THE TEACHING OF ARITHMETIC

F. F. POTTER, M.A., B.Sc.

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LONDON
SIR ISAAC PITMAN & SONS, LTD.
PARKER STREET, KINGSWAY, W.C.2
BATH, MELBOURNE, TORONTO, NEW YORK
1922

PRINTED IN BATH, ENGLAND
BY SIR ISAAC PITMAN AND SONS, LTD.

INTRODUCTION

The teaching of arithmetic has, like the teaching of other things, improved during the last forty years. The improvement has been a steady growth due almost entirely to the teachers themselves. It is not the residuum of a series of fashionable innovations hastily adopted and quietly discarded, nor is it the kind of evolution which has left, in turn, our Handwork, Drawing, Language Teaching and Infant School Method in a state of chaos, waiting reverently for a pronouncement from some distinguished native or foreign dogmatist.

Briefly stated, the teacher of arithmetic to-day wishes his pupils to be able to do everything that he himself did at school and to do it equally well, omitting only those units, measures and processes that he has never found useful ; he insists from the earliest stage that the problems his pupils solve should be such as actually occur from day to day in an ordinary man's life, or better still, in the life of a boy or girl ; and that with this in view, many exercises should be based on the current geography, or geometry, or handwork of his class.

Tables of weights and measures of merely historical interest are left embalmed in dictionaries and encyclopaedias, and are now history, not arithmetic ; and in spite of the copious and capable treatment of the metric system encouraged in this book, I am bound to declare my feeling that there is something wrong in attempting for forty years (some of the older teachers will remember the early " charts ") to teach children a foreign system of weights and measures which the adults of the country have never had any intention of adopting for general purposes. I am not speaking here of measurements in the exact sciences ; there it was adopted by adults, and not thrust like other lost causes on the small children of the country.

A certain number of "tables" must be learnt, and learnt very thoroughly; no teacher of arithmetic has ever questioned this; he knows that failing this necessary training his pupils will later in life be mistaken for imbeciles. But to-day he shows his pupils by diagrams and by experiments with measurable stuff, with scales and weights and standard measures, how such tables are built up, and makes them share in the responsibility for their accuracy. If a boy forgets some item, all is not lost; he sets his reason to supply his lapse of memory.

No practical teacher has any sympathy with an idea which not many years ago spread among some examiners, that accuracy in calculation was quite a minor point, if the "method" was right. It is quite true that there is something wrong with an engineer who calculates the strength of an iron bridge to the tenth decimal of an ounce; but there are two things wrong with him if, in calculating the tenth place or any other figure he declares that six fives are thirty-five. Much has been written during the last ten years on the possibility of "training," but everyone now acknowledges that accuracy in calculation can be improved by *practice* in calculation, and in this book the author gives generous help to teachers who include accuracy in their scheme of arithmetic.

Mr. Potter's experience among engineers and with technical classes has led him to deal at some length with those excellent problems where pupils find all their own data by measuring, weighing, and reference to books. Such exercises, apart from their immediate stimulus to the pupils' intelligence and initiative, help to smooth the passage from the ordinary school to the technical class.

The very practical bias of the whole book is modified by the author's recognition of the fact that some boys and girls are in a sense pure mathematicians by nature, and are interested in numbers, and in the properties of number; they are continually constructing their own little theories of number, and have a craving for knowledge rather than skill. They are never left behind by their class-mates in applied arithmetic; but they take up a different attitude towards squares and square roots, tests of divisibility, and numerical puzzles, and as they are the mathematicians of the future, their special tastes should be satisfied and developed.

The author of this book has for many years had special opportunities in training colleges and schools for studying the teaching of arithmetic. Formerly his daily work included the teaching of mathematics in all the stages from the application of calculus to engineering down to "demonstration" lessons on simple arithmetic to small boys; he knows, as few men do, the difficulties that teachers are always meeting; and he has submitted many of his pages to the keen discussion of other successful teachers. The result is a book that teachers and students will find valuable and interesting; and its earnest effort to invigorate the teaching of arithmetic deserves a great success.

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July, 1922.

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CHAPTER I

ARITHMETIC IN THE INFANT SCHOOL

(By Miss A. Sweaney, B.A., *Vice-Principal, Cheshire County Training College, Crewe*)

Formal lessons in number are of no value to the child until he has had a sound and varied experience of number, and until he has acquired a considerable vocabulary in the language of number. It is the work of a teacher of children in the Infants' School (that is, generally speaking, of children four and a half to seven years old) to supply this experience and vocabulary.

The method is the same whether the child is one of thirty or forty, or whether he is the single tyrant of a household. Grown-ups call the method "The Playway," but to the child play is life itself; he is playing from morning until night; he teaches himself the beginnings of number, that is if he is not interfered with and if his environment is sufficiently varied to arouse his continual curiosity. A child knows whether his apple is *bigger* than his neighbour's; that he is *four* years old; that John is the *tallest* boy in the class; that he has *three* brothers in the big boys' school; that he was *first* in the line; that the clock strikes *twelve* at home-time, and *seven* when he must go to bed; that dogs and horses have *four* legs, and that little boys have *two*; that tables and chairs must be drawn with *four* legs, and that he has *five* fingers on each hand.

A child is continually meeting with these and similar number facts and words both at home and at school; he knows them; the words *big*, *three*, *first*, and *tallest* have a real meaning to him; in the schoolroom he should meet with more of these real experiences unobtrusively put in his way by his teacher; the main point to bear in mind is that the teacher should excite a curiosity in the child so that the language she gives to him supplies a want. If the child is allowed to work with a will at real games and toys, his teacher will be able to give reality to the language of number; and she will be able to emphasize whatever term she chooses for emphasis in whatever lesson she will. There are stories of giants with *one* eye, of *three* blind mice, of *three* bears, of Cinderella and her *two* sisters, stories, games, and songs without end; these stories are acted and shown in pictures. There are as many as *ten* little nigger boys, and this old favourite gives a natural opportunity of allowing the children to count *ten* backwards as well as forwards; the grimy little fingers of a child have often been described as ten little nigger boys; when one hand is clean and the other dirty, then there are *five* little nigger-boys and *five* little white boys, and so on. Birthdays are a never-ending source of interest and much can be made of them, for each boy on his birthday must know how old he is. Thus, in many ways his knowledge of number in language and experience grows.

At this earliest stage there should be no attempt to generalize about terms such as *three*, *big*, *first*; and this is not the time for dots and strokes, or sticks and counters, and other neutral kinds of apparatus, unless these rather dull objects in themselves become transformed into table-legs and the like. For example, in a period of free play in a well-supplied Babies' Room, a boy of four brought three sticks to his teacher and asked for help in making a table. The teacher said simply, "I can't make a table with only *three* legs, because such a table would not stand up; please bring me *four*." The child turned away and regarded in a puzzled manner his three sticks; he put them back in the box whence he had taken them and which contained scores of sticks; next he carefully picked up *two* sticks and then *two* more; it was easy to see that he knew he had overcome his difficulty. From his procedure it was clear to the most inexperienced observer that the child was teaching himself that *two* and *two* make *four*, but that he had not yet grasped that *three* and *one* made *four*.

This is just one of the many daily incidents in this boy's playroom; the Babies' Room was its official title. There were many objects and toys, and there were boxes containing scores of beads, tablets, and sticks. Healthy English boys or girls aged four and a half, of normal health and intelligence, will not find much employment for their eager brains and fingers in three beads, yet children in the past have been conceded twenty minutes free play with three beads! Let the child play by all means, but let it be real play. The teacher must earnestly try to see life from the child's point of view, and to give reality to all the activities of the child, and must take care that the child knows the meaning of the terms that he uses.

Counting. Children love counting and many can chant far beyond twenty on their fifth birthday. Counting is to little children a kind of recitation, one of the first they learn; many a nursery rhyme means nothing more to them than counting up to *twenty*. But counting often becomes more interesting than a nursery rhyme because there is a kind of game attached to it; for as you say *one, two, three, four, five,* and so on, you must touch different railings or climb up different stairs; it comes to have a more definite meaning when a child can count all the chairs at the table and see whether there is one for each person; the laying of a table for a doll's tea-party is really a lesson in counting. All teachers can give meaning to counting in games; skipping may be used to provide a variety of counting experiences; even during the troublesome time of marking a register the children can sometimes try to count how many are present in the classroom.

Values of Number. The second step in the teaching of number is to systematize the games and work of the classroom in order to teach the values of numbers up to *five*. Some children learn this by themselves and know it quite well on their fifth birthday; others need definite help and direction. Teachers choose for themselves the time when they begin this systemization. Greater uniformity and speed of progress can be hoped for in a class of six-year-olds than in a class of five-year-olds; but whatever the child's age at which directed experience or definite teaching of number is begun, the principles underlying the teaching are the same. First teach by experience. It is as important in the Infants' School as in the standards that the number problems to be solved by the child should be drawn out of his own experience.

In teaching the values of three and two, the teacher must grant that the normal child, like the normal grown-up, does not count to know whether there are two or three apples on the plate; he *sees* two or three. Dewey says "that it requires a considerable power of mental abstraction to count three," so do not expect it too soon of a child. Let him see that when *one* is taken from *three*, *two* remain; and that when *two* are taken away, *one* remains; the material used may be apples, or balls, or skittles, or what you will; the utmost the teacher can do is to help the child *to say what he sees*. Leave the idea of unity alone; that will grow and look after itself for a time; abstract conceptions of unity do not belong to the Infants' School; and abstract conceptions of two and three come later. A normal child, if he is allowed to throw balls into a basket, can soon say whether he sees one, two, or three balls in the basket.

It is well now to associate regularly the figures 1, 2, and 3 with their values, so that a child may learn to read three for "3," as he is learning to say "cat" for "c-a-t." This can be impressed by a series of pictures, for to a child "three" will always be three of something: three cats, three apples, and so on. At this stage it is helpful to begin what are known as number pictures; later on, perhaps almost immediately, the child can be allowed to arrange number patterns as he pleases; but it is as well to have one set of pictures or patterns to begin with, and to refer to, when the child is in later difficulties. The most reasonable set of pictures up to five, seems to be as follows—

1 2 3 4 5

These patterns can be associated continually with illustrations of different objects, and with symbols. Folk-songs and nursery rhymes are useful. The song, "The Twelve Days of Christmas," provides pictures of objects up to twelve, and this song can complete itself for the time being, on the sixth day or the seventh, or on any day up to the twelfth. The objects associated with the number up to five are: *one* partridge, *two* turtle-doves, *three* French hens, *four* singing-birds, and *five* gold rings. A very fascinating set of pictures might be made out of the old series, "One old ox opening oysters," "Two toads totally tired, trying to trot to Tutbury," "Three fairies flying to France for fashions." Yet here there is a danger. Children must be interested in the facts of number, not in the fictions of fairy-tales; whatever the objects chosen to be depicted, they are only illustrations to be associated with the symbols and number-patterns shown above.

So, after much experience, a child sees numbers up to the value of three; he learns to put *two* and *two* together and call it *four*; he learns various ways of arranging *five*;

and he has long ago learnt how to count past *ten*. The teaching of numbers up to five is not difficult.

Some teachers find it hard to bring real, interesting experience of number into the classroom. The following so-called "apparatus" was invented by a student, and is the type of thing that can be used for occasional lessons. A picture of an outstretched branch of a tree was drawn on grey paper; the ground was covered with snow; five tiny sparrows were cut out of cardboard; by means of a strip of cardboard pasted behind each sparrow, these birds could be placed in the picture either on the branch or on the ground. A suitable story accompanied the flitting of these birds here and there, and the children were continually analysing the number 5. Other numbers could be dealt with in a similar way.

One thing must be borne in mind in making up apparatus to teach number, and that is that it is better to deal with like objects even though "unlike objects must ultimately be recognized as forming one group." The following problem was given by a student to a class of children: "Now, what do three potatoes, two carrots, and one tomato make?" The answer proffered by an inspector who was present was "vegetable stew."

The real difficulties begin with the third step in teaching the value of numbers after five. While these attempts to teach five are being made, the child is teaching himself values and word numbers far beyond five. His counting games are going on; he has found, perhaps, a new way of chanting 2, 4, 6, 8, 10, up to 20; that is, he can count in two's. He has played dominoes, and from the picture of the dots he knows how to place 9 by 9, as well as 3 by 3, and so on. His understanding of number is far beyond his powers of expression in number, just as a baby who is beginning to talk understands the meanings of very many words he has not the power to use.

Grouping Numbers. A teacher must decide how she is going to approach the bigger numbers; let us say, first, the numbers between five and ten. She must choose whether "6" shall be from the outset $5 + 1$ or three "2's," or two "3's." This is an important decision; the child, perhaps, is not aware of it, but the teacher must be. If the ultimate aim of teaching arithmetic is to be kept in mind from these earliest days, as it should be, namely, that it is "nothing more than that of leading the child to figure quickly and accurately in the common problems of his experience," then it is best to avoid the addition of ones. The child already knows from the language of counting, that 6 follows 5; as far as language goes, the number series up to 10 is safely established. It seems reasonable then, as we must continually think of bigger numbers in factors, to begin by regarding "six" as two "threes." This falls in with the traditional multiplication table. It is easier to deal with two groups than with three groups; *eight* as two "fours" is seen in the legs of two chairs, or horses; *ten* is two "fives," and the illustration is ever present in a pair of hands. Teachers by the way, need not be afraid to use the hands as illustration; such a method does not necessarily encourage "counting on the fingers." As the child's understanding of each number grows a new table can be built up alongside the traditional one, namely,

$$1 \times 2 = 2$$
$$2 \times 2 = 4$$
$$3 \times 2 = 6$$
$$4 \times 2 = 8 \quad \text{and so on.}$$

The child is not ready yet for drill in tables, but he is learning what is meant by saying "two threes are six," and "three twos are six," and he can count up from two in twos. This chant of his is beginning to mean something; now he can begin to count up from one in twos, and from three in threes. Number patterns show the difference between odd and even numbers. *Seven* and *nine* obviously cannot be approached by the table method; their place, however, is known in the series. It is known, in counting, that *seven* comes after *six* and before *eight*. If we make use of the " pattern " of seven (Fig. 1), or of the domino seven (Fig. 2), *seven* is obviously *four* and *three*.

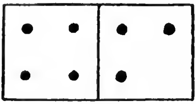


FIG. 1.

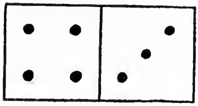


FIG. 2.

In teaching numbers up to ten, always remember to group the objects. Grown-ups generally count these numbers in groups ; a handful of nine counters thrown upon a table will group itself to the eye in three " threes," or *five* and *four*. A young student asked to give a lesson on " nine," and intending to show it as *eight* and *one*, had the following diagram on the board—

○ ○ ○ ○ ○ ○ ○ = 8 pennies
○ = 1 penny

○ ○ ○ ○ ○ ○ ○ ○ = 9 pennies

She did not at first understand where she was wrong when the mistake was hinted at. If counting in ones is to be avoided, then these " pennies " must be grouped.

It will be found that many a child who has begun to deal with numbers as big as *seven*, *eight*, *nine*, can add small numbers together without any *apparent* reference to the concrete. Some teachers who have thus succeeded in their aim have been known to be frightened of the result. When a child states simply, 6 and 3 are 9, or that 7 and 4 are 11, they shake his confidence in himself by asking unnecessary questions such as, " How do you know ? " " Did you guess that ? " Perhaps the child does not know the meaning of the word " guess." It is as well to give him credit for his powers. Such a teacher is more interested in her method than in its result, and she has lost sight of her aim in teaching arithmetic ; she reminds one of the hen who clucks in helpless dismay when she sees the ducklings she has hatched swimming across the pond. Many a child's mind has emerged from the cumbersome weight of counters and sticks, in spite of the well-meaning efforts of his teacher to increase her "apparatus" and vary her "methods" of holding him captive. Many children under seven years of age have learned how to add, and they have a good understanding of what they are doing when they say that 5 and 4 are 9.

We should remember that, even in the infants' school, "the notion of number is not the result of immediate sense perception, but the product of an activity of our minds." We may begin our teaching of definite number by *seeing* one, two, or three, but we cannot *see* nine. And from the philosopher's point of view, "the general proposition that two and two always make four, can obviously be known with certainty by consideration of a single instance, and gains nothing by the enumeration of other cases in which it has been found to be true." In other words, when a child knows that *two* and *two* make *four*, it is a sheer waste of time to keep him proving it by means of apparatus. He must learn the use he can make of this knowledge and apply it in a practical way.

Our methods of dealing with the normal child have in the immediate past been a little too much influenced by the methods of dealing with the slower minds of the sub-normal. The present generation of grandfathers learned how to count and figure accurately when they were at school, and they are crying out at the present time against the inaccuracy and unreliability of present-day teaching. They had no Froebel's gifts or Tillich's bricks or Montessori apparatus, and other aids. These things are used with great success in teaching their grand-children, but the more successful the teaching the sooner are these props to be dispensed with. If a teacher finds a child, even at this early stage, soaring into what is called the abstract, let her give him practice to strengthen his wings. This, of course, means individual teaching; but where there is practical handwork at which the children are working in groups it can be done. In practical work children can learn how to use and apply the knowledge they have acquired.

Practical Arithmetic. The business of weighing and measuring has gone on alongside that of counting, and new words are entering into the child's vocabulary, such as dozen, pair, shilling, feet, pound, ounces, inches, pennyworth, and so on. These words must now grow in meaning to the child, and the need for a unit of measurement or of weight be made clear to him. One teacher, in order to give practice in measuring used one wall of her classroom for the purpose of recording the heights of the children. Each was measured on his sixth birthday. The children learned to do the measuring with the twelve-inch ruler, and they used the terms feet and inches. Lines were drawn on the walls to indicate the different heights. The children were very interested in studying these heights or lengths, and they compared them and talked about them among themselves. The event of a birthday occurs on an average nearly once a week, and the ceremony of measuring was always repeated so that the children had constant practice. The birthday child had always to state the day of the week, the date of the month, and the year.

A child's aid could be used in making, for example, certain covers for the desks to be used during clay-modelling. Each cover must fit the desk; one child may suggest putting the material to the desk and measuring, but a child who has had practice in measuring his own height up a wall might offer another method, for he would understand the use of a twelve-inch ruler, and would soon recognize the need for knowing the two dimensions. A teacher should seize at once on the chance that the cutting out of covers for desks offers of a practical arithmetic lesson for children in Standard I.

The permanent shop is now a common feature in an infants' school. Much use can be made of this. It is always a sad thing to see children called upon to roll up their clay or plasticine models of tiny spoons, eggs, carrots, and the like, and to hear them told to replace the plasticine in their boxes.

Some of these objects could be placed regularly in the shop with other suitable things made of other materials. These articles give opportunities for practice in buying and selling. Paper coins are often used, but there should be a sensible pair of scales. During such recreative work the child will find his own difficulties and in many cases solve them.

Addition. The next formal step in number is the teaching of the values of numbers beyond ten. The teacher in the infants' school must choose her way of showing the child how to add such numbers as 8 and 6, 9 and 8. Here the first difficulty of 10 has to be overcome; it is, perhaps, easier to understand the components of 20 or of 30 than of 10. When considering 10 it is more difficult to make use of language, for the names of eleven and twelve do not help. It is better that a child should learn—language aiding where it can—that 10 *and* 4 *are* 14, 10 *and* 7 *are* 17, before teaching that 8 *and* 6 *are* 14, and 9 *and* 8 *are* 17. The best way of teaching the composition of 10 is to count 8 *and* 6 as corresponding to 8 *and* 2 *and* 4, that is to 10 *and* 4 *are* 14. If a child knows that 7 *and* 3 *are* 10, and 10 *and* 2 *are* 12, he has already the knowledge necessary to teach himself that 7 *and* 5 *are* 12. Thus the value of 10 *and* 2 must be taught before the value of 7 *and* 5; 10 *and* 3 before 7 *and* 6; and 10 *and* 4 before 9 *and* 5, and so on.

This may seem a cumbersome method considered thus, and it is as well to compare it with the ways used by grown-ups in counting. Some have never recovered from the injury done them in their early teaching, and they count in ones on their finger as a rule, in spite of the possession of a fine reasoning and mathematical faculty. This leads to inaccuracy and slowness in the mechanical mental processes; they probably think of numbers, let us say the workable numbers, up to a hundred as in a series, and rarely in factors. Certain numbers like 61 and 47 mean very little more than numbers in a series; but numbers like 48, 56, 72, are very often thought of as 6 times 8, 7 times 8, or in terms of other factors. Then, there are other people who add a long column of figures without any apparent calculation, just as the majority of people, including children, have memorized the values up to 10. But most people break a number at the multiple of 10; for example, 47 and 8 is 50 and 5. This is the most satisfactory method of teaching a child how to add.

Other points to bear in mind are that 7 *and* 7 should always be *two sevens*, and 8 *and* 8, *two eights*, and so on; that it is easier to add up if the bigger number is put first; that as 9 is one less than 10 it might be better to add 10 and take 1 away, for a child soon learns to add in tens; or he may take the 1 off first, and then add ten. These little dodges, and there are many others, the children often discover for themselves.

These are some of the principles of adding that will be kept in mind by the teacher and that will underlie her use of games, handwork, measuring, drawing, and weighing. A child's experience is growing all the time, and he is daily finding out how useful number is to him in such activities as measuring out the playground for a game, building a house, making an engine, giving out counters, scoring for games. He must be given daily some opportunity of applying his powers of calculation to some useful, purposeful end. He must also learn to recognize the need for accuracy. Definite lessons in handwork and drawing should be planned with this end in view—scoring games are easy to find and to invent.

The child learns to read numbers as he learns to read. He has to find the right page in his book. But as all problems of the infants' school can be worked mentally, there is, as yet, no actual need for notation. He easily learns the meaning of the signs $+$, $-$, $\times$, $\div$, $=$. It is obvious that if these signs are to be used at all they must be used accurately.

By this time he has had a little experience of tables; he has seen simple tables on the board and has built up the greater part of the twice times table and some part of the three-times table. Although he understands the twice-times table he may not yet have memorized it. Table-drill belongs to the standards. But he might be taught how to refer to the tables on the board. It is necessary, in order to secure speed and efficiency, to memorize the multiplication table, but later a child should know how to refer to tables which it is unnecessary to memorize.

Subtraction. Subtraction in the infants' school is closely allied to addition, and can be taught with it. If the child understands that 5 *and* 3 *are* 8, then he easily understands that 3 *from* 8 *is* 5, and that 5 *from* 8 *is* 3. If the method of teaching addition is allied to that of teaching subtraction then 9 *from* 16 would be taught as 9 *from* 10, 1, *and* 1 *and* 6 *are* 7; *answer* 7. This plan is seldom adopted in the written work of the standards because it is slow; nevertheless, it is the easiest one to adopt when a child adds 9 *and* 7 as 9 *and* 1 *and* 6.

Division. Division sums are difficult things to deal with in the infants' school, and a teacher must be very careful in giving a meaning to the processes of division. All work in division must be practical. Let us examine one instance: 15 *divided by* 3 *is* 5. But this means nothing as yet to the child. The two things he may understand are that (*a*) if he had fifteen nuts to share among three children, each would have 5 nuts; and that (*b*) he could take 3 nuts away from 15 nuts 5 times. These processes should be taught separately ; the first, containing the idea of sharing, is the more easily understood. A child has continual practice in this kind of division, while sharing out counters for games, in shopping, in handwork, and the like. Sums in division worked on paper do not belong to the infants' school, and it is disheartening to find some teachers writing $\frac{2)9}{4 + 1}$ on the blackboard.

When counters are used they should be in bags containing a definite number. There might be bags of 10, of 20, of 50. When they have been in use they should not be collected in a haphazard way and heaped indifferently into the bags, but they should be most carefully replaced, so that no mistake can occur when, for example, a bag of 50 has to be divided into bags of 10, because perhaps there are not enough of the bags of 10 to go round the class.

Handwork and Arithmetic. Practical work in money problems and weighing problems should precede any formal teaching. One single piece of constructive work, carefully and accurately carried out, supplies material for innumerable practical lessons in number. Children love to build a house into which they can creep ; they make a tent out of a clothes-horse and a dust sheet, or they outline a square in bricks on the ground ; or they build a rampart of sand. Barrie let the children build Wendy a house. Children in an infants' school can build Wendy's house too ; in this house there are *nine* boys to look after, that is, Wendy has *ten* to provide for, including herself. A suitable house can be contrived ; the children will discover a way, and they will provide the right number of chairs, tables, plates, spoons, napkins, and other articles for ten children. The plates and spoons can be bought at the shop ; they have been made by the children in the different classes ; the napkins, perhaps, are pieces of paper suitable for paper-folding ; each of the ten children must learn to fold the napkin neatly ; plates that are lost or broken can be renewed at the shop at a reasonable price. Money transactions take place and paper money is circulated. Peas and beans can be weighed on real scales and distributed among hungry Peters and John Napoleons. Water can be measured in exact quantities in graduated cans. Suitable cloth can be measured and cut to fit the table—a stock of different sizes can be found in the shop. Baskets, mats and other objects of the children's handwork gain reality, and find their way to Wendy's house via the shop. All this is true play to children, and they must find in the teacher one who enriches the play by her presence and her suggestions, not one who spoils it by asking them to do too many sums. While the children are playing, the teacher should organize their experience in counting, adding, sharing, weighing, measuring, buying, selling ; they are feeling the need for, and learning the use of, these processes.

Throughout all the teaching of numbers the teacher has to be keen and always ready to take every opportunity of impressing a fact while the child is playing ; that is while the child is getting acquainted in his own way with the tiny facts and problems of his own life. Children themselves will often guide the teacher in her choice of game, story or handwork, but she herself must have a definite plan in her mind which she follows in the play and which she modifies or adapts according to the needs of the children.

CHAPTER II

THE SIMPLE RULES—ADDITION AND MULTIPLICATION

FOR some years in every child's life, the scholar is concerned mainly with the so-called "Rules" or Processes of Addition, Subtraction, Multiplication, and Division. It is customary and natural to begin with the "Simple Rules," i.e. with the manipulation of numbers, before proceeding to the "Rules" in Money and Measures, Fractions and Decimals.

Before these formal processes can be begun it is essential that the child should have a real idea of number and numbers, within certain reasonable limits. This idea is usually gained incidentally by children either at home or in the ordinary infant school. The average child can count without much help from teachers or elaborate apparatus, though the power to perceive readily the number of units in a group is acquired but slowly. Many elaborate methods and much ingenious apparatus have been put forward to teach and enlarge the perception of number in young children. Many interesting theories have been offered as to how this perception of number arises in the normal mind. Nearly all teachers of young children have some pet method for teaching number, and it may even be possible that no particular method is more useful than another, for in this, as in every branch of teaching, the enthusiasm and personality of the teacher is of far greater influence than the details and apparatus of his method.

Sooner or later, the child escapes from these elaborate theories and methods of analysis and synthesis of numbers, and has to face the hum-drum drudgery of "adding up," "subtracting," "multiplying," and "dividing"—for drudgery a great deal of it must ever be, however much we try to lighten and disguise the labour. And the drudgery arises from the fact that teachers are concerned most of all, at this stage, with the formation of *habits.* The child needs good habits of adding, subtracting, etc., just as much as good physical habits, a fact which is apt to be overlooked by the newer race of teachers. In their zeal to cultivate what they term "intelligence" in arithmetic, they are apt to forget that speed and accuracy in calculation are as essential to-day as ever. Now speed and accuracy in the Rules of Arithmetic are largely a matter of habit, and habit depends upon practice. For most learners this necessary practice involves much effort before even such a simple process as ordinary addition can become habitual and automatic.

But habits, even in arithmetic, may be good or bad; and it is here that the teacher should be a safe guide, for it is usually just as easy to form the good habit and to teach the reasonable method as it is to perpetuate some of the bad methods which still linger in schools. Let us state here a fact which most young teachers and some old teachers find it hard to believe: The methods of calculation which *they* learned in their school days are not of necessity the best or most reasonable even though they personally find them the easiest to use and, perhaps, to teach. What is required from all teachers is an impartial examination of all usual methods and a reasoned conclusion as to which method attains its object in the simplest and most straightforward manner.

With these few introductory remarks, we will begin our examination of the Simple Rules and their teaching.

Addition. Adding speedily and accurately is a habit formed only by constant practice. Let us examine how a column of figures is usually "totted up" by an adult.

$$\begin{array}{r} 6 \\ 5 \\ 7 \\ 3 \\ 2 \\ 4 \\ \hline 27 \\ \hline \end{array}$$

Here the stages beginning from the bottom are 4, 6, 9, 16, 21, 27, the numbers being voiced or thought.

Few adults should find it necessary to say "4 and 2 are 6, 6 and 3 are 9, 9 and 7 are 16." Some, of course, with the bad habits acquired in childhood, will still find it necessary to say four, five-six, seven-eight-nine, ten-eleven-twelve-thirteen-fourteen-fifteen-sixteen, etc., i.e. they will laboriously count the units in each number—we have even met educated men who count upon their *fingers*.

Now this normal process (4, 6, 9, 16, 21, 27) is acquired slowly and is the result of much patient practice and teaching. But to establish the normal process should always be our aim.

Many people suffer throughout their lives through using bad methods of addition. Nearly all these methods are based upon the lengthy and laboured counting of units. Many of them are relics from previous generations of teachers who laboured under the vicious "payment by results" system, and consequently were obliged to resort to mechanical devices to produce a mechanical accuracy. And, unfortunately, these methods still survive in some schools and the poor scholar finds himself encumbered with clumsy props and methods which he seldom manages to discard. Such are the methods of "tapping" or "dotting" each figure and counting the dots or taps; of ornamenting each figure with tails thus  and counting all the tails singly; of putting down strokes at the side and counting and crossing off each stroke; of performing a tattoo with right or left hand or both, on table, knee, forehead, or nose, gravely counting in units as before. All these clumsy devices, seen all too frequently even among adults, are habits which *need not have been acquired*. Some, such as finger-counting, certainly have the sanctity of a long history in human development, but because primitive man yesterday and to-day made and still makes great use of his fingers and toes in simple calculations, this is no valid reason why we in the twentieth century should continue to employ primitive devices. The formation of many of these bad habits is undoubtedly due to the attempt to do formal *written* sums in addition too early in the child's life before the necessary mental facility has been acquired.

Mental and Oral Practice in Rapid Addition. The keynote of success in rapid addition is constant *oral* and *mental* practice, before formal written work is attempted. This may take many forms.

(1) Practice in *group* counting, e.g.—

2, 4, 6, 8, etc., proceeding by 2;
3, 6, 9, 12, etc., proceeding by 3.

The clock face is a ready example of "fives" up to 60.

This may be continued and extended to any desired limit, and forms the very best introduction to the multiplication table.

(2) Similar *group* counting but with a *varying starting point*, e.g.—

1, 3, 5, 7, proceeding by 2;
2, 5, 8, 11, proceeding by 3.

Both (1) and (2) may later be performed *backwards* as well as forwards, thus providing useful practice in subtraction.

(3) Regular addition of *alternate groups*, e.g.—

(i) 2, 4, 7, 9, 12, 14, etc., adding first 2 and then 3;
(ii) 3, 7, 12, 16, 21, 25, etc., adding 4 first and then 5.

(4) The *ring* of figures.

This ancient device is as useful as any for rapid addition. It is especially valuable where a large class has to be taught. The ring may be constructed of any numbers in any order.

1
5 6
4 9 3
8 7
2

The usual method is for the teacher to write a ring on the blackboard and to touch certain numbers, the children adding together the numbers as indicated. The addition may be done mentally by all the class or orally by selected individuals, and the answers may be given orally or written down. The device provides unlimited practice in rapid addition, and should be frequently employed.

(5) A *row* of figures.

A series of figures such as : 1, 2, 3, 4, 5, 6, 7, 8, 9, may be used in the same way as the *ring*, the class adding the numbers pointed to ; but the constant horizontal motion of the eyes is more tiring to the children than the use of the *ring* described above.

(6) Rapid addition of a *series of numbers* as dictated by the teacher.

Here the teacher calls out a series of numbers, e.g. 2, 5, 6, 4, 3, etc., and the children add them mentally, thus: 2, 7, 13, 17, 20, etc.

This is a harder exercise, for the child has no visual images to aid it.

(7) Oral practice in *complementary* addition.

Examples of this are—

(*a*) What will make each of these up to 10 ?
2, 3, 4, 5, 6, etc. (*Ans.* 8, 7, 6, 5, 4, etc.)

(*b*) What will make each of these up to 20 ?
18, 15, 11, 9, 5, etc. (*Ans.* 2, 5, 9, 11, 15, etc.)

This is, of course, subtraction disguised.

The constant use of exercises such as the above is essential if the habit of adding speedily and accurately is to be formed. The range and speed of the exercises can be varied according to the age and ability of the class.

(8) Mental addition of numbers where the *units figures together are equal to* 10.

Examples should be given such as—

$14 + 6$, $14 + 16$, $14 + 26$, etc.
$15 + 5$, $15 + 15$, $15 + 25$, etc.

(9) Ready application of the *principle* $a + b = b + a$. The children should be taught to recognize, for example, that

$$13 + 8 = 18 + 3 \text{ (for } 3 + 8 = 8 + 3)$$

and similarly that

$$22 + 9 = 29 + 2.$$

Such knowledge frequently makes an addition easier as well as forming a useful check.

The Formal Addition Sum. Immediately numbers of two or more digits are employed the difficulty of the "carrying" figure arises. The actual process is soon learned. Most adults "carry" the figure mentally, but for some children it seems necessary to write it down. Later when long additions are made, it is generally written down for purposes of checking.

Usually, however, the process is as follows: Using example (*a*): 2, 10, 14, 21. *Put down* 1 *and carry* 2. 2, 4, 5, 8, 9. *Ans.* 91.

The actual meaning of the *carried* 2, i.e. that 21 units equals two tens and one unit, can be illustrated by ball-frame or by another convenient apparatus or objects.

(*a*)

```
 17
 34
 18
 22
 --
 91
 ==
```

A few teachers prefer to have the whole displayed in full, thus showing the actual value of each column as in example (*b*). In this method there is no bother of a *carrying* figure, though a second addition has to be made at the end. The method is useful and convenient when long columns have to be added.

(*b*)

```
  35
  27
  19
  84
 ---
  25
 140
 ---
 165
 ===
```

Aids and Devices in Addition. (*a*) Consider example (*c*). Some teachers insist on proceeding by means of the *ten*. Thus, example (*c*) would be worked 5 and 5 are 10 and 1 are 11, 11 and 4 are 15, 15 and 5 are 20 and 2 are 22, 22 and 3 are 25. Here both 6 and 7 are broken up to arrange the amounts in *tens*. The additional labour of decomposing these numbers seems unnecessary even for

(*c*)

```
  3
  7
  4
  6
  5
 --
 25
 ==
```

children. It is far more sensible to teach the children to keep their eyes well in advance of their reckoning and to count figures that together make 10 as ten wherever possible. Thus, in example (*c*), $6 + 4 = 10$, and $7 + 3 = 10$. Hence, the normal steps in the addition should be 5, 15, 25, so that the answer is reached in two steps instead of four.

(*b*) In long additions it is useful to mark each hundred as reached, in any column of figures. This saves the labour of repeating orally or mentally, e.g. the lengthy "one hundred and five," "one hundred and eleven," etc., for if the hundred is marked with a dot or a stroke of the pencil, the counting of the *new* hundred can proceed easily; thus, "five," "eleven," etc.

(*c*) Though *vertical* addition is more usual, *horizontal* addition is important and should receive constant practice. The arrangement is a little more difficult for children, since *units* and *tens* are no longer placed vertically under one another.

(*d*) *Checks* are all-important. Every child should be encouraged to form the *habit* of checking every addition sum. In checking, additions should always be made in the reverse direction, i.e. "down" instead of "up," or from left to right instead of from right to left.

(*e*) The familiar exercise *add rows and columns* as in example (*d*), provides both "vertical" and "horizontal" practice while the addition of the separate totals serves as a check.

	(i)	(ii)	(iii)	Totals.	
(*a*)	234	527	631	1392	
(*b*)	29	135	67	231	
(*c*)	243	189	521	953	
(*d*)	59	542	432	1033	
Totals	565	1393	1651	3609 3609	Grand Total

More Advanced Work in Addition. (1) Addition of *two columns* simultaneously.

In these exercises *tens* and *units* are added as one column, and then *hundreds* and *thousands* as one column. The method certainly saves time where much addition has to be done. Consider the example given. Added as one column, the steps are 26, 71, 104, 133, 174. If this is considered too difficult the following method should be tried, adding first tens and then units thus—

41
29
33
45
26
174

26, 66, 71, 101, 104, 124, 133, 173, 174.

(2) Simple addition used in the *formation of squares.*

From any given whole number which is a square the next consecutive square can always be formed by simple addition. The following examples will show the method—

$$(12)^2 + 12 + 13 = 169 = (13)^2$$
$$(13)^2 + 13 + 14 = 196 = (14)^2$$
$$(14)^2 + 14 + 15 = 225 = (15)^2$$
Similarly $$(30)^2 + 30 + 31 = 961 = (31)^2$$

The mathematical reader will recognize this as the simplest application of—

$$(n^2) + (n) + (n + 1) = (n^2 + 2n + 1) = (n + 1)^2$$

(3) The addition of a series of numbers in *arithmetical progression.*

This need have no terrors for teachers. Let those who fear attempting to teach it try the following simple lesson with a class of boys aged twelve or thirteen years.

Examine a simple number series, e.g.—

1, 4, 7, 10, 13, 16, 19.

Note the characteristic of such series, i.e. that they proceed by a regular addition (or subtraction).

What is the average of 1 and 19 ? (10.)
,, ,, ,, 4 ,, 16 ? (10.)
,, ,, ,, 7 ,, 13 ? (10.)
What is the average of all *seven* numbers ? (10.)
What then is the sum ? (7 × 10 or 70.)

Apply the same process to the following—

2, 4, 6, 8, 10, 12. (Average 7. Sum $7 \times 6 = 42$.)

10, 15, 20, 25, 30, 35. (Average $22\frac{1}{2}$. Sum $22\frac{1}{2} \times 6 = 135$.)

The method of working is clear—

(i) Determine the *average.* This is always in the *middle* of the series. In number it is also by symmetry, $\frac{1}{2}$ (First No. + Last No.) or $\frac{1}{2}$ (Second No. + Last-but-one No.), or the mean of any pairs of numbers symmetrically placed about the middle.

(ii) Having found the *average*, multiply this by the *number* of terms. All similar series can be summed in the same way.

Sum of series = (No. of terms) × (average value of terms).

The more formidable algebraic statements,

$$s = \frac{n}{2}(a + l) \text{ or } s = \frac{n}{2}\{2a + (n - 1)d\}$$

are but generalized forms of this simple truth—

The sum of an arithmetic series = (No. of terms) × (average value of terms).

(4) The sum of *consecutive odd numbers*, e.g.

$$1 + 3 + 5 + 7 + 9, \text{ etc.}$$

This can be summed either by the method shown in example (3) or by first principles.

Its interesting property, known in the time of the ancient Greeks, if not earlier, will be discovered by any normal scholar—

$$\begin{aligned} 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

The sum is always the *square* of the *number of terms.* Thus the sum of the first *ten* odd numbers is $(10)^2$ or 100, and the sum of the first 12 odd numbers is $(12)^2$ or 144.

The property is easily demonstrated by squares drawn or cut from paper, thus—

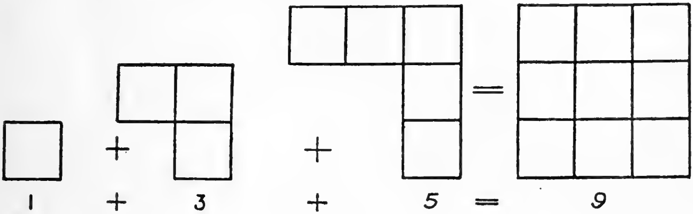


Fig. 3.

The separate pieces may be cut out, or imagined, to fit together as the complete square 9, and the same may be done with any series of odd numbers beginning with 1.

(5) The formation of *magic squares.*

This is a fascinating subject to most children. Fig. 4 shows the numbers from 1 to 9 arranged as a *magic* square, i.e. as a square in which rows, columns and diagonals all have the same total, viz., 15. An infinite number of other squares may be constructed from any given square by performing upon each number of the original square the same series of arithmetical operations. Thus, multiplying each number in Fig. 4 by 2 and also adding 1, we get another magic square (Fig. 5). The subject has engaged the attention of mathematicians for centuries. A good account of the history and the formation of these squares will be found in W. W. R. Ball's *Mathematical Recreations*, and in any good encyclopaedia.

2	7	6
9	5	1
4	3	8

Fig. 4.

5	15	13
19	11	3
9	7	17

Fig. 5.

Multiplication is usually defined as the shortened addition of equal quantities. Thus, if we wish to find the sum of 234 repeated 9 times we should not write down 234 nine times in a column and then *add* but should use our knowledge of 4×9, 3×9, and 2×9, and work by *multiplication.*

There are many cases on record of a child who, set to perform some such addition as $456 + 456 + 456 + 456$, has discovered the short cut without help. Those who are enthusiastic for *heuristic* methods might try the experiment if they can be sure of finding a child who has never heard of multiplication.

The above definition of multiplication should, of course, never be taught as of universal application, for the mathematical teacher will recognize that it hardly covers such cases as $\frac{3}{4} \times \frac{2}{5}$, -2×-3, $\sqrt{5} \times \sqrt{7}$, $\sqrt{-4} \times \sqrt{-5}$, etc. The fact that simple multiplication is merely addition of a special kind is, however, the common-sense basis for beginners, and should be made the groundwork of all teaching of the subject. Proficiency in multiplication depends upon a ready knowledge of the *tables.*

Unfortunately the traditional method of repeating and memorizing the "multiplication table" has obscured the consideration of multiplication as the addition of equals. For long years the drone of "*Twice* 1 are 2, *twice* 2 are 4, *twice* 3 are 6," etc., has been heard in our schools, but now the alternative form of "Two twos are 4, 3 twos are 6, 4 twos are 8, "etc., is gradually being adopted.

The difference between the two will be seen clearly if they are placed side by side—

1. The "Twice-times" Table.	2. The Table of Twos.
Twice one are two	One two is two
Twice two are four	Two twos are four
Twice three are six	Three twos are six

The second method enables every child to construct any required table without assistance, by the process of equal additions. The method of counting in equal groups has already been recommended in the teaching of addition, and forms the very best introduction to tables.

But the mere construction of tables by the method of equal additions or by any other method is only preliminary work, useful and necessary though it be. The much more important and more tiresome work still remains, viz., the tables when constructed must be *memorized*.

Young teachers who forget this all-important work of memorizing the tables will realize their mistake after bitter experience. A teacher should not be afraid of being caught at the drudgery of repeating tables. Yet he must also beware of another grievous mistake, namely, that of causing the class always to repeat in the same mechanical order. Such repetition leads to much waste of time, for a child who has learned tables solely in this way has to mumble through from 7×1 in order to reach 7×9.

But memorizing there must be—and repetition in varying order (cf. counting by 6 *forward* to 72 and *backward* from 72) is as convenient a method as any other. Practice in the products in *any* order must also be given, for the desired and necessary result is that the child shall respond automatically and without the slightest hesitation to the question, $6 \times 7 = ?$ or $4 \times 9 = ?$ etc. Without this, speed and accuracy in ordinary multiplication sums will never be acquired.

The tables need not be mechanically taught in consecutive order from the table of "twos" to the table of "twelves." The thoughtless teacher alone would do this; for the table of "fives" is easier than the table of "fours," and the table of "elevens" is easier than the table of "nines" or "eights," while the table of "tens" is the easiest of all.

It is customary to-day to summarize the multiplication tables in the form of the *number square* (Fig. 6, p. 28).

Now this square contains all the products of the tables as customarily taught from 1 × 1 to 12 × 12. It can, however, be both extended and reduced.

It may be extended as follows. Children may be asked to continue the table, e.g. of sixes beyond 72, and continue 78, 84, 90, 96, 102, etc. They may also construct the table of 13, e.g. 13, 26, 39, 52, etc., or 14, e.g. 14, 28, 42, 56, etc.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

FIG. 6.

A convenient grouping of the tables in order of difficulty has been found to be—

(1) Tables of *tens* and *fives*.
(2) Tables of *twos*, *fours*, *eights*.
(3) Tables of *threes*, *sixes*, *nines*, *twelves*.
(4) Tables of *elevens* and *sevens*.

The table of 14 is useful in converting stones (Avoir.) to pounds (Avoir.). So, too, the table of 16 is useful in converting pounds to ounces. Ability to multiply and divide in *one* process by such numbers as 13, 14, 15, 16, saves valuable time.

But, if multiplication is confined to multipliers of not more than *one* digit, then the above table may be very much reduced. Remembering the fact that $a \times b = b \times a$, e.g. $8 \times 3 = 3 \times 8$, then the only products really required are set out in Fig. 7.

2	2 4							
3	2 6	3 9						
4	2 8	3 12	4 16					
5	2 10	3 15	4 20	5 25				
6	2 12	3 18	4 24	5 30	6 36			
7	2 14	3 21	4 28	5 35	6 42	7 49		
8	2 16	3 24	4 32	5 40	6 48	7 56	8 64	
9	2 18	3 27	4 36	5 45	6 54	7 63	8 72	9 81

FIG. 7.

These thirty-six products are all that need really be memorized, for every multiplication sum can be worked by these and these alone.

Formal Multiplication. It is in multiplication that we first meet the fundamental laws of arithmetic in actual use. These laws are usually termed—

(*a*) The commutative law ;
(*b*) The distributive law.

The Commutative Law. This states that additions and multiplications can be performed in any order, i.e.

$$x + y = y + x, \text{ and } x \times y = y \times x.$$

Now this truth appears so obvious when we are dealing with numbers that proof seems unnecessary, e.g. even the non-mathematical recognize that $3 + 9 = 9 + 3$, or that three nines have the same product as nine threes. Children can demonstrate for themselves with dots or squared paper such a truth as $4 \times 5 = 5 \times 4$.

It is, however, the application of this truth which is of value in actual multiplication. Thus both 987×234 and 234×987 represent the same product, but most people would prefer to use 234 for the multiplier and not 987. This selection of the most convenient multiplier is a good test of intelligent work even in "mechanical" sums. We shall find numerous examples of useful applications of the commutative law in later chapters. One more illustration is given here to convince teachers of its use.

$3\frac{3}{4}$ yds. at 9d. per yd. is most easily worked as
9 yds. at $3\frac{3}{4}$d. per yd., i.e. $3\frac{3}{4}$d. $\times$ 9 (not 9d. $\times$ $3\frac{3}{4}$).

The Distributive Law. In symbols: $a(b + c + d \ . \ . \ .) = ab + ac + ad \ . \ . \ .$

In words: If a number (a) be multiplied by the sum of several other numbers ($b + c + d \ . \ . \ .$) the product is the same as the sum of the partial products obtained separately.

This, we shall see, is the basis of long multiplication, and is the groundwork of many methods in "Practice."

Factor Law. To these two laws is sometimes added a third, the Factor Law, which states that multiplication can be performed either with the whole number or by consecutive multiplications with its factors. Thus, if the multiplier is 24 we may either multiply by 24 in the ordinary way or first by 4 and then by 6, or first by 3 and then by 8, etc.

This again may easily be demonstrated in numbers by the use of either dots or squared paper (Fig. 8).

Here clearly $4 \times 12 = 4 \times 4 \times 3$.

Combining this with the commutative law, we see that

$$247 \times 45 = 247 \times 9 \times 5 \text{ or } 247 \times 5 \times 9$$

i.e. the *order* of the factor multiplications does not affect the answer.

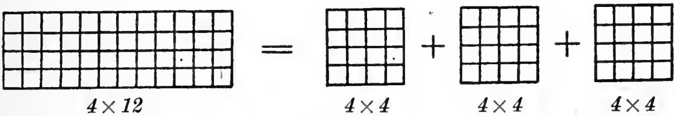


Fig. 8.

Short Multiplication. This is usually set down in the form shown, and seldom gives much trouble. All that is necessary is a ready knowledge of the tables and an ability to multiply and add in the *carrying* figure without writing down the separate answers. If required, oral practice in this may be given, e.g.

$$2345 \times 9 \qquad \begin{array}{r} 2345 \\ 9 \\ \hline 21105 \\ \hline \hline \end{array}$$

$$(6 \times 7) + 5 = ? \quad (5 \times 9) + 6 = ?$$

The process is usually confined to multipliers not greater than 12. But with older scholars it may be extended to 13, 14, 15, 16, and even further. Ability to multiply by such numbers in one line is frequently a means of saving time. Thus, a child might find the number of ounces in a hundredweight by direct multiplication by 16, e.g. $\begin{array}{r} 112 \\ 16 \\ \hline 1792 \end{array}$ working as follows: 16 twos are 32, 16 ones are 16 and 3 are 19, 16 ones are 16 and 1 are 17.

The process depends upon the distributive law, i.e. $2345 \times 9 = (5 \times 9) + (40 \times 9) + (300 \times 9) + (2000 \times 9)$.

If desired, the process may be demonstrated by writing an example out in full as shown.

T.Th.	Th.	H.	T.	U.
	2	3	4	5
				9
			4	5
		3	6	
	2	7		
1	8			
2	1	1	0	5

Factor Multiplication is merely a succession of short multiplications. It is seldom employed with ordinary numbers, though useful in the multiplication of money and compound quantities. With numbers it is frequently the more lengthy method. The reader may compare the following—

(a)		(b)	
	234		234
	42		6
	468		1404
	936		7
	9828		9828

Clearly in (*a*) multiplication by 2 and 4 is easier than multiplication in (*b*) by 6 and 7.

Particular cases of factor multiplication are, however, extremely useful. Such are—

Multiplication by 30 (10 × 3), etc.
" " 300 (100 × 3), etc.
" " 3,000 (1,000 × 3), etc.

Here all that is necessary is to acquire the trick for multiplication by 10, 100, 1,000, etc. Usually these are taught in the rough and ready form of "add a 0" or "add two 0's," etc. It will aid the teaching of decimals very considerably if these rules are made more rational. The idea of *place value* should be used and the child should see that to multiply 23 by 10 is to move each figure one place to the left.

Thus—

H.	T.	U.		H.	T.	U.
	2	3	multiplied by 10 becomes	2	3	

hence the rule for adding 0 in the vacant unit's place.

Multiplication and addition may be combined in one process, thus the example (2345 × 9) + 6789 may be worked directly as follows—

6789	Say :	9 fives are 45 and 9 are 5**4**.
2345 × 9		9 fours are 36 and 5 are 41 and 8 are 4**9**.
		9 threes are 27 and 4 are 31 and 7 are 3**8**.
27894		9 twos are 18 and 3 are 21 and 6 are **27**.

Multiplication and subtraction may be combined in one process, thus 9345 − (1479 × 5) may be worked as follows—

9345	Say :	5 nines are 45 *and* **0** are 45.
1479 × 5		5 sevens are 35 and 4 are 39 *and* **5** are 44.
		5 fours are 20 and 4 are 24 *and* **9** are 33.
1950		5 ones are 5 and 3 are 8 *and* **1** are 9.

This combined multiplication and subtraction is the basis of the *Italian* or *Austrian* method of long division (see page 60).

Long Multiplication. This process again depends upon the distributive law, thus 987 × 654 is the same as 987 × (600 + 50 + 4). The work is usually arranged either as (*a*) or (*b*), according as we begin to multiply by the left-hand or the right-hand figure of the multiplier.

(*a*)		(*b*)
987		987
654		654
5922ØØ		3948
4935Ø		4935Ø
3948		5922ØØ
645498		645498

The crossed 0's are usually omitted in practice.

Both arrangements are in common use. From the point of view of *difficulty* it is immaterial whether the child is taught to "slope" his products to the *right*, i.e. to begin with 6 as the multiplier as in (*a*), or to the *left* as in (*b*). The first arrangement is certainly more useful in the modern method of multiplying decimals.

987
654
42
4835
544028
4532
36
645498

The above convenient arrangements have not always been in use. Teachers who are interested in the history of arithmetic and the evolution of its processes may examine the elegant arrangement here shown of the above example, and try to detect the principles upon which it is constructed. Such arrangements were much in favour two or three centuries ago when the appearance of the work was considered more important than the process itself.

Halving and Doubling. Every multiplication sum may be performed by the following simple process of "halving and doubling."

Example: 73 × 59.

Arrange in two columns (*a*) and (*b*)—

(*a*)	(*b*)
73	59
~~36~~	~~118~~
~~18~~	~~236~~
9	472
~~4~~	~~944~~
~~2~~	~~1888~~
1	3776

Numbers in column (*a*) are obtained by continued "halving," remaining units being rejected.

Numbers in column (*b*) are obtained by continual "doubling."

All numbers in column (*b*) are struck out if corresponding to an *even* number in (*a*), e.g. in the above example, 36, 18, 4, 2 cause the rejection of 118, 236, 944, 1888.

The remaining numbers in column (*b*) when added give the required answer to 73 × 59, i.e. 4307. The process requires only the ability to multiply and divide by 2, and to perform a simple addition. No further knowledge of tables is necessary and the process never fails. Those who are curious will find a mathematical explanation in the following theorem.

Every number is either some power of 2 or can be formed by the addition of powers of 2, i.e. every number is made up of one or more terms of the following series—

$$2^1,\ 2^0,\ 2^2,\ 2^3,\ 2^4,\ 2^5, \text{ etc.}$$

Thus 73 (in above example) $= 2^0 + 2^3 + 2^6$
$= 1 + 8 + 64$

Now the numbers in column (*b*) are seen to consist of the following in order—

$$59 \times 1,\ 59 \times 2,\ 59 \times 2^2,\ 59 \times 2^3,\ 59 \times 2^4, \text{ etc., e.g.}$$

$$\begin{aligned} & 59 + 472 + 3776 \\ = {} & 59\ (1 + 2^3 + 2^6) \\ = {} & 59\ (1 + 8 + 64) \\ = {} & 59 \times 73 \end{aligned}$$

The process appears at intervals in the periodical press, each time being greeted with interest by those to whom it is new. When last seen by the author it was proclaimed as "Bolshevik" multiplication, and was stated to be in common use among the Russian peasants.

Miscellaneous Methods. Multiplication is not to be regarded merely as a mechanical process, for it constantly gives scope for the use of intelligence and is a veritable paradise of "short cuts" for all scholars and students who do not allow it to degenerate into a mechanical routine.

A few of the more usual devices are grouped here for the convenience of teachers. It should not be necessary to state that these should *not* be made the subject of any *formal* lessons or taught as stereotyped "short cuts," but that children should be encouraged to invent these and similar methods for themselves.

(1) Cases where multiplication is simplified by *subtraction*.

These depend upon such facts as $9 = 10 - 1$, $99 = 100 - 1$, $990 = 1000 - 10$, etc. The method can be understood from the following examples.

(*a*) Multiply 5432 by 99.

```
543200 = 5432 × 100
  5432 = 5432 × 1
------
537768 = 5432 × 99
======
```

(*b*) Multiply 9876 by 990.

```
9876000 = 9876 × 1000
  98760 = 9876 × 10
-------
9777240 = 9876 × 990
=======
```

(*c*) Multiply 2468 by 195.

```
  2468
------
493600 = 2468 × 200
 12340 = 2468 × 5
------
481260 = 2468 × 195
======
```

(*d*) Multiply 3579 by 97.

```
357900 = 3579 × 100
 10737 = 3579 × 3
------
347163
======
```

(2) Cases in which the labour of multiplication and the number of partial products may be lessened by an examination of the digits of the multiplier.

These all depend upon some such relation as that noticeable in the digits of 357. Here it may be noted that $35 = 5 \times 7$. Hence the sum may be worked by means of two partial products instead of three, as shown in (*a*) below.

(*a*) Multiply 9753 by 357.

```
   9753
    357
   ----
  68271 = 9753 × 7
 341355 = 68271 × 50 = 9753 × 350
-------
3481821
=======
```

(*b*) Multiply 1357 by 3612.

```
   1357
   3612
   ----
  16284 = 1357 × 12
 48852  = 16284 × 300 = 1357 × 3600
-------
4901484
=======
```

(*c*) Multiply 147036 by 56147.

$$\begin{array}{rl} 147036 & \\ 56147 & \\ \hline 1029252 & = 147036 \times 7 \\ 2058504 & = 1029252 \times 2 = 147036 \times 140 \\ 8234016 & = 2058504 \times 400 = 147036 \times 56000 \\ \hline 8255630292 & \\ \hline\hline \end{array}$$

The method illustrated by the above examples decreases considerably the number of figures necessary to the working. Children should be taught to employ it in the simplest cases. Thus they should recognize that multiplying by 84 requires multiplication by 4 and then by 2 only. Similarly, multiplication by 93 can be worked by two multiplications by 3. Such methods also form convenient alternative forms for checking ordinary working.

(3) Cases where multiplication is avoided by simple division.

The process is employed with such multipliers as 5, 25, 125, $3\frac{1}{3}$, $33\frac{1}{3}$, $333\frac{1}{3}$, etc.

The fact that $5 = \frac{10}{2}$, $25 = \frac{100}{4}$, $3\frac{1}{3} = \frac{10}{3}$, etc., explains the method.

Examples.

(*a*) Multiply 987 by 25.

$$\begin{array}{rl} 98700 & = 987 \times 100 \\ 24675 & = 98700 \div 4 \\ & = 987 \times 25 \end{array}$$

(*b*) Multiply 468 by 125.

$$\begin{array}{rl} 468000 & = 468 \times 1000 \\ 58500 & = 468000 \div 8 \\ & = 468 \times 125 \end{array}$$

(*c*) Multiply 7534 by $2\frac{1}{2}$.

$$\begin{array}{rl} 75340 & = 7534 \times 10 \\ 18835 & = 75340 \div 4 \\ & = 7534 \times 2\frac{1}{2} \end{array}$$

(*d*) Multiply 5814 by $33\frac{1}{3}$.

$$\begin{array}{rl} 581400 & = 5814 \times 100 \\ 193800 & = 581400 \div 3 \\ & = 5814 \times 33\frac{1}{3} \end{array}$$

(4) Cases where multiplication is assisted by the use of simple algebraic identities.

