the part B can do in 1 da. (and so on).

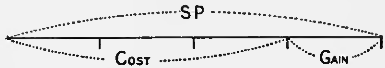
Days both < Part both 1 da. < Part A 1 da. < Days A, 2
Days both < Part both 1 da. < Part B 1 da. < Days B. 3

"Solve. If A requires 2 da., what part can he do in 1 da.? $\frac{1}{2}$. If B requires 3 da., what part can he do in 1 da.? $\frac{1}{3}$. If A can do $\frac{1}{2}$ of it in 1 da. and B can do $\frac{1}{3}$ of it in 1 da., what part can they both do in 1 da.? $\frac{5}{6}$. If both can do $\frac{5}{6}$ of it in 1 da., in how many days can they do $\frac{6}{6}$ of it? $1\frac{1}{5}$ da."

Graphic Aid. The situation can often be studied to advantage from a drawing. *See § 12.*

ILL. 3. T. "An article sold for \$8 at a gain of $\frac{1}{3}$, what was the cost?

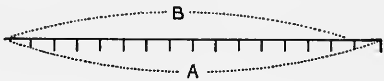
"'Sold at a gain of $\frac{1}{3}$' means 'the selling price is the cost plus $\frac{1}{3}$ the cost.' Draw a diagram to represent the terms.



"Solve. If the gain is $\frac{1}{3}$ the cost, what is the selling price? $\frac{4}{3}$ the cost. If $\frac{4}{3}$ the cost is \$8, what is $\frac{1}{3}$ the cost? \$2. If $\frac{1}{3}$ the cost is \$2, what is $\frac{3}{3}$ the cost? \$6."

ILL. 4. T. "A is $6\frac{2}{3}$% taller than B. B is how many per cent shorter than A?"

"What does this mean? A's height is B's height plus $\frac{1}{15}$ B's height. B's shortage is what part of A's height? Draw a diagram to represent the terms.



"Solve. If B's shortage is $\frac{1}{15}$ B's height and A's height is $\frac{16}{15}$ B's height, what part of A's height is B's shortage? $\frac{1}{16}$ or $6\frac{1}{4}$%."

When drawings are used, fractions may usually be omitted in the solutions.

Thus: In ILL. 3, If 4 parts are \$8, how much is 1 part? \$2. If 1 part is \$2, how much are 3 parts? \$6.

38. Into Complex Problems. While a complex problem can always be separated into simple problems, it may sometimes be separated to advantage into complex problems.

ILL. At what rate will \$200 gain \$24 in 2 yr. at simple interest?
What is the interest of \$200 for 2 yr. at 1%? \$4. If the interest is \$4 at 1%, at how many per cent is the interest \$24? 6.

39. Abbreviated Solutions. After pupils understand that complex problems must be separated into component problems, they may cast each component problem and its answer into a statement more or less abbreviated instead of into question and answer.

Ill. If the cost of 1 lb. is 12 ¢, and the selling price is 25 ¢, the gain on 1 lb. is 13 ¢. If the gain on 1 lb. is 13 ¢, the gain on 10 lb. is $ 1.30. Or, still shorter, the gain on 1 lb. is 13 ¢, the gain on 10 lb. is $ 1.30.

It is often best in problems involving only multiplication and division to express the answers to component problems without performing the operation.

Ill. What is the simple interest of $ 720 for 157 days at 7 %? Use cancellation method.

P, $ 720		1.57
T, 157 da.		14
R, 7 %	$\overset{2}{\cancel{720}} \times \frac{7}{100} \times \frac{1}{\cancel{360}} \times 157$	628
I, ?		157
I, $ 21.98 *ans.*		21.98

Multiplying $ 720 by $\frac{7}{100}$ gives the interest for 1 yr. at 7 %; dividing by 360, for 1 da.; multiplying by 157, for 157 da.

40. Model Analysis. The model analysis of a complex problem consists of the model analyses of the simple problems into which the complex problems may be separated. Its use is not recommended because of verbiage.

Ill. Since the cost of 2 apples is 6 ¢, the cost of 1 apple is $\frac{1}{2}$ of 6 ¢ or 3 ¢. Since the cost of 1 apple is 3 ¢, the cost of 5 apples is 5 times 3 ¢ or 15 ¢.

41. Exercises. **1.** A, B, and C eat 8 loaves of bread, each the same amount; A furnishes 3 loaves and B 5 loaves; C pays 24 ¢ for what he eats. How much should A receive? Make the analytic diagram. **2.** State and solve the simple problems. **3.** Give the model analysis. **4.** A buys a chair and a table for $ 35; the cost of the chair is $\frac{3}{4}$ of the cost of the table. What is the cost of each? Solve by the aid of a drawing: (*a*) using fractions; (*b*) not using fractions.

LESSON 9. Written Problems — Arrangement

42. Arrangement. In order to grasp the situations involved in a problem, it is helpful to write what is given and what is required. Since the answer to each component problem is to be used in the solution of another, it is helpful to write each answer as soon as it is found. The work which cannot be performed mentally may appear at the right. No denominations need appear in the scratch work if they are kept in the statements.

Write as concisely as possible what is given and what is required in a vertical line with a short horizontal line below, expressing the denominations. Write below the line the answer to each component problem as soon as it is found, expressing the denominations. At the right put all work that is not performed mentally, omitting the denominations. After the answer write *ans.*

43. Simple Problems.

Addition		Subtraction	
Ap T, 38		Had, $358	
Pr T, 96		Spent, $299	
T, ?		Left, ?	
T, 134 *ans.*		Left, $59 *ans.*	
Multiplication		**Quotition**	
C 1 H, $216	216	C 1 H, $216	19
C 19 H, ?	19	En C, $4104	216)4104
C 19 H, $4104 *ans.*	1944	H, ?	216
	216	H, 19 *ans.*	1944
	4104		1944

Partition

C 19 H, $4104	216
C 1 H, ?	19)4104
C 1 H, $216 *ans.*	38
	etc.

NOTE. Observe the awkwardness of retaining the denominations in division.

Quotition	*Partition*
19	$516
$216)$4104	19)$4104

Explanations. The author prefers question and answer. *See § 34.*

Scratch Work. In addition and subtraction it is unnecessary to rewrite the terms. In multiplication and division it is unnecessary to use the denominations because they appear at the left. To require the denominations is to require what is never done in practice, to insist upon distinctions which are wearisome, and to spend energy which may be better applied. *See the note above.*

44. Complex Problems. 1. A man gave $5760 cash and 128 cows at $56 in exchange for land at $64 an acre. How many acres did he get?

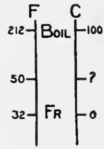
Pd, $5760 and	128	202
128 C @ $56	56	64)12928
1 A, $64	768	128
A, ?	640	128
Cows, $ 7168	7168	128
Land, $12928	5760	
Acres, 202 *ans.*	12928	

EXPL. What is the value of 128 cows @ $56? $7168. If the value of the cows is $7168 and the cash $5760, what is the value of the land? $12928. If the value of 1 A. is $64, how many acres for $12928? 202.

2. What is the reading of the centigrade thermometer when the reading of the Fahrenheit is 50°?

Fr. C, 0°; F, 32°
Boil C, 100°; F, 212°
F 50°, C ?

180° F, 100° C
1° F, $\frac{5}{9}$° C
Ab. Fr., 18°
18° F, 10° C *ans.*



Expl. If the freezing point of F is 32° and the boiling point 212°, what is the difference? 180°. If the freezing point of C is 0° and the boiling point 100°, what is the difference? 100°. If 180° F is 100° C, what is 1° F? $\frac{5}{9}$° C. If the freezing point of F is 32° and the reading is 50°, how much above freezing is the reading? 18°, etc.

45. Teaching. In writing what is given and what is required, pupils are prone to spend too much time in expressing terms fully. To counteract this it is well for the teacher to read a problem at ordinary speed, while no one writes, that the pupils may understand it as a whole; and then to read it slowly, requiring every one to finish writing as soon as he finishes reading. The object is to be as concise as possible. Another good plan is for the teacher to read the problem just as he wishes the pupil to write it, and then to call upon some one to state it in full.

46. Proofs. Pupils should not be allowed to consider a solution complete until they have proved the answer both approximately and exactly.

Approximate. Consider whether the answer is reasonable.

Ill. Problem, § 44. 200 A. at $60 would be worth $12000; 202 A. must be approximately correct.

Exact. There are four methods: *First*, review the work with care. This is the common method; it has been used in each of the foregoing problems. *Second*, solve the problem in a different way. This is of value when the problem can be separated into different sets of component problems. *Third*, discover whether the answer meets the conditions of the problem. This is of value when the problem is algebraic in nature. *Fourth*, form and solve a second problem in which the required term of the first is made a given term of the second, and some given term of the first is made the required term of the second. This method is cumbrous for practice, but valuable as an exercise for the pupil. Observe how the simple problems of § 43 in multiplication, quotition, and partition prove each other.

Ill. *Fourth Method.* T. "From the solution in § 44, make another problem in which the number of acres shall be given to find the cash payment. A man gave 128 cows at $ 56, and a sum in cash for 202 A. of land at $ 64 an acre. How much cash did he pay?"

47. Discussion. For written work, the requirement through all the grades of writing what is given and what is required, is of supreme importance, because it fixes the attention upon the situations. The time lost in making the statements is usually more than gained in determining the operations.

48. Exercises. 1. At 396 lb. a day, a ship's crew consume 11088 lb. of beef in 28 da. Solve the problem arising from the omission of 11088 lb. **2.** From the omission of 396 lb. **3.** From the omission of 28 da. **4.** From the land problem in § 44, make and solve the complex problem by the use of 202 A. as a known term, and the number of cows as the required term. **5.** Solve the last problem in § 46. **6.** Show that your answer is approximately correct. **7.** Explain your proof that the answer is exactly correct.

LESSON 10. PROBLEMS — BY ALGEBRA

49. Solutions by Algebra. A solution by algebra differs from a solution by arithmetic in two respects. By algebra, the unknown terms are represented by x, y, z; by arithmetic, they are expressed in full. By algebra, the equations are solved by the laws of algebra; by arithmetic, they are solved by analysis.

ILL. An article is sold for \$60 at a gain of $\frac{1}{5}$. What is the cost?

Algebra		**Arithmetic**
S, \$60	*Let* x = *the cost*	S, \$60
G, $\frac{1}{5}$ C	$\frac{x}{5}$ = *the gain*	G, $\frac{1}{5}$ C
C, ?		C, ?
C, \$50 *ans.*	$\frac{6x}{5} = 60$	$\frac{6}{5}$ C, \$60
	$6x = 300$	$\frac{1}{5}$ C, \$10
	$x = 50$	C, \$50 *ans.*

The difference up to the formation of the equation is the use of x in the one and of *cost* in the other. By algebra the equation is solved, Clearing of fractions, $6x = 300$; dividing by the coefficient of x, $x = 50$. By arithmetic the equation is solved, If $\frac{6}{5}$ of the cost is \$60, what is $\frac{1}{5}$ of the cost? \$10. If $\frac{1}{5}$ of the cost is \$10, what is $\frac{5}{5}$ of the cost? \$50.

Use in Arithmetic. Solutions by algebra are valuable for the indirect cases in percentage and interest, and for all other cases in which an operation must be performed upon the required term.

50. Formulas. The relation of the required term of a problem to the given terms may be expressed by an equation in which the required term is the left-hand member and combinations of the given terms the right-hand member. In solutions by formula, the first step is to get the formula; it must be recalled from memory, it must be taken from a book, or it must be derived. The second step is to substitute the given values and to solve the equation.

Ill. 1. What is the circumference of a circle whose radius is 6 in.? Recall the formula.

R, 6 in.		3.1416
C, ?	$C = 2\pi R$	12
C, 37.6992 in. *ans.*		37.6992

Ill. 2. How far will a body fall from rest in 2 sec.? Get the formula from physics.

T, 2 sec.	
D, ?	$D = 16\frac{1}{12} \times T^2$
D, $64\frac{1}{3}$ ft. *ans.*	

Ill. 3. Problem of p. 7, case 6.

A, 53 sh.	(1) $P = B \times R$	$R = \frac{A - D}{A + D}$
D, 47 sh.	(2) $A = B + P$	
R, ?	(3) $D = B - P$	$= \frac{53 - 47}{53 + 47}$
R, 6 %. *ans.*	$2B = A + D$	
	$2P = A - D$	$= 6\%$

Proof. *Second Method* (§ 46). If the original no. plus the no. purchased is 53, and the original no. minus the no. purchased is 47, what is twice the original no.? 100. What is twice the no. purchased? 6. If the original no. is 50, and the no. purchased 3, what is the rate? 6 %.

Use in Arithmetic. Formulas may be used in problems in which the relations of the terms must be found by geometry, by physics, or by involved processes.

51. Rules. A rule is the translation of a formula into language free from algebraic expressions.

Ill. 1. $C = 2\pi r$ is translated, Given the radius of a circle to find its circumference, multiply twice the radius by 3.1416.

Ill. 2. $D = 16\frac{1}{12} \times T^2$ is translated, Given the time of a body fallen from rest to find the distance, multiply $16\frac{1}{12}$ ft. by the square of the number of seconds.

Ill. 3. $R = \frac{A-D}{A+D}$ is translated, Given the amount and the difference to find the rate, divide the difference between the amount and difference by the sum of the amount and difference.

The teacher will find the exercise of translating from formula to rule excellent to strengthen his command of expression.

Ill. $A = \sqrt{s(s-a)(s-b)(s-c)}$. Translate.

Given the sides of a triangle to get the area, find the continued product of the half sum of the three sides and the remainders found by subtracting each side from the half sum separately, and extract the square root of the result.

Use in Arithmetic. Formerly, rules were used in solving most of the problems of arithmetic. At present, their use is restricted for the most part to mensuration. The rule is stated from memory and its directions are followed.

Ill. What is the radius of a circle whose area is 78.54 sq. in.?

A., 78.54 sq. in.	25.
R, ?	3‸1416.)78‸5400.
R, 5 in. *ans.*	62832
	157080
	157080

Given the area of a circle to find the radius, divide the area by 3.1416 and extract the square root of the quotient.

52. Mult. and Div. Probs. Problems which involve no other operation than multiplication and division are made up of a number of different terms and have two values for each term.

Ill. If 3 men can pick 240 bbl. of apples in 8 da., how many men will be required to pick 480 bbl. in 4 da.? The terms are *men*, *barrels*, *days*. The values for men are 3 and x; for barrels, 240 and 480; for days, 8 and 4.

53. Relation of Terms. The relation of two terms is ascertained by multiplying one of them by a number and noting the effect upon the other.

Men and Work. What is the effect on work of multiplying no. of men by a number? To multiply the work by that number. Thus, twice the no. of men do twice the work. The no. of men is proportional to the work, or the no. of men varies as the work.

Men and Time. What is the effect on time of multiplying no. of men by a number? To divide time by that number. Thus, twice the no. of men require half the time. The no. of men is inversely proportional to the time, or the no. of men varies inversely as the time.

Area and Radius. What is the effect upon the radius of multiplying the area by a number? To multiply the square of the radius by that number. Thus $A = \pi R^2$ and $2A = \pi \times 2R^2$. The area of a circle is proportional to the square of the radius, or the area of a circle varies as the square of the radius.

54. Exercises. **1.** After losing a third of his sheep a man had 166 left. How many did he have at first? *Solve by algebra.* **2.** In what time will a sum of money double at 6 % simple interest? *Solve by algebra.* **3.** On a lever the weight is 54 lb., the power is 12 lb., and the power's distance from the fulcrum is 9 in. What is the weight's distance? *Solve by formula.* **4.** What is the surface of a sphere whose radius is 6 in.? *Solve by rule.* **5.** What is the relation of the distance fallen by a body from rest to the time in seconds?

LESSON 11. PROBLEMS — BY PROPORTION

55. Two-Term Problems. Find the relation of the terms by multiplying one of them by a number and noting the effect on the other, and form the proportion indicated.

ILL. 1. At 3 for 5¢ how many apples can be bought for 30¢?

3 ap, 5¢
x ap, 30¢
No. ap, 18 *ans.*

The no. of apples is proportional to their cost.

$$3 : x = 5 : 30$$

$$x = \frac{3 \times 30}{5} = 18$$

ILL. 2. If 2 men require 10 da. for a job, how many days do 5 men require?

2 men, 10 da.
5 men, x da.
No. days, 4 *ans.*

The no. of men is inversely proportional to the time.

$$2 : 5 = x : 10$$

$$x = \frac{2 \times 10}{5} = 4$$

ILL. 3. If the area of a circle whose radius is 5 in. is 78.54 sq. in., what is the area of a circle whose radius is 10 in.?

5 in., 78.54 sq. in.
10 in., x sq. in.
Area, 314.16 sq. in. *ans.*

The area of a circle is proportional to the square of the radius.

$$78.54 : x = 5^2 : 10^2$$

$$x = \frac{78.54 \times 100}{25} = 314.16$$

ILL. 4. If the area of a circle whose radius is 5 in. is 78.54 sq. in., what is the radius of a circle whose area is 314.16 sq. in.?

5 in., 78.54 sq. in.
x in., 314.16 sq. in.
Radius, 10 in. *ans.*

The area of a circle is proportional to the square of the radius.

$$78.54 : 314.16 = 5^2 : x^2$$

$$x^2 = \frac{314.16 \times 25}{78.54} = 100$$

56. N–Term Problems. Problems of more than two terms give rise to compound proportion. This subject is usually omitted from arithmetics.

Find the relation of the required term to each of the other terms and make a proportion for each relation.

Ill. *Problem of* § *52.*

3 men, 240 bbl., 8 da.
x men, 480 bbl., 4 da.

No. men 12 *ans.*

$$3 : x = \begin{cases} 240 : 480 \\ 4 : 8 \end{cases}$$

$$x = \frac{3 \times 480 \times 8}{240 \times 4} = 12$$

The no. of men is proportional to the no. of barrels and inversely proportional to the no. of days. This means that so far as the no. of barrels is concerned $3 : x = 240 : 480$; so far as the no. of days is concerned, $3 : x = 4 : 8$; so far as both are concerned, $3 : x = 240 \times 4 : 480 \times 8$.

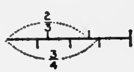
57. Proportional Parts. The statement that a whole is divided into parts proportional to given numbers means that the ratio of the two sums equals the ratio of each part of the first sum to the corresponding part of the second sum.

Ill. 1. Divide 1728 into parts proportional to 3, 4, 5.

12; 3, 4, 5 — $12 : 1728 = 3 : x$
1728; x, y, z — $12 : 1728 = 4 : y$
Parts, 432, 576, 720 *ans.* — $12 : 1728 = 5 : z$

Ill. 2. Divide 68 into parts proportional to $\frac{2}{3}$ and $\frac{3}{4}$.

$\frac{17}{12}$; $\frac{8}{12}$, $\frac{9}{12}$ — $17 : 68 = 8 : x$
68; x, y — $17 : 68 = 9 : y$
Parts, 32, 36 *ans.*



58. Use in Arithmetic. There is no great need for proportion to solve the problems of arithmetic. It seems necessary, however, to teach the subject in the elementary schools because its terms are so often used in common speech. All mechanical methods which avoid a study of relations should be avoided.

59. Two-Term Problems. Find what one value of a term has been multiplied by to give the other value, and multiply or divide the given value of the other term as the relation indicates.

Ill. 1. § 55. 5 ¢ has been multiplied by 6; multiplying cost by a number multiplies apples by that number; 3 apples must be multiplied by 6. — *Ans.* 18 apples.

Ill. 2. § 55. 2 men has been multiplied by $\frac{5}{2}$; multiplying men by a number divides days by that number; 10 da. must be divided by $\frac{5}{2}$. — *Ans.* 4 da.

Ill. 3. § 55. 5 in. has been multiplied by 2; multiplying the radius of a circle by a number multiplies the area by the square of that number; 78.54 sq. in. must be multiplied by 4. — *Ans.* 314.16 sq. in.

Ill. 4. § 55. 78.54 sq. in. has been multiplied by 4; multiplying the area of a circle by a number multiplies the radius by the square root of that number; 5 in. must be multiplied by 2. — *Ans.* 10 in.

60. N-Term Problems. Compare each of the terms with the required term as in two-term problems.

Ill. § 56. 240 bbl. has been multiplied by 2; multiplying barrels by a number multiplies men by that number; 3 men must be multiplied by 2. — *Ans.* 6 men. 8 da. has been multiplied by $\frac{1}{2}$; multiplying days by a number divides no. of men by that number; 6 men must be divided by $\frac{1}{2}$. — *Ans.* 12 men.

61. Proportional Parts. Find what one sum has been multiplied by to make the other sum.

Ill. 1. § 57. 12 has been multiplied by 144; multiplying the sum by a number multiplies each of the parts by that number; 3, 4, 5, must be multiplied by 144. — *Ans.* 432, 576, 720.

Ill. 2. § 57. $\frac{17}{12}$ has been multiplied by 48; $\frac{2}{3}$ and $\frac{3}{4}$ must be multiplied by 48. — *Ans.* 32, 36.

62. Use in Arithmetic. The method of variation is of great value in all problems which involve multiplication and division because its use requires strict attention to the relations of the terms. It is of special value for problems in mensuration which have to do with similarity. Thus, A is 6 ft. tall; it is proposed to make his statue 12 ft. tall. A's little finger is 3 in. long; to paint a statue of A's size costs $2; the weight of a statue of A's size is 1000 lb. What will be the length of the little finger of the statue? What will it cost to paint the statue? What will be the weight of the statue?

A's height has been multiplied by 2; multiplying a linear part by a number multiplies a linear part by that number, multiplies a surface part by the square of that number, and multiplies a solid part by the cube of that number. 2 in., the length of the little finger, must be multiplied by 2; $2, the cost of painting a statue of A's size, must be multiplied by the square of 2; 1000 lb., the weight of the statue, must be multiplied by the cube of 2.

63. Problems in General. All problems may be solved by stating and solving their simple problems, by algebra, by formula, and by rule. Problems involving multiplication and division may also be solved by proportion and by variation. The secret of success by each methodis to grasp the situations. Arrangement of the work is an important factor.

64. Exercises. **1.** If a body falls from rest $64\frac{1}{3}$ ft. in 2 sec., how far will it fall in 6 sec.? The distance varies as the square of the time in seconds. Solve by *proportion*. **2.** Solve by *variation*. **3.** If 3 boys earn $3 in 3 da., how many boys will earn $100 in 100 da.? Solve by stating *simple problems*. **4.** Solve by *algebra*. **5.** Solve by *formula*. **6.** Solve by *rule*. **7.** Solve by *proportion*. **8.** Solve by *variation*. **9.** State with reasons which method you prefer for No. **3**.

LESSON 12. LESSON PLANS

65. Preparation. Before conducting a class exercise the teacher should have a definite plan. He should know exactly what he is going to do and exactly how he is going to do it. He will then look forward with pleasure to the exercise and will know at its close what changes to make the next time he presents a similar exercise.

In making the plan the principal things to consider are:

1. *The object and scope of the exercise.*

2. *The logical steps demanded by the subject.*

3. *The knowledge which pupils must have before they are ready to take up the subject.*

4. *The means which must be used to induce the pupils to take the logical steps.*

Object and Scope. The object of a class exercise may be to develop a new subject, to drill upon a subject previously developed, or to determine how well a subject has been mastered. This gives rise to development exercises, drill exercises, and test exercises.

By scope is meant where a subject shall begin, where it shall end, and how fully it shall be treated. Important factors are the degree of maturity of the child, the time allowed for the exercise, and the requirements of the course of study.

Logical Steps. By logical steps are meant the steps that must be taken in the order of their dependence. *See* § *10.*

Knowledge. The teacher should consider not only what knowledge pupils must have before they are ready to take up a subject, but also whether they actually possess this knowledge. Otherwise he will present what has been presented before or he will present what is irrelevant.

Means. This topic demands the teacher's principal study. He must consider the teachings of psychology, logic, and experience to determine how the mind acts; the teachings of school management to fix upon class government and mechanical movements; works on methods and history of education to test his theories; and in fine, everything which he has studied bearing upon the subject to add to his efficiency.

66. Development Exercises. These exercises have to do with new subject matter.

ILL. *Combinations of 4's in Addition.* First lesson.

Object and Scope. To get the pupils to repeat from memory the table of 4's in addition.

Steps. To discover the results of the new combinations by counting objects, to repeat the table with the objects in sight, and to repeat the table with the objects out of sight.

Knowledge. Counting and the tables of 1's, 2's, and 3's.

Means. To show the need, state that there is frequent occasion to find the sum of 4 and each of the digits, and that it saves time to memorize the results. To help the pupils to satisfy this need, have them write the combinations which they already know by numerals and the new combinations by dots so arranged that their number may be recognized by form, have them find the sums by counting the dots, have them repeat the table with the dots in sight, and have them repeat the table with the dots out of sight.

1	2	3
4	4	4

67. Drill Exercises. Like finger exercises upon the piano, drill exercises in arithmetic are to give ability to do accurately and rapidly what can already be done less accurately and less rapidly.

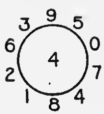
Ill. *Combinations of 4's in Addition.* Second lesson.

Object and Scope. To get pupils to call the results of the combinations of 4's in addition arranged in miscellaneous order, with accuracy and at the rate of three a second.

Steps. To call the results from memory, and in case a result is missed, to repeat the entire table.

Knowledge. Ability to repeat the table of 4's from memory.

Means. To get all the combinations before the pupils, use the circle device and the device of writing the addends of each combination in a vertical line. To get speed, tap on the board slowly and require the pupils one by one and also in concert to call the results in unison with the sounds, gradually increasing the speed until the rate of three combinations a second is attained. If this speed is not reached the first day, repeat the exercise at intervals for months or even terms. For additional practice, state simple problems requiring the answers instantly without any form of explanation.



7 1 5 9 3 8 0 4 2 6
4, 4, 4, 4, 4, 4, 4, 4, 4, 4

68. Tests. The setting of proper tests requires much skill. Not only must the object and scope be determined with great care, but also the means must be studied with unusual attention. An hour spent in the preparation of a test will often save several hours in the grading of papers and will insure a better measurement of the ability of the pupils.

Ill. *Oral Test. Combinations of 4's in Addition.* A lesson after the subject is well mastered.

Object and Scope. To determine whether each pupil can call the results of the combinations of 4's in addition accurately and without hesitation.

Steps and Knowledge. Same as in the drill exercise.

Means. A pupil should be able to read the results as rapidly when they are expressed by the combinations as when they are expressed by the common method. Write a half dozen of the results by the common method, and then all of the combinations. Require each pupil to read all the results at the same rate he reads those expressed by the common method.

8 0 6 4 2 9 3 1 5 7
12, 10, 11, 13, 9, 6, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4.

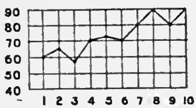
III. *Written Test.* The forty-five Combinations in Addition.

Object and Scope. To determine how rapidly pupils can write the results of the 45 combinations in addition.

Steps and Knowledge. Same as in the drill exercise.

Means. Give each pupil a paper on which the 45 combinations have been written twice in miscellaneous order, and require him to write as many of the answers as possible in one minute.

Graphs. Below is a graph showing the records of ten pupils in the written test.



This graph shows that pupil No. 1 wrote 60 correct answers in a minute; pupil No. 2, 65; and so on.

69. Exercises. 1. Plan a development exercise on the combinations of 4's in multiplication.* **2.** A drill exercise. **3.** An oral test. **4.** A written test on the 45 combinations in multiplication. **5.** Prepare a graph showing the records of ten pupils in the multiplication test. No. 1, 60 answers in a minute; 2, 80; 3, 75; 4, 60; 5, 72; 6, 85; 7, 80; 8, 90; 9, 84; 10, 50.

* **Suggestion.** 1_{4}, $2_{4's}$, $3_{4's}$, $\begin{smallmatrix}4 & 4\\ 4 & 4\end{smallmatrix}$, $\begin{smallmatrix}4 & & 4\\ & 4 & \\ 4 & & 4\end{smallmatrix}$, . . .

LESSON 13. IN GENERAL

70. Mental and Written. There is a tendency on the part of both teacher and pupil to use the pencil too freely. The greater part of the computations made by persons in business and persons in the trades and professions are mental. Often, the use of a pencil is a sign of weakness. The first step in the solution of problems is to grasp the situation; the second, is to perform the operations. In the schoolroom at least 80% of these computations should be mental.

ILL. Pupils should be able to work such examples as the following mentally:

$\$7.38 + \$9.64$; $\$10.50 - \$6.84$; $\$7.85 \times 12$; $\$57.65 \div 12$; $6\frac{2}{3} + 8\frac{3}{4}$; $9\frac{2}{3} - 4\frac{3}{4}$; $49 \times \frac{2}{3}$; $49 \div \frac{2}{3}$; $\frac{2}{3} \times \frac{5}{7}$; $6\frac{2}{3} \times 8$; $\frac{1}{2}$ to $\frac{7}{8}$ of numbers to 100; $\frac{2}{3} \div \frac{4}{5}$; $3\frac{1}{2} \div 4\frac{1}{3}$; 6% of 500; $\frac{1}{8}$% of \$1000; 528×25; $1728 \div 25$; . . .

71. Economy of Time. *Operations Easy.* Usually, drill in the solution of problems should be separate from drill in the performance of the fundamental operations. In the former, the numbers should be such as not to distract the attention from the consideration of the relations. The greatest care must be exercised that problem work shall not degenerate into an exercise in multiplication and division.

ILL. If an article is sold for \$525.65 at a gain of $27\frac{2}{3}$% what is the cost?

The division of 525.65 by $1.27\frac{2}{3}$ is so difficult that the attention is likely to be focused upon the process of division rather than upon the discovery that division is necessary—\$540 and 8% are better numbers in general.

Operations Omitted. For a quick review of complex problems, pupils may be asked to state component problems and to name the answers by letters without computations.

Ill. In what time will \$530.50 amount to \$641.30 at 5%?

If the principal is \$530.50 and the amount is \$641.30, what is the interest? \$$a$. What is the interest of \$530.50 for 1 yr. at 5%? \$$b$. If the entire interest is \$$a$ and the interest for 1 yr. is \$$b$ what is the number of years? $\frac{a}{b}$.

Component Problems by Diagrams. For the discovery of whether pupils understand the relations, they may be required to prepare analytical diagrams as in the analysis of sentences. See § 37.

Ill. *Problem above.*

T
- En Interest
  - A, \$641.30
  - P, \$530.50
- I for 1 yr.
  - P, \$530.50
  - R, 5%

72. Expression. *By the Teacher.* Before a thought is expressed it exists in the mind without words as an impulse. When words are selected, the result may be satisfying to the speaker but unintelligible to the hearer. Hence, it is of prime importance that the teacher should use with accuracy the technical terms and forms of phrasing peculiar to each subject. Following are expressions having a tendency to arrest development which the author has heard from teachers in the schoolroom.

It is criminal for a teacher to give long development exercises which are both inaccurate and silly. Hours and hours are wasted by many teachers in this way, and pupils of intelligence come to despise both the subject and the teacher. *No explanation at all is better than a foolish explanation.*

Ill. 1. "If 1 apple costs 3¢ the cost of 5 apples will be as many cents as 5 multiplied by 3¢ or 15¢." Such an expression has a tendency to make a fool of the pupil. It is impossible to multiply 5 by 3¢; the phrasing is bad.

Ill. 2 "Since 12 is $\frac{2}{3}$, $\frac{1}{3}$ is $\frac{1}{2}$ of 12." *12* = $\frac{2}{3}$ is a false statement, $\frac{1}{3}$ = $\frac{1}{2}$ *of 12* is false.

Ill. 3. "If the interest of 1 yr. amounts to $12, the time it gains $24 is $\frac{1}{12}$ of 24 which is 2 yr." 'The interest *of* 1 yr.' is ridiculous, 'amount' in interest problems means technically 'the principal plus the interest,' '$\frac{1}{12}$ of 24' is 2 and not 2 yr., it is incorrect to take $\frac{1}{12}$ of $24, for then the denomination of the answer would be dollars.

Ill. 4. "To find the least common denominator of two fractions by inspection, compare the successive **factors** of the **largest number** with the smaller until a **factor** of the smaller is found." The correct expression is, "Compare the successive **multiples** of the **larger denominator** with the smaller until a **multiple** of the smaller is found."

By the pupil. When a pupil is taking up a new topic, his major effort is to master the thought. He should not be corrected for the use of fragmentary and crude expressions until quite late. For a time it is enough if he hears invariably the correct forms from the teacher. He will adopt them as soon as he gets a stronger grip upon the thought. He should not be required to give the theoretical explanations of the fundamental operations. He will understand such explanations more or less clearly if they are skilfully given by the teacher, but he has not the power of expression to make them himself.

73. Arrangement. The arrangement of the work is of prime importance.

The teacher should always paragraph properly. The rule is simple. The first word of a paragraph, even when it is a numeral or a letter, should begin three or four letters farther to the right than the first word of each of the other lines. Thus:

> 1. Draw two vertical lines. Begin the first word of each paragraph on the right-hand vertical.
> 2. Begin every other line on the left-hand vertical.

If a drawing is used, it should be put in the center of the page from right to left and no writing should appear on either side.

