HAT RESEARCH SAYS
O THE TEACHER

9

Teaching High-School Mathematics

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merican Educational Research Association
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SCHOOLING is what happens to children and youth under the guidance of classroom teachers. If the teachers are well prepared, the teaching is likely to be effective in helping pupils attain the goals of the school program. But the most effective teacher is one who keeps his planning and instruction in tune with the useful and constructive findings of educational research.

Research may be useful to the classroom teacher in at least three ways: (1) by helping him develop an alert, sensitive attitude to the advancing edge of human knowledge, (2) by supplying him with facts whereby he can improve his own work, and (3) by stimulating him to go on beyond existing research findings to discover additional facts for himself.

The problem of the typical classroom teacher is to keep pace with the continually advancing field of educational research. He must know where and how to find research and then he must be able to read with understanding what he finds. The problem is further complicated by the varying degrees of reliability among research studies. These complications are so serious that many classroom teachers do not have the benefits of research and many research studies have little effect on everyday practice.

The bridging of this gap seems to be one of the most important problems in today's education. For this reason the NEA Department of Classroom Teachers and the American Educational Research Association have joined together to produce a series of pamphlets on "what research says to the teacher." The cost of printing these publications has been met by the Department of Classroom Teachers of the National Education Association. The authors are well-known research leaders from among the membership of the AERA. The layout and editing of the series have been done by the NEA Research Division.

The Department of Classroom Teachers and the AERA are indebted to the individual authors of this series. All of them have made personal sacrifices to prepare their manuscripts; none has received an honorarium. Their contributions are unselfish gifts to the progress of education.

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Teaching High-School Mathematics

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EXPLANATION

The author has attempted to draw from research material on mathematics education the items which promise to be of most help to classroom teachers. It is not a complete summary of research. In some instances opinion has been given which is believed to represent the views of most experts. The interpretations and recommendations are those which the author, Howard F. Fehr of Teachers College, Columbia University, believes to be soundly supported by research. His original manuscript was reviewed by Lynwood Wren, George Peabody College for Teachers; John J. Kinsella, New York University; Irvin H. Brune, Iowa State Teachers College; and M. H. Ahrendt, executive secretary, National Council of Teachers of Mathematics. Changes were made by the author on the basis of the suggestions of the reviewers and of the staff of the NEA Research Division.

TEACHING HIGH-SCHOOL MATHEMATICS

IMPARTIAL RESEARCH does not attempt to prove what is good or what is bad for society, since society will decide this for itself. But research should discover pertinent facts and efficient methods by which society can obtain what it wants. Thus, research attempts to answer questions such as: If we proceed in this manner, what will be the result? If we desire to achieve a defined outcome of teaching, what procedures are both necessary and sufficient? It attempts to resolve controversy by producing facts, relations, and laws. Research can hardly answer the question as to whether it is desirable or is not desirable to teach factoring in algebra, but it can and should answer the question as to whether or not students can learn the art of factoring in a meaningful way at a given mental age, and whether the factoring that is learned has or does not have later significance in the further study of mathematics.

Only when research findings cause a change in classroom teaching can we look forward to a changed product of our educational system. Only when research is reported in an operational manner, that is, in a way that classroom teachers and school administrators can see how to apply it in the daily teaching of mathematics, does it have genuine possibilities of being used.

METHODS OF TEACHING

For some students, some methods of teaching are more effective than others. If we teach algebra in a verbalized, abstract, deductive manner, defining the terms and concepts and then showing how to use these concepts in problems and exercises, the students will learn the algebra. If we teach the same material from an experiential, nonverbalized, concrete, inductive point of view, guiding students thru experience involving applications of the concept studied to their own discovery and verbalization of the concept, they also learn the algebra in about the same length of time. Furthermore, most students three months later

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Meaning and Skills

There are those who say, "Show them how to do their mathematics, and the meaning will come." Others deny this and say, "Only that is truly learned which is understood at the time it is taught." The question of which comes first, the "how" or the "why" of mathematics has been under much investigation in recent years. In all cases, the organizational, relational, concept-building approach has proved most effective for retention and application. In trigonometry there is opportunity for much rote learning as well as for concrete application. The amount of learning retained by the students follows the Ebbinghaus curve closely, for after 30 weeks students can recall only 37 percent of the knowledge they once had. However, the more meaningful and associated the learning is, the greater is the retention, while rote memorization of formulas leads to almost no recall.

If this is so, then the extreme rote memory method of teaching should be abandoned in favor of memory based on active experi-

Research does not attempt to answer all questions

mental learning followed by a deductive organization of the knowledge thus gained. In fact, if rote deductive mature procedures are the initial phase of learning, students develop severe handicaps in, and blocks toward, the creative thinking required of them in adult life. When a student is given the sequence 2, 3, 5, 7, 11, 13, 17—and asked to discover the way the sequence is being constructed, he can frequently continue the prime numbers 19, 23, 29—but he cannot give you the rule. He has made a *nonverbalized* discovery. Given time, he will make his own verbalization which, with the teacher's aid, he can put into a mature English expression, as his own personally created law which he retains a long, long time. Students who are allowed to discover (do plausible reasoning) learn best.

This indicates that open-book learning, if it is real learning, is just as good as closed-book learning. In fact, students who study plane geometry with the book always open for their reference (except on tests of achievement) do better work in deductive reasoning, solving originals, and proving theorems, than those students who must do all their class work with the book closed. We must teach students how to use books effectively, especially if we desire them to continue to gain knowledge in later life from the study and cross-reference books.

Sensory Learning

Experiential and experimental methods of teaching make use of visual aids. The use of films and filmstrips has grown tremendously in the teaching of mathematics, and yet we can see no evidence of better learning of mathematics thereby. Why? The answer is in both the type of film produced and the criteria used for selecting films for classroom learning. First of all, the film must present correct mathematics. A visual impression is strong, and an incorrect concept can easily be planted in the mind of the student. The film itself should (a) arouse interest, (b) develop the material under a sound psychology of learning, (c) summarize the important learnings, and (d) offer a challenge to further learning.