(*a*) The formation of "squares."

The identity in use here is

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \text{ or its simpler form—}$$
$$(a \pm 1)^2 = a^2 \pm 2a + 1.$$

Examples.

$$\begin{aligned}(26)^2 &= (20)^2 + 240 + (6)^2\\ &= 400 + 240 + 36\\ &= 676\\ (99)^2 &= (100)^2 - 200 + 1\\ &= 10000 - 200 + 1\\ &= 9801\end{aligned}$$

This method is especially useful for squaring any number ending in 5.

$$\begin{aligned}\text{Thus } (65)^2 &= (60)^2 + 600 + 25\\ &= (4200) + 25\\ &= (6 \times 7)\ 100 + 25\\ \text{Similarly } (85)^2 &= (8 \times 9)\ 100 + 25 = 7225\\ (115)^2 &= (11 \times 12)\ 100 + 25 = 13225\end{aligned}$$

(*b*) The use of $a^2 - b^2 = (a + b)\ (a - b)$ in evaluating the difference between two squares, e.g.

$$\begin{aligned}&= (103)^2 - (87)^2\\ &= (103 + 87)\ (103 - 87)\\ &= 190 \times 16\\ &= 3040\end{aligned}$$

The amount of labour saved is obvious.

In later work this simple algebraic identity has numerous practical applications. One only may be mentioned: its use in finding the area of the cross section of a circular pipe (Fig. 9).

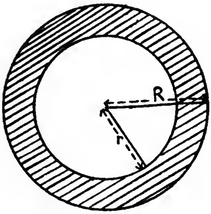


Fig. 9.

$$\begin{aligned}\text{Shaded area} &= \pi R^2 - \pi r^2\\ &= \pi(R + r)\ (R - r).\end{aligned}$$

which reduces the multiplication to a single process and eliminates the original subtraction.

(*c*) Applications of the identity

$$(a + b)(c + d) = ac + ad + bc + bd$$

may also be frequently made, thus—

$$\text{(i) } 103 \times 104 = (100 + 3)(100 + 4)$$
$$= 10000 + 700 + 12$$
$$= 10712$$
$$\text{(ii) } 12\tfrac{1}{2} \times 10\tfrac{1}{2} = (12 + \tfrac{1}{2})(10 + \tfrac{1}{2})$$
$$= 120 + 6 + 5 + \tfrac{1}{4}$$
$$= 131\tfrac{1}{4}$$

These are but a few of the methods by which the mechanical labour of multiplication may frequently be lightened. The wise teacher will encourage the use of all such *short* methods wherever possible, and will insist upon alternative methods being used to check each answer.

CHAPTER III

THE SIMPLE RULES—SUBTRACTION AND DIVISION

SUBTRACTION is a vexatious subject for most teachers. Much controversial and heated arguments have been expended by teachers in proclaiming the virtues of the various methods of subtracting. Yet the subject is as difficult as ever for both teachers and pupils, and no agreement on the best method for children seems likely to be reached. Circular 807 ("Suggestions for the Teaching of Arithmetic") briefly states that "the methods of teaching subtraction are various, but it is safe to say that no method will prove satisfactory which does not readily permit of practical illustration." The battle rages chiefly upon the comparative merits of "Decomposition" and "Equal Additions." Infant teachers in general prefer the former as being more readily illustrated, while teachers of older scholars mostly prefer the method of "equal additions," claiming that it is at once reasonable and rapid. Perhaps much of the confusion and hostility which arise would be avoided if formal subtraction were left entirely to the upper school, and if infant teachers confined themselves to informal subtraction of smaller numbers and quantities, for here again most of the difficulties arise from the attempt to teach rigid and formal methods to children before their mental capabilities are sufficiently developed.

Let us examine some of these methods of Formal Subtraction.

Consider the Example (*a*).

(*a*)	2345
	1987
	358

In many schools the ancient dirge may still be heard as follows—

"Seven from five we can't; borrow ten; seven from fifteen, eight. Pay back one.

Nine from four we can't; borrow ten. Nine from fourteen, five. Pay back one; one and nine are ten. Ten from three we can't; borrow ten; ten from thirteen, three. Pay back one. Two from two, nothing."

It is safe to say that this lengthy rigmarole includes nearly every bad point in any teaching of subtraction. We will work the same example by Decomposition and by Equal Additions.

Decomposition. In this method the bigger number is broken up into a more convenient form.

	Th.	H.	T.	U.		Th.	H.	T.	U.
Instead of	2	3	4	5	we use	1	12	13	15
	1	9	8	7		1	9	8	7

What we actually say (or think) as we work on this plan is something like the following: Seven from fifteen, eight; eight from thirteen, five, etc.

It will be noted that there is "borrowing" in that help for the unit and tens, etc., is obtained from the tens and hundreds, etc., but no "paying back." Indeed, the absurdity of "borrowing" from *one* quantity and "paying back" to *another* quantity and still hoping to get the correct answer should be patent to every teacher.

The method, using simple numbers, is one favoured by teachers of younger children, and it certainly lends itself easily to concrete demonstration.

Where zeros occur as in subtracting 27 from 100, the method is more difficult for children, but any trouble may be avoided by first subtracting 1 from each, i.e. 100 − 27 may be easily worked as 99 − 26.

It is clear that once the decomposition is performed it is immaterial whether we begin to work the sum at the right hand or at the left, i.e. in the example first worked we may begin by subtracting the *thousands* first instead of the units. Some teachers would prefer to base their method on this procedure and would work somewhat as follows—

2345
1987
———

Keeping careful watch on the requirements of the column to the right hand, we proceed, beginning at the left hand:

(*a*) 1 from 2, 1, but we need this 1 for the next column, therefore we put nothing in the answer.

(*b*) 9 from 13, 4, but we need 1 for the next column, therefore we put down 3.

(*c*) 8 from 14, 6, but we need 1 for the next column, therefore we put down 5.

(*d*) 7 from 15, 8.

Equal Additions. The method depends upon the simple axiom that if equals be added to unequals, the original difference remains constant, e.g. 9 − 5 = 19 − 15 = 119 − 115, etc.

The ages of two children provide an instance, for as long as both are alive the difference remains constant whatever their individual ages.

In working the sum (*a*) above:

Instead of using	2345 1987 ———	we arrange figures thus:			H.	T.	U.
				2	13	14	15
				2	10	9	7

To form the second example we have clearly added 1,110 to each line of the original. What we *say* (or think) as we work on this plan is as follows: 7 from 15, 8; 1 and 8 are 9, 9 from 14, 5; 1 and 9 are 10, 10 from 13, 3; 2 from 2, 0. Here there is neither "borrowing" nor "paying back."

This method is very popular with teachers of older scholars, though it is not quite so easily understood by younger scholars. But either "Decomposition" or "Equal Additions" may become firmly fixed as habitual methods in the average child by the help of any teacher with the necessary patience and persistence.

Complementary Addition. The process of subtraction may be examined from two points of view. We may find the difference between two numbers either by *taking away* one from the other or by *adding on* to the smaller until we make up the larger. Thus 9 − 6.

9 ~~111111~~111 6 ~~111111~~	} Difference 3 by taking away.
9 111111111 6 111111⋮⋮⋮	} Difference 3 by making up.

It is this latter aspect of adding on the complement or Complementary Addition which is (or should be) the normal process of working a subtraction sum. Returning to our original example, we work as follows—

7 *and* **8** are 15 (not 7 *from* 15 *leave* 8). Put down 8 and carry 1. 1 and 8 are 9, 9 *and* **5** are 14; put down 5 and carry 1. 1 and 9 are 10, 10 *and* **3** are 13; put down 3 and carry 1. 1 *and* 1 are 2, 2 and 0 are 2.

```
2345
1987
————
 358
————
```

Here there is "carrying" as in addition for the process *is* addition.

This is the straightforward natural process of working a subtraction sum, and should be universal. It is usually known as the method of *Complementary Addition*, and is termed, by some, *Interrogative Addition.*

The word "subtraction" need not be mentioned at all, for the whole process may be taught as addition in a new form. Thus, the above example may be arranged as follows:

```
1987
****
————
2345
————
```

in the form of an addition sum where the answer is given and the missing line has to be found.

Proceeding Through the Ten. Some teachers, especially those who teach routine methods either of Decomposition or Equal Additions, insist that the child shall in every case proceed through the *ten*. Thus, they would work example (*a*) not as 7 from 15, 8, but in *two* steps: (i) 7 from 10, 3; (ii) 3 and 5, 8. And similarly in every other case where a *ten* is added or borrowed.

(*a*) 2345
987
——

This step, however, is an additional prop which though helpful to the weaker children in a class should not be necessary to the majority. After all, the possible differences are distinctly limited in number, for in any subtraction sum we can only range from 1 – 0 to 18 – 9. The following table contains all subtractions necessary in a formal sum—

(*a*) 18–9	(*j*) 9–9, 8, 7, 6, 5, 4, 3, 2, 1, 0
(*b*) 17–8, 9	(*k*) 8–8, 7, 6, 5, 4, 3, 2, 1, 0
(*c*) 16–7, 8, 9	(*l*) 7–7, 6, 5, 4, 3, 2, 1, 0
(*d*) 15–6, 7, 8, 9	(*m*) 6–6, 5, 4, 3, 2, 1, 0
(*e*) 14–5, 6, 7, 8, 9	(*n*) 5–5, 4, 3, 2, 1, 0
(*f*) 13–4, 5, 6, 7, 8, 9	(*o*) 4–4, 3, 2, 1, 0
(*g*) 12–3, 4, 5, 6, 7, 8, 9	(*p*) 3–3, 2, 1, 0
(*h*) 11–2, 3, 4, 5, 6, 7, 8, 9	(*q*) 2–2, 1, 0
(*i*) 10–1, 2, 3, 4, 5, 6, 7, 8, 9	(*r*) 1–1, 0

If these differences receive frequent oral practice then such a response as "7 from 15, 8" will come speedily and automatically, and all the additional labour of "proceeding through the *ten*" may be dispensed with.

The above table of differences may be arranged in a square formation (Fig. 10, p. 45). The square should be constructed and used for both addition and subtraction. The *addition* of a number at the head of a column and another at the beginning of a row is found at the intersection of row and column as with the three numbers shaded. Conversely it may be used for all subtractions up to 18 – 9.

Preliminary Oral Work. It is safe to say that the more preliminary *oral* practice in subtraction that a class receives, the sooner it acquires any particular "method" of working a written example.

This preliminary work may take many forms. The essential object is, of course, that all the usual results required, as set out in the above table, must by practice become automatically ready when needed. It is necessary to explain and demonstrate to a child that such a difference as 14 – 8 *is* 6, if he cannot readily give it, but the final aim must always be to register this and all similar facts in a child's memory. Concrete demonstration and explanation will help, but *practice* alone will fix it.

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

FIG. 10.

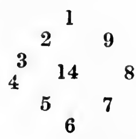
Other forms of Preliminary Oral Work are—

(*a*) *Counting backwards by equal groups*, e.g.

60, 54, 48, 42, 36, etc., decreasing by 6.
50, 47, 44, 41, 38, etc., decreasing by 3.

(*b*) *The Ring of Figures.*

Any convenient number may be placed in the centre, e.g. 14. The teacher points to any number and the class gives the necessary complement to make 14. Thus the teacher points to 8 and the scholars should be ready with the answer 6.



(*c*) *Continued Subtraction* from a larger number.

For this purpose the ring of figures may be used or the numbers dictated. The class starts with a number, e.g. 50. The teacher points to (or dictates) a series of numbers, such as 7, 4, 2, 5, etc., the class follow mentally with 43, 39, 37, 32, etc., for as many steps as required.

(*d*) *Oral Practice in Complements.*

This is the most necessary and useful of all oral work in subtraction.

The one constant idea is the *making up*, e.g.

(i) What will make each of the following up to 10 ?
1, 2, 3, 4, 5, 6, etc.

(ii) What will make each of the following up to 20 ?
15, 13, 11, 9, 7, etc.

(iii) What will make each of the following up to 100 ?
70, 50, 30, 95, 85, 75, etc.

These examples may be extended as much as required, but the form of the question should never vary. It should never take the form "From 100 take 55."

Additional Suggestions for the Teaching of Subtraction. (*a*) Formal methods should not be introduced too early in the child's life. Considerable proficiency in *oral* work should be acquired before any routine method is taught.

(*b*) It is essential that all teachers in the same school should agree to teach the same method. It is grossly unfair to the children to be taught one method in one class and another method in the next.

(*c*) Such terms as Subtrahend and Minuend no doubt have their place in mathematics, but it is absurd pedantry to bother little children with such names.

(*d*) The habit must always be formed of *checking* any subtraction sum by the corresponding addition.

(*e*) Practise subtraction in any direction, e.g. work 625 – 347 without placing 347 *underneath* 625. Practise also subtraction of the top number in a column from the bottom one, e.g.

```
1456
 ***
----
2325
====
```

(*f*) Where a number of zeros occur the work is lessened by first subtracting 1 from each quantity, thus—

1000 456	is much easier for a child when worked as	999 455

(*g*) *Mental Subtraction of Larger Numbers.* This useful process may easily be taught to older scholars. Consider the example 543 – 387. The mental steps are subtractions of 300, 80, and 7. The corresponding stages of the answer are 243, 163, 156. Similarly the difference between 1256 and 784 may be reached mentally by the following stages: 556, 476, 472, where the subtractions in order have been 700, 80, and 4.

Addition and Subtraction in One Operation. This important process deserves much more attention in schools than it has usually received.

Consider the following problem—

In a school of four rooms the total number on the books is 220. The numbers in each room on a certain day are 79, 35, 47, and 41. How many are absent?

This may be worked in *one* process as follows.—Arrange the sum as set out. Work thus:

1, 8, 13, 22, *and* **8** are 30; carry 3.

3, 7, 11, 14, 21, *and* **1**. *Ans.* 18.

```
220
---
 79
 35
 47
 41
---
 18
===
```

Worked on this method the problem involves *one* process only, not *two* (first an *addition* sum and then a *subtraction* sum).

The work might also be arranged as an addition sum with the last line missing, as indicated. The

79
35
47
41
**
220

method would then be (working downwards from the top)—

9, 14, 21, 22 *and* **8** are 30 ; carry 3.

3, 10, 13, 17, 21, *and* **1** are 22. *Ans.* 18.

This method is continually employed by accountants in finding balances.

The same addition sum will provide several examples.

Thus from	123	we have	***	123	123	123	
	45		45	**	45	45	
	678		678	678	***	678	
	91		91	91	91	**	
	937		937	937	937	937	

Each of the four missing lines may be obtained by addition in one operation as above.

Division. This process is usually placed last of the Four Rules in the ordinary text-book. This is undoubtedly its logical position, for a good knowledge of addition, subtraction, and multiplication is necessary in order to understand the underlying theory.

In the first place, division may be regarded as shortened subtraction, just as multiplication has been considered as shortened addition. Thus the question involved in $51 \div 3 = ?$ may be worded *How many times can* 3 *be subtracted from* 51 ?

Division might equally well be defined as shortened addition, for the question above may be worded *How many* 3*'s must I add together to make* 51 ?

Again Division might be viewed as *interrogative* multiplication and the same question worded *By what must I multiply* 3 *so as to obtain the answer* 51 ?

The main aspects of the division process are given by :

(*a*) How many groups of 3 are contained in 51 ? (Quotition.)

(*b*) Divide 51 into 3 equal parts. (Partition.)

These various aspects of division show clearly the complementary nature of the four main processes of calculation.

The group idea (as in *How many times is* 3 *contained in* 51 ?) is sometimes called the *measuring* aspect, while the other main view (as in *What is one-third of* 51 ?) is usually called the *sharing* aspect. These two words, "measuring" and "sharing," are sufficient to name these two fundamental aspects of Division for children, though pedants still exist in every kind of school who would prefer the more pretentious terms of "Quotition" and "Partition."

These different ways of regarding Division are very important for they control the character of the answer, and especially the nature and interpretation of the remainder. This point will be closely examined in connection with compound division.

Yet, after all, they represent only *view-points* and *purposes* of division, and *do not affect the process itself.* This fact cannot be emphasized too much. The *process* remains the same though its meaning may vary, and it is the *process* which must be taught.

Let us accordingly turn at once to the teaching of division.

So long as divisors do not exceed the limits of the multiplication table it is customary to work by the process called *short* division. Success here depends, as in multiplication, upon a ready knowledge of tables. Hence constant oral practice on the division aspect of the tables is necessary, e.g.

(*a*) How many 3's in 12 ?
How many 4's in 12 ?
How many 6's in 12 ?
How many 2's in 12 ?

(*b*) What is $\frac{1}{2}$ of 12 ?
What is $\frac{1}{3}$ of 12 ?
What is $\frac{1}{4}$ of 12 ?
What is $\frac{1}{6}$ of 12 ?

Work of this kind is necessary in all table practice.

Short Division. The actual process may be demonstrated by working a "short" division by the long method, thus—

Short 7)4321
617, rem. 2

Long

```
   617
7)4321
  42
  --
   12
    7
   --
    51
    49
    --
     2
```

The long method shows the underlying theory of the Division process.

We begin with the largest quantity possible (in this case *hundreds*), and the answer is 6 *hundred* and 1 *hundred* remainder. This remainder is converted to *tens*, and with the 2 tens provides **12** for further division. Thus the sum proceeds, the remainder at each stage being converted to a lower denomination so that division may continue.

But where tables are well known, division by simple numbers seldom gives much trouble, especially if checking by multiplication is habitually employed.

The writing down of the *remainder* in a division sum is frequently confused and often incorrect.

Consider a very simple example: $12 \div 5$

If simple numbers are dealt with, the natural and obvious answer to a child is 2 and 2 over. Now some teachers insist that this shall be written $2 + 2$. Similarly $100 \div 12$ would be written as $8 + 4$. This form of written remainder is not only incorrect but is mathematically absurd.

Returning to our simple example, let us re-word it in a concrete form. Let us state the question—

(*a*) "How many pieces of tape 5 inches long can be cut from one measuring 12 inches, and what length is left?"

The answer here is 2 *pieces* and 2 *inches* left. But the question might be worded thus—

(*b*) Divide 12 inches of tape into 5 equal parts.

Here each part is $2\frac{2}{5}$ inches long and there is obviously no remainder.

The answer "2 and $\frac{2}{5}$ times over" is as absurd as 2 + 2. Thus we see that the form of our remainder depends entirely upon the nature of the problem and the purpose of the division. Hence, teachers who insist upon the remainder being given *always* in fractional form are not of necessity any more correct than those who adhere to the common and faulty "Q + R" method.

Division by Factors. This process follows naturally from "short division," and the actual working lends itself readily to concrete and graphical demonstration, though it is to be feared that far too many teachers are content to rely upon diffuse verbal explanations.

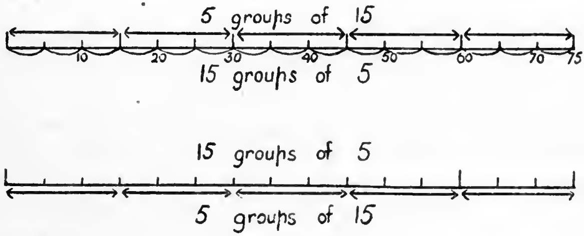


FIG. 11.

Consider a simple example without remainder, viz., 75 ÷ 15.

In concrete form we may word this in two forms: (*a*) Divide 75 pencils into 15 equal groups, or (*b*) divide 75 pencils into groups of 15. Children working (*a*) will see the advantage of dividing the 75 pencils first into 5 equal groups and then subdividing each group into 3 smaller equal groups. Children working (*b*) may work by first dividing the pencils into groups of 5 and then grouping these sets of 5 into larger groups of 15. Among the explanations most popular with teachers the "group" explanation (*b*) is most commonly found.

The process can be illustrated on a squared blackboard or squared paper as in Fig. 11.

This graphic work may be done by the class and saves much oral description. It is especially useful in indicating the nature and in calculating the remainder after division by factors.

Finding the Remainder in Division by Factors. This has always been recognized as the greatest difficulty in the teaching of division by factors.

It is customary to explain the nature of the remainder on the basis of successive division into groups. Thus the example—

$$2519 \div 35$$

is explained as follows—

```
7)2519 units
  ----
 5)359 groups of 7 + 6 units
   ---
    71 groups of 35 + 4 groups of 7 + 6 units
```

Ans. 71, *rem.* 34.

To an adult mind the explanation of this process seems simple enough when set out as above, but many teachers will remember the confidence with which they set out to teach this calculation to a class in thirty-five or forty minutes, and the humility with which they left the class at the end of the period. The process is *not* an easy one to teach or to learn, and much practice is necessary before children acquire any facility in the calculation.

With small numbers the process is capable of concrete demonstration with counters and small objects. Too often, however, the limitations of large classes compel the teacher to depend upon demonstration by words and symbols. Even so, much may be accomplished by a patient teacher. But, as in every other process, speed and accuracy will only be attained by practice.

The remainder after division by factors may be ascertained by graphical methods. Paper ruled in inches and tenths may be employed for class work.

Example: $41 \div 14$ (Fig. 12.)

The 41 units are first divided into groups of 7 and 6 squares remain. The groups of 7 are next re-arranged as groups of 14. Both quotient and remainder are seen clearly when set out thus.

This method is suitable either for black-board demonstration or for individual work, and gives a very clear conception of the process.

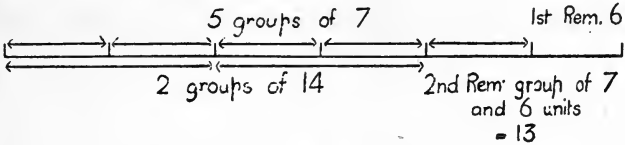


FIG. 12.

An Alternative View of Factor Division. The above explanation of the remainder after division by factors is not the only possible one. It is based, as we have seen,

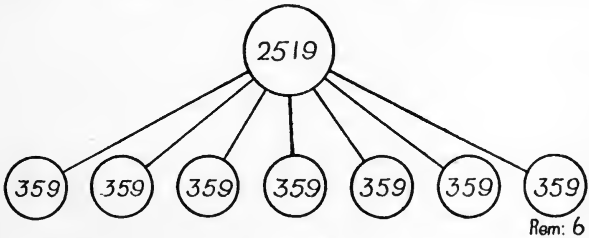


FIG. 13.

upon the idea of "groups." The process is also capable of very clear exposition upon the sharing or partitive basis.

Using our original example, 2519÷35, we may illustrate the first division by diagram (Fig. 13). Thus division by 7 gives us 7 sets of 359 with 6 over (*not* 359 groups of 7).

7)2519
5)359, rem. 6 ⎫
71, rem. 4 ⎭ 34

The second division, i.e. by 5, consists of finding a fifth part of one of these groups of 359 (Fig. 14).

We have then that $\frac{1}{35}$ (or $\frac{1}{5}$ of $\frac{1}{7}$) of 2519 is 71. Clearly if *each* group of 359 is divided up thus there will be 4 units over in each and the total remainder is (7 groups of 4) + 6 = 34 as before.

This explanation of division by factors is the only one possible when the process is applied to compound quantities. Let us take as an example—

Divide £6 4s. 9d. by 15 correct to the nearest penny.

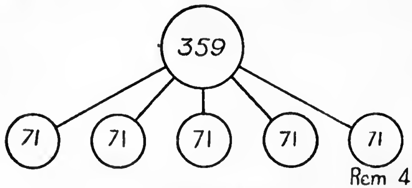


FIG. 14.

Using factors 5 and 3, the sum appears as follows—

	£	s.	d.		
5)	6	4	11		
3)	1	4	11, rem. 4d.	Total rem. 1s. 2d.	
		8	3, rem. 2d.		

Here the remainder is (2d. × 5) + 4d. = 1s. 2d.

The "group" explanation is clearly out of the question, for we cannot call £1 4s. 11d. "groups of five."

The diagram (Fig. 15), however, will make the process clear.

Similarly there will be a remainder of 2d. after *each* £1 4s. 11d. is divided into 3 equal parts. Hence the total remainder is (2d. × 5) + 4d. = 1s. 2d.

Division by factors is constantly employed in dealing with English weights and measures.

Operations in which it may be employed are—

Changing ounces to pounds, division by 16; pounds to quarters, division by 28; yards to chains, division by 22; hours to days, division by 24. All these provide useful practice in division by factors.

Other calculations in which division by factors may be used advantageously are—

Changing sq. in. to sq. ft., division by 144; cu. in. to cu. ft., division by 1728; cu. ft. to cu. yds., division by 27; sq. yds. to sq. poles, division by $30\frac{1}{4}$.

Happily this last calculation is not so frequently taught in schools now, but many older teachers will recall the clumsy process involving (*a*) multiplication of divisor and dividend by 4; (*b*) division by 121; (*c*) calculation

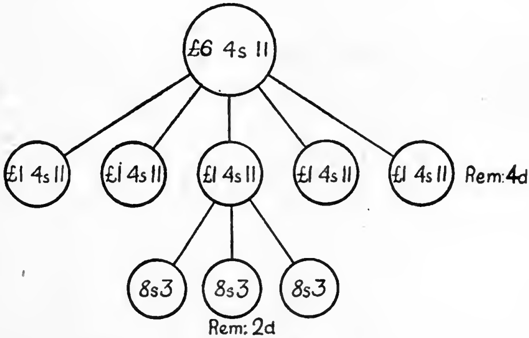


Fig. 15.

of remainder in (*b*); (*d*) division of (*c*) by 4 to get the true remainder.

It is more usual now to proceed direct from square yards to square chains or acres.

More Difficult Division by Factors. So far we have dealt with division by factors where the divisor is split into *two* factors only. The process may be extended to more than *two* factors. It is questionable, however, if this extension is necessary for the average child. The whole process of factor division has, perhaps, received too much attention from teachers.

Possibly the difficulty of teaching the subject has caused it to assume an unmerited importance.

If thought necessary, however, harder examples may be set to older scholars. Thus in the example: *Change* 20,000 *yds. to miles*, the work could be done by the factors of 1,760, i.e. 10 × 11 × 16, and the sum would appear when written out fully as follows—

$$\left.\begin{array}{r} 110\left\{\begin{array}{l} 10\underline{)20000} \\ 11\underline{)2000}(\;+\;9 \end{array}\right\}\;90 \\ 16\underline{)181}(\;+\;5 \\ \underline{\underline{11}} \end{array}\right\}\;640$$

Ans., 11 *miles* 640 *yds.*

Similarly in the example: *Change* 2,000 *lbs. to cwts.*, the factors of 112, i.e. 7 × 4 × 4, might be used in any order—

$$\left.\begin{array}{r} 28\left\{\begin{array}{l} 7\underline{)2000}(5 \\ 4\underline{)285}(1\;+\;5 \end{array}\right\}\;12 \\ 4\underline{)71}(3 \\ \underline{\underline{17}} \end{array}\right\}\;96$$

Ans., 17 *cwt.* 96 *lbs.*

The teacher who is interested in the process and perhaps over-confident of his ability to teach it, may test his powers of calculating the remainder in some such example as the following—

$$\begin{array}{r} 2\underline{)148199}(1 \\ 3\underline{)74099}(2 \\ 4\underline{)24699}(3 \\ 5\underline{)6174}(4 \\ \underline{\underline{1234}} \end{array}$$

There remains one form of factor division which is frequently not recognized in its contracted form and so is taught as a "trick." This form includes all divisors such as 20, 30, 40, etc., 200, 300, 400, etc.

Thus, in the example, *Change* 1,275 *sq. poles to roods,* we work the calculation 1275 ÷ 40 as follows—

```
In Full—                              In Contracted Form—
10)1275(5  }                           4\0)1\275
   ----    }  35 sq. poles                 -----
 4)127(3   }                                31  roods 35 sq. poles
   ---
   31 roods
```

This shortened form for division by such numbers as 20, 30, 40, etc., is most useful, and must receive constant practice.

Older children may well be set interesting puzzles of the "missing figure" type. Thus given the divisors, quotient, and remainders they may be required to calculate the original dividend as in the example—

```
3) . . . . . .
   -----------
  5) . . . . .   rem. 2
     ---------
    7) . . . .   rem. 4
       -------
       4 5 6     rem. 6
```

Long Division. This is the last of the fundamental processes to be taught to children, and it is certainly the most difficult to teach as well as the most tedious to perform. Perhaps its importance is over-rated in the elementary school, for frequently in after-life it is avoided whenever possible. In commercial life we rely for our long division answers upon tables and ready reckoners, while in technology, our slide-rules will always give us an answer to a reasonable degree of approximation.

Head teachers and others who control the Arithmetical teaching of Elementary Schools are gradually realizing that most of the drudgery necessary to master long division arises from the attempt to teach it *too soon.* Not so very long ago it was a grim and compulsory feature of the work of every Standard III. This evil tradition still survives, but more and more teachers are postponing the teaching of long division until the child reaches Standard IV or even Standard V. Those who do so are all delighted with the result, for the process can then be taught in half the time and with a quarter of the labour on the part of teacher and child.

The Theory underlying the Process. This need not detain us long. We have already seen on page 50 that every division sum can be worked on the long method.

All that is necessary is to start with the largest possible quantity in the dividend, and to convert the remainder at each stage into a lower power of ten so as to allow further division.

Consider the example—

$$34567 \div 89$$

```
      388
   ______
89)34567
   267
   ---
    786
    712
    ---
     747
     712
     ---
      35
```

First Stage. Dividend 345 *hundreds*, quotient 3 *hundreds*, remainder 78 *hundreds*.

Second Stage. Dividend 786 *tens*, quotient 8 *tens*, remainder 74 *tens*.

Third Stage. Dividend 747 *units*, quotient 8 *units*, remainder 35 *units*.

It is, however, not in the *theory* of the process but in its *practice* that the difficulties occur. Readers who have taught this process, or who can recall their own struggles as children, will regard with sympathy the difficulties of the learner. These difficulties may be grouped as follows—

(*a*) The difficulty of determining the actual value of the quotient figure, i.e. its *intrinsic* or digital value.

(*b*) The difficulty of the *place-values* of the quotient figure, especially when a " won't go " occurs and a zero has to be put in the quotient. Most teachers know from sad experience how many of their class are likely to give an answer as 29 which should be 209 or 290.

These difficulties are usually dealt with in easy stages. The examples set should employ graduated divisors and should begin with a single quotient figure.

Divisors of two figures are sufficient at first, and in the earlier stages should consist of numbers approximately equal to an exact number of tens, e.g.

(*a*) 21, 31, 41, etc. ; 22, 32, 42, etc.
(*b*) 19, 29, 39, etc. ; 18, 28, 38, etc.

The divisors should be extended gradually to include any case between the limits of exact tens.

Many successful teachers insist on all the work being based on the limits of exact tens between which the divisor lies. The following example illustrates the method—

$$234 \div 46.$$

The divisor lies between 40 and 50.

The quotient is either $23 \div 4$, i.e. **5**, or $23 \div 5$, i.e. **4.** *It could not be* 3 *or* 6 or any other figure but 4 or 5, and a single mental trial determines the exact figure.

Some such method must be employed to limit the child's choice of quotient-figure, for nothing is more disturbing to the teacher to find a child diligently trying every quotient figure from 9 downwards when 4 and 5 may be the only possible figures.

(*c*) The difficulty of the place-value of the quotient figures and of zeros in the answer is minimized if the practice is taught of writing the answer *over* the dividend instead of at the right-hand side, as in the example worked above—

$$\begin{array}{r} 388 \\ 89\overline{)34567} \end{array}$$

The *first* quotient figure found should be placed correctly. If the habit is well-formed from the start even zeros will cause no trouble.

Italian or Austrian Method of Long Division. This method, which is commonly known in England as the Italian method, though sometimes called in America the Austrian method, is best explained by means of an example set out fully—

Ordinary Method—	Italian Method—
269 87)23456 174 605 522 836 783 53	269 87)23456 605 836 53

The method consists of combined mental multiplication and complementary addition. Remainders *only* are written down at each stage.

Work as follows—

(*a*)	Twice 7, 14 and **0,** 14, carry 1. ↑ ↓ Twice 8, 16 and 1, 17 and **6,** 23.	First remainder 60.
(*b*)	Six 7's, 42 and **3,** 45, carry 4. ↑ ↓ Six 8's, 48 and 4, 52 and **8,** 60.	Second remainder, 83.
(*c*)	Nine 7's, 63 and **3,** 66, carry 6. ↑ ↓ Nine 8's, 72 and 6, 78, and **5,** 83.	Third remainder, 53.

One word of warning is necessary here. The method appears short and simple, and indeed to an adult, with very little practice, it is easy and rapid ; yet to attempt to teach the method too soon to children is to court disaster.

Again, the method requires more concentration and is consequently more liable to error. Errors in the ordinary method of long division are easy to discover, for every figure and product is shown, but in the Italian method mistakes are very difficult to detect, and if any occur it is necessary to begin the example anew.

Further, it is sometimes actually the longer method, for with such a quotient as 7878 all four multiplications would have to be performed mentally in the *Italian* method, whereas in the ordinary form the products corresponding to the second 7 and the second 8 would be copied from previous lines.

A Method Alternative to the Italian. The following method, given by Mr. Workman,[1] is really just as short as the Italian, and does not suffer from the same defects.

Consider the example—

$$100{,}000{,}000 \div 78.$$

The first step is to construct a table of multiples of 78 from 1 to 9 times thus—

78, 156, 234, 312, 390, 468, 546, 624, 702.

These multipliers are written on a strip of paper and the strip is placed at the proper position under the dividend, and the remainder written down. Thus, 702 would be placed beneath 1000 and 298 written down.

The method is especially valuable in long sums with a large quotient. It can be applied readily to division of decimals, and to the conversion of fractions to decimals. Thus the above example might be used to convert $\frac{1}{78}$ to a decimal to any required number of places.

A generation ago, when teachers were compelled to

[1] *The Tutorial Arithmetic,* by W. P. Workman, M.A., B.Sc. (University Tutorial Press.)

work for "results" and with a large class, the method of writing down the nine multiples at the side of the slate or paper was not unknown, though, as the ordinary examples seldom contained all nine different figures in the quotient, much of the labour of constructing this table of multiples was wasted. Its use, however, lessens the risk of inaccuracy.

Finally, the nature of the remainder after long division must be known. Questions of the following type are helpful in the understanding of remainders.

(*a*) What number must be taken from 100 to leave a number exactly divisible by 7? (*Ans.*, 2, or $7 + 2$, or $14 + 2$, etc.)

(*b*) What number must be added to 100 to make a number divisible by 7? (*Ans.*, 5, or $7 + 5$, or $14 + 5$, etc.)

(*c*) What is the greatest number less than 1,000, which is divisible by 15? ($1000 \div 15 = 66$ *and* **10** *over*, hence the required number is $1000 - 10 = 990$.)

(*d*) What is the number nearest to 500 which is exactly divisible by 9. ($500 \div 9 = 55$ and **5** over, hence the nearest number is *not* $500 - 5$ but $500 + 4$.)

Short Methods in Division. (*a*) Cases where division is avoided by multiplication. The theory of this method lies in the fact that $5 \times 2 = 10$, $25 \times 4 = 100$, $125 \times 8 = 1000$, etc.

(i) To divide by 5; divide by 10 and multiply by 2, e.g. $4375 \div 5 = 437{\cdot}5 \times 2 = 875$.

(ii) To divide by 25; divide by 100 and multiply by 4, e.g. $4375 \div 25 = 43{\cdot}75 \times 4 = 175$.

(iii) To divide by 125; divide by 1000 and multiply by 8, e.g. $1375 \div 125 = 1{\cdot}375 \times 8 = 11$.

Most children can make use of these processes without any reference to decimals.

The student or teacher who is interested should investigate the following rules—

(*a*) To divide by 75, add to the dividend one-third of itself and then divide by 100; or multiply by 4 and divide by 300.

(*b*) To divide by 15, subtract from the dividend one-third of itself and then divide by 10; or multiply by 2 and divide by 30.

Other such rules may be devised.

Many ingenious rules for division have been formed by accountants and others constantly employed in dealing with figures. Numerous examples will be found in a fascinating little book, *Short Cuts and Byways in Arithmetic*, by Cecil Burch.[1]

General Remarks. In division we have four distinct quantities: (*a*) Divisor, (*b*) Dividend, (*c*) Quotient, (*d*) Remainder.

The straight-forward process is, given (*a*) and (*b*), find (*c*) and (*d*).

It is essential that the child should know the connections between these four quantities. He should learn, first, that every division sum can be checked by multiplication according to the equation, Dividend = (Divisor × Quotient) + Remainder. He should also learn that divisor and quotient are interchangeable (so long as the remainder is less than either). Hence, a division sum can often be checked by a second division, this time using the quotient as divisor.

Again, these four quantities enable numerous problems to be set. The missing quantities in the following table represent the chief problems possible in this connection—

Divisor.	Dividend.	Quotient.	Remainder.
23	6841	?	?
27	?	13	25
?	4981	27	?
?	1000	12	28

[1] Published by Blackie and Son, Ltd.

CHAPTER IV

TEACHING THE SIMPLE RULES

In the previous chapters the fundamental processes, Addition, Subtraction, Multiplication, Division, have been discussed.

It by no means follows that the rules should be *taught* in this order. The memories of older teachers will carry them back to a period when such an order was officially prescribed in the Government "Code." Thus Standard I laboured for a whole year at Addition and Subtraction, Standard II at Multiplication and Short Division, and Standard III at Long Division. The famous Scheme B represented the first official attempt to break down this rigid method, by which one rule was treated exhaustively before the next was attempted. The necessity for some such reform was very great, for conscientious but uninspired teachers had reduced arithmetical teaching to the soulless routine of the same kind of sum day in and day out, week in and week out, until mechanical accuracy became a habit and the required percentage of passes could be guaranteed on the eventful day of the annual examination.

From the lead given by "Scheme B" the modern *spiral* method of treatment has been gradually evolved. This is so widely in use at the present time that only a brief description is necessary.

In the spiral method all four rules are treated simultaneously as far as possible, beginning with very simple numbers and quantities. These numbers and quantities are gradually increased in size and complexity so that the child returns again and again to each rule, but each time with harder numbers and quantities and fresh applications.

This method is more natural and less tiring to the pupil. Teachers who work thus, instead of proceeding in the older way of dealing thoroughly with one rule at a time, claim that the child loses nothing in accuracy and gains considerably in the power of intelligent application of rules to problems. The old method, however, was not entirely bad, and the "spiral" method in careless hands may result in very much being *attempted* and very little being *thoroughly done.* The child may be whirled round the spiral so carelessly that mathematical giddiness and unsteadiness result. The common complaint of the older teachers that the modern race of scholars is neither as thorough nor as accurate as that of a former generation has a basis of truth. There has certainly been a distinct tendency to scamp the drudgery so necessary to form the habit of accuracy, and to devote all the arithmetic period to short, snappy, catch-questions to which a quick and ready wit can, without effort, obtain the necessary answer. The answers to most of these artfully concocted puzzles usually "come out" conveniently in whole numbers. The little class books of examples so popular of recent years have to some extent encouraged this vicious tendency, as some of these books in their frantic eagerness to be "up to date" have lost all solidity, and must be described in the words of Bacon as "flashy things."

The Mechanical Sum. In this endeavour to teach arithmetic entirely through "real," "practical," "concrete," "interesting" examples—an endeavour in every way commendable—there lies however the danger that the actual abstract number processes which form the basis of all accurate computation, may be neglected. Concrete methods, interesting subject matter, real and practical examples, all aid the scholar to *understand* the problem and to determine the nature of the process to be applied, but once this stage is reached then all the concrete, interesting, practical accompaniments of the "sum" are so many trimmings to be ruthlessly torn away that the mind may concentrate on the essential process—the *manipulation of number.* It is the professional custom of some of the modern race of educational experts who throng our elementary schools, especially the lecturers and methodists from the Training Colleges, to hold up their hands or to turn up their eyes in pious horror or distress if they meet with a class that is working plain calculations unadorned by any attractive verbiage. "Mechanical sums," they exclaim, "how dreadfully out of date!" But let us ask them "Is not Accuracy and Speed in simple computations as important to-day as ever? Is it old-fashioned for a child to be ready, quick and accurate in his figuring?"

We will examine this question of the "mechanical" sum a little further. Such an example as "Add together 27, 43, 51" is usually termed a "*mechanical*" sum while, when put in the form "John had 27 marbles, Dick had 43 marbles, and Harry had 51 marbles; how many had they together?" it is thought by some to be a *problem.* In reality both are *calculations*, and the second form results in the same mechanical process as the first. In each form the process (addition) is obvious. It is only the *form* that alters, being in the first case imperative: "Do this"; and in the second case interrogative, i.e. "How many?" Even this *interrogative* form is not necessary for we may re-write "How many, etc.?" in the form "Find the number they had together," which is as direct and imperative as the first form. Yet the second form is preferable since the process becomes more *pointed* and *purposive.* And this leads us to the crux of the matter. In every sum a process or several processes have to be applied. If the particular process (Addition, Subtraction, etc.) is fairly *obvious*, the sum is not a problem so much as a calculation. In a real problem the child has first to determine which process he shall use, and then to perform the necessary calculations.

The difficulty has always been to strike the balance between the "sum" that is an obvious *calculation* and the "sum" that is presented in *problematic* form. In the past the tendency—a very natural one—was to concentrate on the *calculation* to the neglect of the problem, and so scholars laboured at addition, etc., rather than at the ordinary problems and applications of addition. The *process* was the object—children must all *do* addition and then *do* subtraction after they had *done* addition, and so on. The mechanical sum flourished; the problem was introduced under protest; and too often its problematic nature was ruined by a careful preliminary explanation by the teacher.

But times have altered. No longer is there the ominous figure of a Government Examiner looming in the background, in whose notebook will be entered, once a year, the ability of the teacher and the class in addition, subtraction, etc. No longer do we begin by "grinding" mechanical processes into little children before they have realized the need and use of them. Every "rule" is now "approached" in the most attractive way. Concrete methods—common objects, cardboard, coloured paper, coins, foot rules, weights and measures, are all employed in the modern school. The array of *props* now used to support and encourage the child's mathematical intelligence is a vast and imposing one; so imposing, perhaps, that we hardly need wonder that the child sometimes finds progress *without* all these props a little difficult. Young teachers are at times so dazzled by the splendid array of "helps" and "aids" and concrete illustrations of arithmetic, that to use a common phrase they cannot see "the wood for the trees," and so they are apt to forget that behind all this material reality there lies a higher reality, the *abstract* idea of *number* and the *manipulation* of number which is a reality of *mind*. This is the ultimate end to which all illustrations and demonstrations are but a means; this must be our goal, even though the pathway at times be dull and dreary: *constant, steady practice in calculation* is essential in every school both of to-day and to-morrow.

One extract from the "Suggestions for the Teaching of Arithmetic" (Circular 807) is here quoted, to convince both the very modern teacher, who is inclined to despise the calculation, and the timid teacher, who knows the necessity of calculation, but fears the adverse opinion of higher authority. The extract should remove any lingering doubt—

"It is also important that children should be frequently set to work quickly sums which present no difficulty as to method; in this way they may be led to combine speed with accuracy. The amount of time spent in this practice will vary from school to school, but it may often amount to one lesson per week."

Mental Arithmetic. The paragraph devoted to mental arithmetic in the "Suggestions" issued by the Board of Education opens with the remark: "All arithmetic is really 'mental.'" To a non-teacher such an observation is absurd, yet to an experienced teacher the point is a good one, for the tendency in the past, a tendency largely due to the absence of suitable material, has been to divorce mental arithmetic from arithmetic as a whole. The subject received separate treatment, and even had separate text-books, chiefly quaint collections of useful tricks and "rules of thumb." Now, however, with the spread of cheap writing materials and with the multiplicity of little class-books of examples, it is to be feared that mental arithmetic does not always receive due attention, with the result that many children are unable to perform the simplest calculation without the help of writing materials.

While, then, admitting the truth emphasized authoritatively as follows: "It cannot be made too clear that mental arithmetic and written arithmetic do not differ in essence" ("Suggestions," Circular 807). We must, for teaching purposes, keep the distinction clearly before us, if mental arithmetic is to receive its true share of treatment.

To a teacher, the natural use of mental arithmetic is for the revision of rules and processes already taught and for the development of new ones. There are very few arithmetic lessons in which ten minutes preliminary oral work is out of place, while every lesson on a new topic must of necessity have an oral beginning. The easy stages of such a lesson are—

(*a*) Examples that can be worked orally ;

(*b*) Examples that can be worked orally with the help of the blackboard ;

(*c*) Examples needing the help of writing materials.

In the above sequence we proceed naturally from easy examples to difficult examples, and from combined class-work to independent effort. Young teachers need a warning that (*a*) and (*b*) alone do not form a satisfactory "first" lesson. Time should always be allowed for (*c*), which requires individual effort and in itself forms the best test of the success of the oral demonstration. But besides this natural use of "mental" or "oral" arithmetic, the subject has other definite functions in the teaching of arithmetic.

It should be constantly used as arithmetical *drill* in teaching the fundamental units, relations, and tables. Numerous examples of this drill have already been suggested, and the more varied the practice the less necessity is there for mere mechanical repetition of tables, etc. In this connection simple sketches of common objects will be found very helpful both to teacher and scholars. The sketch forms a focus for the child's thoughts, and enables the teacher to frame a definite sequence of questions superior to the usual impromptu efforts of inexperienced teachers.

One example may be given. Fig. 16.)

Examples.

1. What is the value of the chalk at ½d. per stick ?
2. If a box could be bought for 3s., how much is this per stick ?

FIG. 16

3. If 7½ doz. pieces have been used, how many pieces are left ?
4. If I use half-a-dozen sticks a week, how long will the box last ?
5. If I use 2 sticks per day, how long will the gross last ?
6. If the chalk weighs 1 lb., how many sticks is this to the ounce ?

Thumb-nail sketches and simple illustrations for oral work form a feature of many of the excellent little class-books of arithmetic now in use. " Pure " mathematicians may scoff at such aids, but in the hands of a capable and enthusiastic teacher these sketches provide an inexhaustible supply of varied mental problems.

Again, though it may be fashionable to despise and decry mental arithmetic of the older type with its intricate " rules of thumb," there are, nevertheless, many rules and even " tricks " in common use that must be systematically taught. Among these may be put the calculation of the price per dozen from the price per single article, and *vice versa* ; the connection between the price per cwt. and the price per ton ; the decimalization of English money at sight, the approximate conversion of kilometres to miles, litres to pints, kilograms to lbs., etc. This mental arithmetic of practical short cuts is as valuable to-day as ever, and the child has a right to know it. The rules must be taught and practised systematically. The method of teaching such rules will vary ; time will hardly allow of all being developed heuristically, thus giving the child the joy of discovery.

Mental arithmetic as usually conducted orally with a class of any size where " hands up " are allowed for answers, suffers from obvious disadvantages. The method is not conducive to steady independent effort, and the lesson often becomes snappy and scrappy. The continual flourish of hands is disturbing, and the bulk of the work may fall to energetic volunteers, the dull children being neglected and the lazy ones escaping. The remedy for this is simple. Let the children be periodically required to *write down answers only* to questions worked mentally. The questions are preferably written or printed, so that each child has a copy. No written working is allowed and a time limit is always given. The advantages are many. Quiet independent effort is secured ; individual ability and progress is noted ; a wider range of questions is possible, and the teacher's voice is saved. " This practice . . . might well become universal in Elementary Schools."

Lastly, mental arithmetic still has definite uses even in the ordinary written work. Wherever possible in calculations or problems a preliminary approximate or *rough* answer should be made mentally, and noted as a check upon the written work. Some teachers insist that such shall be written at the head of every example. The practice is sound. Thus in the example

$$8{\cdot}197 \times 6{\cdot}9$$

a preliminary calculation gives $8 \times 7 = 56$, and is written

Rough Answer or R.A. = 56

The aid of such a calculation in the fixing of the decimal point in the answer needs no comment.

The Balance of Oral and Written Work. Much controversy has raged round the subject of the age at which the child should begin to do formal *written* work in arithmetic. Perhaps most of the differences of opinion have arisen from the common and dangerous habit of attempting to generalize upon a subject which after all is entirely individual. We cannot decide from a few particular examples that *all* children of a specified age *ought* to do this or that, but should state more modestly that some children of the given age will, and some will not, reach the given standard. It is almost impossible to dogmatize on the amount of formal written arithmetic that should be done in any particular junior class or standard. The present tendency is to postpone the written work until it is intelligible to the pupil and does not become a dull mechanical performance. Teachers are gradually realizing that in this matter of written arithmetic they have tried to force an unnatural pace. It is a curious but well-marked tendency of the pedagogue of all ages to take an unnatural delight in the "precocious" scholar. We can imagine that his delight and satisfaction would pass all bounds if he met with an infant's class whose teacher boasted that the children could do the arithmetical work of Standard I or Standard II. We find in all towns teachers of this type who strive to show that *their* children of ten are equal to any other children of eleven. This fatal rivalry is at the bottom of much of the mischief of formal arithmetical teaching in schools. Children in some of our infant schools are still daily engaged in the dull grind of some formal written arithmetical process. The lame excuse sometimes offered is that the teachers of the standards require this drill in "setting out." Teachers of infants and juniors are only slowly arriving at the point of view that in attempting to make the child reach the arithmetical proficiency of eight or nine years they have prevented it from making the most of its sixth or seventh year, and that by introducing this written work *too soon* they are thereby magnifying instead of minimizing the drudgery of arithmetic. For the fact is that in the past we have often attempted formal written work too early in the child's life. "Think of the progress that a child of the same age during the same period can make in motor co-ordination as shown in games, dancing, modelling, constructing, and other activities; think of the number of words that he can thus add to his usable vocabulary; think of the number of songs that he can learn and the number of nursery rhymes that he can memorize; think of the possible increase in his store of nature knowledge; and then compare any of these with his meagre achievements in arithmetic. Does it not give rise to the suspicion that the child's mind is at that period not quite ripe for that particular kind of training."[1]

Teachers who have realized this and have been courageous enough to act up to the conviction have had no cause to regret the result. In many infant schools to-day no formal sums are ever attempted in writing and yet these same scholars in senior classes are in no way deficient in arithmetical intelligence and calculating ability. At least one London senior school has carried the idea further still, for there *no* written arithmetic is attempted for the first three years. During these three years oral lessons of twenty minutes' length are the rule, and the time thus saved is devoted to reading. All the work is done *orally*, and rules are only introduced when the size of the numbers makes their manipulation by definite methods necessary. In this particular school there is no doubt of the increased arithmetical intelligence of the scholars, nor are they below the average arithmetical ability in the later standards, for oral work has the great advantage that a very much wider range of subject matter and methods can be covered. The scholar's ability to figure and to calculate mechanically according to some rigid process on paper is *not* and *never can be* a measure of his mathematical intelligence. The two are utterly separated, and the difficulty has always been to decide the pace at which each can be encouraged and developed, but the danger of attempting too soon formal written sums that must be set out in a rigid prescribed framework is a very grave one.

[1] Board of Education: Special Reports on Educational Subjects. "The Teaching of Mathematics in London Elementary Schools" (Dr. P. B. Ballard).

The natural view is that tersely and simply stated in Circular 807: "Oral arithmetic and written arithmetic do not differ in essence"—writing materials are only necessary when the size of the numbers makes mental retention unduly laborious. The use of writing materials in arithmetic should thus grow out of this felt need. Thus the natural stages are—

(*a*) Continuous oral work in all common processes and quantities, with the occasional help of the blackboard in the retention of the steps of a problem, but with simple numbers and quantities.

(*b*) The same continuous work in all the usual processes and applications but with numbers and quantities gradually increased in size until the help of writing material for each individual becomes a necessity.

Writing Materials. One final caution in this matter of written arithmetic is necessary. Writing materials in use in schools have evolved very rapidly during the last hundred years, through the finger and sand-tray, the slate and slate pencil, to pen, ink, and paper. The latter method was tending before 1914 to extreme development. Neatness, cleanliness, tidiness, and orderliness of setting out were becoming objects of scholastic worship, and were totally obscuring the more vital aspect of arithmetical teaching. It was the old, old error of mistaking "form" for "matter"; shadow for substance. Lessons—*arithmetical* lessons—were gravely given on "how to use a round ruler." The cramping effect of all this artificial rigidity of arithmetical teaching too often passed unnoticed, and least of all did it strike the teachers that *pens* (usually with scratchy steel nibs) and *ink*, were the most absurd and unsuitable materials for use in the arithmetical work of children. Almost any other kind of writing material is preferable to these. Let the teacher who doubts it try the effect of the humble lead pencil and paper instead of the conventional pen-and-ink. Our American cousins have long realized the disadvantages of pen and ink, and lead-pencil work in arithmetic is much more common in the United States than with us. The height of the absurdity is seen when a class struggles to draw diagrams in mensuration, and to do all kinds of graphic work, with this same scratchy steel pen.

The effects of the war in our schools brought forward a striking illustration of this same point. The scarcity and consequent dearness of exercise books and paper caused a very general return to the slates of a former generation. (We must remember that most of our young teachers have learned in their hygiene lectures at college that these same slates are clumsy, out of date, and old-fashioned, as well as being filthy, loathsome, greasy germ-carriers, spreading disease and pestilence everywhere.) These same young teachers have, however, in many cases welcomed the return of slates for these, they say, enable them to get *so much more work done in arithmetic*. And this is largely truth, for too much pen-and-ink work has a cramping, clogging, baneful effect upon a child's arithmetical energy.

If the use of slates is considered undesirable much may be accomplished with an ordinary lead pencil and cheap form of " jotter " or writing tablet. It is too much to expect of any child that *all* its work in arithmetic shall be worthy of careful preservation in ink in the usual " exercise book."

CHAPTER V

MONEY AND MONEY CALCULATIONS

MONEY sums have always figured very largely in the arithmetical curriculum of schools, especially of elementary schools. It is possible indeed that too much attention in the past has been paid to money calculations, yet to neglect this branch of essentially practical mathematics would be grossly unfair to the wage-earning and salary-earning classes from which the vast majority of scholars are drawn. Money is an essential factor of the intercommunication of civilized peoples, and without it we should be obliged to return to the primitive condition of barter and service. For this reason, then, we must continue to teach money sums in all our schools.

Only the very wealthy and very lazy can scorn learning to add, subtract, multiply, and divide money, and for every child *money* forms the first and most important practical application of arithmetic.

With a sigh we must notice regretfully that most of this chapter is essentially English. In no other modern coinage system do such elaborate and intricate processes have to be mastered. If we were not so accustomed and hardened to the "rules" we should be aghast at their complexity. But patiently and persistently we pursue our drudgery, and only at times when tired and disheartened do we pause and dream of an age and a school, where 10 and not 12 pennies, would make a shilling, and 10 and not 20 shillings would make a pound. It is difficult for teachers to realize what time would be saved, what energy, both of teacher and taught, would be thus set free. This for a teacher is the most powerful argument in favour of a reformed coinage. But revolutions are seldom accomplished easily, and are always opposed by a tremendous weight of real, if conservative, opposition.

Thus, for the present, English teachers must labour on with their complex money rules, and extend their labours to all our other insular weights and measures. It is to help these English teachers that this chapter is written.

The Beginnings. Children become familiar with money, and the significance of money, at a very early age. Farthings, halfpennies, and pennies are met with more or less frequently even by the poorer children, while the running of errands for parents and neighbours frequently gives familiarity with larger sums and with the commoner silver coins. With gold coins, even in times of peace, we cannot assume that they have such familiarity, but with copper and silver coins we have a good working basis for preliminary exercises in money calculations. These provide a satisfactory starting point. Imitation coins are now largely in use in schools, and just as the earliest number lessons deal with the composition and decomposition of *numbers*, so our earliest work in money deals with the relative values of common coins such as the penny, the sixpence, the shilling, the florin, and the half-crown. These give ample scope for oral examples and exercises long before any formal money sums need be attempted, and they may be taken as early as possible, for no longer is the teacher obliged to postpone "money sums" until the four simple rules have been exhaustively treated.

In modern schools, the child of to-day, as soon as he has grasped the properties of the *number* 12 and its various parts, applies the knowledge to *money*, and deals with the fact that 12 pence are one shilling just as he applies it at the same time to the fact that 12 inches are one foot. Thus we find children even in Standard I readily using all four rules in money involving small numbers of shillings and pence, the amount perhaps at first not exceeding ten shillings. So, too, as soon as the child has investigated, through his practical work and his drawing, the subject of "halves" and "quarters" and their relation to wholes, he applies his first knowledge of fractions to the manipulation of halfpennies and farthings. This work, carefully carried out, illustrates the "spiral" method at its best, and is in strong contrast to the older method in which "money sums" were not dealt with until huge "number" sums had been mastered.

Shopping. It is natural that, in the attempt to connect arithmetic with ordinary daily life, the shopping lesson should have developed. This method of teaching money and quantity through the ordinary business of retail buying and selling has great possibilities, but not a few drawbacks. If the subject is skilfully and sympathetically treated the children's imaginations will bridge all the pretence of imitation shop, imitation articles, and imitation coins. Endless variety is possible in the lessons themselves, for the purchases may involve not only the manipulation of money but also the use of, and the knowledge of, all the ordinary simple weights and measures, such as yards, pounds, and ounces, pints and quarts. It is essential that the shopping lessons should be regular, systematic, and carefully graded. Each lesson should have a purpose. The desultory, occasional lesson, given in haphazard fashion without any definite aim is generally too shallow and superficial to serve any useful purpose.