74. Crutches. The teacher makes a serious mistake if he instructs or permits pupils, when they are beginning a topic, to write figures showing how many are to be carried, or to use signs of operations when terms are written in a vertical line, because in many cases pupils never discard such crutches. Thus:

68	$\overset{7}{8}3$	68		$7)3\overset{2}{\not 2}\overset{4}{\not 2}(46$
$+\ 7_{1}4$	-27	$\times\ \ 9$	$7)3\overset{2}{7}8$	28
142	56	$61^{7}2$	54	42
				42

The small figures should never be written, even by the teacher for purposes of explanation. Many high school graduates who enter training school use these small figures habitually in performing all operations. Nothing that can be said or done seems to be effective to prevent them from perpetuating this practice when they become teachers.

The illustration at the right shows how one of these graduates divides by 7. The work is eloquent of improper instruction. Not only does she use crutches, but she also employs long division in dividing by a number of one order.

The use of the signs is entirely unnecessary, is opposed to practice, and is confusing in algebra.

75. Records. Each pupil should have one grade and only one for each unit's work in a term, and the name of the unit should be written in connection with the grade. The average of a great number of grades with no statement for what each grade was given is of little value. One grade for each unit together with the name of the unit gives more information of what a pupil has done during a term than a hundred miscellaneous unmarked grades.

ILL. In the 4 A grade (N. Y. City, 1912), the units are notation, counting, addition, subtraction, multiplication, division, measurements, fractions, problems. Below is a good record of what two pupils have accomplished.

NAMES	NOT.	COUN.	ADD.	SUB.	MULT.	DIV.	MEAS.	FR.	PROB.
Brown, Henry B.	*A*	*A*	*B*	*C*	*B*	*C*	*C*	*C*	*B*
Camp, Mary S.	*A*	*A*	*A*	*B*	*A*	*A*	*B*	*A*	*A*

76. Exercises. **1.** Give the correct expression for Ill. 1, § 72. **2.** For Ill. 2, § 72. **3.** For Ill. 3, § 72. **4.** For Ill. 4, § 72. **5.** Criticise this exercise in subtraction of fractions and rewrite correctly: $8\frac{2}{3} - 5\frac{3}{4} = \frac{8}{12} + \frac{12}{12} = \frac{20}{12} - \frac{9}{12} = \frac{11}{12} = 2\frac{11}{12}$.

TEACHING ARITHMETIC

PART II. SUBJECT MATTER

LESSON 14. NOTATION AND NUMERATION

77. How Many. *Through the Ear.* The first need in mathematics is to express how many there are in a group or to measure a group with reference to how many. The steps are the same as in all measurement. *See § 9.*

The first step is to assert that there are as many individuals as in a well-known group and to name the concept. Thus, there are as many as there are fingers on both hands or **ten**.

The second step is to use number with the standard. Thus, there are **two** tens.

The third step is to select larger standards. Thus, ten tens are a **hundred**; ten hundreds, a **thousand**; a thousand thousands, a **million**; a thousand millions, a **billion**; and so on.

The fourth step is to use two or more standards. Thus, there are 'two *tens* three,' 'five *hundreds* seven *tens* six,' 'three hundred twenty-five *millions* sixty *thousands* five.'

Through the Eye. The primary method is to display as many objects as there are individuals. Thus, **𝍸 𝍸 //**.

The Arabic method of writing numbers less than ten is to use a distinct symbol for each number. Thus, 0, 1, 2, ...

The Arabic method of writing numbers less than a thousand is to write the number of hundreds, of tens, and of units by the plan of writing numbers less than ten and to express the names of the groups by position. Thus, 536 means '5 *hundreds* 3 *tens* 6.'

The French method of writing numbers greater than a thousand is to write the number of thousands, of millions, of billions, of trillions, and of higher groups by the plan of writing numbers less than a thousand and to express the names of the groups by position. Thus, 57,876,205 means '57 *millions* 876 *thousands* 205 *units*.'

78. Through Ten. A concept is more vivid than its name. Thus, 'as many as a man has fingers on the hand' is more vivid than 'five.' The teacher begins with the concept but passes quickly to the names and uses them in counting objects, motions, and sounds.

ORAL. T. "How many sticks (**//**)? There are as many sticks as a man has eyes; one for this eye and one for this eye, or **two**. How many sticks (**///**)? Two and one or **three**. How many sticks (**////**)? As many as a horse has legs, or **four**."

WRITTEN. T. "How shall we write two? We will make a mark for each individual, **//**. Take pencil and paper. Write three, four, five. For five, **/////** would do, but we will draw the last mark across the other four so that we can recognize five without counting, **𝍸**. Write six, seven; **𝍸 /**, **𝍸 //**.

"There is a shorter way of writing numbers. To-day we will learn the short way of writing **////**. Upon your desk there is a paper upon which 4 is written (each part about a half inch long) and upon the board there is also a 4 (each part about 6 in. long). Trace 4 in the air (with his side to the class he traces and the pupils imitate his movements). With the blunt end of your pencil trace the 4 on your paper. Begin here. Below this 4, with the sharp end of your pencil, write another 4."

79. Through a Thousand. The teacher has at least 2000 sticks about $4'' \times \frac{1}{8}'' \times \frac{1}{8}''$ and a box of short rubber bands. The pupils put the sticks into bundles of ten, the tens into bundles of a hundred, and count the bundles and single sticks left over. In reading and writing in the abstract they visualize these objects.

Tens Units. T. "Here are more than ten sticks (he holds 64 loose sticks). How shall we find how many there are? Take them (he distributes the sticks and rubber bands), put them into bundles of ten with a band around each, and bring to me the bundles of ten and the single sticks left over. Now, who can tell how many sticks there are? 6 *tens* and 4. Instead of '6 *tens*' we say '*6ty*.' What shall we call 7 tens? 8 tens? 9 tens? 5 tens? Say *fifty*, not fivety. 4 tens? 3 tens? Say *thirty*, not threety. 2 tens? Say *twenty*, not twoty. Count these sticks by tens. Ten, twenty, thirty, . . . How many sticks did we have? Sixty-four.

"Count (he takes up a bundle of ten and one stick at a time). Ten, ten-one, ten-two, ten-three, . . . ten-nine. Instead of ten-one say *eleven;* instead of ten-two, *twelve;* instead of ten-three, *thirteen.* What shall we call ten-four? ten-five? Say *fifteen*, not fiveteen."

T. "How shall we write sixty-four? We might write 6*ty* 4; we will omit the *ty* and write 64. Mary may take from the box eighty-three sticks (she counts ten, twenty, . . .). You may all write eighty-three. 83. What is understood after 8? Read 97. What is understood after 9? Jane may take sixty sticks from the box. You may all write sixty. Yes, 60; 6*ty* naught."

Hundreds Tens Units. T. "Here are more than a hundred sticks (he holds 385 loose sticks). How shall we find how many there are?"

The teacher proceeds in the same way as with the tens and units.

80. Beyond a Thousand. *French Plan.* The teacher has drawings, as below, upon the board and a bundle of a thousand sticks shaped as a 4-in. cube upon his table. He asks the pupils to visualize ten thousand as a row of 10 such bundles; a hundred thousand, as a layer of 10 such rows; and a million, as a block of 10 such layers. He asks them to visualize billions and the higher groups in a similar way.

THROUGH A MILLION. T. "Here are more than a thousand sticks. How shall we find the number? Suppose we put them into bundles of a thousand, and find 2 such bundles with 364 left over. How shall we write this number? '2 *thousand* 364' would do, but let us omit *thousand* and write 2,364, expressing the name of the group by position. Write '2 *thousand* 6.' 2,6 will not do because no place is left for hundreds and tens. There are 2 *thousands* no *hundreds* no *tens* 6 *units*. We write 2,006. That is, the number less than a thousand must be written with three figures. Write 8 with three figures. Yes, 008. Write zero with three figures. Yes, 000, 'no *hundreds no* tens *no* units.' Write 968 *thousand* 16. Yes, 968,016.

"How can you picture to yourself ten thousand? As a row of 10 bundles of a thousand. Show me on the floor in the corner of the room how long this row would be (40 in.). How can you picture a hundred thousand? As a layer of 10 such rows. Show me how wide the layer would be (40 in.). How can you picture 10 hundred-thousand, or a million? As a block of 10 such layers. Show me how high the block would be (40 in.)."

HIGHER GROUPS. T. "Suppose there are many more than a million sticks, how can we represent the number? Put the sticks into bundles of a thousand, the thousands into bundles of 1000 thousands or bundles of a million, the millions into bundles of 1000 millions or bundles of a billion, and so on, making large bundles of trillions, quadrillions, quintillions, and so on. Picture for me 10 millions; 10 ten-millions; 10 hundred-millions or a billion. 10 millions would be a row of 10 of the large blocks; the row would be $33\frac{1}{3}$ ft. or longer than the width of this room. 100 millions would be a layer of 10 such rows; the layer would be $33\frac{1}{3}$ ft. or wider than the length of this room. A billion would be a block of 10 such layers; the height would be $33\frac{1}{3}$ ft. or more than twice the height of this room."

T. "Read 37000068782. Beginning at the right, point off the number into periods of three figures each (37,000,068,782). Beginning at the right numerate by periods to get the name of the highest group (units, thousands, millions, billions). Read the no. of each group and call the name of the group but do not read the no. of a group when it is zero (37 billion 68 thousand 782)."

T. "Write 37 billion 68 thousand 782. Write the no. of the highest group and express the name of the group by position (37,). Write the no. of the next highest group, using three figures, and express the name of the group by position (37,000,). And so proceed (37,000,068,782)."

81. Uniformity. The above plan may be modified for each grade to suit the course of study. Uniformity of treatment throughout would be a great help to pupils.

82. Exercises. **1.** Teach counting, reading, and writing of hundreds tens units as in § 79. **2.** Teach from ten-thousand through a hundred-thousand. **3.** Teach from a million through a billion. **4.** An order is the place for writing one figure; a period, the place for writing three figures. Numerate 37,000,068: (*a*) by orders; (*b*) by periods. **5.** Name the periods forwards and backwards through quintillions. **6.** Why is it necessary to be able to name these periods both ways?

LESSON 15. Notation and Numeration

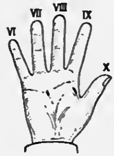
83. Roman Plan. Numbers are written by means of capital letters. For units, for tens, and for hundreds the additive principle is used three times with the same letter, then the subtractive principle with a new letter, and then the new letter, as suggested by the fingers on one hand. (The fingers increase in length from the little finger to the forefinger and then decrease). The additive principle is used three times with the last letter, the subtractive principle with a new letter, and the new letter, as suggested by the fingers on the other hand.

This method requires two letters for units, **I** and **V**; two letters for tens, **X** and **L**; and two letters for hundreds, **C** and **D**. One thousand is **M**; a *number* of thousands is the *number* under a horizontal line.

	Units	Tens	Hundreds	Thousands
1	I	X	C	M or $\overline{\text{I}}$
2	II	XX	CC	MM or $\overline{\text{II}}$
3	III	XXX	CCC	MMM or $\overline{\text{III}}$
4	IV	XL	CD	$\overline{\text{IV}}$
5	V	L	D	$\overline{\text{V}}$
6	VI	LX	DC	$\overline{\text{VI}}$
7	VII	LXX	DCC	$\overline{\text{VII}}$
8	VIII	LXXX	DCCC	$\overline{\text{VIII}}$
9	IX	XC	CM	$\overline{\text{IX}}$

68 million 539 thousand 754, $\overline{\overline{\text{LXVIII}}}$ $\overline{\text{DXXXIX}}$ DCCLIV.

Through Ten. Pupils should be taught to read and write numbers as suggested by the additive and subtractive principles and then as in common practice. They should feel the necessity of a new letter as soon as the forefinger is reached.



T. "On the face of a clock you have seen numbers written by the Roman plan. To-day we are going to study the plan through ten. Touching the little finger of your left hand and then each of the other fingers count, one, two, three, one from five, five. Touching the fingers of your right hand, beginning with the little finger count, five and one, five and two, five and three, one from ten, ten.

"Take pencil and paper. For *one*, write capital I; write *two;* write *three.* Before we can write *one from five* we must have a new letter for five; it is capital V. Write *one from five.* I from V would do but we omit *from*, IV. Write *five; five and one*, V and I would do but we omit *and*, VI; write *five and two; five and three.* Before we can write *one from ten* we must have a new letter for ten; it is capital X. Write *one from ten;* write *ten.* Read I, II, III, IV, V, VI, VII, VIII, IX, X, as above; one, two, three, one from five . . .; read calling the usual names; one, two, three, four . . . ."

Through a Hundred. Pupils should be taught to develop tens in the same way as units. In writing numbers of two orders they should think of the tens as suggested by the additive and subtractive principles and then of the units in the same way.

T. "Touching your fingers count, one ten, two tens, three tens, one ten from five tens, five tens; five tens and one ten, five tens and two tens, five tens and three tens, one ten from ten tens, ten tens. (He proceeds as with the units.)

"Read XL, XC, L, C, XXX, LX by the names as above; by the common names. What shall we think in writing eleven? 1 ten 1. Twelve? 1 ten 2. Forty-eight? 4 tens, 5 and 3. Sixty-nine? 5 tens and 1 ten, 1 from ten."

Through a Thousand. Pupils should be taught to develop hundreds in the same way as units.

T. "Touching your fingers count, one hundred, two hundreds, three hundreds, one hundred from five hundreds, five hundreds; five hundreds and one hundred, five hundreds and two hundreds, five hundreds and three hundreds, one hundred from ten hundreds, ten hundreds. (He proceeds as with the units.)

"What shall we think in writing 974? 1 hundred from 10 hundreds, 5 tens and 2 tens, 1 from 5."

Beyond a Thousand. Pupils should be taught that a bar over an expression multiplies the value of the expression by 1000.

T. "For 1 thousand, write M or $\overline{I}$; for 2 thousand, MM or $\overline{II}$; for 3 thousand, MMM or $\overline{III}$. How shall we write 4 thousand? $\overline{IV}$. 17 thousand? $\overline{XVII}$. 8 thousand thousand thousand or 8 billion? $\overline{\overline{\overline{VIII}}}$."

84. English Plan. Numbers through 999 million 999 thousand 999 are read and written by the English plan exactly the same as by the French plan. The next number greater than this is read 1000 million by the English plan and 1 billion by the French plan. This is due to the fact that the English count a million million as a billion, a million billion as a trillion, and so on, while the French count a thousand million as a billion, a thousand billion as a trillion, and so on.

Ill. 376,823456,025479,000006 is read 376 trillion 823456 billion, 25479 million 6. For reading, numbers must be pointed off into periods of six figures each; for writing, the number of each group except the highest must be expressed by six figures.

85. History. The Arabic notation, so called because the Arabs introduced it into Europe, was used by the Hindus as early as 200 B.C. At first, the names of the groups were represented by their initial letters, but about 400 A.D the names of the groups were expressed by position. At this time the symbol, 0, was introduced. Thus, from 200 B.C. to 400 A.D., six hundred fifty-eight was written 6 h 5 t 8 u; six hundred eight was written 6 h 8 u. Since 400 A.D., 6 h 5 t 8 u has been written 658; 6 h 8 u, has been written 608.

The Arabic notation was introduced into Europe in the twelfth century, but did not come into general use until the sixteenth century. Nothing was added to what the Hindus practised until the fourteenth century. Up to this time no single word expressed more than a thousand, and the best way of reading 2,724,345,968,273 was 2 thou thou thou thou 724 thou thou thou 345 thou thou 968 thou 273. The French suggested that a thousand thousands be called a million, that a thousand millions be called a billion, and so on. The Italians suggested that a thousand thousands be called a million, that a million millions be called a billion, that a million billions be called a trillion, and so on. Thus, the French read the number above '2 trillion 724 billion 345 million 968 thousand 273,' and the English, Germans, and other nations of northern Europe, read it '2 billion 724345 million 968273.'

The Roman notation is supposed to have been invented by the Etruscans. It was used by Europeans exclusively until the twelfth century and quite generally until the fourteenth. It is now used for ornamental purposes only.

86. Exercises. **1.** Teach Roman numerals from X through L. **2.** From L through C. **3.** From C through M. **4.** Beyond M. **5.** Why is the French plan for writing numbers superior to the Roman for practical purposes? **6.** State the rule for reading numbers by the English plan. **7.** State the rule for writing numbers by the English plan. **8.** Read 3000000287695400027; (*a*) by the French plan; (*b*) by the English plan.

LESSON 16. ADDITION

87. Needs. A whole may be separated into parts, each of which is made up of like individuals. Thus, 5¢ = 3¢ + 2¢. Let us classify the needs which arise from this statement by the omission of each term in succession (§ 7). If the whole is wanting, the requirement becomes No. 1 and gives rise to **addition**; if one of the parts is wanting, the requirement becomes No. 2 and gives rise to **subtraction**.

1. What = 3¢ + 2¢?
2. 5¢ = what + 2¢? or What = 5¢ − 2¢?

Addition is the process of finding the number of individuals in the whole from the numbers of individuals in the parts. The whole is the **sum** or **amount**, the parts are **addends.**

Subtraction is the process of finding the number of individuals in one of two parts from the number of individuals in the whole and the number of individuals in the other part. The whole is the **minuend**, the given part is the **subtrahend**, the required part is the **remainder.**

88. Means. The primary means in addition is to take as many objects as there are individuals in each addend and to find the result by counting.

The improved means is to find the sum of the numbers in each order separately, by calling from memory the sum of two digits, by increasing the result by a third digit, by increasing the last result by a fourth digit, and so on.

This necessitates memorizing the sums of the digits taken two at a time, and learning how to increase a number by each of the digits.

89. Memorizing the Combinations. The combinations of the digits taken two at a time are 45 in number:

1 2 3 4 5 6 7 8 9 2 3 4 5 6 7
1, 1, 1, 1, 1, 1, 1, 1, 1; 2, 2, 2, 2, 2, 2,
8 9 3 4 5 6 7 8 9 4 5 6 7 8 9
2, 2; 3, 3, 3, 3, 3, 3, 3; 4, 4, 4, 4, 4, 4;
5 6 7 8 9 6 7 8 9 7 8 9 8 9 9
5, 5, 5, 5, 5; 6, 6, 6, 6; 7, 7, 7; 8, 8; 9.

The steps in mastering the combinations are to find the sums by counting and to memorize the results. To assist in finding the sums, the domino chart is valuable. It separates the known combinations from the unknown, groups the individuals in such a way that their number can be found without counting, and keeps all of the combinations in the field at the same time. To assist in memorizing the results, the combinations and sums must be repeated as found in the tables and must be called miscellaneously. The sums must become as intimately associated with the addends as the names of objects with the objects themselves. There must be no counting upon the fingers and no form of computation.

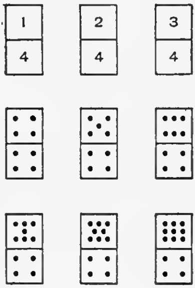
A pupil should not be regarded as proficient until he can write the sums accurately at the rate of 60 a minute and speak them accurately at the rate of 3 a second. To gain these ends may require drills at intervals during the entire school course.

Ill. *Combinations of 4's.* Development lesson. For plan see § 66.

T. "To-day we are going to find the sums when each digit is increased by four.

"A boy had 7 apples and obtained 4 more. How many did he then have? Use sticks for apples. Eleven is right, but it takes too long to proceed in this way. Who can suggest a shorter way? We will find the results and then memorize them.

"You may draw on paper three rows of rectangles, three in a row (he draws them on board). Let us write all the combinations of 4's, one in each rectangle. We know 1 and 4, 2 and 4, and 3 and 4, and we will write them in this way (teacher and pupils fill out the first row). We will represent 4 and 4, 5 and 4, . . . 9 and 4 by dots (teacher and pupils fill out the other two rows).



"By counting Mary may find the sum of 4 and 4, 5 and 4, and so on.

"You may all look at your charts and Jane may give the table in this way: 1 and 4 are 5, 2 and 4 are 6, and so on. Watch carefully and see whether she makes a mistake.

"You may put your charts in your desks and I will cover mine. Who can give the table? Henry may try. He missed 8 and 4. Look at my chart and count (the teacher uncovers his chart). How many are 8 and 4? Henry may repeat the table again (the teacher covers his chart). The class together may repeat the table."

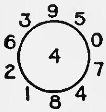
Ill. *Combinations of 4's.* Drill lesson. For plan see § 67.

T. "You are able to repeat the table of 4's. To-day we are going to call the results in miscellaneous order as rapidly as possible.

"As I point, Nellie may say, '9 and 4 are 13, 5 and 4 are 9,' and so on. You missed 7 and 4. Give the table of 4's. How many are 7 and 4?

"As I point, Susie may say, '13, 9,' and so on.

"I am going to tap upon the board with my pointer and at the word *now* I want John to say, '13, 9,' and so on, giving a result at each sound."



7 1 5 9 3 8 0 4 2 6
4, 4, 4, 4, 4, 4, 4, 4, 4, 4.

"James may call the sums keeping in time with the sounds (teacher taps upon the board at a uniform rate). Class all together, the same.

"Give me the answers to these problems without explaining. If I have 9¢ and get 4¢, how much shall I then have? If I have 7¢ and get 4¢? 4¢ and get 4¢?"

Ill. *Combinations of 4's.* Oral test. For plan see § 68.

T. "I wish to discover whether you know the 4's as well as you ought to know them.

8 0 6 4 2 9 3 1 5 7
12, 10, 11, 13, 9, 6, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4.

"John may read the first six numbers as rapidly as possible. Read the whole line at the same rate as the first six: 12, 10, 11, 13, 9, 6, 12, 4, 10, . . ." If John slackens in speed at any combination his work is unsatisfactory.

90. Increasing by a Digit. In grades above the second year pupils should be drilled at intervals throughout the course in counting from 1, 2, and so on, by 9's, by 8's, by 7's, by 6's, by 5's, by 4's, by 3's, and by 2's. A pupil is unsatisfactory until he can count by each digit as rapidly as by 1.

Ill. T. "Mary, count to 10 by 1's as fast as you can. Now, beginning with 1 count by 7's to 99 at the same rate. You count by 7's much more slowly than by 1's. I want you to practice counting by 7's several times a day for a week. Class, together, count by 7's beginning with 1; keep up with the pointer." (The teacher beats with the pointer up and down at a rate suited to the slowest in the class until 99 is reached, then more rapidly, then still more rapidly.)

91. Column Addition. A pupil should never be allowed to call the answer to an example in addition until he knows that his result is correct. Speed is of no value without accuracy. A clerk who sends out bills with incorrect footings cannot retain his position.

Common Check. A good check is to add each column from the bottom up and from the top down, and not to add the next column until the sums agree.

ILL. T. "We will take an example in addition. Nellie may work at the board and others on paper.

		Nellie's Work		
87				
48	28✓	28✓✓	29	30
69	27✓	27✓		
74				
278				

"Add units' column from the bottom up and place the sum at the right (13, 21, 28); add units' column from the top down; if the sum is the same as before, check it; if not the same, place the second sum to the right of the first (15, 24, 28, check). Nellie, I see that you get 29. Add again (she gets 30). Add aloud (at last she gets 28 several times and checks the result each time). Write 8 in units' column.

"Add tens' column in the same way (9, 15, 19, 27)."

Proof by Nines. Expert accountants prove addition by excess of 9's occasionally in cases of perplexity. Pupils may practice this proof from the sixth year on if time permits. They should prove the following principles inductively (§ 16).

To find the excess of 9's of a number, add its digits, add the digits of the sum, and so proceed until the last sum is less than 9. Count 9 as 0.

To prove addition, add the excesses of 9's of the addends; the excess of 9's of the result should be the excess of 9's of the original sum.

9687	*23*√	**3**
4832	*26*√	**8**
6975	*28*√	**0**
8369	*29*√	**8**
29863		**1**√

EXPL. 14, 21, **3**; 12, 15, 17, **8**; 13, 18, **0**; 11, 17, **8**; 8, 16, 19, **1**; 10, 16, 19, **1**√.

Double Check. Pupils should be given considerable practice in adding numbers arranged as in statistics.

	1 A	1 B	2 A	2 B	3 A	3 B	TOTAL
M.	32	34	29	35	36	35	
T.	52	49	48	49	51	49	
W.	28	29	24	25	25	27	
T.	33	31	32	33	33	31	
F.	22	19	26	24	24	26	
Total							

ILL. "Fill in the blanks, adding both horizontally and vertically. For a double check add the totals both ways."

92. Mental Work. Pupils should gain the ability to increase numbers by numbers of two or three orders mentally. The accomplishment is of value in all walks of life. Drills should be given with the addends in sight and with the addends out of sight. See § 70.

568	587
72	426

T. "Find the sum but do not write the answer. The best way is to begin at the left. Observe that 5 must be increased by 1 because

$\genfrac{}{}{0pt}{}{6}{7}$ is more than 9; say, **6**. Observe that $\genfrac{}{}{0pt}{}{8}{7}$ must be increased by 1 because $\genfrac{}{}{0pt}{}{8}{2}$ is more than 9; say, **4**; say, **0**. Find the sum in the 2d."

T. "Increase $2.54 by 69¢. Say, $2.54, $3.14 ($2.54 + 60¢), $3.23. Increase $5.67 by $2.85.—*Ans.* $7.67, $8.47, $8.52."

93. Written Problems. *See § 43.*

94. Exercises. 1. Give a development lesson on the combinations of 6's. **2.** A drill lesson. **3.** An oral test. **4.** Take the oral test you have just set. Do you know the combinations of 6's as well as you should known them? **5.** Take the test of § 90. Can you count by 7's as easily as by 1's? **6.** Copy the example of § 91 under *double check* and fill in the blanks. Explain the double check. **7.** Prove inductively the first principle of § 91, *under proof by 9's*. **8.** Prove inductively the second principle. **9.** Find the sum in the example below using the common check. **10.** Prove by 9's. **11.** Prove by finding the sums of the addends in sets of six and by adding the sums.

$786.95
89.67
437.69
87.58
6.39
896.79
88.77
76.98
97.86
738.59
46.73
538.97
9.87
8.69
238.49
569.78
49.99
8.78

LESSON 17. Subtraction

95. Needs. The need of subtraction has been developed in § 87. It manifests itself in two forms, to find what must be added to one number to make another, and to find the result when one number is diminished by another. Thus, 5¢ = what + 3¢? What = 5¢ − 3¢?

96. Means. The process of finding what must be added to one number to make another is well illustrated by the process of making change, in which it is customary to add to the purchase price by cents until a multiple of five is reached and then to add by multiples of five. In column subtraction, enough is added to the subtrahend to make units' digit the same as units' digit of the minuend, then enough is added to the result to makes tens' figure the same as tens' figure of the minuend, and so on. This necessitates ability to call one of the addends in each combination in addition from the sum and the other addend.

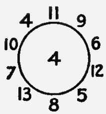
The process of taking one number away from another requires the changing of the numbers so as to make each digit of the subtrahend equal to or less than the corresponding digit of the minuend. This necessitates the mastery of the combinations in subtraction whose minuends are the *sums* of the combinations in addition and whose subtrahends are the *addends* of the combinations in addition. They are 90 in number.

Pupils should be drilled upon the combinations from both points of view, but should be taught only one method of column subtraction.

97. The Combinations. The combinations in subtraction are known as soon as the combinations in addition are known but drill is necessary to accustom the mind to the new phase. The pupil is unsatisfactory until he can call three results a second.

Ill. *Combinations of 4's in Subtraction.* Drill exercise.

T. "We are going to drill on the combinations of 4's in subtraction. Draw a circle; on the outside write the sums of the combinations of 4's in addition; at the center write 4.



11 7 5 13 6 9 10 12 4 8
4, 4, 4, 4, 4, 4, 4, 4, 4, 4.

"Henry, read from the circle in this way, 4 and 7 are 11, 4 and 5 are 9, and so on. Mary, read in this way, 11 minus 4 are 7, 9 minus 4 are 5, and so on. You missed 12 minus 4. 4 and how many are 12? John, read in this way, 7, 5, and so on. As I tap on the board Susan may call the results, 7, 5, and so on in perfect time (he taps at the rate of two a second). Nellie, read the remainders at the right, 7, 3, and so on in perfect time (he taps at the rate of two a second).

"Give me the answer without explanation. A girl has 4¢ and wants to spend 13¢; how much does she need? A boy had 12¢ and spent 4¢; how much did he have left?"

98. Column Subtraction. *Austrian Method.* It is well to introduce the work by exercises in making change.

T. "On each desk there is an envelope containing a toy nickel and five toy cents. I am going to buy things of you and ask you to make change. I buy a top for 4¢ and pay with a dime. Make the change (they all do it). John, make the change. 4 (he names the purchase price), 5 (he puts down a cent), 10 (he puts down a nickel)."

T. "On each desk there is an envelope containing toy money. I buy a cap for 59¢ and pay with a dollar bill. Make the change (they all do it). Make the change, John. 59 (he names the purchase price), 60 (he puts down 1 ¢), 85 (he puts down a quarter), 95 (he puts down a dime), $ 1 (he puts down a nickel). How much change? 41 ¢ (1 + 25 + 10 + 5)."

T. "To-day we are going to find what must be added to one number to make another or to subtract.

89	83	8786
37	59	3879

"James and Henry at the board. What must be added to 37 to make 89? How shall we proceed? To 37 add enough to make a number whose units' digit is 9 or to 7 enough to make 9. Yes, add 2 (the sum is 39). To 39 add enough to make 89 or to 3 tens enough to make 8 tens. Yes, add 5 tens. What is the answer? 52.

"A harder example. What must be added to 59 to make 83? How shall we proceed? To 59 add enough to make a number whose units' digit is 3 or to 9 enough to make 1 ten 3. Yes, add 4 (the sum is 63). To 63 add enough to make 83 or to 6 tens enough to make 8 tens. Yes, add 2 tens. What is the answer? 24.

"Again. This time say, 9 and 4 are 13 (write 4), 6 and 2 are 8 (write 2). Without rewriting add 59 and 24 to see if the sum *is* 83."

T. "From 8786 subtract 3879. Say 9 and 7 are 16 (write 7), 8 and 0 are 8 (write 0), 8 and 9 are 17 (write 9), 4 and 4 are 8 (write 4). Prove your answer."

First Italian Method. The first Italian method depends upon the principle that taking one from any order and adding its equivalent to the next lower order does not change the value of the number.

8006		7	9	9	(16)
3879		3	8	7	9

ILL. The numbers are changed in the imagination as at the right to make the digit in each order of the subtrahend equal to or less than the number in the corresponding order of the minuend. The pupil says, "9 from 16, 7; 7 from 9, 2; 8 from 9, 1; 3 from 7, 4."

Second Italian Method. The second Italian method depends upon the principle that adding 10 to any order of the minuend and 1 to the next higher order of the subtrahend cannot affect the remainder.

8006		8	(10)	(10)	(16)
3879		4	9	8	9

Ill. The numbers are changed in the imagination as at the right to make the digit in each order of the subtrahend equal to or less than the number in the corresponding order of the minuend. The pupil says, "9 from 16, 7; 8 from 10, 2; 9 from 10, 1; 4 from 8, 4."

99. Ledger Balances. The Austrian method is of value in finding ledger balances.

		36	89			225	
		95	38			387	50
		28	67				
		79	58				
	Balance	*371*	*98*				
		612	50			612	50
					Brought forward	*371*	*98*

Ill. The sum of the larger side is 612.50. We want to find what must be added to the smaller side to make this number. 15, 23, 32 (sum of units), 40 (enough must be added to 32 to make a number whose units' digit is 0 or to make 40. Write 8); 9 (the 4 of 40 plus 5), 15, 18, 26, 35 (write 9); and so on.

A check on the work is to add both sides; the sums should be the same.

100. Mental Subtraction. Pupils should be drilled until they can call quickly the difference between numbers of two or three orders and numbers of two orders, and until they can subtract readily numbers of two orders without use of a pencil. See § 70.

Ill. *Minuend 100.* Call what must be added to tens' digit to make 9 and what must be added to units' digit to make 10. Thus, 100 – 37, 6ty 3; 100 – 58, 4ty 2.

ILL. *Minuend less than 100.* Call what must be added to tens' digit or tens' digit plus one to make tens' digit of the minuend, and what must be added to units' digit to make the least number ending in units' digit of the minuend. Thus, 83 – 42, 4ty 1; 83 – 19, 6ty 4.

101. Written Problems. *See § 43.*

102. Discussion. The three methods of column subtraction have been used since 200 B.C. Two of them were brought into prominence by the Italians and the other by the Austrians. Each method has enthusiastic supporters. The advocates of the Austrian method claim that it embodies the primary notion of subtraction and that it does away with borrowing.

103. Exercises. **1.** Explain the derivation of the terms, addend, subtrahend, minuend. **2.** Show that there are 90 combinations in subtraction derived from the combinations in addition. **3.** Set an oral test for determining whether a pupil knows the combinations of 6's as well as he should. **4.** Try the test yourself. What is the result? **5.** Describe a written test for determining the proficiency of a pupil in writing the results of the 90 combinations. See § 68. **6.** Say the words which a pupil should use in subtracting 76083456 from 87692063: (*a*) by the Austrian method; (*b*) by the First Italian; (*c*) by the Second Italian. **7.** State with reasons which method you prefer.