There are sensory devices other than films that can be used effectively in teaching high-school mathematics, their effectiveness depending upon the application of sound psychological principles. In general: *When a teacher has developed a psychological theory of sensory learning and combines with it well-defined, sharply envisaged goals of learning mathematics, he will make correct and effective use of sensory experience.*

The learning appears to mature from the concrete physical world to the abstract mathematical concept thru the following levels or stages: (a) *sensory-motor skills,* illustrated by acquiring the use of a pair of compasses in drawing circles; (b) *perceptual-motor skills,* illustrated by the use of a protractor in measuring the size and drawing angles to a given size; (c) *mental association,* illustrated by associating the name (exponent) or symbol $\{(\ )^n\}$ to a given situation (a^4 means $a \cdot a \cdot a \cdot a$)—this is the first step away from direct experience toward mind action; (d) *concept formation,* illustrated by the formation of a mental image and relating this image to other images thru definitions, laws of operation, applications, discriminations, abstractions, and generalizations—in this manner the concept of "proof" is established; and (e) *problem-solving,* illustrated by the skilful mental manipulation and reorganization of mathematical concepts. Thus, mathematics learning proceeds about as follows:

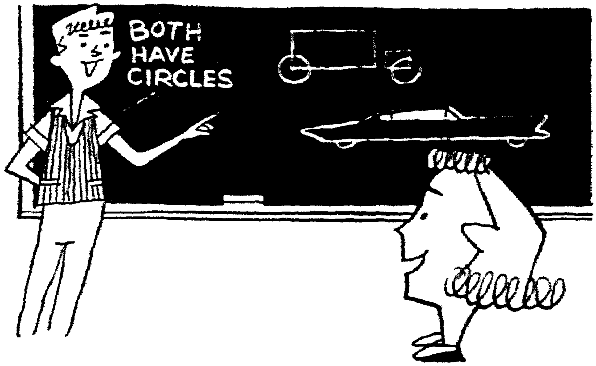
1. Experience in time, space, and quantities
2. Formation of concepts by abstraction and discrimination (plausible reasoning)
3. Building a structure of knowledge thru interrelated concepts
4. Using the knowledge (by recall and reorganization) to solve problems.

The teacher's task is to guide the student from sensory experience to meaningful abstract mathematical reasoning.

Transfer of Learning

Can mathematics be taught so that its procedures and concepts transfer to the solution of problems in assumedly nonmathematical situations? Under proper methods of teaching, the answer appears to be Yes. Mathematics contains within its development a logic that is applicable to the development of arithmetic, algebra, geometry, or any of the many branches of mathematics study. To understand this logic is one of the primary goals of mathematics instruction. Quite a number of investigations have shown that not only is this logic learned to a more significant degree, but it also becomes applicable to a wider variety of problems than just mathematical ones, if we study it in connection with areas affecting our everyday living.

Math has many concrete applications



Using the method of accepting certain assumptions, certain undefined terms, and definitions, and applying the logic of implicative statements in social situations, students become sensitive to the difficulties and purposes of this type of reasoning. Since the applications are to their own experience rather than to abstractions, such as line, point, and parallel, students grasp more readily the significance of each of these aspects of proof. They are then more discerning and more critical in applying the same procedures to proof in geometry. In fact this procedure has resulted in no loss of geometric knowledge and in a considerable gain in critical reasoning in life problems.

This method of teaching involves the following steps:

1. The student selects significant words and phrases in a situation, shows their importance in the situation, and attempts to define them.
2. The student attempts to identify stated and implicit assumptions essential to the conclusion.
3. The student attempts to evaluate the assumptions, accepting only those that past experience make plausible. The other assumptions he rejects.
4. The student requires evidence (logical and previously established fact) to support the conclusion. Lacking this, he rejects the conclusion or holds it for further investigation.
5. The student applies this method in establishing a mathematical structure.

Under such an approach to learning, not only is there a gain in transfer of thinking for bright students, but for average students also there are significant gains in critical thinking and creative power. Perhaps the best example of the type of transfer possible is the case of a high-school girl studying both geometry and English. The geometry was taught for transfer. In the English class the study of Silas Marner had just been completed. The teacher asked for the theme of the story. One student began by saying, "Silas was accused of stealing money, and he ran away." The girl student interrupted by saying, "You cannot give a particular case and have it applicable to all cases. The theme is, if a man is unjustly accused of a crime, he cannot prove his innocence by fleeing from the scene."

Historical Methods

It has been suggested that, if a person studies the historical development of mathematics from its first crude ideas to its modern abstract form, he will gain deeper understanding and greater cultural appreciation of the subject. Heretofore, most high-school textbooks have merely inserted a picture, or a space-filling statement in the text. However, when a deliberate attempt is made to fuse the historical development with the subjectmatter content, the suggestion becomes fact. Indeed, compared with students who have no historical approach, those who do, show a deeper understanding of the principles and concepts involved, an ability to apply the principles to solving problems, a genuine appreciation of mathematics as an integral element of the culture in which it develops, and greater respect for knowledge as man's guide to improvement of his life.

THE CURRICULUM OF MATHEMATICS

The goals or intended outcomes of the study of mathematics should determine the content of the high-school courses in this field. In 1940 two separate investigating committees published reports of their research that on the surface appear to be quite contradictory. Recent studies have confirmed that a convergence of both these philosophical researches gives promise of the best curriculum. The chart on page 9 summarizes findings.

A COMPARISON OF THE TWO PHILOSOPHIES OF MATHEMATICS EDUCATION: DICHOTOMY OR CONVERGENCE

Mathematics in General Education		The Place of Mathematics in Secondary Education
Child Society Universe	Order of Importance	Physical universe Society Child
Personal living Personal—social living Social—civic relationships Economic—career relationships which call for Social sensitivity—esthetics Tolerance—cooperativeness— Self-direction—creativeness— Reflective thinking	Objectives	Ability to think clearly Information, concepts, principles Fundamental skills Attitudes Interests and appreciations
Formulation and solution Data Approximation Function Proof Symbolism Operation	Subjectmatter	Field of number Geometric form, space perception Graphic representation Elementary analysis Logical thinking Relational thinking Symbolic representation
Changing values and Problem-solving ability	Primary Values	Permanent values and Organized subjectmatter
Field psychology—analysis—insight	Principal Learning Aspects	Association and generalization
A philosophic-configurational integrated learning program Difficult to evaluate	Teaching Implications	A well-structured and organized subjectmatter program Easily evaluated

The philosophy of the report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics (right-hand column of chart) has merit in that it is explicit in the mathematics content, it is readily adaptable to the present training of classroom teachers and organization of school programs, it meets most local needs, it has a continuity of subjectmatter, and it suggests correlation with other subjects of instruction. Yet, by its very organization it lends itself to mechanical and rote teaching. The Progressive Education

Association report gives more basic concepts, but fails to show classroom teachers how to help students acquire and use these concepts. Modern pure mathematics, however, has validated the PEA report as one containing the essentials for mastering the subject, and more and more, the basic ideas of solution, function, data as evidence, operation, proof, symbolism, and logic will govern the content and teaching of mathematics.