We could continue to enlarge upon the benefits of this natural and pleasant method of dealing with money calculations, but must not refuse to recognize the very real limitations to the use of the method in elementary schools. The teacher in these schools is in nearly every case faced with two very grave difficulties: (*a*) a large class, and (*b*) a lack of space. The writer has seen many young teachers after such lessons in a chastened and even despondent mood. They have discovered the difficulty of keeping sixty children working mentally when two only were engaged in buying and selling. They have struggled to maintain the interest by a constant change of buyers and sellers; by requiring all the class to make the calculations and to give the correct change for each transaction ; and by insisting that every child shall make out a bill for each transaction ; but at the end they have confessed that it was much harder work and gave much less satisfactory results than they had anticipated.

The cause of all such difficulties is, of course, the fact that the method is not a *class* method at all, but an *individual* method. Buying and selling, normally, is the concern of two people only—the buyer and the seller. Hence, what we require is not *one* shop with *sixty* children, mostly uninterested onlookers, but *thirty* shops with *thirty* buyers and *thirty* sellers, if we must confine the exercise to one " period " of about forty minutes. But it is not at all necessary to make a *formal lesson* of " shopping " one to be conducted much on the lines of a formal history lesson or geography lesson. A far more sensible plan is to look upon the " shop " as a permanent piece of school mathematical apparatus. But this involves, of necessity, some more or less *permanent* form of shop. It is in this guise only that the " shop " and " shopping " attains its greatest usefulness in schools.

Such a permanent feature of a school can be seen in full working order at Werrington Industrial School, near Stoke-on-Trent. I am indebted to Mr. J. Douglas Johnstone, the superintendent, for the brief account that follows.

It should be noted in connection with this description that all the pupils at this school are in residence—a fact which enables the " shop " to be used more frequently and more incidentally than would be possible in a day school.

A SCHOOL SHOP

" It is generally conceded in these days that it is applied and concrete arithmetic that has to be taught, and practical work in this subject now occupies a prominent position in the curricula of many schools. Such practical work frequently takes the form of exercises in " Shopping " and the school shop is but a step further on.

"There are two entrance porches to our school-room and, as one of them was seldom used as such, we resolved to use it as a shop. A long narrow table was installed as the counter, and an old desk top placed at one end. The scales, weights and measures in school-use received a permanent place in the centre of the counter, and the fittings were completed by the conversion of grocery boxes into shelves.

"Getting the stock ready was a most interesting and instructive time. We collected jam jars, bottles of all kinds, tea and cocoa packets, sugar and currant bags, starch boxes, match boxes, salt and syrup tins—in fact, anything we could fill with substitutes for the real things. Jam jars were covered with gummed papers, and labelled "Raspberry," "Marmalade," "Damson," etc. Pickle bottles were filled with round pebbles and water; the various packets filled with sand and sealed up; and the whole arranged on the counter and the shelves in a business-like fashion.

"The juniors supplied us with hand-made envelopes and paper bags, and pebbles sorted in sizes. These were used as peas, lump sugar, sweets, currants or raisins according to size, while the biggest of all were labelled "Potatoes" or "Eggs." One class produced some very creditable plasticine bananas, pies, biscuits, bars of chocolate, cakes—even chops. The same material was used for butter, margarine, cheese and lard, and weighed out when ordered. A good supply of sand was obtained, stored in biscuit tins, and used as flour, coffee, sugar, etc. For milk, vinegar, and paraffin oil we used water which we kept in a bucket. Articles of clothing, games, pocket knives, balls, toys, etc., were borrowed from the boys, while some of the articles made in the manual training classes were added to the stock.

"As soon as the collection of the stock and the arrangement of the shop were well in hand, a few of the senior boys were set to work preparing price tickets, bill heads, and the price list. When all was ready the staff was chosen. We appointed a manager from the seniors and an assistant manager from the juniors. They were responsible for the shop, its arrangement, and cleanliness. Others on the staff were two assistants, a clerk, and a commissionaire.

"The actual shopping was a matter of no little difficulty, but we ultimately adopted the following method. A time table was drawn up to arrange that each class in the school would have the use of the shop on one half-day every week. Three or four customers only were allowed to go shopping at once, so that their temporary absence from their class did not materially affect their class work. The teacher in the early stages was in frequent attendance, but later on it was found possible to leave a monitor in charge, the teacher looking in occasionally to see that the shopping was being conducted in a courteous and business-like manner, and in *English*. The commissionaire stood at the door and gave each customer, as he entered, a sum of cardboard money with which to make his purchases. On leaving, the customer surrendered his goods, his change and his receipted bill to the commissionaire, who checked them and thus played the part of the careful parent.

"The customers could make a list of their intended purchases if they liked, but were limited to three or four items, one of which had to involve weighing or measuring. One assistant made up the order and wrapped up the goods while another made out the bill. The customer took the bill to the desk and tendered enough money to the clerk to settle it. The latter receipted the bill and handed it back with any change there might be. The customer checked his change, and the clerk entered the amount received in his cash book. At closing time the cash in hand had to tally with the total in the cash book. Sometimes customers were allowed to take the place of different members of the staff, the latter returning to their class. Different boys were chosen as assistants, clerk, and commissionaire, each time the shop was open, notice being given to them a week before hand so that they could be coached by those who had previously held those positions.

"We hope to open a Post Office branch where dummy stamps, postal orders, etc., can be bought ; letters, parcels, and telegrams sent ; and an imitation Savings Bank instituted.

"It is possible that in the near future the shop may become more real. Many of the things that boys buy may be stocked and sold—without unduly interfering with legitimate external businesses—provided that the profits are devoted to matters concerning the boys, such as the Sports Fund or School Treat, or the acquisition of a lantern or cinematograph. We have in mind such things as garden seeds, note books, string, marbles, pencils, football studs, balls, books, and boys' magazines and periodicals. They could also be trained in the use of catalogues. If it were possible—as it is in institution schools—to obtain samples of suits, caps, collars, boots, ties, shirts, socks, etc., the boys could be trained to discriminate in the quality of goods, and to expect a certain standard for a certain outlay.

"In conclusion, if it were desired to make a really permanent shop or general store—whether for imitation or real goods or for both—the making of the counter, shelves, show cards, and other fittings would provide an excellent course for the manual training class."

Formal Money Sums. Sooner or later the more serious work of formal methods in money operations must be faced. It has been customary for many years to base these operations upon a good knowledge of money tables, notably the "pence" table. This table usually proceeds through the 12's and the multiples of 10, thus—

12	pence	are	1s.	
20	,,	,,	1s.	8d.
24	,,	,,	2s.	
30	,,	,,	2s.	6d.

ending generally at 144 pence are 12s. In that fascinating little volume on *Infant Education or Remarks on the Importance of Educating the Infant Poor from the age of eighteen months to seven years*, by Samuel Wilderspin (of date about 1825), we find the following quaint doggerel—

Twenty pence are one and eightpence,
  That we can't afford to lose;
Thirty pence are two and sixpence,
  That will buy a pair of shoes.
Forty pence are three and fourpence,
  That is paid for certain fees;
Fifty pence are four and twopence,
  That will buy five pounds of cheese.
Sixty pence will make five shillings,
  Which we learn is just a crown.
Seventy pence are five and tenpence,
  This is known throughout the town:
Eighty pence are six and eightpence,
  That sum once my father spent;
Ninety pence are seven and sixpence
  That for a quarter's schooling went.
A hundred pence are eight and fourpence,
  Which is taught in th' infant school,
Eightpence more make just nine shillings,
  So we end this pretty rule.

This is an excellent sample of the rhymes which these "poor infants" of a century ago sang in chorus as they danced round the trees in the playground or sat in the swings of the schoolroom. By this method, says the author, in charming naivety, " the children are gradually improved and delighted, for they call it play, and it is of little consequence what they call it, so long as they are edified, exercised, and made happy." Our only regret is that the writer does not remember to give us the melody to which the lines were sung. A comparison of this tune with modern " table tunes " still heard all over the country would be interesting.

The following table leaves us rather more breathless. It was in use at Stratford Infant School, Bow, London, in the year 1825, and was penned by the headmaster, Mr. James Carroll—

> Four farthings just one penny make,
> Enough to buy two half-penny cakes:
> And to allow, I am most willing,
> That twelve pence always make a shilling,
> And that five shillings make a crown,
> Twenty a sov'reign, same a pound.
> Some have no cash, some have to spare—
> Some who have wealth for none will care,
> Some through misfortune are (see! lo!)
> When money's gone, are filled with woe.
> But I know better than to grieve,
> If I have none I will not thieve:
> I'll be content whate'er's my lot,
> Nor for misfortunes care a *groat*.
> There is a Providence whose care,
> Whose sovereign love I crave to share;
> His love is *gold without alloy*:
> And those possessed have *endless joy*.

Mr. James Carroll was certainly a better moralist than poet, and this quaint combination of moral instruction and arithmetic is a " correlation " not usual to-day.

The traditional pence table is usually memorized by the ordinary method of continual repetition, and its knowledge serves a useful purpose, for every teacher knows that the difficulty of dealing with pence is greater than the difficulty of dealing with farthings or shillings. But some teachers prefer to use only the " Table of

Twelves," e.g. 12, 1s.; 24, 2s.; 36, 3s.; etc., and not the intermediate tens, e.g. 20, 1s. 8d.; 30, 2s. 6d.; thus, in changing 43 pence to shillings the child says at once 43 = 3 twelves and 7, i.e. 3s. 7d., and *not* 40 pence are 3s. 4d., 43 pence 3s. 7d. The difference between the two methods is a minor one. What is far more important is this that the pence table should not be taught by mechanical repetition *before* the child has investigated, constructed, and memorized the table of twelves in number.

Reduction of Money. The additional labour involved, both for teachers and scholars, by our miscellaneous system of money, weights and measures, is nowhere more clearly seen than in the process commonly known as *Reduction*. Here at once the arbitrary connection of our money units—4, 12, 20—strikes us most forcibly, for it has added a fifth process to the ordinary operations of addition, subtraction, multiplication, and division. In numerous problems and processes it becomes necessary to change our units and to proceed from pounds to shillings or shillings to pounds; from pounds to pence or pence to pounds, etc. The number of common coins in use makes reduction a very useful and necessary process. As in every other process, much may be memorized. Thus the child should discover and memorize such facts as—

£1 = 240 pence = 480 half-pence = 960 farthings
£1 = 8 half-crowns = 10 florins = 40 sixpences
= 80 threepences

He should be able to count by 2s. 6d. to £1, thus: 2s. 6d., 5s., 7s. 6d., 10s., etc., and by $1\frac{1}{2}$d. to 1s., thus: $1\frac{1}{2}$d., 3d., $4\frac{1}{2}$d., 6d., etc. Constant oral practice and drill in these and similar money relations is necessary for ready manipulation of English money.

The general type—reduction of pounds to farthings, or farthings to pounds, involves multiplication or division by 4, 12, 20. Before discussing the formal process as usually set out, let us note that the traditional word *reduction* is somewhat pedantic and hardly correct, for if we use the same word "reduce" in the two following examples—

(*a*) Reduce £52 to farthings;
(*b*) Reduce 10,000 farthings to £ *s. d.*;

the same word expresses clearly two different ideas and involves two different processes, for (*a*) is multiplication and (*b*) is division. The difference may be further expressed thus: In (*a*) we *reduce* the denomination but *increase* the number of coins; in (*b*) we *reduce* the number of coins but *increase* their value. The ordinary English word "change" will serve our purpose equally well, and we may express both processes just as simply and more correctly in the forms—

(*a*) Express £52 in farthings,
or Change £52 to farthings.
(*b*) Express 10,000 farthings in £ *s. d.*,
or Change 10,000 farthings to £ *s. d.*

It is certain that in the past *reduction* as a formal process in money, weights, and measures, has received too much attention. Sums such as the following: "Change 100,000 farthings to £ *s. d.*" very seldom occur in adult life. Where they *are* met with, a ready-reckoner is usually consulted if the numbers are large. In our weights and measures the absurdity of long reduction sums is clearly seen, for *never* (except at school) is it necessary to change 1,000,000 inches to miles, or 12,345,678 secs. to weeks, days, hours, mins., secs., or to express 10 tons 12 cwt. 14 lb. 15 oz. in ounces. The only defence for such unreal examples is that they serve as a means of teaching the connections between the various units and for practice in calculation, both of which purposes may be achieved in other and better ways.

To treat *reduction* as an end in itself and to burden the scholar with long mechanical and pointless reductions is a mistake of which every teacher should beware.

We will now discuss a few examples in the Reduction of Money.

Reduction by Multiplication, i.e. to lower denominations.

(*a*) Express £13 14s. 10d. in pence. (*b*) Express £15 9s. $10\frac{3}{4}$d. in farthings.

The work is usually set out as follows—

	£	*s.*	*d.*			£	*s.*	*d.*
(*a*)	13	14	10		(*b*)	15	9	$10\frac{3}{4}$
	20					20		
	274*s.*					309*s.*		
	12					12		
	3298*d.*					3718*d.*		
						4		
						14875*f.*		

Alternative arrangements of setting out these examples are sometimes seen as follows—

	£	*s.*	*d.*		£	*s.*	*d.*	*f.*
(*a*)	13	14	10	(*b*)	15	9	10	3
		260	3288			300	3708	14872
		274	3298			309	3718	14875

This alternative arrangement is slightly more compact than the traditional one first described.

Reduction by Division. (*a*) Express 10,000 farthings in £ *s. d.*

(i)
```
   4)10000f.
  12)2500d.
 2,0)20,8s. 4d.
     £10 8s. 4d.
```

(ii)
```
10000f.
 9600f. = £10
  400f. = 100d. = 8s. 4d.
     Ans. £10 8s. 4d.
```

(*b*) Express 567 pence in £ *s. d.*

(i) 12)567
2,0)4,7*s.* 3*d.*
£2 7*s.* 3*d.*

(ii) 567 pence
480 pence = £2
87 pence = 7*s.* 3*d.*

Method (ii), using the larger multiples and proceeding by subtraction, may frequently be employed with advantage.

The general form of an example in reduction is all too often the bald and uninteresting direction "Reduce, etc." Yet no calculation is thrown more easily into concrete form, and so presented as a real interesting and *pointed* calculation. A few suggestions only are given here. Numerous other examples will be found in the author's series of *Common Sense Arithmetics.*[1]

(1) How many bags marked "copper 5s." will be necessary to hold 7,500 pennies?

(2) What will a halfpenny newspaper cost for a year of 313 week days?

(3) A business firm uses 1,000 two-penny stamps and 1,500 three-halfpenny stamps per week. Find their cost.

(4) The subscription to a club was 2s. 6d. How much would be received from 125 members?

More complex examples may be taken from a day's cinema receipts, sale of concert tickets, analysis of a subscription list, analysis of a church collection, calculation of a wage-sheet on the basis of payment by the hour, etc., etc. With such a wealth of material ready to hand, purposeless calculations need seldom be set.

[1] *Common Sense Arithmetic*, in eight parts (Sir Isaac Pitman & Sons, Ltd.).

CHAPTER VI

THE FOUR RULES IN MONEY

Addition of Money. This process is required in every walk of life. Like all other fundamental processes, it is a mechanical art requiring much practice before rapid and accurate work becomes habitual. But since it *is* a habit it is essential that the teacher should examine carefully the various minor differences in methods of adding up a sum of money, for it is the teacher's duty to examine carefully and impartially all methods in use, and to urge his pupils to cultivate calculating habits that seem best and simplest. And again, let it be noted by young teachers, the methods *they* learnt in childhood are not of necessity the *best*. Lastly, the thoughtful teacher will understand that the age of a method or its widespread acceptance is not of necessity a criterion of its excellence. For while it is true that many good methods in our arithmetic survive because they *are* good and have been consciously and deliberately chosen, on the other hand many bad methods persist through sheer inertia and conservatism.

Let us examine the chief methods in common use.

Addition of Farthings. These we may add in two ways—

d.
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{3}{4}$
$\frac{1}{4}$
$\frac{1}{2}$

(1) As *numbers* and then *reduce* thus: 2, 3, 6, 7, 9; 9 farthings $2\frac{1}{4}$d.

(2) As *farthings*, completing pence as we proceed, thus: $\frac{1}{2}$d., $\frac{3}{4}$d., $1\frac{1}{2}$d., $1\frac{3}{4}$d., $2\frac{1}{4}$d.

Between these two methods there is little to choose: (1) is perhaps more usually taught, but (2) is just as rapid.

A few teachers advocate a logical extension of notation to avoid the fractional form of farthings. Instead of the usual three groups £ s. d., these would have four, thus: £ s. d. f. Hence £17 11s. $9\frac{3}{4}$d. would be written and manipulated as £17 11s. 9d. 3f. The extension is interesting but hardly necessary. It may form a useful prop for children but is not met with in ordinary life.

Where reduction is necessary as in method (1) above, one caution is advisable. The reduction should in general be done mentally. "It is certain that many children are accustomed to rely too much upon written work. This undue reliance should be checked, e.g. a child advanced enough to work money sums on paper should be able to see that 19 farthings are equivalent to $4\frac{3}{4}$d., and should therefore not find it necessary to reduce formally the farthings to pence. The marginal work in exercise books often contains much which it is needless to write down; this work, therefore, should be closely scrutinized by the teacher, and operations like the reduction which has been mentioned should be disallowed."[1]

Addition of Pence. As in the addition of farthings, so here again two methods are used.

d.
9
5
8
7
3

(1) We may add the *numbers* and "*reduce*," thus: 3, 10, 18, 23, 32; 32 pence are 2s. 8d.

(2) We may add the *pence*, completing the shillings as we proceed, thus: 3d., 10d., 1s. 6d., 1s. 11d., 2s. 8d.

Usually we are taught the first method in childhood, but sooner or later find ourselves unconsciously using both; the one to check the other. Children may with profit be taught to do the same.

s.
15
17
9
8
19

Addition of Shillings. Here the choice of method is easier. We can as in method (2) above "make up" the pounds as we proceed, thus: 19s., £1 7s., £1 16s., £2 13s., £3 8s., but this is undoubtedly slower than the addition of the numbers, especially for children. Hence, most people work first in *number* and then "reduce," thus: 9, 17, 26, 33, 38, 48, 58, 68; 68s. are £3 8s. Some teachers introduce a

[1] "Suggestions for the Teaching of Arithmetic" (Circular 807).

combination of the two methods to save the "reduction" of the shillings at the end of the process. This is avoided by adding the units as numbers, but considering each "ten" as half a sovereign and adding these in pairs, thus reaching the number of complete pounds without formal reduction.

Addition of Pounds. This presents no differences from the ordinary addition of number.

General Addition of Pounds, Shillings, Pence, and Farthings. The only new point introduced in the general type of sum is the "carrying" figure from column to column. This is usually carried mentally, but the carrying figure or figures may be *written* down in the case of lengthy additions. The figures thus written down are useful in checking.

The Teaching of Addition of Money. Speed and accuracy are within the reach of any normal child by (1) constant and regular practice in carefully graded examples; (2) the formation of the habit of "checking" the result.

The grading of examples usually adopted to-day is generally based upon the size and the number of the quantities. A common gradation is—

(*a*) Addition of the simplest shillings and pence involving no sum above 10s. or £1.

(*b*) The same but involving also halfpennies and farthings.

(*c*) Addition of simple pounds and shillings.

(*d*) Extension of (*a*), (*b*), and (*c*) to the general type.

The exact point at which halfpennies and farthings should be introduced is debatable. Some teachers introduce them at once; others prefer to deal first with pence, and afterwards with pence and farthings.

Practice can be given in *horizontal* as well as *vertical* addition of money. The horizontal form is harder and the strain on the eyes is greater, but it is sometimes necessary in after-life. The well-known arrangement of a block of items with the direction " Add rows and columns and check totals " will supply the necessary practice.

The continual " carrying forward " of a total from column to column is also a useful form for practice. Thus:

£	*s.*	*d.*		£	*s.*	*d.*		£	*s.*	*d.*
12	9	8	car. forwd. →				car. forwd. →			
15	6	7		17	15	8		15	6	9
17	19	11		19	14	11		27	13	3
15	13	4		87	15	6		14	15	6
5	8	9		72	19	8		16	18	4
22	15	6		36	13	11		12	7	11
←				←						

The aim of the teacher should always be to set types and examples approximating as closely as possible to those met in real life.

Subtraction of Money. Any of the methods discussed in the subtraction of number, i.e. Decomposition, Equal Additions, Complementary Addition, may be adapted to the subtraction of money. In ordinary retail transactions the method of complementary addition " or making up " the difference is the one *always* used. The problem is further simplified by the fact that the bigger sum to be " made up " is always a " round " sum such as an exact number of shillings, half-a-crown, half-a-sovereign, £1, £5, etc.

These " shopping " transactions provide a good starting point for practising subtraction on this " making-up " method. If goods to the value of 7s. $7\frac{1}{2}$d. are paid for with a 10s. note, the shopkeeper invariably reckons the change, usually in front of the customer, by the " making-up " method. He says " *And* $4\frac{1}{2}$*d.* is 8s., *and* 2*s.* is 10s."

Much well-graded work should be given on this method, e.g.

(*a*) What will make each of the following amounts up to 1s.? $4\frac{1}{2}$d., $6\frac{1}{4}$d., $7\frac{3}{4}$d., $8\frac{1}{2}$d., $10\frac{1}{2}$d., etc.

(*b*) What will make each of these up to 2s. 6d.? 2s. 3d., 1s. 11d., 1s. 9d., 9d., 1s. $10\frac{1}{2}$d., 1s. $2\frac{1}{4}$d., $7\frac{1}{2}$d.

(*c*) What will make each of these up to 10s. ? 7s. 6d., 5s. 9d., 3s. 3d., 8s. 4d., 6s. 8d., 3s. 7d.

(*d*) What will make each of these up to £1 ? 18s. 6d., 13s. 9d., 7s. 6d., 4s. 3d., 16s. 8d., 12s. 10d., 8s. 4d., 15s. 11d., 13s. 7d., 11s. 2d., 18s. $10\frac{1}{2}$d., 12s. $4\frac{1}{2}$d., 6s. $7\frac{1}{2}$d.

Formal Methods. *Example.* From £12 7s. $9\frac{1}{2}$d. take £9 16s. $10\frac{3}{4}$d.

(*a*) *By Decomposition.* In this method there is continual "borrowing" and "decomposing" and the example

£	*s.*	*d.*		£	*s.*	*d.*	
12	7	$9\frac{1}{2}$	is actually	11	26	20	$1\frac{1}{2}$
9	16	$10\frac{3}{4}$	worked as	9	16	10	$\frac{3}{4}$
				2	10	10	$\frac{3}{4}$

The usual working is as follows—

Farthings. $\frac{3}{4}$d. from **$\frac{1}{2}$d.** impossible ; "we can't" ; borrow 1d. $\frac{3}{4}$d. from $1\frac{1}{2}$d. leaves **$\frac{3}{4}$d.**

Pence. 10d. from **8d.** impossible ; borrow 1s. 10d. from 1s. leaves 2d., and 8d. is **10d.**

Shillings. 16s. from **6s.** impossible ; borrow £1. 16s. from £1 leaves 4s., and 6s. is **10s.**

Pounds. £9 from **£11** leaves **£2.**

Answer, **£2 10s. $10\frac{3}{4}$d.**

(*b*) *By Equal Additions.*

£	*s.*	*d.*		£	*s.*	*d.*	
12	7	$9\frac{1}{2}$	is actually	12	27	21	$1\frac{1}{2}$
9	16	$10\frac{3}{4}$	worked as	10	17	11	$\frac{3}{4}$
				2	10	10	$\frac{3}{4}$

The usual working is as follows—

Farthings. $\frac{3}{4}$d. from $\frac{1}{2}$d. impossible ; *add* 1d. $\frac{3}{4}$d. from $1\frac{1}{2}$d. leaves **$\frac{3}{4}$d.**

Pence. Add 1d., 1d. and 10d. is 11d., 11d. from 9d. impossible ; add 1s. 11d. from 1s. leaves 1d. and 9d. is **10d.**

Shillings. Add 1s., 1s. and 16s. is 17s. ; 17s. from 7s.

impossible ; add £1. 17s. from £1 leaves 3s., and 7s. is **10s.**

Pounds. Add £1, £1 and £9 is £10. £10 from £12 leaves **£2.**

(*c*) *By Complementary Addition.* In this the problem is viewed as "What must I add to £9 16s. $10\frac{3}{4}$d. to make £12 7s. $9\frac{1}{2}$d. ?" The actual working does not differ much from that followed in "equal additions."

Farthings. $\frac{3}{4}$d. *and* **$\frac{3}{4}$d.** make $1\frac{1}{2}$d. Write down $\frac{3}{4}$d. and carry 1d. (as in addition).

£	s.	d.
12	7	$9\frac{1}{2}$
9	16	$10\frac{3}{4}$
2	10	$10\frac{3}{4}$

Pence. 1d. and 10d. is 11d. *and* **10d.** is 1s. 9d. (or : and 1d. is 1s. and 10d. is 1s. 9d.). Write down 10d. and carry 1s.

Shillings. 1s. and 16s. are 17s. 17s. *and* **10s.** are 27s. (= £1 7s.). Write down 10s. and carry £1.

Pounds. £1 and £9 are £10 *and* **£2** are £12. Write down £2.

Each of these three methods gives good results with sufficient practice. The first (decomposition) is popular with some teachers because it can be readily "demonstrated" by practical means. The last (complementary addition) is the one usually employed by adults who have to deal with money accounts.

Complementary addition saves much labour in the following type of sum.

(*d*) *Addition and Subtraction in One Operation.* Consider the example : "Mother spent 3s. 5d. at the baker's, 4s. 11d. at the butcher's, and 6s. 7d. at the draper's. What change would she have out of £1 ?"

£	s.	d.
1	–	–
	3	5
	4	11
	6	7
	5	1

This may be done in *one* operation of addition. Work as follows—

Pence. 7, 18, 23 *and* **1d.** makes 2s. Write down 1d. and carry 2s. *as in addition.*

Shillings. 2, 12, 15 *and* **5s.** makes 20s. Write down 5s. *Answer*, **5s. 1d.**

This is the method always used in finding *balances.* For older children the question may be put in this form : Find the *balance* in the following account.

£	*s.*	*d.*
12	19	11
13	15	6
27	12	9
10	17	10
**	**	*
75	–	–

The work should always be done as above in *one* operation. The advantage of the method is apparent. Only one sum—addition—is necessary to obtain the result instead of two, viz., first addition then subtraction. The method deserves much more attention than it usually receives.

The Teaching of Subtraction. As with addition, examples in subtraction should be most carefully graded, introducing one difficulty at a time.

Teachers should not be misled by the ease with which "decomposition" can be illustrated and demonstrated practically to little children. It is fairly well established that either of the other two methods give better results in speed and accuracy for equal amounts of practice. What is however most essential is that in the earlier years there should be *consistency* of method as the child proceeds from standard to standard, i.e. he should not work by "decomposition" in Standard I, and by "equal additions" in Standard II.

Multiplication of Money. It is customary to grade work in the multiplication of money in something like the following stages—

(*a*) Short multiplication by numbers not exceeding 12.

(*b*) Multiplication by factors for multipliers easily factorizable, or closely allied to numbers easily factorizable.

(*c*) Multiplication involving larger numbers and general methods.

We will discuss each of these in turn.

Short Multiplication. Type £2 3s. $4\frac{1}{2}$d. × 11. The work is usually set out thus :

£	s.	d.
2	3	$4\frac{1}{2}$
		11
23	17	$1\frac{1}{2}$

All that is necessary for this work is a ready knowledge of the ordinary multiplication " tables " and an ability to change rapidly from one denomination to another, e.g. in the above : $\frac{1}{2}$d. × 11 = $5\frac{1}{2}$d., 49d. = 4s. 1d., 37s. = £1 17s.

Factor Multiplication. Types (i) £3 4s. $5\frac{1}{4}$d. × 35. (ii) £5 4s. $3\frac{1}{2}$d. × 43.

The work is usually set out as follows—

(i)

£	s.	d.	
3	4	$5\frac{1}{4}$	
		5	
16	2	$2\frac{1}{4}$	= 5 times
		7	
112	15	$3\frac{3}{4}$	= 35 times

(ii)

£	s.	d.	
5	4	$3\frac{1}{2}$	
		6	
31	5	9	= 6 times
		7	
219	–	3	= 42 times
5	4	$3\frac{1}{2}$	= 1 time
224	4	$6\frac{1}{2}$	= 43 times

Success here depends upon a ready knowledge of factors. The actual working is done by successive applications of " short " multiplication.

Practice should be extended to include *subtraction* as well as *addition*, e.g. just as 43 in (ii) above was worked as (6 × 7) + 1, so 41 might be worked as (6 × 7) – 1.

The calculation may frequently be set as : " Multiply £1 3s. $7\frac{1}{2}$d. by 30 and use your answer to find (*a*) £1 3s. $7\frac{1}{2}$d. × 31 and (*b*) £1 3s. $7\frac{1}{2}$d. × 29."

It should be noted that the *order* in which the factors are used does not affect the answer (Commutative Law), hence it is useful, where "copying" is suspected, to divide the class into the usual groups of A and B alternately, and to require the A's to work with one order of factors and the B's with another.

General Method. Type: £2 5s. 8½d. × 345.

In this type of example we have several methods, all of which are in common use.

(1) *Traditional Method.*

The method is self-explanatory. It becomes somewhat long and cumbersome for a multiplier of more than three figures.

£	*s.*	*d.*				
2	11	8½				(*a*)
		10				
25	17	1	=	10 times		(*b*)
		10				
258	10	10	=	100	,,	(*c*)
		3				
775	12	6	=	300	,,	[(c) × 3]
103	8	4	=	40	,,	[(*b*) × 4]
12	18	6½	=	5	,,	[(*a*) × 5]
891	19	4½	=	345	,,	

An improved arrangement of this method of working is as follows—

	A. £	*s.*	*d.*				B. £	*s.*	*d.*
(1)	2	11	8½	×	5	=	12	18	6½
(10)	25	17	1	×	4	=	103	8	4
(100)	258	10	10	×	3	=	775	12	6
					345		891	19	4½

In the method column A is first found by continued multiplication by 10, to any desired multiple. From these by horizontal multiplication the totals in column B are found and added as before. The arrangement is neater and more compact.

The gain in neatness is seen in working an example with a larger multiplier, e.g. £1 9s. $10\frac{1}{2}$d. × 5432.

	£	s.	d.			£	s.	d.
(1)	1	9	$10\frac{1}{2}$	×	2	2	19	9
(10)	14	18	9	×	3	44	16	3
(100)	149	7	6	×	4	597	10	–
(1000)	1493	15	–	×	5	7468	15	–
					5432	8114	1	–

(2) *Method of Reduction.* In this method, as illustrated by the worked example, the working runs extremely smoothly, and mechanical accuracy is easily obtained.

	£ s. d.
	1 9 $10\frac{1}{2}$
	5432
($\frac{1}{2}$d. × 5432)	2716 *d.* (A)
(10d. × 5432)	54320 *d.*
	12)57036 *d.*
	4753 *s.* **0** pence (B)
(9s. × 5432)	48888 *s.*
	2,0)5364,1 *s.*
	2682 £. and **1s.** (C)
(£1 × 5432)	5432 £.
	8114 £. (D)
	Ans. £8114 1 –

This method is often found in use in higher classes. The actual "working" is sometimes done in the margin and the answer only is shown, as in the example.

£	s.	d.
1	9	$10\frac{1}{2}$
		5432
8114	1	–

When it is thoroughly grasped, the writer has found this method to be more rapid than any other.

The worked example explains the method. The steps are—

(i) Multiply the farthings and reduce to pence (A).
(ii) ,, pence ,, ,, shillings (B).
(iii) ,, shillings ,, ,, pounds (C).
(iv) ,, pounds to complete the sum (D).

The method is compact, easily taught and understood, and by it multiplications are worked quickly and easily.

(3) *The Method of "Denominational Units."* This interesting method deserves very careful consideration by teachers. Its existence is due, I believe, to Mr. R. Hargreaves, and it is illustrated by him in his text-book of Arithmetic.[1]

Type £2 11s. $8\frac{1}{2}$d. × 345 (see page 96). The working is set out as follows—

	A. £	A. s.	A. d.	£	s.	d.
				2	11	$8\frac{1}{2}$
				345		
				690		
(345 at 1s.)	17	5	–	189	15	–
(345 at 1d.)	1	8	9	11	10	–
(345 at $\frac{1}{4}$d.)		7	$2\frac{1}{4}$		14	$4\frac{1}{2}$
				891	19	$4\frac{1}{2}$

Step 1. Multiply £2 × 345 (= £690).

Step 2. Write down in column A the value of **1s. × 345,** i.e. £17 5s.; then by division by 12, find the value of 1d. × 345, i.e. £1 8s. 9d.; then by further division by 4, find the value of $\frac{1}{4}$d. × 345, i.e. 7s. $2\frac{1}{4}$d.

These are the *denominational units.*

The calculation is completed by £17 5s. × 11, £1 8s. 9d. × 8, 7s. $2\frac{1}{4}$d. × 2, and addition of these in the ordinary way.

This method should be compared with the next and last general method.

(4) *The Method of Practice.* Type £2 11s. $8\frac{1}{2}$d. × 345 (see above).

[1] *Arithmetic,* by R. Hargreaves, M.A. (C.U.P.)

The working is usually set out as follows—

	£	s.	d.	
	345	–	–	(Cost at £1 ea.)
			2	
	690	–	–	(,, £2 ,,)
10s. = $\frac{1}{2}$ of £1	172	10	–	(,, 10s. ,,)
1s. = $\frac{1}{10}$ of 10s.	17	5	–	(,, 1s. ,,)
6d. = $\frac{1}{2}$ of 1s.	8	12	6	(,, 6d. ,,)
2d. = $\frac{1}{3}$ of 6d.	2	17	6	(,, 2d. ,,)
$\frac{1}{2}$d. = $\frac{1}{4}$ of 2d.		14	4$\frac{1}{2}$	(,, $\frac{1}{2}$d. ,,)
	891	19	4$\frac{1}{2}$	

The work might have been shortened by taking 1s. 8d. as $\frac{1}{6}$ of 10s., but this would have involved a final division of $\frac{1}{2}$d. = $\frac{1}{40}$ of 1s. 8d.

This method of Practice depends to some extent upon a knowledge of fractions and a recognition of "aliquot" parts. Hence in the past it has been customary to defer this useful method of multiplication until some acquaintance with fractions can be assumed. "Practice" in all its varieties will be fully discussed in a later chapter.

All the above interesting, ingenious, and useful methods for multiplying a sum of money have been successfully taught, and the whole subject forms a characteristic feature of English arithmetic. Steadily, however, there has arisen a more modern point of view, that a great deal of unnecessary labour has been expended in this branch of the subject. It is recognized that complex multiplications of awkward sums of money with huge multipliers do not commonly occur in ordinary life. In any particular business where they *do* occur, a ready reckoner would certainly be used. Thus the time spent in the multiplication of a sum involving all four units—pounds, shillings, pence, and farthings—has been lessened, and multipliers have been made more reasonable in size. The only defence of such a calculation as

£157 18s. 10$\frac{1}{2}$d. × 56789

is that it forms a severe test of the child's power of concentration, as well as of his speed and accuracy in calculation. If we turn from the severely formal methods outlined above to the money transactions of ordinary life we at once notice that it is the multiplication of *small sums* of money which most frequently occur, and consequently it is to *these* that the child's attention should be constantly directed.

Division of Money. Before reading this section the reader is advised to re-read the section on Division of Numbers, for the actual *process* of division never alters.

It is customary to deal with Division of Money in the following stages—

I. Division of money by *number* (the "partitive" aspect of division).

II. Division of money by *money* (the "quotitive" aspect of division).

Thus we may

(1) Divide £1 10s. by 30. Answer 1s.

(2) Divide £1 10s. by 2s. 6d. Answer 12.

Division of Money by Number falls into easy stages—

(*a*) Short division;

(*b*) Factor division;

(*c*) Long division.

We will discuss briefly each of these in turn.

Short Division. Type £2 3s. 4d. ÷ 5. The ordinary setting out is here shown.

	£	s.	d.
5)	2	3	4
		8	8

As in short multiplication of money, all that is necessary is a good knowledge of the tables, and an ability to change quickly pounds to shillings, shillings to pence, and pence to farthings.

The general principle of division is followed here as in all other division, i.e. first divide the group of *highest* denomination (pounds). Convert any pounds remaining into the next lower denomination, i.e. shillings, so as to be able to continue the division. Continue the process as far as directed, i.e. divide to the nearest shilling, or nearest penny, or nearest farthing.

Any division of money may be performed so that *theoretically* there is no remainder, i.e. £1 2s. 6d. ÷ 7 = 3s. $2\frac{4}{7}$d. with no remainder. In practice, however, the answer would be either—

(*a*) 3s. 2d. with remainder 4d.,
or (*b*) 3s. $2\frac{1}{2}$d. with remainder $\frac{1}{2}$d.,

according as the division is worked carried to pence or farthings.

Factor Division. This requires only the successive applications of the process of "short" division. Type £26 15s. 6d. ÷ 21. The work is usually set out as here shown.

	£	*s.*	*d.*
7)	26	15	6
3)	3	16	6
	1	5	6

Remainder after Factor Division of Money. This has been fully discussed in the section on Division of Number. Let us take one more example. Type: Share equally £102 13s. 8d. to the nearest penny among 25 people.

	£	*s.*	*d.*	
5)	102	13	8	
5)	20	10	8	rem. 4d.
	4	2	1	rem. 3d.

Total remainder = (3d. × 5) + 4d. = **1s. 7d.**

For a full explanation of this total remainder the reader should refer to p. 54.

This same example can, of course, be worked so that *theoretically* there is no remainder, i.e.—

	£	*s.*	*d.*
5)	102	13	8
5)	20	10	8·8
	4	2	1·76

Practically, of course, it is impossible to give any person ·76 of 1d.

Factor-division of money provides an interesting type of arithmetical example, but possibly its whole importance out of school has been considerably over-rated. It is a process not frequently used in adult life.

Long Division of Money. This is the most difficult process of all the money rules for the ordinary child to master. Its operations are numerous and perplexing. It combines all the ordinary difficulties and pitfalls of Long Division of Number with additional complexities due to the presence of different units: pounds, shillings, pence, and farthings.

Following the ordinary process and beginning with the group of largest denomination, we first divide the *pounds*. Any remaining pounds are converted to shillings, and the process continues by division of the shillings. Each remainder in turn is reduced to a lower denomination to enable division to be continued.

Type £87 12s. 8d. ÷ 39.

The working is usually set out in one of the following ways—

(*a*)

```
      £   s.  d.
  39)87  12   8(2
     78
     —
      9
      20
     ——
  39)192(4
     156
     ——
      36
       12
     ——
   39)440(11
      429
      ——
       11
        4
       —
    39)44(¼
       39
       —
        5
       ══
```

Ans. **£2 4s. $11\frac{1}{4}$d. rem. 5f.**

(*b*)

```
        £   s.  d.
    39)87  12   8(£2
           9
   39)192(4
        36
   39)440(11
        11
    39)44(1
         5
        ══
```

Ans. **£2 4s. $11\frac{1}{4}$d.**

Method (*a*) shows the arrangement in full as it would be worked by a beginner; (*b*) shows the same arrangement condensed by use of the Italian method (see p. 60).

(*c*) The above working may be arranged in much more compact form as follows—

```
       £        s.         d.
       2        4          11¼
     ----------------------------
  39)87        12          8
     78    ┌→ 180     ┌→  432
     --    │   ---    │   ---
      9 ←──┘   192    │   440    ┌→ 44
               156    │   429    │  39
               ---    │   ---    │  --
                36 ←──┘    11 ←──┘   5
                                    ==
```

The general complexity of the process of Long Division of Money is now recognized, and more and more teachers are learning from experience the wisdom of not attempting to teach this subject too soon.

Division of Money by Money. As soon as division is applied to concrete quantities its two aspects commonly known as "partition" and "quotition" or "sharing" and "measuring" result in two types of calculations.

Division by *number*, fully discussed above, corresponds to the "partition" aspect. In such division, e.g. £4 6s. 8d. ÷ 10, the calculation could be worded: "Divide £4 6s. 8d. into 10 equal *parts*," or "Find a *tenth part* of £4 6s. 8d."

Division by *money* corresponds to the *quotition* aspect. In such a calculation, e.g. £3 2s. 6d. ÷ 12s. 6d., the calculation could be worded: "*How many times* can 12s. 6d. be taken from £3 2s. 6d.?" or "*How many times* is 12s. 6d. *contained in* £3 2s. 6d.?" This type of division of money, requiring the division of one sum of money by another, is sometimes termed "concrete division."

The one essential point in the method is that *no division* *is possible* until each sum of money is expressed in terms of the same unit.

The *reduction* method is usually the one first taught.

Type: "How many yards at 2s. 11d. per yd. can be bought for £2 6s. 8d.?"

Here the child must learn that both sums must be expressed in *pence* before division can take place. The work is usually set out as follows—

s.	*d.*	£	*s.*	*d*
2	11	)2	6	8
12		20		
35		46		
		12		

35)560(16
35
210
210

Ans. **16 yds.**

The method gives plenty of scope for choice of the most suitable unit to which to reduce both quantities; thus, in the example £78 3s. 9d. ÷ £5 4s. 3d., it will suffice to reduce each sum to *threepences* and not *pence*. Similarly in £25 ÷ 12s. 6d. the child should reduce each to half-crowns. If the right unit is not thus chosen the calculation is needlessly increased.

The *remainder* in this concrete division sometimes proves a difficulty to the child.

A typical example is: "How many people can have 10s. 6d. each out of £25, and what sum will be left?"

Reducing both sums of money to *sixpences* the actual division to be performed is 1000 ÷ 21.

Using factors—

7)1000
3)142, rem. 6
47, rem. 1

} Total rem. 13

The *value* of the remainder (13) always depends upon the units, i.e. *sixpences*, in which the division is performed.

Hence the answer above is "47 people, and 13 *sixpences* or 6s. 6d. left."

Concrete division as set out above and worked by reduction methods is a common and useful type of calculation, but it is as well to remember that later in the child's life many of the examples will be solved much more easily by fractional methods. Thus the example £3 2s. 6d. ÷ 12s. 6d. is simply worked as £$3\frac{1}{8}$ ÷ £$\frac{5}{8}$, i.e. $\frac{25}{8} \times \frac{}{5}$.

Finally, teachers should put all examples of concrete division in as concrete and real a form as possible. It is hardly fair to leave the child in doubt as to whether the required answer is *number* or *money*. Thus, even in mechanical work for speed and accuracy the form "*How many times* is 17s. 6d. contained in £105?" is preferable to the bald direction "Divide £105 by 17s. 6d."

Bills and Invoices. These have long been a traditional feature of the arithmetic of elementary schools. In the past the term usually employed was "Bills of Parcels," though some of the "parcels" dealt with under this heading would have been of most unwieldy size.

The general tendency has been to restrict these too often to the ordinary accounts between customers and retail shopkeepers. They may be extended, with advantage, to many other types of "bills." "Bills such as the parent may be expected to receive or make out will be useful" (Circular 807). Now the ordinary householder in the course of a year receives bills of many varieties. In addition to the ordinary tradesman's accounts there are gas bills and coal bills, plumber's bills and paperhanger's bills, not to mention lawyer's and doctor's bills. All of these, except perhaps the last, give sufficient details to form a sensible and reasonable sum. The wise teacher, then, will not restrict bills to the perpetual butcher-baker-grocer-draper type of example, but will extend their range to include all ordinary examples usually met with.

The direction " Make out with suitable heading and receipt " sometimes accompanies the list of items. This exercise is useful as an exercise in *commercial English*, but as *arithmetic* it is frequently a waste of valuable time for bill-headings are usually printed and in most busy establishments the bill is stamped " PAID." The child who is made continually to draw out an elaborate form with pen and ink, and even to write " Received with thanks " over a carefully ruled facsimile of a postage stamp, is wasting energy misguided by a thoughtless pedant. If such commercial reality *must* be introduced much time may be saved by obtaining a set of ordinary account-slips or bill-heads from an enterprising tradesman (who will no doubt scent advertising possibilities), or, failing this, a suitable form may be " graphed " or otherwise duplicated.

More important, however, than this showy, if fictitious, reality, is the " checking " of bills received. Happy is the householder whose bills are always correct when received. All such recipients should acquire the habit of checking bills as received, and since tradesmen are as liable to error as other human beings, mistakes are sometimes found.

Let the arithmetical exercise in bills then sometimes take the following form. " Find the mistake or mistakes in the following bill as received and give the correct total."

Clearly the work is not lessened, for each item must be tested in turn.

CHAPTER VII

ENGLISH WEIGHTS AND MEASURES

THE teaching of English weights and measures has always been the most formidable task in the arithmetic of English schools. Older teachers will recall the time when all the "tables" of weights and measures had to be committed to memory and formal calculations in reduction, addition, subtraction, multiplication, and division had to be taught for every "table," and all this within the space of one year.

Much of this mechanical drudgery has disappeared, though remnants still persist in unexpected corners of our schools. On the whole, however, the change in the teacher's attitude towards this section of arithmetic is one of the most hopeful signs of the spread of common-sense and practical methods in arithmetic. "Tables" are still taught but the *approach* is different, and everywhere, alike in boys' schools and in girls' schools, we see honest attempts to teach the *units* themselves as realities through *actual use* and not merely as a string of words and numbers in a formal table. This older method of "tables and sums" was liable to the common fallacy of pedagogy: the teacher was apt to mistake *words* for *realities*. Much lip-service was rendered to the tables of weights and measures; much patient labour was expended in the formal manipulation of numbers representing these quantities; but very little time was devoted to the study of the units themselves and of their actual applications.

Practical Arithmetic. To-day, concrete methods and "practical" arithmetic are almost universal. This term "practical" is used in arithmetical language with so many significations that some analysis of its meanings may be found useful.

In the widest sense of the word all arithmetic which is directly connected with life outside the school is *practical.* Thus, sums involving the ordinary calculations of buying and selling may be intensely practical, though no objects or apparatus be employed. For the same reason we should condemn such an example as: "Find the cost of 12 tons 13 cwt. 2 qrs. 14 lbs. 12 oz. 9 dr. at £3 4s. 6d. per cwt." as unpractical on the ground that such a calculation would never occur in real life. This simple test: "Is such a calculation of common occurrence in ordinary life?" is one which should be constantly applied by teachers. It will frequently prevent them from setting examples which are out of touch with the larger world.

The word "practical," is also applied in a more definite sense. It is used to designate all methods now used "to give children definite ideas of the various units and quantities which occur in arithmetical calculations."[1] Thus we teach "money" through cardboard coins and "shops"; we teach inches, feet and yards through actual measurement; we investigate square inches and square feet, cubic inches and cubic feet, through actual construction and use; we "weigh" and "measure" with actual weights and actual measures: all of which work is intensely practical in the sense of being concrete, and dealing with realities instead of names.

Again, we apply the term practical to the use of objects, apparatus, graphical and pictorial illustration, as aids to the explanation and development of rules and processes of calculation. Thus we have practical methods of drawing, paper-cutting and folding, for teaching the meaning and manipulation of fractions; for teaching the meaning of the simpler decimal places; for developing most of the "mensuration" rules, and for "proving" many useful formulae. Most graphical and "squared paper" work is practical in this sense though some of it

[1] Circular 807, "Suggestions."

can hardly be termed practical, using the word in its first and general sense. This is a confusion commonly made; the mere use of pencil and ruler and lines and curves and diagrams in contrast to ordinary "number" calculations does not of necessity always deserve to be termed *practical.*

Finally, the term "practical," in connection with arithmetic and other branches of mathematics, is employed in a very definite sense when applied to their *definite practical applications.* "Practical" used with this meaning might well be replaced by *applied.* In this sense all so-called "mensuration" is very definitely practical *when applied* but very strongly unpractical, artificial, and ineffective when taught in the older way by the memorizing of rules, and the working of calculations the data for which is supplied by teacher or text-book.

The All-practical or Figureless Example. It is on this question of "data" that the whole matter of the really *practical* example turns. In actual life, the *data*—the figures which form the basis of the calculation—have usually to be *found* before the sum can be worked. In the schoolroom they are usually *supplied.* If the work of the child then, especially in mensuration, is really practical, he must constantly be obliged to collect his own data. The teacher must regularly set "sums without figures," as well as the usual sum with all necessary data supplied. It is the old, old difference—too often lost sight of—between knowing *how* to do a thing and *doing* it. One simple illustration will suffice: A child may be quite proficient in finding the area of a triangle from any given dimensions, but may fail when requested to find the area of a definite triangle, i.e. a triangular object, or a triangle drawn on paper, no dimensions of which are given, all having to be found by actual measurement.

These *figureless* sums or *all-practical* examples may take very many forms. The extent to which they are used depends upon the enthusiasm and ingenuity of the teacher. It is to be feared that the teacher will always exist who is content to grind daily at the monotonous arithmetical mill, feeding the class with an endless string of figures and quantities and being satisfied with a reasonable percentage of accurate answers. This is certainly the easier and lazier method ; for the all-practical method is largely an individual method where each child, or a very small group of children, works at a separate quest. The actual form of the quest will vary according to school and class. With older scholars " a group of three or four children may with advantage be set to carry out a detailed survey of the playground, or to calculate from their own measurements the number of bricks in some piece of brickwork and the cost thereof at a given rate, or to calculate from their own measurements the cost of papering a room with paper of a given breadth, or to ascertain from their own data the weight of rain which falls on a given area on some particular day. Any task of this kind may afford occupation for several arithmetic lessons of the most valuable kind."[1]

Most teachers will have already prepared their own all-practical examples, but for the sake of the younger teachers we print in an appendix to this book, by the courtesy of the Superintendent, Mr. J. D. Johnstone, a set of examples in actual use at Werrington Industrial School, Stoke-on-Trent.

The History of English Weights and Measures. A knowledge of syllabuses and methods at present in common use in our elementary schools reveals to us the excellent developments that have been made within a generation in the teaching of English weights and measures. Much mechanical teaching of course still persists, and much aimless reduction through the whole length and breadth of a " table " is still the introduction of many children to what ought to be a real and fascinating

[1] " Suggestions for the Teaching of Arithmetic " (Circular 807).

part of arithmetic. We should not forget that English weights and measures are essentially part of our history. It is customary to rail at their complexities and to contrast them most unfavourably with the smooth fluency of a decimal system, but a fuller knowledge of the subject shows us that behind all the apparent absurdities, inconsistencies, and incongruities of our weights and measures lies a long and wonderful history. We can lighten the labour of memorizing and using these arbitrary units to a great extent if we will but make use of this history. In our anxiety to teach "tables" and "sums" we are apt to lose sight of this excellent aid. The enthusiastic teacher will frequently find that an incidental remark on the history of a unit or a symbol such as "hundredweight" will fix a fact in a way that no humdrum repetition can ever do.

Such opportunities are too numerous to mention. Only a few hints can be given here. Let the teacher, who is sceptical of the possibility of thus connecting history with arithmetic, test his own ability to answer the following questions—

(*a*) How did the custom first arise of using *Latin* terms and symbols £ s. d. for "pounds," "shillings," and "pence"?

(*b*) Has the "penny" always been a copper coin?

(*c*) Is it an accident that the same word "pound" is used for both money and weight?

(*d*) Why is the term "sovereign" used for the gold coin £1?

(*e*) When were guineas last minted?

(*f*) Why is 12 such a favourite number in our tables?

(*g*) When had the term "pennyweight" a literal meaning?

(*h*) Does the derivation of the word "inch" suggest its origin?

(*i*) Chaucer uses "yarde" for "stick." What is the connection?

(*j*) What is the historical connection between the furlong and the acre ?

(*k*) Has the English " mile " any historical connection with the Roman " mille " ?

(*l*) How has the custom arisen of having one " mile " for land and another for sea ?

(*m*) How is it that the awkward length, $5\frac{1}{2}$ yards, became a recognized unit and was termed the " rod," " pole," or " perch " ?

(*n*) How is it that the *hundred*weight is written " cwt.," and yet contains more than 100 lbs. ?

(*o*) When did the term " avoirdupois " first come into common use in England ?

Teachers who are interested in these and similar questions should add to their professional library some such little book as *British Weights and Measures*, by Colonel Sir C. M. Watson, R.E. (John Murray). Knowledge of the above kind imparted gradually and quietly without any formal fuss of collective instruction will add a new interest to a subject always difficult for children. In addition to the interest, the mental attitude in thus linking present with past and in noting a definite continuity between the customs and units employed yesterday and to-day, is in itself valuable.

In connection with exercises on time and the calendar, special opportunities for this historical treatment arise. The facts usually taught by arbitrary rhyme—

> Thirty days hath September,
> April, June, and November.
> All the rest have thirty-one,
> Excepting February alone.

cover in themselves a wonderful story. The history of the origin of our present calendar could hardly be exhausted in several lessons, yet the main lines of the development are straightforward enough for children to understand and appreciate. The information will be found in any good encyclopaedia.

Having discussed the possibilities of these two methods of approach to our English mixture of weights and measures, (*a*) the practical, (*b*) the historical, we now proceed to discuss the teaching of each table in some detail.

Length

Take barley corns of moderate length
And three you'll find will make an inch.
Twelve inches make a foot—if strength
Permits, I'll leap it and not flinch.

Three feet's a yard, and understood
By those possessed with sense and soul;
Five yards and half will make a rood,
And also will a perch or pole.

Forty such poles a furlong make,
And eight such furlongs make a mile,
O'er hedge, or ditch, or seas, or lake;
O'er railing, fence, or gate or stile.

Three miles a league by sea or land,
And twenty leagues are one degree;
Just four times ninety degrees a band
Will make to girt the earth and sea.

But what's the girt of hell or heaven?
(No natural thought or eye can see)
To neither girt nor length is given;
'Tis without space—Immensity!!!

Still shall the good and truly wise
The seat of heaven with safety find,
Because 'tis seen with inward eyes,
The state resides within their mind.

Thus sang the pious infants of Mr. James Carroll at the Stratford Infant School, Bow, London, E., less than a hundred years ago.[1] No doubt these same infants proceeded, at a suitable age, when the "table" was firmly fixed in their memory, to "reduce" millions of inches to miles, furlongs, poles, yards, feet, inches, or even to leagues, etc. Such was the traditional

[1] Wilderspin: *Infant Education*; Appendix to Third Edition.

method: first, the memorizing of the "table," then mechanical figuring involving reduction, addition, subtraction, etc. Trained teachers were few; apparatus scanty; class rooms almost unknown; and so "learning by heart" predominated. But the modern teacher begins differently.

Feet and Inches. As soon as the properties of the number "12" are grasped, they may be applied to feet and inches. The fundamental fact: 12 inches = 1 foot is now taught through the use of the ordinary *ruler.* The earliest exercises involve the measurement of lines and edges to the nearest inch and the drawing of lines of specified length. The ordinary ruler contains far too many graduations for a beginner, but this difficulty can be overcome by constructing simple rulers or scales showing inches only, or inches, halves, and quarters. These simple scales in stiff paper or cardboard may well be made by an older class for use with the younger children.

Measurement in ordinary life is always for a definite purpose, generally to obtain data for calculations as to quantities and costs. This fact must always be remembered by the teacher. The actual practical work should as far as possible be accompanied by some calculation however simple, e.g. "Here is a candle. If it burns 1 inch every hour how long will it last?" This combination of practical work and calculation is absolutely necessary if the measuring, etc., is to maintain its interest. Aimless, pointless measuring of books and other objects soon leads to a lack of zest.

Once the class has become somewhat familiar with actual inches and feet, practice in *eye-estimation* should be constantly given. The importance of such practice cannot be over-estimated; the habit is of constant use in after life. Ability to estimate at *sight* small or great distances is largely a matter of practice for persons of ordinary eyesight. The work may be collective or individual, and usually takes the following form: "Write down what you *think* is the length of this (any convenient object). Check by measurement."

Yards. As soon as feet are known as realities the extension of the knowledge to yards is easy. Actual measurement may be continued by means of the ordinary "tape measure" used in needlework or even by string knotted at intervals of a foot or yard.

To aid in the eye-estimation of yards, many schools have a yard-length painted on blackboard or wall in some conspicuous position to which ready reference may be made.

The draper, it should be noted, seldom uses feet and inches, but measures and sells by the yard, half yard, and quarter yard. This may serve as the basis of useful "shopping" sums, especially in girls' classes.

As soon as inches, feet, and yards are understood the learner has a groundwork for the simplest elements of *scale-drawing*, i.e. he may begin with such problems as—

(*a*) If 1 inch represents 1 foot, draw a line to represent 1 yard.

(*b*) This line (given) represents 5 feet. Find what 1 inch represents on the same scale.

These are among the simplest exercises. Later the work may be extended to the ordinary method of working, i.e. actual measurement, freehand dimensioned sketch, accurate reproduction to given scale.

Chains, Furlongs, Miles. With large units the difficulties of practical work are many. Chains, furlongs, miles, are usually taught before the child is capable of doing much work involving the use of these units out of doors. Thus the teacher is compelled to resort to descriptive and other methods in teaching these units.

The *Furlong.* Historically this provides the best starting point. Its derivation from the ancient art of ploughing—furlong = furrow-long—is well known, and usually interests even town children. Less well known is the fact that in Early and Mediaeval England the *furlong* was the "*acre's length*," while the corresponding "acre's breadth" was one-tenth of the length. If we take the acre's length or furlong as 220 yards, we see that the acre's breadth is 22 yards, and corresponds to the modern *chain*. This provides the best means of teaching furlong, chain, and acre. A simple diagram such as that given in Fig. 17 will form a useful image for all three units.