LESSON 18. MULTIPLICATION

104. Needs. A whole may be separated into parts each of which is made up of the same number of like individuals. Thus, 6¢ = 2¢ + 2¢ + 2¢, or 6¢ = 2¢ 3 times, or 6¢ = 3 times 2¢, or 6¢ = 3 × 2¢.

Let us classify the needs which arise from the statement by the omission of each term in succession (§ 7). If the whole is wanting, the requirement becomes No. 1 and gives rise to **multiplication**; if the number of equal parts is wanting, the requirement becomes No. 2 and gives rise to that case in division which is known as **quotition**; if one of the equal parts is wanting, the requirement becomes No. 3 and gives rise to that case in division which is known as **partition**.

1. What = 3 × 2¢?
2. 6¢ = what × 2¢? or What = 6¢ ÷ 2¢?
3. 6¢ = 3 × what? or What = 6¢ ÷ 3? or What = $\frac{1}{3}$ of 6¢?

Multiplication is the process of finding the number of individuals in the whole from the number of individuals in one of the equal parts and the number of the equal parts. The whole is the **product**; one of the equal parts, the **multiplicand**; the number of equal parts, the **multiplier**.

The multiplier must always be abstract. '3 × 2¢' is read '3 times 2¢'; '2¢ × 3' is read '2¢ multiplied by 3.'

Quotition is the process of finding the number of equal parts from the number of individuals in the whole and the number of individuals in one of the equal parts.

Partition is the process of finding the number of individuals in one of the equal parts from the number of individuals in the whole and the number of equal parts. Division is the process of finding one of two numbers from their product and the other number. It embraces quotition and partition. The whole or product is the **dividend**; the given number, the **divisor**; the required number, the **quotient**.

105. Means. Multiplication. The primary method is counting. A shorter method is addition. The improved method is to call from memory the products of the digit in each order of the multiplicand by the digit in each order of the multiplier, and to add.

106. The Combinations. The improved method necessitates memorizing the products of the nine digits taken two at a time. They are 45 in number. *See § 89.* The steps in mastering the combinations are to find the products by addition and to memorize them.

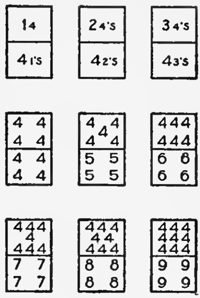
Ill. *Combinations of 4's in Multiplication.* Plan of development exercise. *See § 66.*

Object and Scope. To get the pupils to repeat from memory the table of 4's in multiplication.

Steps. To discover the results of the new combinations by addition, to repeat the table with the objects in sight, and to repeat the table with the objects out of sight.

Knowledge. Addition by 4's and the tables of 1's, 2's, and 3's.

Means. To show the need, state that there is frequent occasion for finding the product of 4 and each of the digits, and that it saves time to memorize the results. To help the pupils to satisfy this need, have them write the combinations which they already know in this way, 1 4, 2 4's, 3 4's, and the new combinations by 4's so arranged that their number may be recognized by form, have them find the sums by adding the 4's, have them repeat the table with the 4's in sight, and have them repeat the table with the 4's out of sight.



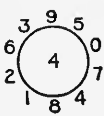
Ill. *Combinations of 4's in Multiplication.* Plan of drill exercise. *See § 67.*

Object and Scope. To get pupils to call the results of the combinations of 4's in multiplication arranged in miscellaneous order, with accuracy and at the rate of two a second.

Steps. To call the results from memory, and in case a result is missed, to repeat the entire table.

Knowledge. Ability to repeat the table of 4's from memory.

Means. To get all the combinations before the pupils, use the circle device and the device of writing the terms of each combination in a vertical line. To get speed, tap on the board slowly and require the pupils one by one and in concert to call the results in unison with the sounds, gradually increasing the speed until the rate of two combinations a second is attained. If this speed is not reached the first day, repeat the exercise at intervals for months or even terms. For additional practice, state simple problems, requiring the answers instantly without any form of explanation.



8 2 0 5 9 4 6 1 7 3
4, 4, 4, 4, 4, 4, 4, 4, 4, 4.

107. Carrying. The improved method necessitates increasing the product of two digits by a number from *one* to a number one less than the multiplier. For a time daily drills, and later drills at intervals throughout the course, are of value.

ILL. *Carrying with 4's.* First form of drill. T. "Henry, increase the products from the circle (§ 106) by 1. Say, 37, 21, 1, . . . Mary, increase the products by 2. Say, 38, 22, 2, . . . Joseph, increase the products by 3. Say, 39, 23, 3, . . ."

ILL. *Carrying with 4's.* Second form of drill. T. "John, beginning at the right find mentally the product of each digit by 4, add mentally the tens of the preceding product, and speak the answers. Do not write anything. Thus, 32, 15, 29, 2, . . ."

40738	12659
4	4

108. Multiplier One Order. The pupil should strive to have the product of each combination recalled and the addition made subconsciously as in the second form of drill in the last paragraph. After sufficient practice it will seem to him that he is simply calling the results of another's work.

ILL. *Multiplier 2.* T. "We are going to multiply large numbers by 2. John, at the board; others, pencil and paper. Multiply 768 by 2.

Teacher's Work	*John's Work*
768	768
2	768
1536	1536

"John's work is right. 768 × 2 means to find the sum of two 768's. I want you to use a shorter way. Write 768 and below it 2 showing that there are two 768's. Instead of adding 8 and 8, think 2 8's are 16 and say, **16**; instead of adding 6 and 6 and 1, think 2 6's are 12, and 1 are 13, and say **13**; and so on."

ILL. *Multiplier 3.* T. "We are going to multiply large numbers by 3. Multiply 768 by 3. Proceed as in multiplying 768 by 2. Prove your answer by going over the work a second time. Every one is right. I am greatly pleased.'

109. Multiplier 10. To multiply by 10, move each figure one place to the left. This seems to be a better rule than 'annex a cipher,' because a cipher is not annexed in multiplying by a number of two orders and because annexing a cipher does not multiply a decimal by 10.

Ill. T. "We know how to multiply by 9. How shall we multiply by 10? Let us multiply 23 by 10.

Hundreds	Tens	Units
	2	3
2	3	

"What is 3 units × 10? 3 tens. What is 2 tens × 10? 2 hundreds. What is 2 tens 3 units × 10? 2 hundreds 3 tens or 230. What is the rule for multiplying by 10? Move each figure one place to the left."

110. Multiplier Several Orders. Multiply by the digit in each order separately, placing the right-hand figure of each partial product under that digit of the multiplier which produces it. If one or both factors are multiples of 10, multiply without regard to the ciphers and annex the disregarded ciphers to the product.

To prove, go over the work a second time. To prove by 9's, multiply the excess of 9's in the multiplicand by the excess of 9's in the multiplier. The excess of 9's in the result should be the excess of 9's in the original product. Proof by 9's is recommended from the sixth year on.

3768	**6**	43000	**7**
386	**8**	52600	**4**
22608		258	
30144		86	
11304		215	
1454448	$\bar{3}$✓	2261800000	$\bar{1}$✓

Ill. *Multiplier Two Orders.* Development exercise. See § 66.

Object and Scope. To teach the multiplication of 264 by 24.

Steps. See § 10.

Knowledge. How to multiply by a number of one order and how to multiply by 10.

Means. Ask the pupils to multiply 264 by 24 and see what they will do. Show that 24 is 4 + 20, that they may find 4 264's and 20 264's, and add the results. They know how to find 4 264's; have them do it. Show that to find 20 264's they must multiply by 2 and at the same time move each figure one place to the left. Have them finish the work. Have them prove the work by going over it again.

111. Mental Multiplication. Pupils should be able to call all products to 100 instantly and to multiply by a number of one order mentally.

Ill. *Products to 100.* 2 × 11 = 22, 2 × 12 = 24, . . . 2 × 49 = 98, 2 × 50 = 100; 3 × 11 = 33, . . . 3 × 33 = 99; 4 × 11 = 44, . . . 4 × 25 = 100; 5 × 11 = 55, . . . 5 × 19 = 95, 5 × 20 = 100; . . .

Ill. *General Case.* 368 × 8. Say 2400; 480, 2880; 64, 2944. $5.37 × 6. Say, $30; $1.80, $31.80; 42¢, $32.22.

Ill. *Twenty-Nine, Etc.* 39¢ × 7. Say, $2.80, 7¢, $2.73.

Ill. *Aliquot Parts.* 732 × 25. Multiply by 100 and divide by 4.

112. Written Problems. *See § 43.*

113. Exercises. 1. Teach the combinations of 6's in multiplication — development exercise. **2.** Drill exercise. **3.** Oral test. **4.** Explain a written test on the 45 combinations to measure efficiency. **5.** Try the exercises of § 107. How proficient are you? **6.** In the example of § 110 do you say from habit in multiplying by 6, 48, 40, 46, 22? If you use more words it will pay you to change your habit. **7.** Discover inductively the rule for the proof by 9's. **8.** Explain the proof by 9's in the examples of § 110. **9.** The commutative law of multiplication is, The product of two numbers is the same whichever number is the multiplier. Prove the law (take 3 rows of 4 dots each).

LESSON 19. DIVISION

114. Needs. The need of division has been developed in § 104. It manifests itself in two phases, to find the number of equal parts (quotition), and to find one of the equal parts (partition). Each phase is expressed in several forms.

Quotition	*Partition*
6¢ = 2¢ multiplied by what?	6¢ = what multiplied by 3?
What = 6¢ divided by 2¢?	What = 6¢ divided by 3?
How many times does 6¢ contain 2¢?	What = 1 of the three equal parts of 6¢?
How many times is 2¢ contained in 6¢?	What = $\frac{1}{3}$ of 6¢?

In quotition, both dividend and divisor must be of the same denomination; in partition, the divisor must always be abstract.

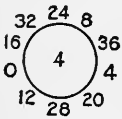
115. Means. The primary method is counting. A shorter method is subtraction. The improved method is to form partial dividends, to divide each partial dividend, and to add the quotients.

The improved method necessitates the mastery of the combinations in division whose dividends are the products of the 45 combinations in multiplication and whose divisors are the factors of the combinations in multiplication. They are 90 in number.

ILL. In the division of 364 by 7, the partial dividends are 36 tens and 14 units. The division of 36 tens by 7 necessitates knowing 35 ÷ 7; the division of 14 units by 7 necessitates knowing 14 ÷ 7.

116. The Combinations. The combinations in division are known as soon as the combinations in multiplication are known, but drill is necessary to accustom the mind to the new phase. The pupil is unsatisfactory until he can call two results a second.

ILL. *Combinations of 4's in Division.* Drill exercise. T. "We are going to drill on the combinations of 4's in division. Draw a circle; on the outside write the products of the combinations in multiplication; at the center write 4.



4)36; 4)20; 4)4; 4)28; 4)16;

4)32; 4)12; 4)0; 4)24; 4)8.

"John, read from the circle in this way, 8 4's are 32, 6 4's are 24, and so on. Jane, read in this way, 32 contains 4 8 times, 24 contains 4 6 times, and so on. You missed 36 contains 4. Give the multiplication table of 4's. How many times does 36 contain 4? Mary, read in this way, 4 is contained in 32 8 times, 4 is contained in 24 6 times, and so on. Joseph, read in this way, $32 \div 4 = 8$, $24 \div 4 = 6$, and so on. Henry, read in this way, 8, 6, 2, and so on. As I tap on the board, Peter may call the results, 8, 6, 2, and so on in perfect time (he taps at the rate of two a second). Susan, read the quotients at the right, 9, 5, and so on in perfect time (he taps at the rate of two a second).

"Nellie, read from the circle in this way, $\frac{1}{4}$ of 32 is 8, $\frac{1}{4}$ of 24 is 6, and so on.

"William, give me the answer without explanation. If a boy rides 4 mi. an hour, in how many hours will he ride 28 mi.? 36 mi.? 12 mi.? Frances, If a boy rides 36 mi. in 4 hr., how many miles does he ride in 1 hr.? If he rides 20 mi. in 4 hr., how many miles in 1 hr.?"

117. Partial Dividends. *First Form.* The improved method necessitates finding quotients when the dividends are not multiples of the divisors. For a time daily drills, and later drills at intervals throughout the course are of value. Charts like the following are recommended.

They help pupils to associate each dividend with the dividend which is a multiple of the divisor, and they present all possible dividends in the same field.

Drill for 4's				Drill for 6's					
4	5	6	7	**6**	7	8	9	10	11
8	9	10	11	**12**	13	14	15	16	17
12	13	14	15	**18**	19	20	21	22	23
16	17	18	19	**24**	25	26	27	28	29
20	21	22	23	**30**	31	32	33	34	35
24	25	26	27	**36**	37	38	39	40	41
28	29	30	31	**42**	43	44	45	46	47
32	33	34	35	**48**	49	50	51	52	53
36	37	38	39	**54**	55	56	57	58	59

Ill. In the 4's, the teacher points to 30, 19, 21, . . . and the pupil declares instantly 7 with 2, 4 with 3, 5 with 1, . . .

Second Form. Gratifying results will be obtained by daily drills in forming partial dividends before short division is introduced.

Take the first or first two digits for the first partial dividend, use the remainder with the next digit for the second partial dividend, the remainder with the next digit for the third partial dividend, and so on. Speak no words except the partial dividends. Do no writing.

2)978 3)13761 4)27696

Ill. 1st. Ex. Say **9**; think 4 with 1 and say **17**; think 8 with 1 and say **18**; think 9.

2d. Ex. Say **13**; think 4 with 1 and say **17**; think 5 with 2 and say **26**; think 8 with 2 and say **21**; think 7.

118. Short Division. If the preparation suggested in the last paragraph has been made, short division will be performed as quickly as multiplication by a number of one order.

Ill. T. "To-day we are going to divide large numbers by 2. Divide 973 by 2.

```
2)973             486
  486 with 1        2
                  972
```

"Say **9**; think 4 with 1, write 4, and say **17**: think 8 with 1, write 8, and say **13**; think and write 6 with 1. How many times does 973 contain 2? 486 times with 1 remaining. How can we prove the answer? Multiply 486 by 2, add 1, and see if the answer is 973.

"Let us see why this process gives the right result. The first divisor is 9 groups of what? 9 hundreds; 9 hundreds ÷ 2 are 4 hundreds with 1 hundred remaining; we write the 4 in hundreds' order. The 1 hundred remaining and the 7 tens make 17 tens; 17 tens ÷ 2 are 8 tens with 1 ten remaining, and so on."

119. Long Division. Examples should be selected in the order of difficulty (§ 8). The bases of classification are 'number of figures in the divisor,' and 'ease of finding quotient figures.' Thus, 41 is an easier divisor than 212, or than 14.

To find the first quotient figure, divide the number denoted by the first or first two figures of the dividend by the number denoted by the first figure of the divisor. Multiply mentally the number denoted by the first two figures of the divisor by the result. If the quotient is too large try the next smaller and so proceed.

Divisor Two Orders. The method of procedure is the same as in short division.

Ill. T. "To-day we are going to divide by numbers of two orders.

"Divide 3134 by 41.

```
     76       |         Proof
 41)3143      |     76        3116
    287       |     41          27
     273      |     76        3143
     246      |   304
      27      |   3116
```

"What is the first partial dividend? 314. Yes, because this is the least part of the dividend that can be divided by 41. What is 314 ÷ 41? An easy way to find the quotient is to divide 31 by 4. The quotient appears to be 7 but to make sure that 7 is not too large we will multiply 41 by 7 mentally. The product is 287 or less than 314. Where shall we place 7? Yes, over 4, the last figure of the first partial dividend.

"How shall we get the next partial dividend? Annex the next figure of the dividend to the remainder. What is the remainder? It will be necessary to multiply 41 by 7 and subtract. The next partial dividend is 273. And so on.

"What is 3143 ÷ 41? 76 with 27. Prove your answer. State a problem in which this division is necessary."

Divisor N Orders. To prove, go over the work a second time; or multiply the divisor by the quotient and add the remainder.

To prove by 9's, multiply the excess of 9's of the divisor by the excess of 9's of the quotient and add the excess of 9's of the remainder. The excess of 9's of the result should be the excess of 9's of the dividend. Proof by 9's is recommended from the sixth year on.

Ill. T. "Divide 184570 by 2976.

```
          62
2976)184570        6
     17856         8
     -----        --
      6010        48
      5952         4
      ----        --
        58         7✓
```

"What is the first partial dividend? 18457. Yes, because this is the least part of the dividend that can be divided by 2976. Mary, explain finding the first quotient figure.

'18 ÷ 2 = 9, 29 × 9 = 261, **9** is too large; 29 × 8 = 232, **8** is too large; 29 × 7 = 203, **7** is too large; 29 × 6 = 174, **6** is right.'

"Prove by 9's. 9, 15, **6** (excess of the divisor); **8** (excess of the quotient); **48** (the product); 13, **4** (excess of the rem.); 52, **7** (excess of the result); 9, 13, 18, 25, **7** (excess of the dividend); check."

Divisor a Multiple of 10. From the first, pupils should be taught to divide by a multiple of 10 in such a way as to avoid the repetition of ciphers.

Divide by the highest power of 10 which is a factor of the divisor, and then divide the quotient by the other factor of the divisor. The true remainder is the second remainder multiplied by the power of 10, plus the first remainder.

ILL. T. "Divide 272568 by 7000.

Teacher's Work	*Mary's Work*
7)272.568× 38 with 6568	38 7000)272568 21000 62568 56000 6568

"Mary's work is right but it is too long. Divide by 1000 and then by 7. Dividing by 1000 the quotient is 272 and the rem. 568 (make a cross and a point); dividing 272 by 7 the quotient is 38 with a rem. 6; the true remainder is 1000 × 6 + 568 or 6568."

120. Austrian Method. By the Austrian method the product of each digit of the divisor by the quotient digit is subtracted as soon as it is found and the remainder alone is written. It is shorter than the common method but is not recommended because it affords greater opportunity for error.

ILL. T. "Divide 184570 by 2976 by the Austrian plan.

"The first partial dividend is 18457 and the first quotient is 6. We will multiply 2976 by 6, subtracting the product of each digit from 18457 as we proceed.

```
        62
2976)184570
       6010
         58
```

"6 6's are 36, and 1 are 37, write **1**; 6 7's are 42, and 3 are 45, and 0 are 45, write **0**; 6 9's are 54, and 4 are 58, and 6 are 64, write **6**; 6 2's are 12, and 6 are 18, and 0 are 18.

"The next partial dividend is 6010. 2 6's are 12, and 8 are 20, write **8**; 2 7's are 14, and 2 are 16, and 5 are 21, write **5**; 2 9's are 18, and 2 are 20, and 0 are 20; 2 2's are 4, and 2 are 6, and 0 are 6."

121. Mental Division. Pupils should be able to call all the factors of numbers to 100 instantly and to divide by a number of one order mentally.

ILL. *Factors to 100.* See § 132.

ILL. *General Case.* Proceed as in short division. Thus, 439 ÷ 5. 43 ÷ 5, 8 with 3; 39 ÷ 5, 7 with 4. *Ans.* 87 with 4.

ILL. *Aliquot Parts.* 475 ÷ 25. Multiply by 4 and divide by 100.

122. Sequence of Signs. To avoid confusion it is agreed that the sign '×' or '÷' shall be used before the sign '+' or the sign '−.' There is no agreement as to whether '×' or '÷' shall be used first. The exact meaning should be expressed by means of a parenthesis. Some writers employ these signs with the understanding that they are to be used in the order of their occurrence.

ILL. T. "What is the value of $6 + 8 \times 2$? 28 if the signs are used as they occur, or 22 if the sign '×' is used first. It is agreed to use '×' or '÷' before '+' or '−.' 22 is the right answer.

"What is the value of $16 \div 8 \times 2$? 4 if the signs are used as they occur, or 1 if the sign '×' is used first. There is no agreement here. If such an expression occurs in a test use the signs as they occur. The proper way is to use a parenthesis. $(16 \div 8) \times 2 = 4$; $16 \div (8 \times 2) = 1$."

123. Operations Combined. Pupils enjoy greatly the four operations combined within the limits of the multiplication table. A comma should be written after each number to prevent confusion.

ILL. T. "Give me the final result instantly. Write no figure. 6, × 8, + 1, ÷ 7, × 8, + 9, ÷ 5, − 1, ÷ 4. (As the teacher says '6 multiplied by 8,' the child thinks 48; as he says 'plus 1,' the pupil thinks 49; and so on)."

124. Explanation of Problems. *See Lessons 6, 7, and 8.*

125. Written Problems. *See Lesson 9.*

126. Exercises. 1. Read: 3 × 8¢; 8¢ × 3; explain how these readings come about. **2.** Set an oral test on the combinations of 7 in division. Take the test and determine your own efficiency. **3.** Set a written test on the 90 combinations in division. Take the test and determine your own efficiency. **4.** What is the objection to the expression, '6 goes into 12 2 times'? **5.** Make a chart for 7's for the first form of drill in finding partial dividends. **6.** Drill the class on the second form. **7.** Teach short division by 7. **8.** Teach long division by a divisor of four orders. **9.** Prove inductively the rule for proof by 9's.

LESSON 20. Factoring

127. Need. The product of two or more numbers is called a **multiple** of each number, and each number is called a **factor** of the product. The need of finding multiples arises in the preparation of fractions for addition and subtraction; the need of factoring arises in the reduction of fractions to their lowest terms and in cancellation. Factoring is the life of Arithmetic.

128. Composites and Primes. Every composite number is made up of prime numbers. It is worth while for pupils to grasp this thought quite early.

Ill. *Classification.* T. "Let us classify numbers with reference to their factors."

1, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~.

"Write the numbers through 12. What are all the factors of 3? 1 and 3. What are all the factors of 4? 1, 2, and 4. What difference do you notice? 4 has other factors besides itself and 1; 3 has not. Cross every number that has other factors besides itself and 1; they are **composite** numbers; the others are **prime** numbers. What is a composite number? A prime number? State all the prime numbers to 12."

129. Finding Primes. Pupils should know the primes through 100 in connection with mental work in multiplication and division. They find the sieve method taught by Eratosthenes more than 2000 years ago both interesting and valuable.

Ill. T. "To-day we are going to sift composites from primes. You may all write the numbers from 2 through 31 in rows of six numbers each."

2	3	~~4~~	5	~~6~~	7
~~8~~	~~9~~	~~10~~	11	~~12~~	13
~~14~~	~~15~~	~~16~~	17	~~18~~	19
~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~
~~26~~	~~27~~	~~28~~	29	~~30~~	31

"How shall we find the higher multiples of 2? By crossing every 2d no. after 2; do so. To find the higher multiples of 3 cross every 3d no. after 3. To hit the multiples of 4 will it be necessary to cross every 4th no. after 4? No, because every multiple of 4 is a multiple of 2 and has been already crossed. Cross the higher multiples of 5. To hit the multiples of 6 will it be necessary to cross every 6th no. after 6? No, because they are multiples of 2 and have been already crossed; it is never necessary to count by a composite number. Cross all the multiples of 7. 7 × 7 is 49. If any no. less than 49 is divided by 7 what will be the quotient as compared with 7? Less than 7. Yes, and we have now crossed all the composites because we have counted by every number less than 7. Name all the primes to 30."

"Who can tell how to separate composites from primes? Write the numbers from 2. Cross every 2d no. after 2, every 3d no. after 3, and so on, counting by primes only until a prime has been used whose product when multiplied by itself is more than the largest number written. By the sieve method, find all the primes to 50."

Law of Primes. The association of primes with multiples of 6 assists in making their acquaintance.

Ill. T. "What seems to be true of prime numbers with reference to multiples of 6? 5 is one less than **6**, 7 is one more than **6**; 11 is one less than **12**, 13 is one more than **12**; every prime no. except 2 and 3 seems to be one less or one more than a multiple of 6. Is it also true that every number which is one less or one more than a multiple of 6 is a prime number?"

130. Finding Prime Factors. In reducing fractions to their lowest terms and in finding the least common multiple of large numbers, it becomes necessary to find the prime factors of a number. The subject should be taught after the addition and subtraction of fractions in ordinary use.

Test the divisibility of the number by the successive prime numbers beginning with 2 until that prime has been tried whose product when multiplied by itself is greater than the number. If a factor is found, treat the quotient in the same way, and so proceed. If no factor is found, the number is a prime.

ILL. T. "We are going to find the prime factors of numbers.

$$\begin{array}{r|l} 3 & 231 \\ \hline 7 & 77 \\ \hline & 11 \end{array}$$

"How shall we find the prime factors of 231? What is the smallest prime number? 2. Is it a factor of 231? No. What is the next? 3. Is it a factor of 231? Yes. Divide 231 by 3. What is the first prime number which is a factor of 77? 7. Express 231 by its prime factors. $231 = 3 \times 7 \times 11$.

"Find the prime factors of 101. 101 is not a multiple of 2, nor 3, nor 5, nor 7, nor 11. Is it necessary to try any prime no. greater than 11? No, because 11×11 is 121, a number greater than 101. 101 is a prime number."

131. Rules for Divisibility. Pupils should develop inductively the rules for divisibility and then memorize them. They will have occasion to use them constantly. *See* § *16*.

A number is a multiple of 2 if its last digit is a multiple of 2; of 3 if the sum of its digits is a multiple of 3; of 5 if the last digit is a multiple of 5; of 11 if the difference between the sum of its digits in the odd orders and the sum of its digits in the even orders is a multiple of 11. There is no serviceable rule for 7.

A number is a multiple of 4 if the number denoted by its last two digits is a multiple of 4; of 8 if the number denoted by its last three digits is a multiple of 8; of 9 if the sum of its digits is a multiple of 9.

132. Through a Hundred. Pupils should be drilled on naming the factors of numbers to a hundred because people of all occupations use these numbers daily.

ILL. T. "Call the numbers from 100 backwards and give the factors of each in pairs. **100**, 2 and 50, 4 and 25, 5 and 20, 10 and 10; **99**, 3 and 33, 9 and 11; **98**, 2 and 49, 7 and 14; **97**, prime; and so on."

133. Greatest Common Divisor. There is little occasion in arithmetic for finding G. C. D. The subject should be presented in the later grades in connection with finding the L. C. M. by factoring. *See § 135.*

134. Least Common Multiple. The L. C. M. of numbers must be found as a preparation for the addition and subtraction of fractions. The work should be done even in the lower grades by inspection. The occasion rarely arises for the use of fractions whose least common denominator cannot readily be found by this method. To provide for the unusual, finding the L. C. M. of large numbers should be taught in the advanced grades by the method of factoring.

By Inspection. To find the L. C. M. of two numbers by inspection, compare the successive multiples of the larger with the smaller until a multiple of the smaller is found. To find the least common multiple of more than two numbers, find the least common multiple of two of them, the least common multiple of the result and a third, and so on.

ILL. T. "We are going to find the least common multiple of numbers. What is the L. C. M. of two or more numbers? The least number that will contain each of them without a remainder. Find the L. C. M. of 6 and 8. Yes, 24 is right. How did you get it, Mary? The L. C. M. is the least multiple of 8 that is also a multiple of 6; 8 is not a multiple of 6, 16 is not, 24 is. Mary compared the multiples of 8 with 6. Would it be right to compare the multiples of 6 with 8? Yes, but more comparisons would be necessary; 6 is not a multiple of 8, 12 is not, 18 is not, but 24 is. What is the rule for finding the L. C. M. of two numbers?

"What is the L. C. M. of 6, 8, and 9? The L. C. M. of 6 and 8 is 24. The L. C. M. of 6, 8, and 9 must be the L. C. M. of 9 and 24. 24 is not a multiple of 9, 48 is not, but 72 is. The L. C. M. of 6, 8, and 9 is 72. What is the rule for finding the L. C. M. of more than two numbers?"

135. L. C. M. and G. C. D. of Large Numbers. It is necessary to provide for those rare cases of finding the L. C. M. and G. C. D. which cannot be treated by inspection. The factoring method appeals most strongly to the intelligence. A skillful arrangement of the prime factors is of service.

Write the numbers in a vertical line. Find the prime factors of the first number. Discover whether the prime factor in the 1st column is a factor of the 2d number; if so write it in the column of the 1st prime factor and write the quotient as scratch work. Discover whether the prime factor in the 2d column is a factor of the quotient; and so on.

To find the L. C. M. take a factor from every column. To find the G. C. D. take a factor from every complete column.

Ill. T. "Sometimes numbers are so large that it is not easy to find their L. C. M. and G. C. D. by inspection. How shall we find these properties of 42, 28, and 154? We will find the prime factors of each number, arranging them in such a way that each factor not found in a preceding column shall have a column of its own. Write the numbers in a vertical line.

						Scratch
$42 =$	2	$\times 3$	$\times 7$			
$28 =$	2		$\times 7$	$\times 2$		14
$154 =$	2		$\times 7$		$\times 11$	2
L. C. M. $=$	2	$\times 3$	$\times 7$	$\times 2$	$\times 11 = 924$	77
G. C. D. $=$	2		$\times 7 = 14$			11

"$42 = 2 \times 3 \times 7$. The 2 of the 1st column is a factor of 28; write it in the 1st column and write its quotient, 14, at the side. 3 of the

2d column is not a factor of 14. 7 of the 3d column is a factor of 14; write it in the 3d column; its quotient, 2, is a prime number; write it in the 4th column.

"2 of the 1st column is a factor of 154; write it in the 1st column and write its quotient, 77, at one side, and so on as before.

"The L. C. M. of 42, 28 and 154 must contain every prime factor of each number or must be made up of a factor from every column.

"The G. C. D. of 42, 28 and 154 must contain every prime factor that is common to all or must be made up of a factor from every complete column."

136. Fractions to Lowest Terms. Divide both terms by a common factor, divide both terms of the result by a common factor, and so proceed. When no common factor can be found by inspection, find the prime factors of one of the terms, and test the other term by these factors one by one.

$$\frac{2541}{5753} = \frac{231}{523}$$

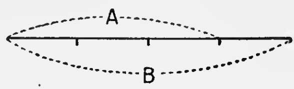
ILL. By inspection, 11 is found to be a common factor of 2541 and 5753. By inspection, no common factor of 231 and 523 is found. $231 = 3 \times 7 \times 11$; if 231 and 523 have a common factor it must be 3, or 7, or 11; 3 is not a factor of 523, 7 is not, 11 is not. The fraction is in its lowest terms.

137. Exercises. **1.** Find the prime factors of 529 and write the trial divisors which you use. **2.** Is 211 a prime number? Write the trial divisors which you use. **3.** Discover inductively a rule for the divisibility of a number by 11. **4.** Try to discover inductively a rule for the divisibility of a number by 7. What troubles? **5.** Why should pupils *not* be taught to find L. C. M. by dividing by any prime number that is a factor of two of them, etc.? **6.** Why should pupils *not* be taught to find G. C. D. by dividing the greater by the smaller, etc. (the Euclidean method)? **7.** In reducing fractions to their lowest terms, is it necessary to find the G. C. D. of their terms?

LESSON 21. FRACTIONS

138. What Part. The need arises of measuring a part in terms of the whole. It is satisfied by selecting as a standard one of the equal portions of the whole and by stating how many of the standard make the part. As in other measurements (§ 9) the expression is abbreviated by naming the standard. The result is a **fraction**, the number of the standard in the part is the **numerator**, the number of the standard in the whole is the **denominator**. A fraction is written by placing the numerator over the denominator and by separating the terms by a line.

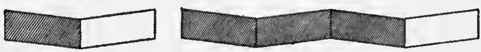
ILL. A is what part of B?



A common measure of A and B is one of the 4 equal parts of B. A is 3 of the 4 equal parts of B. Let us call 'one of 4 equal parts' a **fourth**; A is 3 fourths of B.

139. Teaching Fractions. A concept is more vivid than its name. Thus, 'one of four equal parts' is more vivid than 'a fourth.' This fact should be kept well in mind by the teacher.

ILL. T. "We are going to learn how to express a part of a whole.



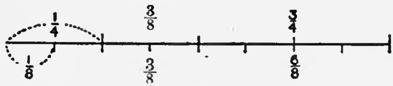
"Fold a piece of paper into two equal parts and shade one of them. What part of the paper is shaded? One of the two equal parts of the whole. One of two equal parts we call a **half**. The shaded part is one half of the whole. One of three equal parts is a **third**. What is one of four equal parts? Yes, a **fourth** or a **quarter**. One of five equal parts? One of six equal parts?

"Fold a piece of paper into 4 equal parts and shade 3 of them. What portion is shaded? 3 of the 4 equal parts or 3 fourths. Write this, $\frac{3}{4}$. What does the 3 show? The no. of equal parts in the shaded portion. It is called the **numerator**. What does the 4 show? The no. of equal parts in the whole. It is called the **denominator**. What is the numerator? The denominator?"

140. Comparisons. The conception of fractions is strengthened by comparisons. Representation of fractions by one-line diagrams as in Lesson 3 should be taught before reductions and the operations.

ILL. See ILL. § 12.

141. Cardinal Principles. The cardinal principles of fractions are: multiplying the numerator, multiplying the denominator; dividing the numerator, dividing the denominator; multiplying both numerator and denominator by the same number, dividing both numerator and denominator by the same number. They can easily be taught in two lessons by aid of the following diagram. The mnemonic rules on p. 20 should be called for at intervals.