A primary factor in the change of objective in mathematics education has been the change in the function of the secondary school. In 1900 the high school was primarily a preparatory school for college and for business. Today, it is a school for general education, preparing future citizens for their many roles in our democratic society. Thus, the content of mathematics provides for varied interests, needs, and abilities of students.

Preparatory Mathematics

The traditional program of college preparatory mathematics has been under attack as failing to develop necessary skills and concepts needed later in college studies. One major criticism has been the almost watertight compartmentalism of algebra, geometry, and trigonometry into separate units of study. The relatedness of these subjects and the elements of mathematics structure they have in common are thus forfeited. To show the oneness of mathematical thinking and yet to distinguish within this thinking the separate areas of study (number, space, analysis, algebra, geometry, etc.) has been a difficult and not completely resolved problem. Certain promising experiments are underway.

Modern pure mathematics is quite different from classical algebra and geometry, and far more powerful in its methods and concepts. Some of these concepts can be introduced into high-school mathematics in an intuitive manner. Among these concepts are: *set* (aggregate of numbers or of points); *variable; function* (as a mapping in one-one correspondence of one variable or another); and *axiomatics*. These ideas belong to mathematics, and not to geometry alone, number alone, or algebra alone. Hence, they can serve as unifying concepts. They are applied to algebra in the study of the theory of solution of equations; in geometry as a basis of considering the relativeness of assumptions and theorems, and of truth and falsity; in both subjects as a relating of geometric terms with algebraic expressions; e.g., in a plane: point $\longleftrightarrow$ (x,y); line $\longleftrightarrow$ $ax+by+c=0$; distance $\longleftrightarrow$ $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$; parallel $\longleftrightarrow$ $m=m^1$; perpendicular $\longleftrightarrow$ $m=\frac{1}{m^1}$; etc. The purpose of this integration and comparison is not to eliminate the separate subjects, but to show their interrelatedness and the universality of the method of mathematical proof.

To meet the need of future mathematicians and scientists, several changes in content have been instituted. It has been found impossible to defend instruction in traditional demonstrative geometry at the tenth-year level either from the viewpoint of general education or from that of pure mathematics instruction. The attempts to reform this instruction have not been successful, largely because traditional geometry was retained as the core. A tenth-year program should contain several areas of mathematics with more emphasis on methods of thinking (in mathematics and in life), the meaning of probability and simple statistical inference, the valid use of the inductive method, and the deductive method of proof.

Much of the traditional course in plane geometry concerned itself with constructions made under the restriction that only an unmarked straightedge and pair of compasses may be used. In most textbooks no attempt is made to justify the restrictions to the use of these instruments. Nor is any attempt made to discuss the possible and impossible constructions under these restrictions. The fact that Euclid used collapsible compasses (those that come together to form a zero angle once they are lifted from the drawing board) while the modern pairs of compasses are rigid is never discussed. There is also a trend toward decreasing the number of required construction problems in the study of plane geometry. The texts of 1950 have 25 percent fewer construction problems than those of 1900. Altho classroom teachers are not certain of the purpose of constructions in the study of geometry, most of them prefer to adhere to the traditional limitation of instruments.

In view of the lack of understanding of the value or purpose of the limitations in the use of instruments, the fact that modern mathematics concerns itself not at all with this problem, and the fact that there are other mathematical instruments of greater practical value and employing mathematical principles, it seems wise today to admit these other instruments into the study of geometry.

Mathematics study in the secondary school has preparatory value, not only for collegiate study of pure mathematics, but also for engineering and applied science. The content of mathematics should then be judged also in light of its subsequent use in college courses other than mathematics alone. To this end studies have been made of the mathematics necessary to learn meaningfully and successfully college general physics, engineering physics, supervision of coal mines, meteorology, chemical engineering, agriculture, and so on.

For the study of general science courses of a descriptive nature, as now given in many colleges, very little mathematics is needed. A year of algebra and a year of geometry will give more than the required mathematical concepts, facts, and skills used in these courses. However, it has been shown that these descriptive courses fail to give genuine insight into the nature of modern science because they deliberately avoid a mathematical treatment. Modern physics, chemistry, meteorology, and geology are essentially mathematical in their mode of development. Most of the mathematics needed for this development occurs in the four years of college preparatory secondary-school mathematics. Students who know series, sequences, probability, solution of formulas and equations, and the fundamentals of trigonometry, and have a knowledge of derivative and integral can study a mathematical treatment of these college courses. The mathematical treatment gives them not only the general education in these fields, but it also prepares them for special study in these fields if they wish it.

For the preparation for the special professional fields, it is quite significant that, altho much of the mechanics of algebra, geometry, and trigonometry as now taught occurs quite rarely, the concepts of the operations, variable, function, graph, proof, and so on, become quite essential. The high-school mathematics must teach concepts—the why and the relations—as well as the skills of operation. The essential requirement in all professional fields is an ability to read and grasp the formulation of a problem, to set up the fundamental mathematical relations involved as functions or equations, to express the conditions and the natural scientific laws involved in mathematical form, to derive necessary solutions from these mathematical representations, and to check the solutions in the problem. Approximate numerical solutions obtained by trial and error, by graphs, or by numerical processes are quite common. This demands a knowledge of the approximate nature of measurement and the theory of computations with approximate data, both of which are seldom found in present secondary-school mathematics courses. The numerical solution is essential in all engineering problems.

Mathematics Education of the Slow Learner

Before 1920 to 1923 most students, incapable of studying traditional high-school mathematics, dropped out of school. With the raising of the mandatory age for remaining in school and a change in philosophy of high-school education, brought about by the Seven Cardinal Principles, a new problem was posed for high-school teachers of mathematics: What mathematics is essential and can be learned by students with low scholastic ability? Tests show that 25 years ago and continuing right up until today, noncollege preparatory students entered high school with great deficiency in arithmetic knowledge. On the basis of standard tests, about 75 percent of these students are below ninth-grade level both in computational skills and in arithmetical reasoning; quite a few students manifest only fifth-grade achievement.

These students show a variation in achievement ranging over five years. They are particularly deficient in computations which require sustained attention (as in multiplication and division) and reasoning (as in fractions and placing the decimal point in a product). By careful and considerate teaching, significant improvement in ability to compute can be made. In fact, the students with the greatest deficiencies make the greatest gains in computation. However, the students with the highest IQ's make the greatest gain in ability to reason.