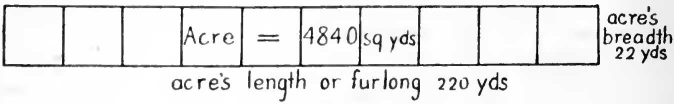


FIG. 17.

From the above simple diagram many facts are apparent, such as—

10 chains = 1 furlong
10 sq. chains = 1 acre
= (22 × 220) sq. yds.
= 4,840 sq. yds.
1 sq. chain = 484 sq. yds.

The *Chain*. This modern name for the old acre's breadth may also be taught independently as the length of an ordinary cricket-pitch, from wicket to wicket. In classes of older children the actual *chain* of 100 links may be used in playground or field, and the convenience of *links* in calculating areas may be discovered in actual practice.

Rod, Pole, Perch. These are not essential in the table of length, which can well proceed directly from yards to chains. Where the *rod* is taught, however, it is important to place it in its right historical setting; "$5\frac{1}{2}$" seems a very awkward and arbitrary number until we note that it is arrived at in a very simple way, for it is *one quarter* of the old acre's breadth, viz., 22 yards ÷ 4. This is a reasonable explanation of its probable evolution, since a pole $5\frac{1}{2}$ yards long is probably the longest one that could be conveniently handled.

The *Mile.* This corresponds very nearly to the Roman mile of 1,000 (double-step) paces. The actual difference forms itself the basis of an interesting calculation. The actual statutory length of 1,760 yards is usually well known by children. They are, however, not so ready with its equivalent of 5,280 feet, though numerous calculations require this.

The actual "mile" is somewhat large for a child to visualize, yet many aids are ready to hand. Thus, most children have seen the familiar mile posts and the older milestones on our main roads. Of great use, too, is the time-measurement of longer distances. Thus, a mile is a quarter-of-an-hour's walk for an adult, or a twenty minute's walk for a boy.

With the mile, the *half-mile* and the *quarter-mile* must be taught as actual realities.

Additional Units. The above units and their connection constitute the table of length as usually taught. A few additional measures may be mentioned.

The *Knot* or *Nautical* Mile. This is fixed by statute at 6,080 feet. Children need not commit this number to memory. It should be supplied when required. The ratio of the *knot* to the ordinary mile is readily approximated for 6080 : 5280 is roughly 8 : 7. This provides a ready rule for converting knots to miles and miles to knots—

Knots to miles : Add on $\frac{1}{7}$; thus 35 knots = 40 miles approx.
Miles to knots : Deduct $\frac{1}{8}$: thus 24 miles = 21 knots.

The *Fathom* of 6 feet is learned without any difficulty. It is approximately the full arm stretch of a tall man.

Formal Calculations in Length. While, ideally, all sums should be preceded by actual measurement and the collection of the data for the example yet the care necessary to ensure correctness of data consumes a great deal of time, and so the teacher has to amplify the calculations arising out of the actual practical work by setting numerous exercises—ranging from mechanical sums to problems.

It has long been the habit to fix every table by exercises in reduction. Generally, the examples set have been very mechanical and frequently most clumsy. The following are typical examples of what still prevails in schools (and even in examinations conducted by well-known examining bodies)—

"Reduce 98765432 inches to miles, etc."

"Reduce 5 miles 7 fur. 6 ch. 21 yds. 2 ft. 11 in. to inches."

Yet the conviction that much reduction work, such as the above, has no *practical* application whatever, arises but slowly, for teachers in the mass are among the most conservative of people. Let the reader, who is already bristling with deep indignation and ready to defend reduction, ask himself (or herself) the simple question: "What trade or profession ever uses customarily the *whole* of the table of length?" Engineers work with feet and inches; drapers use yards and fractions of a yard; carpenters and builders need to be proficient in inches, feet and yards; land surveyors measure in yards and chains, and calculate in acres and square miles, while landowners may need yards, poles, and chains. But not one of them habitually uses the whole of the table. We must then find some other defence for the custom of teaching children to reduce millions of inches to miles, etc. The usual defence is that such examples in rules and tables are useful to cultivate speed and accuracy. To this we may answer that speed and accuracy may be cultivated by more useful types of mechanical sums such as the much maligned tots. We could also answer that it has not yet been satisfactorily proved that the accuracy which reduction cultivates is to any extent *transferable*.

It does not follow that the child who is accurate in reducing miles to inches is of necessity accurate in his applications of the method of unity; much less does it follow that the same child must of necessity be accurate in spelling or grammar. We must then conclude regretfully that this general abstract virtue of accuracy requires some qualification. We *may* and *can* form the habit of accuracy in this or that particular subject, but as for Accuracy as a quality separated from any definite subject, of this we are more doubtful. It is now necessary for the defenders of long reduction sums to prove the usefulness not of Accuracy but of accuracy-in-doing-reduction. "But," the persistent objector rejoins, "if they do nothing else, they teach 'concentration.'" The voice sounds harsh and grim: "concentration" is an uncomfortable virtue that some teachers inculcate with much keen relish. They consider it their bounden duty to frighten and force a class to "concentrate" upon some soulless task of drudgery. To these we must point out that the value of concentration like that of accuracy depends entirely upon the subject of concentration. One last defence still remains: "But *we* were taught reduction when *we* went to school." To such argument, reply is needless. We (or our forefathers) were once taught that the earth was the centre of the universe, but we do not now believe that. Yet, we fear, this last type of argument is responsible for much of the persistence of obsolete types of examples throughout arithmetic. Thus, text-books, otherwise excellent, will continue to include long reduction sums, and particularly so long as examiners and examining bodies, otherwise admirable, include year by year such examples in their "tests," though what department of mathematical intelligence they thus hope to test we are unable to discover.

"Then you would not teach 'reduction' at all?" has been the innocent question sometimes put to the writer after he has expounded the above views (usually with some heat) to an obstinate teacher who refuses (or fears) to depart from tradition. Wrong again! Reduction *must* be taught, but it should be of the sensible variety. The natural reduction is from any *one* unit to that above or below it in the table. Thus plenty of practice can be given in changing feet to yards, yards to feet; inches to feet, feet to inches; miles to furlongs, chains to furlongs, etc., but inches to leagues—never. So, too, it may be necessary at times to proceed *direct* from miles to feet or from feet to miles, but this may usually be performed by the use of the number 5280 and not by the use of the whole table as shown in the traditional method below.

Since, however, we cannot at a bound escape from the necessity of having to teach the old long reduction we must endeavour to arrive at the best method of "setting-out" the calculation.

Type: Change 100,000 inches to miles, etc. The work proceeds by division, using factors where possible, and is usually set out as here shown.

```
          Inches.
      12)100000
          ------
        3)8333 ft. 4 in.
          -----
    ( 11)2777 yds. 2 ft.
 22{      ----
    (  2)252 and 5 yds.
          ---
      10)126 chains
          ---
        8)12 f. 6 ch.
          --
           1 m. 4 f.
```

Ans. **1 m. 4 f. 6 ch. 5 yds. 2 ft. 4 in.**

Type: Reduce 2 m. 3 f. 7 ch. 14 yds. 2 ft. 7 in. to inches.

(*a*) *Traditional* Method—

```
2 m. 3 f. 7 ch. 14 yds. 2 ft. 7 in.
   8
  ——
  19 f.
  10
 ———
 197 ch.
  22
 ———
 394
394
  14
————
4348 yds.
   3
—————
13046 ft.
   12
——————
156559 in.
══════
```

(*b*) *Traditional*, shortened by omission of multipliers—

```
2 m. 3 f. 7 ch. 14 yds. 2 ft. 7 in.
    19 f.
   197 ch.
   394
  394
    14
  ————
  4348 yds.
 13046 ft.
156559 in.
══════
```

(*c*) *Alternative* arrangement—

```
 m.         f.        ch.        yds.          ft.            in.
  2          3          7          14            2              7
 16 <——> 16     |—> 190    |—> 394     |—> 13044    |—> 156552
             ——   |     ———   |   394     |    —————    |    ——————
             19 <—|     197 <—|   ———     |    13046 <—|    156559
                                  4348 <——|                 ══════
```

(*d*) Alternative or " Multiple " Method—

1 mile =	5280 ft.		2 miles =	10560 ft.
1 f. =	660 ft.		3 f. =	1980 ft.
1 ch. =	66 ft.		7 ch. =	462 ft.
1 yd. =	3 ft.		14 yds. =	42 ft.
			2 ft. =	2 ft.
				13046

13046 ft. 7 in. = 156559 in.

This neat arrangement is always possible in all reduction sums. We work a simpler example as an additional illustration.

Express 9 yds. 2 ft. 7 in. in inches. Our work may be concisely arranged thus:

36 × 9	324
12 × 2	24
7	7
	355.

[This particular example may, of course, be worked as 10 yds. – 5 in., i.e. (360 – 5) inches.]

Similar common-sense methods should always be encouraged. Thus the example: A gun has a range of 12,000 yds. Express this in miles, etc., should *not* be treated *formally* by continuous division by $5\frac{1}{2}$, 40, 8 or by 22, 10, 8, but by some method such as the following—

	yds.	
	12000	
1760	10560	= 6 miles
	1440	
220	1320	= 6 fur.
	120	

Ans. **6 miles 6 fur. 120 yds.**
or **$6\frac{3}{4}$ miles nearly.**

"*Reduction*" *of Poles.* The conversion of poles to yards or yards to poles involves the awkward multiplier or divisor "$5\frac{1}{2}$." The calculation is seldom required except in examinations, and is treated much more simply by fractional methods. Formerly it was included in reduction, and many readers will recall the time and trouble of learning to apply the "rule-of-thumb," i.e. "multiply by 11 and divide by 2," or "multiply by 2 and divide by 11." This awkward work may, of course, be avoided altogether by proceeding through the "chain."

Finally, it should be noted that longer distances in practice are nearly always given in miles and fractions or decimals of a mile.

In schools where reduction of length is considered of sufficient importance to receive systematic treatment, a table of equivalents such as the following will be found very useful in the ordinary type of reduction example.

	Miles.	Furlgs.	Chains.	Yards.	Feet.	Inches.
Miles	1	8	80	1760	5280	63360
Furlgs.		1	10	220	660	7920
Chains			1	22	66	792
Yards				1	3	36
Feet					1	12

The construction of this table forms an excellent exercise for every child. Once constructed it can be used as a ready reckoner.

Ex. (*a*) Reduce 4 m. 5 f. 7 ch. 15 yds. to yds. Using the table, the work may be arranged as follows—

	Yds.
1760 × 4	7040
220 × 5	1100
22 × 7	154
	15
	8309

(*b*) Reduce 2 m. 3 f. 4 ch. 5 yds. 2 ft. 6 in. to inches.

	Inches.
63360 × 2	126720
7920 × 3	23760
792 × 4	3168
36 × 5	180
12 × 2	24
	6
	153858

The Four Rules in "Length." In everyday life lengths, after being measured, have frequently to be added, subtracted, multiplied, and divided.

Ideally, the child's methods of performing these operations should grow naturally out of his practical work, and out of the data thus collected. Thus such a simple problem as: "Find the distance round the class room" (rectangular) will need two operations, simple addition of lengths and simple multiplication by two.

In practice, however, many harassed and anxious teachers fear to rely solely upon this and return to the older method of mechanical "drill" in "Length." Thus we may still see much formal addition, etc., of lengths. Methods have, of course, been evolved similar to those employed in dealing with money, but far too many of the examples are long, mechanical, and pointless, and consequently dull and tedious to a child. Such an example as—

Multiply 2 m. 3 f. 4 ch. 12 yds. 2 ft. 7 in. by 29

has no real application, and need seldom be set. In this work as in all other arithmetical work common-sense is required by both teachers and examiners. In ordinary life lengths are seldom given in more than two or three units. This then should be the basis of our formal "drill." We may add, subtract, multiply and divide, using miles, furlongs, chains; or miles, furlongs, poles; or yards, feet, inches, but the whole range from miles to inches has seldom if ever to be manipulated in real life.

Problems in Length. The applications of Length in ordinary life are so numerous and ready to hand that no calculation need ever be set in other than concrete form. Clothes, furnishings, walls, floors, gardens, roads, fields, every form of locomotion, of necessity involve some of the length units. All that is necessary is that the problems and calculations set shall be real and of *ordinary* *occurrence.* The teacher should compare the following two aspects of the *same* calculation.

(*a*) The bald, uninteresting, pointless " Do this " form :

" Divide 1 mile by 7 ft. 4 in."

(*b*) The real, definite, interrogative, " What is this ? " form :

" A bicycle wheel is 88 in. round. How many turns does it make per mile ? "

It is safe to say that the artificial separation of *processes* from *problems* is largely disappearing from the modern teaching of arithmetic, and the disappearance is unregretted. The bad old days of preparation for the common form of test of four sums—three mechanical, one " problem "—with the resultant deadly " grind " and consequent mathematical rigidity and atrophy, have gone—never, we hope, to return.

The difference possible in six short years in a London school is shown very clearly in two tests set out in Appendix IV of a Special Report on *The Teaching of Mathematics in London Public Elementary Schools* (by Dr. P. B. Ballard), issued by the Board of Education in 1911 (Wymans, 2d.). The tests were set to the same standard (IV) in 1904 and in 1910. The differences in the two tests is astonishing. One is rigid, mechanical, and colourless, and is printed in four brief lines. The other occupies twenty-seven lines, and pulsates with life and reality. Many, but perhaps not *all*, other schools, if they could search their records, might discover the same differences.

CHAPTER VIII

ENGLISH WEIGHTS AND MEASURES (Continued)

THE English units of weight have a long and varying history. They take their rise in immemorial custom. They make the usual progress from custom to law, so that again and again in our history legal attempts have been made to fix and modify customary weights and weighing practices, until in the twentieth century we have reached some sort of agreement, and the ordinary "Avoirdupois" Table has become the accepted table with its units standardized by law.

Avoirdupois weight is the *only* system of weights and weighing that the average child need learn, for it is the only system in common use. It is true that problems involving *Troy Weight* and *Apothecaries' Weight* occur in the traditional arithmetical text-books and in some examination papers, but these have no place in an ordinary arithmetic syllabus. They are systems that are used by only a few people in strictly limited walks of life, and hence for the great majority of scholars they are of no practical value. Should an occasional scholar become, in later life, a goldsmith or a chemist, we may be sure that he will quickly acquire the necessary "table." Consequently it is absurd to teach these extra tables of weight with the excuse that they *might* be needed. Time and energy thus saved is valuable in the more important sections of the arithmetic syllabus.

In dealing with the units of weight we must follow the same general plan that we have outlined for the treatment of the table of *length*. Thus we shall not begin in the old way of causing the whole table of weight to be first committed to memory as mere words. We shall rather teach the common units as real things. Thus the learner will actually weigh and estimate weight and so obtain a grasp of the fundamental units of weight and of their relations to one another. Incidentally with this practical work, we shall introduce simple calculations in weight involving all four rules and so, finally, by the most natural path we shall reach the stage of ordinary everyday problems involving a sensible combination of money and weight. Nor shall we attempt to teach the whole of this in any one particular year or part of a year, but gradually, step by step, we shall introduce the units and the corresponding simple calculations. We shall begin with the most familiar units : *pounds* and *ounces*, and later shall proceed to *hundredweights* and *tons*. If we are wise, we shall omit *drams*, which are very seldom used, but shall give considerable attention to common fractions of pounds and ounces such as halves and quarters.

Practical Work. As with all weights and measures, practical work in class is limited by the size of the units. The practical teaching of pounds and ounces will give no trouble. Stones, quarters, and hundredweights may also be handled practically, in schools which possess some form of weighing machine. But tons must be left largely to the imagination and out-of-school observation of the child, for the handling of a ton is impossible in the schoolroom.

The apparatus need not be elaborate ; indeed the simplest form is best for every purpose not requiring minute and scientific accuracy. For *pounds* and *ounces*, some simple form of lever balance may be used. Ideally, each child should have its own simple balance and set of weights. This, however, is not commonly possible in the average class, so the teacher is usually obliged to rely upon one, or at the best, a few balances. The provision of sets of weights gives less difficulty, for these can easily be duplicated in some handy form such as lead or sand, while the actual *making* of such a set of weights is a valuable exercise in itself. The ordinary composition of a " set " of weights should be noted. The child who has noticed that it is not necessary to have a weight for every multiple of an ounce up to 15, and that with weights of 1 oz., 2 oz., 4 oz., 8 oz., it is possible to weigh out any number of ounces up to 15, has made considerable mathematical progress. Later, he may, with profit, examine a set of weights from 1 lb. to 1 cwt., e.g. 1 lb., 2 lb., 4 lb., 7 lb., 14 lb., 28 lb., 56 lb. Interesting class-room problems on these " sets " may be given, e.g. " Which of the above weights would you use to weigh out 40 lbs. of flour ? "

Teachers who are interested in the theory of number will note that weights forming the simple series 1, 2, 4, 8, 16, 32, etc., are sufficient to weigh any given multiple up to 63, etc., using all the weights in *one* scale pan and proceeding by addition only, thus $11 = 1 + 2 + 8$; $23 = 1 + 2 + 4 + 16$, and so on for any number up to 63. If *both* scale pans are used for weights (a very common practice), then the following series is sufficient: 1, 3, 9, 27, etc. Every number up to 40 can be made by the addition or subtraction of some or all of these. Thus, $11 = 9 + 3 - 1$; $23 = 27 - (1 + 3)$, and so on. Hence if the weights were pounds and we wished to weigh out 23 lbs. of flour, we should place the 27-lb. weight in one pan and the 1-lb. and 3-lb. weight in the other, and add the flour to balance.

Later, the child may examine in the same way a set of *metric* or *decimal* weights and discover that their composition is based upon the numbers 1, 2, 2, 5, e.g. 1 gr., 2 gr., 2 gr., 5 gr., 10 gr., 20 gr., 20 gr., 50 gr., and so on for multiples and sub-multiples of a gram. These suffice for any weight within the range of their sum; thus 78 grams might be weighed as $50 + 20 + 5 + 2 + 1$.

The practical work for beginners is usually associated with shopping transactions, and is used with the object of accustoming the children to the actual weights (pounds and ounces). Preliminary estimations should be made wherever possible, though any very great skill in this is but slowly acquired. Solids, especially substances in dry powder form or in very small pieces, are usually best for beginners. Later the teacher may introduce the weighing of liquids, though here it is necessary that the principle of counterpoise should be grasped, for the mechanical difficulties in the weighing of liquids are considerably greater than in dealing with solids. It should, however, be possible for every child to make an attempt, at some period of his school career, to find the weight of a pint of water or of a pint of milk.

Remembering our remarks on the extreme importance of giving all practical work a definite point and purpose, we shall be careful to make our directions as purposive as possible. Thus we might be content to say: "Find the weight of this reading book," and a thoughtful child would be quite correct in questioning the utility of such work. It would be better if we worded our directions in a slightly less peremptory and more attractive form. "A boy was carrying 3 dozen of these reading books. What weight was he carrying?" The actual work is the same; the weighing must still be done; but the approach, and the consequent interest, is very different.

Much valuable work may be done in the estimating of weight and the practice of individual judgment by what may be paradoxically called "weighing without weights." An example will illustrate: A child is provided with a bag containing two pounds of sugar, a simple balance, and *no weights*. He is asked to divide the sugar into four separate half-pounds, first by trial and then using the balance to correct his judgment. The practical work necessary clearly involves halving and balancing first to get pounds, and secondly to get half-pounds. A more difficult problem would be: "Here are two lbs. of sugar; using the balance but without using weights measure out one and a half pounds.

Instances of useful, purposeful, practical weighing will occur to every enthusiastic teacher. Many hints will be found in the list of all-practical examples printed in the Appendix, while more will be found in the author's

Common-sense Arithmetics, especially in Books II, III, and IV. But most teachers will prefer to make their own collection according to the apparatus and needs of the class. One caution only is necessary: The *collective* treatment of these problems, with the teacher doing most of the work and the class largely passive, is the *least* effective method. For better results are obtained by setting the problems to individuals or to small groups. The tasks may be written or typed on cards for quick distribution at any convenient time. Thus, the many odd periods of leisure which fall to the quicker children may be filled up and the children be kept busy and interested.

The Theory of Weight. There is little time in the ordinary school curricula for much formal instruction on the underlying theory of weight. The intricacies of mass and weight; of gravity; of specific gravity and density, cannot be grasped by children under the age of fourteen. It is true that some schools are successful in teaching older scholars to perform practical work and calculations involving specific gravities and densities, but the fundamental notions of gravity and mutual attraction between bodies are hardly ever grasped by young scholars. Place may be found in the later classes for useful lessons on levers as illustrated and applied in the simple balance, and in the steelyard and weighing machine; and the entirely different principle underlying the spring balance may with profit be investigated. To work of this nature the word *science* is usually applied, but if it is work of any real value it must be mainly *practical arithmetic*, and might as well be termed such and included in the arithmetic syllabus.

General Notes on the Table of Weight. Avoirdupois weight is the legal system for weighing *all* articles, though by the Weights and Measures Act of 1878, gold, silver, etc., *may* be (and usually are) sold by the *ounce Troy*, while *drugs* may be sold by apothecaries' weight.

It is a point of interest to notice that the common basis of all three systems is the *grain*, which fact shows the natural origin of our English system in the weight of average grains of corn. This may be compared with the origin of the units of *length*, where the *inch* was formerly the length of three barley corns of average size. Children may be set to find out for themselves the interesting fact that three English pennies together weigh approximately one ounce. It should be noted that the *Troy* ounce and pound are not the same as the *Avoirdupois* ounce and pound, but that the pound and ounce used by grocer and chemist are *exactly* the same.

The *stone* is a very variable measure though, according to the table, it should be fourteen pounds.

The Hundredweight. The history of the extra 12 lbs., i.e. (100 + 12) lbs., to the so-called "hundred weight" (written cwt.) is always interesting to children. It should be noted later in calculation that 112 is a very convenient number for calculation since it contains numerous factors, e.g. 7, 8, 14, 16, 28, 56. In visualizing the cwt. the homely sack of coals or coke is always a convenient illustration, while the connection 20 cwt. = 1 ton is easily made by analogy with 20 shillings = £1.

The Ton. This may be visualized from the ordinary coal cart loaded with 20 cwt. bags. Later it may be connected with volume as being the weight of nearly 36 cu. ft. of water, i.e. a tank 4 ft. × 3 ft. × 3 ft. would hold just over one ton of water. Similarly, one ton of coal in the loose form occupies approximately 40 to 42 cu. ft.

Weight Calculations and Problems. As with every other table, quantities involving units of weight may be added, subtracted, multiplied, divided, and reduced; and some or all of these operations may be combined in weight and money. Thus we have the foundation of a great variety of "sums" from ordinary mechanical drill to the most interesting problems. But the sensible teacher will not devote too much attention to the mechanical manipulation of quantities, though proficiency in this is perhaps the easiest of all to obtain.

Note again as with the table of length that the *whole* table is seldom in use at any one time by any one person. Thus, a coal merchant will talk of, and deal in, tons, cwts., and perhaps quarters of coal, but he would seldom deal with a quantity such as 12 tons 3 cwts. 2 qrs. 17 lb. 6 oz. of coal. Yet similar absurd statements are frequently found in books and examination papers both old and new. The quantities are usually dealt with, in after life, in small groups, e.g. tons and cwts., cwts. and qrs., cwts. and lbs., lbs. and ozs. It follows, then, that reduction through the whole range from drams to tons is seldom required in practice.

Reduction. Much oral practice must be given in proceeding from one unit to the next higher or lower, e.g. changing oz. to lbs. and lbs. to oz.; changing stones or quarters to lbs. and lbs. to stones or quarters; changing cwts. to quarters and quarters to cwts.; changing tons to cwts. and cwts. to tons. These reductions involving one step only are constantly required.

Reduction of Quantities *downwards*, i.e. to lower denominations, may be treated by either of the following methods.

Example: Reduce 2 tons 3 cwts. 3 qrs. 7 lbs. to lbs.

Method (*a*). By continuous multiplication—

```
   T.    c.    q.    lb.
    2    3     3     7
   20
   ——
   43 c.
    4
  ———
  175 q
   28
 ————
 1407
 350
 ————
 4907 lbs.
 ════
```

Method (*b*). This is based upon the Table of Equivalents as here set out.

	t.	c.	qr.	lb.	oz.
t.	1	20	80	2240	35840
c.		1	4	112	1792
qr.			1	28	448
lb.				1	16
oz.					1

Using these, the above example may be set down concisely as follows—

				lbs.
2240	× 2	=		4480
112	× 3	=		336
28	× 3	=		84
				7
				4907

This table of equivalents may be constructed by each child. It will serve for all ordinary reductions from tons to ounces.

Reduction of Quantities *upwards*, i.e. to higher denominations, is usually treated by continuous division through all the steps of the table, thus—

Example: Change 20,000 lbs. to tons, etc.—

```
4)20000      }
  -----      } 8 lbs.
7)5000(2     }
  ----
4)714(2 qrs.
  ---
2,0)17,8(18 cwt.
    ---
    8 tons
```

Ans. 8 tons 18 cwt. 2 qrs. 8 lbs.

This is the conventional treatment of the process. A more rapid treatment is to proceed at once to tons, and to give the answer in tons, or tons and lbs., thus—

```
2240)20000(8
     17920
     -----
      2080
```

Ans. 9 tons nearly,
or 8 tons 2,080 lbs.,
or 9 tons less 160 lbs.

The form of the answer will depend upon that of the question.

Addition and Subtraction of Weight. Examples involving these processes need little explanation. They will be worked by methods similar to those in use for money, length, etc. As always, care should be taken to choose examples as real as possible from occurrences within the child's experience. Thus truck-loads and cart-loads provide plenty of subtractions for we have (*a*) Wt. of Load + Wt. of Vehicle; (*b*) Tare or Wt. of Vehicle, and the actual weight of the load is found by subtraction.

Multiplication and Division of Weight. Any of the methods used for the multiplication and division of money are applicable to multiplication and division in weight. It should be remembered, however, that examples such as: Multiply (or divide) 2 tons 3 cwts. 2 qrs. 17 lbs. 12 ozs. 13 drs. by 37 may give plenty of humdrum figuring practice, but such calculations very seldom occur in actual life.

Concrete Division in Weight. As with similar division in other compound quantities, the basis of the process is reduction. Careful choice of the most convenient unit is the chief requirement.

Example: Iron piping runs 43 lbs. per ft. length. How many feet is this per ton? Here the convenient unit would be 1 lb. and the division would be as follows:—

```
43)2240(52
   215
   ---
    90
    86
    --
     4
    ==
```

Ans. 52 ft. or 52 ft. 1 in. approx. per ton.

More conventional and less real are examples of the type: 2 tons 14 cwts. 2 qrs. 21 lbs. ÷ 1 cwts. 2 qrs. 7 lbs.

Examples in Compound Practice involving Weight will be taken in a later chapter.

Weights and Costs. The combination of weight and cost is of such common occurrence in ordinary life that its importance in ordinary school arithmetic cannot be over-estimated. It is true that the trader usually consults a ready-reckoner if he wishes to determine the price per cwt. from the price per lb., or the price per lb. from the price per cwt., etc., but many simple and useful rules can be taught in schools in this connection. It is here, too, that many short cuts may be systematically practised. We illustrate a few in common use.

To find the cost of a given number of pounds. Example: Find the cost of 35 lbs. at 7d. a lb. Remembering the identity $a \times b = b \times a$, we note that 7d. × 35 = 35d. × 7 = 2s. 11d. × 7 = £1 0s. 5d.

Example: Find the cost of $6\frac{3}{4}$ lbs. at 11d. a lb. As in the previous example, 11d. × $6\frac{3}{4}$ = $6\frac{3}{4}$d. × 11 = **6s. $2\frac{1}{4}$d.**

To find cost per lb. from cost per oz. and vice versa. The links here are: $\frac{1}{4}$d. per oz. = 4d. per lb.; $\frac{3}{4}$d. per oz. = 1s. per lb.

Thus—

3s. per lb.	= $\frac{3}{4}$d. × 3 per oz.,	i.e. $2\frac{1}{4}$d. per oz.
10s. „	= $\frac{3}{4}$d. × 10 „	i.e. $7\frac{1}{2}$d. „
£1 „	= $\frac{3}{4}$d. × 20 „	i.e. 1s. 3d. per oz.

and

$2\frac{1}{2}$d. an oz.	= $\frac{10}{3}$s. per lb.,	i.e. 3s. 4d. per lb.
$4\frac{1}{2}$d. an oz.	= $\frac{18}{3}$s. „	i.e. 6s. per lb.

To proceed from cost per lb. to cost per cwt. and vice versa. Almost every trader does this by his own particular method, but a few hints may be given. Useful data are—

1d. per lb.	=	112pence or	9s. 4d. per cwt.
½d. ,,	=		4s. 8d. ,,
¼d. ,,	=		2s. 4d. ,,

Thus—

1 cwt. at 5d. a lb. costs 9s. 4d. × 5, i.e. £2 6s. 8d.
1 cwt. at 4½d. ,, 4s. 8d. × 9, i.e. £2 2s.
1 cwt. at 1¾d. ,, 2s. 4d. × 7, i.e. 16s. 4d.

Again—

2 cwt. at 2½d. a lb. is the same as
1 cwt. at 5d. a lb., and costs 9s. 4d. × 5 = £2 6s. 8d.

Similarly—

3½ cwt. at 4d. a lb. is the same as
14 cwt. at 1d. a lb., and costs 9s. 4d. × 14 = £6 10s. 8d.

The fact that ¼d. per lb. is 2s. 4d. per cwt. has been made the basis of a good working rule. "To obtain the price of 1 cwt. in shillings multiply the number of farthings per lb. by 2⅓."

Example—

7½d. per lb. = 30 farthings per lb.
30 × 2⅓ = 70

Ans. **70s. per cwt.**

The rule may, of course, be reversed, i.e. "To obtain the price in farthings per lb. take $\frac{3}{7}$ of the number of shillings per cwt.

Example—

21s. per cwt.
21 × $\frac{3}{7}$ = 9, 9 farthings = 2¼d.

Ans. **2¼d. per lb.**

To proceed from price per cwt. to price per ton and vice versa. This is the simplest of all where the price per cwt. is in shillings, for 1s. per cwt. = £1 per ton.

Thus—

1s. 4d. per cwt. = £1 6s. 8d. per ton,

and

35s. or £1¾ per ton = 1s. 9d. per cwt.

This may be extended still further: at £1 per ton, 1 cwt. costs 1s., and 1 quarter costs 3d. Thus 2 tons 4 cwts. 3 qrs. at £1 a ton will cost £2 4s. 9d., hence the sum "Find the cost of 2 tons 4 cwts. 3 qrs. at £3 10s. per ton may be worked very simply as follows—

£	*s.*	*d.*		
2	4	9		(cost at £1 per ton)
		$3\frac{1}{2}$		
6	14	3	=	3 times
1	2	$4\frac{1}{2}$	=	$\frac{1}{2}$ „
7	16	$7\frac{1}{2}$	=	$3\frac{1}{2}$ „

Capacity. The idea involved in the measurement of capacity is closely akin to that of the measurement of volume. Many articles in common use, especially those in liquid form, are handled and measured more easily according to their *bulk* than according to their *weight*. Now a *gallon* of water and a *cubic foot* of water both represent the same fundamental notion. They state in different ways the three-dimensional space or bulk that a certain quantity of water is capable of filling. But the gallon and its kindred measures in the Table of Capacity represent a simpler, possibly older, method of measuring bulk than those of the more scientific Table of Volume.

Like every other unit in our English system of weights and measures the pint (and consequently the quart and gallon) is a measure which has varied considerably in the course of time, though now it is standardized by law under the imposing title of "Imperial Pint." Few people, however, are prepared without notice to say exactly what an imperial pint is. It is interesting to note that legally it is dependent upon the unit of *weight*, for the gallon is defined as containing "10 imperial standard pounds of distilled water weighed in air against brass weights, the water and the air at a temperature of 62° F.

and the barometer at 30 inches." We hasten to add for the benefit of young teachers that to impart the foregoing definition to a class of beginners is to put the cart before the horse, for the essential part of the connection between the gallon and the pound is best discovered by the child through the medium of his practical work, and is not to be taught as an arbitrary fact to be accepted on the teacher's authority.

The Table of Capacity. For young scholars the essential parts of the table are easily grouped in two sections: (*a*) Two pints are one quart, four quarts are one gallon; (*b*) two gallons are one peck, four pecks are one bushel, eight bushels are one quarter.

Frequently, of course, the whole "table" is taught together; yet, if we examine these two sections, we find that they present divergent aspects, for we can speak of a pint of milk but not of a peck of milk, and we talk of a gallon but never of a bushel of petrol. This is due to the fact that the table has two main applications, one to *liquid* and the other to *dry* goods.

The "wet" or "liquid" part of the above table stops short at the *gallon*, while the "dry" section employs gallons, pecks, bushels, and quarters, and less often uses pints, quarts, and gallons, though seedsmen and corn-dealers and some fruiterers still sell by the pint. Thus the table as usually taught is actually formed by the coalescence of two distinct tables. As a matter of fact it is not so many years since *three* tables of capacity were taught in schools.

(1) *Dry Measure* (for corn, seeds, etc.). Pints, quarts, pottles, gallons, pecks, bushels, quarters.

(2) *Ale and Beer Measure.* Gills, pints, quarts, gallons, firkins, hogsheads.

(3) *Wine, Oil, and Spirit Measure.* Pints, quarts, gallons, tierces, hogsheads, puncheons, pipes, tuns.

An examination of these will show that the only common elements are the *pint*, the *quart*, and the *gallon*, and these are all the units of capacity that the average scholar need know, though in rural districts and market towns some acquaintance with pecks and bushels and quarters is necessary. It is a welcome sign of progress that the obsolete and unusual measures of capacity such as the pottle, the firkin, the hogshead, the puncheon, and the tun are no longer taught generally in schools. Even the commoner "gill" is by no means a fixed measure, for a "gill" of beer in Cheshire always means *half* a pint, though the "table" says "*Four* gills, one pint."

Omitting, then, these unnecessary parts, the remnant as set out above forms the simplest table of all our weights and measures. The rhythmical alternation of the multiples: 2, 4, 2, 4 gives an easy method of remembering the connection of the units from pints to bushels.

Practical Work in Capacity. The units may be learnt within certain limits by actual manipulation of measures. Thus a child may discover for himself that two pints make one quart, though he could hardly discover that eight bushels make one quarter.

One caution to young teachers: Liquids are messy—"dry" goods are preferable, so that work with sand or seed is better than work with water or milk.

But whatever the medium, children can discover for themselves the capacity of common utensils such as cups and bowls, bottles, jugs and glasses, and should be able to estimate with rough correctness the capacity of a given receptacle. It is a sign of general, rather than mathematical, education to recognize a half-pint glass or a pint-pot, or to know that the ordinary whisky bottle counts six to the gallon. Gallons are a little more difficult to recognize though the two-gallon jar or water-can or petrol tin is common enough to be known by all.

There is, however, one section of practical work which should never be omitted, and that is the investigation of the connection between the measures of weight and capacity. The connection is usually given in the form of the common rhyme—

A pint of pure water
Weighs one pound and a quarter.

This simple fact should, wherever possible, be made the subject of a *practical* lesson. The question: "How would you teach a child to find the weight of a pint of water?" has been set to students in training colleges. A student with a fund of humour suggested that the child should *drink* the water, and be weighed before and after. Many students to whom the question is given outline methods just as absurd if less humorous. It is common for them to state that the work should be performed with a vessel holding *exactly* one pint; that this vessel should be weighed empty and full, and the result obtained by subtraction. The smallest amount of actual experience would convince them of the difficulty of manipulating such a vessel *full to the brim.* They would also find that the empty vessel could be *counterpoised* accurately, and need not itself be weighed. Far better results can be obtained by using a vessel holding more than a pint, provided that a pint-measure, or some accurate means of running exactly a pint into the larger vessel was available. Further, the child should also learn, as an important point of scientific method, that it is never safe to rely upon the results of *one* experiment only. The exercise should be repeated carefully several times, and the average of the results should be found.

From this fact several other most important connections may be made such as: 1 gal. of water weighs 10 lbs.

Later, in dealing with volume, when the pupil has learnt that 1 cubic foot of water weighs approximately 62·5 lbs., the important fact may be deduced that 1 cu. ft. of water contains approximately $6\frac{1}{4}$ gals. This connection of capacity, weight and volume in English units is of very great practical use. Thus, the number of gallons which would fill an ordinary rectangular cistern or tank can easily be calculated approximately.

Calculations in Capacity. It is possible in capacity, as with the tables of length and weight, to set calculations and problems in the table of capacity both separately and in connection with money.

Too much formal manipulation of capacity quantities is, however, artificial and unnecessary. Thus, the calculation "Reduce 12345 pints to quarters, bushels, etc.," may give excellent *table* practice, but its real applications are very few indeed. Such a problem, however, as: "A milkman delivers daily 63 pints and 39 half-pints of milk. How many gallons does he deliver in a week?" gives table practice in a much more interesting and sensible form.

Utility should be the keynote of all calculations and problems in capacity as in all weights and measures; the exercises should always deal with *real* quantities and ordinary everyday experiences.

CHAPTER IX

TIME AND ITS MEASUREMENT

THE table of time measurement is the table that, above all others, needs sensible and common-sense treatment in schools. It differs from the other tables in the important fact that it is not peculiarly English, but is in use all over the world. The explanation of its international character is simple ; its units, such as the day and year, are not matters of arbitrary choice as are the foot or the pound and the gallon, but are regulated strictly by the movements of the solar system. It is true that the subdivisions of the primary units, e.g. the division of the year into months and the day into hours, have an arbitrary aspect and so must be committed to memory in arbitrary form, but the basis of time measurement is astronomical. It follows that the whole subject of time measurement is intimately connected with the more mathematical side of geography, and the two subjects may be developed side by side. Again, following our usual plan in this book, we shall not neglect the historical side. We shall find opportunity in our historical talks with young children to indicate the gradual development of the means of measuring time from sticks and shadows to the more elaborate sun-dial ; from the primitive sand glass, through the development of the clock and the watch running mechanically by force of weight, onward to the most intricate and delicate electrically controlled clock. Later, we may review the efforts of mankind to correct and regulate their calendar so as to " keep time " with the earth's yearly motion. The objection may be raised that all this is history and geography and not mathematics. To such objection we would reply that " time " is a subject so unique and important to human beings that its study can never be confined to the arithmetic lesson nor deadened by dry-as-dust formal calculations. Therefore, before we set out on the humdrum well-worn path of "reducing" "seconds to years" or performing "long division of time," or some equally unpractical calculation, let us try to set down what a person of ordinary education may reasonably be expected to know with regard to *time*. If we can agree upon some kind of minimum requirement, we shall have established a solid basis for our "teaching."

(1) The person of normal education should be able to "tell" the time, i.e. he should read at sight any ordinary clock face whether in ordinary or Roman figures.

(2) He should be able to reckon rapidly the time interval between any given times within the day. Thus he should readily give the time elapsing between 9.30 a.m. and 4.10 p.m.

(3) He should be able to read ordinary railway and other time-tables with intelligence and speed.

(4) He should possess a few definite distance-time standards such as a mile being approximately a quarter-of-an-hour's walk, or a kilometre being a good ten-minutes' walk.

(5) He should know the sequence of the months and their varying lengths.

(6) He should be clear as to which years are leap years.

(7) He should have clear ideas as to the dates of the seasons.

(8) He should know the ordinary quarter-days and the bank holidays of the calendar.

(9) He should be able to give readily the number of days or weeks elapsing between any given dates, e.g. from 28th April to 14th July.

The above requirements represent a minimum. An intelligent person should also have some clear and definite ideas on the connection of longitude and time; on the meaning of standard time; on the 24-hour basis of

European time ; on the necessity of a date-line to sailors ; on the connection between full moon and spring-tide, and on the necessity for the provision of a leap year.

Again the teacher of arithmetic may object that this is mostly general knowledge. We repeat that most of it is the essential background of time-measure, and that to attempt the old mechanical grind of " sums " in " time " without this background is to court failure, for the work will be remote from life, dull and uninteresting, and so educationally unsound and even harmful.

We will now outline what seems to be the natural sequence in the teaching of time measure.

"Telling the Time." Most children readily learn to " tell the time "—some children at an unusually early age. It is customary in most schools to provide a clock-face and movable " hands," by which constant practice in this useful art can be given. By this means pupils soon grasp the salient facts that 60 minutes make 1 hour ; 24 hours make 1 day. The transition from " telling the time " to the interpreting of time-table " times " will require some attention, for children must learn to recognize at once the identity between such times as 7.40 and " twenty minutes to eight."

" Seconds " may be investigated on any watch or time-piece that has a " seconds " hand and thus the complete table from seconds to days may be constructed.

Simple Calculations in Time. These should be as real as possible. The child should be able to determine the time elapsing, say from 8.30 a.m. to 3.15 p.m. ; he may similarly determine from the times of sunrise and sunset the lengths of the longest and shortest days in England ; he may be asked to find the " lighting up " times for a week from a list of times of " sunset " and instructions to deduct half an hour from each ; he may investigate the daily differences in the times of " high tide " if he is in a locality where such occur ; he may investigate the delay in the daily rising of the moon ; and a thousand other simple little calculations involving chiefly hours and minutes, the data for which may be obtained from any almanac or year book.

Again he may find out from the school time-table the number of hours per week he spends in school and by multiplication he may estimate the number of hours he spends in school in a year; or he may find the total number of hours that his father works per week or that a shop is open per week, from data supplied or collected.

If problems of this kind are constantly supplied to the scholar there will be little need for mechanical examples such as: "Change 1,000,000 seconds to weeks, days, hours, etc.," or "Divide 19 weeks 6 days 23 hours 59 min. 49 sec. by 93."

Time-table Exercises. These may take many forms. The work should follow the ordinary common-sense order. Thus the child may be set to find the best train in the day from, for example, London to Liverpool, or from his own town to any other; he may find the most convenient train for reaching a given destination before a given time; he may find the time allowed for "connections" when different trains have to be taken on a journey; he may find out the longest time he could conveniently spend in London on a day's trip, or the least time in which he could make a given return journey. All these are problems of everyday occurrence in ordinary life. He may also investigate the amount of time to be spent in "stops" on a given journey; knowing the mileage he may be asked to find the average running speed in miles per hour after allowing for "stops"; with a complete time-table he might be asked to determine the longest non-stop run in the day on a given main line; Again, he might compare, for example, the speeds of the Scotch expresses, on the East Coast and West Coast routes. But many other such exercises will suggest themselves to the alert teacher.

Larger Units of Time. The table proper ceases at weeks for we cannot proceed by simple multiples either from weeks to months or weeks to years. But years and months form a most important section of our idea of time, even if they do not fit easily into conventional types of sums.

The connection between years and days, viz., 365 days = 1 year can only be taught by reference to the earth's motion round the sun just as the length of the day depends upon the earth's motion upon its own axis.

The means by which any leap year may be recognized may easily be taught, but too often the reason for the *necessity* of having leap years if we would "keep time" with the sun is only vaguely and obscurely alluded to. Yet children should know this just as well as they know how to determine whether any given year is a leap year.

By division of 365 by 7, it is at once apparent that the ordinary year contains 52 weeks and 1 day. This explains why in ordinary years New Year's Day and all other anniversaries move forward one day in the week each year. Again, it will supply the answer to such a question as: "When could there be 53 Sundays in a year?" to which the answer is of course that there will be 53 Sundays in any ordinary year which begins on a Sunday or in any leap year which begins on a Saturday or Sunday.

The various attempts in history to regulate and synchronize the accepted "year" and "seasons" with the earth's annual motion form a fascinating story that no child should miss. He should learn something of the efforts of Julius Caesar to regulate matters by instituting "leap years," and of the additional correction applied by Pope Gregory some sixteen centuries later. He should learn of our own adoption of the Gregorian corrections in 1752 and of the strong objections of the ignorant populace who firmly believed, when it was decreed that the calendar should suddenly jump from 2nd Sept. to 14th Sept. that they were being defrauded of eleven days of life (and wages). Once the scholars realize the difference between "old style" and "new style" they will grasp the reason for Russia's Christmas day being thirteen days later than our own. These facts, though perhaps too difficult for younger scholars, should certainly find a place in the history and geography of the older pupils.

The Months. The lengths of the months are the most arbitrary feature of the calendar. From time immemorial it has been the custom to learn the varying lengths of the months through some such rhymes as—

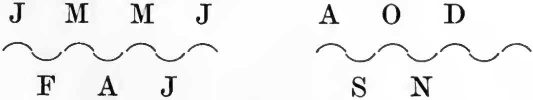
(*a*) Thirty days hath September,
April, June, and November.
All the rest have thirty-one,
Excepting February alone,
Which has but twenty-eight days clear
Though twenty-nine in each leap year.

or—

(*b*) April and June both pass too soon,
Likewise, remember, September, November.

The rhyme is perhaps the easiest means of committing such arbitrary facts to memory.

The lengths of the months may, however, be easily recalled by the use of the knuckles and the hollows between the knuckles. Thus—



It will be seen that all the months with 31 days correspond with the knuckles and the shorter months with the hollows between.

If, however, the teacher will take the trouble to probe into the origin of these thirties and thirty-ones, he will find that an underlying scheme exists, in spite of the apparently haphazard arrangement.

Simple problems to help the children remember the various lengths may be set such as—

(*a*) Which is the longer six months, 1st Jan. to 1st July, or 1st July to 1st Jan. ?

(*b*) Which is the longest period of three months, Jan.-Feb.-Mar., Apr.-May-June, July-Aug.-Sept., Oct.-Nov.-Dec. ?

In solving many problems it is necessary to reckon the number of days or weeks which elapse from one given date to another given date. In this the simple common-sense rule of subtraction requires that only *one* of the two given dates shall be included in the reckoning unless the word "inclusive" is expressly mentioned.

Thus the number of days from 28th April to 4th July may be reckoned as follows—

(*a*) 2 (remainder of April)
31 (whole of May)
30 (whole of June)
4 (part of July)
—
67

or—

(*b*) 3 (April days from *beginning* of April 28th)
31 (whole of May)
30 (whole of June)
3 (July days to *beginning* of 4th)
—
67

The calculation clearly may be made from the *beginning* of 28th April to the *beginning* of 4th July, or from the *end* of 28th April to the *end* of 4th July. If the word "inclusive" were stated in the above problem the answer would of course be **68** days.

Reasoning similar to that of (*a*) or (*b*) above will also show that from 9 a.m. on 28th April to 9 a.m. on 4th July is also 67 days. Numerous problems involve such reckoning. A few are here suggested.

(1) *Postal Times for Long Distances*—

Place.	*Date of Posting.*	*Approx. Time taken.*	*Probable Date of Arrival.*
Australia	12th Mar.	31 days	?
Japan	11th Aug.	26 days	?

Information necessary for such examples as these is usually displayed at any Post Office.

(2) *Departure and Arrival of Steamers*—

(*a*) Leave London 19th Jan.
Arrive Bombay 7th Feb. } Length of journey ?

(*b*) Leave London 2nd Apr.
Arrive Melbourne 43 days later } Date of Arrival ?

In normal times the *sailings* of any steamship company will provide the necessary data.

(3) *Length of Military Service*—

Date of enlistment : 10th Nov., 1915.
Date of discharge : 19th Feb., 1917. } Find length of service in years and days.

(4) *Holidays*—

Fourteen days holiday began at 6 a.m. on 27th April. On what date did it end ?

Each teacher will be able to set similar examples suitable for any particular class or school.

The Calendar and the Church. We have already noted the labours of Pope Gregory XIII to regulate the length of the year—labours resulting in the famous Gregorian correction of 1582. For many centuries before this, however, the study of arithmetic in its astronomical aspect had been considered of special ecclesiastical importance, for thus only was it possible to fix the dates of the important Church festivals such as Easter Day. Even to-day we find a wonderful and ingenious set of numerical tables prefixed to every Book of Common Prayer of the Church. It is not suggested, of course, that these should be introduced into schools and arithmetic lessons, but for their historical interest they are worthy of examination by all teachers. It is remarkable that few people are able to indicate how Easter Day is fixed in any particular year. It is easily remembered as the first Sunday after the first full moon after 21st March.

Longitude and Time. The relation between longitude and time should be given special consideration. The factor that governs differences in time (the rotation of the earth) may be termed geographical, but the arithmetical applications are quite definite. The usual problem is of the following type—

(*a*) Given the difference in longitude of two places to find the difference in their "times"; or—

(*b*) Given the difference in "time" of two places, to find their difference in longitude, and its subsidiary problem:— Given the *actual* time at any one place to find through longitude the actual time at any other place.

The cardinal fact (to be demonstrated as concretely as possible) is that a rotation through 360° of longitude takes 24 hours (approximately). Now $360 \div 24 = 15$, from which we deduce the correspondence between a difference of 15 degrees of longitude and 1 hour of time. From this, by further division, the child learns that a difference of 1 degree of longitude corresponds to 4 minutes in time.

The second fundamental fact is the meaning of "noon" and the necessity for some "standard time." In England, of course, Greenwich time must be taken as the standard.

The third (and perhaps for children the most difficult) point to grasp is the effect of longitude, or east or west, in rendering the "time" *earlier* or *later*. Remembering, however, that the rotation is always *towards* the east, we at once deduce the fact that all places east of our own position experience *noon* or have the sun "crossing the meridian" *earlier* than we do, hence their actual times are *in front* of ours. Thus, if it is noon at Greenwich it is approximately 2 p.m. at Petrograd by sun-time. From this it follows that the sun-time of all places *westward* of our position lags *behind* our own, thus the difference between New York time and our own is approximately five and a half hours, and so, when actually noon with us, the American clocks may show 6.30 a.m.

But the problem is by no means thus completed, though reduced to orderly dimensions, for many practical details occur. The children should know that the "clocks" on an Atlantic liner are altered every night and should be able to suggest whether they must be "put back" or "put on" according to the direction of the voyage. They should know further that the captain keeps an accurate chronometer which indicates Greenwich time and that by using this and observing the actual "noon" wherever he may be, he is able to determine accurately his "longitude."

Finally, they should, in thought, take a continuous voyage round the world and thus arrive at the necessity for a "date line" if the days of the month are to be kept correct.

The Moon and its "Times." Place is generally found in the school geography scheme for some instruction on the phases of the moon. The class should at the same time be taught the connection between "moon" and "month." Thus they may be set to find the interval between successive full moons from the dates and times as supplied in the ordinary almanack. They will, in this way, find more accurately the length of the lunar month, and in the higher classes they may be asked to calculate the *average* length of a lunar month, which is frequently incorrectly taught as twenty-eight days.

"Time" and the Pendulum. The use of a swinging pendulum for the purpose of beating-time is of such universal application that place should be found in the science scheme for an investigation of the factor controlling the time of swing of a simple pendulum. The facts that neither the weight of the bob nor the amplitude of swing (within certain limits) affects the time-of-swing can be demonstrated by simple means to a large class, but the matter should not end here. The actual effect of the *length* of pendulum upon the *time* should be systematically studied by the older boys. Corresponding "lengths" and "times" should be plotted over a good range of experiments; the law $t \propto \sqrt{l}$ should be deduced—not from two or three examples only, but from many experiments—and the graph should be used to determine the length in inches or centimetres of the "seconds" pendulum.

Problems on "Clocks." It has long been the custom to include in arithmetic books and syllabuses problems based upon the motion of the "hands" of a clock.

Simple problems on fast or slow clocks, or clocks gaining or losing at a known rate, may legitimately be used, though with caution, in schools. The great danger is that they may easily become too elaborate and involved.

There is, however, one type of "clock sum" formerly of wide popularity, but now generally discarded as being of very small practical significance, though forming a concise kind of sum dear to the heart of many teachers. The type to which we refer may be illustrated by the following—

"At what time between 5 p.m. and 6 p.m. will the hands of a clock be { together, or
opposite, or
at right angles, or
ten minutes apart, etc., etc. ?"

Common-sense at once informs us that the interest in such problems is a purely mathematical interest, for the ordinary man is not anxious to know the answer to these puzzles, but is more concerned with how much work (or how little) he can accomplish between 5 p.m. and 6 p.m. Since, however, such problems still appear in text-books and even in some examination papers, we make this the excuse for the following few hints on the method of approach.

(1) *Arithmetical Method.* The basis of this method is a clear idea of the difference between *minute-spaces* on the

dial and actual *minutes of time;* together with a grasp of the *relative* motion of the two hands. Thus in any one hour the minute hand travels 60 spaces and the hour hand 5 spaces, and the relative rates of motion are in the ratio 60 : 5 or 12 : 1; actually the minute hand gains 55 spaces on the hour-hand in 60 minutes of time.

This gives the *key* to the arithmetical solution. Consider the first problem above—

"At what time between 5 and 6 p.m. will the hands of a clock be together?"

A reference to the clock face shows that the "gain" must be 25 spaces.

But, if 55 spaces are gained in 60 minutes,

Then 25 ,, ,, ,, $\frac{60}{55}$ of 25 minutes,

or $1\frac{1}{11}$ of 25,

or $27\frac{3}{11}$ minutes.

(2) *Algebraic Method.* A diagram (Fig. 18) will show the simplicity of the method.

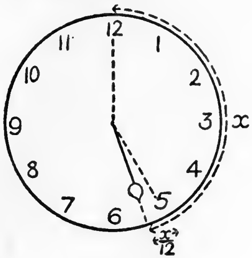


FIG. 18.

Consider the same problem above.

Let the minute hand travel x spaces to reach the hour hand. In this time the hour hand travels $\frac{x}{12}$ spaces.

Then from diagram—

$$x - \frac{x}{12} = 25 \text{ min.}$$

$$\text{or } \frac{11}{12}x = 25 \text{ min.,}$$

$$\text{whence } x = 25 \times \frac{12}{11} \text{ or } 27\tfrac{3}{11} \text{ mins.}$$

To a class familiar with simple equations this method is less mechanical than the arithmetical method.

(3) *Graphic Method.* Since we are dealing with uniform rates of motion all ordinary clock sums can be illustrated and solved by the methods of the straight line graph.

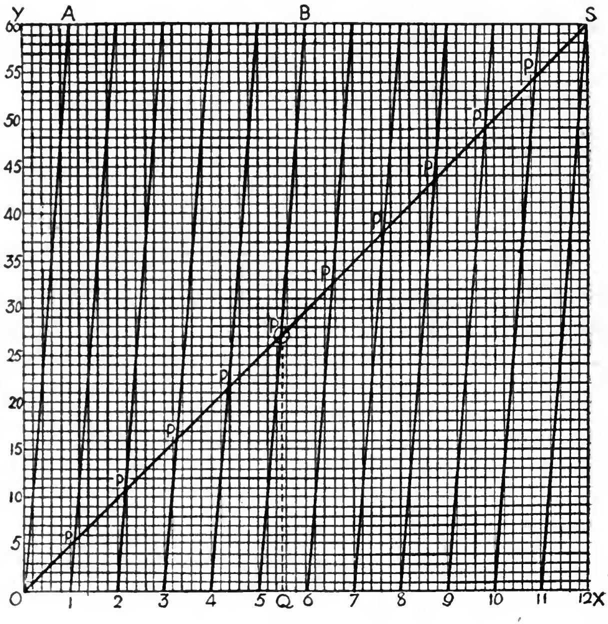


FIG. 19.

The graph (Fig. 19) contains the solution of all ordinary problems in "clocks."

The axis *OX* represents 60 minute *spaces* and is convenient for "*hours*."

The axis *OY* represents 60 minutes of time.

Thus the line *OS* represents the motion of the *minute* hand during every hour while the twelve successive parallel lines represent the motion of the *hour* hand which only moves 5 spaces per hour.

Now the points P are the intersections of the graphs, in this case we consider the point P between 5 o'clock and 6 o'clock. PQ drawn tells at what time between 5 and 6 o'clock the hands of a clock are together.

The actual length of PQ can be read off or may easily be determined from the principle of similar triangles, thus—

$$\frac{PQ}{OY} = \frac{PQ}{SX} = \frac{OP}{OS} = \frac{AB}{AS} = \frac{5}{11}$$

$$\therefore\ PQ = \frac{5}{11} \text{ of } OY = \frac{5}{11} \text{ of } 60 = \frac{300}{11} = \mathbf{27\tfrac{3}{11}\ minutes.}$$

The mathematical student who is interested in graphic solutions may find numerous other interpretations of the time represented by the lines PQ. Thus it also represents one of the two instants between 6 and 7 o'clock when the hands are exactly 5 minute spaces apart, viz., $27\frac{3}{11}$ min. past 6; or it represents one of the two instants between 8 and 9 o'clock when the hands are at right angles to each other, viz., $27\frac{3}{11}$ min. past 8; or it represents the time between 11 and 12 o'clock when the hands are pointing in opposite directions in one straight line, viz., $27\frac{3}{11}$ mins. past 11.

This one example is sufficient to show that the graph as drawn gives solutions to all the usual problems.

In general, however, we must repeat the warning: "Clock" sums of the above kind are not of sufficient practical interest to warrant their inclusion in an ordinary school syllabus.

CHAPTER X

SQUARE MEASURE AND AREA

THIS subject is usually divided into two sections—

(1) Square Measure, with calculations and problems (largely " Reduction ").

(2) Mensuration of Plane Areas.

Though modern methods are in favour of the common-sense plan of treating these two sections concurrently, it will be more convenient in this book to deal with each section separately.

Square Measure. This subject has suffered in the past from the usual traditional treatment. The table was first learned by heart and was then followed by " sums," frequently elaborate reductions and calculations such as " Reduce 12345678 sq. inches to square miles," or " Reduce 4 sq. m. 527 acres 3 roods 27 poles 25 sq. yds. 7 sq. ft. 113 sq. in. to sq. inches." Practical work was seldom attempted; no acquaintance with actual units or areas was thought necessary, though children often acquired most astonishing mechanical skill in calculation, especially in such manipulations as multiplying or dividing by such awkward numbers as $30\frac{1}{4}$.