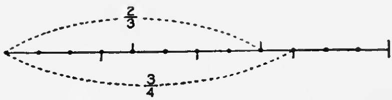


ILL. See ILL. § 20. The teacher spends one lesson on the principles involving multiplication and one lesson on the principles involving division, and drills upon all the principles at intervals.

142. Reductions. Fractions are reduced to lowest terms in order to grasp their values as clearly as possible.

They are reduced to higher terms as a preparation for addition and subtraction. Each form of reduction should be taught as the need arises. The reductions of fractions with large denominators should not be presented until pupils have mastered the operations upon fractions within the field of the multiplication tables, and until they have been well grounded in factoring.

Ill. *To Higher Terms.* T. "The other day we found the sum of $\frac{3}{4}$ and $\frac{2}{3}$ by diagram. Let us find the sum again and study the work.



"We find that $\frac{3}{4}+\frac{2}{3}=\frac{9}{12}+\frac{8}{12}$. What is the first thing to do in adding these fractions? To reduce them to equivalent fractions having their least common denominator. This denominator will be the least multiple of 4 that is also a multiple of 3. 4 is not a multiple of 3, 8 is not, but 12 is. How shall we get a fraction equal to $\frac{3}{4}$ whose denominator is 12? Divide 12 by 4 and multiply both terms by the quotient. What right have we to do this? Multiplying both terms of a fraction by the same number does not change the value of the fraction. In the same way, reduce $\frac{2}{3}$ to 12ths. Who can tell me how to reduce fractions to equivalent fractions having their least common denominator?"

Ill. *To Lower Terms.* T. "In the addition of $\frac{5}{16}$ and $\frac{7}{16}$ we obtain $\frac{12}{16}$. Let us see if we can find a simpler expression.

"What does $\frac{12}{16}$ mean? 12 of 16 equal parts. Draw a diagram and see if you can find a simpler expression. John is right, $\frac{6}{8}$ or 6 of 8 equal parts is simpler. Why is it that $\frac{12}{16}=\frac{6}{8}$? Dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction. Is there a simpler expression than $\frac{6}{8}$? Yes, $\frac{3}{4}$. Who can give me a rule for reducing a fraction to its lowest terms?"

Ill. *Large Denominators. See § 136.*

143. Addition and Subtraction. Only the addition and subtraction of fractions whose least common denominators can be found by inspection should be taught in the lower grades. These operations upon fractions with large denominators may be taught in the higher grades to impart a feeling of power.

The following forms are suggested. The L. C. D. is written below the main line with a short line above. This marks the name of the column. It is then unnecessary to write the denominator of each fraction separately. After a time, pupils should do this work without writing anything except the answers.

Prove every answer.

For finding L. C. D. by inspection, see § 134. See also Ill. 4, § 72.

Addition			Subtraction	
				12
$6\frac{2}{3}$	8	8	$6\frac{2}{3}$	8
				20
$2\frac{3}{4}$	9	9	$2\frac{3}{4}$	**9**
$9\frac{5}{12}$	$\frac{17}{12}$	12	3^{11}	$\frac{11}{12}$

Ill. *Subtraction. Austrian Method.* To 2 and 9 *12ths* add enough to make a number ending in 8 *12ths*, or to 9 *12ths* enough to make 1 unit and 8 *12ths* or 20 *12ths*; add 11 *12ths*. In practice, say 8 *12ths* and 12 *12ths* are 20 *12ths*, 9 *12ths* and 11 *12ths* are 20 *12ths* (write 11 *12ths*); 3 and 3 are 6 (write 3). See § 98.

Ill. *Large Denominators.* Turn to § 135. Suppose the denominators are 42, 28, and 154. The L. C. D. is 924. To find the quotient of 924 divided by each denominator do not perform the division, but take a factor from each column not represented by the divisor. Thus, $924 \div 42 = 2 \times 11$ or 22; $924 \div 28 = 3 \times 11$ or 33; $924 \div 154 = 3 \times 2$ or 6.

144. Multiplication and Division. *In General.* The teaching of the multiplication and division of fractions has been illustrated on pp. 20 and 21. The abbreviation afforded by cancellation should be clearly taught. Two principles are involved. The product of two numbers is the same whichever is used as the multiplier (commutative law), and dividing both terms of a fraction by the same number does not change the value of the fraction.

Prove every answer.

ILL. T. "You know the rule for multiplying fractions. Multiply the numerators for a new numerator and the denominators for a new denominator. We are going to learn how to shorten the process when there is a common factor in a numerator and a denominator.

"Multiply $\frac{9}{25}$ by $\frac{5}{8}$. Yes, $\frac{45}{200}$ or $\frac{9}{40}$. Instead of dividing both terms of $\frac{45}{200}$ by 5 *after* multiplying, try the effect of dividing 5 and 25 by 5 *before* multiplying. The result is the same. Why? Since the numerators are to be multiplied, the 9 and 5 may be regarded as changing places, $\frac{5}{25} \times \frac{9}{8}$. Then we may divide both terms of $\frac{5}{25}$ by 5. How shall we modify our rule for multiplying fractions? Add to it *canceling when possible.* Give me the complete rule."

Mixed Numbers. If only one of the terms is a mixed number, pupils even in the lower grades should perform the operation without reducing the mixed number to an improper fraction.

MULTIPLICATION	DIVISION
$\begin{array}{r} 368\frac{2}{3} \\ \underline{7} \\ 2580\frac{2}{3} \end{array}$	$7\overline{)2580\frac{2}{3}}$ $368\frac{2}{3}$
$\begin{array}{r} 461 \\ \underline{2\frac{3}{4}} \\ 345\frac{3}{4} \\ \underline{922} \\ 1267\frac{3}{4} \end{array}$ *1383*	$2\frac{3}{4})1267\frac{3}{4}$ $11\overline{)5071}$ 461

ILL. T. "Multiply $368\frac{2}{3}$ by 7. Write as on the board. Say, 7 times $\frac{2}{3}$ are $4\frac{2}{3}$ (write $\frac{2}{3}$), 60 (write 0), etc.

"Divide $2580\frac{2}{3}$ by 7. Write as on the board. Say **25**, think 3 with 4 and say **48**; think 6 with 6 and say **60**; think 8 with 4 and say $4\frac{2}{3}$ or $\frac{14}{3}$; think $\frac{2}{3}$ ($\frac{14}{3} \div 7$) and write $\frac{2}{3}$.

"Multiply 461 by $2\frac{3}{4}$. Write as on the board. We must multiply by $\frac{3}{4}$ and then by 2. To multiply by $\frac{3}{4}$, we multiply by 3 placing the product to one side, and then divide the result by 4. Do not write any more figures than are on the board.

"Divide $1267\frac{3}{4}$ by $2\frac{3}{4}$. Write as on the board. Multiply both dividend and divisor by 4 to make the divisor an integer. Do not write any more figures than are on the board."

Mixed Numbers. If both of the terms are mixed numbers, pupils in the upper grades may perform the operations without reducing the mixed numbers to improper fractions. In division, after the divisor has been changed to an integer, pupils in the upper grades should practice dividing by using the factors of the divisor.

Multiplication		Division
$37\frac{3}{4}$	*74*	
$18\frac{2}{3}$	*54*	$18\frac{2}{3})704\frac{2}{3}$
$\frac{1}{2}$		$56)2114$
$24\frac{2}{3}$		$8)2114$
$13\frac{1}{2}$		$7)264\frac{1}{4}$
296		$37\frac{3}{4}$
37		
$704\frac{2}{3}$		

Ill. T. "$\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$; $37 \times \frac{2}{3} = 24\frac{2}{3}$; $\frac{3}{4} \times 18 = 13\frac{1}{2}$; $37 \times 8 = 296$; $37 \times 10 = 370$."

Ill. *Div.* T. "Those who sit at my right may divide 2114 by 56; the others may divide by 8 and then divide the result by 7. Which is the easier process?"

Remainders. The quotient is usually declared as a mixed number. To express it as a whole number with a remainder, for the whole number take the whole number of the mixed quotient; for the remainder, take the product of the divisor by the fraction of the mixed quotient.

ILL. T. "How many lengths of $5\frac{1}{2}$ yd. each may be cut from a rope 23 yd. long and how long will be the remainder?

"Mary's answer is 4 lengths and 2 yd. Prove it. 4 lengths are 22 yd.; 22 yd. + 2 yd. are 24 yd. Something is wrong. $23 \div 5\frac{1}{2} = 4\frac{2}{11}$; there are $4\frac{2}{11}$ lengths, but not 4 lengths and 2 yd. The remainder is $\frac{2}{11}$ of a length or $\frac{2}{11}$ of $\frac{11}{2}$ yd. or 1 yd."

145. Simple and Complex. The terms of a fraction may both be integers or not both integers. This gives rise to **simple** fractions and **complex** fractions. A fraction of a fraction is called a **compound** fraction.

To simplify complex fractions, simplify each term separately, and divide the numerator by the denominator. This subject should be omitted from elementary schools.

146. Mental Work. Many drills should be given in performing the operations mentally. Multiplication and division demand special attention.

ILL. T. "How shall we multiply a fraction by an integer mentally? Multiply the numerator or divide the denominator. $4 \times \frac{5}{23}$? $\frac{20}{23}$ (multiply the nu.). $4 \times \frac{5}{24}$? $\frac{5}{6}$ (divide the den.).

"How shall we divide a fraction by an integer mentally? Divide the numerator or multiply the denominator. $\frac{8}{9} \div 2$? $\frac{4}{9}$ (divide the nu.). $\frac{7}{9} \div 2$? $\frac{7}{18}$ (multiply the den.)."

147. Problems. *See Part III.*

148. Exercises. **1.** Set an oral test for the sixth year on the four operations with fractions. **2.** A written test. **3.** Give a first lesson on the addition of fractions. **4.** On the multiplication of fractions. **5.** On the division of fractions. **6.** How many times can $8\frac{3}{4}$ gal. be taken from 100 gal. and how many gallons will remain? **7.** Prove the answer. **8.** Explain as to a class.

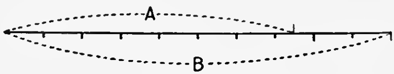
LESSON 22. Decimals

149. Need and Means. In measuring a part in terms of the whole the need arises of a uniform set of standards in place of half, third, fourth, and so on. This need is satisfied by extending the decimal plan of expressing how many. Just as one of the 10 equal parts of a hundred is **ten,** and one of the 10 equal parts of ten is a **unit,** in like manner one of the 10 equal parts of a unit is a **tenth,** one of the 10 equal parts of a tenth is a **hundredth,** and so on. It only remains to use a decimal point to mark where units end and tenths begin in order to express both integers and fractions by the same plan.

A decimal is a fraction expressed in tenths, hundredths, thousandths, and so on, by aid of a point.

Ill. T. "We are going to find a new way of expressing a part. Let us divide a whole into 10 equal parts, each small part into 10 equal parts, and so on.

"What is one of the 10 equal parts of a whole? A *tenth.* One of the 10 equal parts of a tenth? A *hundredth.* One of the 10 equal parts of a hundredth? A *thousandth.* You may draw a diagram like mine on the board. *A* is what part of *B*? *7 tenths 5 hundredths* or *75 hundredths.*



438.75

"Let us find a shorter way of writing this. You may write 438 and place after it a period, which we will call a decimal point, to show that 8 is in units' order, and after that you may write seven five.

"What is one of the 10 equal parts of a hundred? *Ten.* One of the 10 equal parts of ten? A *unit.* One of the 10 equal parts of a unit? A *tenth.* Then, in what order is the 7? *Tenths.* What is one of the 10 equal parts of a tenth? A *hundredth.* Then, in what order is the 5? *Hundredths.* Read 438.75. 438 and 7 tenths 5 hundredths or 438 and 75 hundredths.

"Write: 6 tenths, *.6*; 4 tenths, *.4*; 3 tenths 2 hundredths, *.32*; 4 hundredths, *.04*, Read: .2, *2 tenths;* .03, *3 hundredths;* .27, *27 hundredths* or *2 tenths 7 hundredths.*"

150. Reading. Pupils should be taught to read decimals by calling the name of each standard and also by calling the name of the lowest standard. They should memorize the names of the first six standards by number and find the names of lower standards by numerating.

Ill. T. "Read .638 calling the name of each order. 6 *tenths* 3 *hundredths* 8 *thousandths.* Read calling the name of the lowest order only. 638 *thousandths.*

"Commonly, we read the name of the lowest order only. I want you to memorize the numbers of the orders by name. Thus, the 1st is tenths; the 2d, hundredths; the 3d, thousandths; the fourth, ten-thousandths; the fifth, hundred-thousandths; the sixth, millionths. Read .00007, 7 *hundred-thousandths*, because 7 is in the fifth order. Read .00000758; 758 hundred-millionths, because the 6th order is millionths, the next ten-millionths, and the next hundred-millionths."

151. Writing. Pupils should be taught to write the number of the lowest standard, and to express the name of the standard by reviving in memory the names of the first six standards and by finding the names of other standards by numerating.

Ill. T. "Write 6 *thousandths;* .006. Yes, write 6, the number of thousandths, remember that thousandths is the 3d order and place the decimal point so as to show 3 decimal places. Write 6 *ten-millionths.* Write 6, the number of ten-millionths, remember that millionths is the 6th order, find that ten-millionths must be the 7th, and place the decimal point so as to show 7 decimal places."

152. Addition and Subtraction. Since decimals are written by the same plan as integers, their addition and subtraction may be performed by the same plan.

Prove every answer.

Addition	Subtraction
.97	2.06
.438	.087
1.408	1.973

Ill. T. "In writing integers for addition and subtraction, where did we place digits of the same order? In the same column. We must do the same with decimals. Be sure to place the decimal points in the same column."

153. Multiplication and Division. Since decimals are written by the same plan as integers, their multiplication and division may be performed by the same plan. Special attention must be paid to the decimal points.

Multiplication		Division	
5.375	**2**	.68	
.68	**5**	$5_{\times}375.)3_{\times}655.00$	**2**
43000		3 2250	**5**
32250		43000	**10**
3.65500	**1** ✓	43000	**1** ✓

Multiplication by an Integer. Multiply as in integers and point off as many decimal places in the product as there are decimal places in the multiplicand.

Ill. T. "Multiply .28 by 4. Nellie gets 112. This cannot be right because .28 is less than 1 and the product must be less than 1 × 4. She has neglected the decimal point. The answer should be 112 *hundredths* just as 28 apples × 4 is 112 *apples*. 112 *hundredths* is 1.12. There must be the same number of decimal places in the product as in the multiplicand. Who will give me the rule for multiplying a decimal by an integer?"

Multiplication by a Decimal. Multiply as in integers and point off as many decimal places in the product as there are decimal places in both multiplicand and multiplier.

ILL. *First Lesson. See p. 17.*

ILL. *Later Lesson.* "Be sure to prove your answer by going over the work again. Check the decimal point by finding the product of the first significant digit in each term. In the above, $5 \times .6 = 3$. The decimal point is in the right place."

Division by an Integer. Divide as in integers and point off as many decimal places in the quotient as there are decimal places in the dividend.

ILL. *See Multiplication by an Integer above.*

Division by a Decimal. Move the decimal point in both dividend and divisor so as to make the divisor an integer and proceed as in the division of a decimal by an integer.

ILL. T. "Divide .76 by .4. Mary gets .19. This cannot be right because $.76 \div 4 = .19$. She has neglected the decimal point of the divisor.

Teacher's Work	*Mary's Work*
×4.) ×7.6	.4) .76
1.9	.19

"How shall we proceed? We will make the divisor an integer. How can we do this? By moving the decimal point one place to the right in both dividend and divisor. We may do this because multiplying both dividend and divisor by the same number does not change the value of the quotient. Cross out the old points and write the new. Finish the work. What is the rule for dividing a decimal by a decimal?"

154. Reductions. It would be possible to dispense with fractions other than decimals just as it would be possible to dispense with systems of weights and measures other than the decimal, but as long as both common fractions and decimals are used the need will arise of changing from one form to the other.

In the lower grades these reductions should be made with those fractions only which can be reduced to decimals exactly, viz., with fractions whose denominators have no other prime factors than 2 and 5.

In the higher grades, the reduction of other fractions may be introduced. It is then necessary to show clearly how results are expressed, both exactly and approximately, and that a common fraction appended to a decimal cannot occupy a decimal place but must be of the denomination of the figure which it immediately follows:

Com. Frac. to Dec.	Dec. to Com. Frac.
$7\overline{)3.0000}$ $.4285\frac{5}{7}$	$.4\frac{2}{7} = \frac{30}{7} \div 10 = \frac{3}{7}$
$\frac{3}{7} = .4\frac{2}{7}$ ex.; $.4$ approx.	$.42\frac{6}{7} = \frac{300}{7} \div 100 = \frac{3}{7}$
$\frac{3}{7} = .42\frac{6}{7}$ ex.; $.43$ approx.	$.428\frac{4}{7} = \frac{3000}{7} \div 1000 = \frac{3}{7}$

Ill. *Lower grades.* T. "After a part has been expressed by a common fraction it is sometimes necessary to express it by a decimal and *vice versa.*

"Change $\frac{3}{4}$ to a decimal. $\frac{3}{4}$ means $3 \div 4$. The easiest way is to perform the division. Do so. $\frac{3}{4} = .75$.

"Change .75 to a common fraction. .75 means $\frac{75}{100}$. Reduce it to its lowest terms. $\frac{75}{100} = \frac{3}{4}$."

Ill. *Upper grades.* T. "Reduce $\frac{3}{7}$ to a decimal. Does the division terminate? No. If there is any prime factor in the denominator which is not found in 10 the division can never end. What shall we do? To get the exact result, we may stop dividing at any point and write after the last figure of the quotient the fraction whose numerator is the last remainder and whose denominator is the divisor. To get an approximate result, we may stop dividing at any point, increasing the last figure of the quotient by 1 if the next figure of the quotient would be 5 or more.

"Let us examine our first result, $.4\frac{3}{7}$. Is the $\frac{3}{7}$ tenths, or hundredths? We divided 3 by 7 or 30 tenths by 7. Hence, our result must be $4\frac{3}{7}$ tenths just as 30 apples divided by 7 is $4\frac{3}{7}$ apples. Remember that a common fraction can never occupy a decimal order but is of the denomination of the figure which it immediately follows."

The teacher takes the pupil into his confidence and discusses all that is suggested in the examples worked above.

155. Degree of Approximation. The nature of every problem suggests the number of decimal places that should be used in its solution. The attention of pupils in the later grades should be called to this matter.

Ill. T. "You should always consider the number of decimal places that must be used in a computation.

"We find the interest of \$1 to be \$$.185\frac{5}{6}$. To find the interest of \$2596, to what place should $\frac{5}{6}$ be reduced? We want the answer true to cents; we agree to count 5 mills or more as 1¢. Hence, we must carry the answer to the 3d decimal place. The highest order of 2596 is the 3d place to the left of units; the 3d to the left of units must be multiplied by the 6th to the right of units to make the 3d to the right of units. The multiplicand must be \$.185833.

"We find from the tables that the amount of \$1 is \$1.6958814, what is the amount of \$6.75? Reasoning as before, we find that \$1.695 may be multiplied by 6.75."

156. Use of Factors. It is often of advantage to multiply or to divide by the factors of a number instead of by the number itself. This device is of special value in division where the quotient is desired without consideration of the remainder.

132.7098 ÷ 48	272568 ÷ 7000
8)132.7098	7)272.568×
6)16.5887+	38.938+
2.7647+	*See P. 87.*

157. Mental Work. The decimal equivalents of the business fractions should be memorized and applied :

$\frac{1}{2}$; $\frac{1}{3}$, $\frac{2}{3}$; $\frac{1}{4}$, $\frac{3}{4}$; $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$; $\frac{1}{6}$, $\frac{5}{6}$; $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.

158. Problems. *See Part III.*

159. History. Decimals were invented in the sixteenth century, but the decimal point was not introduced until the next century. For a hundred years, the number of a decimal order was expressed by an Arabic or Roman numeral above or to the right of the order.

$$\overset{0}{5}\,\overset{1}{9}\,\overset{2}{6}\,\overset{3}{7}\;;\quad 5_{(0)}9_{(1)}6_{(2)}7_{(3)}\;;\quad \overset{\text{O}}{5}\,\overset{\text{I}}{9}\,\overset{\text{II}}{6}\,\overset{\text{III}}{7}\;;\quad 5.967$$

160. Exercises. **1.** Explain the plan of writing dollars, cents, and mills. **2.** Read 400.008 and .408. **3.** Give the rule for the use of *and* in reading decimals. **4.** Add $.4\frac{2}{3}$ and .86. **5.** In No. 4, why cannot 6 be added to $\frac{2}{3}$, making the answer $1.26\frac{2}{3}$? **6.** Multiply $14.2\frac{2}{7}$ by $.82\frac{1}{3}$ and get the exact answer. **7.** In No. 6, express the multiplicand and multiplier without the use of common fractions with just enough decimal places in each to get the answer true to two decimal places. Three decimal places are to be found in the answer, but only two are to be retained.

LESSON 23. Denominate Numbers

161. How Much. The need of expressing how much arises as early as the need of expressing how many. It is satisfied by measuring as explained in § 9. The results of the measurements are **denominate** numbers.

The relative place of denominate numbers appears in the following classification.

A number may have its unit expressed or not expressed. This gives rise to **concrete** numbers and **abstract** numbers. Concrete, 2 gallons, 2 men; abstract, 2.

A concrete number may have its unit in the tables expressing how much or may not have its unit there. This gives rise to **denominate** numbers and to not-denominate numbers. Denominate number, 2 gallons.

A denominate number may have its value expressed by one unit or by more than one unit. This gives rise to **simple** denominate numbers and **compound** denominate numbers. Simple den. no., 2 gallons; compound den. no., 2 gallons 3 quarts.

162. Teaching. Teachers should develop the tables in accordance with the laws of measurement in such a way as to excite interest, and should then insist upon their thorough mastery. The old plan was to require the memorizing of the tables with no explanations. Let us not err in the other direction by giving developments and requiring no memorizing.

Ill. *Long Measure. See Ill. § 9.*

Ill. *Square Measure.* T. "How large is this blackboard? It is 8 ft. long and 4 ft. wide. Yes, that is a good way of expressing it. I am going to teach you another way.

"Here is a piece of paper 1 ft. long and 1 ft. wide. We call its area a **square foot**. Let us find how many such pieces will cover the board. Yes, we might put this piece on the board as many times as necessary and count the times, but there is a better way. If the board is 8 ft. long how many such pieces would make 1 strip? If the board is 4 ft. wide how many such strips would there be? What is the area of the board? 32 square feet.

"Suppose we want to express the area in square inches. Find how many square inches make a square foot. How did you get 144, Mary? 'There are 12 sq. in. in 1 strip and 12 strips.' Excellent."

He continues in a similar way with the other units and insists upon the memorizing of the table.

ILL. *Time.* T. "How many days are there in a week? What do you mean by a day? The time from sunrise to sunrise. Yes, or the time it takes the earth to make one rotation on its axis (he explains the phenomenon).

"How old are you, John? What do you mean by a year? We really mean the time it takes the earth to make one revolution about the sun (he explains the phenomenon).

"How many days make a year? How do you suppose this was found out in the first place? The shadow of a pole was measured every day and the number of sunrises counted from the time the shadow was the shortest until it was the same length again. We could test the length of a year in the same way, but we have better methods. The ancients found only 360 days, but we know now that there are 365 days and a fraction of a day ($\frac{1}{4}$ da. less 674 seconds)."

In a similar way, he creates interest in each of the other units.

ILL. *Weight.* T. "How much do you weigh, Henry? What do you mean by a pound? The pound used by the grocer is the weight of 7000 grains of wheat. Here is a bag of wheat. I am going to ask you to count out 7000 grains (he distributes the wheat). As soon as any one gets 100 grains he may bring them to Henry, who will put them into a bag. John may make a mark on the board for every hundred. How many hundreds shall we need? At last I have 7000 grains of wheat. Here are scales. Joseph, you may weigh the wheat and see how the weight agrees with the count." And so on.

Neglected Weights. Apothecaries' weight and apothecaries' fluid weight are excluded from some courses of study. It seems a pity because they are used in the household constantly. They can be made both helpful and interesting.

ILL. T. "What is the least amount of a fluid as of water? Yes, a drop. How many drops of water make a teaspoonful? To-night you may find out.

"How many drops did you find, John? 100. Joseph? 120. I think the number is from 100 to 120. The druggists count 60. Who will find out and tell us to-morrow how many teaspoonfuls of water weigh an ounce? Henry, we shall depend on you.

"How many did you find, Henry? 6. The druggists count 8. How many ounces of water make a pint? You may find out later. There is an old saying that I want you to remember, 'A pint's a pound the world round.' This means a pint of water weighs a pound. It is nearly correct. 16 oz. of water make a pound. Write the table of Apothecaries' fluid weight. 60 drops make a spoonful, 8 spoonfuls make an ounce, 16 ounces make a pint, 8 pints make a gallon."

Instruction by Problems. In the upper grades important facts may be brought out by a series of problems.

ILL. T. "Let us find a rule for correcting the calendar.

"A year lacks 674 sec. of 365¼ days. What objection is there to counting 365 da. to the year? After a time summer would come in the dead of winter. What is the correction for ¼ da.? One year out of every 4 is counted as a leap year with 366 da. *Every year that is a multiple of 4 is a leap year.*

"This plan makes every year 674 sec. too long. How many days of error are there in 100 yr.? *Ans.* ¾ da. nearly. How is this error corrected? Out of every 400 years, three years that would be counted as leap years by the first plan are counted as common years. *To be a leap year every century year must be a multiple of 400.*

"By the last plan how many seconds of error are there in 100 years? *Ans.* 2600. In how many years will there be a day's error? *Ans.* 3323.

"*Memorize these rules.*"

163. Reductions. Every exercise in reducing from higher to lower denominations or from lower to higher denominations should be treated as a problem. The teacher should keep in mind that very little of this work is done outside of the schoolroom.

930 in., ? integers of h. den.	4 rd. 3 yd. 2 ft. 6 in., ? inches
930 in. = 77 ft. 6 in. 77 ft. = 25 yd. 2 ft. 25 yd. = 4 rd. 3 yd. 930 in. = 4 rd. 3 yd. 2 ft. 6 in. *Ans.*	4 rd. 3 yd. = 25 yd. 25 yd. 2 ft = 77 ft. 77 ft. 6 in. = 930 in. *Ans.*

Expl. How many feet in 930 in.? 77 ft. 6 in. How many yards in 77 ft.? 25 yd. 2 ft. How many rods in 25 yd.? 4 rd. 3 yd.

For the remainder in 25 yd. ÷ $5\frac{1}{2}$ yd. see § 144, remainders.

Expl. How many yards in 4 rd. 3 yd.? 25. How many feet in 25 yd. 2 ft.? 77. How many inches in 77 ft. 6 in.? 930.

164. Operations. The operations should be treated exactly as the operations upon integers. Attention is called to increasing a given date by a number of days and to finding the number of days from one date to another as is necessary in problems in interest.

June 29 + 90 da., ? date	June 29 to Sept 27, ? days
June 119 July 89 Aug. 58 Sept. 27 *Ans.*	June, 1 July, 31 Aug., 31 Sept., 27 Days, 90 *Ans.*

Expl. June 29 + 90 da. is June 119 or July 89 (June has 30 da.) . . .

Expl. In June there is one day left; in July there are 31 days; . . .

165. Mental Work. Nearly all of the work should be mental. *See § 70.*

166. The Metric System. The need arises of a uniform system of weights and measures. It is satisfied by the decimal plan; ten units of one denomination make one of the next higher. The names of the prefixes below units are taken from the Latin and the names of the prefixes above units are taken from the Greek; they are abbreviated by their initial letters, small for the Latin and capital for the Greek. The units are determined systematically from the distance from the equator to the north pole; they are abbreviated by their initials. No period is placed after an abbreviation.

The metric system bears the same relation to English weights and measures that decimals bear to common fractions. Each system has its advantages; they are not all in favor of the metric.

Teaching Lower Grades. The nickel (5 ¢ piece) should be made the basis of teaching the metric system. Pupils should not think in English denominations and then give the metric equivalents, but should proceed as if there were no English denominations.

Ill. T. "Yesterday I asked each of you to bring a nickel to class. From it we are going to learn a new system of weights and measures.

"The nickel is 2 millimeters thick. How thick is the cover of your arithmetic? How thick is your penholder? We express very short lengths in millimeters. When you study physics and chemistry later on, you will have much to do with this new system. Talk with the folks at home about it.

"Place 5 nickels in a pile. What is the thickness of the pile? 10 millimeters is called a centimeter. How long is your little finger? How long is your penholder?

"The nickel is nearly 2 centimeters in diameter, exactly 2.1. Place 5 nickels in a row. The row will be 10.5 centimeters long. Draw a line as long as the row; erase .5 centimeters or 5 millimeters of it. The line is 1 decimeter long. How long is your arm?

" Henry, draw on the blackboard a line 10 decimeters long; it is a meter long. What is the width of this room?

" 10 meters make 1 dekameter, 10 dekameters make 1 hectometer, 10 hectometers make 1 kilometer. Meters and kilometers are the units in common use. What is the length of a city block in meters? About 75. How many blocks make a kilometer? About $12\frac{1}{2}$. How far is it from 10th St. to 135th St.? About 10 kilometers.

" Who will give me the table for long measure? 10 mm = 1 cm, etc. Does this table remind you of any other? 10 mills = 1 cent, etc."

The teacher continues the exercise and proceeds in a similar way with the measures of weight (the nickel weighs 5 grams) and of capacity. At the end, he insists upon thorough memorizing of the tables.

Teaching Higher Grades. A thorough mastery of the metric system may be gained by a study of the following table. From the knowledge of how each unit is obtained and the value of the meter, each of the English equivalents may be computed.

Table	Unit	Ab.	How Obtained	English Equivalent
Length	meter	m	.0000001 distance from the equator to the pole	39.37 in.; 1 Km = $\frac{5}{8}$ mi. (ap.)
Weight	gram	g	wt. 1 cu cm of water	15.432 gr.; 1 Kg = $2\frac{1}{5}$ lb. (ap.)
Capacity	liter	l	1 cu dm	1 qt. (ap.)
Land	are	a	10 m × 10 m	$\frac{1}{40}$ A. (ap.)
Wood	stere	s	1 m × 1 m × 1 m	$\frac{1}{4}$ cd. (ap.)

Ill. T. "Kilometer is used in the metric system where mile is used in the English. A meter is 39.37 in. Compute the value of a kilometer in terms of the fraction of a mile.

"Kilogram is used in the metric system where pound is used in the English. A gram is the weight of 1 cu cm of water; a cubic foot of water weighs $62\frac{1}{2}$ lb. Compute the weight of a kilogram in terms of pounds."

History. The metric system was invented in France under direction of Napoleon and is now used by most peoples except those who speak English. In the United States it was legalized in 1866 but its use at present is confined chiefly to scientific works.

167. Exercises. **1.** Teach cubic measure. **2.** Which is heavier and by how many grains, a pound of feathers or a pound of gold? **3.** An ounce of feathers or an ounce of gold? **4.** Was 1900 a leap year? Why? **5.** Below are given the usual solutions of the problems of § 163. Criticise them. **6.** Teach the table of weight in the metric system, using a nickel as the basis. **7.** Solve the last problem of § 166.

12 | 930 in.	4 rd. 3 yd. 2 ft. 6 in.
3 | 77 ft. 6 in.	$5\frac{1}{2}$
$5\frac{1}{2}$ | 25 yd. 2 ft.	25 yd.
4 rd. 3 yd.	3
930 in. = 4 rd. 3 yd. 2 ft. 6 in.	77 ft.
	12
	930 in.

LESSON 24. Mensuration

168. Forms. In accordance with the laws of logical division pupils may be led to create for themselves the various geometrical forms. They will respond with delight and enthusiasm, and will gain a grasp of the subject which will be a constant joy.

Major Forms. A subject is made more vivid by considering it from different points of view.

Ill. T. "We may consider that which has no dimension, that which has one dimension, that which has two dimensions, and that which has three dimensions. This gives rise to **points, lines, surfaces,** and **solids.** This block (he shows a cube) is a solid, its faces are surfaces, its edges are lines, and its vertices are points.

"We may consider points, lines, and surfaces as moving and leaving paths. What is the path of a moving point? What is the path of a moving line? What is the path of a moving surface?

"What is the intersection of two plane surfaces? What is the intersection of two lines?"

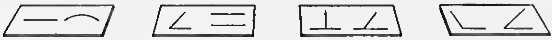
Lines and Angles. The teacher develops horizontal, perpendicular, vertical, and oblique lines as below.

Ill. T. "A point may move constantly in the same direction or constantly in a different direction. This gives rise to **straight** lines and **curved** lines.

"Draw straight lines in pairs and find what is true of their meeting. They meet or do not meet. This gives rise to **angles** and to **parallel** lines.

"Draw lines in pairs that meet. What is true of their adjacent angles? They are equal or are not equal. This gives rise to **right-angles** and **oblique** angles.

"Compare oblique angles with right-angles. An oblique angle is greater than a right angle or less than a right angle. This gives rise to **obtuse** angles and **acute** angles."

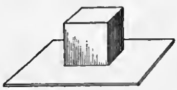


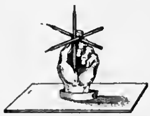
Dimensions. Pupils should not only use the word, *dimension*, but should know what it means.