Students in general mathematics have difficulty in learning algebra. They can learn with a satisfactory degree of success how to evaluate formulas, use signed numbers, and solve simple equations. They have great difficulty with making formulas, operating with general numbers, and solving simple word problems, and the solution of literal equations is beyond the ability of most students. *In concrete geometry these students fare best of all.* They can memorize the name and recognize geometric figures, recall simple relations of lines and angles, and do geometric drawings using T-squares, triangles, protractors, and drawing boards. Of all the mathematics they study, they enjoy this geometric work most.

Certain psychological characteristics of nonacademic minded students suggest to the classroom teacher means of teaching and selection of subjectmatter that has interest and motivates learning for these students. Because of their past continued failure in the study of arithmetic, they are, in the majority of cases, frightened and inhibited by the ordinary presentation of mathematics. This attitude can be changed by building self-respect thru permitting initial success with simple material, by giving everyday practical applications, and by showing sympathetic understanding of their difficulties. The classroom teacher must recognize that the primary distinguishing feature is the lack of ability in the higher mental processes. The slow student has difficulty in forming associations with words and ideas, he cannot readily generalize from a series of experiences, his memory is not strong, he has little taste for the abstract, and he learns best by starting with concrete physical situations.

These students have shorter attention spans; they cannot concentrate on problems over a long period of time. Hence, the classroom teacher's presentation must be short (five to 10 minutes at most), the students must have a variety of activity: laboratory activity, group discussion, short concentrated study or practice, board work, and showing of films. There must be frequent review of materials previously studied. These students learn, but they learn at a much slower rate than the more mentally able.

Students do not always like or desire to study the mathematics they will need in post-school life. The mathematics necessary for personal use, use in the home, for intelligent reading and communication, for everyday occupations, and for good citizenship reveal a desirable body of mathematics as a basic minimum for all persons to learn. This content has been published as a check-list of 29 items in several publications (see bibliography). However, as modern life grows more mathematical, the basic 29 competencies are not sufficient, nor can all the mathematics needed in general education be taught in one year. Investigations of citizens as consumers show that the following topics of consumer mathematics are essential: budgeting, buying on the instalment plan, cost of operating a car and a home, how to read the financial section of a newspaper, how to save and invest money, how to use graphs, the nature of measurement, the computation of taxes, the interpretation of simple statistics, and how to solve quantitative problems.

These topics are best learned when placed in a concrete practical situation that has face value to the student. For this purpose the use of community resources is invaluable. Two procedures are successful: the one consisting of field trips or going into the community, the other of making use of human and other resources of the community by bringing them into the class. For seeing mathematics at work, the institutions of most value to students are a local bank, the city treasury and tax department, a lumber mill, an engineering factory, a retail department store, a city water or sewage disposal plant, an insurance or loan office, and the post office. In all such visits the learning is effective only if it is carefully planned so that the mathematical features are searched for, meaningfully prepared questions are asked and answered, and the total mathematical aspects are organized into subsequent classroom instruction.

Less effective, but still of value, is bringing to the class an authority or specialist who will talk about and illustrate his particular mathematical needs. Such specialists include Social Security representatives, actuaries from insurance companies, accountants, engineers, architects, tax assessors, machinists, and research scientists. These persons frequently bring models and literature with them that have high motivation value.

In one community, a most effective project was the interviewing of many people who use mathematics. Forms were prepared for interviewing and obtaining one or more genuine mathematics problems that these people had to solve in the daily routine of their work. In a few weeks the students had gathered hundreds of genuine problems involving mathematics from the simplest arithmetic to differential equations. The persons giving these problems, among others, were housewives, secretaries, laundry operators, bankers, retail clerks, engineers, chemists, and executives. These problems, where applicable, formed the basis for instruction in the class work thruout the year and proved a highly effective learning procedure.

Desirable Outcomes of Learning

Besides the nature of proof, one of the most frequently mentioned desired outcomes of the study of solid geometry has been the development of space perception. Space perception is defined as ability to do successfully the items on various space relations tests. This perception is considered a great need in architectural and engineering study. When over a thousand students, selected at random, are examined, it is found that the study of the traditional course in solid geometry, or of excerpts from this course imbedded in the plane geometry course, does not develop these space perceptions. Students who do not study solid geometry make *more gain* in space perception (as measured by space relation tests) than those who do study the traditional course. To develop space perception, we must teach a type of geometry different from the traditional Euclidean course. While no experimentation has yet been announced, it appears that a space coordinate geometry combined with descriptive and projective geometry is the type of study conducive to filling the needs of scientific engineering students.

An attempt, in a single year of study, both to learn the mathematics of plane geometry and to develop the ability to do critical thinking in democratic life in most cases has proved to be possible. A number of experimental classes show an achievement in plane geometry at least equal to that of the straight geometry class and, moreover, a superior achievement in applying logic to life problems. Most research workers agree that critical thinking in everyday affairs can be achieved in high-school study and that the study of proof in plane geometry is aided by and abets this critical thinking, if sufficient time is allowed for the study. The applications in life must be studied by taking time from the regular study of geometry. The questions of how much of our usual geometric study is unnecessary repetition and what values can be gained by devoting more time to the study of the nature of mathematical proof both in geometry and in algebra have not been reported.

Modern logic has, in its study of axiomatic procedures, gone far beyond the classical logic. This logic has had a pronounced effect on the teaching objectives of geometry. The usual traditional approach in high-school geometry is to state a theorem and give its proof, and then go on to more theorems. These theorems become facts, and most students remember the facts rather than the nature of proof as an outcome of their study. When is a theorem proved? When are the conditional and biconditional relations valid? How are the various logical forms of a proposition, such as the theorem, the converse(s), the opposite(s), and the contrapositive(s) related to each other? These questions have proved to be significant if students are to penetrate into the real nature of mathematical proof. Students must be habituated to constantly framing the other forms of any given proposition and examining them for their truth or falsity.

The study of algebra has frequently been regarded as a task of learning manipulative procedures on literal and numerical expressions. It has never been regarded or presented in high-school textbooks as a logical deductive science of relationships in various number systems, or as a logical extension of arithmetic to generalized relationships between any number to be considered. The results show that stated goals of algebra study are not being met. The mechanics tests fairly high, but when concepts and understandings of number system, binomial, equation, variable, term, factor, and exponent are evaluated, less than 20 percent of our first-year algebra students show reasonable mastery. The ability to use algebra to analyze and solve problems shows the lowest attainment of all, yet this is probably one of the most desirable outcomes.