Since, however, the measurement of area of any kind requires certain units of measurement, we will begin with a few remarks on the fundamental units. Again, as with other tables, it should be carefully noted that the units, though in definite connection, are grouped in distinct sets. Thus the surveyor and the farmer deal with acres, roods, poles, but less often with square feet and square inches; the house furnisher and decorator are familiar enough with square yards and square feet, though they have but little use for square miles or acres.

It is difficult, outside the traditional text-book and school example, to find a single concrete instance where the *whole* table from square inches to square miles is required at one and the same time.

The Square Inch. This should be a reality to every beginner. Square inches or inch-squares may be cut out of cardboard, handled by every member of the class, and used in the beginnings of the measurement of area. The connection: 144 sq. inches = 1 sq. ft., may be discovered by actual manipulation of cardboard squares.

The Square Foot. This, too, should be examined in concrete form. The children should *see* a foot-square in paper or cardboard, and should establish for themselves the connection between the square inch and the square foot.

The Square Yard. Here concrete illustration becomes unwieldy even in cardboard, but a square yard may be drawn on a wall or board and the relation 9 square feet = 1 square yard may be discovered.

Square Pole = $30\frac{1}{4}$ sq. yds. Less frequently required are acres, roods, square chains, square poles. Taking the smallest unit first, the connection between square yards and square poles is also best shown by drawing to scale (Fig. 20).

The square yards should be drawn to scale by every child and actually counted to check the statement $30\frac{1}{4}$ sq. yds. = 1 sq. pole.

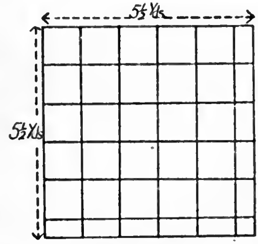


FIG. 20.

Square Chain. This, though sometimes omitted from the table, is important as forming the best link between square yards and acres. A square chain clearly contains (22 × 22) or 484 square yards, and ten square chains or 4,840 square yards make an acre. The historical connection between the pole, the chain or "acre's breadth," the furlong or "acre's-length," and the acre has already been discussed.

It is possible to show on one diagram the connection between all these units.

From the diagram (Fig. 21) we see—

$$\begin{aligned} 16 \text{ sq. poles} &= 1 \text{ sq. chain} \\ 2\tfrac{1}{2} \text{ sq. chains} &= 1 \text{ rood} \\ \left.\begin{array}{l} 4 \text{ roods or} \\ 10 \text{ sq. chs.} \end{array}\right\} &= 1 \text{ acre} \end{aligned}$$

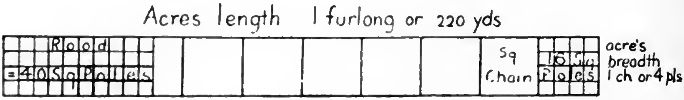


FIG. 21.

Those who are interested in graphical representation may observe that an area one mile square may be made up of 640 such acre-strips arranged in 80 rows with 8 strips to each row.

This whole subject of the intimate connection between the larger units of length and area may form the subject of an excellent lesson in drawing to scale for older scholars.

Plenty of practice in the eye-estimation of area may be given in the smaller units, but, as always, the estimation of larger units needs considerable skill. A practised farmer can tell the approximate number of acres in a field at sight and may perhaps give a fair estimate of the number of sq. rods in a garden patch, but children cannot be expected to acquire this skill. Yet some help may be given. A square of 70 yards side is very little larger than an acre, while a small-sized hockey field, say (100 × 50) yards, is also a fair approximation to an acre. With the spread of "allotments" since 1914 more practice may be given in the measurement and estimation of square yards and square poles, though sometimes local measurement of "rods," etc., does not always agree with the standard definition. Thus in Cheshire a rood implies a square of 64 square yards.

The Fundamental Processes in Square Measure. Square Measure opens out many possibilities for calculations dear to the heart of the older and more conventional arithmetician. It is, of course, possible, using the square units as discussed above, for anyone to set out an imposing array of calculations requiring their addition, subtraction, multiplication, and division, and so we still find in the traditional text-book and examination papers examples such as the following—

(*a*) Reduce 5 acs. 3 rds. 37 sq. poles 25 sq. yds. 7 sq. ft. 119 sq. ins. to inches.

(*b*) Reduce 3367251 sq. ins. to acres, etc.

(*c*) Add together 4 acs. 3 rds. 29 sq. poles 17 sq. yds. 5 sq. ft. 113 sq. ins. and 7 acs. 2 rds. 33 sq. poles 15 sq. yds. 7 sq. ft. 132 sq. ins.

(*d*) Divide 7 acs. 3 rds. 39 sq. poles 27 sq. yds. 8 sq. ft. 109 sq. ins. by 57.

(*e*) Divide 1 sq. miles 520 acs. 3 rds. 39 sq. poles 40 sq. yds. by 64 acs. 2 rds. 30 sq. poles 15 sq. yds.

(*f*) Find the cost of 212 acs. 3 rds. 37 sq. poles 15 sq. yds. 2 sq. ft. 36 sq. ins. at £26 5s. an acre.

Now any intelligent person examining impartially such examples as the above is obliged to confess that such calculations are not needed half a dozen times in any ordinary person's life. Such examples, neatly and correctly worked, may be exhibited with pride by some teachers as examples of their pupils' industry, yet the majority must confess that, as far as real life is concerned, they are largely useless. It is necessary for every teacher, then, to make a careful selection of such parts of the above calculations as he considers really necessary. To discard work which for so long held an honoured place in the pedagogue's esteem needs some courage, for schools and syllabuses are still largely under the tyranny of text-books and examinations ; but, sooner or later, the question of clearing out the lumber-room of school arithmetic must be resolutely faced, for the demands on the time and energy of the pupil, made by an ever-increasing and expanding curriculum, grow yearly more insistent. It is thus essential that calculations such as the above, occurring but seldom in after life, shall no longer be allowed to clog and consume the mental effort of children. Square measure offers a good starting point for the teacher anxious for reform. All involved and complicated calculations can be discarded. The present generation of adults will remember perhaps the labour, temper, and tears expended over the teaching and learning of the process of changing square yards to square poles or square poles to square yards, involving multiplication or division by $30\frac{1}{4}$. This process was always taught as involving either multiplication by 4 and division by 121, or the converse. Since the operation was traditionally taught before the child knew anything of the manipulation of fractions it follows that the process was to the pupil nothing but a trick which (according to the teacher's authority) would produce the right answer. If this and similar operations must be taught—and some teachers will discard them dubiously, possibly even with regret—it is just to the scholar to postpone such work until he has some idea of multiplication and division by fractional quantities.

It is true that the difficulty in schools may be avoided by proceeding through square chains to acres, and using the relations 484 (or 22^2) sq. yds. = 1 sq. ch., 10 sq. chs. = 1 acre; but this, though avoiding the nuisance of operating with $30\frac{1}{4}$, is still a cumbersome procedure.

The conscientious teacher may ask in anxious perplexity: "What are we to do then if we wish to change square yards to acres?" In answer we would offer—

(*a*) In actual life we very seldom *need* to change square yards to acres. If we should need to do so, simple division by 4,840 will give the answer in the sensible form of acres and fractions of an acre.

(*b*) It follows from (*a*) that such calculations are based upon the artificial requirements of traditional books and examinations and are, therefore, not of much importance.

(*c*) It is also to be observed that if such did occur in real life the *important* part of the answer is the nearest *whole* number of *acres* in the given number of square yards and, therefore, meticulous accuracy in the exact number of odd square roods, square poles, square yards, is unnecessary and pedantic.

(*d*) This *nearest whole number* is usually obtainable by direct methods, e.g. in finding the number of acres in 30,000 square yards, since 4,840 sq. yds. = 1 acre, a first approximation is 6 acres (4,840 is 5,000 approx.).

Actually by division the answer is seen to be $6\frac{1}{5}$ acres nearly, which is a reasonable answer to such a problem.

```
4840)30000(6
     29040
     -----
       960
```

Let us compare this with the traditional method—

```
            30000 sq. yds.
                4
          -----------
      11)120000(1 )    89
         ---------  }  --  =  22¼ sq. yds.
      11)10909(8  )    4
         --------
      4,0)99,1 p.
          ----
         4)24 r. 31 p.
           --
            6 a.
           ====
```

Ans. **6 ac. 0 r. 31 p. $22\frac{1}{4}$ sq. yds.**

If we would further test the appeal to common sense let us present each answer, viz., $6\frac{1}{5}$ acres approx. and 6 acs. 0 rds. 31 sq. poles $22\frac{1}{4}$ sq. yds., to a hard-headed farmer of our acquaintance, and ask him which is more understandable.

In leaving for the present this subject of square measure, we would, in view of what has been stated above, ask every teacher and especially every head-teacher, to include in the arithmetic syllabus only such operations, calculations and examples in square measure as are reasonably likely to occur in actual life, and to add their effort to the struggle which is always necessary to break through the conservatism and inertia of custom.

The Measurement of Area. With the measurement of area we begin that series of topics long known traditionally in the schools as *mensuration*. It is in this branch of applied arithmetic that most improvement has been noticeable in recent years, both in the method of teaching the rules and formulae, and in the character of the problems set. It is in mensuration that practical work finds its most natural place. The older method of dealing with this subject is worthy of note, if only as a warning against mechanical and vicious teaching methods. Formerly, the scholars accepted a formula such as " Area of a triangle $= \frac{1}{2}$ (base $\times$ height) " on the teacher's authority and dutifully committed it to memory; now in the best schools they are set to work to discover and demonstrate the fact for themselves. They once applied the formula to innumerable calculations such as: " Find the area of a triangle whose base is 9 ft. and height $8\frac{1}{2}$ ft." Now they are frequently set to work a real problem such as: " Find as accurately as you can the area of this triangle," for which they must first carefully obtain dimensions by actual measurement. Similarly, other common areas are dealt with: no longer do we proceed deductively from the accepted formula but our teaching is both inductive and deductive, and the child learns clearly the fundamental truth of practical mathematics, that accurate calculation is useless if based upon inaccurate data, and that in actual life this data has usually to be obtained with patient care before any calculation is possible.

Rectangular Area. It is customary to begin mensuration with a simple lesson on rectangular area. Though the idea of surface or area is a distinctly abstract and mathematical conception, children readily grasp the method of its measurement if not its philosophical significance. A successful lesson may be given to quite young children on the following lines—

(*a*) The square inch—taught by examination of inch squares cut in cardboard or paper.

(*b*) The measurement of some simple rectangular area by actual superposition on it of the cardboard squares.

(*c*) The development of the simple rule for quickly *calculating* the number of inch squares, i.e. $A = L \times B$.

(*d*) Finding simple areas (rectangles cut in cardboard or paper, book covers, window panes, etc.) by first measuring length and breadth.

(*c*) Representation of simple areas on squared paper or squared blackboard.

(*f*) The discovery of the relation of one square inch to one square foot by actual measurement of a piece of cardboard one foot square.

Having done this, the children will have an idea of (*a*) How area is measured; (*b*) The fundamental units used.

Later the *square yard* may be investigated in terms of square feet and applied to similar simple problems and calculations.

As soon as the pupil has grasped the above preliminary notions, the possible developments of the subject of rectangular areas are innumerable. We mention only a few which find a place in nearly every arithmetic syllabus.

Domestic Areas. (*a*) *Interiors*—

Floors: carpeting and staining, borders and surrounds.

Walls: papering, colouring, whitewashing, painting of friezes, picture rails, etc.

Doors and windows: painting and glazing.

Hearths and fireplaces: pictures and picture frames.

Halls and staircases.

(*b*) *Exteriors*—

Walls: bricks, plaster, stucco.

Roofs: tiles, slates, corrugated iron, felting, tarring.

Lawns and paths.
Gardens and flower beds.
Greenhouses and outhouses.
Yards and paving.

In this selection of topics for ordinary school arithmetic in rectangular areas common custom is undoubtedly right. The work is in touch with actuality and may be made as practical as desired, and most of it has the advantage of being suitable alike for boys and girls. The only units that need be employed are square yards, square feet, and square inches. Beyond the formula: Area

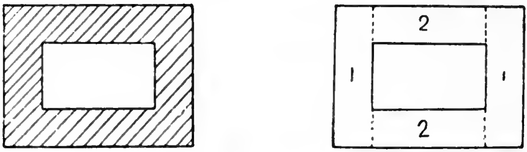


FIG. 22.

= Length × Breadth, only a few extensions are necessary, such as Length $= \frac{\text{Area}}{\text{Breadth}}$, Breadth $= \frac{\text{Area}}{\text{Length}}$, and a few new terms such as Perimeter. The scope for the "All-practical" example or the "figureless sum" is here very wide indeed. We add a few notes on some of the above topics for guidance.

Many of the examples concerning borders surrounding carpets, picture mounts and frames, paths round lawns, etc., involve an area that is clearly the difference of two other areas. Thus in the Fig. 22 the shaded area is clearly the difference of the areas of the outer and inner rectangles. Children are usually inclined to find such an area by the summation of parts as in Fig. 22, e.g. they will find the required area by adding areas 1 + 1 + 2 + 2. The "difference" method should be pointed out as being frequently shorter, and indeed both methods may be regularly used, the one to check the other.

Floors. It should be remembered here that the staining or painting of floors is not usually charged for by the square inch; that ordinary looms weave carpet of standard width; and that seamless squares are to-day as common in the smaller houses as the "made-up" carpet, these seamless squares also being made in standard sizes. Many of the older examples are therefore only pseudo-practical, and an example such as the following thoroughly carried out may be as valuable as pages of mechanical calculation.

"From a carpet-catalogue select a suitable seamless "square" for this room (dimensioned plan provided), and estimate the amount of linoleum surround necessary. Using your catalogue, estimate also the probable cost."

Much valuable practical work connected with floors may be done by teacher and class using diagrams drawn to a good scale; and rolls or strips of paper to represent carpet or linoleum. Here, as always, we repeat the necessity of keeping in touch with life, and of constructing live examples instead of relying upon text-book examples usually concocted to "come out" in convenient whole answers.

Walls of a Room. Problems relating to the walls of a room provide many opportunities for demonstration and class teaching. The four walls of a room may be represented by the four sides of a cardboard box which may be easily flattened out to form one long rectangle, the area of which is given by Perimeter × Height.

Until the principle has been thoroughly grasped the areas required in a few examples should be represented by a diagram and dimensions inserted.

The teacher should endeavour to be as practical as possible. Paper-hanging is a trade, and wall-paper is usually bought in standard lengths and widths. The usual "piece" of paper is a roll 12 yards long and approximately 21 inches wide. Its area is clearly $(12 \times \frac{7}{12})$ sq. yds. or 7 sq. yds. Now cost of papering is usually estimated on the number of whole pieces necessary. Hence the most practical solution is to find the wall area in square yards and to divide this by 7 to determine the number of pieces. Thus, suppose the wall area to be papered $= 65\frac{1}{2}$ sq. yds., the number of pieces required $= \frac{65\frac{1}{2}}{7} = 9\frac{5}{14}$, and clearly 10 pieces must be bought.

It is well to avoid examples in which usual widths of paper are given or in which the cost of paper is to be reckoned per square foot. These may form "sums" capable of numerical answers but they are not very real.

Doors, Windows, Hearths, Fireplaces call for no special comment. Allowance has usually to be made for some of these in papering walls or covering floors.

Pictures, Picture Mounts, Picture Frames supply a foundation for useful work, either actual practical work or calculations, or both.

Picture Rails, Friezes, Wainscoting, Skirting Board offer useful exercises in perimeters.

Halls and Staircases, Landings, etc. These will give good practice in mixed rectangular areas such as in Fig. 23, while plenty of examples are possible involving lengths of carpet for hall or stairs.

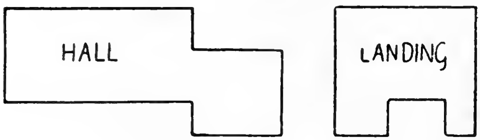


FIG. 23.

Outside Walls and Roofs. Examples should be as real as possible, and may involve both area and cost. The calculation of the length of roof-slope may require a knowledge of the properties of a right-angled triangle, and may well be postponed until the "Three-square" theorem of Pythagoras has been investigated. In roofing with such material as felt, we frequently use the common principle

$$\text{Length Required} = \frac{\text{Total Area}}{\text{Width of Material}}$$

Lawns and Paths, Gardens, Beds and Borders. These are favourite examples for practice in Area and Costs. In dealing with paths the *difference* method is frequently the easier. Thus, in Fig. 24, the total area of the paths is given by the difference $(a \times b) - (c \times d)$. Endless variety is possible. Paths may be gravelled, cemented, pebbled, paved, edged with box or tiles, etc., all of which make useful "sums," while lawns may be turfed, sown, rolled, cut, provided we endeavour to be sensible in our quantities and do not propose in our examples to turf a lawn with sods 4 feet square or to cut a small suburban lawn with a six-foot mower. But indeed all examples, artificially propounded as sums, are liable to these faults, and the best examples in this section will be those which arise naturally out of the school gardening operations.

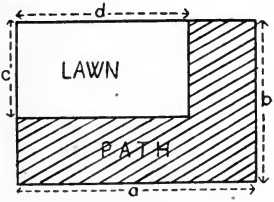


FIG. 24.

Outhouses, Sheds, Greenhouses, Cold Frames will provide excellent work in finding area and also volume.

Nor do these domestic examples exhaust our topics, for the school room provides plenty of examples ready to hand. Indeed, the rectangular area is almost universally employed by mankind. Thus, in the schoolroom, examples of every type may be suggested from the simple problem: "Calculate how much floor space per child there is in this room," to the most elaborate examples involving block-flooring, etc.

We will conclude this section with one or two teaching hints.

(1) The examples may be "all-practical," where

directions only are given by the teacher, and all dimensions required have to be found. This is undoubtedly the best type of work, though it takes up a great deal of time, and is therefore unpopular among the teachers of the "four-sums-right-each-lesson" type.

2. If book examples are given, careful selection and grading are necessary.

3. The practice of insisting upon a sketch or drawing (preferably drawn to scale) for examples in area is entirely sound. The sketches are a help to every scholar.

Diagrammatic Representation. This aspect of area and drawing deserves attention, for it forms a handy visible form of demonstration, and thus affords a most valuable teaching aid. We indicate here a few possible uses.

(*a*) It may be used to demonstrate fundamental relations, e.g. we can show by sketches that

$$144 \text{ sq. in.} = 1 \text{ sq. ft.}$$
$$\text{or } \quad 9 \text{ sq. ft.} = 1 \text{ sq. yd.}$$

It has already been used to show the connection between acre and furlong, etc.

Again a square unit ruled in tenths (Fig. 25) forms a splendid basis for teaching decimals to two places or for teaching percentages and their relation to fractions and decimals.

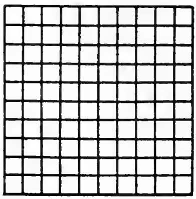


FIG. 25.

(*b*) Area may be used to develop and illustrate new rules and processes such as the multiplication and division of fractions by fractions. The square pole is a striking illustration of fractional multiplication, for a drawing makes it clear that $5\frac{1}{2} \times 5\frac{1}{2}$ *is* $30\frac{1}{4}$. So, too, we may read off on squared paper in square units the answer to decimal multiplication as in Fig. 26, which shows clearly the answer to $1{\cdot}6 \times 2{\cdot}3$.

(*c*) It forms the readiest means of demonstrating such useful algebraic formula as

$$(a + b)(c + d) = ac + ad + bc + bd$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

all of which will be indicated graphically as they occur in this book.

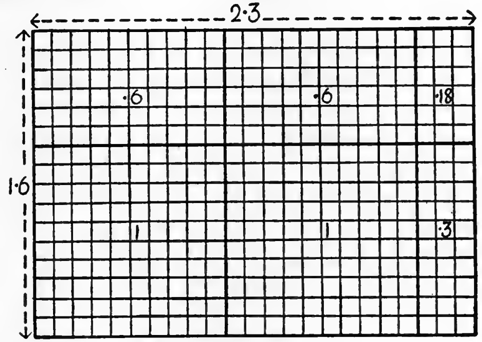


FIG. 26.

(*d*) Rectangles are invaluable, too, for graphic representation of geographical facts. Thus monthly rainfall

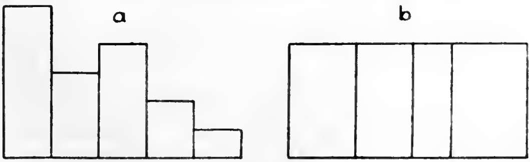


FIG. 27.

is best examined when set out as a series of rectangles of constant width but of varying heights. Similar use of rectangles either as (*a*) constant width but varying heights (Fig. 27*a*) or (*b*) constant height but varying widths (Fig. 27*b*), enables statistics to be examined readily and deductions to be drawn rapidly. The construction of such diagrams from statistics should be constantly required from older scholars. It will be noted in the chapter on "graphs" that quantities which are most correctly represented graphically by rectangular areas are essentially of a discrete nature. It is, of course, always possible to draw a continuous line through such a series of values plotted graphically, but it will be pointed out that this continuous line for discontinuous quantities has no real mathematical significance, since interpolation is not possible.

Most of the numerous excellent little books on practical geography contain copious examples and statistics suitable for graphic representation in rectangular area.

In addition to teaching the rule for rectangular area, it is customary to teach the average child a few methods of dealing with common areas not rectangular. Chief among these may be placed the triangle.

The Area of a Triangle. The simple rule: area of a triangle $= \frac{1}{2}$ (Base $\times$ Height) may be developed in several ways.

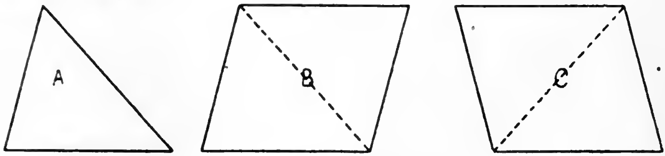


FIG. 28.

(1) We may base our demonstration upon the theorem that every triangle is half some parallelogram, e.g. A is $\frac{1}{2}$ B or $\frac{1}{2}$ C. This may be investigated by drawing and cutting or folding in paper (Fig. 28). Since the area of a parallelogram is (base $\times$ height) then area of triangle is clearly $\frac{1}{2}$ (base $\times$ height). This method, however, depends upon a knowledge of the mensuration of the parallelogram which must consequently be investigated first.

(2) We may avoid reference to parallelogram by the following method or by method (3).

(*a*) Cut out in paper two identical triangles (A and B in Fig. 29).

(*b*) Cut triangle B into two right angled triangles 1 and 2 along the perpendicular as dotted.

(*c*) Rearrange the pieces 1 and 2 round A as in C (Fig. 29).

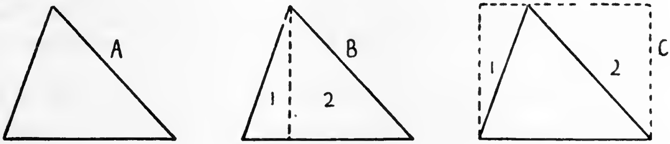


FIG. 29.

(*d*) The two equal triangles thus form *one* rectangle whose area is (base × height). Hence the area of A or B is clearly $\frac{1}{2}$ (base × height).

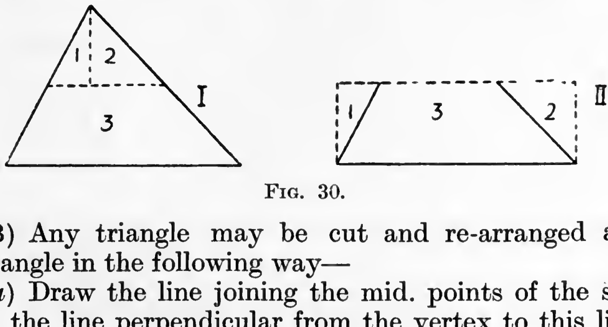


FIG. 30.

(3) Any triangle may be cut and re-arranged as a rectangle in the following way—

(*a*) Draw the line joining the mid. points of the sides and the line perpendicular from the vertex to this line.

(*b*) Cut the triangle into three pieces (1, 2, 3) by cutting along these dotted lines (Fig. 30 (i)).

(*c*) Re-arrange these three pieces to form a rectangle as in Fig. 30 (ii).

(*d*) Area of this rectangle is clearly

Base × Half-height of Original Triangle

or $$\frac{\text{Base} \times \text{Height}}{2}$$

The rigid mathematician may be inclined to scoff at such demonstrations. They are not put forward as proofs, however, but as convenient means of demonstrating to large classes and young minds an accepted geometrical truth.

The rule may be tested in various ways, among them being the following—

(*a*) The area may be drawn on squared paper and the approximate area calculated directly by counting the squares.

(*b*) Three results for any triangle may be obtained by taking each side in turn as base and measuring the corresponding heights. The three areas thus calculated may be compared.

Once the rule is grasped we may proceed as usual to further examples in—

(*a*) Calculation of triangular areas from dimensions given.

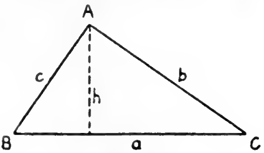


Fig. 31.

(*b*) All-practical examples involving the actual measurement of the dimensions of triangular areas. (Convenient forms are triangles cut in paper or cardboard—no dimensions marked.)

It is to be feared that far too much practice is given under (*a*) and far too little under (*b*). Thus scholars again and again will state glibly how the area of any triangle may be found, and will yet fail lamentably when asked to find the area of a *particular* triangle—no dimensions being supplied.

The simplified formula in the case of the right-angled triangle should be noted—

$$\text{Area} = \frac{\text{Product of Sides enclosing the right angle.}}{2}$$

This frequently saves time and trouble, e.g. if A is a right angle (Fig. 31). Area is most easily found as $\frac{1}{2}bc$ and not as $\frac{1}{2}ah$.

Area of a Parallelogram. This may be dealt with either before or after the triangle as preferred. Similar methods of drawing and paper cutting will convert any parallelogram into an equivalent rectangle as in Figs. 32, (i) and (ii).

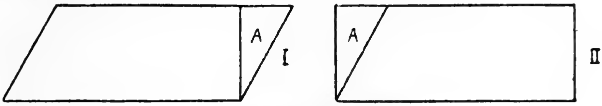


FIG. 32.

The rule follows : Area = Base × Height
or Area = Any side × Perp. distance between it and the opposite parallel side.

The rule is, however, not frequently required in real life and hence need not receive too much attention.

The Right-angled Trapezium. This is an area occurring frequently in domestic architecture. Its formula is a simple one, and does not always receive the attention it merits.

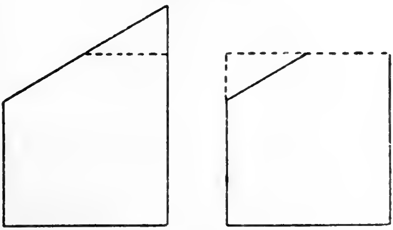


FIG. 33.

By the ordinary methods of drawing, cutting and rearranging (Fig. 33), we may convert any such area into its equivalent rectangle of which the area is clearly

Base × Mean Height

Applications are endless : areas, sides of houses, outhouses, lean-to sheds, greenhouses, cold frames, swimming baths, are readily calculated by use of this formula and hence such areas need not be divided, as is so often done, into triangle + rectangle. If a, b and c (Fig. 34) are known or measured the area can be at once written down as

$$b\left(\frac{a + c}{2}\right).$$

The *Mean Height* can, of course, be always calculated as the average of a and c, i.e. $\frac{a + c}{2}$.

The right-angled trapezium is the basis of that excellent method for calculating an irregular area known to students of practical mathematics as the Mid-Ordinate Rule, whereby such an area is calculated as a summation: i.e. $\Sigma y dx$ (where y is a mean ordinate and dx the thickness of the selected strip). The actual integration process $\int y dx$ is of course capable of graphic representation in area as the limiting case of $\Sigma y.dx$. Approximate integrations from a series of tabulated values of x and y, where the differences are comparatively small, may always be readily calculated on the Mid-Ordinate method.

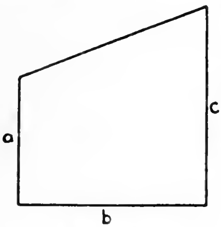


FIG. 34.

There are many schools, however, in which the teaching emphasizes the more general case of the trapezium, as defined by the parallelism of two opposite sides only, and neglects to pay any special attention to the right-angled trapezium as treated above. This is to be regretted, for the general form of a trapezium as shown in Fig. 35 is not nearly of so common occurrence in actual life as the right-angled form.

The general case may be dealt with by either of the methods indicated below.

Method (A) (Fig. 35) reduces the figure directly to a rectangle by the cutting and rotating of two small triangles to new positions as indicated.

A class should be able to deduce the rule—

Area = (Mean of Parallel Sides) (Distance between them).

Method (B) (Fig. 35) derives the formula also very simply. Either diagonal will divide the trapezium into two triangles, of which the area of one is $\frac{a \times h}{2}$, and of the other $\frac{b \times h}{2}$, and thus of the whole $\left(\frac{a+b}{2}\right) h$, which may be expressed in words as before.

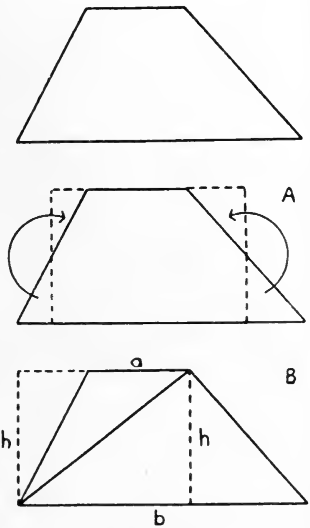


FIG. 35.

Two important uses of the trapezium in connection with finding the area of a circular ring or the curved surface of a conical frustum will be noted in a later chapter.

An Irregular Rectilinear Figure. Armed with methods for dealing with triangles and trapeziums, the pupil is now ready to develop methods for dealing with any irregular quadrilateral or polygon.

The following methods of approach are suggested.

(1) An area of irregular shape may be "graphed" on paper or cut out in paper or cardboard, and distributed so that each child has a copy. Some form of duplicating will ensure readier means of checking the calculations, but it is not necessary that every child should have an identical area. The area should be of a reasonable size, such as may be measured to some degree of accuracy in square inches or square centimetres.

(2) If asked to propose a method, most classes are ready with the suggestion of dividing the area into

convenient triangles, as shown in Fig. 36. The area of each of these triangles may then be found by the ordinary rule for a triangle—

$$\text{Area} = \tfrac{1}{2}\,(\text{Base} \times \text{Height})$$

The practical work then necessary consists of the measurement of the bases and of the heights as dotted. It will be found in practice that children frequently show an extraordinary lack of appreciation of the most convenient base for each triangle.

(3) If the measurement of an area such as that shown above is intended as a preliminary to actual field work

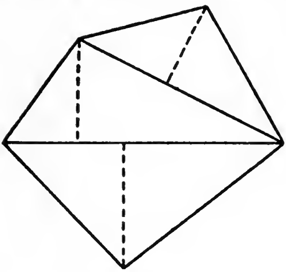


Fig. 36.

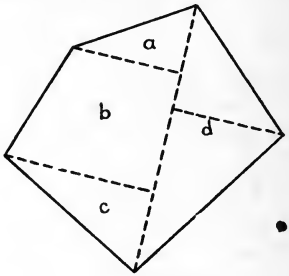


Fig. 37.

then it is preferable to follow the ordinary "survey" method, and to reduce the figure to simple right-angled triangles and right-angled trapeziums by working from the most convenient base line. Fig. 37 illustrates the ordinary method.

The work concludes by the summation of the areas thus created.

The method is simple, systematic, and finds favour among children. The treatment of a *small* area preferably drawn on paper on this survey method is desirable and even necessary as a preliminary to actual outdoor surveying practice. Having thus grappled with a small area, the pupil has a grasp of the essential procedure, and is not so likely to be overwhelmed on the actual field by the size of the measurements and the practical difficulties of obtaining them. Out-door work in mathematics will be discussed in a later chapter, and further survey methods indicated.

Areas of Curvilinear Figures. The course we have so far sketched in the measurement of area will suffice for all ordinary areas bounded by straight lines.

Unfortunately areas in real life are most wilfully irregular, often refusing to be confined within the orderly limits of straight lines. Such areas are discreetly avoided by the precise pedagogue and the conventional text-book. They are unpopular because they do not make good "sums" which "come out" and provide answers that can be marked in bulk and quickly pronounced right or wrong. It is, however, the business of arithmetic to meet the actual needs of life and to supply approximate methods where actual things refuse to fit into its rigid rules.

For the sake then of those teachers in "higher tops," continuation classes, central schools, etc., we offer a few remarks on the treatment of these areas.

(1) "*Counting the Squares*" *Method.* It is possible to transfer any area or its replica (to scale) to squared paper. The area may then be found approximately by the usual method of counting squares and parts of squares. The method, however, is cumbersome and laborious, though the work may be lightened, and duplication sometimes saved, by the use of squared *tracing* paper which is placed over the given area.

The method may well be applied to map areas, if *accurate* maps, such as ordnance maps, are used, but there is something ludicrous in the sight of a scholar attempting to approximate to the area of a section of a small-scale and inaccurate map by the method of "counting squares," an error of two or three squares meaning a difference of two or three hundred square miles in the area to be thus measured.

(2) *The Mid-Ordinate Method.* This method of common practice is easily mastered by the average pupil.

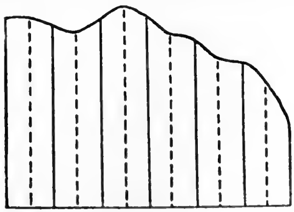


FIG. 38.

The diagram (Fig. 38) shows the treatment of a section of such an area. The dotted lines represent the mean heights. The area is divided into any convenient number of strips by parallel lines, and the area of each strip is estimated to be approximately: Width × Mean Height.

The method is obviously based on the formula for the area, the right-angled trapezium.

The work may be tabulated conveniently in a small space, as in Fig. 39.

Strip.	Mean Ht.	Width.	Area.	
1				
2				
3				
4				
5				
6				
etc.				
				Total.

FIG. 39.

The degree of approximate correctness clearly depends upon the number of strips taken, and, as the width of the strips continually diminishes the method approaches that of the actual process of integration $\int y dx$ as applied to the area of a regular curve.

(3) *The Method of "Simpson's Rule."* Older scholars may with profit apply the method of "Simpson's Rule" to irregular areas. The method usually gives slightly more accurate results than that of the Mid-Ordinate rule.

As, however, this book is intended mainly for teachers in schools for younger scholars, we must refer teachers to any ordinary practical or technical mathematics text-book for a statement of Simpson's rule and a demonstration of its use.

(4) *The Method of the Planimeter.* This, the most ingenious and accurate method of all, may be readily applied. Teachers who are interested should borrow a planimeter and use it. Those who have not done so will learn more of this instrument in a few minutes' actual use than they can from the most elaborate and careful description it is possible to give.

(5) *Pseudo-practical Methods.* Any irregular area may be reduced with care and ingenuity to a polygon of approximately equal area, and this polygon may then by ordinary "parallel" methods be reduced to an equivalent rectangle or triangle. The method is seldom used in practice.

Any area (or its replica to scale) may be cut out of the paper on which it is represented and weighed. By comparison with the weight of a known area of the same paper its area may be determined approximately.

So much however depends upon the accuracy of the drawing and of the cutting that the answer is seldom to be relied upon.

CHAPTER XI

FACTORS AND PRIME NUMBERS

SCHOLARS occupied with the work of mastering the ordinary processes of arithmetic and their application to problems of every-day experience, are not, as a rule, able to devote very much time to the pure theory of number. Nor, perhaps, is this to be regretted, for the subject is one mainly of mathematical interest; yet there are certain fundamental notions that, theoretical in themselves, are nevertheless of much practical value. Among these ideas we may place prime numbers, factors, and the cognate conceptions of measures and multiples. Ability to detect factors saves labour in calculations of nearly every description.

Factors. The notion of a factor is usually acquired by the child without much help from the teacher. He will have employed convenient factors of multipliers and divisors in his multiplication and division by factors and from this work he will be able to give readily convenient pairs of factors of given numbers within the limits of the multiplication table. Without any formal definitions being taught, he will also be able to recognize factors as being common to two or more numbers. This ability is essential for rapid work in fractions such as "cancelling," "reducing to lowest terms"; indeed, practically all processes in fractions depend to some extent upon factors, measures, or multiples. Thus it has been customary in the past to preface formal methods in fractions by a considerable amount of formal practice in factors and in greatest common factors and least common multiples. What the conscientious teacher has perhaps too often forgotten, however, is that all this work is primarily a *means* to a further end, and is not an end in itself. The result of this confusion of means and ends has been that G.C.F. and L.C.M. have frequently received a treatment much too elaborate and rigid, resulting in difficult examples having no practical value whatever. The wise teacher will do well to try to view this matter in its right perspective and avoid elaborate and unpractical examples in these "rules."

Prime Factors. It is customary to begin this subject by requiring children to split up a composite number into its prime factors. Rapidity in this work depends upon ability to apply the simpler tests of divisibility. These again are best taught or acquired incidentally. A few notes on the more common tests may be helpful to teachers.

(1) *Divisibility by* 2. This needs no explanation. All that is necessary is that the units digit shall be even.

(2) *Divisibility by* 4. This depends upon the fact that any number of *hundreds* is divisible by 4. Hence the necessary condition is that the number formed by the last two digits shall divide by 4.

(3) *Divisibility by* 8. As any number of thousands is divisible by 8 we have the necessary condition that the number formed by the last three figures must be divisible by 8.

Thus : 1254 is divisible by 2, but not by 4 or 8.
1284 is divisible by 2 and 4, but not by 8.
1384 is divisible by 2, 4, and 8.

(4) *Divisibility by* 5 *and* 10. These tests need no comment. We may combine 10 with any of the previous tests and so arrive at tests for divisibility by 20, 40, or 80.

(5) *Divisibility by* 3 *and* 9. These are to children the most interesting tests of the series. As usually stated, a number can be divided by 3 if the sum of its digits divides by 3, while an exactly similar statement is true for divisibility by 9. It should be noted that the two statements are not identical, for a number may be divided

by 3 and not by 9, though if it is divisible by 9 it is also divisible by 3. The explanation of these rules to children is not usually attempted. Avoiding all general algebraic statements, however, some such example as the following might be worked with a class of older scholars—

$$\begin{aligned} 234 &= (100 \times 2) + (10 \times 3) + 4 \\ &= (99 \times 2) + (9 \times 3) + 2 + 3 + 4 \end{aligned}$$

But the numbers within the brackets divide by 9. Therefore 234 divides by 9 because 2 + 3 + 4 does.

A simple extension of this method enables the remainder after division by 9 to be given without dividing in full. Thus the remainder after dividing 1234 by 9 is 1, for it must be the same remainder as is obtained by dividing 1 + 2 + 3 + 4 by 9.

The old method of proving the answer to a multiplication sum by the process of "casting out the nines," a trick beloved by teachers of a former generation, depended upon this remainder property after division by 9. The trick is slowly falling into disuse through the modern tendency to avoid long mechanical examples. We illustrate the method for the sake of those to whom it may not be familiar and also for the purpose of indicating some of the cases in which the proof does not hold.

Example—**789 × 456.**

789	Remainder 6	Product 36. Remainder 0.
456	Remainder 6	
4734		
3945		
3156		
359784	Remainder 0.	

The rule is straightforward. Cast out the nines from the two original numbers: 789, 456. Multiply the remainders (if any) and cast the nines from their product. The final remainder (e.g. 0 above) must be the same as the remainder after casting the nines from the product.

The proof we leave to the reader. It will be found in any of the older text-books.

The defects of the so-called "test" are easily seen. Any of the figures in the product can be interchanged or re-arranged in any order and the "test" will still be satisfied. This, the product 359874 in the above sum, answers to the test but is wrong. The "proof" cannot detect this common type of mistake. Again, it cannot detect *wrong position* of a partial product. Thus in the example

$$789 \times 4506$$

a child might forget the effect of the zero and so have his second partial product in the wrong position, thus obtaining a wrong answer as in the example. "Casting out the nines" will not reveal the mistake. The reason is obvious: 394500–39450, which is the amount of error, must be a number itself divisible by 9.

$$\begin{array}{r} 789 \\ 4506 \\ \hline 4734 \\ 3945 \\ 3156 \\ \hline 359784 \\ \hline\hline \end{array}$$

Finally, if 9 should be written for 0 or 0 for 9 in the product, or if a zero is omitted from the product, the test does not show the mistake.

Divisibility by 11. This well-known rule is somewhat more complex and is not usually given to children.

Extensions of the Tests. Simple extensions of the above tests may be made, e.g.—

(*a*) If a number divides by 2 and 3 it also divides by 6.
(*b*) If a number divides by 3 and 4 it also divides by 12.
(*c*) If a number divides by 3 and 5 it also divides by 15.

Similarly 3 and 8 imply 24, 2 and 9 imply 18 (but 3 and 6 do **not**).

Children should also recognize at sight simple multiples of 13, 14, 16, 17, 19, and 25, among others.

The simple process of resolving a number into its prime factors presents no difficulty. Examples set to children should involve only such factors as can be recognized by the ordinary simple tests of division or by inspection. Thus the example 7854 is legitimate, for by successive division it may be resolved into 2 × 3 × 7 × 11 × 17, thus—

$$\begin{array}{r|l} 2 & 7854 \\ \hline 3 & 3927 \\ \hline 7 & 1309 \\ \hline 11 & 187 \\ \hline & 17 \end{array}$$

No child, however, could be expected to factorize easily 4199, which is the product of 13 × 17 × 19.

Most examples given to children will consist of powers of a factor, thus 112 in prime factors is 2 × 2 × 2 × 2 × 7, or $2^4 \times 7$. This repetition of a factor sometimes confuses a child. We may note here that to express a number in prime factors does not disclose all its possible *divisors*, thus 112 in prime factors is 2 × 2 × 2 × 2 × 7, but *any* combination of two or more of these will give a factor or divisor, e.g. 2, 4, 8, 16; 14, 28, 56 are all *divisors* of 112. This suggests that the problem "Find the prime factors of . . ." may usefully be varied to "Find all the divisors of . . ."

The subject of a number and its possible divisors is of some historic interest. Thus, any number that is the sum of all its possible divisors (including unity) was termed a *Perfect* Number. Examples of Perfect Numbers are—

6, 28, 496, 8128;

$$\text{e.g.}\quad 6 = 1 + 2 + 3 \text{ (sum of divisors)}$$
$$28 = 1 + 2 + 4 + 7 + 14$$

Readers who are interested may test 496 and 8128 in the same way. Again, any pair of numbers each of which is the sum of the divisors of the other were termed *Amicable* Numbers. Well-known examples are 220 and 284. Thus the divisors of 220 are—

1, 2, 4, 10, 11, 20, 22, 44, 55, 110,

and the sum of these divisors is 284, while the divisors of 284, viz., 1, 2, 4, 71, 142, have a total of 220.

Practice in the resolution of a number into its prime factors will soon enable a child to recognize at sight the simpler prime numbers. He should also, by further practice, be able to recognize numbers which, though composite themselves, are *prime* to each other, since they possess no factors in common.

In finding whether a number is or is not a prime number most children quickly grasp the fact that it is unnecessary to try any divisors greater than the square root of the given number.

The actual method of finding out all the primes between given limits consists of striking out all multiples. The child may perform the operation for the numbers 1–100 by use of the simple number square as in Fig. 40.

The numbers are struck out systematically :

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	93	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

FIG. 40.

(1) Multiples of 2, e.g. 4, 6, 8, etc., are struck out.
(2) ,, 3, e.g. 6, 9, 12, etc., are struck out.
(3) ,, 5, e.g. 10, 15, 20, etc., are struck out.
(4) ,, 7, e.g. 14, are struck out.

Finally the primes only will be left, e.g. 1, 2, 3, 5, 7, 11, 13, etc.

The method of "sifting out" the primes depends upon the same principle of multiples.

1	11	21	31	41	51	61	71	81	91	101	111	121	131
3	13	23	33	43	53	63	73	83	93	103	113	123	133
5	15	25	35	45	55	65	75	85	95	105	115	125	135
7	17	27	37	47	57	67	77	87	97	107	117	127	137
9	19	29	39	49	59	69	79	89	99	109	119	129	139

Etc., etc.

FIG. 41.

The odd numbers are written down in columns of five numbers as in the diagram (Fig. 41). It will be noted that any given set of multiples are grouped in the table in a constantly recurring order. Thus the multiples of 3 recur in each set of 3 columns in the order shown in Fig. 42. If then a "sieve" be constructed of paper or cardboard with holes in position of squares marked × and applied to the columns, all multiples of 3 can be rapidly struck out when the sieve is applied to the numbers.

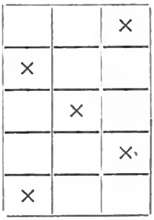


FIG. 42.

A "sieve" similar to that shown in Fig. 43 will serve for all multiples of 7.

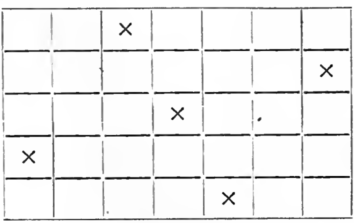


FIG. 43.

Similar sieves may be constructed for any desired series of multiples.

"Aliquot Parts." Though usually dealt with when studying fractions, the subject of "aliquot parts" is a simple concrete application of factors to money and quantity. Thus 10s. may be termed a factor or sub-multiple of 20s. in the same sense that 10 is a factor of 20 and the definition of factor as usually taught is satisfied in the case of the money as in the numbers. Much practice may be given in these concrete factor-parts, of quantities without the continual mention of the somewhat pedantic term "aliquot." Thus, we may ask the children for the factor-parts of 1 foot, e.g. 6 in., 4 in., 3 in., 2 in., $1\frac{1}{2}$ in., etc., or of 1 cwt., e.g. 2 qrs., 1 qr., 1 stone, 16 lbs., etc.; we may ask for the smallest sum of money that can be paid either in florins or half-crowns, for the smallest length that can be cut into pieces of 18 in. or 24 in.; all this work may be done without reference to special rules or special terminology. If the teacher will thus introduce "Aliquot Parts" as simple concrete applications of factors he will prepare the way for later work in Practice, and will also avoid the criticism that his arithmetical teaching proceeds by rules each in a rigid water-tight compartment.

Problems in the Theory of Number. To the mathematician the Theory of Number has always proved a most fascinating field of study. Its problems are infinitely varied; many have been solved with difficulty; many are still unsolved, though most remarkable discoveries have been made in the properties of numbers. Much of this work is of course entirely out of place with young scholars and students, yet when the difficult and unsuitable parts have been omitted, there is a remainder of an elementary yet most attractive nature, which will suggest interesting and profitable quests for older scholars. We suggest a few topics and problems which will be found useful.

Divisibility. (*a*) Complete the following numbers according to directions—

235*	so as to be divisible by			9
456*	,,	,,	,,	6
2*75	,,	,,	,,	15 (three solutions)

(*b*) Why must the product of any three consecutive numbers divide by 6 ? What will be the corresponding divisor for four consecutive numbers ?

(*c*) Why must the difference of any two numbers composed of the same figures in different orders be divisible by 9 ?

Squares. These provide endless problems of interest:

(1) What is the smallest number by which 56 may be multiplied so that the result is a perfect square ?

(2) Any *odd* number may be expressed as the difference of two other squares, thus $13 = 7^2 - 6^2$, $19 = 10^2 - 9^2$, $37 = 19^2 - 18^2$. Why is this ?

(3) Certain square numbers can be expressed as the sum of two other squares, thus—

$$5^2 = 4^2 + 3^2$$
$$13^2 = 12^2 + 5^2$$
$$17^2 = 15^2 + 8^2$$

Can you find any others that are not multiples of these ? (The general solution to this interesting problem is indicated in a later chapter.)

(4) Every number that is a perfect cube can be expressed as the difference of two squares thus—

$$2^3 = 3^2 - 1^2$$
$$3^3 = 6^2 - 3^2$$
$$4^3 = 10^2 - 6^2$$

Try to continue this series and express 12^3 as the difference of two squares ($12^3 = 78^2 - 66^2$).

(5) Every square number is either a multiple of 5 or is one more or one less than a multiple of 5. Thus

no square number can end in 2, 3, 7, or 8. Why is this ?

The above problems indicate but a few of the interesting problems in pure number theory. These and similar problems, if set to older children occasionally, will serve to fill many odd moments in a profitable manner. It is not suggested that they should be propounded as class exercises in the ordinary school routine ; this would destroy much of their charm. They can, however, be given, as suitable occasions arise, to individual scholars to ponder over. For these they will brighten up the dull round of class work and will cause them perhaps to reflect upon the "eeriness" of number. Mathematical seed thus sown by the teacher-enthusiast may bear fruit of untold worth.

Numerous examples in number-theory, set out in popular and attractive form will be found in *Mathematical Amusements*, by H. E. Dudeney. Popular periodical literature also frequently contains "posers" and puzzles, which will attract all but the dullest pupils.

G.C.F. and L.C.M. It has long been the custom in schools, to devote considerable time to the teaching of processes of finding the Greatest Common Factor and Least Common Multiple, as a necessary introduction to fractions. From this point of view the custom cannot be criticized. Both Greatest Common Factor and Least Common Multiple play an important part in ordinary fractional manipulations.

The mistake in the past, however, has been that these rules have been taught as if they were of intrinsic importance to the ordinary child. Thus some teachers still persist in setting examples such as finding the G.C.F. of 23562 and 27846, though we seldom have to reduce to lowest terms such a fraction as $\frac{23562}{27846}$ even in the most complex of fractions.

The subject has its humorous aspect also, for the child who can diligently and accurately apply the rule or trick to clumsy numbers such as those in the above example is also known to " cancel " a much simpler fraction such as $\frac{72}{256}$ by some such method as the following—

$$\frac{\begin{array}{c}9\\ 18\\ 36\\ 72\end{array}}{\begin{array}{c}256\\ 128\\ 64\\ 32\end{array}}$$

thus showing that the connection of G.C.F. with " cancelling " has never been taught to him.

The point we would insist upon is that these two processes of finding G.C.F. and L.C.M. are *subsidiary* processes and are not to be taught to children as if they were important ends in themselves. They are processes having but few applications in ordinary life. Let the teacher who doubts this try to put the example—

" Find the G.C.F. of 23562 and 27846 "

into reasonable concrete form in terms of everyday experience.

Even in working with fractions we may overrate the importance of these rules for the modern tendency in all calculations is to avoid clumsy denominators in fractional manipulations, especially in addition and subtraction of quantities, and to substitute decimals for the fractions.

If, then, by these means we avoid all needlessly complex work in fractions, it follows that we have proportionately reduced the practical importance of the processes known as G.C.F. and L.C.M.

Yet the tradition dies hard. Teachers cling to these two processes. They can easily be " taught " and " learnt " ; they fit into nice compact rigid frameworks ; they form convenient types for " mechanical " sums ; and so we suppose they will linger long in schools, even though they remain tricks seldom understood by children. The height of absurdity seems to be reached by the teacher who without spark of humour teaches his young scholars to apply these tricks to *fractions themselves*, gravely and solemnly propounding some such conundrum as " Find the G.C.F. and L.C.M. of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$ etc.

It follows from what has been said that the only types of examples in G.C.F. or L.C.M. upon which much time need be spent are those involving small numbers which can easily be resolved into factors.

H.C.F. or G.C.M. The initials G.C.M. (Greatest Common Measure) are for some reason or other now displaced by the more fashionable H.C.F. It is true that confusion between Measure and Multiple frequently arose when G.C.M. and L.C.M. were used together. But it should be noted that Greatest Common Measure and Highest Common Factor are not quite identical for " Highest " has a definite algebraic significance which " Greatest " has not. Thus x^2 is of a higher degree than x though not necessarily greater. It is for this reason that we prefer to use the initials G.C.F.

The Teaching of G.C.F. or H.C.F. Suggested stages are as follows—

(1) Oral examples in numbers and quantities, e.g.

What is the greatest number common to 4 and 6, to 6, 10, 12, to 12, 16, 20, to 15, 25, 30, etc. ?

What is the greatest length contained in 1 ft. 6 in. and in 2 ft. ? etc.

What is the largest sum of money contained exactly in 7s. 6d. and in 10s. ? etc.

(2) More difficult examples requiring the use of writing materials but which can be solved by finding the prime factors of the numbers, e.g.

Find the G.C.F. of 90 and 162; of 216 and 516; of 65, 91, and 143, etc.

The method of finding G.C.F. by factorization is the most sensible method for children. All that is necessary is ability to resolve each number readily and accurately into its prime factors. The required greatest factor is easily seen from an examination of these prime factors.

Example—

Find the G.C.F. of 384, 480, 576.

Factorizing—

$$384 = 2^7 \times 3$$
$$480 = 2^5 \times 3 \times 5$$
$$576 = 2^6 \times 3^2$$

An examination of the factors shows that the G.C.F. is $2^5 \times 3$, or 32×3, i.e. 96.

The method is applicable to any number of quantities, whereas the formal method is only directly applicable to two numbers at a time.

Questions set to children involving the finding of a Greatest Common Measure or Factor should in general be such as can be easily solved by the above method.

The Formal Method for G.C.F. Such an example as the following—

"Find the G.C.F. of 42336 and 53088"

has little practical significance and may therefore well be omitted in most schools. But many teachers cling to the older traditions and, therefore, some remarks on the formal method for this process will be looked for.

As usually set out the work is arranged as in (*a*), or more neatly as in (*b*)—

```
(a) 42336)53088(
          42336
          -----
          10752)42336(3
                32256
                -----
                10080)10752(
                      10080
                      -----
                        672)10080(15
                            10080
                            =====
```

Ans. **672**

(*b*)

3	42336	53088	1
	32256	42336	
15	10080	10752	1
	10080	10080	
		672	

Ans. **672.**

The work may be shortened at any stage by rejecting a factor obviously not common to both.

No trick is more easily taught, and all that is necessary is careful division at each stage. Many teachers prefer to begin by writing out the work in full as in (*a*). Later the figures are condensed and tabulated as in (*b*).

It is doubtful whether the above general method is ever, to most children, anything more than a trick which they never trouble to understand but which they know from experience will produce the answer the teacher requires. Even students-in-training frequently fail badly when asked to explain the method mathematically. Yet the mathematical theory which is the basis of this process may be stated very briefly.

"If a number is a factor of two other numbers it is also a factor of their sum, or their difference, or the sum or the difference of any multiples of them."

In symbols—

Let F be a factor of P and Q such that $P = aF$ and $Q = bF$.

Then F is clearly a factor of $mP \pm nQ$, i.e. $maF \pm nbF$.

We leave to the reader the application of this principle to each step of the G.C.F. process as worked out above.

This same general method is, however, capable of very simple graphical demonstration.

Consider the example—

Find the G.C.F. of 119 and 49.

Worked by the general method we have—

2	49	119	2
	42	98	
	7	21	3
	—	21	

This we may illustrate: (*a*) by Lengths; (*b*) by Areas.

By Lengths. Take two strips of paper, 119 mm. and 49 mm. long respectively (Fig. 44).

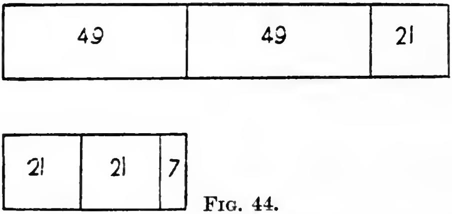


Fig. 44.

Mark off the shorter strip as many times as possible from the longer strip as in the diagram. The remainder is seen to be 21. Next mark off this remainder 21 as many times as possible from the shorter strip. The second remainder is seen to be 7. This second remainder 7 is exactly contained in the first remainder 21, and is therefore the factor required.

By Areas. Example —

Find the biggest square tile which can be used to pave a hearth 9 ft. 11 in. (119 in.) long and 4 ft. 1 in. (49 in.) wide.

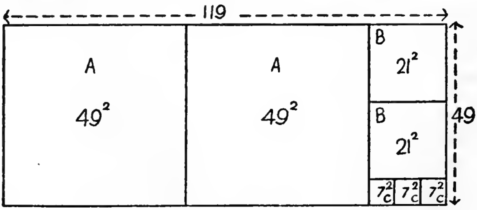


Fig. 45.

Draw the hearth to any convenient scale as in the diagram (Fig. 45).

Proceed by marking off the biggest possible squares, i.e.

(1) Mark off A, A, each 49×49.

(2) Mark off B, B, each 21×21.

(3) Mark off C, C, C, each 7×7.

Square c is clearly the tile of size required.

The diagram is self-explanatory. All such questions involving G.C.F. may be illustrated similarly.

The formal method applies directly only to two numbers or quantities. For more than two numbers the process must be repeated.

Too often in the past examples set to scholars in G.C.F. have involved two numbers only with the unfortunate result that an example such as: Find the G.C.F. of 24, 27, 36, 42, has frequently resulted in the child finding not *G.C.F.* but *L.C.M.*

Applications of G.C.F. The most general application of G.C.F. is in the cancelling of fractions or in their reduction to lowest terms. Other examples involving the process do not commonly occur, though many pseudo-practical examples may be constructed.

Examples in areas involving the finding of the greatest possible square tile, block, or paving stone are frequent in arithmetic books. Such examples may also include volumes as in the following—

"What is the largest possible cubic block that can be used to build a rectangular pile 143 in. long, 78 in. wide, and 39 in. high?"

To such an example the teacher should always append "and how many such cubes would be required?"

In general the examples that can be set on this process are always somewhat artificial, and it remains true that G.C.F., as a process, is a process subsidiary to other calculations. It is possible that its intrinsic importance has in the past been overrated by teachers.