Ill. T. "Let us find what is meant by one dimension, two dimensions, and three dimensions.

"Look at an edge of this cube; it has only one dimension. Look at a face; at a point on the face how many lines can be drawn each of which is perpendicular to each of the others? Only two. The face has two dimensions.

"Hold three pencils in such a way that each shall be perpendicular to each of the others. Can you hold four pencils in such a way? Look at three edges of a cube meeting at a point. What is true of these edges? Each is perpendicular to each of the others. The cube has three dimensions because at every point within it three lines may be perpendicular each to each of the others. Can you image a form that has more than three dimensions?"





Polygons. Quadrilaterals, and triangles according to their sides, have been developed in § 4 and § 5.

Ill. *Triangles.* T. "Let us classify triangles according to their angles.

"How many right angles can a triangle have? (They determine by experiment.) How many obtuse angles? How many acute angles? This gives rise to **right-angle** triangles, **obtuse angle** triangles, and **acute angle** triangles."

Ill. *Regular Polygons.* T. "The sides and angles of a polygon may all be equal or not all equal. This gives rise to **regular** and **irregular** polygons.

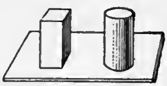
"Draw a regular polygon of three sides, of four sides, of five sides, of six sides, . . . of an infinite number of sides. This gives rise to **equilateral** triangles, **squares**, regular **pentagons**, regular **hexagons**, . . . **circles**. Give me two definitions of a circle. A circle is a regular polygon of an infinite number of sides. A circle is a plane surface bounded by a curved line every point of which is equally distant from the center."

Altitudes. Pupils should understand clearly that a triangle may have three altitudes — one for each base.

T. "The altitude of a triangle or quadrilateral is a line drawn from any vertex perpendicular to the opposite side. Draw an acute angle triangle, an obtuse angle triangle, a parallelogram, and a trapezoid; draw the altitude of each by a dotted line."

Pyramids and Prisms. The common forms should be classified.

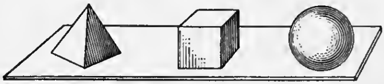
Ill. T. "A solid may have a polygon for its base and equal triangles for its faces, or a polygon for its base and equal rectangles for its faces, or various other forms, but we shall consider the first two kinds only. This gives rise to **pyramids**, or to **cones** when the base of the pyramid is a circle; and to **prisms**, or to **cylinders** when the base of the prism is a circle. A pyramid or cylinder is named from its base. Here are (he shows the forms) a square pyramid and a cone, a square prism and a cylinder."



Regular Solids. Attention should be called to the regular tetrahedron and to the cube as leading up to the sphere.

Ill. T. "A solid may have all of its faces equal polygons. Here are a regular tetrahedron, a cube, and a sphere. The word, *tetrahe-*

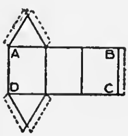
dron is the Greek for *four-base.* Define a regular tetrahedron; a cube. Give two definitions for a sphere. A sphere is a regular solid with an infinite number of faces. A sphere is a solid bounded by a curved surface, every point of which is equally distant from the center."



169. Constructions. Pupils should be required to make solids from pasteboard. The teacher who is not willing to direct this work will do well to omit solids from the course.

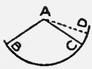
Ill. *Prisms and Cylinders.* T. "We will make a triangular prism. On the pasteboard upon your desk draw two parallel lines 4 in. apart (he draws upon the board as he instructs). Lay off 3 distances of 2 inches on each line; connect the points of division; construct equilateral triangles. Prepare flaps for pasting as represented by the dotted lines. Cut entirely through outside lines and partly through inside lines. Fold and paste the flaps on the inside.

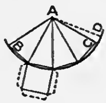
"We will make a cylinder. On the sheet of paper upon your desk draw two parallel lines 4 in. apart. Lay off 6 in. on each line; connect the points of division; prepare the flap. Cut entirely through outside lines. Roll, and paste the flap on the inside. You do not need a base."



Ill. *Pyramids and Cones.* "We will make a cone. Use the sheet of paper upon your desk. From any point as a center with a radius of 5 in. draw an arc of a circle. Lay off any convenient distance, as BC; draw AD for the flap. Cut entirely through the outside lines; roll the form until AB coincides with AC, and paste the flap on the inside.

"We will make a square pyramid. Proceed as in drawing the form for a cone, using the pasteboard upon your desk, and making *BC* the same length as before. Divide *BC* into 4 equal parts and draw the chords. Connect the points of division with *A*. Construct a square on one of the sides. Prepare flaps for pasting as represented by dotted lines. Cut entirely through outside lines and partly through inside lines. Fold, and paste the flaps on the inside."





170. Circumference of Circle. On p. 13, the circumference has been found to be $3\frac{1}{7} \times D$ or πD.

171. Areas. *Rectangle.* The area of a rectangle is the number of square units in one strip multiplied by the number of the strips.

Ill. T. He proceeds as suggested in § 162, Ill.

Parallelogram. The area of a parallelogram is the area of a rectangle which has the same base and altitude.

Ill. T. He proceeds as in § 13, Ill.

Triangle. The area of a triangle is half the area of a rectangle which has the same base and altitude.

Ill. T. "Find by paper cutting (§ 14, Ex. 5) the relation of a triangle to a parallelogram which has the same base and altitude. What is the relation of a parallelogram to the rectangle which has the same base and altitude? What is the area of a triangle?"

He shows the same by paper folding (§ 14, Ex. 6).

Trapezoid. The area of a trapezoid is half the sum of the areas of two rectangles one of which has the upper base and the other the lower base and both the altitude of the trapezoid.

Ill. T. "Into what does the diagonal of a trapezoid divide the figure? What is the area of a trapezoid?"

Circle. The area of a circle is the area of a square whose side is the diameter, multiplied by $\frac{1}{4}\pi$. Pupils should visualize a circle as inscribed within a square and should remember that the ratio of the circle to the square is $3\frac{1}{7} : 4$.

ILL. T. "Inscribe a circle within a square. Divide the circle into equal parts by radii. The circle is the sum of an infinite number of small triangles; the sum of the bases of the triangle is the circumference of the circle, πD; the altitude of the triangles is the radius, $\frac{1}{2}D$. Hence, the area of the circle is half the area of a rectangle whose base is πD and whose altitude is $\frac{1}{2}D$, or $\frac{1}{2} \times \pi D \times \frac{1}{2}D$, or $\frac{1}{4}\pi D^2$."

172. Convex Surfaces. The convex surface of a solid is all the surface except its bases. The forms of prisms, cylinders, pyramids, and cones, cut out ready for pasting, suggest the rules.

Prisms. The convex surface of a prism or cylinder is the area of a rectangle whose base is the perimeter of the base of the solid and whose altitude is the altitude of the solid.

Pyramids. The convex surface of a pyramid or cone is half the area of a rectangle whose base is the perimeter of the solid and whose altitude is the slant height of the solid.

Spheres. The surface of a sphere is π times the area of a square whose side is the diameter.

ILL. "The surface of a sphere is 4 times the area of a circle which has the same diameter (§ 13, ILL.). The area of the circle is $\frac{1}{4}\pi D^2$. Hence, the surface of the sphere is $4 \times \frac{1}{4}\pi D^2$, or πD^2."

173. Volumes. The volume of a solid is the number of cubic units in its contents.

Prisms. The volume of a prism or cylinder is the number of cubic units in one layer multiplied by the number of the layers.

Ill. T. He follows the plan suggested for visualizing in § 80, Ill. The cylinder is a variety of the prism.

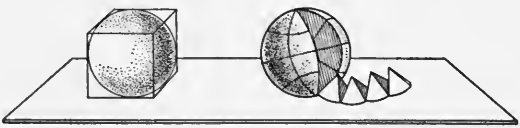
Pyramids. The volume of a pyramid or cone is one third the volume of a prism which has the same base and altitude.

Ill. T. "To find the relation of a pyramid to a prism, I am going to ask some one to construct a square prism and a square pyramid with the same base and altitude, the prism with *one* base and the pyramid with *no* base. Make the pyramid first; the side of the base of the prism will then be known. We will compare their volumes by filling them with sand. John, we shall depend on you."

"These forms are well made. John may stand before the class and make the comparison. What do you find? 'The volume of the pyramid is one third the volume of a prism which has the same base and altitude?' John has done well."

Spheres. The volume of a sphere is the volume of a cube whose edge is the diameter, multiplied by $\frac{1}{6}\pi$. Pupils should visualize a sphere as inscribed within a cube and should remember that the ratio of the sphere to the cube is $3\frac{1}{7} : 6$.

Ill. *Sphere.* T. "Inscribe a sphere within a cube. Image the sphere as divided into equal pyramids whose vertices are at the center. The sphere is the sum of an infinite number of small pyramids; the sum of the bases of the pyramids is the surface of the sphere, πD^2; the altitude of the pyramids is the radius of the sphere, $\frac{1}{2}D$. Hence, the volume of the sphere is one third the volume of a prism whose base is πD^2 and whose altitude is $\frac{1}{2}D$, or $\frac{1}{3} \times \pi D^2 \times \frac{1}{2}D$, or $\frac{1}{6}\pi D^3$."



174. Similarity. If a form is enlarged and the likeness is preserved in every respect the original and the result are said to be similar.

Multiplying a linear part by a number multiplies every linear part by that number; multiplies every surface part by the square of that number; multiplies every solid part by the cube of that number.

ILL. *See § 62.*

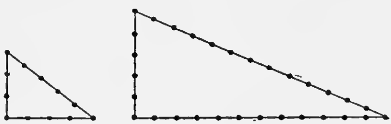
175. Right-Angle Triangles. The right angle triangle is the basis of many problems. The square of the hypotenuse is equal to the sum of the squares of the other two sides. The ancients determined this law experimentally; pupils in the elementary schools must be content to do likewise.

ILL. T. "Let us discover the relation of the hypotenuse of a right angle triangle to the other two sides.

"Draw two lines at right angles; lay off 4 equal spaces on one and 3 equal spaces on the other; find how many spaces there are on the hypotenuse by measurement.

"Do the same, making 12 and 5 equal parts on the lines.

"Do you find a common relation between the hypotenuse and the other sides? $3^2 + 4^2 = 5^2$; $5^2 + 12^2 = 13^2$. What seems to be the relation of the hypotenuse of a right-angle triangle to the other sides?



176. Exercises. **1.** Teach horizontal, vertical, and oblique lines. **2.** Draw an oblique triangle and indicate its three altitudes by dotted lines. **3.** Construct a 3-in. cube of pasteboard. **4.** Construct a regular tetrahedron of pasteboard. **5.** Develop the rule for convex surface of a pyramid. **6.** Teach the rule for the volume of a rectangular prism.

LESSON 25. INVOLUTION, EVOLUTION AND LOGARITION

177. Needs. A product may be separated into a number of equal factors. Thus, $8 = 2 \times 2 \times 2$, or $8 = 2^3$.

Let us classify the needs which arise from this statement by the omission of each term in succession (§ 7). If the product is wanting, the requirement becomes No. 1 and gives rise to **involution**; if the equal factor is wanting, the requirement becomes No. 2 and gives rise to **evolution**; if the number of times the equal factor occurs is wanting, the requirement becomes No. 3 and gives rise to **logarition.***

1. What $= 2^3$?
2. $8 = \text{what}^3$? or What $= \sqrt[3]{8}$?
3. $8 = 2^{\text{what}}$? or What $= \log 8_2$?

178. Involution. Involution is the process of finding the product from the equal factor and the number of times the factor occurs. The product is the **power**, the equal factor is the **base**, the number of times the factor occurs is the **index** of the power or the **exponent.**

ILL. T. "There is a short way of expressing that the same number is used several times as a factor.

"Write $8 = 2 \times 2 \times 2$. The number of times 2 is used as a factor is written over and a little to the right of 2. Thus, $8 = 2^3$. 2 is called the **base** and 3 is called the **exponent**. 2^3 is read '2 to the 3d power' or 'the cube of 2.' What does 3^2 mean? It is read '3 to the 2d power' or 'the square of 3.' How is 2^5 read? What does it mean? What is its value? What is 2 called? What is 5 called?

* A word suggested for the phrase, the process of finding logarithms.

"The process of finding the product of equal numbers is called **involution**; the word itself means *rolled up*. The product is called the *power*. In what way is involution performed? By multiplication."

Use. The principal use of involution is to afford an abbreviated form of expression. If a number is to be raised to a high power, the law for multiplying when the bases are the same may be used to advantage.

ILL. The amount of \$1 for 20 yr. at 6% compound interest is $\$(1.06)^{20}$. To make 20 separate multiplications would be a long process. The result can be found by 5 multiplications. Thus, $1.06 \times 1.06 = 1.1236$ or $(1.06)^2$; $1.1236^2 = 1.2625$ or $(1.06)^4$; $1.2625^2 = = 1.5938$ or $(1.06)^8$; $1.5938^2 = 2.5404$ or $(1.06)^{16}$; $2.5404 \times 1.2625 = 3.2071$ or $(1.06)^{16} \times (1.06)^4$ or $(1.06)^{20}$.

179. Evolution. Evolution is the process of finding the equal factor from the product and the number of times the factor occurs. The product is not named, the equal factor is the **root**, the number of times the factor occurs is the **index** of the root.

ILL. T. "It is sometimes necessary to find one of the equal factors of a number. Can you give me an illustration? If the cube of a number is 8, what is the number? Yes, or if the volume of a cube is 8 cu. in., what is its edge?

"Find the number whose 5th power is 32. How did you get 2, Henry? 'I found the prime factors of 32. $32 = 2^5$, or 2 is one of the 5 equal factors of 32.' Excellent.

"It is convenient to give names to the process and to the terms. The process is called **evolution**; the word itself means *unrolled*. The equal factor sought is called the *root*; why is *root* a good name for the term? The number of equal factors is called the **index** of the root. To find the number whose 5th power is 32 is to find the '5th root of 32.' The 'fifth root of 32' is written $\sqrt[5]{32}$. The sign, $\sqrt{\ }$, is a modification of *r*, the initial of root. When the index is 2 it is not written."

The Process for Pupils. A root which can be exactly expressed by the decimal notation can be found by factoring. Every root can be found by trial. It is recommended that the extraction of roots be limited in elementary schools to these two methods.

Ill. *By Factoring.* T. "Find the number whose square is 576 or find the square root of 576. By factoring, we find $576 = 2^2 \times 2^2 \times 2^2 \times 3^2 = 24^2$; $\sqrt{576} = 24$. Find $\sqrt[3]{9261}$."

Ill. *By Trial.* T. "Find $\sqrt{231}$. $231 = 3 \times 7 \times 11$; $\sqrt{231}$ cannot be found by factoring. What shall we do?

"We will find by trial two numbers differing by 1 unit, such that the square of one shall be less than 231 and the square of the other greater than 231. The smaller will be $\sqrt{231}$ true to units. $15^2 = 225$; $16^2 = 256$; $\sqrt{231} = 15+$.

"We will find by trial two numbers each of which is '15 +' differing by 1 tenth, such that the square of one shall be less than 231 and the square of the other greater than 231. The smaller will be $\sqrt{231}$ true to tenths. $15.1^2 = 228.01$; $15.2^2 = 231.04$; $\sqrt{231} = 15.1+$.

"In a similar way we can find the answer true to any required place. Can the answer be expressed exactly by the decimal notation?

"Find $\sqrt[3]{2.5}$ to one decimal place."

3 | 9261 3 | 3087 3 | 1029 7 | 343 7 | 49 7 $\sqrt[3]{9261} = 3 \times 7$	1.3 1.3 1.69 1.3 507 169 2.197	1.4 1.4 1.96 1.4 784 196 2.744
	$\sqrt[3]{2.5} = 1.3+$	

The Process for Teachers. To extract the *n*th root of an integer, point off into periods of *n* figures each beginning with units' place, extract the root of the number denoted by the first two periods, then the root of the number denoted by the first three periods, and so on.

As a guide, raise $a + b$ to the *n*th power and proceed as the formula indicates.

Ill. Extract the cube root of 1860867.

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$$
$$= a^3 + (3\,a^2 + 3\,ab + b^2)b$$

	1′860′867 | 123	
300	860	
60		$a = 10$
4		$b = 2$
364	728	
43200	132867	
1080		$a = 120$
9		$b = 3$
44289	132867	

Preparation. As a guide we raise $a + b$ to the 3d power and factor. We separate the number into periods of three figures each because the cube root of the number denoted by the first period will give the first figure of the root, the cube root of the number denoted by the first two periods will give the first two figures of the root, and so on. (The teacher should satisfy himself of these facts inductively.)

First Extraction. We extract the cube root of 1′860 to obtain the first two figures of the root.

If we extract the cube root of a^3, we get a, the first term of the root in the guide; hence, if we extract the cube root of 1, we must get the first figure of the root in this example; the cube root of 1 is 1, or the first figure is 1; $a = 10$ because 1 is not 1 unit but 1 thousand. Subtracting the value of a^3, we get 860 which equals $3\,a^2b + 3\,ab^2 + b^3$.

If we divide $3\,a^2b$ by $3\,a^2$, we get b, the second term of the root in the guide; hence, if we divide what corresponds to $3\,a^2b$, or the greater part of 860, by what corresponds to $3\,a^2$, we must get the second figure of the root in this example; $3\,a^2 = 300$; $860 \div 300 = 2$, or $b = 2$.

If we multiply what is within the parenthesis by b, we get the rest of the power in the guide; hence, if we multiply what corresponds to what is within the parenthesis by what corresponds to b, we should get the rest of the power in this example if it is a perfect power; $3\,a^2 = 300$; $3\,ab = 60$; $b^2 = 4$; the parenthesis $= 364$; multiplying by b, or 2, and subtracting, we get 132. Hence, the cube root of 1′860 is 12 +, and the remainder found by subtracting the cube of 12 from 1860 is 132.

Second Extraction. We extract the cube root of 1860′867 to get the first three figures of the root; we may regard the whole as 1860 *thousands* 867 *units.*

We know that the new a is 120 because 1860 is not 1860 units but 1860 thousands, and that the remainder after subtracting the cube of 120 from 1860867 is 132867 which $= 3\,a^2b + 3\,ab^2 + b^3$. Hence, we start in at once to find the new b.

If we divide $3\,a^2b$ by $3\,a^2$, we get b, the second term of the root in the guide; hence, if we divide what corresponds to $3\,a^2b$, or the greater part of 132867, by what corresponds to $3\,a^2$, we must get the next figure of the root in this example; $3\,a^2 = 43200$; and so on.

The Process for Teachers. To extract the root of a fraction extract the root of the numerator and then the root of the denominator. To extract the root of a decimal, point off into periods beginning with the decimal point and proceed as with integers, pointing off as many decimal places in the root as there are decimal periods in the decimal.

ILL. The square root of $\frac{25}{36} = \frac{5}{6}$; the square root of $\frac{2}{3} = \sqrt{.66'66^+}$. The square root of $.00365 = \sqrt{.00'36'50}$. The teacher should discover why the decimal is pointed off from the left rather than from the right.

Use. The principal use of evolution in arithmetic is in connection with a few problems in mensuration involving right-angle triangles, areas, and volumes.

180. Logarition. Logarition is the process of finding the number of times the equal factor occurs from the product and the equal factor. The product is the **antilogarithm**, the number of times the equal factor occurs is the **logarithm**, and the equal factor is the **base.**

ILL. How many times must 2 be used as a factor to make 8? Or, to what power must 2 be raised to make 8? This question is written, what is $\log 8_2$? It is read, what is the logarithm of 8 in the system whose base is 2? *Ans.* 3.

Use. Tables have been prepared stating the powers to which 10 must be raised to produce all numbers. By their aid, numbers may be multiplied, divided, raised to powers, and depressed to roots with marvelous speed and ease. Logarithms were discovered by Napier in the seventeenth century. It is said, "The miraculous powers of modern calculation are due to three inventions: the Hindu notation, decimal fractions, and logarithms."

181. Exercises. **1.** Extract the square root of 2 true to two decimal places by trial. **2.** True to 4 decimal places by the formula method. **3.** Show inductively that if an integer is pointed off into periods of two figures each, the square root of the first period gives the first figure of the root, the square root of the first two periods gives the first two figures of the root, and so on. **4.** Show deductively that in extracting the cube root, a decimal must be pointed off from the left. **5.** Show deductively that there are as many decimal places in a root as there are decimal periods in the power.

LESSON 26. Algebra in Arithmetic

182. Needs. The needs arise of a briefer notation, and of a simpler method of solving equations than is afforded by arithmetic. How to satisfy these needs as fully as is desirable in elementary schools is discussed in this lesson. *See* § *49.*

183. Numbers by Letters. The first letters of the alphabet, a, b, c, are used for known numbers, and the last letters, x, y, z, for unknown numbers.

Ill. "If you take a certain number, multiply by 12, add 10, divide by 2, and subtract 5, the result is 24; find the number. If the result is 24 after 5 is subtracted what is the number? 29. If the result is 29 after a number is divided by 2 what is the number? 58. And so on; the answer is 4.

"A shorter way is to represent the required number by x. Think, x, $12x$, $12x + 10$, $6x + 5$, $6x$, $6x = 24$, $x = 4$."

184. Signs + and −. A quality may be considered in two opposite phases, as heat and cold, up and down, east and west. One of these phases is called **positive** and represented by '+'; the other is called **negative** and represented by '−.'

Ill. T. "Let us agree that opposites may be represented by the signs, '+' and '−', and called **positive** and **negative.**

"Name two opposites. To the right and to the left; they may be represented by '+' and '−'. What would $+5$ in. mean? 5 in. to the right. -3 in.? 3 in. to the left.

"Have you had any such use of '+' and '−' before? Are not addition and subtraction opposites? Is not one represented by '+' and the other by '−'?"

185. Coefficients. A number may be used several times as an addend or several times as a subtrahend. The number of times is called the **coefficient.** A positive coefficient shows how many times a base is used as an addend; a negative coefficient shows how many times a base is used as a subtrahend.

ILL. T. "Write by the use of signs that a is to be added 3 times. $a + a + a$ is written $+ 3a$ or $3a$; when the sign is '+' it is usually omitted; $+ 3$ is called a **positive coefficient**; *coefficient* means *working together with.* Write that a is to be subtracted 3 times. $- a - a - a$ is written $- 3a$; $- 3$ is called a **negative coefficient.**

"What does $+ 3a$ mean? What does $- 3a$ mean? What does a positive coefficient show? A negative coefficient?

"Have you had any such use before? Does not 3×4 mean $4 + 4 + 4$ or that 4 is to be added 3 times? $+ 3$ may be regarded as the coefficient of 4."

186. Numbers with Direction. The need arises of performing the fundamental operations upon numbers with direction.

Addition. To add when the signs are alike write the sum and use the common sign; to add when the signs are unlike write the difference and use the sign of the greater.

ILL. T. "We want a rule for adding numbers with signs.

$$\begin{array}{cccc} +5 & -5 & +5 & -5 \\ \underline{+2} & \underline{-2} & \underline{-2} & \underline{+2} \end{array}$$

"Suppose that '+' means to the right and '−' to the left. Find the sums of the numbers on the board in this way. I draw a line and take the point A for the starting point. I will add $+ 5$ and $- 2$. $+ 5$ means 5 places to the right, — one, two, three, four, five; $- 2$ means 2 places to the left, — one, two; my last mark is 3 places to the right of A or $+ 3$; the sum of $+ 5$ and $- 2$ is $+ 3$.

"What seems to be the rule for addition when the signs are alike? When the signs are unlike?

"Let us see why this rule holds. With like signs, a number of operations is to be united with a number of the same operations and the sum must denote the same operation. With unlike signs, a number of additions cancels the same number of subtractions and the sign of the result is the sign of the greater coefficient."

Subtraction. To subtract change the sign of the subtrahend and proceed as in addition.

It would be very confusing if a minus sign were written before each subtrahend to indicate subtraction. $\begin{smallmatrix}+2\\-+5\end{smallmatrix}$ would be a monstrosity. *See* § *74.*

Ill. T. "We want a rule for subtracting numbers with signs.

$$\begin{array}{cccc} +2 & -2 & -2 & +2 \\ \underline{+5} & \underline{-5} & \underline{+5} & \underline{-5} \end{array}$$

"If you go to the south, what do you do to your position? You subtract from your north position and add to your south position. If you go to the north, what do you do to your position? You subtract from your south position and add to your north position. It seems, then, that to subtract a number with either sign is to add the number with its sign changed. Apply this rule to the numbers on the board. To subtract + 5 from + 2, change the subtrahend and proceed as in addition. What is the rule for subtraction?

"Let us look at this in another way. To subtract + 5 is the opposite of to subtract − 5; the opposite of to subtract − 5 is to add − 5; hence to subtract + 5 is to add − 5."

Multiplication. The product of like signs is plus; the product of unlike signs is minus.

Ill. T. "We want a rule for multiplying numbers with signs.

"-3×-4 means that -4 is to be subtracted 3 times. -4 from $0 = +4$; -4 from $+4 = +8$; -4 from $+8 = +12$; $-3 \times -4 = +12$. In the same way, find the value of $+3 \times +4$. Finish the sentence; the product of numbers with like signs is ——.

"In the same way, find the value of $+3 \times -4$; of $-3 \times +4$. Finish the sentence; the product of numbers with unlike signs is ——."

Division. The quotient of like signs is plus; the quotient of unlike signs is minus.

Ill. T. "We want a rule for dividing numbers with signs.

"Either of two factors is equal to their product divided by the other factor. $+3 \times -4 = -12$. From this law what must be $-12 \div -4$? $-12 \div +3$?

"$-3 \times -4 = +12$. What must be $+12 \div -3$? $+12 \div -4$?

"$+3 \times +4 = +12$. What must be $+12 \div +3$? $+12 \div +4$?

"Finish the sentences; the quotient of numbers with like signs is ——; the quotient of numbers with unlike signs is ——."

187. Removing Parentheses. To remove a parenthesis, multiply every term within it by its coefficient.

Ill. T. "A parenthesis about an expression means that every term within it is to be subjected to the same operation. Thus, $-5(6x-2)$ means that $6x-2$ is to be multiplied by -5. Remove the parenthesis; $-5(6x-2) = -30x+10$. What is the rule for removing a parenthesis?

"A bar is sometimes used in place of a parenthesis. Find the value of $-\overline{6x-2}$. This means that $6x-2$ is to be multiplied by -1. $-\overline{6x-2} = -6x+2$."

188. From nx to find x. Given the value of nx to find x, divide both members of the equation by the coefficient of x.

Ill. T. "$-3x = -12$. How shall we find the value of x?

"To find x we must divide $-3x$ by -3. Dividing both members of an equation by the same number cannot affect the balance. Divide both members of the equation by the coefficient of x. $x = 4$."

189. From x^n to find x. Given the value of x^n to find x, extract the nth root of both members of the equation.

Ill. T. "$x^2 = 16$. How shall we find the value of x?

"To find x we must extract the square root of x^2. Extracting the same root of both members of an equation cannot affect the balance. Extract the square root of both members. $x = 4$. $x^3 = 27$; find x."

190. Transposing. To transpose a term from one member of an equation to the other, change its sign.

Ill. T. "$2x - 3 = 9$. It is necessary to transpose -3 to the right-hand member of the equation so that $2x$ alone will be in the left-hand member. How shall we do it?

"What must be added to -3 to make zero? $+3$. Adding the same number to both sides cannot affect the balance. Adding $+3$ to both sides what do you get? $2x = 9 + 3$. What is the rule?"

191. Clearing of Fractions. To clear of fractions, multiply both members of an equation by the least common denominator.

Ill. T. "$\frac{3x}{2} - 10 = \frac{2x}{3}$. It is necessary to clear of fractions.

"What is the least common denominator? Multiplying both members of an equation by the same number cannot affect the balance. Multiply both members by 6. Say, 2 is contained in 6, 3 times; 3 times $3x$ is $9x$; 6 times -10 is -60; 3 is contained in 6, 2 times; 2 times $2x$ is $4x$. *Ans.* $9x - 60 = 4x$. What is the rule for clearing of fractions?

"Clear of fractions (I) on the board. The L. C. D. is 5; 5 times $3x$ is $15x$; 5 is contained in 5, 1 time; 1 times -1 is -1; -1 times $6x$ is $-6x$; -1 times -2 is $+2$; 5 times 4 is 20. *Ans.* $15x - 6x + 2 = 20$. I want you to see why we say, '1 times -1.' $-\frac{6x-2}{}$ means that $6x - 2$ is to be multiplied by -1 (§ 187); the 1 is not written but must be understood.

"Clear of fractions (II) on the board. Look out for the '$-$' before the fraction. 5 is contained in 10, 2 times; 2 times -3 is -6; -6 times x is $-6x$; -6 times -4 is $+24$."

(I) $3x - \frac{6x-2}{5} = 4.$ (II) $\frac{x}{2} - \frac{3(x-4)}{5} = \frac{2x-15}{2}.$

192. Discussion. Only enough of algebra should be taught in the elementary schools to give pupils something of an idea of its power and to provide for the indirect cases in percentage and interest and for the formulæ of mensuration. The latter, for $\frac{1}{4}\pi D^2$ and $\frac{1}{6}\pi D^3$, require the handling of equations with the 2d power of x and no other power of x, and equations with the 3d power of x and no other power of x.

193. Exercises. **1.** By counting as in § 186, find the sum of -5 and $+2$. **2.** If income is constant, savings and spendings are opposites; what is the effect on savings of subtracting from spendings? **3.** What is the effect on spendings of subtracting from earnings? **4.** In equation (II) of § 191 find the value of x and explain every step. **5.** Solve the equation, $\frac{1}{4}\pi D^2 = 78.54$, and explain every step (count π as 3.1416). **6.** Solve the equation, $\frac{1}{6}\pi D^3 = 113.0976$, and explain every step.

LESSON 27. Percentage

194. Development. In business a part is usually expressed as a number of the hundred equal portions of a whole. Hundredths is called **per cent** and the fraction is expressed by aid of the symbol, %.

Ill. T. "We are going to consider another way of expressing hundredths.

"Write 6 hundredths as a common fraction. $\frac{6}{100}$. Write small circles in place of the numerator and denominator and write 6 to the left. 6%. This means that the numerator is 6 and the denominator 100; it is read *6 per cent.*"

195. Reductions. When the denominator of a fraction is 2, 3, 4, 5, 6, or 8, it is customary to use the common fraction in place of its per cent equivalent. Pupils should find the equivalents and master them so thoroughly that an expression in either form will suggest the other instantly without any form of computation.

$\frac{3}{5}$	$83\frac{1}{3}$%	$\frac{7}{8}$	$62\frac{1}{2}$%	$\frac{1}{2}$	75 %	$\frac{3}{4}$	50%	$\frac{5}{6}$	$16\frac{2}{3}$%
$\frac{5}{8}$	$87\frac{1}{2}$%	$\frac{1}{3}$	$12\frac{1}{2}$%	$\frac{2}{5}$	$33\frac{1}{3}$%	$\frac{3}{8}$	60%	$\frac{1}{4}$	40 %
$\frac{4}{5}$	25 %	$\frac{1}{8}$	$66\frac{2}{3}$%	$\frac{2}{3}$	$37\frac{1}{2}$%	$\frac{1}{5}$	20%	$\frac{1}{6}$	80 %

Ill. T. "I asked you to bring in on paper the per cent equivalents of the business fractions. Here they are on this chart. As I point call the equivalent instantly (he points at the rate of one a second). You fail on $\frac{5}{6}$, $\frac{5}{8}$, and $\frac{7}{8}$. Give these special attention; you must not compute their values."

196. Terms. In the full expression of a part there are three terms, the whole, the fraction of the whole, and the part. They are called **base, rate,** and **percentage.** The base plus the percentage is the **amount**; the base minus the percentage is the **difference.**

ILL. T. "3 = 6% of 50; 53 = 50 + 6% of 50; 47 = 50 − 6% of 50. 3 is the **percentage**; 6%, the **rate**; 50, the **base**; 53, the **amount**; 47, the **difference.**"

197. Direct Cases. The direct cases are to find the part, the whole plus the part, and the whole minus the part, from the whole and the fraction of the whole — P, A, D from B and R (p. 7).

At least 99% of all problems in percentage outside of the schoolroom are of these types. Pupils should work with them for a long time before considering the other types, and until they can find results rapidly and accurately both *without* the pencil and *with* the pencil.

It is best both for mental and for written work to find 1% and to modify the result. It is worth while in business problems to be able to call the facts in the following table without stopping to compute them.

1% of $100 = $1	1% of $10,000 = $100
1% of $1000 = $10	1% of $100,000 = $1000

1% of $1,000,000 = $10,000

Mental. Before taking up written exercises pupils should gain facility in mental computations. It should be remembered that the pencil is required only when the operations are difficult. See § 70.

Exercises like the following are as valuable in preparation for the solution of problems as examples in abstract multiplication and division.