That this state of affairs can be changed is evident if we teach concepts and understandings as well as skills. Students who are taught to transfer their understanding of algebraic concepts to the analysis and interpretation of numerical data make gains both in understanding and in algebraic skills. In achieving this desired result, perhaps less time should be spent on the mechanics of factoring, special products, combining fractions, and substituting in formulas, and the time thus saved placed on building the concepts of these operations, proportion and variation, indirect measurement, and statistics. The outstanding characteristic contributing to success is the use of student discovery thru experiment. Students who discover an algebraic principle or law from an experiment cannot always immediately verbalize the result, but must be guided to a mature expression of their discovery.

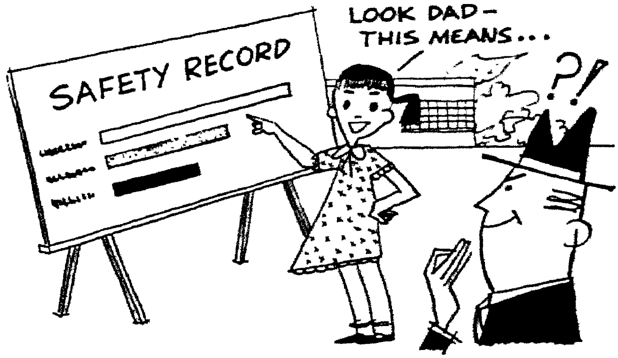
Since 1900, teachers have been urged to integrate the study of mathematics and science. There have been a few attempts to teach mathematics and physics as a combined course, but these experiments have given little help. The more promising way is to use physics in the problems and explanations of mathematics instruction, and to use mathematics in the development and study of physical laws. The most important aspect of mathematics for this purpose is "function." The essential concept here is relatedness; that is, one set of numbers is related to another set thru a law of association of the individual numbers of one set with that of the other. Then, when the one set of numbers represents a measure of one physical quantity and the second set that of another physical quantity, the mathematics function is an explanation of the relatedness between the physical quantities.

Under this concept, there must be developed the subconcepts of variable, dependence, variation, equality, inequality, continuity, discontinuity, and realm and domain of definition. These subconcepts are the basis for understanding and applying the concept of function to the relations of quantities in our environment. The teaching of function as one of the unifying themes between the branches of mathematics, and between mathematics and other sciences, is one of the major changes in mathematics instruction in the last 30 years.

New Considerations

Statistical reasoning is fast becoming an integral part of the common activities of man. It enters into the understanding and the explanation of problems in safety, longevity, human variability, cost of living, occupational choice, income distribution, advertising, age distribution, change in family size, testing, measuring, public opinion in public affairs, games of chance, sports, consumer preferences, scientific investigations, conservation of resources, and many other areas that affect the living of all of us. For the interpretation of these areas there is not so much a need for the numerical computation of statistics, such as the mean, median, mode, correlation, variation, or statistical parameters, as there is need for knowing what these concepts imply. It is in the correct interpretation of presented statistics that laymen can take intelligent action.

Statistical understanding is a MUST today



The two important aspects of statistics for the citizen are: (a) in a distribution of attributes arising from chance causes, there is a coexistence of stability (a central tendency) accompanied by a variation (dispersion about the central tendency); and (b) by proper sampling, both the stability and the variation of the whole population can be predicted to a high degree of probability. Since the solution of group problems in a democratic society demands mass participation, and since the process of solution often entails the use of statistical reasoning, the participation is possible only if large numbers of people are prepared to make the statistical approach. Thus, statistics must become a part of the common learnings of all secondary-school students.

The examination of current textbooks in secondary-school mathematics indicates that the only statistics that is common is graphic presentation. There is little or no attention given to collection and interpretation of data. There is not available, for high-school use, a body of tested technics and an organization of statistics subjectmatter. The teacher must manufacture his own course. In such a course students must collect, organize, and analyze their own statistical data. They must apply mathematical methods to the concepts of stability, variability, and sampling. For example, if the weights of all 14-year-old boys in a school are placed in a distribution and the arithmetic mean weight is found to be 124 pounds with a standard deviation of 7 pounds, does this distribution apply to all 14-year-old boys? Most probably it does not. Is a boy in this group weighing 138 pounds overweight? A correct concept of variability in chance causes will indicate to parents that *overweight* for an individual is a phenomenon that has little to do with the average weight of all individuals. When all facts are considered, the boy may actually be underweight. Parents who insist that their children, who are under the average weight for their age, should gorge themselves with fat-producing foods are in dire need of statistical reasoning.

A course in mathematical statistics can well replace a half year of traditional study of mathematics without loss to a foundation for later advanced study of mathematics.

Mathematics and the Core

A core program of education attempts to develop problem-solving ability by utilizing all organized subjectmatter knowledge in an integrated application to large areas of human activity. On the other hand, a program in mathematics per se attempts to develop an organized structure of knowledge by abstracting from human experiences the quantitative aspects that are common and related by similar laws. Can mathematics be learned in a core program? The answer is Yes and No; Yes, by providing genuine situations for application and study, and No, since any deductive structure demands all the bricks from the foundation up for its understanding and use, even tho only the top floor of the edifice is needed.

The organization of mathematics around concrete problem situations works best in the junior high school. The concepts, especially those of arithmetic, are as important in problem-solving as are the skills. Students can learn as much mathematics in a *problems program* as in a traditional program, provided they study a regular mathematics textbook as well as other reference material. In the study of large problem areas at the junior high-school level, mathematics concepts are essential to the solution of problems, and the amount of mathematics needed is reflected in the reports of national committees and commissions on mathematics study.

Modern Mathematics and the Secondary-School Program

While the major responsibility of the high school is to develop a useful mathematics for its students who study mathematics, it also has a secondary but very powerful task of identifying and preparing future mathematicians. This would be a simple task if mathematics did not change. However, it does, and modern mathematics is so utterly different from the traditional in its language, symbolization, and processes of solution that in turn it demands a different orientation for the secondary-school students who will go on to this type of advanced study. Some of the changes in concept and language are a matter of debate. Others are in the general realm of agreement. In particular, while initially we retain the present language, we apply the more modern manner of speaking before the learning is complete. For the various types of numbers we shall eventually use the word *set* as in *set of integers, set of real number,* and *set of rational points on a line.* In changing fractions to lower or higher terms, in changing equations thru the use of axioms, in changing axes by translation or rotation, we shall finally use the more general idea "transformation." In discussing function as change, as variation, as an analytic expression, e.g., $y = 3x + 4$, we shall sooner or later use the language "set of points" or set of ordered pairs $(x, 3x + 4)$ for every allowable x. These changes in approach to mathematics are available to the classroom teacher in some very recent college textbooks.