Method of Finding L.C.M. The idea underlying this term is akin to, and complementary with, that of G.C.F., and the methods of finding both are usually taught concurrently.

As with G.C.F., oral work with simple numbers and quantities forms the best beginning and should be constantly employed to make the fundamental notion clear.

Examples—

What is the smallest number which will contain both 3 and 4 an exact number of times ? (12.)

What other numbers will also contain 3 and 4 exactly ? (24, 36, 48, etc.)

What is the smallest sum of money which can be paid exactly in florins or half-crowns ? (10s.)

What other sums can also be paid in florins or in half-crowns ? (£1, £1 10s., £2, etc.)

What is the smallest distance which can be measured exactly with a 2 ft. rule or a yard stick or a 5 ft. tape measure ? (30 ft.)

What other distances can also be thus measured exactly ? (60 ft., 90ft., 120 ft., etc.)

These should be varied as much as possible using the simplest convenient numbers and quantities.

The following table (Fig. 46) shows common multiples and least common multiples of the numbers 1–10 with the range 1–40.

.	2	.	4	.	6	.	8	.	10	.	12	.	14	.	16	.	18	.	20	.	22	.	24	.	26	.	28	.	30	.	32	.	34	.	36	.	38	.	40
.	.	3	.	.	6	.	.	9	.	.	12	.	.	15	.	.	18	.	.	21	.	.	24	.	.	27	.	.	30	.	.	33	.	.	36	.	.	39	.
.	.	.	4	.	.	.	8	.	.	.	12	.	.	.	16	.	.	.	20	.	.	.	24	.	.	.	28	.	.	.	32	.	.	.	36	.	.	.	40
.	.	.	.	5	.	.	.	.	10	.	.	.	.	15	.	.	.	.	20	.	.	.	.	25	.	.	.	.	30	.	.	.	.	35	.	.	.	.	40
.	.	.	.	.	6	.	.	.	.	.	12	.	.	.	.	.	18	.	.	.	.	.	24	.	.	.	.	.	30	.	.	.	.	.	36	.	.	.	.
.	.	.	.	.	.	7	.	.	.	.	.	.	14	.	.	.	.	.	.	21	.	.	.	.	.	.	28	.	.	.	.	.	.	35	.	.	.	.	.
.	.	.	.	.	.	.	8	.	.	.	.	.	.	.	16	.	.	.	.	.	.	.	24	.	.	.	.	.	.	.	32	.	.	.	.	.	.	.	40
.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	.	18	.	.	.	.	.	.	.	.	27	.	.	.	.	.	.	.	.	36	.	.	.	.
.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.	.	20	.	.	.	.	.	.	.	.	.	30	.	.	.	.	.	.	.	.	.	40

Fig. 46.

Thus at a glance we see that the Least Common Multiple of 2, 3, 4, 6, is 12, while Common Multiples are 24 and 36.

So, too, 24 is seen to be the L.C.M. of 2, 3, 4, 6 and 8.

A similar table should be constructed and used by beginners.

The next stage should concern examples easily factorized thus—

Find the L.C.M. of 15, 20, 24, 25.

Factorizing—

$$\begin{aligned} 15 &= 3 \times 5 \\ 20 &= 2 \times 2 \times 5 \\ 24 &= 2 \times 2 \times 2 \times 3 \\ 25 &= 5 \times 5 \end{aligned}$$

Taking the factors in order the L.C.M. is obtained as $3 \times 5^2 \times 2^3 = 600$.

The underlying theory is straightforward. The examples set to children, especially to beginners, should all be solved in this manner.

The Formal Method. As usually set out, the above example worked formally would appear as follows—

$$\begin{array}{r|l} 2 & 15,\ 20,\ 24,\ 25 \ldots (1) \\ \hline 2 & 15,\ 10,\ 12,\ 25 \ldots (2) \\ \hline 3 & 15,\ \not{5},\ 6,\ 25 \ldots (3) \\ \hline & \not{5},\ \quad 2,\ 25 \ldots (4) \end{array}$$

Ans. $2 \times 2 \times 3 \times 2 \times 25 = 600$

The method is clearly one in which the prime factors necessary for the answer are found by successive division. At any stage a number that is clearly a factor of another number in the same line may be discarded as with " 5 " in lines 3 and 4 of the above for any multiple which contains 25 must also contain 5.

The choice of divisors should not be haphazard, and beginners, for safety, should divide by prime factors only. As an example that may serve as a warning, the reader should study the following, which is incorrect, and endeavour to detect the error.

```
6)18, 28, 108, 210
  ----------------
7) 3, 28,  18,  35
  ----------------
     2) 4,  18,   5
       -----------
        2,   9,   5
```

Ans. **6 × 7 × 2 × 2 × 9 × 5 = 7,560,** the right answer being half this, viz., **3,780.**

The method is a dangerous weapon in careless hands. It is not, however, necessary for the method to be taught at all, for examples in L.C.M. should be such as can be solved either by inspection or by the previous method of factorizing.

Practical Applications of L.C.M. As with G.C.F., the most obvious use of L.C.M. is in the manipulation of fractions, especially in their addition and subtraction. Thus, if we desire to add $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, we proceed through the L.C.M. of 2, 3, and 5, and add them as $\frac{15}{30} + \frac{10}{30} + \frac{6}{30}$. A moment's thought will show the close connection between G.C.F. and L.C.M., for though the process apparently depends upon L.C.M. yet $\frac{1}{30}$ is actually the G.C.F. of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$.

Apart from the use of L.C.M. in fractional manipulation, the range of problems involving its use is somewhat greater than with G.C.F. Most of these problems deal with what may be termed recurring concurrences, such as the well-known problems of bells tolling at different intervals; of clocks, etc., working at different rates; of rolling wheels of different diameters; of rotating shafts with different velocities. The problem to be solved is usually a simple L.C.M.: "After what $\left\{\begin{array}{l}\text{time}\\ \text{distance}\end{array}\right\}$ will ——— occur together?" or "At what intervals will ——— recur?"

The same idea may be extended to volume as in the simple case: "What is the smallest perfect cube that can be built with the ordinary brick (3 in. $\times$ $4\frac{1}{2}$ in. $\times$ 9 in.)?

Many well-known puzzle problems depend upon L.C.M. for their solution. Among such are those of the following type—

(*a*) What is the least possible number of stairs in a flight such that if I go up two at a time or three at a time or four at a time or five at a time, there is always an odd one at the top?

The answer is clearly—

$$(\text{L.C.M. of } 2, 3, 4, 5) + 1, \text{ i.e. } 61 \text{ stairs.}$$

The same would be true of course for 121, or 181, or for $(60n + 1)$ stairs.

(*b*) What is the least possible number of stairs in a flight such that if I go up two at a time there is one over; three at a time, two over; four at a time, three over; five at a time, four over?

The answer is—

$$(\text{L.C.M. of } 2, 3, 4, 5) - 1, \text{ i.e. } 59 \text{ stairs.}$$

One important connection between the G.C.F. and the L.C.M. of two numbers is worthy of note as being frequently useful in solutions—

Let P and Q have a G.C.F. $= F$ such that $P = aF$ and $Q = bF$ where a and b have no factor in common.

The L.C.M. of P and Q is clearly abF, and the product of L.C.M. $\times$ G.C.F., i.e. $abF \times F$ is the same as the product $P \times Q$ or $aF \times bF$. *Thus the product of two numbers is always the same as the product of their L.C.M. and G.C.F.* This enables the L.C.M. to be found directly, if the G.C.F. is known, for L.C.M. $= \dfrac{\text{Product}}{\text{G.C.F.}}$

The truth does not hold for more than two numbers.

CHAPTER XII

THE TEACHING OF FRACTIONS

It has been usual for a long period to be spent on the manipulation of "Vulgar Fractions" after exhaustive treatment of the simple and compound rules. With the break-up of the old rigidity in the treatment of the "rules of arithmetic," however, it is customary to teach and to use fractional ideas at a very early age. Historically this modern practice of ensuring early acquaintance with fractions is sound, for according to the history of civilization it is certain that "fractions" in calculations were known and used at a very early period. (The Egyptians of the earliest epochs were expert in some of the simpler manipulations.) This ancient foundation is possibly at the basis of the common fault of the traditional treatment of fractions, which is usually far too elaborate and involved and, therefore, takes time that should be given to simpler decimal manipulations by which the required result is obtained more easily.

The result of this undue emphasis is seen in many directions. We here indicate a few.

(*a*) Many results to ordinary problems are given in clumsy fractional form. Thus we see $9\frac{17}{87}$ feet instead of the simpler and almost equally accurate 9·2 feet or 9 ft. $2\frac{1}{2}$ in. The difference in the difficulty of converting $\frac{17}{87}$ feet to inches and ·2 ft. to inches is too obvious to need comment. So, too, the zeal for fractional sums may lead us to denominators much larger than 87, forgetful of the fact that few people in real life will trouble to evaluate (except in a roughly approximate manner) such a sum as £$9\frac{97}{389}$. Yet *exercises* involving fractional manipulations such as these may be found in any school and text-book. These frequently result in denominators of a size incomprehensible to children.

(*b*) Another grave fault of too much fractional teaching is seen in the tendency to make the children turn all decimals into vulgar fractions before working, with the result that work is continually set with common decimals such as ·25, ·125, ·0625, etc., and general work in decimals is neglected. Thus the child who is asked to evaluate

$$\cdot 865 \text{ of } £9$$

is apt to proceed laboriously through the vulgar fraction $\frac{865}{1000}$, which is carefully cancelled to $\frac{173}{200}$, and the answer finally obtained by further laborious evaluation of $£\frac{173}{200} \times 9$.

All this fractional work is, of course, unnecessary, and is easily avoided by simple continued multiplication by 9, 20, 12, as follows—

$$\begin{array}{r} \cdot 865 \\ 9 \\ \hline £7\cdot 785 \\ 20 \\ \hline \text{s.}15\cdot 700 \\ 12 \\ \hline \text{d. } 8\cdot 4 \\ \hline\hline \end{array} \qquad \textit{Ans. } £7 \ 15\text{s. } 8\text{d. approx.}$$

So, too, the converse process is treated by fractional methods when simple division is all that is required. Thus, in working the example—

Express 1 ft. 10 in. as a decimal of 1 yard, many children proceed through fractions first, thus: $\frac{1 \text{ ft. } 10 \text{ in.}}{1 \text{ yd.}} = \frac{22}{36} = \frac{11}{18}$, and they then convert $\frac{11}{18}$ to a decimal instead of attacking the problem directly by division by 12 and 3 thus—

$$\begin{array}{rl} 12\overline{)22} & \text{inches} \\ 3\overline{)\ 1\cdot 833} & \ldots \text{ ft.} \\ \cdot 611 & \ldots \text{ yds.} \end{array}$$

Nor does the "fractional" evil stop with this tendency to convert all decimals to vulgar fractions rather than to manipulate them *as* decimals. It extends into the whole subject of percentage, with the consequent narrowing of the field of examples to such percentages as are expressed easily as vulgar fractions, e.g. $2\frac{1}{2}\%$, 4%, 5%, 10%, $12\frac{1}{2}\%$, 20%, 25%, etc. The result is that such an example as—

Find 28% of £7 10s. is frequently treated thus—

$$\frac{\overset{7}{\cancel{28}}}{\underset{\underset{5}{\cancel{25}}}{\cancel{100}}} \times \frac{\overset{3}{\cancel{15}}}{2} = £\frac{21}{10} = £2\ \ 2\text{s.}$$

while it might more simply be worked as—

$$£(\cdot 28 \times 7\cdot 5) \text{ or } £(\cdot 14 \times 15) = £(2\cdot 1) = £2\ \ 2\text{s.}$$

or by simple multiplication by 7 and a division by 2, as—

$$\begin{array}{rl} \cdot 28 & \\ 7\frac{1}{2} & \\ \hline 1\cdot 96 & = 7 \text{ times} \\ \cdot 14 & = \frac{1}{2} \quad ,, \\ \hline £2\cdot 1 & = 7\frac{1}{2} \quad ,, \\ \hline\hline \end{array}$$

The advantage of decimal treatment is clear in such an example as—

Express 3s. $4\frac{1}{2}$d. as a decimal and as a percentage of £1.

3s. $4\frac{1}{2}$d. = 40·5 pence

Dividing by 12 and 20, we have—

$$\begin{array}{r} 12\overline{)40\cdot 5} \text{ pence} \\ 2,0\overline{)3\cdot 375} \text{ shillings} \\ \cdot 16875 \text{ pounds} \end{array}$$

Hence 3s. $4\frac{1}{2}$d. is clearly £·16875 or 16·875 % of £1.

Here both answers required are obtained at once. This simple decimal treatment should be compared with the fractional method requiring—

(i) The simplification, reduction to lowest terms, and decimalizing of the fraction $\dfrac{3s.\ 4\frac{1}{2}d.}{£1\ \ 0s.\ 0d.}$

(ii) The evaluation of $\dfrac{3s.\ 4\frac{1}{2}d.}{£1\ \ 0s.\ 0d.} \times 100.$

(*c*) Too much reliance on " fractions " is also a frequent cause of clumsy calculation in mensuration and practical work. Thus, children when asked to find the area of a triangular piece of paper will frequently make all necessary measurements using fractions and perform a calculation such as—

$$\text{Area} = \tfrac{1}{2}\,(9\tfrac{7}{12} \times 3\tfrac{5}{8})\text{ sq in.}$$

when decimal measurements in inches and tenths would probably be just as accurate while the calculation would be simpler and the result more easily interpreted.

Similarly far too many children, when asked to find the circumference of a circle of 18 inches radius, will proceed through fractions thus—

$$\text{Circumference} = 2\pi r = \frac{36 \times 22}{7} = \frac{792}{7} = 113\tfrac{1}{7}$$

involving a needless multiplication 36 × 22, when the answer is readily obtained in inches or feet as follows—

(*a*)		(*b*)	
36 in.		3 ft	
$3\frac{1}{7}$		$3\frac{1}{7}$	
108		$9\frac{3}{7}$ ft.	
$5\frac{1}{7}$			
$113\frac{1}{7}$ in.			

A much graver evil is the tendency to devote too much time to what are known as *complex* fractions, especially those which have been wittily described as " skyscrapers."

The following examples taken from a well-known text-book are typical complex fractions—

(*a*) Evaluate $1 \div [4 - 1 \div \{2 - 1 \div (1 - \frac{5}{13})\}]$

(*b*) Simplify $\dfrac{1\frac{1}{8} \div 1\frac{1}{5}}{\frac{2}{3} \text{ of } \frac{5}{9} \div 10\frac{1}{3}} \times \dfrac{1\frac{1}{2} \text{ of } 4\frac{1}{9}}{6\frac{15}{16} \text{ of } 5\frac{1}{3}}$ of £1

(*c*) Find $\dfrac{1}{4 - \dfrac{1}{2 - \dfrac{1}{1 - \frac{5}{13}}}}$ of 3 cwt. 16 lb.

Now, as mathematical amusements, these may be excellent; to many minds their solution will undoubtedly give some pleasure. But as practical problems, suitable in kind for the average pupil of an elementary school and for application in the after life of the average adult, their presence cannot be defended. The average life is too short and crowded to be bothered with these amusements of the schoolmen and the professors. The strict grim educationist who measures the importance of any subject of the curriculum by its "disciplinary" rather than by its "useful" value may insist on their retention, but the modern teacher will lighten his arithmetical cargo by avoiding the long, involved, cumbersome complex fractions.

Of course, such manipulations *can* be taught by a persistent teacher, yet they involve such a background of mathematical theory and convention that at best they become for the pupil but a carefully memorized mechanical process.

Most thoughtful teachers will admit in their more impartial moments that such fractional manipulations as those illustrated above are hardly worth the time and trouble expended upon them alike by teachers and pupils. To the plea that the text-books print such examples the answer is obvious. They can always be avoided or better still a text-book that does not print them can be selected.

To the second objection—that examiners frequently set such examples—the answer is more difficult, for it is an unfortunate fact that examiners in the very elementary subjects sometimes appear to be at least a generation behind in their subject, and thus act as a heavy and unnecessary drag upon educational progress.

But a knowledge of fractions must still be taught in every school, and so after this somewhat lengthy introduction we will proceed to the main business of this chapter, which is to indicate the main lines of a suitable development of the subject and the best methods of approach to each separate topic.

Introductory Fractions. To the teacher who makes full use of ordinary concrete aids, the teaching of the meaning and writing of simple fractions presents no serious difficulties.

Paper-cutting and folding, squared paper, and squared blackboard, ruler graduations, simple drawing exercises, and many other means may be employed to teach fundamental notions. In actual notation the child has a familiar concrete instance in the method of indicating farthings, i.e. $\frac{1}{4}$d., $\frac{1}{2}$d., $\frac{3}{4}$d. He will appreciate that $\frac{1}{2}$d. could also be written as $\frac{2}{4}$d. This is the simplest illustration of the theory of fractions and ratio.

He will begin by finding meanings in length and area, on paper and on ruler-edge, for such symbols as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. He can draw areas and divide them by cutting or drawing, to show such simple facts as—

(a) $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ (c) $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$

(b) $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$ (d) $\frac{3}{4} \div \frac{3}{8} = 2$

All these and countless similar examples, can be demonstrated by such a simple piece of apparatus as a square of plain paper of 4 in. edge.

So, too, on the edge of the ruler, as commonly graduated, he can perform numerous little calculations involving the addition, subtraction, multiplication and division of the above fractions. No formal methods of setting out are necessary. Each problem has a concrete interpretation on paper or ruler edge.

Again we may combine simple fractions with easy drawing exercises. Thus a child may be required to draw a line 6 inches long; to mark off one quarter of it; to write down the length of the remainder; to find how many pieces of $1\frac{1}{4}$ inch can be cut from it and what length remains; or he may be asked to draw a line half as long again as the given line, etc.

Following halves, quarters, eighths, and perhaps sixteenths treated concretely as outlined above, we may next deal with *thirds*, *sixths*, and *twelfths* by similar methods. Here, the ruler graduations are very useful, for we may make numerous references to these fractions of *one foot* as well as of one inch. Thus such an example as $\frac{1}{3} + \frac{1}{6}$ has a simple interpretation as 4 in. + 2 in. (if 1 ft. is taken as the unit), which is clearly 6 in. or $\frac{1}{2}$ of 1 foot. Hence, we may infer that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ in this case.

"Fifths" and "tenths" may similarly be treated with the help of inches and tenths of inches. If preferred, these fractions may be left until decimals receive preliminary treatment, but they may be treated equally well as vulgar fractions.

These simple fractions—their meaning and relation—may be taught long before any precise rules for addition, subtraction, etc., of fractions are known. With this concrete and natural approach to the world of fractions, the ordinary technical terms "numerator" and "denominator" and their meaning are acquired easily and incidentally. Indeed to *begin* by teaching definitions of these terms is to waste valuable time and effort.

Innumerable applications arise naturally. For example, a child readily learns to evaluate such things as—

$$\tfrac{3}{4} \text{ of £1 or } \tfrac{3}{8} \text{ of 10s.,}$$

and recognizes the equivalence of $\frac{3}{4}$ lb. of tea and of $\frac{1}{2}$ lb. + $\frac{1}{4}$ lb.; he is sure that $\frac{3}{4}$ lb. = 12 oz., and might therefore be written equally well as $\frac{12}{16}$. Again he may find the cost of $2\frac{1}{2}$ lbs. of sugar at 7d. a lb. and of $\frac{3}{4}$ of a yard of calico at 10d. a yard. So, too, he can find the length of $\frac{2}{3}$ of a yard as 2 feet from which it is also clear that $\frac{2}{3}$ of 7 yards ought to be seven times as long, or 14 feet.

In these simple applications of fractions there is excellent opportunity for calling attention to short cuts. Thus, if we require to find $\frac{4}{5}$ of £3 10s., we might easily deduct $\frac{1}{5}$, i.e.

£	s.	d.
3	10	–
	14	–
£2	16	–

Similarly, $\frac{9}{10}$ of 370 is clearly 370 – 37 (by deducting $\frac{1}{10}$) as well as $\frac{370 \times 9}{10}$ or $37 \times 9 = 333$.

Again, such shopping items as $3\frac{3}{4}$ yards at 11d. a yard lose all their fractional terrors if we use the simple truth that $a \times b = b \times a$, i.e. $3\frac{3}{4}$ yards at 11d. a yard cost as much as 11 yards at $3\frac{3}{4}$d., and this cost is found by simple multiplication, no awkward evaluation such as $\frac{3}{4}$ of 11d. being required. This principle is not duly recognized, and the writer has found that even adults will frequently traverse the longer and more difficult path.

One other simple yet essential transformation at this preliminary stage must also receive regular treatment in view of later requirements. This is the change from improper fraction to mixed number, and from mixed number to improper fraction. The method is acquired easily and as easily demonstrated.

Thus the fact that $1\frac{1}{4} = \frac{5}{4}$ or that $\frac{3}{2} = 1\frac{1}{2}$, and all similar facts are easily seen on ruler edge or by drawing exercises. Constant oral practice in these two converse manipulations is necessary for successful work in formal fractions.

Theory of Fractions. One cardinal fact influences the whole field of fractions and their manipulations. It is that any fraction can be expressed in an infinite number of ways, i.e.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}, \text{ and so on } \textit{ad inf.}$$

This is usually expressed in the form "a fraction is unaltered if numerator and denominator be multiplied or divided by the same number." This fact, so obvious to the mathematical mind, is of such paramount importance in mathematics that it must receive definite treatment with beginners. It might well be called the Golden Rule of Fractions, for it is the basis of all general operations such as "reducing to lowest terms" and "cancelling"; it is employed in fundamental processes such as addition and subtraction, and its uses throughout the whole range of mathematics in ratio, equations, simplifications, etc., are too numerous to specify.

The principle is easily demonstrated. The pupil will already be familiar with simple concrete instances. He will have seen in drawing squares and rectangles or on his ruler-edge that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$, etc. He will know that $\frac{3}{4}$ of 1s. is also $\frac{9}{12}$ of 1s. Conversely he will see that $\frac{8 \text{ pence}}{12 \text{ pence}}$ is better expressed as $\frac{2}{3}$, and he will have noted some simple cases as that $\frac{1}{3}$ and $\frac{1}{4}$ can both be expressed in the same name-parts, i.e. as $\frac{4}{12}$ and $\frac{3}{12}$; or that $\frac{1}{2}$ of 1 ft. $-$ $\frac{1}{3}$ of 1 ft. leaves 2 inches which is $\frac{1}{6}$ of 1 ft., and might have been worked as $\frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.

The more intelligent pupils might reason more abstractly from a unit. Since 1 by the definition of a fraction may be expressed in any infinite number of ways, for example $\frac{4}{4}$, $\frac{6}{6}$, $\frac{8}{8}$, $\frac{10}{10}$, $\frac{24}{24}$, etc., it follows that $\frac{1}{2}$ may be expressed as $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{12}{24}$, etc.

Exercises on this important point are capable of great variety.

1. *Paper-folding.* It is possible by continued folding to divide a strip of paper into 2^n parts. This is useful as showing the equivalence and relation of halves, quarters, eighths, sixteenths, etc. The same strip by careful folding first into three may be creased to show thirds, sixths, twelfths, etc., and the whole may be used for any number of demonstrations.

2. *Drawing Exercises.* These can be set as often as desired. The pupil may draw lines or rectangles to show that $\frac{2}{3}$ is the same as $\frac{4}{6}$, or that $\frac{15}{20}$ may equally well be expressed as $\frac{3}{4}$, and that this again is the same as $\frac{9}{12}$. Again, he may show in lines or rectangles that $\frac{1}{2}$ contains $\frac{1}{12}$ six times. Squared paper is invaluable in these graphic exercises.

3. *Calculations and Applications.* These may take the form of mechanical drill or of problems.

(*a*) *Reduction to Lowest Terms.* This important process is constantly used. Regular practice, oral and written, should be given. The child usually has no trouble in learning the method of cancelling common factors of numerator and denominator. Care should be taken that the largest factor should be looked for. Nothing looks so clumsy as reductions such as—

2
~~4~~
~~8~~
~~16~~
~~32~~
—
~~48~~
~~24~~
~~12~~
~~6~~
3

where one division by 16 would have given the required answer.

(*b*) *Comparison of Fractions.* This involves the "multiplication" aspect of the rule. Thus the child may be taught to compare two fractions by expressing them in the same denomination.

Taking as example the exercise, "Which is greater, $\frac{3}{4}$ or $\frac{4}{5}$?" and using 20ths, we have $\frac{15}{20} : \frac{16}{20}$. These fractions can be compared at once. Children may be set to arrange in order of magnitude three fractions such as $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{6}$, and the question may be still further extended by asking them to find the difference between the greatest and the least of the fractions given. This general method of comparing fractions by reducing each to the same denominator forms a convenient and natural introduction to formal addition and subtraction of fractions. Frequently, however, other methods may be adopted, shortening work considerably. Thus, in comparing two fractions such as $\frac{4}{17}$ and $\frac{12}{53}$, we may express $\frac{4}{17}$ as $\frac{12}{51}$, and then compare $\frac{12}{51} : \frac{12}{53}$. Children find this method of equal numerators a little harder in practice, but it is frequently useful. The more general method is also capable of "short cuts" in comparing two fractions. Thus, in dealing with two fractions such as $\frac{4}{7}$ and $\frac{5}{9}$, the new denominator may be ignored and the question settled by simple cross multiplication, i.e.

$$\frac{4}{7} > \frac{5}{9} \text{ because } 36 > 35.$$

(*c*) *Problems and Applications.* Since such a statement as $\frac{2}{3} = \frac{10}{15}$ may also be viewed as a simple equation, we may set a variety of "missing figure" puzzles such as $\frac{2}{3} = \frac{?}{15}$, $\frac{2}{3} = \frac{10}{?}$, $\frac{10}{15} = \frac{?}{3}$, $\frac{10}{15} = \frac{2}{?}$. These "missing-figure" problems are easily constructed. Any method of solution should be allowed and the whole subject of equivalent fractions may thus be linked up with the subject of ratio and equations. The whole subject of proportion and similarity is another aspect of this equivalence of fractions or ratios.

We have thus outlined a fairly extensive course of preliminary fractions which may be begun as early in the child's life as is thought desirable. If such a course is patiently carried out and spread over two or three years the formal methods and manipulations will be taught with far less trouble and in far less time than they would otherwise require. For the sake of the inexperienced teachers we would insist once more upon the need in all this preliminary work for (*a*) simple denominators; (*b*) ample practical work—drawing, etc.; (*c*) concrete applications throughout.

If this groundwork is faithfully covered there will be no need of the old pedantry of teaching "Numeration and Notation" of Fractions in one lesson, or of committing to memory formidable definitions of Vulgar Fraction, Numerator, Denominator, Improper Fraction, Mixed Number, etc., and the teacher of older scholars will be amazed at the understanding shown by the pupils of the subject of fractions and their manipulation.

Addition of Fractions. Fractions can be added readily if expressed in like terms; we can add $\frac{1}{2}$ and $\frac{1}{5}$, if we express both in tenths, i.e. as $\frac{5}{10}$ and $\frac{2}{10}$, our answer being $\frac{7}{10}$. We might have added them equally well as twentieths, i.e. as $\frac{10}{20}$ and $\frac{4}{20}$, giving us $\frac{14}{20}$, but this would have involved reduction afterwards. The rule is clear: "Express all fractions to be added, as fractions having the same denominator." In practice the rule becomes the simple one of finding the L.C.M. of the denominators.

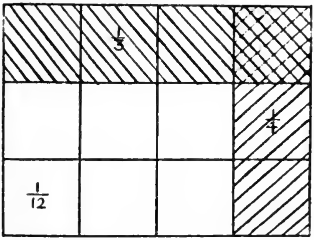


FIG. 47.

The rule is usually readily grasped but, when necessary, it may be easily demonstrated by drawing or by ruler graduations. Thus, in Fig. 47, if a rectangle 4 × 3 represents a unit, then a horizontal row represents $\frac{1}{3}$ or $\frac{4}{12}$, and a vertical row represents $\frac{1}{4}$ or $\frac{3}{12}$, and clearly

$\frac{1}{3} + \frac{1}{4}$ is $\frac{7}{12}$ as shaded. The same addition may be performed on the ruler edge in fractions of an inch (graduations showing "twelfths") or in fractions of a foot.

Formal Setting Out.

$$\text{e.g. } \frac{1}{2} + \frac{2}{3} + \frac{4}{5}$$

$$\text{L.C.M. of denominators} = 30.$$

$$\therefore \text{ Sum is now } \frac{15}{30} + \frac{20}{30} + \frac{24}{30} \quad . \quad . \quad (a)$$

$$\text{or } \frac{15 + 20 + 24}{30} \quad . \quad . \quad (b)$$

$$\text{i.e } \frac{59}{30} \text{ or } 1\frac{29}{30}$$

Line (*a*) is not usually written; the example is generally set out in form (*b*).

The following alternative arrangement of working is worthy of note as being frequently more convenient—

	30
$\frac{1}{2}$	15
$\frac{2}{3}$	20
$\frac{4}{5}$	24
	$\frac{59}{30} = 1\frac{29}{30}$

The addition is easier, being vertical instead of horizontal, and the repetition of the sign + is avoided. The method is in use on the Continent.

All that is necessary for success in teaching addition of fractions is practice in carefully graded examples. Mixed numbers may be introduced at any point. They present no new difficulties, for fractions and whole numbers are added separately before final combination.

Subtraction of Fractions. This rule like the rules for any kind of subtraction has its own special difficulties.

The basis of the rule is identical with that for addition of fractions, and is capable of demonstration by the same methods. Thus we cannot subtract $\frac{2}{3}$ from $\frac{3}{4}$ directly, but can do so if the fractions are expressed as $\frac{8}{12}$ and $\frac{9}{12}$.

The process may be set out in either of the forms already noted, e.g. (*a*) or (*b*)—

(*a*)

$$\begin{aligned} &\frac{3}{4}-\frac{2}{3} \\ =\ &\frac{9}{12}-\frac{8}{12} \\ =\ &\frac{9-8}{12} \\ =\ &\frac{1}{12} \end{aligned}$$

(*b*)

$$\begin{array}{c|c} & 12 \\ \hline \frac{3}{4} & 9 \\ \frac{2}{3} & 8 \\ \hline & \frac{1}{12} \end{array}$$

The subtraction of mixed numbers has, however, considerable difficulties for the beginners, and generally the teacher finds it convenient to insist on a particular method. The following methods are all in common use. The reader should examine each impartially, striving not to be biased by the particular method which he or she uses personally.

Example: $3\frac{1}{3} - 2\frac{3}{4}$

Decomposition. Work as follows:

$$\begin{aligned} 3\frac{1}{3}-2\frac{3}{4} &= 3\frac{4}{12}-2\frac{9}{12} \\ &= 2\frac{16}{12}-2\frac{9}{12} \\ &= \frac{7}{12} \end{aligned}$$

Equal Additions. Work as follows:

$$\begin{aligned} 3\frac{1}{3}-2\frac{3}{4} &= 3\frac{4}{12}-2\frac{9}{12} \\ &= 3\frac{16}{12}-3\frac{9}{12} \quad \left(\begin{array}{l}\text{adding } \frac{12}{12} \text{ to one} \\ \text{and 1 to the other}\end{array}\right) \\ &= \frac{7}{12} \end{aligned}$$

Method using Negative Quantities. Work as follows—

$$3\frac{1}{3} - 2\frac{3}{4} = 1\,\frac{4-9}{12} = 1 - \frac{5}{12} = \frac{7}{12}$$

The last method is perhaps the most convenient for dealing with examples involving both addition and subtraction. Thus—

$$3\frac{1}{2} - 1\frac{5}{6} + 2\frac{1}{9}$$

May be worked as—

$$4\,\frac{9-15+2}{18} = 4\,\frac{11-15}{18} = 4 - \frac{4}{18}$$

$$= 3\frac{14}{18} = 3\frac{7}{9}$$

Multiplication and Division of Fractions. No two "tricks" in arithmetic are so readily acquired by children as multiplication and division of fractions, and no two processes are more difficult to demonstrate and explain. Yet they must always be taught, for they are constantly required in arithmetical calculations. So useful indeed are the two rules that we are obliged to confess that here the teacher is compelled to choose between facility to perform a "trick" with accurate rapidity, and the ability to "explain" or to "prove" the same. We think the great majority of children (and very many teachers) will work the sums accurately and yet stumble at explaining these processes. Yet some reasonable demonstration is possible, and even necessary, if teachers are not to be charged with the old crime of handing on "rules of thumb," the reasons for which are not clearly understood. We will discuss each in turn.

Multiplication and Division of Integers. Some teachers prefer to subdivide each process into two, i.e. to teach first multiplication of fractions by whole numbers and then by fractions and similarly with division. This subdivision is not really necessary, for the same general methods which cover multiplication and division, for example by $\frac{3}{4}$, will also cover multiplication and division by $\frac{3}{1}$, i.e. by 3. The manipulation of whole number multipliers and divisors, however, introduces one aspect of cancelling worthy of explanation. Taking the example $\frac{3}{4} \times 4$, the work may be performed in two ways—

(*a*) We may increase the numerator

Fig. 48.

i.e. increase the *number* of parts (still fourths) from 3 to 12 as seen in Fig. 48 ; or

(*b*) We may increase the *size* of the parts (still retaining their number—three) as seen in Fig. 49—

Fig. 49.

making each *part* four times as large. Thus the answer to (*a*) is $\frac{12}{4}$, and to (*b*) 3 units.

Method (*b*) explains "cancelling" in this example.

Division by an integer may be performed similarly in two ways. Thus $\frac{3}{5} \div 3$ may be expressed as (*a*) $\frac{1}{5}$ (by division of the *number* of parts), or (*b*) $\frac{3}{15}$ (by division of the *size* of the parts). Clearly method (*b*) is preferable, where possible, for multiplication and method (*a*) for division by integers.

Multiplication of Fractions by Fractions. When multiplying fractions by fractions, all previous conceptions of multiplication must be modified. So far, we have considered multiplication in its *normal* aspect, i.e., as involving the addition of equal quantities and the process has given an answer always larger than the original quantity multiplied.

Addition of equals will clearly not apply to such an example as $\frac{5}{7} \times \frac{3}{4}$, for the answer is *less* than $\frac{5}{7}$ instead of greater.

Without discussing any elaborate theories of multiplication, a little thought will show us the link between multiplication by whole numbers and multiplication by fractions. The link is found in the little word "of," for just as 20×3 means that the answer required is the total of three *of* the groups of twenty, so $20 \times \frac{3}{4}$ may be interpreted as requiring us to find three-quarters *of* the group of twenty.

This explanation of fractional multiplication covers every case. Thus $1\frac{1}{3} \times 2\frac{1}{4}$ means that we must find the value of $2\frac{1}{4}$ of the group $1\frac{1}{3}$, or (since $b \times a = a \times b$) we may equally well interpret the requirement as $1\frac{1}{3}$ groups of $2\frac{1}{4}$.

This identity of " × " and "of" is soon taught. The child can draw lines or rectangles to discover that $\frac{1}{4}$ of $\frac{1}{2}$ or $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$, a result reached by the ordinary rule, thus $\frac{1}{4 \times 2}$.

So, too, he can discover on his ruler edge that $\frac{2}{3}$ of $\frac{3}{4}$ is $\frac{1}{2}$ or that $\frac{3}{4}$ of $\frac{2}{3}$ is the same as $\frac{3}{4}$ of $\frac{4}{6}$, i.e. $\frac{3}{6}$ or $\frac{1}{2}$, which again may be reached by cancelling and multiplying in the usual way.

The close identity of "of" and "×" is also seen by comparing answers to pairs of problems such as—

(*a*) $\frac{1}{2}$ of 3 inches; $\frac{1}{2}$ inch × 3.
(*b*) $\frac{3}{4}$ of 4 inches; $\frac{3}{4}$ inch × 4.
(*c*) $\frac{2}{3}$ of 3 shillings; $\frac{2}{3}$s. × 3.
(*d*) $\frac{3}{4}$ of £2; £$\frac{3}{4}$ × 2.

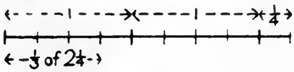


Fig. 50.

Interpreting, then, the problem $1\frac{1}{3} \times 2\frac{1}{4}$ as meaning $1\frac{1}{3}$ *of* the group $2\frac{1}{4}$, we may represent this group by drawing or by concrete objects, and so find an answer.

(*a*) Using lines, a group of $2\frac{1}{4}$ can be arranged as in Fig. 50.

Thus $1\frac{1}{3}$ such groups can be shown as in Fig. 51—

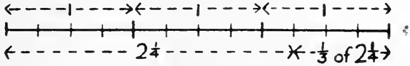


FIG. 51.

and the answer is clearly 3.

Using the "trick," we work as follows—

$$2\frac{1}{4} \times 1\frac{1}{3} = \frac{\overset{3}{\cancel{9}}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{3}} = 3$$

The same kind of drawing will demonstrate that the group $1\frac{1}{3}$, if repeated $2\frac{1}{4}$ times, will also give the result.

(*b*) Using rectangles or squares. Draw the group $1\frac{1}{3}$ in any convenient units (Fig. 52). Repeating this

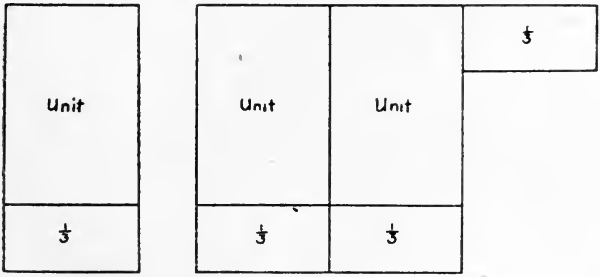


FIG. 52.

group $2\frac{1}{4}$ times, we can see clearly that $2\frac{1}{4}$ of these groups is equivalent to 3 of the original units.

Such drawing exercises are useful, if the results are compared in every case with the answers obtained by the rule.

Many teachers of the older generation, however, are rather shy when graphic methods such as the above are proposed, and would prefer the older method of explanation by argument.

The reasoning is simple though somewhat abstract for children.

Consider the example : $\frac{2}{5} \times \frac{3}{7}$

Here the multiplier may be considered *not* as $\frac{3}{7}$ of a unit but as the seventh part of 3 or as $3 \div 7$. If then we multiply by 3 and so obtain $\frac{6}{5}$, this answer is clearly too big. We have multiplied by 3 instead of by a seventh part of 3. Hence to correct we divide $\frac{6}{5}$ by 7 and so obtain the answer $\frac{6}{35}$.

We may reverse the steps of the argument as follows—

$\frac{1}{5}$ of any quantity is clearly the same as that quantity divided by 5. Hence $\frac{1}{5}$ of $\frac{3}{7}$ is the same as $\frac{3}{7} \div 5$ or $\frac{3}{35}$. Therefore $\frac{2}{5}$ of $\frac{3}{7}$ must be twice as much, i.e. $\frac{6}{35}$.

Both arguments reach a result agreeing with that obtained by the "trick," viz.,

$$\frac{2 \times 3}{5 \times 7} = \frac{6}{35}$$

Algebraic Methods in the Multiplication of Fractions. Examples such as : $3\frac{1}{3} \times 2\frac{1}{2}$ can be worked on a binomial basis thus—

$$\left(3 + \frac{1}{3}\right)\left(2 + \frac{1}{2}\right)$$

$$= 6 + \frac{2}{3} + 1\frac{1}{2} + \frac{1}{6}$$

$$= 8\frac{1}{3}$$

This is a particular case of the identity $(a + b)(c + d) = ac + ad + bc + bd$, and is readily illustrated as an area (Fig. 53).

This method is frequently of use in the calculation of areas, e.g. Find the area of a room $14\frac{1}{2}$ ft. long and $11\frac{1}{2}$ ft. wide.

Traditional method.

$$\text{Area } \frac{29}{2} \times \frac{23}{2} = \frac{667}{4} = 166\frac{3}{4} \text{ sq. ft.}$$

Method outlined above.

$$\text{Area } \left(14 + \frac{1}{2}\right)\left(11 + \frac{1}{2}\right) = 154 + 7 + 5\frac{1}{2} + \frac{1}{4}$$

$$= \mathbf{166\frac{3}{4}} \text{ sq. ft.}$$

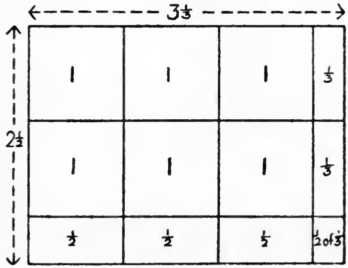


FIG. 53.

Division of Fractions. Proceeding on the same lines as those of the preceding section, we might consider the subject of division of fractions first by using whole-number divisors. This stage, however, is no more necessary than the corresponding stage in multiplication, for the general method of division by a fraction, e.g. $\frac{4}{5} \div \frac{3}{4}$, will also cover division by an integer, e.g. $\frac{4}{5} \div \frac{3}{1}$. Since, however, the pupils are familiar with whole-number divisors, we may discuss the process of dividing a fraction by a whole number in its two aspects involving (*a*) actual division of the *number* of parts, i.e. dividing the numerator, or, (*b*) division into *smaller parts*, i.e. multiplying the denominator.

$$\frac{4}{5} \div 2 = \frac{2}{5} \text{ illustrates } (a)$$

$$\frac{4}{5} \div 3 = \frac{4}{15} \text{ illustrates } (b)$$

Each aspect may be exhibited graphically in lines or areas, thus (Fig. 54)—

$$(a)\ \frac{4}{5} \div 2 = \frac{2}{5}$$

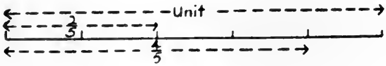


FIG. 54.

(Fig. 55)—

$$(b)\ \frac{4}{5} \div 3 = \frac{4}{15}$$

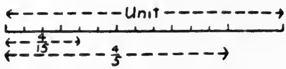


FIG. 55.

Division of Fractions by Fractions. The process is best approached through the idea of groups. On the analogy that 20 ÷ 4 has one meaning (among others) that may be expressed as, "How many times can the

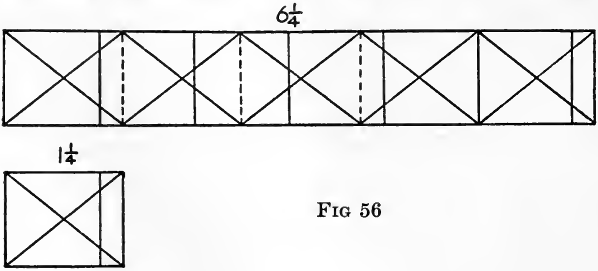


FIG 56

group 4 be taken out of the group 20 ?" we may similarly interpret $6\frac{1}{4} \div 1\frac{1}{4}$ to mean, "How many times can the group $1\frac{1}{4}$ be taken from (or out of) the group $6\frac{1}{4}$?"

Adopting the usual method of representation by lines or areas, we may represent $6\frac{1}{4}$ and $1\frac{1}{4}$ in any convenient linear or square unit (Fig. 56).

By inspection or by actual measurement we find that the group $1\frac{1}{4}$ is contained 5 times in the larger group of $6\frac{1}{4}$. This answer would be obtained by the usual process thus—

$$\frac{\overset{5}{\cancel{25}}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{5}} \qquad \textit{Ans. } \mathbf{5}$$

Similarly we may demonstrate $2\frac{4}{5} \div \frac{7}{10}$ (Fig. 57)—

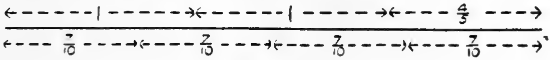


FIG. 57.

Here, $\frac{7}{10}$ is contained 4 times in the group $2\frac{4}{5}$. The ordinary process would give the answer thus—

$$\frac{\overset{2}{\cancel{14}}}{\cancel{5}} \times \frac{\overset{2}{\cancel{10}}}{\cancel{7}} = 4.$$

Cases in which the answer is not an integer may also be demonstrated graphically. Thus in $3\frac{1}{2} \div 2\frac{1}{3}$, the groups $3\frac{1}{2}$ and $2\frac{1}{3}$ are first drawn to a convenient scale.

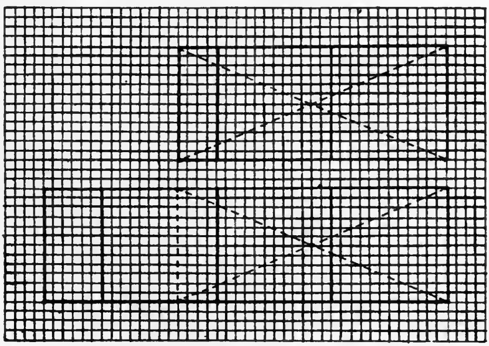


FIG. 58.

Inspection or actual measurement confirms that the second group is contained in the first group $1\frac{1}{2}$ times. This agrees with the answer given by the normal process—

$$\frac{\cancel{7}}{2} \times \frac{3}{\cancel{7}} = \frac{3}{2} = 1\frac{1}{2}$$

Other Concrete Instances of Fractional Division. These are of endless variety. Thus with the help of a ruler the child may answer questions similar to—

(i) How many pieces of $\frac{3}{4}$ inch can I cut from 6 ins. of tape? ($6 \div \frac{3}{4} = 8$ pieces.)

(ii) How many times is $1\frac{1}{2}$ inches contained in $7\frac{1}{2}$ ins.? ($7\frac{1}{2} \div 1\frac{1}{2} = 5$ times.)

(iii) How many pieces each $1\frac{1}{4}$ inches long will measure 5 ins. altogether?

(iv) How many bottles holding $1\frac{1}{3}$ pints can be filled from 1 gallon? (This can be worked on the ruler edge, taking 1 in. to represent 1 pint.)

(v) How many $\frac{3}{4}$ lb. packets can be made out of 9 lbs. of tea? (Letting 1 in. represent 1 lb.)

Indeed, all previous instances of "concrete" division may now be viewed in the new light of fractional division.

Thus $\frac{\text{17s. 6d.}}{\text{1s. 3d.}}$ may now be viewed as $17\frac{1}{2} \div 1\frac{1}{4}$, and worked as $\frac{35}{2} \times \frac{4}{5}$, i.e. 14.

This aspect of concrete division of quantity by quantity may be still further developed.

Thus $\frac{\text{17s. 6d.}}{\text{1s. 3d.}}$ has been previously worked by reducing both quantities to some convenient unit and performing ordinary division, e.g. a child might reduce to threepences and finally divide 70 by 5. Similarly, in fractions, we may reduce to the same unit before division. Thus in

$$\frac{\text{17s. 6d.}}{\text{1s. 3d.}} \text{ or } \frac{17\frac{1}{2}}{1\frac{1}{4}} \text{ or } \frac{\text{35 halves}}{\text{5 quarters}} \text{ we may}$$

change 35 halves to quarters by multiplying by 2, and so divide $\frac{\text{70 quarters}}{\text{5 quarters}}$.

Similarly we might explain—

$$\frac{5\frac{1}{3}\text{ feet}}{1\frac{1}{2}\text{ feet}} \text{ as } \frac{16 \text{ thirds}}{3 \text{ halves}} \text{ or } \frac{(16 \times 2) \text{ sixths}}{(3 \times 3) \text{ sixths}}$$

The reason for the rule "invert and multiply" may now begin to be clear to the reader who has followed this patiently.

We will work one more example—

$$\frac{2\frac{1}{4}}{\frac{7}{9}} = \frac{9 \text{ fourths}}{7 \text{ ninths}} = \frac{(9 \times 9) \text{ thirty-sixths}}{(7 \times 4) \text{ thirty-sixths}}$$

The above explanation of division by fractions is a reasonable if lengthy explanation of the rule. We may, however, reach the explanation by a much more direct path.

Explanation of the Rule by making the Divisor Unity. Consider the example above—

$$2\tfrac{1}{4} \div \tfrac{7}{9}.$$

Writing this in complex form and remembering the golden rule of fractions, we multiply numerator and denominator by the same fraction $\frac{9}{7}$ and the "trick" requires no further demonstration—

$$\frac{2\frac{1}{4}}{\frac{7}{9}} = \frac{\frac{9}{4}}{\frac{7}{9}} = \frac{\frac{9}{4} \times \frac{9}{7}}{\frac{7}{9} \times \frac{9}{7}} = \frac{\frac{9}{4} \times \frac{9}{7}}{1} = \frac{9}{4} \times \frac{9}{7}$$

This is the most concise explanation of the common rule. In symbols we may state it tersely in a form easily remembered—

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times \frac{d}{c}\right) \div \left(\frac{c}{d} \times \frac{d}{c}\right) = \frac{a}{b} \times \frac{d}{c}$$

Explanation by Argument. The older generation preferred to base the explanation on integers.

Thus to explain $3\frac{1}{3} \div \frac{7}{4}$ they would argue as follows:

(*a*) Divide $3\frac{1}{3}$ by 7; answer $\frac{10}{21}$.

(*b*) This answer is *too small* for we had to divide *not* by 7 but only by a *quarter* of 7 or $\frac{7}{4}$. Therefore we correct our answer by multiplying by 4 and obtain $\frac{40}{21}$, a result usually found by "inverting and multiplying," i.e. $\frac{10}{3} \div \frac{7}{4} = \frac{10}{3} \times \frac{4}{7} = \frac{40}{21}$.

This somewhat lengthy exposition of a well-known rule may be tedious to the chance reader who is not a teacher. "Why," we think we hear him protest, "I learnt the trick in two minutes: I know it is right; and no one ever asked me to explain it." We admit this ultra-pragmatic view. No trick is more easily acquired and none so laborious to explain. The shrewd teacher will always "cut" the verbiage and get to the business. Our excuse for the undue extent of our exposition lies in the plea that what works will not satisfy everyone. To many the question, "Why does it 'work' and give the right answer?" is even more important.

To young teachers we would add one warning: "Never try to demonstrate too much—you may only confuse. The accurate habit *must* be formed. The light of reason may dawn later."

Miscellaneous Manipulations and Applications of Fractions

Brackets and Continued Fractions. Two particular innocent-looking traps have caused oceans of tears in our schools and yet with unholy glee the "disciplinarian" continues to set them. We refer to examples of the following types—

(*a*) $\frac{2}{3}$ of $\frac{3}{4} + \frac{1}{2}$. (*b*) $2\frac{1}{3} - (1\frac{7}{8} - \frac{9}{10})$.

Let us be honest and ask ourselves, "Are these really necessary?" We do not think so. If we *must* set examples of type (*a*), let us be straight-forward and help the child by writing in the form—

$$(\tfrac{2}{3} \text{ of } \tfrac{3}{4}) + \tfrac{1}{2}, \text{ or } \tfrac{2}{3} \text{ of } (\tfrac{3}{4} + \tfrac{1}{2}),$$

each of which different meanings are established by the friendly bracket.

So again the manipulation of brackets as required in examples of type (*b*) cannot be expected of children. At best the answer will be obtained by a mechanically memorized trick. The algebraic background necessary for true appreciation is seldom possessed by the child.

At the risk of shocking the older generation of teachers, we advise the rejection of all such types.

We should also avoid, as a general rule, complex fractions, especially those in the form of continued fractions we have already alluded to. Yet continued fractions have a long history and a fascination of their own, and we wish that those teachers who still insist on setting these examples to their scholars would occasionally reverse the form. Thus a pupil who has made many tedious journeys from the bottom to the top of examples such as

$$1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{4 + \frac{1}{4}}}}$$

might be set the task of turning a given fraction into a continued fraction. The process is easily seen from an example, thus—

$$\frac{13}{23} = \frac{1}{\frac{23}{13}} = \frac{1}{1 + \frac{10}{13}} = \frac{1}{1 + \frac{1}{\frac{13}{10}}}$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \frac{3}{10}}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{3}}}}$$

which, of course, a mathematician would write as a "continued fraction" in the form $\frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{3}$

The actual denominators are usually obtained by simple continued division of the type used in the *long* method of finding G.C.F. Thus the above could be arranged as—

```
13)23(1
   13
   --
   10)13(1
      10
      --
       3)10(3
          9
          --
          1)3(3
            ==
```

The successive quotients give the successive denominators.

Though we should be shocked to find anyone trying to teach the above as an ordinary class topic, we must recognize that children delight in finding methods for working "both ways." Our mathematical engine in school is too often treated as if it only ran one way and had no reversible gear. The hard-working and tired teacher who finds it difficult to supply the necessary mathematical food for a class of eager youngsters might even in these reversible operations find salvation; for on the principle of setting a thief to catch a thief, we can imagine him setting one half of his class to *construct* continued fractions which he makes the other half *solve.*

But we must return from this interesting field to the limited paths of utility.

Among the applications of fractions with which every child must be made familiar are two converse exercises—

(*a*) Finding a given fraction of a given quantity.

(*b*) Expressing one quantity as a fraction of another.

To Find a Given Fraction of a Given Quantity.

Example: Find $\frac{3}{7}$ of £1 11s. 6d.

This may be worked either in form (i) or form (ii).

(i)

	£	*s.*	*d.*
	1	11	6
			3
7)	4	14	6
		13	6

(ii)

	£	*s.*	*d.*
7)	1	11	6
		4	6
			3
		13	6

The order of the operations is clearly immaterial, though where answers are *fractional*, method (i) is preferable, i.e. multiply before dividing. Children should be encouraged to select the simpler method in each case. Thus, (ii) above is certainly simpler for this example.

Some examples may be treated purely by fractional methods, e.g. Find $\frac{3}{8}$ of £2 10s., which can be worked as £$\frac{3}{8} \times 2\frac{1}{2}$ = £$\frac{15}{16}$ = 18s. 9d.

Examples in which numerator and denominator differ by 1 may be worked by subtraction.

Thus the example, "Find $\frac{8}{9}$ of 13s. 6d.," might be worked thus—

	s.	*d*
9)	13	6
	1	6
		8
	12	–

as above, but could be worked more easily (on the principle that $\frac{8}{9} = 1 - \frac{1}{9}$) as (13s. 6d.) – (1s. 6d.). Multiplication and division are thus avoided.

Similarly such an example as 2 ft. 6 in. $\times 2\frac{11}{12}$ could be worked fractionally. Thus—

$$2\frac{1}{2} \times 2\frac{11}{12} = \frac{5}{2} \times \frac{35}{12} = \frac{175}{24} = 7\frac{7}{24} = 7 \text{ ft. } 3\frac{1}{2} \text{ in.}$$

but is much more readily worked as—

ft.	in.	
2	6	$\times (3 - \frac{1}{12})$
	3	
7	6	$= 3$
	$2\frac{1}{2}$	$= \frac{1}{12}$
7	$3\frac{1}{2}$	

Those interested in "short cuts" may also note that this example might have been worked as—

ft.	in.	
2	11	$\times 2\frac{1}{2}$
	2	
5	10	$(= 2)$
1	$5\frac{1}{2}$	$(= \frac{1}{2})$
7	$3\frac{1}{2}$	$(= 2\frac{1}{2})$

This is another application of the fact that $a \times b = b \times a$. The application of this method of avoiding improper fractions to the case of multiplication by π has already been noticed elsewhere.

To Express one Quantity as the Fraction of Another. This is closely akin to the process of concrete division. The child will have worked many examples of the form £2 2s. ÷ 1s. 9d., the answers usually being whole numbers. He will now "reduce" the reciprocal form $\frac{\text{1s. 9d.}}{\text{£2 2s. 0d.}}$ to a fraction. The same method as for concrete division will apply, i.e. changing both terms of the fraction to the same units. Thus, using the most obvious unit 3d., the child might obtain as the first stage $\frac{7}{168}$ which finally "reduces" to $\frac{1}{24}$.

The general method will solve every case though labour may occasionally be saved by the use of fractions throughout. Thus in the example $\frac{\text{2 qr. 7 lbs.}}{\text{3 cwt. 1 qr. 14 lbs.}}$ we might work as follows—

$$\text{Fraction} = \frac{2\frac{1}{4}}{13\frac{1}{2}} = \frac{1}{6}$$

Converting Factors. We have such approximate relations as 5 miles = 8 kilometres, $2\frac{1}{5}$ lbs. = 1 kilogram, 7 pints = 4 litres. These and all similar relations can be expressed in fractional form, and a table constructed thus—

To Change :	Multiply by :
Miles to km.	$\frac{8}{5}$
Km. to miles	$\frac{5}{8}$
Lbs. to kg.	$\frac{5}{11}$
Kg. to lbs.	$\frac{11}{5}$
Pts. to litres	$\frac{4}{7}$
Litres to pts.	$\frac{7}{4}$

The pupil may similarly work out fractions for converting miles an hour to feet per second, etc.

Unit Costs and Speeds. *Finding Unit Costs.* E.g. Find the cost per yard where $11\frac{1}{2}$ yds. cost 3s. 10d.

Work thus: cost $= \frac{46}{11\frac{1}{2}}$ pence $= (46 \times \frac{2}{23})$d. $=$ 4d. per yd.

Similarly, if the local rates on £24 10s. were £1 12s. 8d., to find the rate per £1, work thus—

$$\text{Rate in shillings per £1} = \frac{32\frac{2}{3}}{24\frac{1}{2}} = \frac{\overset{2}{\cancel{98}}}{3} \times \frac{2}{\cancel{49}} = 1\tfrac{1}{3}\text{s.} = 1\text{s. } 4\text{d.}$$

Finding Speed per Hour, etc. Examples: Express the following speeds in miles per hour: (i) $13\frac{1}{2}$ miles in $1\frac{1}{2}$ hours. (ii) $78\frac{3}{4}$ miles in $2\frac{1}{4}$ hrs.

$$\text{(i)} \quad \frac{13\frac{1}{2}}{1\frac{1}{2}} = \frac{\overset{9}{\cancel{27}}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \quad \text{i.e. 9 m.p.h.}$$

$$\text{(ii)} \quad \frac{78\frac{3}{4}}{2\frac{1}{4}} = \frac{\overset{35}{\cancel{315}}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{9}} \quad \text{i.e. 35 m.p.h.}$$

Fractions in Engineering and Technical Mathematics. Teachers of trade and technical classes would do well to note that the fractions commonly used in engineering are largely restricted to halves, quarters, eighths, sixteenths, etc. Thus the workman readily understands what $\frac{1}{2}$-in. nut or a $\frac{5}{8}$-in. spanner means. Nearly all simple tools are made in standard ranges, proceeding in sizes by eighths or sixteenths of an inch.