ILL. *Fractions.* T. "What is $\frac{2}{5}$ of 30? What is the result when 30 is increased by $\frac{2}{5}$ of itself? What is the result when 30 is diminished by $\frac{2}{5}$ of itself? What is the result when a no. is increased by $\frac{2}{5}$ of itself? $\frac{7}{5}$ *no.* What is the result when a no. is diminished by $\frac{2}{5}$ of itself? $\frac{3}{5}$ *no.*"

Ill. *Per Cents.* T. "What is 6 % of $ 500? Find 1 % and multiply by 6; think, 1 % of $ 500 is $ 5, and say, $ 30. What is the result when $ 500 is increased by 6 % of itself? What is the result when $ 500 is diminished by 6 % of itself? What is the result when a no. is increased by 6 % of itself? *106 % no.* What is the result when a no. is diminished by 6 % of itself? *94 % no.*"

Ill. *Equivalents.* T. "What is 87½ % of $ 48? Think, ⅛ of $ 48 is $ 6, and say, $ 42. What is the result when $ 48 is increased by 87½ % of itself? What is the result when $ 48 is diminished by 87½ % of itself? What is the result when a no. is increased by 87½ % of itself? $\frac{15}{8}$ *no.* What is the result when a no. is diminished by 87½ % of itself? $\frac{1}{8}$ *no.*"

Ill. *Frac. Per Cents.* T. "You were asked to memorize the facts on the board. Clean the board. What is 1 % of $ 10,000? of $ 1,000,000? (he drills expecting an answer each second). What is ⅛ % of $ 10,000? Think, $ 100, and say, $ 12.50." (Such exercises are valuable because brokerage is usually reckoned as ⅛ % of the par value.)

Written. Examples of the same nature as the foregoing should be given to afford practice when the computations cannot be readily made without the pencil. Attention is called to the value of finding 1 % as the first step.

$ 2.86×50	$ 286.50
37	.37
20055	20055
8595	8595
$ 106.005	$ 106.005

Ill. To find 37 % of $ 286.50, we move the decimal point to find 1 %, and then multiply by 37. This practice conforms to the plan in mental work and assists in fixing the place of the decimal point.

Since 30 × 2 is 60, the answer cannot be so small as $10 nor so great as $1060; it must be $106. As a check, the pupil should be required to fix the point as just suggested, and also by counting the places.

198. Indirect Cases. The indirect cases usually presented in arithmetic are the inverse of the direct cases.

DIRECT	INDIRECT
P from B & R; $P = BR$	$\begin{cases} R \text{ from } B \text{ \& } P;\ R = \frac{P}{B} \\ B \text{ from } P \text{ \& } R;\ B = \frac{P}{R} \end{cases}$
A from B & R; $A = B + BR$	B from A & R; $B = \frac{A}{1 + R}$
D from B & R; $D = B - BR$	B from D & R; $B = \frac{D}{1 - R}$

ILL. T. "You may copy from the board the cases which you have already had. They are called **direct**. From the first case write a new problem in which the answer to the first is given in the second and the required term in the second is 6%; do the same making the required term 50 (§ 46). From the 2d and 3d direct cases, do the same, making the answer the given term and 50 the required term. These four new cases are called **indirect**."

DIRECT	INDIRECT
What is 6% of 50? *Ans.* 3	{ 3 is what % of 50? { 3 is 6% of what?
What is 50 + 6% of 50? *Ans.* 53	53 is 6% more than what?
What is 50 − 6% of 50? *Ans.* 47	47 is 6% less than what?

Solutions in General. The indirect cases can be solved by algebra, by formula, by rule, or by analysis. The 2d and 3d cases can also be solved by proportion, or by variation (Lesson 11).

By algebra, the values of the known terms may be substituted in the formula for the direct relation and the equation solved, or the required term may be represented by x and the equation formed by reasoning. See § 49.

By formula, the required term is made the left-hand member of an equation derived from the direct formula before the substitutions are made. See § 50.

By rule, the translation of the formula obtained by the last method is stated from memory. See § 51.

By analysis, the problem is separated into its simple problems. See § 36.

Each method has its advocates. The author prefers the analysis method, or the 2d algebra method, because by them it is not necessary to decide upon the technical name for each term, a decision which is sometimes quite perplexing. Thus, in the problem below, by these methods it is not necessary to decide that 47 is the difference.

Ill. If a number is diminished by 6 % of itself, the result is 47. What is the number?

1st Algebra		2d Algebra	
D, 47 R, 6 % B, ? ——— $B = 50$. *Ans.*	$D = B - BR$ $47 = B - .06\,B$ $47 = .94\,B$ $B = \frac{47}{.94} = 50$	No. dec., 47 Dec., 6 % no. No. ? ——— No., 50. *Ans.*	Let $x =$ no. $.06\,x =$ dec. $.94\,x =$ no. dec. $.94\,x = 47$ $x = 50$
Formula		**Rule**	
D, 47 R, 6 % B, ? ——— B, 50. *Ans.*	$D = B - BR$ $D = B(1 - R)$ $B = \frac{D}{1 - R}$ $= \frac{47}{.94} = 50$	D, 47 R, 6 % B, ? ——— B, 50. *Ans.*	Given the difference and the rate to find the base, divide the difference by 1 minus the rate. $47 \div .94 = 50$

Analysis in General. It is convenient to designate the three direct cases as Case 1, and the four indirect cases as Case 2, Case 3, Case 4, and Case 5.

Ill. *Case 2.* T. "12 is what part of 20? *Ans.* $\frac{12}{20}$ of 20 or $\frac{3}{5}$ of 20. "12 is what % of 20? *Ans.* 12 is $\frac{12}{20}$ of 20 or $\frac{3}{5}$ of 20 or 60 % of 20."

Ill. *Case 3.* T. "12 is $\frac{3}{5}$ of what? *Ans.* If 12 is $\frac{3}{5}$ no., what is $\frac{1}{5}$ no.? *4.* If 4 is $\frac{1}{5}$ no., what is $\frac{5}{5}$ no.? *20.*

"12 is 3 % of what? *Ans.* If 12 is 3 % no., what is 1 % no.? *4.* If 4 is *1 % no.*, what is 100 % no.? *400.*"

Ill. *Cases 4 & 5.* T. "What no. increased by $\frac{2}{5}$ of itself becomes 14? *Ans.* If a number is increased by $\frac{2}{5}$ of itself, what does it become? *$\frac{7}{5}$ no.* If *$\frac{7}{5}$ no.* is 14, etc.

"What no. increased by 6 % of itself becomes 53? *Ans.* If a no. is increased by 6 % of itself, what does it become? *106 % no.* If 106 % no. is 53, what is 1 % no.? $\frac{1}{2}$. If 1 % no., is $\frac{1}{2}$, etc.

"What no. decreased by $\frac{2}{5}$ of itself becomes 21?

"What no. decreased by 6 % of itself becomes 47?"

Analysis in Particular. Cases 4 and 5 may be regarded as another way of stating Case 3 and may be treated as Case 3. Pupils should practice changing from one form of expression to the other.

Ill. T. "Change to an expression *without* more or less: 16 is $\frac{1}{3}$ more than 12. *Ans.* 16 is $\frac{4}{3}$ of 12. 8 is $\frac{1}{3}$ less than 12. *Ans.* 8 is $\frac{2}{3}$ of 12. A's money is $\frac{1}{4}$ more than B's. *Ans.* A's money is $\frac{5}{4}$ as much as B's. In a mixture of water and vinegar the water is $\frac{1}{3}$ more than the vinegar. *Ans.* In a mixture of water and vinegar the water is $\frac{4}{3}$ as much as the vinegar.

"Change to an expression *with* more or less: A's weight is $\frac{3}{5}$ of B's. *Ans.* A's weight is $\frac{2}{5}$ less than B's. A's salary this year is $\frac{6}{5}$ as much as last year. *Ans.* A's salary this year is $\frac{1}{5}$ more than last year.

"Solve as Case 3. If a no. is increased by 6 % of itself the result is 53. What is the no.? *Ans.* If 106 % of a no. is 53, what is 1 % no.? etc. In a mixture of 14 gal. of water and vinegar the water is $\frac{1}{3}$ more than the vinegar. How much is the vinegar? *Ans.* If the water is $\frac{4}{3}$ as much as the vinegar, how much is the mixture? $\frac{7}{3}$ as much as the vinegar. If $\frac{7}{3}$ as much as the vinegar is 14 gal., etc."

Analysis Written Problems. A problem can be stated in full from the proper expression of what is given and what is required.

CASE 2. Had 210 sheep and sold 54. Sold what % flock? F, 210 sh Sold, 84 sh Sold, ? % F —— Sold, 40 % F *Ans.* $\frac{84}{210} = \frac{12}{30}$ $= \frac{2}{5}$ $= 40\%$	CASE 3. Sold 84 sheep which was 40 % flock. Flock what? Sold, 84 sh or $\frac{2}{5}$ F F, ? —— $\frac{1}{5}$ F, 42 sh F, 210 sh *Ans.* 42 210
CASE 4. Bought 40% flock; then had 294 sheep. Flock what? Aft. P, 294 sh or $\frac{7}{5}$ F F, ? —— $\frac{1}{5}$ F, 42 sh F, 210 sh *Ans.* 42 210	CASE 5. Sold 40% flock; then had 126 sheep. Flock what? Aft. S, 126 sh or $\frac{3}{5}$ F F, ? —— $\frac{1}{5}$ F, 42 sh F, 210 sh *Ans.* 42 210

Case 2. If he had 210 sheep and sold 84 sheep, what per cent of his flock did he sell? $\frac{84}{210}$ or $\frac{12}{30}$ or $\frac{2}{5}$ or 40%.

Case 4. If he purchased $\frac{2}{5}$ as many sheep as were in the flock, how many sheep did he then have? $\frac{7}{5}$ flock. If $\frac{7}{5}$ flock is 294 sheep, etc.

199. Exercises. **1.** Try the reductions of § 195. Can you call one a second? **2.** Can you name the results in § 197 in 5 seconds? **3.** In § 198 show how each of the indirect formulas is derived from its direct form. **4.** State the problem of Case 3, p. 143. **5.** Solve it by the 1st algebra method. **6.** Solve it by the 2d algebra method. **7.** Solve it by the formula method. **8.** Solve it by rule. **9.** State with reasons which method you prefer. **10.** Which of the five cases should receive chief attention? Why? **11.** How many cases are there in percentage? See p. 7. **12.** State a problem for every case.

LESSON 28. PERCENTAGE

200. Important Suggestion. In solutions, what the per cent is of should be expressed after every per cent. It is usually omitted in the statement of a problem, and if it is omitted in the solution also, clear thinking is impossible.

201. Profit and Loss. Unless otherwise specified gain or loss is reckoned as some per cent of the cost.

ILL. *Development.* T. "Shall we regard gain and loss as some per cent of the cost or some per cent of the selling price? As some per cent of the cost because the cost comes first; a man cannot sell a thing until he gets it. A merchant sells goods at a gain of 40%. Supply the omission. *Ans.* A merchant sells goods at a gain of 40% of the cost."

ILL. *Direct Cases.* T. "Goods that cost 50¢ are sold at a gain of 40%. What is the gain? the selling price? *Ans.* If the cost is 50¢ and the gain is 40% of the cost, what is the gain? 20¢. Be sure to use after 40% what 40% is of.

"Butter that costs 24¢ is sold at a loss of $12\frac{1}{2}$%. What is the loss? the selling price?"

ILL. *Indirect Cases.* T. "Make the statement without the use of the words, gain or loss. An article is sold for 60¢ at a gain of $33\frac{1}{3}$%. *Ans.* An article is sold for 60¢ or for $\frac{4}{3}$ cost. An article is sold for 40¢ at a loss of $16\frac{2}{3}$%. *Ans.* An article is sold for 40¢ or for $\frac{5}{6}$ cost.

"A book that cost $6 was sold for $4. What per cent was lost?

"On goods sold at a profit of $66\frac{2}{3}$% the gain was $24. What was the selling price? *Ans.* If $\frac{2}{3}$ cost was $24, etc.

"Goods are sold for $2.50 at a gain of 25%. What is the cost?

"A coat was sold for $15 at a loss of $16\frac{2}{3}$%. What was the loss in dollars?"

Written Problems. Be sure to write after each per cent expression what it is of.

Case 1. A man bought goods for $528.60 and sold them at a gain of 6%. What was the gain?

C, $528.60
G, 6% C
G, ?

5.28×60
6
31.716

G, $31.72 *Ans.*

Case 2. A man bought goods for $528.60 and sold them at a gain of $31.716. What % did he gain?

C, $528.60
G, $31.716
G, ? % C

.06
528×6.)31×7.16
31 7 16

G, 6% C *Ans.*

Case 3. A man sold goods at a gain of 6% or at a gain of $31.716. What was the cost?

G, $31.716 or 6% C
C, ?

6)31.716
5.286

C, $528.60 *Ans.*

Case 4. A man sold goods for $560.316 at a gain of 6%. What was the cost?

S, $560.316 or 106% C
C, ?

5.286
106)560.316
530

C, $528.60 *Ans.*

Expl. *Case 4.* If 106% cost is $560.316, what is 1% cost? $5.286. If 1% cost is $5.286, what is the cost? $528.60.

Difficult Problems. Problems involving two or more bases are often given in examinations of teachers. They should not be taught in the elementary schools. The secret of success in their solution is to state what the per cent is of after each per cent.

Ill. 1. At what per cent must a merchant mark goods so that he can make a discount of 10% and yet make a gain of 26%?

As Case 4 and Case 5 this means, At what per cent of the cost must a merchant mark goods so that he can make a discount of 10% of the marked price and yet make a gain of 26% of the cost?

As Case 3 this means, At what per cent of the cost must a merchant mark goods so that he can sell them for 90% of the marked price or for 126% of the cost?

If 90% marked price is 126% cost, what is 1% marked price? etc.

Ill. 2. A man sold two articles at the same price. On one he gained 20% and on the other he lost 20%. What per cent did he gain or lose on the whole?

This means, On the first he gained 20% cost of first and on the second he lost 20% cost of second. What per cent of the cost of both did he gain or lose? As Case 3 it means, He sold the first for $\frac{6}{5}$ cost of first and the second for $\frac{4}{5}$ cost of second, etc.

The cost of each, the cost of both, the selling price of each, and the selling price of both must be expressed by the same unit. If SP is $\frac{6}{5}$ C_1, what is C_1? $\frac{5}{6}$ SP. If SP is $\frac{4}{5}$ C_2, what is C_2? $\frac{5}{4}$ SP. What is cost of both? $\frac{25}{12}$ SP. What is selling price of both? $\frac{24}{12}$ SP. What is the loss on both? $\frac{1}{12}$ SP., etc.

202. Commission. When an agent buys, his commission is some per cent of his buying price; when he sells, his commission is some per cent of his selling price; when he collects, his commission is some per cent of his collection.

Ill. T. "A person may engage an agent to transact business. In what capacities may the agent serve? He may buy, or sell, or collect.

"Give me an illustration of an agent buying. A grain dealer in N. Y. City employs an agent in Chicago to purchase 1000 bu. of wheat. The agent's commission is 1%. What is the commission 1% of? 1% of what the agent pays for the wheat. Why?

"Give me an illustration of an agent selling. A farmer sends 100 bbl. of apples to a merchant in N. Y. City, directing him to sell them. The agent's commission is 10%. 10% of what? Of what the merchant sells them for. Why?

"Give me an illustration of an agent collecting. An owner of an apartment house employs an agent to collect the rents. The agent's commission is $2\frac{1}{2}$%. $2\frac{1}{2}$% of what? Of what he collects. Why?"

Difficult Problems. Problems in which the commissions are of different bases are often given in examinations of teachers. They should not be taught in the elementary schools. The secret of success in their solution is to state what the commission is of after every per cent.

Ill. An agent sells goods for $4800, charging $3\frac{1}{4}$% commission. After paying $25 charges he invests the balance in raw material, retaining a commission of $2\frac{1}{2}$%. How much does the agent pay for raw material?

Agt's SP,	$4800	48.00x	102½)4619
1st com,	3¼% Agt's SP	3¼	
Ch,	$25	12	45.0634
2d com,	2½% Agt's C	144	205)9238.
Agt's C,	?	156	820
	———	4800	1038
1st com,	$ 156	4644	1025
Rem,	$4644		etc.
Af. ch,	$4619 or 102½% Agt's C		
Agt's C,	$4506.34 *Ans.*		

Expl. If the agt's SP is $4800 and the 1st com. is 3¼% of agt's SP, what is the 1st com.? If the agt's SP is $4800 and the com. $156, what is the rem.? If the rem. is $4644 and the charges $25, what is left after the charges? If the 2nd com. is 2½% agt's C, what is the amount with the com.? If 102½% agt's C is $4619, etc.

203. Commercial Discount. The net price is the continued product of the list price and the remainders found by subtracting each discount from 100%.

Ill. *Development.* T. "Manufacturers and wholesale dealers usually publish a catalogue of their goods with a fixed price after each article. Here is such a list (he passes it around). For the sake of making quick sales or to meet the market, how can they change these prices? It would be very expensive to publish a new catalogue. They offer a single discount or several successive discounts from the list price.

"The first discount is some per cent of the list price, the second is some per cent of the 1st remainder, the third is some per cent of the 2d remainder, and so on.

"What is the net price with a single discount of 25%? *Ans.* 75% L; it is the list minus 25% list.

"What is the net price with two discounts of 25% and 20%? *Ans.* 60% L. The rem. after the 1st dis. is 75% L; the rem. after the 2d dis. is '80% or ⅘ of 75% L' or 60% L.

"What is the net price with three discounts of 25%, 20% and 10%? *Ans.* 54% L. The rem. after the 2d dis. is 60% L; the rem. after the 3d dis. is '90% or $\frac{9}{10}$ of 60% L,' or 54% L."

Ill. *Direct Cases.* T. "What is the net price of a bill of goods listed at $100 with two discounts of 10% and 5%? *Ans.* The net price is 95% of 90% L or 85.5% L; 85.5% of $100 is $85.50.

"Work it in another way. *Ans.* The 1st dis. is 10% of $100 or $10; the 1st rem. is $100 − $10 or $90; the 2nd dis. is 5% of $90 or $4.50; the 2d rem. is $90 − $4.50 or $85.50.

"What is the difference in the net price between three discounts of 20%, 10%, and 5%, and three discounts of 5%, 10%, and 20%? *Ans.* One is 80% of 90% of 95% L, and the other is 95% of 90% of 80% L. The product is the same in whatever order the multiplication is performed. There is no difference."

204. Stocks. *Development.* Several persons may form a company to do business as a single individual, and may become incorporated by obtaining a legal charter from the secretary of state. Such a corporation is a **stock company**, their holdings is **stock**, the equal parts into which the stock is divided are **shares**, and the paper showing how many shares have been sold at one time is a **certificate** of stock.

Ill. July 1, 1911, several individuals living in N. Y. City obtained a charter under the name of the Hygiea Ice Co. for the production of artificial ice. After electing John Smith president and Henry Brown treasurer, they printed 2000 certificates of stock and bound them into books.

Cer. No.______	Cer. No.______ New York City, N. Y.
No. Sh.______	No. Sh.______ ______, 19__
Date______	*This is to certify that______*
Name______	*is the owner of______shares of*
______	*stock of the Hygiea Ice Co., office 169 Broadway, N. Y. City. Capital stock,*
Address______	*1000 shares; par value of 1 sh., $100.*
______	______ Pres. ______ Treas.

They sold all of the stock at $ 80 a share, and thus obtained a capital of $ 80,000 for the development of the business.

A certificate duly made out and signed is given to each purchaser, and the stub duly filled in is retained by the company.

ILL. Oct. 15, 1911, the company sold 50 shares to George W. Williams, Yonkers, N. Y., and issued to him certificate No. 100. Write the certificate and the stub and detach the stub.

If the company is prosperous, it issues dividends at equal intervals, as annually or semiannually, declared as some per cent of the par value, that is, some per cent of the value printed on the certificate.

ILL. During the first year the company made $ 10,000 clear of all expenses. They retained $ 2000, and paid out $ 8000 in dividends.

What per cent of the par value was the dividend? What was the dividend in dollars on 1 share? How much was Williams' dividend? What effect on the market value of the stock should be expected from this large dividend?

If the owner of a certificate of stock desires to sell, he may offer it to different individuals, but usually he takes it to a dealer in stocks (**broker**) and pays a commission (**brokerage**) for making the sale.

ILL. Oct. 15, 1912, Williams decided to sell his stock. He wrote on the back of his certificate, "Transfer to the order of ———— George W. Williams," and gave it to a broker, who sold it to Henry Wilson for $ 102 a share. The broker retained $ $\frac{1}{8}$ per share for brokerage and 2 ¢ a share for state tax, and sent Williams the balance. He wrote the name of Henry Wilson on the line left for the name of the purchaser, and sent the stock to the home office, where a new certificate was issued and sent to Henry Wilson.

Make the transfer indorsement on the certificate. How much did Williams make on his investment? Consider cost, dividend, selling price, brokerage, and state tax.

Preferred and Common. Sometimes a company issues stock of two kinds, preferred and common. The preferred states that a specified dividend will be paid at fixed intervals if the earnings of the company warrant; the common states that no dividends will be paid until all dividends of the preferred have been satisfied. The preferred is the more secure, but the common often pays the larger dividends.

Forms of Expression. The business terms used in stocks are much abbreviated, and are always some per cent of the par value. For clear thinking, pupils should be required to use the unabbreviated forms, and to express each term as dollars a share.

Business Form	Unabbreviated Form
A 6 % dividend.	A dividend of $6 a share.
5 % stock.	Stock that pays an annual dividend of $5 a share.
Stock @ 90.	Stock @ $90 a share.
Stock @ 90 %.	Stock @ $90 a share.
Brokerage $\frac{1}{8}$.	Brokerage $ $\frac{1}{8}$ a share.
Brokerage $\frac{1}{8}$ %.	Brokerage $ $\frac{1}{8}$ a share.
Bought at a premium of 20 %.	Bought @ $120 a share.
Sold at a discount of 20 %.	Sold @ $80 a share.
$6000 stock.	60 shares of stock.
6 % stock yields an income of 5 %.	Stock pays a dividend of $6 a share or 5 % of the cost of a share.

Written Problems. In the statement of what is given and what is required it is best to use the unabbreviated forms.

Ill. 4. If the first cost of 1 sh. is 89\frac{7}{8}$ and the br. is $$\frac{1}{8}$, what is the entire cost of 1 sh.? $90. If the inc. on 1 sh. is $6, on how many shares is the inc. $540? 90. If the cost of 1 sh. is $90, what is the cost of 90 sh.? $8100.

Ill. 6. If 5% cost is $8, what is 1% cost? $1.60. If 1% cost is $1.60, what is the cost? $160.

Ill. 1. What is the cost of $4800 stock at 110, brokerage $\frac{1}{8}$?

Mv 1 sh, $110
Br 1 sh, $$\frac{1}{8}$
No. sh, 48
C, ?

———

C 1 sh 110\frac{1}{8}$
C, $5286 *Ans.*

$$\begin{array}{r} 110\frac{1}{8} \\ \underline{48} \\ 6 \\ 880 \\ \underline{440} \\ 5286 \end{array}$$

· Ill. 2. How many sh. at 20 % dis. can be purchased for $5128, br. $\frac{1}{8}$?

Mv 1 sh, $80
Br 1 sh, $$\frac{1}{8}$
C, $5128
No. sh, ?

———

C 1 sh, 80\frac{1}{8}$
Sh, 64 *Ans.*

$$\begin{array}{r} 80\tfrac{1}{8})5128 \\ \underline{64} \\ 641)41024 \\ \underline{3846} \\ 2564 \\ \underline{2564} \end{array}$$

Ill. 3. What income will be obtained from $1600 invested in 5 % stock @ 79$\frac{7}{8}$, br. $\frac{1}{8}$ %?

Inv, $1600
Mv 1 sh, 79\frac{7}{8}$
Br 1 sh, $$\frac{1}{8}$
D 1 sh, $5
Td, ?

———

No. sh, 20
Td, $100 *Ans.*

Ill. 4. How much must be inv. in 6 % stock @ 89$\frac{7}{8}$, br. $\frac{1}{8}$, to get an annual inc. of $540?

C 1 sh, $90
D 1 sh, $6
Td, $540
Inv, ?

———

No. sh, 90
Inv, $8100 *Ans.*

Ill. 5. What per cent income will I receive if I buy 6 % stock @ 50?

C 1 sh, $50
D 1 sh, $6
D, ? % C

———

D, 12 % C *Ans.*

Ill. 6. How much must I pay for 8 % stock to make 5 %?

D 1 sh, $8 or 5 % C
C 1 sh, ?

———

C 1 sh, $160 *Ans.*

205. Exercises. **1.** Explain the problems above. **2.** Explain how each full form on p. 150 is derived from the business form. **3.** Translate the statement of each problem on p. 151 from the business form to the full form. Thus, Ill. 1. What is the cost of 48 shares of stock at $110 a share, brokerage $$\frac{1}{8}$ a share? **4.** Consider the par value $50 a share and change every business form on p. 150 to its full form. Thus, "A 6 % dividend" is "A dividend of $3 a share."

LESSON 29. INTEREST

206. Negotiable Notes. *Without Interest.* In order to facilitate the transaction of business one person frequently gives to another a written promise to pay a given sum of money at a given time and place. Pupils should write and memorize the form of such an agreement; they should be careful not to sign their names to business papers prepared in school.

ILL. T. "July 1, 1912, Henry Jones buys a bill of goods of John Smith for $1100. Instead of paying cash he gives a written promise called a **note**, agreeing to pay the bill 3 mos. after date at the Tenth National Bank. In order that this note may be negotiable Jones makes it payable to the order of John Smith.

> *$1100.00* N. Y. CITY, N. Y., *July 1*, 19*12*
>
> *Three months after date, value received, I promise to pay to the order of John Smith, One Thousand One Hundred and $\frac{00}{100}$ Dollars at the Tenth National Bank.*
>
> *No. 1* *Henry Jones*

"Copy this note on a piece of paper 7′′ × 3′′, and mark it No. 1; memorize the form. Explain the features of the note. The note must state the date and place where it is drawn; the date and place where it is to be paid; that value has been received; and by whom, to whom, and how much is to be paid.

"What are the technical terms? Henry Jones is the **drawer** or **maker** or **payer**; John Smith is the **drawee** or **payee**; the date of payment is the date of **maturity**; the sum for which the note is drawn is the **face**; the amount to be paid at the date of maturity is the **amount** at maturity.

"What is the date of maturity? Oct. 1. If the note had read '90 days after date,' what would have been the date of maturity? July 91 or Aug. 60 or Sept. 29 (§ 164). What is the amount at maturity? The face or $1100."

With Interest. If a note without interest is not paid when it becomes due, interest begins with the date of maturity. Sometimes, however, the maker agrees to pay interest from the date of the note. In this case, with interest or with interest at a specified rate is added to the note.

Ill. T. "By the sale of these goods Jones expected to make 40% of their cost or $440. He was willing, therefore, to pay something, **interest**, for the use of the money. Suppose he wrote the note for 3 mo. with interest at 4%. Write the note and mark it No. 2.

> *$1100.00* N. Y. City, N. Y., *July 1*, 19*12*
>
> *Three months after date, for value received, I promise to pay to the order of John Smith, One Thousand One Hundred and $\frac{00}{100}$ Dollars at the Tenth National Bank, with interest at 4%.*
>
> *No. 2* *Henry Jones*

"What is the date of maturity? Oct. 1. What is the period for which $1100 is to bear interest? Your answer, 3 mos., is contrary to business usage. The interest period is the actual no. of days from the date of the note, July 1, to its maturity, Oct. 1, or 92 days. See § 164.

"If 'with interest' had been written instead of 'with interest at 4%' what would have been the rate? The legal rate of the state of New York, or 6%."

207. Interest Computed. The cancellation method is a favorite with teachers because it is easily taught. It should be made the basis because it leads at once to valuable modifications.

Cancellation Method. No rule is necessary. The pupil finds the simple problems and their answers. *See § 39.*

ILL. T. "Look at Note No. 2. We must find the interest of \$1100 for 92 days at 4%. Before solving this problem we will solve a few others.

"Find the interest of \$204 for 1 yr. 5 mo. 17 da. at 7%. There are many ways of proceeding. We will find the interest for 1 da. and then multiply by the no. of days. How many days shall we count as a year? In a year there are 674 sec. less than $365\frac{1}{4}$ da. (p. 112), but unless otherwise specified 360 da. are counted to the year in computing interest. How many days are there in 1 yr. 5 mo. 17 da.? 527.

$$204 \times \tfrac{7}{100} \times \tfrac{1}{360} \times 527$$

"Multiplying \$204 by $\frac{7}{100}$ gives the interest for 1 yr. at 7%; dividing by 360, for 1 da.; multiplying by 527, for 527 da. Finish the work; in cancelling, never divide 100 by a common factor. Why not? The interest is \$20.90."

Six Per Cent by Days. To find the interest at 6%, move the decimal point of the principal 3 places to the left, multiply by the number of days and divide by 6. Modify the result for a different rate.

ILL. T. "Let us find a rule for computing interest at 6%. Using the cancellation method, find the interest of P dollars for D days at 6%.

$$P \times \tfrac{6}{100} \times \tfrac{1}{360} \times D = .001\,P \times D \times \tfrac{1}{6}$$

"Who can give me the rule? By this rule what is the interest of \$204 for 1 yr. 5 mo. 17 da. at 6%? \$17.918. At 7%? \$20.90.

17.918	**6**
2.986	**1**
20.904	**7**

"What is the best way of finding interest at 7% from interest at 6%? 7 is 1 more than 6; divide by 6 and add. In dividing 17.918 by 6 the exact quotient is 2.986$\frac{1}{3}$. Is it necessary to write $\frac{1}{3}$? Why not? In case of 2.986$\frac{2}{3}$ what would we have done? Written 7 in place of 6. Why?

"Give me the complete rule."

Six Per Cent Basis $1. Find the interest of $1 for the given time at 6% and multiply by the number of dollars. After the multiplication, modify the result for a different rate. To find the interest of $1, count the interest for 1 yr. 6¢; for 1 month, $\frac{1}{2}$¢; for 1 day, $\frac{1}{6}$ m.

Ill. T. "Let us compute interest by first finding the interest of $1 at 6%.

"What is the interest of $1 for 1 yr. at 6%? 6¢. For 1 mo.? $\frac{1}{2}$¢ ($\frac{1}{12}$ of 6¢). For 1 da.? $\frac{1}{6}$ of a mill ($\frac{1}{30}$ of 5 m.). Memorize these facts.

"Find the interest of $1 for 1 yr. 5 mo. 17 da. at 6%. *See next page.*

"The interest of $1 for 1 yr. at 6% is $ .06; for 5 mo., $.025; for 17 da., $.002$\frac{5}{6}$; for the whole time, $ .087$\frac{5}{6}$.

"If the interest of $1 is $ .087$\frac{5}{6}$, what is the interest of $204? $17.918. If the interest at 6% is $17.918, what is the interest at 7%? $20.90."

Aliquot Part Method. Find the interest for a year at the given rate and modify the result for the years, the months and the days.

Ill. T. "Let us compute interest by first finding the interest for 1 year.

"What is the interest of $204 for 1 yr. at 7%?"

	14.28	1 yr.
1.19		1 mo.
	5.95	5 mo.
.039$\frac{2}{3}$		1 da.
	.674	17 da.
	20.904	

Bankers' Method. To find the interest for 60 days at 6%, move the decimal point of the principal two places to the left. Modify the result for a different time or rate.

ILL. T. "Notes are usually drawn for 30 da., 60 da., 90 da., or 120 da. Let us find a rule for computing interest for 60 da. at 6%.

$$P \times \tfrac{6}{100} \times \tfrac{1}{360} \times 60 = .01\ P$$

"Who can give me the rule? At 6% what is the interest of $225 for 60 da.? $2.25. Of $250 for 90 da.? $3.75 ($2.50, $1.25, $3.75). Of $300 for 30 da.? $1.50 ($3, $1.50)."

CANCELLATION

P, $204
T, 1 yr. 5 mo. 17 da.
R, 7%
I, ?

360
150
17
527

$204 \times \frac{7}{100} \times \frac{1}{360} \times 527$

I, $20.90 *Ans.*

6% BY DAYS

P, $204
T, 1 yr. 5 mo. 17 da.
R, 7%
I, ?

.204×	
527	
107.508	
17.918	6
2.986	1
20.904	7

I, $20.90 *Ans.*

6% BASIS $1

P, $204
T, 1 yr. 5 mo. 17 da.
R, 7%
I, ?

.06	
.025	
.002⅚	
.087⅚	
204	
17.918	6
2.986	1
20.904	7

I, $20.90 *Ans.*

ALIQUOT PARTS

P, $204
T, 1 yr. 5 mo. 17 da.
R, 7%
I, ?

14.28	1 yr.
1.19	1 mo.
5.95	5 mo.
.039⅔	1 da.
.674	17 da.
20.904	

I, $20.90 *Ans.*

BANKERS' METHOD

P, $287.25
T, 92 da.
R, 6%
I, ?

2.87x25	60
1.43 62	30
.09 57	2
4.40 44	

I, $4.40 *Ans.*

BANKERS' METHOD

P, $287.25
T, 120 da.
R, 4½%
I, ?

2.87x25	60
5.74 50	120
1.43 62	1½
4.30 88	4½

I, $4.31 *Ans.*

Discussion. The methods to be used with the class must be decided from the course of study or by the principal of the school. It is important that pupils shall gain the power to compute interest mentally. The cancellation method alone is not sufficient.