THE LEARNING OF MATHEMATICS

The ultimate goal of teaching is that students acquire a set of meaningful concepts that they can use effectively to solve problems. There is sufficient evidence that in the past our students have not done too well in acquiring these concepts, even tho they do acquire temporary skills and pass examinations in computational mathematics. When a problem-solving item is recast into a computational item, the difficulty of the item is significantly lowered, and conversely, if a set of easily solved equations is recast into a verbal problem item, the difficulty level is significantly raised. In fact, good students who do high-grade work in the mechanics of algebra make a very poor showing on a test of quantitative-functional thinking. Students can "do" their mathematics, but they do not know what it is they are doing.

Learning of Concepts

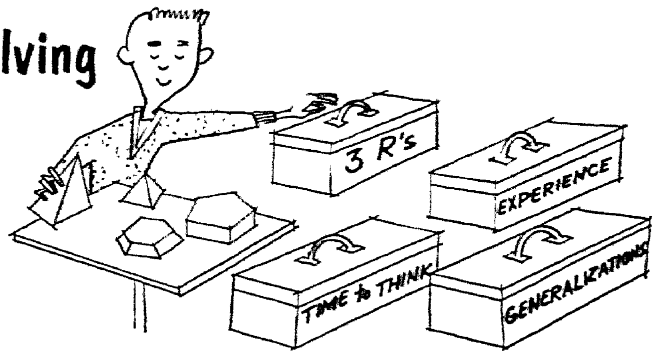
Recognizing that these students do not understand the concepts of algebra (and other mathematics), intensive research has been undertaken to find out why, and how to improve the learning of concepts. A major finding is that students maintain their naive and primitive childhood understandings (of zero and equality, for example) which interfere with the abstract learning. Most students have context constancies; that is, they attach a unique meaning in a unique situation and fail to extend the meaning by abstracting and generalizing. Guidance and cues are necessary if a context idea is to be enlarged into a mathematical concept.

Problem-Solving

The use of concepts finds greatest realization in problem-solving. In mathematics, problem-solving has generally been held synonymous with getting a solution to verbal problems. Thus, the solution to 15 percent of 80 is considered a skill and not a problem. But the solution to: What do I save on an article priced at $80 if I receive a 15-percent discount? is considered a result of problem-solving. To help students solve problems, most textbooks still resort to a standard procedure. A type problem is given, explained by an illustrative example, and followed by 10 to 20 problems involving the same technics for solution. Thus, the student memorizes a procedure and develops a skill, but beyond recognition of a type, does no real thinking. To aid in this type of solution, the student is given from six to eight general steps beginning with (a) read the problem, (b) list what is given, (c) list what is to be found, and so on, to (g) solve the equation, and (h) check the answer.

This procedure has been tested with more modern procedures of problem-solving. In the modern sense, a problem is a situation in which there is need for attaining a goal, but the route to the goal is unknown to the student. To find the route, he must *think*. Once the route is found, the route to the goal is secured and made automatic by practice. Then, the situation ceases to be a problem and becomes a skill or knowledge applicable to the solution of new problems. Thus, all learning at the high-school level is to be regarded as problem-solving accomplished thru thinking. *Learning is thinking. Learning how to learn is accomplished thru problem-solving.*

Problem-solving requires many resources



Under this concept (suggested by psychological research) learning how to combine $2a + 3a$ to a single term is as much a problem as any verbal problem. The student is *not told how* to do it, but *discovers* how to do it by applying previously learned correct concepts. He applies what he has learned about multiplication, about coefficient ($2a$ means $a + a$), about addition, and comes up with his own declaration that $2a + 3a$ is $(2 + 3)\ a$ or $5a$. Then he has solved a problem and he has learned. From here on he generalizes further to $ax + bx$ is $(a + b)\ x$.

Which of the two procedures leads to the best outcome in problem-solving? Every experiment conducted in arithmetic, algebra, geometry, and college mathematics during the last 30 years gives overwhelmingly strong support to the discovery-analysis-organization method in preference to the step procedure. Students who are given a rule and shown how to do some piece of mathematics develop a mental set of waiting for the teacher's explanation and then imitating the explanation. They do not attempt to extricate themselves from a novel situation by their own wits.

Specific suggestions for developing problem-solving ability as revealed thru successful learning are:

1. *Develop a problem consciousness.* Students must gradually become aware that several readings of a problem are necessary to give adequate understanding of the solution the problem requires. They must develop the attitude that "a problem is a situation in which I am supposed to experience difficulty. It is a situation I must explore, bring to bear all my past learning, and do a powerful lot of hard thinking."

2. *Develop wide experience and broad background in mathematical situations.* Without past experience to recall, without a knowledge of how generalizations are made, without specific concepts of operation, function, distance, area, volume, and velocity, a student cannot apply necessary routes toward a solution. One pertinent experience is "reading in mathematics." Reading in mathematics calls for a different approach on the part of students than does reading in history, literature, newspapers, or comics. It calls for building a new and technical vocabulary, for relating the phrases to other phrases in the problem as well as to past knowledge. It is slow and deliberate reading demanding high concentration.

3. *Activate the problem in the classroom.* This can be done in a number of ways. Have the students rephrase the problem in other language or dramatize the problem. The problem can be diagramed on the blackboard or illustrated by concrete objects such as miniature ships, autos, planes, and bowls of mixture. The concrete situation must be symbolized, but in moving from the concrete objects to an abstract symbolization, the student learns how to recognize reference in the world about him for later symbolization and problem-solving.

4. *Develop the ability to ask meaningful questions.* The classroom teacher must set the stage by creating an atmosphere friendly to all genuine inquiry. No student question, no matter how trivial or how deep, can be ignored. A teacher's question must be followed by sufficient time for genuine thinking to take place before a response is given. Most teachers ask too many questions and grant too little time for thoughtful response. A teacher who uses the heuristic method will enable the students to think. But this is not enough. The teacher

must review with the class, the questions asked, in the order he asked them, indicating the purpose of each question so that the members of the class can learn how, on their own part, to ask meaningful questions that tend toward solutions.

5. *Aid students to abandon unsuccessful attacks and try other approaches.* Multiple procedures for attacking problems give deeper insight into the solution of the problem. Encourage students to recall from experience similar situations. Use analogy and induction to gain hints toward a procedure that can be applied to a given problem.