Scale Drawing. The same simple fractions ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$) are in constant use here also. Thus a machine drawing may be full size, half size, quarter size, but is very seldom one-fifth size or one-tenth size.

The whole subject of scale drawing will be discussed more fully in a later chapter.

Representative Fractions. These are important in geography, and are interesting as showing the absolute values of scales, though the latter are usually expressed in more striking, concrete form. Thus one inch to the mile is actually 1 in. to 63,360 ins., and the representative fraction is $\frac{1}{63360}$.

Older scholars may be set the tasks of determining the representative fraction corresponding to any map scale, or of constructing an inch-mile scale for any given representative fraction.

"Fractions of the Remainder" Type. This particular type gives frequent trouble to children. An example is—

Mother spent $\frac{1}{2}$ her money at one shop, $\frac{1}{3}$ of the remainder at another, and $\frac{1}{4}$ of the remainder at another. She had 1s. 6d. left. What had she at first?

The following treatment is suggested. At each step we concentrate our attention on the remainder—

Fraction Spent.	*Fraction Left.*
(*a*) $\frac{1}{2}$	$\frac{1}{2}$
(*b*) $\frac{1}{3}$ of $\frac{1}{2}$	$\frac{2}{3}$ of $\frac{1}{2}$
(*c*) $\frac{1}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$	$\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$

No calculation need be made until the final remainder is written down.

In the above example, accordingly, we evaluate—

$$\frac{\not{3}}{4} \times \frac{\not{2}}{\not{3}} \times \frac{1}{\not{2}} = \frac{1}{4}$$

But if 1s. 6d. = $\frac{1}{4}$, the original sum = 1s. 6d. × 4 = 6s.

Orderly arrangement in two columns as above makes every step clear and reduces the chances of error.

Continued Application of the Same Fraction. These are akin to the remainder type discussed above. We are familiar with the example of the cask of wine or spirits which is diluted by continued application of the poetic principle of taking out a quarter and filling up with water, and we are usually asked to calculate the contents after n such operations.

Such may be worked in a variety of ways either by fractions or by decimals.

Using fractions and arranging in two columns—

Fraction Taken.	*Fraction Left.*
(1) $\frac{1}{4}$	$\frac{3}{4}$
(2) $\frac{1}{4}$ of $\frac{3}{4}$	$\frac{3}{4}$ of $\frac{3}{4}$
(3) $\frac{1}{4}$ of $\frac{3}{4}$ of $\frac{3}{4}$	$\frac{3}{4}$ of $\frac{3}{4}$ of $\frac{3}{4}$
. . . .	. . . .
. . . .	. . . .
(n) $\frac{1}{4}$ of $(\frac{3}{4})^{n-1}$	$(\frac{3}{4})^n$

This treatment makes the method clear to beginners. The older scholars may prefer to state the general problem at once, i.e. Remainder at any stage $=(\frac{3}{4})^n$.

Using decimals, it is possible to apply continuous division and subtraction to find the actual remainder at any stage. Thus, assuming the original contents to be unity, we work as follows—

```
4)1
  ·25
 4)·75       . . . First Remainder
   ·1875
 4)·5625     . . . Second Remainder
   ·140625
   ·421875 . . . Third Remainder
```

The ordinary air-pump exhausts the air in the receiver on this same mathematical principle.

Similar methods may be applied to fractional increases or decreases. Example: A population of 50,000 increases each year by $\frac{1}{10}$ of what it was at the beginning of that year. Find the population after 5 years.

The answer $50{,}000 \times (1\frac{1}{10})^5$ may be reached simply by continued addition of $\frac{1}{10}$ thus—

500000			
50000			
550000	after	1	year
55000			
605000	,,	2	years
60500			
665500	,,	3	,,
66550			
732050	,,	4	,,
73205			
805255	,,	5	,,

The same principle may be applied to the calculation of Compound Interest.

Finally we will discuss one traditional type of problem usually worked by fractional methods. This is the well-known "work" or "pipe" sum. Everyone who has passed through school will recall those surprising automatons "A" and "B" doing their "piece of work," with "C" usually introducing unpleasant complications. "A" and "B" generally worked with most obliging regularity for 8 and 9 hours at a stretch, thereby dutifully performing $\frac{1}{8}$ or $\frac{1}{9}$ of the "piece of work" per hour. The argument may have been poor but the answer was accepted by the teacher.

This type of problem has a long history. All the old Miscellanies and Anthologies of puzzles and problems contained it in some form, and it was certainly known to early Greek and Hindoo mathematicians. It appears in *Propositiones ad acuendos juvenes* (which we translate freely as *Whetstones for youthful wits*). The following is typical. It is taken from the *Palatine Anthology*, date about 310 A.D.—

"Four pipes discharge into a cistern. One fills it in a day, another in two days, the third in three days, and the fourth in four days. If all run together, how soon will they fill the cistern ? "

The usual method of attack is *fractional.*

Thus we find the total filled *per day* by summing the fractions—

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \qquad \text{i.e.} \quad \frac{12 + 6 + 4 + 3}{12}$$

or $2\frac{1}{12}$ times per day.

whence time for filling *once* is deduced as

$\frac{12}{25}$ of a day.

An alternative method of attack may be termed the *multiple* method.

Let us imagine *all* the taps open for all 12 days. (We choose 12 as the L.C.M. of 1, 2, 3, 4.)

The	first	would	have filled it	12	times
,,	second	,,	,,	6	,,
,,	third	,,	,,	4	,,
,,	fourth	,,	,,	3	,,

Altogether they would have filled it 25 times in 12 days, hence they would fill it *once* in $\frac{12}{25}$ of a day.

We will apply the same multiple method to a " work " sum—

A does a piece of work in 4 days, B in 5 days. How long would they take working together ?

In 20 days A would have done 5 such " pieces " and B, 4 such—altogether 9 pieces ; therefore working together they would do *one* piece in $\frac{20}{9}$ days, or $2\frac{2}{9}$ days.

Using this method, most of the ordinary examples of this type become simple mental exercises instead of " fraction " sums.

Some teachers prefer the ordinary algebraic equational treatment. For the sake of these we illustrate the method, using the last example.

Let x be the required number of days. In x days A completes the fraction $\frac{x}{4}$ of the whole and B the fraction $\frac{x}{5}$

$$\therefore \quad \frac{x}{4} + \frac{x}{5} = 1 \quad \text{whence } x = \frac{20}{9} = 2\tfrac{2}{9} \text{ days.}$$

The History of Fractions. Fractions appear to have engaged the attention of mathematicians and philosophers since the beginnings of civilization. The first point of interest is that the ancients appear to have found great difficulty in manipulating any fractions other than what we now term "aliquot parts" or fractions with unit numerators, such as $\frac{1}{2}$ or $\frac{1}{3}$. Consequently early attention seems to have been given to the problem of expressing any fraction as the sum of a series of fractions with 1 for each numerator. Many such can be seen by inspection, thus—

$$\tfrac{5}{6} = \tfrac{1}{2} + \tfrac{1}{3}$$
$$\tfrac{7}{12} = \tfrac{1}{3} + \tfrac{1}{4}$$
$$\tfrac{9}{20} = \tfrac{1}{4} + \tfrac{1}{5}$$

The Ahmes papyrus (*circa* 1000 B.C.) gives a series of results for fractions of the form $\frac{2}{2n+1}$ i.e. with *odd* denominators, though the method by which they were found is not indicated.

Thus $\frac{2}{29}$ is given as $\frac{1}{24} + \frac{1}{58} + \frac{1}{174} + \frac{1}{232}$

It is possible that many results were arrived at by simple deduction as follows—

$$\text{Since } \frac{2}{3}\,[3 \times 1] = \frac{1}{2}\,[1 \times 2] + \frac{1}{6}\,[2 \times 3]$$

$$\text{It follows that } \frac{2}{21}\,[3 \times 7] = \frac{1}{14}\,[7 \times 2] + \frac{1}{42}\,[14 \times 3]$$

$$\text{and that } \frac{2}{27}\,[3 \times 9] = \frac{1}{18}\,[9 \times 2] + \frac{1}{54}\,[18 \times 3]$$

This method of working with unit numerators was apparently common to Egyptian and Greek mathematicians for many centuries until about A.D. 500. Thus we note, in the famous problem about the herd of cattle as propounded by Archimedes (287–212 B.C.) to the Alexandrine mathematicians, that all the fractions employed, viz., $\frac{5}{6}$, $\frac{7}{12}$, $\frac{9}{20}$, $\frac{11}{30}$, $\frac{13}{42}$ consist of a series of pairs of unit fractions, i.e. $\frac{1}{2} + \frac{1}{3}$, $\frac{1}{3} + \frac{1}{4}$, $\frac{1}{4} + \frac{1}{5}$, $\frac{1}{5} + \frac{1}{6}$, and $\frac{1}{6} + \frac{1}{7}$.

For the sake of the curious but non-mathematical reader we give the simple arithmetical method by which any fraction may be converted into a series of "aliquot parts."

Example, $\frac{13}{23}$.

$23 \mid 13($

$\quad 2$

$26(1 \ (= \frac{1}{2})$

23

$\quad 3$

$\quad 8$

$24(1 \ (\frac{1}{8} \text{ of } \frac{1}{2} = \frac{1}{16})$

23

$\quad 1$

23

$23(1 \ (\frac{1}{23} \text{ of } \frac{1}{8} \text{ of } \frac{1}{2} = \frac{1}{368})$

$$\therefore \frac{13}{23} = \frac{1}{2} + \frac{1}{16} + \frac{1}{368}$$

Worked in the above way, Ahmes' example of $\frac{2}{29}$ quoted above might also be found as $\frac{1}{15} + \frac{1}{435}$.

A second prevailing tendency in classical mathematics was to express all fractions with standard denominators. Thus in the Babylonian inscriptions, 30 represents $\frac{1}{2}$ and 20 represents $\frac{1}{3}$, indicating that 60 was the accepted standard for fractions, all fractions being expressed approximately as so many "sixtieths." The choice of

60 is probably due to astronomical reasons. Thus Ptolemy gives π as $3° 8' 30''$ which is $3\frac{17}{120}$. This tendency to use sexagesimal fractions lasted right up to the sixteenth century and disappeared gradually with the introduction of decimals.

The Romans developed a similar system but on a duodecimal basis, all fractions being expressed exactly or approximately in "twelfths."

The method of *writing* fractions has also a varied history. The line of division between numerator and denominator seems to have been introduced by the Arabs. The earlier Hindoo mathematicians used no line. The position of the line does not appear to have been fixed, for all three forms $a - b$, a/b, and $\frac{a}{b}$ occur. The second form is still in common use by printers, though not so convenient for calculations as the ordinary $\frac{a}{b}$. The use of the colon (:) in ratio or fraction dates from the seventeenth century, while the ordinary sign for division (÷) most probably combines both : and —.

Other points of interest in the history of fractions may be found in any History of Mathematics.

CHAPTER XIII

THE TEACHING OF DECIMALS

Decimal methods have not yet achieved the position in our schools which their mathematical simplicity demands. Decimals are still considered "hard," and are consequently avoided by many teachers. They are often treated by mediaeval methods of manipulation, as types of calculation having no real use; and are still looked upon as a tiresome and new-fangled way of dealing with vulgar fractions. They are even taught *as* vulgar fractions with the result that many children habitually change decimals to vulgar fractions before working "sums" in which decimals occur. Traditionally, their teaching has been postponed until vulgar fractions have been thoroughly treated. This has resulted in a vicious tendency to magnify the importance of decimals (such as ·2, ·25, ·5, ·75, ·125, ·375, ·625, ·875) which are capable of being converted into simple vulgar fractions. Not so many years ago the teacher insisted on the memorizing of such decimals as $\cdot 14285\dot{7}$, etc., etc. ($\frac{1}{7}$, $\frac{2}{7}$, etc.), and $\cdot 07692\dot{3}$, etc. ($\frac{1}{13}$, $\frac{2}{13}$, etc.). So, too, when the actual manipulation of decimals could not be dispensed with, any artifice was adopted by which the supposed difficulty of the decimal point could be avoided. This accounts for the widespread use of the "counting places" method in decimal multiplication, and of the "making-the-division-a-whole-number method" in division of decimals.

The reasons for this state of affairs in our English arithmetic are not far to seek. Historically, decimals are new and, therefore, are viewed with distrust by a conservative profession such as teaching, crusted over, as so much of it is, with hoary tradition. Decimals, for many conservative minds, are still intruders, and have yet to prove their right of possession. "Our dear old non-decimal weights and measures, our non-decimal coinage, and our vulgar fractions were good enough for our fore-fathers! Decimals! yes, a quaint amusement—suitable no doubt for a revolutionary foreigner, but not for *us*!" This is a typical English view of the person who states loudly and unashamedly that he "never could 'do' decimals." We do not blame this person. In China they had a better way—they would have hung his *teacher*.

What we wish to do in this chapter is to make clear to every reader,

(*a*) That decimals are easy, simple, and useful.

(*b*) That their manipulation is the simplest possible extension of the simple rules for whole numbers.

(*c*) That they can be taught before, with, or after, the rules for vulgar fractions, but that they can be taught with the *very slightest* reference to vulgar fractions.

(*d*) That their use in calculations is almost universal, especially if we include logarithmic calculation.

(*e*) That they give results more accurate, more easily grasped, and more easily obtained, than any "vulgar fraction" method can ever give.

It is a sign of the times that in Circular 807 ("Suggestions for the Teaching of Arithmetic") the paragraph dealing with Decimal Fractions is the longest paragraph of all. Yet amid this general attention now being directed to decimal methods it is not to be inferred that "vulgar fractions" are to be neglected. Nor are teachers expected to concentrate on decimal methods before the child has made a definite study of the simplest vulgar fractions and their notation.

"According to the traditional practice, vulgar fractions are studied before decimal. To a certain extent this plan is sound, for 'halves' and 'quarters' are easier than the easiest decimals; but once the beginner can deal with these very elementary vulgar fractions, there is no reason why he should not proceed to decimals.

There is, indeed, every reason why he should study decimals before occupying himself with difficult vulgar fractions. Vulgar fractions with large denominators are cumbersome and of limited utility, whilst decimals are comparatively easy to handle and have many practical applications." [1]—(Circular, 807, par. 32.)

The present chapter is but an elaboration and amplification of the sane view thus tersely expressed.

Introductory Decimals. With the break-up of the old rigid treatment of arithmetic by rules, decimals are now introduced in a natural and unobtrusive manner at an early age in the child's school life. Gone for ever, we hope, is the time when "decimals" were introduced by a formidable exposition of their numeration and notation (though we admit with regret that its concrete concomitant—the metric system—is still frequently introduced by a learned, lengthy, and useless disquisition on Greek and Latin terminology, another example of the teachers' fallacy of emphasizing words instead of things.)

We will indicate a simple and natural approach to Decimal Notation and Decimal methods.

Tenths. It is usual to introduce these to beginners through the medium of inches graduated in "tenths." The work follows the usual practical form of drawing and measurement. The meaning of the first decimal place is thus acquired easily and naturally.

As concrete aids for this preliminary work we have—

(*a*) Rulers graduated in inches and tenths.
(*b*) Paper "squared" in inches and tenths.
(*c*) Rulers graduated in centimetres and millimetres.
(*d*) Paper "squared" in centimetres and millimetres.

It would seem more logical to begin at once with metric units, but for young children the larger unit (inch) is preferable, for the centimetre and millimetre being smaller require more careful manipulation and necessitate

[1] "Suggestions for the Teaching of Arithmetic," Circular 807, 1913

closer work with the eyes. Many children's eyes are totally unfitted to deal with the small divisions on paper squared in millimetres. With these aids the work is easy. The child measures lines and objects; draws lines; represents given decimals on squared paper; makes graphic additions, subtractions, multiplications, and divisions of decimals to one place; and without learning any formal rules he is all the time reading and writing decimals. The *tenth* to him will have a visual concrete meaning which no amount of expository theory can ever give. Nor need vulgar fractions be mentioned in this preliminary treatment, except in some simple relations such as $\cdot 5 = \frac{1}{2}$; decimals can thus be taught as useful and complete in themselves and not dependent upon the whole theory of vulgar fractions.

The variety of work involving only inches and tenths or centimetres and millimetres is best seen by an examination of some of the excellent modern series of arithmetical class-books now available. The author in his "Common-sense Series"[1] has worked out a complete scheme beginning in Book I, and continuing through the earlier books of the series.

Hundredths. In teaching hundredths we have other excellent concrete aids ready to hand—

(*a*) Paper squared in inches and tenths.

(*b*) The metre-stick showing centimetres.

Either (*a*) or (*b*) forms an excellent introduction, and both will be used by the energetic teacher.

(*a*) With the square inch we can show in handy form the unit, the tenth, the hundredth (Fig. 59*a*). Many valuable exercises can be based on this alone.

Thus we can illustrate—

1 unit = Ten tenths = One hundred hundredths.

1 tenth = Ten hundredths.

We have, further, a means of showing any decimal of two

[1] *Pitman's Common-sense Arithmetics* (series of eight books).

places. The children can show graphically ·27, ·35, ·89, etc.; they can add or subtract these, and note simple equivalents such as (Fig. 59*b*).

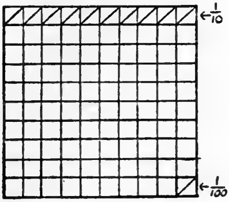


FIG. 59*a*.

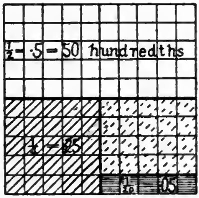


FIG. 59*b*.

$\frac{1}{2}$ = ·5 = 50 hundredths.

$\frac{1}{4}$ = ·25 = 25 hundredths ($\frac{3}{4}$ = 75 hundredths).

$\frac{1}{20}$ = ·05 = 5 hundredths. (This is useful in decimalizing money.)

Using more than one square inch, we can represent "mixed" decimals such as 2·73, 3·45, 8·21, adding, subtracting, multiplying, and dividing these as required.

We can also find a real meaning, in units of area, for such a process as 1·6 × 2·3, though no formal method for multiplication of decimals need be taught at this stage. This example, 1·6 × 2·3, is worked on squared paper elsewhere in this book.

The illustration of a square inch giving hundredths is invaluable later in showing the intimate relation of *hundredths* in decimals to percentage, which, like so many other topics in arithmetic, has suffered in the past from a treatment too much concerned with percentages considered only as *vulgar* fractions.

(*b*) Many teachers prefer to teach "hundredths" by means of the metre-stick, and by measurement in metres and centimetres. The previous method of the square inch makes a direct frontal attack on the meaning and significance of "hundredths," while the metric method attains the same end by more incidental and natural means. As in using inches, the child has learnt incidentally to read and to write "tenths," so in metric measurement he may learn to read and to write "hundredths." It may be a slight mathematical advantage, too, that, using metric measurements instead of square inches, the child-mind is maintained at the level of ordinary *linear* units in both "tenths" and "hundredths" instead of being required to pass suddenly from units of *length* (inches and tenths) to units of *area* (square inches and hundredths).

Following our ordinary practical method, then, we shall ensure that the child measures, draws and calculates in metres and centimetres, using decimal notation throughout. If this plan is followed systematically, we need not worry about any exposition of "hundredths." The child will know clearly in concrete metrical terms what a "hundredth" and a "tenth" really are, and how they are related, though he should not at this early stage be expected to be able to transfer readily his ideas of tenths and hundredths to other concrete units such as Money, Weight, etc.

Thousandths, etc. If "tenths" and "hundredths" have been taught systematically by the methods we have outlined, it may not be necessary to continue the same detailed and concrete treatment in dealing with the third decimal place and thousandths. At this stage the pupil may, perhaps, be able to continue the idea of decimals to any number of places without further practical illustration.

Yet concrete illustration for thousandths is simple. The logical sequence: *length* (inch) for "tenths," *area* (sq. in.) for "hundredths," would appear to demand *volume* or cubic treatment for "thousandths." Though the cubic inch may be rejected as too small when divided into thousandths, yet the ordinary 1,000 c.c. cube is always available to illustrate every decimal place to thousandths (Fig. 60).

Thus—

1 layer of 100 c.c. represents ·1.

1 strip of 10 c.c represents ·01.

1 c.c. represents ·001.

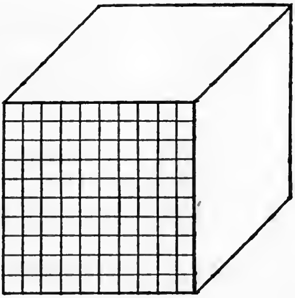


Fig. 60.

If, however, we prefer to confine ourselves to linear units, we may use the metre graduated in centimetres and millimetres to illustrate thousandths, and they can be included in the practical methods of measuring and drawing.

" It is possible also to take a piece of paper ten inches square and to use it to illustrate the value of any decimal of not more than four places."[1]

Let us now summarize what a beginner should learn from a preliminary treatment of decimals and decimal notation as outlined above.

(*a*) He will have ready visual and concrete images and meanings for the most important decimal values.

(*b*) He will be able to perform simple manipulations

[1] Circular 807, " Suggestions."

of decimals in all four rules without having been taught the more formal methods for decimal calculations.

(*c*) He will have grasped the increasing insignificance of the decimal places as we proceed to lower denominations, and will realize that, except where the unit represents a very large quantity, the figures beyond the third or fourth place are not very important.

(*d*) He will be able to think of ·6 or ·7 of a quantity *directly* and not through the indirect path of the corresponding vulgar fractions.

(*e*) Best of all, he will have a working knowledge of the metre and centimetre as realities and not as mere items in a complex and confusing "table" of metric units.

If our preliminary treatment realizes all these hopes we should be thankful. Time will have been well spent and formal calculations in decimals will lose most of their difficulties.

Addition and Subtraction of Decimals. No new rules need here be taught. The child will have discovered that the differences between the addition and subtraction of whole numbers and of decimals are very few. As he has already learned the necessity of keeping *units under units, tens under tens,* etc., so, too, he will have found out the necessity for keeping *point under point, tenths under tenths,* etc. With this precaution, the processes in number and in decimals are almost identical. A little difficulty, perhaps, arises in the manipulation of decimals involving different numbers of figures in the decimal portions as in

$$2{\cdot}3 + 4{\cdot}16 + 2{\cdot}789.$$

Some teachers at the beginning permit "equation of places" and thus the child works by first adding zeros, thus $2{\cdot}300 + 4{\cdot}160 + 2{\cdot}789$. This, however, is a prop that must sooner or later be discarded, though perhaps useful in the few cases where the addition must be performed horizontally. Subtraction similarly is eased by this device. Thus 2·3 − 1·678 is worked as 2·300 − 1·678.

In each case, however, the child can form the habit of *imagining* the zeros as easily as he *writes* them.

The main difficulties in the manipulation of decimal quantities have always occurred in their multiplication and division, and it is necessary, therefore, to deal with these two processes at some length.

At the same time it should be noted that there has been in Elementary Schools a tendency to over-rate the importance of these two processes, especially in their extension to "approximations," for, with the use of logarithms and logarithmic devices (such as slide rules) decimal calculations have been rendered so simple and easy as to be almost universal and to involve but seldom the necessity for actually multiplying or dividing decimals by decimals according to rules taught at school.

Multiplication of Decimals. (*a*) By integers, e.g. 2·345 × 7. This seldom presents any difficulty. The work is identical with short multiplication of whole numbers.

The occurrence of a zero in the product sometimes causes mistakes thus: 2·345 × 6 will sometimes be given as 1·407 instead of 14·07, owing to the zero arising from 5 × 6.

Of vital importance is the multiplication of decimals by 10, 100, etc., briefly described as "moving the point." Thus 1·23 × 10 = 12·3.

The idea of the "*point*" moving is a misleading one. Much better is the idea that the point never moves though the figures do. This is best illustrated by columns thus—

The thick line indicates the constant position of the decimal point. Line (*b*) shows 1·23 (line (*a*)) multiplied by 10. Similarly, line (*c*) shows (*a*) multiplied by 100.

	H.	T.	U.	t.	h.
(*a*)			1	2	3
(*b*)		1	2	3	
(*c*)	1	2	3		

This idea of moving digits and fixed points is essential for a right grasp of decimal manipulation. It applies equally to division by 10, 100, etc. In these cases the original figures move one or more places to the *right* instead of the *left*.

Multiplication of Decimals by Decimals. *Traditional Method.* "Multiply as in ordinary multiplication and mark off in the product as many places as there are in the multiplier and multiplicand together." Thus ran the old rule. Hence 4·23 × 4·5 was worked as shown.

```
 423
  45
----
 2115
1692
-----
19035
```

Since there are *three* places in the two factors altogether, we mark off three in the final product, which gives us 19·035.

The rule is simple and safe, though mechanical. Yet it need not be mechanical, for explanation is easy.

(*a*) *Explanation by Fractions.* $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ or x tenths × y tenths = xy hundredths

(This has been noted already on squared paper.)

Similarly x tenths × y hundredths = xy thousandths, and so for every case.

Thus in example 4·23 × 4·5, we have 423 hundredths multiplied by 45 tenths and our answer 19035 is clearly "thousandths," whence the rule.

(*b*) *Explanation using Indices.* This, though too hard for beginners, will appeal to teachers.

$$\begin{aligned} 4{\cdot}23 &= 423 \times 10^{-2} \\ 4{\cdot}5 &= 45 \times 10^{-1} \\ \therefore\ 4{\cdot}23 \times 4{\cdot}5 &= (423 \times 45) \times 10^{-3} \\ &= 19035 \times 10^{-3} \\ &= 19{\cdot}035 \end{aligned}$$

whence the rule.

Every case may be expressed thus tersely in indices.

The method of "counting places" enjoys widespread popularity, for it is easy to teach, easy to remember, and mechanically safe. Yet objections to it are many. It often appears to the child merely as a trick giving the right answer. This is, perhaps, not the chief objection. The graver charge against it is a mathematical one. It is cumbersome and can never be shortened. Thus using this method in the example 2·34567 × ·13579, it would be necessary to use *every* figure before an answer could be obtained, and finally we should have to mark off *ten* places when clearly ten are unnecessary in any reasonable answer.

Again, by this method the value of each partial product is obscured. Thus in the example, 4·23 × 4·5, as worked by this method, we cannot determine with ease the exact significance or value of the line 1692, i.e. whether it represents tenths, hundredths, or even units, until we have made our final "count up" of decimal places.

$$\begin{array}{r} 423 \\ 45 \\ \hline 2115 \\ 1692 \\ \hline 19035 \end{array}$$

To this, of course, it may be objected by the "practical" mathematician that we need not worry about the *partial* products since all we need is the *total* product.

On the whole it would appear that the most important objection to the method is its inapplicability to contracted forms of working. For children, however, we repeat that it is simple, easy, safe, and need not be mechanical.

Some teachers would approach each example on its merits by first requiring a rough mental approximation to be made. Thus, in such an example as 4·23 × 4·5, the child would be required to estimate the answer as lying between 4 × 4 (i.e. 16) and 5 × 5 (i.e. 25); hence the answer 19035 could not possibly be 1·9035 nor 190·35. The only possibility is 19·035. Using this method the answer is determined by common sense and not by mechanical rule. This method of preliminary approximation has much to recommend it. It certainly prevents the ridiculous answer and should always be employed.

Alternative Method of Decimal Multiplication (including a discussion of the so-called "standard" method).

Of recent years there has been a marked tendency in schools of all types to discard the traditional method of decimal multiplication by " counting the places " in favour of the more modern method now described.

Circular 807 (" Suggestions ") refers to this modern method in the following words (par. 37): " Children who are learning the multiplication of decimals may be trained to begin multiplying by the left-hand figure of the multiplier; this practice facilitates the learning of the contracted methods which are of use at a more advanced stage, and some teachers are in favour of following it from the very beginning."

The basis of the method is the fact already noted in dealing with multiplication and division of decimals by 10, 100, etc., viz., that it is not the *decimal point* but the *digits* which should be thought to move in multiplication and division. This motion of digits to the left or right of a *fixed* decimal point is the only way in which a right conception of " place-value " may be formed.

We will illustrate this point once again by using columns to illustrate the multiplication of 1·36 by 2, 20, ·2, ·02.

			1	3	6		
(*a*)			2	7	2		
(*b*)		2	7	2			
(*c*)				2	7	2	
(*d*)				0	2	7	2

The black line indicates the fixed decimal point.

(*a*) Shows multiplication by 2, i.e. the relative positions of digits unaltered.

(*b*) Shows multiplication by 20, i.e. as for 2, but with the digits moved one place to the left.

(*c*) Shows multiplication by ·2, i.e. as for 2, but with the digits moved one place to the right.

(*d*) Similarly shows multiplication by ·02.

As many examples as are considered necessary may be worked out on this plan, i.e. using the same figure but of different value as multiplier, e.g. 4, ·4, ·04, ·004, etc.; 7, 70, 700, ·7, ·07, ·007.

We may next combine two or more of the steps as in the example,

$$1{\cdot}36 \times 2{\cdot}2,$$

which will now be worked as follows—

```
1·36
2·2
────
2·72  .   .   .  (× by 2)
 ·272 .   .   .  (× by ·2)
────
2·992
════
```

Later we may approach the general case by some such example as—

$$3{\cdot}46 \times 2{\cdot}34.$$

We arrange our work as follows—

```
3·46
2·34
────
6·92   = × by 2
1·038  = × by ·3
 ·1384 = × by ·4
─────
8·0964
═════
```

The advantages are clear—

(1) The decimal point remains unmoved.

(2) Each partial product has its actual value.

(3) The answer is obtained without the application of any mechanical trick.

The key note is clearly the fact that multiplication by the *units* digit does not alter relative place values. Thus, if we begin our multiplication with the units digit (two above) all the rest follows accurately and automatically.

The work may be simplified still further for children by moving the multiplier so that its *unit* digit comes just beneath the last figure to the right of the multiplicand.

Thus, 3·46 × 2·34 may be arranged for multiplication as—

```
3·46
  2·34
------
 ..2
 ...8
 ....4
```

The advantage is shown by the fixing of the position of the first right hand figure of each partial product, while it will be noted that, as before, the position of the decimal point in the partial products and in the total product remains unaltered.

Any example may be similarly treated. Thus the example 27·59 × 23·85 may be arranged thus—

```
27·59
  23·85
```

Here we place the units digit (3) of the multiplier under the last figure (9) of the multiplicand. After this the multiplication proceeds automatically.

(First figures only are shown for each partial product.)

```
27.59
  23·85
-------
.. 8
... 7
.... 2
..... 5
```

The method is easy, speedy, and readily taught, though it may become just as mechanical as the older traditional method.

Multiplication in "Standard Form." It has been noted that multiplication by a single units digit leaves the relative positions of the digits undisturbed. This fact is made the basis of the new rule, which is usually given in some such form as the following—

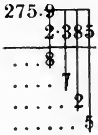
(*a*) Move the point in the multiplier so as to have *one* and only *one* digit to the left of the decimal point (i.e. one units digit). The multiplier is then said to be in standard form.

(*b*) Correct by moving the point in the multiplicand.

(*c*) Multiply, using the figure of most value first and the others in order to the right.

Thus, in working example 27·59 × 23·85, arrange the multiplier in standard form 2·385, and correct this by writing the multiplicand as 275·9.

The example now is 275·9 × 2·385. Set out as before—



It is claimed that the method is reasonable and automatically safer; that it covers all cases and can be readily contracted. Yet the necessity for changing the multiplier into " standard " form does not seem clearly proved, unless it is to indulge the passion of some teachers for a " rule " which can be taught to mechanical perfection.

The absurdity, for instance, of reducing such a multiplication as

·000123 × ·000045

to " standard " form is obvious, for we shall have to deal with either—

·00000000123 × 4·5, or
·0000000045 × 1·23.

Here clearly it is quicker to multiply 123 by 45 and to write our answer as 5535×10^{-12} or $5{\cdot}535 \times 10^{-15}$, which is the ordinary practice.

The passion for standardizing also leads to other errors. The following is typical and appears at length in a recent book as an example of standard form.

Example : 4·3 × ·21.

This is " standardized " as ·43 × 2·1, and worked thus—

```
·43
 2·1
————
·86
·043
————
·903
```

The author should surely have seen that, since 4·3 × ·21 *is the same as* ·21 × 4·3, the example is *already* "standardized," and may be worked directly as follows—

```
·21
  4·3
-----
·84
·063
-----
·903
=====
```

Division of Decimals. *Division by Integers.* This seldom gives any difficulty, as the ordinary methods of short division, factor division, and long division, are followed. The only care necessary in dealing with decimal division is to make certain that the decimal point is correctly placed in the quotient.

We will work a few examples of each type.

(*a*) *Short Division.*

```
2·34561 ÷ 3

3)2·34561
  -------
   ·78187
  =======
```

The process is used for converting simple fractions into decimals.

Thus $\frac{1}{8}$:

```
8)1·000
  -----
   ·125
```

Division, by 10, 100, etc., has already been noted. The point to be noted and emphasized is the movement of the *digits* and *not* of the decimal point. Thus, in full, 43·456 ÷ 10, is—

```
10)43·456
   ------
   4·3456
   ======
```

This process may be extended to any multiple of 10, 100, 1000, etc. Thus, 3·579 ÷ 30 is worked by short division, thus—

$$30\overline{)3{\cdot}579}($$
$$\cdot 1193$$

We may use the process to convert for example $\frac{1}{40}$ to a decimal, thus—

$$40\overline{)1{\cdot}000}($$
$$\cdot 025$$

(*b*) *Division by Factors.* This should be used wherever possible, for it has all the ordinary advantages of factor division with the additional merit that there is no *remainder* to be calculated.

Example: 15·552 ÷ 24.

Worked as—

12)15·552		8)15·552		6)15·552
2)1·296	or	3)1·944	or	4)2·592
·648		·648		·648

The process may be employed to convert harder fractions to decimals in every case where the factors of the denominator are easily seen.

Example: Express $\frac{13}{35}$ as a decimal correct to three places.

Dividing by 5 and 7, we work as here set out—

$$5\overline{)13{\cdot}0000}$$
$$7\overline{)2{\cdot}6000}$$
$$\cdot 3714$$

Answer, ·371.

(*c*) *Long Division of Decimals by Integers.* This process is so closely akin to ordinary long division that it needs no elaboration.

In the older form of setting out, the quotient was placed to the *right* of the dividend. The more modern method of writing the quotient *over* the dividend should always be followed. It fixes automatically the position of the decimal point in the quotient. One example will suffice—

$$204{\cdot}079 \div 437.$$

Older Form.	*Modern Form.*
	·467
437)204·079(·467	437)204·079
1748	1748
2927	2927
2622	2622
3059	3059
3059	3059

Division of Decimals by Decimals. This has long been considered by teachers as the most difficult of the decimal processes to teach. It involves all the difficulties of long division by integers, with the additional trouble of determining the position of the decimal point in the quotient. We will, therefore, discuss each of the common methods in use, indicating their various advantages and disadvantages. The methods group themselves into those requiring some preliminary manipulation of the decimal point of the divisor, and those in which the position of the decimal point is undisturbed. The common methods involving a preliminary adjustment of decimal points are (i) the method based upon first making the divisor a whole number, (ii) the older method of "equating places," and (iii) the more modern "standard" method.

(1) *Method Based upon Making the Divisor a Whole Number.*

Example: $2{\cdot}3456 \div {\cdot}135$.

This is adjusted to $2345{\cdot}6 \div 135$, and division proceeds to any required number of places.

The theory needs no explanation. It is another application of the fundamental rule of fractions, i.e.—

$$\frac{2{\cdot}3456}{{\cdot}135} = \frac{2{\cdot}3456 \times 1000}{{\cdot}135 \times 1000} = \frac{2345{\cdot}6}{135}$$

This is the method most commonly taught to beginners.

Advantages. (*a*) It is easily taught and can be definitely illustrated. Thus, 1·37 metres ÷ ·023 metres is clearly 1370 millimetres ÷ 23 millimetres, and similarly 9·1 inches ÷ 1·3 inches may be viewed as 91 tenths divided by 13 tenths.

(*b*) The difficulty of fixing the position of the decimal point in the quotient is avoided if the quotient be written *over*, and not to the right of the dividend.

Disadvantage. The difficulty of *decimal* division is displaced by division by *integers*, but the true remainder is not obtained at any stage. Let teachers who are inclined to doubt this try some such example as the following—

"How many pieces of 2·25 inches can be cut from 36·7 inches, *and what length is left ?*"

Method of "Equation of Places." This is an older method closely akin to that discussed above. An example will make it clear. In dividing 12·2345 by ·357 we have *four* places in dividend and *three* in divisor. Accordingly we arrange *four* in each, thus "equating" places. This gives us 12·2345 ÷ ·3570.

The decimal point is now ignored and the sum worked as whole numbers, i.e. as 122345 ÷ 3570.

The decimal point appears in the quotient when all figures in the dividend have been exhausted. The method is simple and easy to teach; but it frequently introduces unnecessary zeros, while it suffers from the same "remainder" defect as the first method discussed.

It is, however, of distinct use in what may be termed "complex" decimals.

Example—

$$\frac{12{\cdot}34 \times {\cdot}1279}{{\cdot}0035 \times 18{\cdot}1}$$

Here we first "equate" places thus—

$$\frac{12{\cdot}34 \times {\cdot}1279}{{\cdot}00350 \times 18{\cdot}1} \quad \text{or} \quad \frac{12{\cdot}34 \times {\cdot}1279}{{\cdot}0035 \times 18{\cdot}10}$$

The example may now be worked as whole numbers, i.e. as—

$$\frac{1234 \times 1279}{350 \times 181} \quad \text{or} \quad \frac{1234 \times 1279}{35 \times 1810}$$

The advantage is apparent: no difficulty in manipulating the decimal point in the answer arises until all these figures have been used.

The "Standard" Method. This is akin to the method of decimal multiplication using a "standard multiplier."

Example: 12·789 ÷ ·0237.

The decimal point is first adjusted so that the divisor has only one units digit, i.e. the example becomes 1278·9 ÷ 2·37.

Upholders of the method claim that it is simple and easily taught; that it makes the division closely akin to division by a single "units" digit and so reduces the difficulty of fixing the decimal point in the answer; and finally, that it leads to a very simple contracted method.

The usual method of teaching is to begin with division by unit integers, e.g. 2, 3, 4, 5, etc., and then to proceed to division with divisors having one decimal place, e.g. 2·1, and finally to insist that *all* divisors shall be put into this standard form.

The method has an increasing number of adherents, but is more difficult for beginners, and so is more usually found in classes of older scholars.

Methods involving no Preliminary Movement of Decimal

Points. (*a*) The following method is straightforward, reasonable, and easily taught and learned.

Example: 19·35791 ÷ 1·395.

Arranging in usual form—

$$1{\cdot}39\underline{5})19{\cdot}35\underline{7}91($$

we underline equivalent places in divisor and dividend, i.e. 5 in 1·395 and 7 in 19·35791. The decimal point appears in the quotient after the 7 has been utilized in the dividend. The theory is fairly obvious: we are certain to have whole numbers in our quotient as long as we divide "*thousandths*" into quantities *not less than* "thousandths."

The advantages of this method are —

(i) It involves no preliminary tampering with decimal points.

(ii) It has a common-sense basis.

(iii) It gives the *actual* remainder at any stage.

To illustrate, we will work the example already given by this method, namely:

"How many pieces of 2·25 inches can be cut from 36·7 inches and *what length is left*?"

Adding a zero to 36·7 and underlining 5 and 0, we work thus—

$$\begin{array}{r} 2{\cdot}2\underline{5})36{\cdot}7\underline{0}(16 \\ 22{\cdot}5 \\ \hline 14{\cdot}20 \\ 13{\cdot}50 \\ \hline {\cdot}70 \end{array}$$

The remainder has its actual value, i.e. ·7 inch and the decimal point has remained in correct position throughout the working.

(*b*) The fifth and last method to be discussed is the simplest of all, and yet for some reason the one least commonly found in use in schools.

It is based upon a simple reversal of the "counting the places" method of decimal multiplication.

Example: 13·5678 ÷ 2·754.

We set out the example in the usual way and begin division at once, ignoring the decimal points thus—

```
2·754)13·5678(4·9
      11 016
      ------
      25 518
      24 786
      ------
      . . 732
```

To place the decimal point correctly in the quotient, we argue as follows.

The number of *places* in the dividend is equal to the sum of the number of *places* in the divisor and quotient (from the multiplication rule), hence the number of *places* in the quotient is the *difference* of the number of places in the dividend and the number of places in the divisor.

The method is extraordinarily simple, straightforward, and even automatic: it never fails. We will work a more difficult example: 29·2 ÷ ·3579.

Here, at first sight, we appear to be faced with a *negative* difference for we have fewer places in the dividend than we have in the divisor. The difficulty, however, is only apparent for we may count as many zeros after the last figure in the dividend as necessary, thus—

```
·3579)29·2000(81
      28·632
      ------
        ·5680
        ·3579
        -----
         2101
```

Here we have continued the division until there are (including the zeros) four decimal places in the dividend with four in the divisor. According to rule, there are no decimal places in the quotient which thus begins 81, and may be continued as far as required.

The method is worthy of much more attention than it usually receives. It is, perhaps, the nearest approach to the method used by adults, for the practical worker (having mislaid his logarithms or his "slide-rule") will always, in decimal division, divide as with whole numbers, placing the decimal point in his quotient by common-sense inspection or by counting the places. He seldom goes through the process of "making the divisor a whole number," or of "standardizing" it, etc.

Connection between Decimal and Vulgar Fractions. This section of arithmetic has perhaps received a too detailed treatment in the past, especially when it extended (as it did not so very many years ago) to the teaching of rules-of-thumb for the conversion of "pure recurrers" and "mixed recurrers" to vulgar fractions. These "rules" for recurring decimals are now rightly omitted from ordinary school syllabuses.

All that need be taught is the ability (*a*) to turn simple fractions into equivalent or approximately equivalent decimals; (*b*) to express simple decimals in fractional form.

The first process—the change from *fractions to decimals* —is accomplished by simple division, thus—

$$\text{(i)}\ \frac{3}{8},\quad \begin{array}{r} 8\,\big)\,3{\cdot}000 \\ \hline {\cdot}375 \end{array} \qquad\qquad \text{(ii)}\ \frac{5}{11},\quad \begin{array}{r} 11\,\big)\,5{\cdot}000 \\ \hline {\cdot}4545\ldots \end{array}$$

(i) is exact, (ii) is approximate.

(Children readily recognize simple cases of recurrers, and there is no reason why they should not learn the ordinary notation, i.e. in (ii) above $\cdot\dot{4}\dot{5}$. Recurring decimals from a theoretical and mathematical point of view are a fascinating study, yet their practical applications are so small that we are wise to omit them from a normal arithmetical syllabus.)

Numerous simple equivalents are readily committed to memory (though not necessarily mechanically "learnt by heart") such as $\frac{1}{2}$, ·5; $\frac{1}{4}$, ·25; $\frac{1}{8}$, ·125; $\frac{1}{5}$, ·2; $\frac{1}{3}$, ·3333 . . . or $\cdot\dot{3}$; $\frac{1}{6} = \cdot 1\dot{6}$; $\frac{1}{9} = \cdot\dot{1}$. The older generation of teachers made great play with $\frac{1}{7}$ and other "cyclic recurrers," but this, for children, was work for the memory rather than for the reason.

A judicious use of these simple equivalents lightens the labour of decimalizing fractions very considerably. Thus, if required to decimalize $\frac{3}{40}$ we think of $\frac{3}{4}$, i.e. ·75, and give our answer as ·075. Similarly we might obtain $\frac{3}{32}$ from $\frac{3}{4}$ by simple division by 8, i.e. $·75 \div 8 = ·09375$. Wherever simple factors occur in the denominator some such process may be employed. Thus, to decimalize $\frac{3}{55}$, we may use $\frac{3}{5} \div 11$, i.e. $·6 \div 11$ or $·0\dot{5}\dot{4}$.

Actual long division need be performed only in the case of the larger prime numbers.

(*b*) The second process of changing *decimals to fractions* is not so frequently required in ordinary life, for the general tendency is now to perform all calculations in decimals and not to bother much about the corresponding fractions. Yet a moderate amount of practice is to be desired for children, since it helps to extend their knowledge of the significance of decimal places.

The general process is soon acquired.

Example—

$$·57 = \frac{57}{100}, \quad ·931 = \frac{931}{1000}, \quad ·84 = \frac{84}{100} = \frac{21}{25}$$

The rule is too obvious to need statement. Here again a ready recognition of well-known equivalents will frequently save labour.

Thus ·65 may be worked as $\frac{65}{100}$, but is more readily obtained as $\frac{6\frac{1}{2}}{10}$ by a judicious combination of fraction and decimal.

Other examples of the same method are given—

(*a*) $·325 = \frac{3\frac{1}{4}}{10} = \frac{13}{40}$

(*b*) $·62 = \frac{6\frac{1}{5}}{10} = \frac{31}{50}$

(*c*) $·7125 = \frac{7\frac{1}{8}}{10} = \frac{57}{80}$

(*d*) $·5\dot{3} = \frac{5\frac{1}{3}}{10} = \frac{16}{30} = \frac{8}{15}$

(*e*) $·281\dot{6} = \frac{28\frac{1}{6}}{100} = \frac{169}{600}$

Decimals of Concrete Quantities. These form a feature peculiar to English arithmetic, and consequently must receive special treatment. A child must be able—

(*a*) To evaluate in ordinary English units any concrete quantity expressed in decimal terms.

(*b*) To express in decimal terms any concrete quantity expressed in ordinary units.

Examples of (*a*)—

1. Express £3·1426 in pounds, shillings, and pence.
2. Express 4·3589 tons in tons, cwts., qrs., etc.
3. Express 5·6789 acres in acres, roods, poles, etc.

Examples of (*b*).

Normal and Practical.

1. Express £3 13s. 4½d. in decimal form.
2. Express 7 cwt. 3 qrs. 7 lbs. as the decimal of 1 ton.
3. Express 3 roods 7 poles as the decimal of an acre.

Unreal, Bookish, and sometimes Preposterous.

1. Express 13s. 11¼d. as the decimal of £1 18s. 11¾d.
2. Express 1 day 13 hrs. 5 mins. 31 secs. as the decimal of 1 week 6 days 23 hours 11 mins. 59 secs.
3. Express 6 furlongs 5 ch. 21 yds. 2 ft. 9½ in. as the decimal of 3 miles 5 furlongs 9 ch. 13 yds. 1 ft. 7 in.

These two fundamental processes are easily taught. Examples of the first type (*a*) are worked by the ordinary "reduction" method of continuous multiplication. Thus—

(*a*)	4·3589 tons	4·3589	4 tons
		20	
		7·178	7 cwt.
		4	
		·712	0 qrs.
		28	
		19·936	19·936 lbs.

or **4 tons 7 cwt. 20 lbs.** nearly.

Examples of the second type (*b*) should be worked by the "reduction" method converse to that used in (*a*) above.

Thus—

Express 3 roods 7 poles as the decimal of an acre—

(b) 3 roods 7 poles = 127 poles

division by 40 and 4 follows thus:

$$\begin{array}{r|l} 4,0 & 12,7 \text{ poles} \\ \hline 4 & 3{\cdot}175 \text{ roods} \\ \hline & {\cdot}79375 \text{ acres} \end{array}$$

Too often the child is allowed to "*decimalize*" by proceeding through a clumsy vulgar fraction instead of using this method of simple division. Thus in expressing 1 foot $3\frac{3}{4}$ inches as the decimal of 1 yard, children frequently proceed as follows—

Reduction to inches gives $\frac{15\frac{3}{4}}{36}$, which is converted to $\frac{63}{144}$, cancelled to $\frac{7}{16}$, and finally converted to a decimal. All that is required is simple division by 36 (i.e. 12 and 3) thus—

$$\begin{array}{r|l} 12 & 15{\cdot}75 \text{ inches} \\ \hline 3 & 1{\cdot}3125 \text{ ft.} \\ \hline & {\cdot}4375 \text{ yds.} \end{array}$$

It is seldom necessary to proceed through vulgar fractions.

Decimalization of Money. This subject usually receives special treatment in schools. Perhaps its importance has been unduly emphasized, for its uses are not many. We may be certain that, if any general necessity arose, such as the legal decimalization of our coinage, the conversion of shillings, pence, and farthings to decimals of £1 would not be left to a host of amateur arithmeticians; official tables or ready reckoners, giving all necessary equivalents, would be at once supplied. Teachers, then, should not magnify beyond its normal limits the importance of decimalizing money. The popularity of the type of sum, especially *decimalizing at sight*, may be due to the fact that ingenious and industrious calculators have long since reduced the process to a "rule of thumb" having a concise finality so beloved of pedagogues. As a matter of fact no new methods are necessary. The "reduction" methods already indicated will suffice. Thus, to decimalize 7s. $4\frac{1}{2}$d. we may proceed through division by 12 and 20.

$$\begin{array}{r|l} 12 & 4{\cdot}5 \text{ pence} \\ 2\backslash 0 & 7{\cdot}375 \text{ shillings} \\ & {\cdot}36875 \text{ pounds} \end{array}$$

This answer is exact.

Similarly for 11s. $8\frac{1}{2}$d.

$$\begin{array}{r|l} 12 & 8{\cdot}5 \text{ pence} \\ 20 & 11{\cdot}70833 \text{ shillings} \\ & {\cdot}58541 \text{ pounds} \end{array}$$

This answer is approximate.

This method of reduction cannot be completed "at sight," and accordingly we will develop the usual rule in the way we would prefer to see it taught.

Now a method to be completed "at sight" should be as simple, as rapid, and as accurate as possible. The following, we think, can fairly claim to possess all these advantages.

We shall base the work on two elements only, (*a*) the shilling, (*b*) the farthing.

(*a*) *To Decimalize Shillings.*

$$1\text{s.} = £\tfrac{1}{20} = £{\cdot}05.$$

This can be shown diagrammatically, using the ordinary squared paper.

The *rule* is simple; if every shilling represents 5 hundredths, all that is necessary is that we shall

"Multiply the number of shillings by 5 and call the product *hundredths.*"

Thus 9s. = £·45 (9 × 5 = 45)
13s. = £·65 (13 × 5 = 65)
16s. = £·80 (16 × 5 = 80)

The method is readily taught and readily applied. By some teachers the fact that the florin is ·1 is considered to be a necessary intermediate step ; the introduction of this extra step is entirely unnecessary.

Many teachers prefer to divide the shillings by 2 instead of multiplying by 5, again with the desire to explain the process on the "florin" basis. The author has found the method of multiplying by 5 to be the simplest for children.

(*b*) *To Decimalize Farthings.*

$$\tfrac{1}{4}\text{d.} = £\tfrac{1}{960} = £{\cdot}001 \text{ approx.}$$

Thus having decimalized shillings, we change the remainder to farthings, and call them "*thousandths.*"

Thus $2\frac{3}{4}$d. = ·011 approx.
and $8\frac{1}{2}$d. = ·034 ,,

Since, however, $\frac{1}{4}$d. is not *exactly* £·001 but £·001$\frac{1}{24}$, a small correction may be made.

Clearly $8\frac{1}{2}$d. (above) is actually £·034$\frac{34}{24}$ or more nearly : £·035.

We may easily trace the necessary correction. Sums under 3d. will need no correction.

3d. itself is actually £·012$\frac{12}{24}$ or £·0125 exactly.

6d. is actually £·024$\frac{24}{24}$ or £·025 exactly.

9d. is actually £·036$\frac{36}{24}$ or £·0375 exactly.

Sums above 9d. will clearly need a correction of 2 thousandths, thus $10\frac{1}{4}$d. = £·041$\frac{41}{24}$, or ·043 approx.

The *rule* may now be concisely stated—

(*a*) Multiply the shillings by 5 and call the product *hundredths.*

(*b*) Change the pence and farthings to farthings and call the farthings *thousandths.*

(*c*) Correct by adding 1 thousandth for any sum between 3d. and 9d. and 2 thousandths for sums between 9d. and 1s.

Examples: (*a*) 3s. $1\frac{3}{4}$d.

Steps (usually worked mentally)—

3s. = £·15
$1\frac{3}{4}$d. = £·007 (no correction necessary)
∴ 3s. $1\frac{3}{4}$d. = £·157 correct to three places.

(*b*) 7s. $5\frac{3}{4}$d.

7s. = £·35
$5\frac{3}{4}$d. = £·023
Correction = **£·001**
∴ 7s. $5\frac{3}{4}$d. = £·374 correct to three places.

(*c*) 17s. $11\frac{1}{2}$d.

17s. = £·85
$11\frac{1}{2}$d. = £·046
Correction = **£·002**
∴ 17s. $11\frac{1}{2}$d. = £·898 correct to three places.

In many "practice" and "interest" exercises decimalization to three places is hardly accurate enough, and it is customary in such cases to proceed to five places at least.

To do this with rapidity it is necessary to modify slightly the method just described for decimalizing "at sight" correct to three places.

Consider a sum such as $3\frac{3}{4}$d.

We have noted that this is actually $\cdot015\frac{15}{24}$. Hence we may continue the process to additional "places" by decimalizing $\frac{15}{24}$, i.e. by dividing the number of farthings by 24. But this is identical with the procedure of dividing the pence by **6**, which is the usual method.

Thus $3\frac{3}{4}$d. = £·015 + (3·75 ÷ 6) thousandths
= £·015 + £·000625
= ·015625 exactly

Similarly (*a*) $5\frac{1}{2}$d. = £·022 + $\frac{5\cdot5}{6}$ thousandths
= £·022 + ·0009166
= £·0229166
or £·02292 correct to 5 places

$$(b)\ 9\tfrac{1}{4}\text{d.} = £\cdot037 + \frac{9\cdot5}{6} \text{ thousandths}$$
$$= £\cdot037 + \cdot00158 \ . \ . \ .$$
$$= £\cdot03858 \ . \ . \ . \text{ correct to 5 places}$$

Children may be taught to decimalize successfully *at sight* even to 5 places, but the normal process of reduction as first described is more natural and is as quickly worked.

De-decimalization of Money. The converse process, viz., evaluating decimalized money in £ s. d. may be treated in a similar manner.

Thus we may always proceed by the reduction method of continued multiplication.

To express £·4568 in shillings, pence, etc., we multiply by 20, 12, 4, as in ordinary reduction—

```
 ·4568  pounds
    20
 ------
9·1360  shillings
    12
 ------
 1·632  pence
     4
 ------
 2·528  farthings
 ======
```

	s.	d.	
Answer,	9	$1\tfrac{1}{2}$	
or	**9**	$1\tfrac{3}{4}$	nearly

Any decimal sum known to be correct to three places may be de-decimalized at sight.

Thus (*a*) £·761.

Dividing hundredths by 5 gives 15s. 11 thousandths remaining give $2\tfrac{3}{4}$d.

Hence £·761 = 15s. $2\tfrac{3}{4}$d.

(*b*) £·381.

Dividing " hundredths " by 5 gives 7s. The remaining 31 " thousandths " give 30 farthings (remembering correction of 1 thousandth for sums between 3d. and 9d.).

Hence £·381 = 7s. $7\tfrac{1}{2}$d.

(*c*) £·989.

Dividing " hundredths " by 5 gives 19s. The remaining 39 " thousandths " gives 37 farthings (remembering correction of 2 " thousandths " for sums over 9d.).

Hence £·989 = 19s. 9¼d.

Sums decimalized and correct to five places may also be easily de-decimalized at sight, but the process is seldom necessary.

We conclude this section with a repetition of our belief that this whole subject of decimalizing and de-decimalizing English money is treated with too much importance by teachers. Its main use is in certain calculations that are rendered easier by working with decimalized money, but, we repeat: as soon as our coinage is placed on a decimal basis, all the " decimalizing " necessary will be done for us in tables, while the process termed de-decimalizing will have only an historical interest and will not be required in practice.

Approximate Methods. If the child has followed decimals through the preliminary treatment, he will have some idea of the relative significance of place values and of the rapidly decreasing importance of decimal places as we proceed to the right of the decimal point. Thus he will learn that a statement such as ·3579 of 1d. does not differ to an appreciable extent from ·358 of 1d., while this again for all practical purposes might equally well be written as ·36 of 1d. Again his graphical work will have convinced him that the difference between ·3579 of a square inch and ·36 of a square inch can hardly be detected. In general, he can, if his attention has been systematically directed to the matter, acquire a wholesome distrust of pseudo-accurate decimals tending to many decimal places.

This provides the best starting point. Plenty of exercises may be given of the following type.

Express ·5796 correct to the nearest (*a*) thousandth, (*b*) hundredth, (*c*) tenth.

Alternatively he should recognize that if a decimal is known to be correct to the nearest hundredth as for example ·76, the original may have been *any* decimal between the limits of ·755 and ·765.

He may pursue this a little further and find to what extent his answer is correct if he adds, subtracts, multiplies or divides two such quantities.

Thus suppose we add ·76 and ·38, each known to be correct only to the nearest hundredth—

Actual Answer.	*Least Possible.*	*Greatest Possible.*
·76	·755	·765
·38	·375	·385
1·14	1·130	1·150

This at once means that our answer is not so reliable as our original quantities, for clearly the *actual* answer *might* be 1·131, which is certainly not 1·14 approx.