ILL. A graduate of the N. Y. T. S. for Teachers could not compute the interest of \$100 for 1 yr. at 5% mentally. Given a pencil, she produced the following:

$$\not{1}\not{0}\not{0} \times \frac{5}{\not{1}\not{0}\not{0}} \times \frac{1}{\not{3}\not{6}\not{0}} \times \not{3}\not{6}\not{0} = 5$$

This young woman learned the cancellation method in the elementary school, failed to master any other in the training school, and was helpless without a pencil.

208. Note with Interest, Continued.

ILL. T. "Take note No. 2. What was the interest at maturity? The interest of \$1100 for 92 da. at 4%, or \$11.24. How did you get it, Mary?"

M. "The int. of \$1100 for 1 yr. at 4% is \$44; for 90 da. or $\frac{1}{4}$ yr., \$11; for 10 da. or $\frac{1}{9}$ of 90 da., \$1.22; for 2 da. or $\frac{1}{5}$ of 10 da., \$.24; for 92 da., \$11.24."

T. "How much must Henry Jones pay Oct. 1? \$1111.24. Give me the complete history of this note."

209. Bank Discount. To find bank discount, compute the interest on the amount at maturity for the term of discount at the rate of discount.

ILL. T. "Take note No. 2 again. Let us suppose that on Aug. 1 Smith needed money and obtained it by selling this note to the bank or getting it **discounted**. He wrote his name across its back or **indorsed** it, agreeing thereby to pay it in case Jones should fail to do so, and gave it to the bank. Indorse the note.

"How much did the bank pay for the note? On Oct. 1 it will receive \$1111.24. On Aug. 1 it paid \$1111.24 less the interest at 6% from Aug. 1 to Oct. 1; it simply deducted the interest in advance or the **bank discount**. Bank discount is interest on the amount at maturity for the term of discount at the rate of discount. Memorize this statement.

"We must find the interest of $1111.24 for 61 da. at 6%; it is $11.30. What is the amount at maturity less the bank discount or the **proceeds** or what Smith will receive?

"Complete the history of the note. On Oct. 1, Jones pays the bank $1111.24, the bank writes, 'Paid Oct. 1, 1912, The Tenth National Bank, John Doe, Cashier,' across the face of the note or **cancels** it, and returns it to Jones. Cancel note No. 2.

"Take note No. 1. Suppose that Smith had sold this note to the bank on Aug. 1, how much would he have received for it?"

Note No. 1			Note No. 2		
Face, $1100			Face, $1100		
Date of note, July 1, 1912			Date of note, July 1, 1912		
Date of dis, Aug 1, 1912			Date of dis, Aug 1, 1912		
R of dis, 6%			R of dis, 6%		
Note, without int			Note, with int, 4%		
Proceeds, ?			Proceeds, ?		
Amt at mat,	$1100.00		Amt at mat,	$1111.24	
B dis,	$11.18		B dis,	$11.30	
Proceeds,	$1088.82	*Ans.*	Proceeds,	$1099.94	*Ans.*

210. Exact Interest. Banks of discount sometimes count 360 and sometimes 365 da. to the year, according as it is to their advantage. Thus, they count 360 when they collect interest and 365 when they pay interest. The government counts 365. Unless otherwise directed, count 360.

Ill. T. "Sometimes it is necessary to find **exact** interest or interest counting 365 da. to a year.

"Find the exact interest of $204 from July 1, 1911, to Dec. 18, 1912, at 7%. Can we count the time as 365 + 150 + 17 days? No! In reckoning 365 da. to a year 5 mo. cannot be counted as 150 da. The time is 365 + 170 days or 535 da. Use the cancellation method."

$$204 \times \tfrac{7}{100} \times \tfrac{1}{365} \times 535$$

211. Postal Savings System. Teachers should secure from a post office the pamphlet, "Postal Information," should explain its provisions, should give problems based upon it, should exhibit postal savings cards, ten-cent savings stamps, and savings certificates, and should urge pupils to become depositors.

ILL. T. "Have any of you deposited money in the postal savings bank at the post office? How old must a person be to open an account? 10 yr. How much can you deposit at one time? An exact number of dollars; but you can buy 10-cent stamps and stick them to a card, and when you have a dollar in stamps you can make a deposit. Here is a postal savings card with 5 stamps. How much is it worth?

"When you make a deposit, you receive a savings certificate which bears interest at 2% for every full year the money is on deposit, beginning with the first day of the month following the one in which it is deposited. A person has as many certificates as he makes deposits. Here is a postal savings certificate.

"Jan 2, 1912, I obtained a savings certificate of $3; Oct. 3, 1912, one of $4; and Apr. 30, 1913, one of $2. How much are these certificates worth to-day, counting interest?"

212. Exercises. **1.** Compute the interest of $511 for 11 mo. 11 da. at 5% by the cancellation method. **2.** By the 6% method for days. **3.** By the 6% basis $1 method. **4.** By the aliquot part method. **5.** Find the rule for computing interest at 36%. **6.** By this rule compute the interest in No. 1. **7.** State, with reasons, which method you prefer. **8.** Discuss the bankers' method. **9.** Write note No. 1, making the time 90 days after date, and adding 'with interest.' **10.** Find the proceeds of the note July 20, discounting at 6%. **11.** Prove that for less than a year exact interest is the common interest minus $\frac{1}{73}$ of the common interest.

LESSON 30. INTEREST

213. Bonds. A note given by a village, town, city, county, state, or nation, secured by faith and credit, and bearing interest, is called a **bond.** Bonds are usually treated in arithmetics in connection with stocks, but all which they have in common with stocks is the fact that they are usually bought and sold by brokers. Notes given by individuals and corporations secured by mortgage and bearing interest are also called bonds.

ILL. T. "Suppose the city of Yonkers needed $500,000 on June 1, 1912, to construct a sewer, and the money in the treasury was insufficient, it might issue and sell to the highest bidder 500 notes (**bonds**) of $1000 each, signed by the city officers as provided by law. The bonds in blank might read:

CITY OF YONKERS SEWER LOAN, No. ___

YONKERS, N.Y., *June 1,* 19*12.*

$1000.00

On June 1, 1932, for value rec'd, the City of Yonkers promises to pay the bearer, One Thousand and $\frac{00}{100}$ Dollars, at the First National Bank, with interest at 4% payable on June 1 of each year at said bank. The faith and credit of the City of Yonkers is pledged to the payment of this debt.

_____________ MAYOR

_____________ PRES. OF COUNCIL

_____________ CITY TREASURER

"For the convenience of the purchasers, twenty small notes, called **coupons** (French, *cut*), might be printed on the same paper, each for $40, or the payment of one year's interest, and made payable in order, June 1, 1913, June 1, 1914, and so on. In this case the bond would be a coupon bond. The coupons might be arranged as follows:

No. 20	No. 19	No. 18	No. 17	No. 16
No. 15	No. 14	No. 13	No. 12	No. 11
No. 10	No. 9	No. 8	No. 7	No. 6
No. 5	No. 4	No. 3	No. 2	No. 1

"Suppose this was a coupon bond. Write coupon No. 1, or the note becoming due June 1, 1913. Explain the procedure of Mr. A., who owns one of these bonds, in getting his interest each year."

Bond and Mortgage. A person owning real estate may borrow money by giving to the lender a note called a **bond** (bound) and a sealed instrument in writing transferring the property to him. This instrument is a **deed** or **mortgage** (death-pledge). The lender has this deed recorded to prevent fraud, but is in no sense the owner of the property. If the borrower fails to fulfill his agreement, the property is sold under the direction of the court and the lender is paid.

214. Drafts. If A owes B, B may request A to pay the whole or a part to C. If A honors the request, B may pay C in this way. The paper containing the request is a **draft**. It may or may not bear interest and may be payable on demand or at a specified time.

Ill. T. "George Williams owes the Home Publishing Co. $1000 for books. The latter makes a draft on the former for $500 in favor of Henry Brown, payable 30 da. after sight.

To *George Williams,*	300 BROADWAY, N. Y. CITY,
220 Broadway, N. Y. City.	*Mar. 25, 1913.*

Thirty days after sight, pay to the order of Henry Brown, Five Hundred and $\frac{00}{100}$ Dollars, and charge the same to our account.

$500.00 *Home Publishing Co.,*
James Camp, PRES.

"The draft is sent to Williams, who writes across the face in red ink, 'Accepted, Mar. 26, 1913, George Williams.' It is then sent to Henry Brown, who sells it or collects it when due. Write the draft and its acceptance."

Checks. The most common form of draft is a **check.** A person has money on deposit in bank and pays debts by checks. As a rule, checks are not sent to the bank for acceptance, but sometimes this is necessary when the validity of the check is to be put beyond question. In such an event the check is said to be **certified.** It becomes a draft which has been accepted by the bank. A bank officer writes the acceptance in the usual way.

ILL. T. "We will suppose that you have money on deposit at the Tenth National Bank, 100 Broadway, N. Y. City. Draw a sight draft on the bank for *no dollars* to the order of James Thompson. This draft is a **check.**

"Have this check **certified.** That is, get one of your classmates to accept it as cashier in the name of the bank."

215. Indirect Cases. The direct cases have already been considered; given the principal time and rate to find the interest, and to find the **amount,** that is the principal plus the interest. They should receive 90% of the time devoted to interest.

The indirect cases grow out of the direct cases. Six of them are usually considered in the arithmetics.

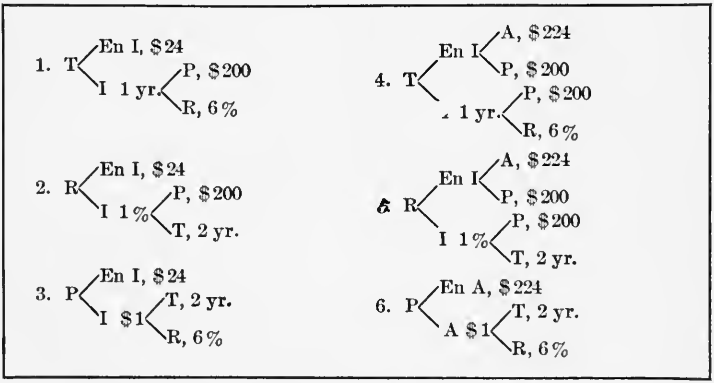
Ill. T. "The interest of $200 for 2 yr. at 6% is $24. We have mastered this case. Find the three others that arise from the omission of each term in succession."

1. In what time will $200 gain $24 at 6%?
2. At what rate will $200 gain $24 in 2 yr.?
3. What principal will gain $24 in 2 yr. at 6%?

"The amount of $200 for 2 yr. at 6% is $224. Find the three other cases."

4. In what time will $200 amount to $224 at 6%?
5. At what rate will $200 amount to $224 in 2 yr.?
6. What principal will amount to $224 in 2 yr. at 6%?

By Analysis. The author prefers this method. The component problems are displayed in the diagrams. *See* § *37.*



Ill. T. "Consider Case 1. From what can we find the no. of years? From the entire interest and the interest for 1 yr. From what can we find the int. for 1 yr.? From the principal and the rate."

Ill. T. "Solve Case 1. What is the interest of $200 for 1 yr. at 6%? $12. If the interest is $12 for 1 yr., in how many years will it be $24? 2.

"Solve No. 5. If the amount is \$224 and the principal is \$200, what is the interest? \$24. What is the interest of \$200 for 2 yr. at 1%? \$4. If the interest is \$4 at 1%, at how many per cent will it be \$24? 6."

By Algebra. The known terms may be substituted in the direct formulæ, or the unknown term may be represented by x. *See p. 141.*

1st Algebra		2d Algebra	
P, \$200 A, \$224 T, 2 yr. R, ? ——— R, 6% *Ans.*	$I = 200 \times 2 \times R$ $224 = 200 + I$ $I = 24$ $24 = 200 \times 2 \times R$ $R = \frac{24}{200 \times 2} = 6\%$	P, \$200 A, \$224 T, 2 yr. R, ? ——— R, 6% *Ans.*	Let $\frac{x}{100} = \text{R}$ $200 \times 2 \times \frac{x}{100}$ $= 4x = \text{I}$ $200 + 4x = 224$ $x = 6$
Formula		**Analysis**	
P, \$200 A, \$224 T, 2 yr. R, ? ——— R, 6% *Ans.*	$I = P \times T \times R$ $A = P + I$ $R = \frac{I}{P \times T}$ $I = A - P$ $R = \frac{A - P}{P \times T} = \frac{24}{400}$	P, \$200 A, \$224 T, 2 yr. R, ? ——— R, 6% *Ans.*	224 200 24 $200 \times .02 = 4$ $24 \div 4 = 6$

216. Different Cases. *For the Teacher.* Let us find all the different cases in interest from the formulæ, $I = P \times T \times R$ and $A = P + I$. *See* § 7.

There are two equations with five quantities. To solve these equations three of the quantities must be known. Hence, for every three known terms there will be two cases.

The combinations of threes in the terms, A, I, P, R, T are AIP, AIR, AIT; APR, APT; ART; IPR, IPT; IRT; PRT. That is, there are 10 times 2, or 20, cases in interest.

Write the 20 cases. Two of them are impossible. Why?

217. Kinds of Interest. *For the Teacher.* Let us classify interest with reference to what may bear interest.

Ill. What is the interest of $ 100 for 4 yr. @ 6 %?

1st Yr.	2d Yr.	3d Yr.	4th Yr	
$ 6	$ 6	$ 6	$ 6	I on P, $ 24
	.36	.36	.36	I on I on P at the end of each year, $ 2.16
		.36	.36	
			.36	
		.0216	.0216	I on all other unpaid I at the end of each year, $ .087696
			.0216	
			.0216	
			.001296	

First Conception. The principal alone may bear interest, **simple interest** ($ 24).

Second Conception. In addition to the above, the $ 6 due at the end of the 1st year may bear interest for 3 yr.; the $ 6 due at the end of the 2d year may bear interest for 2 yr.; the $ 6 due at the end of the 3rd year may bear interest for 1 yr. That is, the principal may bear interest and the interest on the principal at the end of each year may bear interest, **annual interest** ($ 26.16).

Third Conception. In addition to the above, the $ .36 due at the end of the 2d year may bear interest for 2 yr.; each $ .36 due at the end of the 3d year may bear interest for 1 yr.; the $ .0216 due at the end of the 3d year may bear interest for 1 yr. That is, the principal may bear interest, the interest on the principal at the end of each year may bear interest and all other unpaid interest at the end of each year may bear interest, **compound interest** ($ 26.247696).

218. Annual Interest. *For the Teacher.* Annual interest is rarely computed. The above development indicates the method of procedure.

Ill. What is the annual interest of $ 525.26 for 3 yr. 5 mo. 17 da. @ 6 %?

P, $525.26		
T, 3 yr. 5 mo. 17 da.	5.25x26	2 5 17
R, 6	6	1 5 17
An I, ?	31.51 56	5 17
		4 4 21

I on P, $109.17
I on I, $8.30
An I, $117.47 *Ans.*

$525.26 bears interest 3 yr. 5 mo. 17 da.
$31.5156 bears interest 4 yr. 4 mo. 21 da.

219. Compound Interest. *For the Teacher.* Compound interest is rarely computed. For practical use, the third conception is reduced to the form: in compound interest the amount at the end of each year bears interest during the ensuing year.

Ill. What is the amount of $100 for 4 yr. 6 mo. @ 6%?

P, $100
T, 4 yr. 6 mo.
R, 6%
Com I, ?

A, $130.04
Com I, $30.04 *Ans.*

106	1	119.1016	
1.06		1.06	
112.36	2	126.247696	4
1.06		1.03	
119.1016	3	130.035126	

Expl. The amount at the end of the 4th year is $126.247696; the amount of $1 for 6 mo. is $1.03; the amount of $126 + is 126 + times $1.03, etc.

Second Plan. The above is the popular method. A better plan is to find the amount of $1, and to multiply by the number of dollars. Before multiplying it is well to express the amount of the principal by its factors.

Ill. What is the amount of $525 for 20 yr. 7 mo. 17 da. @ 6% compound interest?

P, $525
T, 20 yr. 7 mo. 17 da.
R, 6%
Com A, ?

Com A, $1747.43 *Ans.*

.035
$.002\frac{5}{6}$
$.037\frac{5}{6}$

$$A = 525 \times 1.06^{20} \times 1.037\tfrac{5}{6}$$

EXPL. The amount of \$1 at the end of the 1st yr. is \$1.06; at the end of the 2d, \$$1.06^2$; at the end of the 3d, \$$1.06^3$; . . . at the end of the 20th, \$$1.06^{20}$; the amount of \$1 for 7 mo. 17 da. is \$$1.037\frac{5}{6}$; the amount of \$1 for the whole time, \$$1.06^{20} \times 1.037\frac{5}{6}$.

The value of 1.06^{20} can be found by multiplication as in § 178, from tables, or by the use of logarithms.

Interest Periods. When interest is payable semi-annually or quarterly, it is well to change the problem to an equivalent problem in which the interest is payable annually. It is evident that each interest period may be regarded as a year if the rate is divided by the number of such periods in a year.

ILL. Show the amount of \$525 for 20 yr. 7 mo. 17 da. at 6% interest compounded semiannually. *Ans.* $A = 525 \times 1.03^{41} \times 1.007\frac{5}{6}$.

This means, what is the amount of \$525 for 41 yr. 3 mo. 4 da. (twice 20 yr. 7 mo. 17 da.) at 3% (half of 6%)?

Discussion. In all states, the laws are opposed to the collection of compound interest. The difference between the amount at simple and at compound interest is well illustrated by computing the interest of 1 ¢ at 6% by each method since the beginning of the Christian era. For 1913 years, the interest by one method is \$1.15; by the other, many times the value of a sphere of pure gold whose radius is the distance from the earth to the sun.

220. Savings Banks. Savings banks pay a modified form of compound interest. At the end of each interest period, the interest of the largest sum that has been on deposit during the entire period is added to the amount of the deposit, but interest is computed on an integral number of dollars only.

ILL. T. "Have any of you deposits in a savings bank? Be prepared to tell me to-morrow the rule for computing interest stated in your bankbook. What did you find, James?"

J. "My account is in the People's Savings Bank of Yonkers. The statement is, 'On the first day of Jan. and July of every year, there shall be declared and paid interest on all sums of $5 and upwards, which shall have been deposited for 3 mo. previous to the first day of Jan. and July; but no interest shall be paid on the fractional part of a dollar, nor shall any interest be allowed on any sum withdrawn previous to the first day of Jan. and July for the period which may have elapsed since the last dividend.' Jan. 1, my father opened an account for me with $10; July 1, I shall have $10.20, and next Jan. 1, $10.40."

He develops the subject at length with an imaginary account.

"Henry Brown opened an account with the People's Savings Bank, Jan. 1, 1912, by depositing $150.98; Mar. 10, he withdrew $50; Apr. 1, he deposited $100.40. How much did he have on deposit July 1, 1912, after interest @ 4% was added?

His account appears as follows:

	Deposits	Drafts	Balance
1912, Jan. 1	150.98		150.98
1912, Mar. 10		50.00	100.98
1912, Apr. 1	100.40		201.38
1912, July. 1	3.01 (Int.)		204.39

"The largest whole no. of dollars on deposit during the entire six months is $100; it bears $1 interest during the 1st quarter (1% of $100); the largest whole no. of dollars on deposit during the entire 2d quarter is $201; it bears $2.01 interest during the 2d quarter (1% of $201)."

221. Exercises. **1.** Read the diagrams in § 215. Thus, Case 2, the rate can be found from the entire interest ($24), and the interest at 1%. **2.** Solve Case 2 by analysis. **3.** Solve Case 3 by analysis. **4.** Solve Case 4 by analysis. **5.** Solve Case 6 by 1st algebra method. **6.** By 2d. **7.** By formula. **8.** By analysis. **9.** State, with reasons, the method you prefer. **10.** Verify the answer to the first problem in § 219.

TEACHING ARITHMETIC

PART III. EXERCISES

SECTION 1. Elementary Schools

222. Scope. This section gives an idea of the advance in arithmetic which is made from term to term in the elementary schools of New York City. The exercises for the first two years have been suggested by the author. The exercises for the last six years, with some slight changes, have been given as final tests by those in authority.

In mental tests, the questions were dictated by the teacher and answers only were written by the pupil.

First Year

223. 1st Yr. 1st T. **1.** On your desk there is a package of sticks and some rubber bands. Put them into bundles of ten and tell me how many sticks you have. **2.** Bring me thirty-four sticks. **3.** Read these numbers: 25, 30, 87, 17. **4.** By figures, write: 7, 2, 4. **5.** How far is it from your desk to mine (answer in steps)? **6.** Find the answer by counting objects; there are sticks and pieces of paper on your desks. Jane has 4 dolls and her mother gives her 2 more; how many has she then (Lesson 6)? **7.** As before, John has 2¢ and wants to buy a top for 5¢; how many cents does he lack? **8.** As before, Mary has 4 dolls and 2 dresses for each; how many doll-dresses has she? **9.** As before, if there are 2 plants in a row, how many rows will 10 plants require? **10.** As before, the teacher gives 8 books to 4 girls, to each one the same number; how many books does each girl receive?

224. 1st Yr. 2d T. **1.** From my desk get bundles of ten and single sticks and express forty-seven by objects. **2.** Write 47 by figures. **3.** Write 76 by figures. **4.** Write 20 by figures. **5.** Take 20 sticks from this box, 2 at a time. **6.** I am going to buy things of you and ask you to make change with the toy money on your desks. I buy a top for 4¢ and pay with a dime (p. 72); make the change. **7.** Add: 3, 4, 2; 8, 2, 3. **8.** Draw the objects and find the answer. How much will 6 oranges cost at 3¢ each? **9.** As before, 4 oranges were put on a plate; how many plates were needed for 12 oranges? **10.** As before, 10 hat pins were used in 5 hats; each hat had the same number of pins; how many pins in each hat?

Second Year

225. 2d Yr. 1st T. **1.** On your desks there are bundles of ten, single sticks, and bands. Put the tens into bundles of a hundred and tell me how many sticks you have. **2.** Bring me 321 sticks. **3.** Number to 8 and remember your number. The first eight may take places at the board; draw a circle; write in the center a number one greater than your own; around the circle write the digits through 9 in miscellaneous order; be seated. **4.** John, call all the sums on the board keeping in time with the sound (he taps at the rate of two combinations a second and calls upon another if John fails at any point). **5.** Add: 379, 539. **6.** From 678 subtract 253. **7.** Each of 6 boys has 4 marbles. Make a mark for each boy and write 4 under each. How many marbles in all? **8.** Erase the 4 under each mark. Solve No. 7 again. **9.** There are 20 crayons and some boxes; each box has 4 crayons; how many boxes? Count to 20 by 4's, making a mark at each count. **10.** If I divide 18 little cakes among 6 boys, giving to each boy the same number, how many cakes will each boy get?

226. 2d Yr. 2d T. **1.** Count from 1 to 10 as fast as possible. **2.** At the same rate count from 1 to 97 by 4's. **3.** From 1 to 100 by 3's. **4.** From 1 to 99 by 2's. **5.** Add: 394, 87, 109. **6.** From 925 subtract 288. **7.** Draw a line and mark off $\frac{5}{8}$ of it. **8.** At 3 for 5¢ how much will 12 bananas cost? Make a mark for each group of 3 bananas; it will also stand for each group of 5¢. **9.** At 4 for 5¢ how many peaches can I buy for 25¢? Make a mark for each 5¢. **10.** At 21¢ each how much will 4 books cost? Write 21 4 times and add.

Third Year

227. 3d Yr. 1st T. *Mental.* **1.** Spent 8¢, 5¢, 9¢, 4¢. How much was spent? **2.** Paid 32¢ for 4 lb. of tomatoes. What was the cost of 1 lb.? **3.** If a quart of milk costs 9¢, what will a gallon cost? **4.** How many legs have eight flies? **5.** I have one yard of cloth. If I cut off 10 inches, how many inches will be left? **6.** 20 weeks are how many months? **7.** I had 8 dimes and spent half a dollar. How much money had I left?

Written. **1.** Add: 2701, 1976, 2000, 3006. **2.** From 9654 take 7235. **3.** Multiply 642 by 26. **4.** Divide 664 by 4. **5.** Find $\frac{1}{3}$ of 924. **6.** How many gallon measures will 96 quarts of milk fill? **7.** In 234 feet there are how many yards? **8.** A boy was on board a steamer $1\frac{1}{2}$ days. How many hours was he on board?

228. 3d Yr. 2d T. *Mental.* **1.** Write the Roman number for 99. **2.** A farmer has 37 cows. He buys 62 more. How many has he then? **3.** If 3 oranges cost 5¢, what will 12 oranges cost? **4.** Louis has 36 marbles. His brother has $\frac{2}{3}$ as many. How many has his brother? **5.** John bought 30¢ worth of cake and gave the baker a half of a dollar. How much change was due? **6.** There are 48 boys in the 3-B class. If $\frac{3}{4}$ of them are promoted, how many boys are promoted? **7.** Change 56 days to weeks. **8.** If 6 handkerchiefs cost 36¢, what will 7 handkerchiefs cost?

Written. **1.** Write and add: $365.05, $409.30, $50.75, $800.28, $749.06, $23.53. **2.** A man buys a farm for $6550 and sells it for $9375. Did he gain or lose and how much? **3.** If six suits of clothes cost $29.88, what is the cost of nine suits? **4.** Which is larger, $\frac{1}{4}$ or $\frac{1}{8}$? Diagram. **5.** Show that $\frac{3}{6} = \frac{1}{2}$. Diagram. **6.** Multiply: $96.54 by 59. **7.** A merchant sold 95 yards of cloth to one man, 117 yards to another, and then had 49 yards left. How many yards had he at first? **8.** At 18¢ per dozen, what is the cost of 16 pencils? **9.** How many 5ths in 23 units?

Fourth Year

229. 4th Yr. 1st T. *Mental.* **1.** 7, × 8, + 4, − 10, ÷ 5, − 1, × 9? **2.** Had 56 children in a class; promoted $\frac{7}{8}$ of them. How many were promoted? **3.** Spent 3¢ for a pencil, 5¢ for a book, and 2¢ for a pen. What change did I receive from half a dollar? **4.** How many ounces in 10 lb. of tea? **5.** If 5 books cost $1.75, what will 1 book cost?

Written. **1.** 38704 ÷ 59 = What? **2.** 6041 × 807 = What? **3.** If 25 suits cost $625, what will 1 suit cost? **4.** Sold a house for $4500 and lost $800. What was the cost? **5.** A man earns $1072 a year; he spends $\frac{7}{8}$ of it. How much does he save? **6.** How many days in 100 years if 75 of them have 365 da. each, and 25 have 366 da. each? **7.** What will $5\frac{3}{4}$ lots cost if one lot costs $164? **8.** What is the cost of 40 quarts of milk at 2 cents a pint? **9.** A has $25284 and B has $87611. How much more money has B than A? **10.** Add:

two hundred seventy-five, three hundred eighty-four, twenty six, nine, five thousand two hundred eighty-four, six hundred fifty-eight, ninety-seven, five thousand two hundred sixty-four.

230. **4th yr. 2d T.** *Mental.* **1.** If a yard of silk costs $ 1.25, what will 8 yards cost? **2.** What part of a yard is 18 inches? **3.** Mr. Jones had 36 sheep. Wolves killed $\frac{5}{6}$ of them. How many were left? **4.** What is the number of square feet in the floor of a room 11 ft. by 12 ft.? **5.** Bought books for $25. Sold them and gained $\frac{2}{5}$ of the cost. What was the selling price? **6.** A square field is 10 rods on a side. How far around it? **7.** What is the value of a blackboard 4 feet long by 5 feet wide, at 50 cents a square foot? **8.** At 25 cents a peck, what is the cost of 2 bushels of potatoes? **9.** How many square inches in a 3-inch square? **10.** Reduce $\frac{25}{8}$ to a mixed no.

Written. **1.** Add: 879, 789, 897, 688, 987, 438. **2.** From nine hundred thousand eight hundred six, subtract eight hundred forty-five thousand nine hundred eighty-seven. **3.** Had $ 752; spent $\frac{5}{8}$ of it; how much was left? **4.** If 360 books cost $ 900, what is the cost of 1 book? **5.** Find the cost of fencing a lot 100 ft. long and 50 ft. wide at 50 cents a foot. **6.** How many lots 25×100 can be made from a plot 75×200? **7.** Bought a piano for $ 350 and sold it at a loss of $ 75.50; find the selling price. **8.** A man earned $ 900 a year, spent $ 750 each year, and saved the remainder. How much was saved in 10 years? **9.** Henry weighs $58\frac{3}{4}$ lb., Peter, $65\frac{5}{8}$ lb., and John, $67\frac{1}{2}$ lb. Their father weighs as much as all three together. What is his weight? **10.** How many cubic feet of earth must be removed to make a cellar 36 ft. by 28 ft. by 14 ft.?

Fifth Year

231. **5th yr. 1st T.** *Mental.* **1.** Add: 15, 9, 12, 6, 8, 9. **2.** James has $ $2\frac{1}{2}$ and Harry $ $5\frac{1}{4}$. How much have they together? **3.** If a car goes 4 miles in half an hour, how far will it go in $3\frac{1}{2}$ hours? **4.** A man's salary is \$24 a week. He spends \$15 each week. What part of his salary does he save? **5.** John had 20 marbles, and James $\frac{3}{5}$ as many. How many did both have? **6.** If I burn $\frac{2}{3}$ of a ton of coal in a month, how long will 6 tons last? **7.** $2\frac{1}{2}$ pt. are what part of 5 qt.? **8.** Bought 2 bbl. of apples, one for \$1.25, the other for \$1.50. What change did I receive from a \$5 bill? **9.** A family use 6 eggs each morning. How many eggs do they use in the month of February of this year? **10.** John paid \$16 for his overcoat, which was $\frac{4}{5}$ of the cost of his suit. What did he pay for his suit?

Written. **1.** Paid \$800 for a plot of land 200 feet long and 100 feet wide, which I divided into lots 25 ft. × 100 ft., and sold for \$150 each. How much did I gain? **2.** I had $75\frac{1}{2}$ acres of land and sold $18\frac{3}{4}$ acres to one man, $3\frac{5}{8}$ acres to a second, and $5\frac{5}{6}$ acres to a third. How many acres had I left? **3.** If 1 yard of cloth cost $\$3\frac{1}{3}$, find the cost of $18\frac{3}{4}$ yards. **4.** Make out and receipt the following bill, your mother the purchaser, the date to-day. Bought of R. H. Macy & Co., 34th St. and Broadway, $12\frac{1}{2}$ yd. lace @ \$1.50; 20 yd. linings @ $12\frac{1}{2}$ ¢; 24 yd. ribbon @ $7\frac{1}{2}$ ¢; 15 spools of thread @ 9 ¢. **5.** From a crop of 324 bu. a man sold 243 bu. What part of his crop had he left? **6.** If $\frac{4}{5}$ of a farm is valued at \$86.24, what is the value of the farm?

232. 5th yr. 2d T. *Mental.* **1.** What number must be added to 38 to make 54? **2.** At \$1.25 per yard, what will 36 yd. of silk cost? **3.** At 50 ¢ a day what will board cost me for the month of July? **4.** A man had \$40 and gave away $.12\frac{1}{2}$ of it. How much did he give away? **5.** 4 ounces of tea is what decimal part of a pound? **6.** .75 of a class of 44 were promoted. How many were not promoted? **7.** A man bought a horse for \$96. He sold it at a gain of $.87\frac{1}{2}$. What was the selling price? **8.** If $\frac{2}{3}$ of a doz. cost \$1, how much will 4 articles cost? **9.** If $\frac{3}{4}$ of my money is 63 ¢, what is $\frac{1}{2}$ of it? **10.** At 2 ¢ each how many apples may be bought with 2 dimes and 2 nickels?

Written. **1.** Add twenty-five dollars twenty-five cents, three hundred forty-six dollars forty-eight cents, six thousand two hundred forty-two dollars seventy-five cents, one thousand three hundred nine dollars twenty-nine cents, eighteen hundred dollars four cents. **2.** Subtract $28\frac{4}{5}$ from $301\frac{2}{3}$. **3.** The multiplicand is $2\frac{1}{4}$, the product is $1\frac{1}{2}$. What is the multiplier? **4.** I bought a boat for $240 and sold it at a gain of .20. Find the selling price. **5.** Divide $3\frac{1}{2}$ by $3\frac{1}{3}$ and write the answer in decimal form. **6.** If $2\frac{1}{3}$ yds. of cloth cost $16\frac{4}{5}$ ¢, how many yards can be bought for 63 ¢? **7.** What decimal of a mile is 1056 ft.? **8.** A man earns $12 a week; how much does he earn a year? **9.** Reduce to decimals and add, $\frac{1}{2}$, $3\frac{17}{20}$, $\frac{313}{500}$, $\frac{29}{40}$. **10.** Change to common fractions and arrange in order of value: .005, $.62\frac{1}{2}$, .35, $.33\frac{1}{3}$.