6. *Have students estimate a sensible answer and work backward to the data in the problem.* This is plausible reasoning, one that all mathematicians have used in creating new mathematics. The students can manufacture their own problems and see how the problem comes to be a problem.

7. *Generalize the solution to every problem so that it may have the widest application in solving new problems.* It is generalization that permits transfer to a new situation or to an analogous problem in other subjects. The more students generalize their solutions, the more they verbalize their procedures, and the more they construct a logical organization of each new problem, the better problem-solvers they will become.

The learning of mathematics is not based on one theory of psychology, such as conditioning, associationism, or Gestalt, but on an eclectic agreement of all theories. The following elements have been found sufficient for a satisfactory psychology of learning:

1. There must be a *goal* on the part of the student to learn. The student must be aware of this goal.

2. All cognitive learning involves association. Thus, a^3 means $a \cdot a \cdot a$ is illustrative of much of the learning in mathematics.

3. Trial and error, approximation and clarification, and analysis are all important in discovering routes to a goal or solutions to a problem. It must not be a hit-or-miss guessing situation, but a structural thinking situation.

4. Learning is complete only to the extent to which relationships and their implications have been understood.

5. The learner must be active mentally. He learns what his intelligence is directing him to do.

6. Intrinsic rewards of success and awareness of progress toward a goal strengthen and motivate learning. Punishment and continual failure are deterrents to learning. Praise, success, self-esteem, and status are the best motivation for learning mathematics.

7. Abstractions (discriminating of attributes) and generalizations are essential to effective learning. Mathematics can be learned only in meaningful situations which permit these mental processes.

8. In mathematics, most new learning is a transference of past learning into a reorganization of a new situation. Algebra is generalizing and restructuring arithmetic; geometry is generalizing and restructuring space concepts; trigonometry is generalizing and restructuring the alegbra and geometry; and in all these subjects logic is developed as the structural binder.

9. We learn facts, skills, and understandings, but we also learn "how to learn." The ultimate goal is to project the student thru life on his own power to learn.

10. We also learn attitudes (feelings). If unsuccessful, we learn to dislike mathematics and even those engaged in its teaching or research. From happy experience we learn to like and respect mathematics. For all mathematics students: (a) learning is thinking, and good learning is correct thinking, (b) successful thinking is possible at a given level by all students, (c) successful thinking is dependent upon acquired concepts and relationships, and (d) satisfactions from successful thinking provide the highest and most enduring enrichment for the learner.

The Use of Sensory Aids

The growth of mind is related intimately, from birth on, to the motor and sensory experience the organism shows. Physical activity plays the initial stage in learning fundamentals of number and space. Counting is based on serial motor manipulations. In geometry the child draws lines, then rectangles, then triangles, giving names to these physical constructs and finally retaining the name as an abstract for the purposes of reasoning. How frequently we forget this and delve at the start into pure logical relations!

All initial learning in the classroom should have its origin in some concrete problematic situation in which sensory activity can enter. All learning has a motor base, reinforced by sensory experience, from which concepts are formed by making discriminations. The capacity of a child to initiate this experience and to profit by it is a function of his intelligence. Not all children are endowed with the same basic capacity for initiating experience. Where the experience is lacking, the teacher must aid the learning by supplying proper sensory activity to motivate it.

In the use of models, the teacher must at all times be aware that the ultimate objective is to eliminate the model by replacing it with a body of mathematical knowledge. The model is a means to an end and not the end in itself. When to leave the sensory aspect, because the student has had sufficient experience to proceed to the abstraction of the mathematics to be learned, is a judgment that must be made by the teacher. The teacher's task is to take the student from the sensory to the intellectual in such a way that the intellectual accomplishment has meaning and future application.

INDIVIDUAL DIFFERENCES AND GUIDANCE

Most pupils in the elementary school (thru Grades VII and VIII) are exposed to the same program of mathematics education. At the end of the eighth grade, differentiation in later study is made according to the child's ability, interests, and needs.

Readiness for Learning Mathematics

Standardized prognosis tests have been developed to indicate whether or not a student is intellectually prepared to study the course in algebra, in geometry, or in higher mathematics. These tests have proved highly satisfactory, but they do not tell what other type of mathematics the student who falls below the probable success score is capable of pursuing. Most school systems have differentiated programs in mathematics in the high school.

In the past, the fundamental criterion of selection has been the desire of the parents. It has been shown that a single criterion is not as good as the composite of several measures of the student's competence. The important elements are: (a) prognosis tests, (b) the eighth-grade final arithmetic mark, (c) the eighth-grade teacher's estimate of the student's ability, (d) a general intelligence score, (e) a total scholastic achievement score, and (f) the student's desire. Thorndike has shown, that for a thoro course in ninth-grade algebra a student should have a minimum IQ of 110, yet correlations between achievement in algebra and IQ are generally between .40 and .50. This would seem to indicate that some children, by social and inner drive, interest, and hard work can succeed in an algebra course even tho their intellectual capacity is not as high as teachers deem necessary for success.

Guidance and Mathematics

In the guiding of pupils, the noncollege preparatory type of mathematics must not be considered a "dumping ground" for discipline cases or lower ability students. Guidance consists in finding the type of mathematics study which is most profitable to the individual student and then supplying that type of study for him:

1. *Help students to make their own wise choice.* The teacher or counselor, discusses with the student his school record, his ability, his desires, his possible need, the types of study available, and between them they arrive at a satisfactory choice of study.

2. *Know the students well.* Besides the elements previously listed, the past school record of a student, with indicated trends in achievement, interest, and behavior, aids in guidance. The student's family background, his home life and environment, and his possible place in society are equally valuable in guidance.

3. *Acquaint the student with life's mathematical needs.* Students find the *Guidance Pamphlet in Mathematics for High School Students* exceedingly useful. Industrial pamphlets carry weight and authority with students. Industrial and professional speakers on Career Days stimulate students to make keener judgments on their choice of study.

4. *Shift students from one type of program to another as their interests change or latent possibilities emerge.* A student continuing study that is "over his head" will not profit as much as he could by shifting to a more practical "down to earth" type of study. A general student who suddenly discovers what mathematics is all about needs the challenge of more rigorous and logical study. Schools that provide shifts at several appropriate times thruout the year have the fewest drop-outs.

The Gifted Child

A continuing problem is the identification and education of talented students in the field of mathematics. This involves defining the "gifted child," getting criteria for selecting him, and providing an adequate program of mathematics for him.

Some definitions of talented youth are "the upper 1 percent in intelligence," "the child with an IQ of 137 or higher," "the child with an IQ of 120 or higher." The essential qualifications, so far as mathematics is concerned, are the possession to an unusually high degree of ability to work with ideas, that is, to manipulate and create in abstractions. Classroom teachers have usually failed to identify students who subsequently become famous in mathematics or science chiefly because of three factors: (a) the personal equation of likes and dislikes and confusion of friendliness, obedience, conformity, and beauty for ability and talent; (b) lack of evaluative standards; that is, there was no way to make a valid estimate of memory, curiosity, and abstract thinking to judge these attributes as outstanding; and (c) ignoring chronological age factors; that is, in the case of two equally fine performances, that of the younger person indicates the greater accomplishment.

The recognition of certain characteristics has proved helpful to classroom teachers in selecting and guiding gifted children. The most significant trait is an *extraordinary memory*. Gifted children have a storage capacity for previous learning that is truly amazing. On the contrary, nontalented children have poor retentive powers. Gifted children do *abstract thinking* at a high level. They make brilliant generalizations quickly and accurately. Able students *apply their mathematics* to problems in their environment. They see mathematics in the world about them. They also possess *intellectual curiosity*. They speculate and subject their speculations to analysis and logical procedures. To a very great extent, gifted children show persistent *goal-directed behavior*. They have a capacity for long and deep concentration and stick to a problem until it is solved satisfactorily. Do not confuse genuine concentration with the plodding and muddling that less able students use in arriving at solutions.

A finely distinguishing characteristic is *insight* or *penetrability* into a problem. By intuition and speed of organization capable mathematics students grasp and present solutions to problems that are amazing in their clarity and succinctness. A characteristic of gifted students in all academic fields is the possession of a *high vocabulary with facility of expression*. These students can write and carry on all kinds of communication in a very mature mode. Another characteristic is *virtuosity*. The talented youngster takes an ordinary piece of knowledge and creates out of it new problems and new ways of interpretation that mark the minds of geniuses.

When the capable mathematics students have been identified, they must be given a curriculum commensurate with their abilities. From all studies made on this program, one dominant principle emerges. By the end of the senior year of high-school study, bright students should cover all the essential sequential mathematics that is usually presented thru the first year of collegiate mathematics. Thus, the student is not handicapped in later study by the omission of certain essential areas of mathematics study. In addition, students of this caliber may be given outside reading in books containing approaches to modern mathematics including the study of theory of sets, abstract algebra, finite geometrics, and mathematical logic.

WHAT THE TEACHER SHOULD KNOW

Most studies of teacher knowledge have dealt specifically with the undergraduate preservice training of mathematics teachers. These studies are usually concerned with the subjectmatter and methods courses in mathematics rather than with general education.

Preservice Education

While many of the research studies name specific courses to be pursued, there is no guarantee that any course will give the knowledge that the research worker had in mind. Beyond this, in the proposals there is a stultifying commonness of nineteenth-century collegiate mathematics that would leave the teacher entirely ignorant of the tremendous change and approach in modern twentieth-century mathematics.

More appropriate is the spirit of these research studies which can be summed up in the following criteria. The teacher's mathematics education should:

1. Give the teacher a deep and broad knowledge of the methods, content, and structure of pure mathematics considerably beyond that of secondary-school mathematics
2. Embrace the formal study of several fields of knowledge where mathematics is used, e.g., physics, chemistry, and engineering

3. Initiate an accumulation of mathematics applicable to a wide variety of situations and appropriate for use at all levels of high-school instruction

4. Be professionalized to the extent that, wherever possible, it is linked with secondary-school mathematics

5. Include specific provisions for the study of the basic concepts and mathematical methods of statistics, including knowledge of the applications

6. Give specific attention to the history and development of mathematics with stress on implications for secondary-school teaching

7. Give specific attention to the nature of high-school teaching of nonacademic students with special reference to arithmetic teaching

8. Include technics for operating mathematics laboratories, including the construction and use of mechanical models.

The teaching of mathematics tends to become stagnant because teachers teach the same material in the same way that they were taught. The textbooks are made to satisfy teachers and their felt needs. Thus, traditional algebra, demonstrative geometry, and trigonometry of angle functions still occupy most of the teaching program. Gradually, however, modern approaches to mathematics, its nature, its type of proof, its emphasis on structure and order, are beginning to creep down into the secondary school in some of the newer textbooks and experimental programs. There is a trend toward a breakdown of years of study of one field, such as algebra or geometry, into a more unified study of mathematics. Algebra and geometry are not to be discarded, but they are to be studied simultaneously as different approaches to mathematics structure.

Inservice Education

The reorganization of mathematics in the secondary school poses a real problem for the inservice re-education of the mathematics teachers. This education must include three areas of study: (a) a study of modern mathematics (set theory, topology, decision problems) with particular emphasis upon the teaching of high-school mathematics, (b) the need of mathematics for noncollege bound students with special emphasis on the psychology of teaching these students, and (c) the procedures for revising the constructing programs in mathematics education to meet the changing needs of society (we need more mathematicians and mathematically trained technicians than ever before in the history of our Western culture) and to adopt new knowledge into the program without too great a time lag.

Such re-education and trends, as indicated by the happenings of the last 50 years, can give significant guidance to the best type of study. In this half century, the schools have shifted their *primary function* from that of preparing the major fraction of students for college to providing for the needs, interests, and abilities of all youth. In 1900 there was only one course in mathematics not strictly college preparatory, namely, business arithmetic. This course has shifted since then from a strictly vocational aim for commercial life to general education for personal and consumer use. With this change in purpose, there was also a shift in placement of the course from junior high school to the presentday eleventh and twelfth year of study. The general mathematics of arithmetic, intuitive geometry, and some elements of algebra is now taught in Grades IX and X for noncollege bound students.

In the college preparatory courses, significant changes have occurred in subjectmatter. From a period of *no graphs* (1900) we have moved to a period (1955) when graphs occupy more than 20 percent of the first course in algebra. In the same period, parts of trigonometry have filtered down from Grade XII to find a place in Grades IX, X, and XI. Likewise, some parts of analytic geometry, statistics, and calculus are now included in the senior high-school textbooks. Of even greater effect is a change in requirements for high-school graduation, from three and four years of study (in 1900) to no study of high-school mathematics (in 18 states) and a requirement of at most two years of study (in one state). Of course, college entrance requirements in some cases make the study of mathematics a necessary elective for two or three years. However, in the United States today, only 1 in 12 seniors in the high school is studying twelfth-year college preparatory mathematics.

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