Sixth Year

233. 6th yr. 1st T. *Mental.* **1.** Change $\frac{3}{4}$ bu. to quarts. **2.** Which is greater, $\frac{3}{7}$ or $\frac{8}{21}$? Give the difference. **3.** At 4 ¢ a pint what will you pay for 10 gal. 2 qt. of milk? **4.** Sold for 8 ¢ and lost 2 ¢. What was the per cent of loss? **5.** Have 2 quarters 5 dimes 1 nickel. How many apples can I buy at 5 ¢ each? **6.** A man sells a carriage for $56, which is $\frac{1}{8}$ less than he gets for his horse. What did he receive for the horse? **7.** If $\frac{1}{3}$ of the number of bushels in a bin is 20, how many bushels in the bin? **8.** How many days between March 12 and April 15, 1913? **9.** I had $3.20 and spent 25 % of it. How much did I spend? **10.** An article cost $8 and sold at a gain of $37\frac{1}{2}$ %. What was the selling price?

Written. **1.** Reduce .027 of a ton to ounces. **2.** If a bin contains 5 bu. 3 pk. 2 qt., how much will 5 bins of the same size contain? **3.** A man was born Dec. 25th, 1852. How old was he February 12th, 1881? **4.** I had $1200. I deposited 25 % of this sum in the Bowery Savings Bank, and $33\frac{1}{3}$ % of the remainder in the Metropolitan Savings Bank. How much cash have I? **5.** Make a bill for the following and receipt it; date it to-day; make yourself the purchaser. Bought of A. J. Cammeyer: 10 pairs of men's shoes @ $4.75; 4 pairs of boys' shoes @ $1.47½; 6 pairs of slippers @ .87½¢; 9 pairs of girls' shoes @ $2.43; 8 pairs of women's shoes @ $3.37½. **6.** Paid $450 for a horse and sold it at a profit of 15 %. What was the selling price?

234. 6th yr. 2d T. *Mental.* **1.** It will cost 40¢ to send a ten-word telegram to Boston. Every additional word will cost 3¢. What must I pay for 18 words? **2.** If 9 yards of ribbon cost $2.50, what will 27 yards cost? **3.** A man sold 25 % of his wheat to one buyer, 12½ % to a second, and had 350 bu. left. How many bushels in the crop? **4.** A dealer bought a second-hand sofa for $40 and having spent 10 % on repairs, sold it at a gain of 25 % on the whole cost. For what did he sell it? **5.** What is $\frac{5}{8}$ % of $5600? **6.** A piano was marked $480 but sold at a discount of 10 %. Find the discount. **7.** My selling price for some goods is $30 and my gain is 20 %. What is my money gain? **8.** Lost $3 by selling a desk for $18. What per cent lost? **9.** Find the commission at 5 % on 400 bbl. of fish at $5 a barrel. **10.** My store is worth $4000. I insured it for $\frac{3}{4}$ of its value. Find the premium at ½ %.

Written. **1.** A commission merchant sold 10 bbl. of pears for a farmer at $5 a barrel. His commission was 10 %, and $2.50 was charged for freight. What sum did he return to the farmer? **2.** A farmer raised 517 bu. of potatoes. He sold 20 % of them. How many bushels, pecks, and quarts did he sell? **3.** Find the value of a rectangular piece of land 75 rd. long and 56 rd. wide at $144 per acre. **4.** You sell a 100-lb. keg of horseshoes for $14.05 and gain 25 %. What do they cost you per pound? **5.** The bread from a bbl. of flour (196 lb.) weighs $31\frac{1}{4}$ % more than the flour. What is the weight of the bread? **6.** You pay $ 200 rent a month for your store. How many dollars of sales must you make each month to raise this rent if you average 20 % profit? **7.** Sold a horse for $ 250 and lost 20 %. What would have been the selling price if I had gained 20 %? **8.** Jan. 1, 1913, John Smith bought of Henry Jones a piece of land 100 ft. by 25 ft. and paid for it at 50 ¢ a square foot by a check on the Tenth National Bank, 100 Broadway. Draw the check.

Seventh Year

235. 7th yr. 1st T. *Mental.* **1.** The principal is $ 600; the rate, 3 %; the time, 2 yr. 6 mo. What is the interest? **2.** 25 qt. are $62\frac{1}{2}$ % of how many gallons? **3.** $ 17 are 20 % of how many dollars? **4.** A grocer received 60 bbl. of flour and sold 12 of them. What % had he left? **5.** I collected $ 450 and received a commission of 4 %. What was the commission? **6.** Philadelphia is 75° West longitude. When it is noon in London, what is the time in Philadelphia? **7.** How many dollars in £ 100? How many centimeters in 3 m 70 mm? **8.** A piano listed at $ 400 was sold with discounts of 25 % and 10 %. What was the selling price? **9.** 50 ¢ is what per cent of $\frac{3}{4}$ of a dollar? **10.** By selling a book for $ 4, I lost $ 1. Find the per cent lost.

Written. **1.** On May 3d, 1910, I borrowed $ 640 at simple interest at 5 %. How much do I owe on Sept. 27th, 1913? Count the exact number of days. **2.** Change $ 112 to English money. **3.** What sum of money invested at 5 % per annum will give an annual income of $ 1200? **4.** Two boys have $ 48 between them, one having $ 18 more than the other. How much has each? **5.** A milkman sold 86 qt. of milk. How many liters did he sell? **6.** Cloth costing 1200 francs in France was shipped to America where a duty of 24 % was levied. How much was the total cost in our money? **7.** A man starts from Chicago and some days later finds that his watch is 2 hr. 30 min. slow. In what direction has he been traveling and over how many degrees of longitude has he gone? **8.** A recipe for $1\frac{1}{2}$ lb. of fudge calls for 3 cups of sugar (6 ¢ a pound), 1 tablespoon of butter (32 ¢ a pound), $\frac{3}{4}$ of a cup of milk (9 ¢ a quart), 2 oz. of chocolate (36 ¢ a pound), and 1 teaspoon of vanilla (10 ¢. $1\frac{1}{2}$ oz. bottle). Find the cost of a pound; count 31 tablespoons to a pint or pound, 2 cups to a pint, and 3 teaspoons to a tablespoon.

236. 7th yr. 2d T. *Mental.* **1.** Principal is what when rate is 5%; time is 2 yr. 6 mo.; interest is $12.50? **2.** Principal is $600; time is 3 yr. 4 mo.; interest is $80. What is the rate? **3.** Principal is $900; rate 5%; interest $60. What is the time? **4.** The simple interest on a certain principal is $146. What is the exact interest? **5.** If 11 books cost $1.32, what will 100 books cost? **6.** 9, + 12, ÷ 3, × 12, − 3, ÷ 9, + 4, − 11? **7.** x, × 12, + 9, ÷ 3, + 9, − 12, the result is 20, find x. **8.** The following recipe is for a dozen biscuits. How much of each ingredient must be used to make 15 biscuits? For a dozen, 2 cups of flour, $\frac{2}{3}$ cup of milk, $\frac{1}{2}$ teaspoon salt, $2\frac{1}{2}$ teaspoons of baking powder, 2 tablespoons of shortening. **9.** Find the area of a triangle whose base is 12 in. and whose altitude is 10 in. **10.** What is the ad valorem duty on 100 yd. of silk valued at $4 a yard, duty 25%?

Written. **1.** A merchant had $11,640; he invested $26\frac{2}{3}$% of it in dry goods and $12\frac{1}{2}$% of the remainder in groceries. How much money had he left? **2.** Find the interest of $320 for 1 yr. 8 mo. 27 days at 5%. **3.** Find the amount of $750 from Jan. 12th, 1903, to Dec. 16th, 1905, at 7%. Count the exact number of days. **4.** Multiply six hundred twenty-five thousandths by twenty-five millionths and divide the product by one hundred twenty-five hundred thousandths. **5.** John Smith bought $400 worth of goods from Thomas

Brown on 3-months' credit. Write a bank note for the transaction, dating it to-day. **6.** Find the proceeds at bank discount @ 6 % if the note is discounted 15 days after to-day. **7.** If 29 bales of hay are consumed by 58 cows, how many bales will last 46 cows for the same time? **8.** Sold a wagon for $420, which is 16% less than the cost. What should I have sold it for to gain 25%? **9.** What number increased by $\frac{1}{2}$ of itself equals 402? **10.** Find x: $3x - 5(x + 3) = -21$.

EIGHTH YEAR

237. 8th Yr. 1st T. *Mental.* **1.** Reduce 3 dm 7 m 5 cm to mm. **2.** A house is valued at $6000. It is insured for 76 % of the valuation at 20¢ per $100. What is the premium? **3.** 15, + 17, ÷ 4, square, − 1, × 11 = ? **4.** Reduce 2 T. 5 cwt. to pounds. **5.** Sold a wagon for $72, gaining $12\frac{1}{2}$ %. What was the cost? **6.** How many pint bottles can be filled by 3 gal. 3 qt. of wine? **7.** What is the perimeter of a square whose area is 121 sq. yd.? **8.** What is the circumference of a 28-inch wheel? **9.** What is the ratio of 4 ft. 2 in. to 5 ft.? **10.** State which of the following are multiples, which factors, which powers, and which roots of 8: 16, 4, 64, 24, 2.

Written. **1.** If .625 of a cord of wood cost $3.75, what will .75 of a cord cost? **2.** How much will it cost to carpet a room 14 ft. by 12 ft. with carpet $\frac{3}{4}$ of a yard wide @ $1.25 a yard? Strips to run lengthwise. **3.** I gained $\frac{1}{2}$ % by selling my farm for $1346.70. What was the cost? **4.** A railroad passes through a farm taking a strip $1\frac{1}{2}$ miles long and 66 ft. wide. What is the value of this land at $80 an acre? **5.** A 60-day note of $600, dated Aug. 4th, 1913, was discounted Sept. 1st @ 6 %. Find the proceeds. **6.** If 24 men in 12 days of 9 hours each can do a certain amount of work, how many days of 8 hours each will it take 36 men to do twice that amount of work? **7.** A tank full of water is 8 ft. × 4 ft. × 3 ft. What is the weight of the water? **8.** Find x: $2x - \frac{x-1}{4} = 30$.

238. **8th Yr. 2d T.** *Mental.* **1.** 35, + 40, ÷ 5, × 20, ÷ 4, ÷ 10, − 5 = ? **2.** How many bottles $\frac{2}{3}$ qt. in capacity can be filled from a demijohn containing $4\frac{2}{3}$ qt.? **3.** How many lb. in 1000 Kg.? How many tons? **4.** Sold a book for 60¢, gaining 20 %. What % would I have lost had I sold it for 40¢? **5.** The cost of a bicycle is $36. What shall the marked price be to allow a gain of $16\frac{2}{3}$ %, after falling $33\frac{1}{3}$ % from the marked price? **6.** How many miles in 80 Km? **7.** Solve $8x - 3 = 5x + 21$. **8.** Sold a horse for $225 and gained $25. What per cent gained? **9.** $\frac{1}{2} + \frac{1}{5}$ of my money equals $140. How much money have I? **10.** If 6 men can do a piece of work in 18 days, how long will it take 4 men to do the same work?

Written. **1.** Add 491673, 28674, 847598, 892654, 34567, 67891, 391638, 328695, 64738 and 473876. **2.** Solve by short processes: (*a*) Find the cost of 5400 lb. @ $6.50 per ton; (*b*) What is 250 % of 400? **3.** The surface of a sphere is 1963.5 sq. in. What is its diameter? **4.** Find the cost of carpeting a room 12 ft. by 9 ft. with carpet $\frac{3}{4}$ yd. wide at 95¢ a yd. The strips are to run lengthwise. **5.** A 90-day note for $1000 with interest at 7 % was dated Jan. 17th, 1907, and discounted March 2d, at 6 %. Find the proceeds. **6.** The perimeter of a square piece of land is 16 rods. How much is it worth at 10 cents a square foot? **7.** A cylindrical cistern with a diameter of 5 ft. has 27 inches of water in it. How many gallons are there? **8.** When it is 7 A.M. in New York City, what time is it in London? **9.** I bought 120 meters of lace at 4 francs per meter. For what must I sell it per yard, U. S. money, in order to gain 20 % on the investment? **10.** I have $6600 with which to make an investment. I am offered 6 % stock @ 20 % premium or 5 % stock at 10 % premium. Which is better and by how much annually?

SECTION 2. PRIMARY LICENSE — CITY

239. Scope. This section gives an idea of the preparation necessary for obtaining a certificate to teach arithmetic in elementary schools. With a few exceptions the following tests have been given for License No. 1, a license to teach the requirements of the first six years in the school system of New York City.

240. Algebra. **1.** If an automobile was sold for $ 1025 at a profit of 25 %, how much did it cost? Use x or other algebraic symbol.

2. Illustrate the correct use of the equation in solving two easy examples in percentage.

3. State and solve an easy problem in simple interest to find the rate. Use x.

241. Apperception. **4.** Apply the principle of apperception to a first lesson in decimals.

NOTE. This calls for a knowledge of the five formal steps of the Herbartians: preparation, presentation, comparison, generalization, and application. These steps are taken in § 149. The reader should find them.

242. Cancellation. **5.** State the principle of cancellation and show how it should be taught.

6. After stating two principles upon which the process of cancellation is based in the multiplication of fractions, show how that process should be taught.

243. Decimals. **7.** State the steps in teaching the rule for the multiplication of a decimal by a decimal.

8. State the steps in teaching the rule for the division of a decimal by a decimal.

244. Denominate Numbers. **9.** With respect to a single exercise in linear measurement suitable for the second year, state: (*a*) what is to be measured; (*b*) what it is to be measured with; (*c*) how.

10. How should the subject of cubic measure be presented?

245. Developments. **11.** Enumerate the 45 combinations of digits (taken two at a time in addition) which must be learned in the first two years of school and state how you would teach these combinations.

12. Explain "Only like numbers can be subtracted"; show how this principle applies in the subtraction of integers, of common fractions, of decimals, and of denominate numbers.

13. (*a*) Explain the Austrian method of subtraction, using the following example: subtract 58 from 100. (*b*) State its advantages and its disadvantages.

14. Show the complete form of blackboard demonstration of the process: (*a*) of adding three-figure numbers; (*b*) of multiplying by three-figure numbers; (*c*) of dividing by a one-figure number.

246. Devices. **15.** Suggest a device for helping a pupil to remember what 17 minus 8 is.

16. Suppose a pupil forgets the product of 6 and 7; suggest three devices which may be helpful to him.

247. Diagrams. **17.** Show by a diagram that multiplier and multiplicand (when neither is concrete) can be interchanged without altering the product.

NOTE. This is the commutative law in multiplication.

18. Solve by use of a diagram: How many yards of cloth at $\$\frac{3}{4}$ a yard can be bought for $6?

19. Show graphically that $\frac{3}{5}$ (of 1) is equal to $3 \div 5$.

20. State a problem in profit and loss, given the gain and gain per cent, and requiring the selling price: (*a*) Solve the problem stated. (*b*) Show how to illustrate the solution by a drawing.

248. Drills. **21.** "Since memory is served by multiple associations quite as well as by repetition, the drills employed should be varied in form, in content, and in mode of application." In the light of the foregoing quotation, suggest three distinct types of drill under each of the following heads: (*a*) counting; (*b*) multiplication.

22. Describe three devices or modes of procedure, for enabling the teacher to conduct efficiently a drill in rapid addition, and state the advantages of each.

249. Errors. **23.** Following, there are common errors. Correct each and explain its nature as to a pupil: (*a*) The area of a rectangle is 12 inches multiplied by 6 inches or 72 square inches. (*b*) Since 39 is 1 less than 40, it may be written IXL by the Roman notation. (*c*) 3 is contained in 15¢ 5¢ times. (*d*) $5\frac{1}{2}$ gallons may be taken from 45 gallons 8 times and $\frac{2}{11}$ gallons will remain, because $45 \div 5\frac{1}{2} = 8\frac{2}{11}$. (*e*) $6\% = 144$, $1\% = 24$, $100\% = 2400$.

250. Fractions. **24.** Discover the effect on a fraction of dividing both numerator and denominator by the same number.

25. Discover the rule for multiplying a fraction by a fraction.

26. Discover the rule for dividing a fraction by a fraction.

27. Tell in order the steps involved in getting the pupils to understand the reduction of a common fraction to a decimal. Give illustrations.

28. State and illustrate the steps in simplifying a complex fraction. Define a complex fraction.

29. Find the answer to the following problem: $\frac{1}{2}$ of $\frac{3}{4}$ is 75 % of what number? Show how you would have children understand each step of the process.

30. State a complete problem which involves finding what part one common fraction is of another. State its solution by a diagram.

31. To find a number when the number plus or minus a part of itself is given. State a practical problem of each type designated.

251. Games. **32.** Describe two games or two other recreative exercises appropriate to the first or second year of school and involving counting or other number work.

252. Geometry. **33.** How would you teach finding the area: (*a*) of a rectangle? (*b*) of a parallelogram? (*c*) of an oblique triangle? (*d*) of a trapezoid?

34. How would you teach finding the circumference of a circle?

253. Helps. **35.** A pupil cannot solve this problem: If $\frac{3}{4}$ of a yard costs 24 ¢, how much will 1 yard cost? Help him.

36. At 3 ¢ each how many apples can be bought for 12 ¢? A pupil does not understand why the number of apples for 12 ¢ is the number of times 3 ¢ is contained in 12 ¢. Help him.

254. Inductive Exercises. **37.** What is the gist of the inductive method?

38. Show inductively (as in second year) that adding 10 to both subtrahend and minuend does not change the difference.

39. Discover inductively a rule for the divisibility of a number by 9.

255. Interest. **40.** State and solve a practical problem in simple interest to find the time.

41. (*a*) Compose a problem in bank discount involving the discount of a non-interest bearing note. (*b*) Write the note. (*c*) Give the steps in its solution.

256. L. C. D. **42.** Add $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$. (*a*) Find by inspection the least common denominater. (*b*) Give the rule for finding the least common denominator by inspection.

43. Explain clearly how to find the least common denominator when the denominators are too large for the method by inspection, as in the case of $\frac{1}{24}$, $\frac{1}{36}$, $\frac{1}{84}$.

257. Logical Division. **44.** Discriminate and illustrate logical division; logical definition.

45. In beginning the subject of long division, if the following divisors are to be used, in what order should they be used and why? 24, 17, 29, 27, 80?

46. In presenting the subtraction of one mixed number from another, what is the simplest type of this case? Illustrate this type and three others of increasing difficulty suitable for early lessons in the subject.

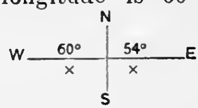
47. State in the order in which they should be taught the types of examples in division which involve one or more decimal numbers. Give reasons for the order chosen.

258. Longitude and Time. **48.** In a first lesson on longitude what points should be brought out? What device should be employed?

49. How would you teach that the time of a place farther west is earlier?

50. What is the difference in time between a place whose longitude is 54° east and a place whose longitude is 60° west?

NOTE. The device required is to locate the places as suggested at the right.



259. Percentage. **51.** Taking some one activity, industry, or experience as a center, construct about it five practical problems involving different applications of percentage.

52. State and solve a problem in trade discount.

53. A man bought 5 % stock and thereby secured an income of 6 % on his investment. (*a*) How much did he pay for the stock? (*b*) Explain as to a pupil how to prove the answer.

260. Proofs. **54.** What is meant by proving an example? by proving a problem?

55. Give reasons why pupils should be taught checking of results in the four operations and tell what these methods are.

56. In the case of the following processes, state and exemplify modes of verifying or checking results which are suitable to pupils below the seventh year: (*a*) addition (two modes); (*b*) finding a whole when a fractional part is given; (*c*) reduction ascending.

57. Solve and prove an original example in the second or third case of percentage.

261. Proportion. **58.** State and solve a practical problem in proportion.

59. (*a*) Give two examples involving ideas of ratio or of proportion, appropriate for the third or fourth year. Give model analysis. (*b*) Write a practical problem in direct proportion and one in inverse proportion. (*c*) Define ratio; define proportion.

262. Unit of Measure. **60.** (*a*) What is meant by unit of measure? (*b*) State and solve a problem in which the number 3 may be used as a unit.

263. Use of Text-book. **61.** State how the text-book should be used in arithmetic — as in the case of fractions.

SECTION 3. PRIMARY LICENSE — STATE

264. Scope. The scope of this section is about the same as that of the last. These tests have been given by the Department of Education of the State of New York, for Training School Certificates or certificates to teach in any elementary school of the state except in certain of the large cities.

265. Aliquot Parts. **1.** Give the aliquot parts that in your judgment should be memorized during the elementary course. When should they be learned? Make up two examples of different types, involving in their solution a knowledge of aliquot parts; assume that the examples are to be solved mentally by pupils of the eighth grade.

NOTE. An aliquot part is understood to mean one of the equal parts of 100; the word means *some times*. Thus, the aliquot parts (of 100) are 50 (one of two parts), $33\frac{1}{3}$ (one of three), 25 (one of four), 20 (one of five), $16\frac{2}{3}$ (one of six), and so on.

266. Analysis. **2.** Of what value are forms of analysis? In what respect are forms of analysis serviceable to you personally in the solution of problems?

3. Give a model analysis: A bookseller sold a book for $2.25 at a loss of 10 per cent; how much did he lose?

4. A man sold $\frac{5}{7}$ of his farm for what the whole of it cost; what per cent did he gain on the part sold? Give a model analysis.

5. Give a logical analysis: A man bought stock paying 4 % dividends, at 20 % discount; what rate of income did he receive on the investment?

6. What must be the marked price of a hat costing $ 6, so that after discounting the price 30 % the dealer may make a profit of $16\frac{2}{3}$ % ? Analyze.

267. Developments. **7.** Outline a development lesson to teach the terms multiplicand, multiplier, and product.

8. Show how to present inductively the idea of a common fraction.

NOTE. 'Inductively' is to be interpreted as 'objectively.' Strictly speaking only principles or laws can be presented *inductively*.

9. When should decimal fractions be first introduced and how should they be presented?

10. Explain how the difference between a draft and a check may be made clear to a class.

11. Explain how the difference between bonds and stocks may be made clear to a class.

268. Proofs. **12.** Write check problems to be used to test the correctness of the solution of the following examples: (*a*) If a train runs 32 miles in one hour, how far will it run in 45 minutes? (*b*) What is the interest on $425 for 2 years and 4 months at 6 %?

269. Teaching. **13.** Write three examples such as you would use in a first lesson on long division and show how you would teach the subject.

14. Show by an outline how you would present addition of fractions.

15. Show how you would teach by means of a rectangle or a circle that $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. Suggest what more you would do to develop the rule for the multiplication of fractions.

NOTE. The complete method (§ 20) is required. The one-line diagram (§ 12) is superior to a rectangle or a circle; fractions with numerators other than 1 are to be preferred.

16. Illustrate how you would explain to a class the reason for inverting the divisor in the division of fractions.

17. State four cases of problems in simple interest and give a plan for teaching one of them.

270. Theory. **18.** How and why should the aim of arithmetic teaching in the primary grades differ from the aim in grammar grades?

19. Discuss the extent to which problems calling for the application of principles should be given in the first three grades.

20. Explain the principal advantages claimed for: (*a*) the Austrian method of teaching subtraction; (*b*) the spiral method of teaching fractions and other topics in arithmetic.

NOTE. The spiral plan calls for teaching the leading principles with fractions which have small denominators, and then in teaching the whole subject again with fractions which have larger denominators, and so on, making each spiral more comprehensive than the one before.

21. State arguments against the use of the Grube method.

NOTE. The Grube method teaches the fundamental operations with each number before taking up the next higher. Thus, the following exercises with 4 are taught before 5: $2+2=4$, $3+1=4$, $1+1+1+1=4$; $4-1=3$, $4-3=1$; $2\times 2=4$, $4\times 1=4$, $1\times 4=4$; $4\div 1=4$, $4\div 4=1$, $4\div 2=2$, $\frac{1}{4}$ of $4=1$, $\frac{1}{2}$ of $4=2$.

22. Discuss the merits of the solution of the examples in simple interest by formula, as compared with the value of solving them by the application of the proper process of reasoning without the use of formula.

SECTION 4. HIGHER LICENSES

271. Scope. This section gives an idea of the preparation necessary to secure a higher license to teach in the schools of New York City. These tests have been given for promotion license (license for the 7th and 8th years), license for head of department, license for assistant to principal, or license for principal.

272. Fundamentals. **1.** Discuss these definitions: A number is a unit or collection of units. A number is an abstract ratio of one quantity to another of the same kind.

NOTE. How many is measured by comparison. There is a coin for this eye and a coin for this eye; there are as many coins as a man has eyes; 'as many as a man has eyes' is named two. Number is an expression of how many (§ 77).

There is an act of comparison for each individual or there are as many times of comparison as individuals. Thus there are '2 coins 1 time' or '1 coin 2 times.' The number of times of comparison is called ratio. A number is a ratio.

An 'act of comparison' or a 'time' is an individual. Hence, the definition, 'Number is an expression of how many' includes the definition, 'Number is a ratio'; the first refers to individuals of any kind while the second refers to individuals alone that are acts of comparison.

2. Describe and criticise the Speer method.

NOTE. The Speer method is based upon the ratio idea of number. Thus, the teacher shows two lines, two surfaces, or two solids, one of which (A) contains the other (B) 3 times. A is how many times B? 3 times. What is the ratio of A to B? 3. If B is the cost of 1 apple, what is A? The cost of 3 apples. B is what part of A? $\frac{1}{3}$. What is the ratio of *B* to *A*? $\frac{1}{3}$. If *A* is the cost of 3 apples, what is *B*? The cost of 1 apple. What is the cost of 3 apples at 5¢ each? The ratio of 3 apples to 1 apple is 3; the sum which bears to 5¢ the ratio of 3 is 15¢; the cost of 3 apples is 15¢.

3. Describe and criticise the McLellan and Dewey method.

NOTE. This method is based upon the ratio idea of number. A ratio involves the quantity to be measured, the unit of measure, and the times contained (number). Thus, in *6¢ : 2¢ = 3*, 6¢ is the quantity to be measured, 2¢ is the unit of measure, and 3 is the number. What is the cost of 3 apples at 5¢ each? Since the unit of measure is 5¢ and the number is 3, the quantity to be measured is 3 times 5¢ or 15¢.

4. "The ratio idea of number should be introduced early, and applied to the work with fractions." — *D. E. Smith.* (*a*) What is meant by the 'ratio idea of number'? (*b*) By giving two typical problems, illustrate the use of the ratio idea in the early teaching of arithmetic. (*c*) By giving three typical problems, illustrate its use in fractions.

5. Give *two* meanings of the expression $\frac{3}{4}$. Show graphically their equivalency.

6. What is meant by unitary analysis in arithmetic? Illustrate by a problem.

NOTE. At 3 for 5¢, what is the cost of 5 apples? Unitary analysis requires the finding of the cost of 1 apple.

7. In arithmetic, give and illustrate the laws of association, commutation and distribution.

NOTE. The commutative (interchangeable) law applies to addition and to multiplication. Addends may be interchanged without affecting their sum; factors, without affecting their product. Thus, $6 + 8 = 8 + 6$; $6 \times 8 = 8 \times 6$.

The associative (bringing together) law applies to addition and to multiplication. Addends may be grouped in any way without affecting their sum; factors, without affecting their product. Thus, $2 + (3 + 4) = 3 + (2 + 4) = 4 + (2 + 3)$; $2 \times (3 \times 4) = 3 \times (2 \times 4) = 4 \times (2 \times 3)$.

The distributive (taking apart) law applies to addition and multiplication combined. A product is the sum of the products of the parts of the multiplicand by the multiplier; a product is the sum of the products of the multiplicand by the parts of the multiplier. Thus, $287 \times 3 = 200 \times 3 + 80 \times 3 + 7 \times 3$; $287 \times 23 = 287 \times 3 + 287 \times 20$.

8. Describe how the following units are derived or fixed: meter; are; liter; grain; yard; gallon; pound; dollar; leap year.

9. Give the unit of the metric system which most nearly corresponds to the following: inch; ton; mile; gallon; grain. Give the equivalent of each.

10. State the arguments in favor of beginning number work with counting, and against the system of beginning with number pictures.

11. Show briefly how the simple equation may be made part of elementary arithmetic, indicating the topics to which it is applicable.

12. Give reasons for or against the use of cases, rules, and formulas in teaching percentage.

13. Give the rule or formula and explain an objective illustration that would make it clear to pupils: (*a*) for the area of a circle; (*b*) for the volume of a sphere.

14. Concerning the calculating of interest name one typical legal enactment or business custom which requires that time and time rate be estimated: (*a*) by compound subtraction and using 360 da. to a year; (*b*) by finding the exact number of days and counting 360 da. to a year; (*c*) by finding the exact number of days and counting 365 da. to a year.

NOTE. (*a*) U. S. rule for partial payments; (*b*) bank discount; (*c*) transaction with U. S. government.

15. Illustrate short process: (*a*) to multiply by 25; (*b*) to multiply by 39; (*c*) to divide by 125; (*d*) to divide by $16\frac{2}{3}$; (*e*) to add fractional units.

16. (*a*) State in the form of a syllogism the argument involved in the explanation: If 8 gal. cost $2.40, how many gallons can be bought for $5.40? (*b*) State two ways in which the argument may be shortened by the omission of a premise.

NOTE. The no. gallons is 8 gal. multiplied by the no. of times $2.40 is contained in $5.40. The no. of times $2.40 is contained in $5.40 is, etc.

17. State the axioms or general laws of number on which rests the process for finding G. C. D. of two numbers by the division of the greater by the less, the divisor by the remainder, etc. (Euclidean method).

NOTE. A factor of each of two numbers is a factor of their sum, or of their difference. A factor of a number is a factor of any multiple of that number.

18. Show how the process of multiplying numbers, involving decimals, may be explained through the fundamental principles of the decimal notation without referring to common fractions.

19. Name five topics in arithmetic that can be taught without giving all the reasons, and explain in each case what device you would use to justify your action to your class.

20. Deduce the formula for changing given temperature $m°$ from F to C; R to F. The freezing point and the boiling point are as follows: F, 32°–212°; C, 0°–100°; R, 0°–80°.

21. What is the test of divisibility of a number by 3? By 4? By 6? By 45?

22. Show how to find the G. C. D. and L. C. M. of several fractions, as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, etc.

NOTE. The G. C. D. of 8 *12ths*, 9 *12ths* and 10 *12ths* is 1 *12th*; the L. C. M. is 360 *12ths* or 30.

23. What is meant by rate of exchange? Explain why it varies.

24. Describe a good method of conducting a recitation in: (*a*) written arithmetic; (*b*) oral arithmetic.

25. Prove that 7, 11, 13 are factors of all numbers composed solely of the first and fourth orders (of the decimal system) taken in equal amounts as 6006, 8008.

273. Problems and Examples. **26.** How much water must be added to a 5 % solution of medicine to get a 1 % solution?

27. If the list price is 60 % more than the cost, what additional % of discount, besides the customary discount of 25 % to the trade, may be allowed for cash payment in order to gain 14 % by the sale?

NOTE. L, $\frac{8}{5}$ C to manufacturer; 1st rem, $\frac{3}{4}$ L; 2d dis, x %; 2d rem or SP, $\frac{100-x}{100} \times \frac{3}{4}$ L; SP, 114 % C to manufacturer; $x = ?$

28. For what sum must I draw my note of 3 mo. to yield $ 1000 at 5 % bank discount?

29. Find the cost of 100 Km of wire, 55 cm in diameter, at 1 franc 24 centimes, per Kg, the specific gravity of the wire being 8.8.

30. (*a*) Find the contents of a cylindrical tank 8 m long and 9 dm in diameter; (*b*) express its contents in liters; (*c*) express in kilograms the weight of water at maximum density which this tank will hold; (*d*) express in kilograms the weight of the same quantity of oil, specific gravity .7; (*e*) translate the answers to (*a*), (*b*), and (*c*) into their approximate English equivalents (cubic inches, quarts, pounds).

31. New York is longitude 73° 58′ 25.5″ W., Sydney, Australia, is 151° 12′ 39″ E. When it is 9 A.M., March 16, at New York, what is the time at Sydney?

32. A certain calculating machine can give products up to ten digits. Explain how to use it to multiply 73,924,583 by 762,343.

33. 2240 lbs. of chalk occupy 15.5 cubic ft. What is the specific gravity of the chalk?

NOTE. The specific gravity of a substance is its weight divided by the weight of an equal volume of water.

34. Find the value of the circulating decimal $.4\dot{2}\dot{6}$ and explain the solution.

NOTE. A circulating (moving in a circle) decimal is a decimal whose last figures repeat without end. Dots are placed over the first and last figures of the part which repeats (the repetend). It is equal to a decimal ending in a common fraction whose numerator is the repetend and whose denominator is as many 9's as there are figures in the repetend. Thus, $.4\dot{2}\dot{6} = .4\frac{26}{99} = \frac{211}{495}$.

35. Simplify: (*a*) $\dfrac{\frac{3}{4} - \frac{1}{2} \div \frac{5}{8} \text{ of } \frac{6}{7}}{\frac{3}{4} + \frac{2}{3} \div \frac{1}{2}(\frac{2}{3} - \frac{3}{8})}$; (*b*) $\dfrac{\frac{5}{8} - \frac{3}{20} \div \frac{2}{3} + \frac{9}{10}}{\frac{1}{2}(\frac{2}{3} - \frac{3}{8}) \div 2\frac{11}{12} \div 6}$.

NOTE. See § 122. $\div 2\frac{11}{12} \div 6$ is ambiguous. Use the signs as they occur.

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