Similarly we might discuss with the child or the class the other manipulations—subtraction, etc.—of decimals approximately correct, and thus lead to the general truth that an answer obtained from approximate data has its accuracy limited and controlled by the accuracy of the data themselves.

If beginners reach this point the rational need for approximate methods has been established.

Addition and Subtraction of decimals to any required decimal place gives no difficulty. Common-sense methods rather than "rules of thumb" should be used. Thus we may add the following correct to two places—

$$1{\cdot}3579 + 2{\cdot}3691 + 3{\cdot}4587 + 2{\cdot}3456.$$

Arranging vertically and working to ensure accuracy to three places, we set out the work usually as follows—

1·357	9
2·369	1
3·458	7
2·345	6
9·531	

And our answer correct to two places is 9·53. To ensure absolute accuracy to two figures, some advocate the retention of *two* extra places.

Subtraction may be similarly dealt with.

Multiplication and Division. It was in these two processes that confusing rules were formerly taught.

Pursuing our treatment as in the previous section, let us investigate to what extent the accuracy of a product (or quotient) depends upon its data.

As before, supposing that ·76 and ·38 are correct to the nearest hundredth, and multiplying them, we have—

Actual Product	·76	×	·38	=	·2888
Least Possible Product	·755	×	·375	=	·283125
Greatest Possible Product	·765	×	·385	=	·294525

Obviously we cannot rely upon all the figures in our product, for the real answer *could* be ·2831 or ·2945, neither of which are adequately expressed by ·2888.

It is absurd, then, in dealing with such subjects as areas, to rely too much upon the accuracy of our answer, when we are using actual measurements, however carefully made, as the basis of our calculations.

Accordingly we need not pursue our calculations beyond the possible limits of accuracy, and so we need to be able to "contract" our work and to use approximate methods. Thus in finding the area 12·34 m. × 7·4 m. to the nearest square metre, all we need is to determine the value of the whole-number part of our product, and perhaps the first decimal place.

Accordingly we may work as here set out. Our answer is seen to be 91 sq. m.

```
12·3 ¦ 4
     ¦ 7·4
———— ¦ —
86·4 ¦
 4·9 ¦
———— ¦ —
91·3 ¦
```

All that is necessary is the rejection of all figures to the right of the dotted line with the precaution of a common-sense regard for any *carrying* figure.

The modern method of decimal multiplication easily lends itself to contraction, especially if the multiplier happens to be in "standard" form.

Approximate Division may be dealt with even more easily.

Simple cases will have been dealt with in "decimalizing" fractions.

Thus, in dealing with $\frac{1}{7}$ as a decimal, the child should realize that the practical difference between $\frac{1}{7}$ = ·143 and $\frac{1}{7}$ = ·142857 is negligible. The pupil should be required to express any fraction known to result in a "pure" or "mixed" recurring decimal correctly to a limited number of places.

The actual "contraction" of decimal division is, if anything, simpler than that for decimal multiplication.

One example will suffice—

$$12{\cdot}3587 \div 4{\cdot}5856$$

```
4·5856)12·3587(2·695
        9·1712 :
       ------- :
        3·1875 :
        2·7514 :
       ------- :
         ·4361 :
          4126 :
         ----- :
           235 :
           229 :
          ---- :
             6 :
```

The process is simple: instead of adding zeros to the dividend, we cut off figures from the divisor, and thus avoid the necessity of all figures to the right of the dotted line.

Any example may be similarly treated, e.g. determine to three significant figures the value of

$$\frac{123{\cdot}57924}{29{\cdot}5867}$$

Since we require *three* significant figures, we will retain for safety *five* figures in the dividend which accordingly we write 12358, and for our divisor we use sufficient figures to divide into this, viz., 2959.

Dividing as in the example already worked, we obtain 417 as the three significant figures—

```
2959)12358(4176
     11836
     -----
     ..522
       296
       ---
       226
       207
       ---
        19
```

By inspection our answer has *one* unit digit, and accordingly we write 4·18 for our quotient of three significant figures.

But perhaps we pay too much attention in schools even to these simple rules, for no one, in actual practice, having a good "slide-rule" available, ever troubles to work out decimals by "approximate" methods. A slide-rule will be found to cover every practical case.

Other Approximations. No boy should leave our schools to-day without a working knowledge of the following approximate methods in decimals—

(*a*) $(1 + a)(1 + b) = 1 + a + b + ab$

$= 1 + a + b$ approx. if a and b are small.

Example: 1·0034 × 1·00051.

$= (1 + \cdot 0034)(1 + \cdot 00051) = 1 + \cdot 0034 + \cdot 00051 = \mathbf{1 \cdot 00391}$

(*b*) $(1 \pm a)(1 \pm b) = 1 \pm a \pm b$.

Example: 1·0053 × ·9948.

$$= (1 + \cdot 0053)(1 - \cdot 0052)$$
$$= 1 + \cdot 0053 - \cdot 0052$$
$$= 1 \cdot 0001 \text{ approx.}$$

Similarly we may extend the method to any number of factors—

(c) $(1 \pm a)(+ \pm b)(1 \pm c)$, etc. $= 1 \pm a \pm b \pm c$, etc., approx.

(d) $(1 + a)(1 + a)(1 + a)$ in factors $= 1 + na$ approx.

(e) $(1 - a)^2$ $= 1 - na$ approx.

Combining the last two and the first three, we have—

(f) $$\frac{1}{(1 + a)^n} = 1 - na \text{ approx.}$$

(g) $$\frac{1 + a}{1 - b} = 1 + a + b \text{ approx.}$$

Examples involving the use of these, especially the use of the binomial approximation, are of such constant occurrence that the process (an easy one) should be mastered by every pupil above the actual beginner.

CHAPTER XIV

THE METRIC SYSTEM

" IN extending the use of decimal fractions to compound quantities the advantage of a decimal system will be apparent to the children, and such parts of the nomenclature of the metric system as apply to measures which they can actually handle or use for themselves may with advantage be taught. In any case the fact that such measures as the gramme and the kilogramme are used throughout great parts of the civilized world may well be mentioned, but a mere lip-knowledge of these measures is of little value, and unless a practical knowledge can be given, it is useless to spend time in teaching the terminology."—Circular 807, " Suggestions."

WE think the author of the above paragraph must have heard some of the lessons frequently given under the title of a " First Lesson in the Metric System." There still exists in many schools a large chart of the Metric System illustrating every conceivable unit, common and uncommon. Armed with this, the novice plunges into a prosy and sketchy " history " of the system, and after a learned disquisition on Greek and Latin prefixes, fills a blackboard with the " Table " (a host of formidable and difficult names), confusing and perplexing the class. Frequently the lesson proceeds without even the production of a metre stick or a reference to the centimetre graduations on the ruler. The result is that the class is nauseated and learns little but words, and the Metric System remains a bugbear of unreality to be avoided and shunned as much as possible.

It is difficult to account for this worship of " Table " and terminology. Perhaps it is that the teacher is misled by what appears to be so simple to an adult mind and yet is so confusing to a child when treated entirely out of the range of his normal experience.

All teachers must grasp this cardinal fact if they wish to be successful: the Metric System cannot be taught in one or two lessons by mere exposition. It must be treated by methods similar to those used for English weights and measures, and must be slowly and regularly absorbed through practical work and concrete applications.

The average child in English schools need only be familiar with the metre, centimetre (and perhaps millimetre), the kilometre, the gramme, the kilogramme, the cubic centimetre, and the litre. Arithmetical exercises involving other units than these are unreal and pedantic. They give, it is true, excellent practice in decimals, but this practice may well be obtained in other and more practical ways. We shall then confine our suggestions mainly to these units, remembering that we are writing for teachers in English schools.

The Metre and Centimetre. These must be taught by methods similar to those used in the teaching of the yard and the inch. They must be known as *things* and not merely as words. Measurement and drawing as well as calculation must be continually employed. The child should measure, draw, and estimate in these units until he has a familiarity with them almost equal to his acquaintance with English units. And this work may be spread over months or even years, and may proceed concurrently with English measurement, while simple calculations in these units will provide ample practice in decimal manipulations to one or two decimal places.

Sooner or later it will be necessary with English children to work out the approximate relation between the English and metre units. This is all too frequently memorized (without any attempt to establish the relations) either in the form (*a*) 1 metre = 39·37 inches, or (*b*) 1 in. = 2·54 cm.

Occasionally the relation (*b*) is proved by the single experiment of drawing a line 4 inches long which is measured in centimetres, etc. This is a useful exercise if repeated with a number of lines of *varying* length, the results being compared and an average found. Little reliability can be placed upon one single measurement.

A much simpler method, but one not so often seen in use, is to place the edges of two rulers together—one edge showing inches and the other centimetres. The correspondences can be seen at a glance and the equivalents worked out.

The Kilometre. This has been concisely defined as a "good ten minutes' walk." It is a little more than 1,000 yards. Its relation to the metre presents no difficulty. Intermediate units are so seldom required that English children need not be bothered with them.

The relation between the mile and the kilometre must be taught so long as both systems are in common use. A rough approximation, 5 miles = 8 kilometres, may serve as the basis for ready "mental" conversion from one unit to another. Somewhat more accurate is the decimal relation, 1 km. = ·621 miles; i.e. to change kilometres to miles it is necesary to *multiply* by ·621 and conversely miles may be expressed as kilometres by *division* by ·621. Beyond these units we do not think it necessary to bother the children, though perhaps the decimetre may also be introduced as a convenient length.

With the teaching of these units the teaching of simple decimal manipulations should always proceed concurrently.

Many teachers think it essential that they should give a lengthy exposition of the *history* of the system. To these we would offer the warning that the "metre" is no *natural* unit but is really as arbitrary as the much-maligned English *yard*.

The Units of Capacity and Weight. The units with which the ordinary child should be familiar are: (*a*) The cubic centimetre and the litre; (*b*) the gramme and the kilogramme.

The necessary knowledge may of course be given by exposition and illustration. Thus the child may make or examine an actual cubic centimetre and cubic decimetre or litre, and he may be told that the weights of each when filled with water are known respectively as the gramme and the kilogram. Some such illustration is an essential preliminary to any "sums" involving these units.

The actual place, however, for gaining real and natural familiarity with such units seems to be in the science room and in the practical science lesson. Here such

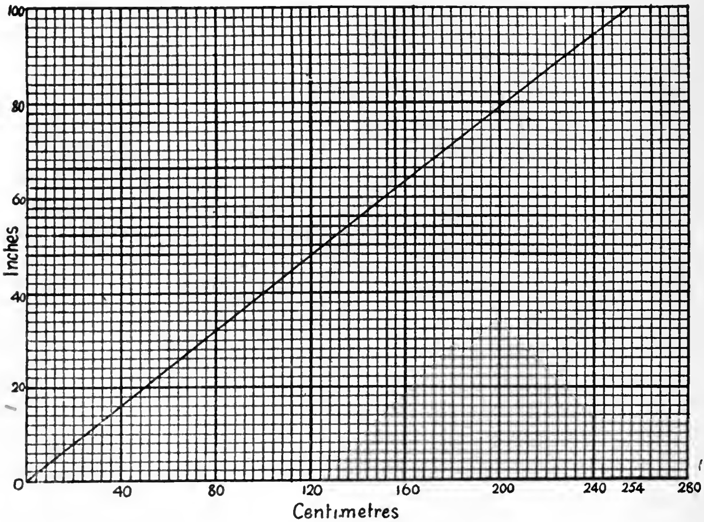


FIG. 61.

things as the litre flask, the 25 cc. or 50 cc. pipette, and the gram weight have a reality to the pupil that no amount of teachers' exposition can otherwise give.

As with all common metric units, the connection of the units of weight and capacity with corresponding English units should be known, either in the approximate forms, 4 litres = 7 pints, 5 kilos = 11 lbs., or in decimal form, 1 litre = ·22 gals. or 1·76 pints, 1 kilo = 2·2 lbs.

The metric ton (1,000 kg.) may also be compared with the English ton.

In teaching these units we shall have supplied a sufficient minimum, and the pupil will gain a knowledge of the common metric units as realities; he will have a clear idea of their relation to corresponding English units; and any examples requiring a bookish and unreal knowledge of tables beyond these units may well be omitted.

Graphical Representation of the Connection between English and Metric Units. These straight-line graphs or " converters " provide many useful exercises. Decimal equivalents form the best basis. Thus to convert inches to centimetres we may use the equivalent 1 in. = 2·54 cm. This may be expressed as 100 in. = 254 cm. Plotting these to suitable scales (Fig. 61), we may construct a ready reckoner to convert any number of inches or centimetres within the limits of the graph. We might of course have used the equivalent, 1 cm. = ·394 in. or 1,000 cm. = 394 in. Similarly equivalents, each providing the foundation for a useful graph, are—

1 m.	=	1·09 yds.	(100 : 109)
1 km.	=	·621 mi.	(1,000 : 621)
1 sq. cm.	=	·155 sq. in.	(1,000 : 155)
1 cub. in.	=	16·4 c.c.	(10 : 164)
1 litre	=	·22 gals.	(100 : 22)
1 kg.	=	2·2 lbs.	(10 : 22)

Decimal Coinage. The decimal and metrical instruction in every school should include some account of the systems of decimal coinage adopted in many of the leading countries. Children should have a ready knowledge at least of American dollars and cents, of the French franc and centime, of the German mark and pfennige, and of their current equivalents in £ s. d.

These equivalents can only be roughly and approximately memorized, and where particular accuracy is required it will be better to give the exact rate of exchange current at the moment.

The custom followed in larger shops and stores of large cities of marking goods in two systems of coinage, e.g. in English and American, or English and French money, will provide plenty of up-to-date examples. Thus, a child may be asked to express the English price 12s. 6d. in equivalent dollars and cents, or francs and centimes, and conversely, given a price-ticket in £ s. d. and in dollars and cents, he may work out the equivalent of 1 dollar in shillings and pence or of £1 in dollars, etc. Using approximate equivalents such as 1 dollar = 4s. 2d. and 1 franc = $9\frac{1}{2}$d., and 1 mark = 1s., an endless variety of examples is possible and much interest can be found in comparing prices of common articles. Thus

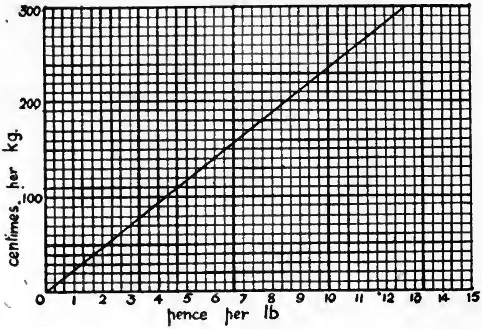


FIG. 62.

sugar at a given price in francs per kilogram may be compared with sugar at a given price in pence per lb. Similarly we may compare fares at centimes per kilometre and at pence per mile. Examples such as these give point and interest to the conversion from one system to another.

With a little preliminary calculation it is possible to draw a graph for any such conversion. We give one as an illustration. To change from centimes per kg. to pence per lb.

Using the equivalents 1 franc = 9·5 pence, and 1 kg. = 2·2 lb.—

$$\begin{aligned} \text{We have 1 franc per kg.} & \\ &= 9{\cdot}5 \text{ pence per } 2{\cdot}2 \text{ lb.} \\ &= \frac{9{\cdot}5}{2{\cdot}2} \text{ pence per 1 lb.} \\ &= 4{\cdot}3 \text{ pence per 1 lb.} \end{aligned}$$

i.e. 100 centimes per kg. = 4·3 pence per lb. These may be set out to any chosen scale and a ready reckoner constructed to any size (Fig. 62).

Similarly we may work out the graphical equivalents between centimes per litre and pence per pint, and many other useful relations.

Arabic Notation and Decimal Fractions. It is to be feared that most people accept our ordinary arithmetical symbols and the decimal basis upon which our notation is planned as an every-day institution, without concerning themselves with its long and fascinating history. It is true that Roman figures survive on clock-faces and sun-dials and in the numbering of chapters in books, but even for these, too, we fear that familiarity as usual breeds indifference if not contempt. If these same teachers will take the trouble to probe—be it ever so little—into the history of arithmetic, they will perhaps return to the humdrum arithmetic lesson with a new reverence and a new enthusiasm for the wonderful possibilities of calculation which this introduction of a decimal system of numeration and notation has made possible.

All modern calculation with its amazing and apparently limitless range is based upon: (*a*) Arabic or decimal notation; (*b*) decimal fractions; (*c*) common logarithms. Remnants of older systems are seen in Roman figure notation, in fractions, in angular measurement, in time measurement, in many of our weights and measures, and in duodecimals. For all other purposes our notation and our arithmetical operations are based upon "ten" and "powers" of ten. Yet this universal decimal arithmetic has been only gradually evolved. It was hardly known to the classical world of Greece and Rome, and filtered through but slowly to modern Europe from the Hindoos and the Arabs. The Arabic or decimal system of notation has actually been in common use in the Western world for less than 500 years, and this accounts in part for the fact that decimal fractions and their manipulations are still looked at with suspicion and fear by many people.

It will astonish many readers to learn on the eminent authority of Mr. W. W. Rouse Ball,[1] that "no instance of a date or number being written in Arabic numerals is known to occur in any English parish register or the Court rolls of any English manor before the sixteenth century; but in the rent roll of St Andrew's Chapter, Scotland, the Arabic numerals were used in 1490."

Yet even in the classical world there seems to have been a vague recognition of the possibilities of a decimal system. Thus Archimedes in his "Sand Reckoner" develops a system of octads, i.e. units such as 10^8, 10^{16}, 10^{24}, etc., by which he estimates the possibility of counting the sand on the Sicilian shore, and finally computes that a world of sand would contain 10^{51} grains. Apollonius, too, appears to have pondered over a decimal notation, though his basis seems to have been sextads, 10^6, 10^{12}, 10^{18}, etc., beginning with the ordinary million.

It is possible that the credit for a decimal system, which is the logical fore-runner of our present system, is due to the Hindoos. It is certain that Arya Bhata, a Hindoo mathematician of the fifth century, had some acquaintance with a decimal system of number as shown in some of his published solutions. But it is in the first chapter of the *Astronomy of the Hindoo Bhaskara* (twelfth century) that we find the first systematic description and use of the decimal system. Here the nine numerals and a zero are found for the first time.

From the Hindoos the system with its numerals had meanwhile spread to the Arabs, and under the most illustrious of the Arab mathematicians, Alkarismi or Al Khwarizmi, it appears to have been established on solid foundations which formed the groundwork for all Italian and mediaeval developments of "algorism" or the art of Alkarismi, as this decimal arithmetic was

[1] *History of Mathematics*, Ch. xi, p. 187.

afterwards termed. First known and used by the mediaeval merchants, by the fourteenth century it appears so well known to scientists and mathematicians generally throughout the Western world, that we find Roger Bacon recommending "Algorism" as a "necessary study for theologians." For an excellent synopsis of the history of the evolution of our ten Arabic symbols as now written, we must refer the reader to Mr. W. W. R. Ball's comprehensive *History of Mathematics*, or to any good encyclopaedia. The history of the zero sign will be found particularly interesting.

To mathematicians of the twentieth century it seems astonishing that the decimal system, found to be so convenient in the manipulation of integers, was not extended more readily and rapidly to fractions. Yet decimal fractions and their manipulations have a history of hardly three hundred years. To us the extension of the decimal idea to the *right* of the units digit seems so obvious that we wonder at the slow evolution of decimal fractions. The main difficulty seems to have been one of a convenient notation. The present notation seems due to Briggs, a mathematician famed for his association with Napier in the epoch-making invention of logarithms some three centuries ago.

Stevinus of Bruges (1548–1620) developed the idea of decimal fractions, but was handicapped by his curious method of writing "tenths," "hundredths," etc. Thus he would write 12·345 as

$$12,\ 3'\ 4''\ 5''' \text{ or as } 12_0\ 3_1\ 4_2\ 5_3.$$

Napier, the inventor of logarithms, used the former of these two methods, and also occasionally used a solidus to separate the integral from the fractional part of a number; thus he would sometimes write a number such as 6·789 as 6/789. It was Briggs who urged Napier to convert all his "natural" logarithms to the more convenient "common" logarithms, and who actually published in 1617 a table of common logarithms, using the point notation as in use to-day. Yet he, too, adopted other methods such as underlining, i.e. a decimal such as 13·579 would be written $13\underline{579}$. Other devices also used are shown in such forms as $13|\underline{579}$ and $1\underset{\circ}{3}579$, and it was not until the eighteenth century that the point was in general use. Even to-day the exact position of the point varies in different countries. Thus 1·23 is printed as 1·23 in England, 1.23 in America and 1,23 on the continent.

The application of the decimal system to weights and measures developed but slowly. The convenience of ten as a basis for calculation in weights and measures was, however, seen very early and was suggested in the *Arithmetic* of Simon Stevinus (Bruges), published in 1585. The reform had to wait two hundred years for the French Revolution when, under the direction of the eminent mathematician Lagrange, the present metric system was formulated. Lagrange, it may be remembered, was Italian born, but was specially exempted by name in the decree of 1793 which banished all foreigners from France. In 1799 the Commission for reform of weights and measures adopted his recommendations both as to units and as to decimal divisions and subdivisions.

It is fashionable amongst teachers to clamour loudly for the adoption in England of the Metric System. The argument most commonly heard is the admitted one of the saving of time in the child's school life. Yet, as we have tried to point out in earlier chapters, the time devoted to English weights and measures may be reduced to a workable minimum. The controversy rages far beyond schools. Reformers reiterate the advantages to industry and commerce that would ensue if we adopted whole-heartedly the decimal system of weights and measures of our neighbours. Scientists have long since cut the knot, and the French decimal system has achieved an international and even universal character as the medium of scientific measurement. It is not our intention here to review the debatable ground—the "pros" and "cons" have long since been laid bare. Nor do we hold any brief for the more conservative element, but at the risk of offending the ardent reformers we feel compelled to point out in fairness to our older English units and systems, one or two grave disadvantages to the adoption of a decimal system as applicable to the ordinary work-a-day world of plain every-day people.

The most natural method of simple division of a unit is expressed in "halves" and "quarters." Now "ten" is unsuitable for this process of primary division. We can halve it but not quarter it conveniently without entering into fractions. Contrast this with any ordinary English unit, the pound, the penny, the load, the pound avoirdupois, etc. An example will make our meaning clearer. Suppose sugar is priced at 25 centimes per kilo or cloth at 2 francs 25 centimes per metre. The buyer of *half* a kilo or *half* a metre is at once at a disadvantage, and has to pay *more* than is correct since 25 cannot be conveniently halved and no smaller coin than the centime exists. It is estimated that the poorer classes in France have, in this way, in a hundred years paid untold millions of francs in small excess prices. It may be pointed out that the English draper with his prices such as 1s. 11¾d. per yard adopts the same device, but here we suspect that a certain shrewd knowledge of psychology may be partly responsible.

But our objection to a decimal basis has very solid support, for in 1821, in America, the great John Quincy Adams, after exhaustive examination, pronounced the decimal system as unsuitable to the ordinary transactions of practical life, and even affirmed that the older systems with their numerous simple binary factors made *mental* arithmetic easier than any decimal system could possibly do. So far the Americans have decimalized their coinage only, and here, as in France, the same disadvantage occurs, for the thirsty Westerner pays his " little bit " for his refreshment if he luckily has a " dime " (10 cents) in his pocket; otherwise he hands over a " quarter " (25 cents) and receives a " dime " change, paying a " big bit " (15 cents) for his drink. Even the introduction of the " nickel " (25 cents) has not quite killed the custom of " big " and " little " bits.

The advantages of the duodecimal and sexagesimal units are seen very clearly in Angular Measurement and in Time Measurement. It is true that a decimal division of the right angle does exist, but the attempt to decimalize the measurement of time has long since been discarded as unworkable.

CHAPTER XV

PRACTICE

THE keynote to this chapter will be found in the following extract from Circular 807, "Suggestions"—

> "It is well to distinguish between practice methods which are often useful and the long practice sums which are often useless or even ridiculous. On the one hand, it is evident that the cost of 19 articles at 9s. 11d. each is best obtained by subtracting 19 pennies from 19 times 10s. rather than by multiplication by 19. On the other, it seems that nothing can be gained by setting the children to calculate the price of 14 tons 18 cwt. 15 lbs. at the price of £17 11s. 8½d. per cwt. Apart from the requirements of a few special businesses such a question is unreal; it serves only as a test of accuracy, and accuracy can be obtained equally well in other ways."

The criticism of the general character of the teaching of this subject of Practice as contained in the above extract is just and well-merited. "Practice" in our schools has been confined to a formal type of example forced into a most rigid framework of lines and spaces, and this beautiful method of calculation, which can be applied to an infinite variety of examples and types of examples, has been slavishly confined to money and costs in the traditional arrangement of "Simple" and "Compound" Practice. The work usually began with countless examples of the "article" variety such as: Find the cost of 456 articles at £3 17s. 10½d. each. We have even heard of a class that did not recognize the sum as Practice simply because the word "article" had been replaced by a more concrete term.

This narrow rigid view of Practice must give place to a broad elastic view of what is undoubtedly one of the most interesting and useful methods of calculation, and one capable of very wide application. It must no longer be thought of, or taught, as a stereotyped kind of sum, but as a method to be employed on every convenient occasion. Practice in this wider sense has been concisely described as "The Tom Tiddler's ground of Arithmetic short cuts" where "every man makes his own rules and all that can be done is to explain principles and to give advice as to particular cases."[1]

The history of "Practice" supports this wider view. According to the *New English Dictionary*, the history of the method is briefly summarized as follows—

"The method of Practice came into use in the sixteenth century, when it was introduced by the merchants and negotiants of Italy for the expediting of business calculations. Early English Arithmetics give a variety of names to the method. *Practica Italica, Italian Usages, Rules of Practice, Brief Rules, The Small Multiplication.* The first English writer to deal with 'practice' appears to have been H. Baker, who in his *Well-Spring of Sciences* (1574), f. 87*b*, has—

"'The third part treateth of certayne briefe rules, called rules of practice. . . . Some there be which call these rules of practice, briefe rules. . . . There be others which call them the small multiplication.'"

Mellis (1596) in his edition of *Recorde's Arithmetic*, Vol. III, p. 406, treats of "Briefe Rules called Rules of Practice." Although these early writers lay stress on the utility of the method in business calculations, it is clear that they also realized the general utility of the method as a method of multiplication in general."[2]

Practice is best viewed not as a new type of sum but as an alternative method of obtaining the results of multiplication. It follows, that as a method it may be applied to *all* multiplication and is not to be confined solely to *costs*.

The process consists of selecting such a system of units for the multiplier as will enable us to convert it into a fraction or series of fractions, and thus we attain the result by the simple process of short division.

[1] *Short Cuts and Byways in Arithmetic*, by Cecil Burch.
[2] *New English Dictionary*, "Practice."

Examples—

(*a*) 13s. 9d. × 37.

Unit £1

	£		
	37		
10s. = $\frac{1}{2}$	18	10	–
3s. 4d. = $\frac{1}{3}$	6	3	4
5d. = $\frac{1}{8}$		15	5
	£25	8	9

(*b*) £7 15s. 0d. × 46.

Unit £10

	£		
	460		
£5 = $\frac{1}{2}$	230	–	–
£2 10s. = $\frac{1}{2}$	115	–	–
5s. = $\frac{1}{10}$	11	10	–
	£356	10	–

(*c*) 1357 × 25.

Unit 100

	135700
25 = $\frac{1}{4}$	33925

(*d*) 287 × $5\frac{1}{2}$.

Unit 10

	2870
5 = $\frac{1}{2}$	1435
$\frac{1}{2}$ = $\frac{1}{10}$	143·5
	1578·5

(*e*) ·234 × ·675.

Unit 1

	·234
(*a*) ·5 = $\frac{1}{2}$	·117
(*b*) ·125 = $\frac{1}{4}$	·02925
(*c*) ·05 = $\frac{1}{10}$ of (*a*)	·0117
	·15795

Advantages of Practice. The advantages of the method are clear. Division is generally easier than multiplication, and, since the operations of division always proceed from left to right in our notation, the process removes any difficulty of determining the position of the decimal point and also facilitates approximation.

The utility of the method depends upon the choice of the original unit for the multiplier. The actual number of steps in the working is immaterial so long as the increased length in the figuring is compensated for by an increased rapidity and ease of working. Thus the whole subject is one not to be " learnt " or " taught " in stereotyped form, but is a personal matter depending upon each individual. Much of the weakness of Practice in the past has been due to the tendency to restrict the method to costs and to insist upon the chosen unit always being £1, thus inflating the importance of " aliquot parts " of £1. It is to counteract tendencies such as these that we shall devote much of what follows in this chapter. The point we earnestly desire to emphasize is that there are various tools for performing multiplication. The pupil should be trained to choose the tool which common sense tells him is best for his purpose at the time, as in the workshop one man may prefer a chisel and another a knife. Both may be right from the individual point of view: so in the classroom. There is no best way in general of doing any multiplication; there is an easiest way for each individual and he ought to do each example in the way easiest *for him*.

Introduction by Aliquot Parts. It is customary to begin our teaching of the Practice method with much drill in " aliquot parts." Now if our methods involving fractions have been sensibly taught we shall have covered much of this ground, and consequently simple practice methods may be used much earlier in a child's life than was formerly thought possible. No longer is it necessary to postpone the introduction of the method until the child has " done " fractions.

Aliquot parts of £1 as far as 12th or 16ths are readily taught. They may be illustrated and compared graphically by the usual rectangular diagram. What is more important is the recognition of " aliquot parts of aliquot parts." Thus the child needs constant practice in such as the following—

What part of 5s. is 1s. 3d.?
What part of 1 ft. 6 ins. is 3 ins.?
What part of 14 lbs. is $3\frac{1}{2}$ lbs.?

In nearly every Practice sum the aliquot part of the bigger unit, i.e. £1, 1 ton, 1 yard, etc., occurs once only; the others are "parts of parts." The method is, of course, closely akin to the fractional manipulation with unit numerators common to early Egyptian and Greek mathematicians. Thus we think of 8s. 6d. in Practice as 5s. + 2s. 6d. + 1s., or $\frac{1}{4} + \frac{1}{8} + \frac{1}{20}$, but actually work it as $\frac{1}{4} + (\frac{1}{2}$ of $\frac{1}{4}) + (\frac{1}{5}$ of $\frac{1}{4})$, and thus proceed by very simple division.

Following the recognition of aliquot parts of all kinds of quantities the beginner needs considerable practice in splitting any quantity into convenient aliquot parts of a larger quantity. It is debatable whether this practice should be definite and preliminary before working examples, or whether it should be incidental and dependent upon each particular example as it occurs. Both methods have been successfully tried. The cautious teacher will perhaps prefer the first method of preliminary drill, but the alternative method gives greater elasticity and seldom results in every child performing the calculation in exactly the same way. We repeat, however, that the skill required in practice is very largely the skill to select the most convenient aliquot parts. Thus 16s. 9d. might be used as 10s. + 5s. + 1s. + 6d. + 3d.; this method would probably be adopted by a beginner. Other and more convenient divisions are 10s. + 5s. + 1s. 3d. + 6d., 10s. + 5s. + 1s. 8d. + 1d., or 10s. + 6s. 8d. + 1d. Children should be encouraged to plan out alternative groupings and to select the easiest (which is not always the shortest).

Simple Practice. As an alternative to the usual custom of beginning with examples involving parts of £1, we here suggest that the teacher shall begin with much smaller units such as 1s. or 6d.

Pence and Farthings. Unit used, 1s.

Example: 53 caps at $7\frac{1}{2}$d. each.

Children may set out the work as follows—

			£	*s.*	*d.*	
	53 at 1s.	=	2	13	–	
6d. = $\frac{1}{2}$ of 1s.	53 at 6d.	=	1	6	6	($\frac{1}{2}$ of £2 13s.)
$1\frac{1}{2}$d. = $\frac{1}{4}$ of 6d.	53 at $1\frac{1}{2}$d.	=		6	$7\frac{1}{2}$	($\frac{1}{4}$ of £1 6s. 6d.)
			£1	13	$1\frac{1}{2}$	*Ans.*

Similarly, 79 ties at $8\frac{3}{4}$d. each.

			£	*s.*	*d.*	
	79 at 1s.	=	3	19	–	
6d. = $\frac{1}{2}$ of 1s.	79 at 6d.	=	1	19	6	($\frac{1}{2}$ of £3 19s.)
2d. = $\frac{1}{3}$ of 6d.	79 at 2d.	=		13	2	($\frac{1}{3}$ of £1 19s. 6d.)
$\frac{3}{4}$d. = $\frac{1}{8}$ of 6d.	79 at $\frac{3}{4}$d.	=		4	$11\frac{1}{4}$	($\frac{1}{8}$ of £1 19s. 6d.)
			£2	17	$7\frac{1}{4}$	*Ans.*

This may be extended to *differences* as well as to *sums* of parts.

Example: 37 yds. at $5\frac{3}{4}$d. a yard.

		s.	*d.*	
37 at 6d.	=	18	6	
37 at $\frac{1}{4}$d.	=		$9\frac{1}{4}$	
By subtraction		17	$8\frac{3}{4}$	*Ans.*

Shillings and Pence. Here we may use as our base unit either 1s. or £1. Thus in the example—

Calculate 365 days pay at 12s. per day, we may work either as in (*a*) or (*b*)—

(*a*) Unit £1.

		£	*s.*	*d.*
365 at £1	=	365	–	–
Pay at 10s. ($\frac{1}{2}$ of £1)	=	182	10	–
Pay at 2s. ($\frac{1}{5}$ of 10s.)	=	36	10	–
By addition	=	£219	–	–

(*b*) Unit 1s.

		£	*s.*	*d.*	
365 at 1s.	=	18	5	–	
365 at 12s.	=	£219	–	–	(by multiplication)

Strict adherers to tradition would no doubt refuse to recognize (*b*) as "Practice."

Example: 30 doz. books at 3s. 6d. each.

Using £1 as the basis, this would be worked as—

		£	*s.*	*d.*	
360 at £1	=	360	–	–	
Cost at 2s. 6d. ($\frac{1}{8}$ of £1)	=	45	–	–	
Cost at 1s. ($\frac{1}{20}$ of £1)	=	18	–	–	
		£63	–	–	*Ans.*

It might be also worked on the basis of 1s., e.g.—

		£	*s.*	*d.*
360 at 1s.	=	18	–	–
360 at 3s.	=	54	–	–
360 at 6d.	=	9	–	–
		£63	–	–

Similarly in finding the cost of 19 lbs. of beef at 2s. 4d. a lb., it is better to use 1s. as the basis and not £1, i.e.—

		£	*s.*	*d.*
19 at 1s.	=		19	–
19 at 2s.	=	1	18	–
19 at 4d. ($\frac{1}{6}$th of 2s.)	=		6	4
		£2	4	4

Beginners should be encouraged to select always the most convenient unit, and should not be obliged to force every example into the same rigid framework of pounds and aliquot parts of a pound. To do this is to perpetuate all the old weaknesses of Practice teaching.

The general case of Simple Practice, i.e. costs involving pounds, shillings, and pence, can now be studied. All that is necessary for success is a careful gradation of examples. Thus, we might work with costs arranged in something like the following order—

(*a*) Pounds and *One* Part only—

E.g. £1 10s., £1 5s., £1 4s., £1 2s. 6d., etc.
£2 10s., £3 5s., £4 4s., £5 6s. 8d., etc.

(*b*) Pounds and *Two* Parts only—

E.g. £1 15s., £1 12s. 6d., £1 9s., £1 8s. 4d.
£2 14s., £3 13s. 4d., £4 7s. 6d., £5 3s. 9d.

(*c*) Involving *Subtraction* of parts—

E.g. 9s. (10s. – 1s.), 19s. 6d. (£1 – 6d.), 1s. 11d. (2s. – 1d.)
9s. 11d. (10s.– 1d.), 3s. 11¾d.(4s. –¼d.), 4s. 10½d. (5s. – 1½d.)

(*d*) Examples involving more than two parts—

E.g. £3 7s. 9½d., etc.

We fear that much of the older teaching *began* instead of ending with examples of the formidable type (*d*) above. Having once reached this stage, we may of course extend our numbers to any desired degree. Thus, we *may* set such an example as: "Find by practice the cost of 98,765 articles at £3 13s. 9¾d. each," but remembering the warning given in the extract at the beginning of the chapter, we shall avoid such futile figuring and confine ourselves to more sensible examples and simpler quantities.

With regard to the "setting out" of examples, it has been usual in the past to insist on an elaborate ruled framework and a formal and fussy tabulating and naming of each line. Let us try to strike a sensible balance and remember that the business man (having mislaid his ready-reckoner), requiring to calculate £1 6s. 6d. × 87, simply sets down something like the following—

£87 (i.e. cost at £1)
£29 (i.e. cost at 6s. 8d.)
————
£116
════

and deducts 87 at 2d., or 14s. 6d.

He has no time or inclination for carefully ruled framework and elaborate side-trimmings.

Decimals may be used in any convenient example. The actual calculation is simplified by the use of decimals, but the answer must be reconverted to £ s. d.

We will conclude this section by working a "Simple" Practice example in the ordinary way, and also by using decimals.

Example: £7 13s. $9\frac{1}{2}$d. × 387.

Ordinary Method.

	£	s.	d.
	£387		
	7		
	2,709		
10s. = $\frac{1}{2}$	193	10	–
3s. 4d. = $\frac{1}{3}$	64	10	–
5d. = $\frac{1}{8}$	8	1	3
$\frac{1}{2}$d. = $\frac{1}{10}$		16	$1\frac{1}{2}$
	£2,975	17	$4\frac{1}{2}$

Using Decimals.

	£387
	7
	2709
10s. = $\frac{1}{2}$	193·5
3s. 4d. = $\frac{1}{3}$	64·5
5d. = $\frac{1}{8}$	8·0625
$\frac{1}{2}$d. = $\frac{1}{10}$	·80625
	£2975·86875

i.e. £2,975 17s. $4\frac{1}{2}$d.

Though it is customary to apply Simple Practice to costs only, it is, of course, applicable to *any* Compound Multiplication. Thus we may work by Practice methods an example such as the following—

11 yds. 1 ft. 9 ins. × 37.

E.g.	yds.	ft.	ins.
	37 yds.	0 ft.	0 ins.
	11		
	407	0	0
1 ft. 6 in. = $\frac{1}{2}$	18	1	6
3 in. = $\frac{1}{6}$	3	0	3
	428 yds.	1 ft.	9 ins.

The reader will by now have noted that most of our Practice methods depend entirely upon the arbitrary connection of the units of our English money, weights and measures.

The Construction and Use of Simple Ready Reckoners. This useful exercise, closely akin to Practice, deserves much more attention in schools than it usually receives.

The child may be set to complete by simple multiplication some such table as the following—

	cost of 43 articles			
¼d. each			10¾	(a)
½d. ,,		1	9½	(b)
1d. ,,		3	7	(c)
3d. ,,		10	9	(d)
6d. ,,	£1	1	6	(e)
1s. ,,	£2	3	–	(f)

The table when constructed may be used for varied examples, thus—

43 at ¾d.	each	= (a) + (b)
43 at 1½d.	,,	= (b) + (c)
43 at 4½d.	,,	= (b) + (c) + (d)
43 at 5d.	,,	= (e) – (c)
43 at 1s. 11d.	,,	= (f) × 2 – (c)

An inexhaustible variety of examples can be worked with one simple table.

Following this we may set the pupil to complete a more elaborate table such as—

Cost of	1	2	5	10	20	50	100
	¼d.						
	½d.						
	1d.						
	3d.						
	6d.						
	1s.						

This when constructed may be made the basis of an endless series of examples, involving combinations of both rows and columns. Work of this nature is of greater intellectual and practical value than the endless repetition of mechanical and clumsy Practice sums, and the child is as much entitled to be taught the construction and use of a common ready reckoner as he is to be taught the use of a foot rule.

The work may easily be extended to costs of compound quantities, and the child may construct and use a table such as the following—

Cost of	1 lb.	2 lb.	4 lb.	7 lb.	14 lb.	28 lb.	56 lb.	112 lb.
	¼d.							
	½d.							
	1d.							
	3d.							
	6d.							
	1s.							

"Compound" Practice. Examples involving costs of Compound Quantities are usually known as "Compound" Practice. They occur perhaps more commonly in books than in the commercial world.

In dealing with these examples there is room for even greater elasticity and ingenuity than in the case of Simple Practice.

Thus to find the cost of 2 cwts. 3 qrs. 7 lbs. at £14 15s. 8d. a cwt., we may work in any of the following ways—

(*a*)

	£	*s.*	*d.*
	4	15	8
			2
	9	11	4
2 qrs. = ½	2	7	10
1 qr. = ½	1	3	11
7 lbs. = ¼		5	11¾
	13	9	0¾

(*b*) Decimalizing the Quantity: 2 cwts. 3 qrs. 7 lbs. = 2·8125 cwts.

	£2·8125	= Cost at £1 per cwt.
	4	
	11.2500	
10s. = ½	1·40625	
5s. = ½	·703125	
6d. = $\frac{1}{10}$	.0703125	
2d. = ⅓	·0234375	
	£13·453125	= £13 9s. 0¾d.

(*c*) Decimalizing the Money: £4 15s. 8d. = 4·78333 ....

	£4·78333
	2
	9·56666 ...
2 qrs. = ½	2·39166 ...
1 qr. = ½	1·19583 ...
7 lbs. = ¼	·29895 ...
	£13·45310 = £13 9s. 0¾d.

The choice of which quantity to decimalize must be determined for each particular case. Generally the object is to avoid a long multiplication at the beginning. Thus in the example: Calculate the cost of 157 tons 12 cwts. of coal at £2 12s. 6d. a ton, we may decimalize £2 12s. 6d.; but this would then require a multiplication by 157, so that it is simpler to decimalize 157 tons 12 cwts. We will work both methods for comparison.

(*a*) *Decimalizing Money:*
£2 12s. 6d. = £2·625.

	£2·625
	157
	18·375
	131·25
	262·5
	412·125
10 cwt. = ½	1·3125
2 cwt. = ⅕	·2625
	£413·7000

(*b*) *Decimalizing Quantity:*
157 tons. 12 cwts. = 157·6.

	£157·6 at £1 per ton
	2
	315·2
10s. = ½	78·8
2s. 6d. = ¼	19·7
	413·7

The figuring saved by method (*b*) is obvious in this example.

Certain Compound Quantities lend themselves easily to the following simple method.

Consider for example: tons, cwts., qrs.

At £1 per ton we note the following equivalents: £1 per ton = 1s. per cwt. = 3d. per qr.

We may use these to work an example such as: Calculate the cost of 13 tons 14 cwts. 3 qrs. at £4 5s. 6d. per ton.

	13 tons	14 cwt.	3 qrs.	
	£1	1s.	3d.	
	£13	14	9	= cost at £1 per ton
			4	
	54	19	0	
5s. = $\frac{1}{4}$	3	8	8·25	
6d. = $\frac{1}{10}$		6	10·425	
	£58	14	6·675	

Ans. £58 14s. 7d. to nearest penny.

Similarly at £1 an acre we have the following—

Acre.	*rood.*	*pole.*
£1	5s.	$1\frac{1}{2}$d.

and at £1 per mile—

Mile.	*furl.*	*pole.*
£1	2s. 6d.	$1\frac{1}{2}$d.

both of which are frequently useful.

But in setting our examples in Practice to a class we shall do well to remember always our first warning—that examples involving large numbers and very complex quantities occur but seldom in actual life. We shall accordingly be wise if we confine our Practice sums to simpler quantities and real cases. The pupil will be more likely to realize the value of Practice as a method instead of being wearied and nauseated by the formidable examples of the older text-books.

Further Applications of Practice. *Money Applications.*

Rates. Find the amount of the rate payable on £24 10s. at 3s. 8d. in the £.

	£	*s.*	*d.*
	24	10	–
3s. 4d. = $\frac{1}{6}$	4	1	8
4d. = $\frac{1}{10}$		8	2
	£4	9	10

Taxes. What is the Income Tax payable on £237 10s. at 2s. 3d. in the £ ?

	£	s.	d.
	237	10	–
2s. = $\frac{1}{10}$	23	15	–
3d. = $\frac{1}{8}$	2	19	$4\frac{1}{2}$
	£26	14	$4\frac{1}{2}$

Duties. Find the duty payable on 1,200 gallons at 11s. 4d. per gallon.

	£	s.	d.
	1,200	–	–
10s. = $\frac{1}{2}$	600	–	–
1s. = $\frac{1}{10}$	60	–	–
4d. = $\frac{1}{3}$	20	–	–
	£680	–	–

Dividends. Find the dividend on £89 10s. at 3s. 2d. in the £.

	£	s.	d.
	89	10	–
2s. = $\frac{1}{10}$	8	19	–
1s. = $\frac{1}{2}$	4	9	6
2d. = $\frac{1}{6}$		14	11
	£14	3	5

Bankruptcy Problems. A bankrupt owes me £339 15s., but pays only 7s. 6d. in the £. What do I receive ?

	£	s.	d.
	339	15	–
5s. = $\frac{1}{4}$	84	18	9
2s. 6d. = $\frac{1}{2}$	42	9	$4\frac{1}{2}$
	£127	8	$1\frac{1}{2}$

Simple Interest. Find the interest on £87 10s. for $2\frac{1}{2}$ years at $4\frac{1}{2}$ per cent (i.e. for 1 year at $2\frac{1}{2} \times 4\frac{1}{2}$, or $11\frac{1}{4}$ per cent).

(*a*) Unit 10%.

	£	*s.*	*d.*
10%	8	15	–
1%		17	6
$\frac{1}{4}$%		4	$4\frac{1}{2}$
	£9	16	$10\frac{1}{2}$

(*b*) Unit 10%

10%	8·75
$1\frac{1}{4}\% = \frac{1}{8}$	1·09375
	9·84375

or £9 16s. $10\frac{1}{2}$d.

(*c*) Unit £100.

	£	*s.*	*d.*
	11	5	–
£50 = $\frac{1}{2}$	5	12	6
£25 = $\frac{1}{2}$	2	16	3
£12 10s. = $\frac{1}{2}$	1	8	$1\frac{1}{2}$
	£9	16	$10\frac{1}{2}$

Compound Interest. Find the amount of £250 in 2 years at $3\frac{3}{4}$ per cent.

Here it will be easier to use decimals.

	£	
	250	
$2\frac{1}{2}\% = \frac{1}{40}$	6·25	
$1\frac{1}{4}\% = \frac{1}{2}$	3·125	
	259·375	= Amount in 1 year
$2\frac{1}{2}\% = \frac{1}{40}$	6·4843 | 7	
$1\frac{1}{4}\% = \frac{1}{2}$	3·2421 | 8	
	269·1015 | 6	

Ans. £269 2s. to nearest shilling.

Application to Decimals. (*a*) Express 16s. 9d. as the decimal of £1.

	£1
10s. = $\frac{1}{2}$	·5
5s. = $\frac{1}{2}$	·25
1s. = $\frac{1}{5}$	·05
6d. = $\frac{1}{2}$	·025
3d. = $\frac{1}{2}$	·0125
	·8375

Many teachers prefer to teach the decimalizing of money in this way to beginners.

(*b*) Evaluate ·45 of £3 15s.

Using Addition.

	£	s.	d.
	3	15	–
·25 = $\frac{1}{4}$		18	9
·2 = $\frac{1}{5}$		15	–
·45	£1	13	9

Using Subtraction.

	£	s.	d.
	3	15	–
·5 = $\frac{1}{2}$	1	17	6
·05 = $\frac{1}{10}$		3	9
·45	£1	13	9

Application to Percentages. (*a*) Find $7\frac{3}{4}$ per cent of £37 10s.

	£	s.	d.
	37	10	–
5% = $\frac{1}{20}$	1	17	6
$2\frac{1}{2}$% = $\frac{1}{2}$		18	9
$\frac{1}{4}$% = $\frac{1}{10}$		1	$10\frac{1}{2}$
	£2	18	$1\frac{1}{2}$ *Ans.*

(*b*) Find 17 per cent of 3,450—

	3450
10% = $\frac{1}{10}$	345
5% = $\frac{1}{2}$	172·5
2% = $\frac{1}{5}$	69
	586·5

Miscellaneous Applications. Ordinary multiplication may be frequently simplified by intelligent application of the method. Thus the methods already noted for ready multiplication by 25, 250, 125, etc., have Practice as a basis.

E.g. 13902 × 25.

	1390200 (× 100)
25 = $\frac{1}{4}$	347550 *Ans.*

E.g. 3579 × 125.

	3579000 (× 1000)
125 = $\frac{1}{8}$	447375 *Ans.*

These may often be extended.

E.g. 4567×259.

$$
\begin{array}{rl|r}
 & & 456\dot{7}000\ (\times\ 1000) \\
\hline
250 = & \frac{1}{4} & 1141750 \\
9 & & 41103 \\
\hline
 & & 1182853
\end{array}
$$

The method also applies to fractional multiplication. E.g. a circle has a diameter of $12\frac{1}{2}$ ins. Find its circumference ($\pi = 3\frac{1}{7}$).

We may work as follows—

$$
\begin{array}{lr}
 & 12{\cdot}5 \\
 & 3 \\
\hline
 & 37{\cdot}5 \\
\text{(Dividing by 7)} & 1{\cdot}7857\ldots \\
\hline
 & 39{\cdot}2857\ldots
\end{array}
\qquad \textit{or} \qquad
\begin{array}{r}
12\frac{1}{2} \\
3\frac{1}{7} \\
\hline
37\frac{1}{2} \\
1\frac{11}{14} \\
\hline
39\frac{2}{7}
\end{array}
$$

Similarly, $127 \times 2\frac{3}{4}$.

$$
\begin{array}{lr}
 & 127 \\
 & 3 \\
\hline
 & 381 \\
\text{Deducting } \frac{1}{4} & 31{\cdot}75 \\
\hline
 & 349{\cdot}25
\end{array}
$$

This might also be worked with 10 as a unit, thus:

$$
\begin{array}{rl|r}
 & & 1270 \\
\hline
2\frac{1}{2} = & \frac{1}{4} & 317{\cdot}5 \\
\frac{1}{4} = & \frac{1}{10} & 31{\cdot}75 \\
\hline
 & & 349{\cdot}25
\end{array}
$$

The method can be applied for all fractions that differ from an integer by a simple part, i.e. $4\frac{1}{6}$ or $2\frac{11}{12}$.

But we might continue indefinitely our applications of this useful method of calculation. Several pages of varied examples will be found in Books V and VI of the author's *Common-sense Arithmetics*.

CHAPTER XVI

SQUARES AND SQUARE ROOTS

THE subject of squares and square roots forms a pleasant change from the more mechanical processes so far discussed. It is full of mathematical interest and at the same time of the widest practical application.

Many points in connection with squares are treated incidentally as the ordinary development of methods and topics proceeds, and the geometrical and arithmetical aspects are developed side by side. Thus the ordinary child, without much teaching, obtains clear ideas of both geometrical squares and square numbers.

We will discuss first the simple process of "squaring" a number. The pupil can usually supply the series of square numbers from 1 to 144. Beyond this he frequently has to resort to multiplication. He may well be taught how to proceed from any given square to the next higher or lower.

This process is the simplest application of the identity $(n + 1)^2 = n^2 + 2n + 1$. Thus from 12^2 or 144 we may proceed to 13^2 by adding first 12 and then 13 (n and $n + 1$). The steps are $144 + 12 + 13$ or 169. Similarly $169 + 13 + 14$ or 196 is $(14)^2$, and $400 + 20 + 21$ is $(21)^2$. This children readily grasp. Equally easy is the converse process of proceeding to the next lower square. Thus, if from 625 we subtract first 25 and then 24 we have 576 or $(24)^2$, and if from 576 we deduct first 24 and then 23 we have 529 or $(23)^2$.

The process of squaring any number ending in 5 is simpler still. The "rule" is simply expressed as "Multiply the tens digit or digits by the next higher number; call the product hundreds and add 25.

Example—

$$(65)^2 = (6 \times 7) \text{ hundreds} + 25 = 4{,}225$$
$$(95)^2 = (9 \times 10) \quad ,, \quad + 25 = 9{,}025$$
$$(115)^2 = (11 \times 12) \quad ,, \quad + 25 = 13{,}225$$

Mathematical readers to whom the rule is new may investigate its simple algebraic basis.

For the general method of squaring a number the simple identities such as—

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

are frequently of use. Thus—

$$(96)^2 = (90)^2 + 2 \times 6 \times 90 + (6)^2$$
$$= 8100 + 1080 + 36$$
$$= 9216$$

or—

$$(96)^2 = (100)^2 + 4^2 - 800$$
$$= 10016 - 800$$
$$= 9216$$

This is of special use in some cases of mixed numbers:

$$\text{E.g.} \quad (8\tfrac{1}{4})^2 = 8^2 + 4 + \tfrac{1}{16}$$
$$= 68\tfrac{1}{16}$$

These useful formulae may be demonstrated to pupils by the drawing and colouring of squares and rectangles. Their numerical applications are endless. The squares of many decimal fractions may be deduced from corresponding integral squares, thus $(2\frac{1}{2})^2$ may be viewed as $(2{\cdot}5)^2$, and on the analogy of $(25)^2 = 625$ we may write $2{\cdot}5 = 6{\cdot}25$. Similarly we may write $({\cdot}25)^2$ as ·0625. We may well set to our pupils questions such as the following. "Find by multiplication $(123)^2$, and deduce from your answer the squares of 1230, 12·3, 1·23, etc."

Graphic Method of Squares and Square Roots. A child that can appreciate the significance of a simple graph should be familiar with the smooth curve connecting the natural numbers with their squares. This is, of course, the positive half of the simple parabola $y = x^2$. Part of the curve is shown in Fig. 63. For the higher values on this curve we may assume that part of the curve between two numbers, e.g. 9 and 10, to be a straight line

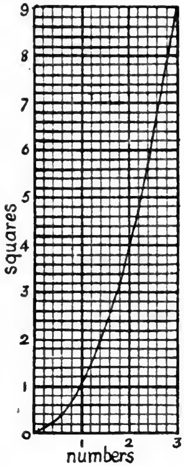


Fig. 63.

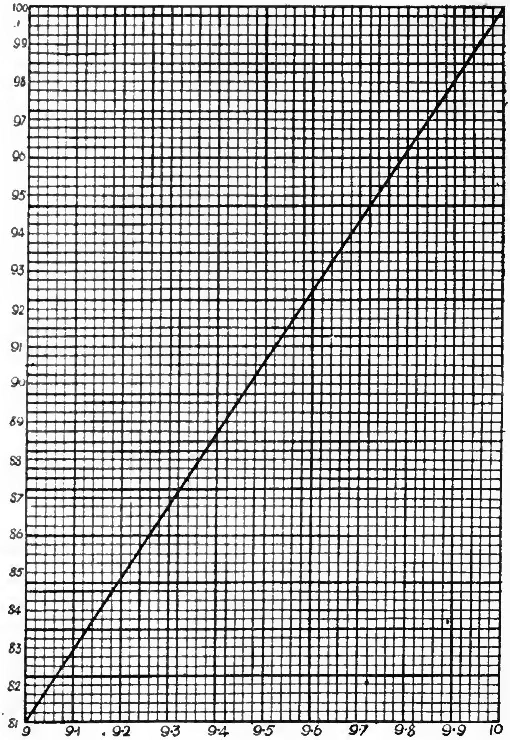


Fig. 64.

This may be reproduced to any enlarged scale (Fig. 64), and the line thus drawn will enable us to read off with considerable accuracy the square of any number between 9 and 10 or the square root of any number between 81 and 100.

This curve $y = x^2$ has so many applications in geometry, physics, mechanics, etc., that it cannot be too well known. The ordinary parabola using the same units on both x and y arcs increases in slope very rapidly. The curve may be flattened by the judicious selection of units. Thus the curve $y = \frac{x^2}{10}$ is only of moderate steepness, and the same effect may of course be obtained by selecting the x unit ten times as large as the y unit. The curve in some form or other may be used to illustrate endless connections, e.g.—

1. Areas and bases of similar triangles ($A \propto b^2$).
2. Areas and sides of squares ($A \propto s^2$).
3. Areas and radii or diameters of circles ($A \propto r^2$ or d^2).
4. Surfaces and radii or diameters of spheres ($s \propto r^2$ or d^2).
5. Distance and time for falling bodies ($s \propto t^2$).
6. Length of pendulum and time of swing ($l \propto t^2$).

Numerous other instances will be met with and should be made the basis of work in graphic squares.

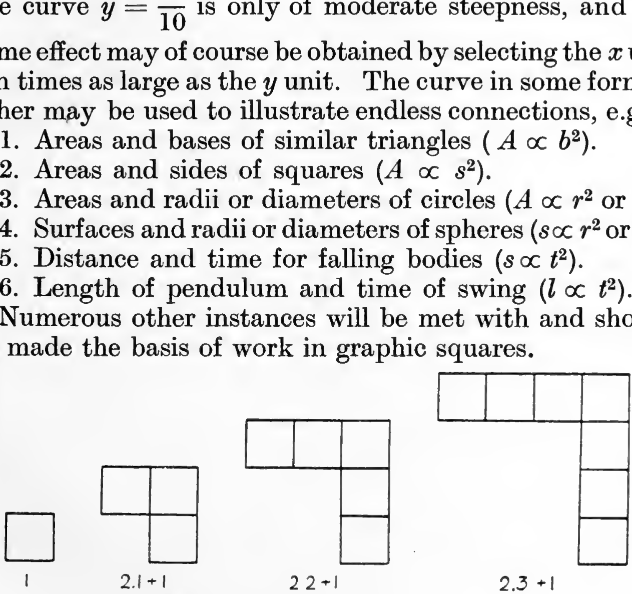


FIG. 65.

Properties of Squares. It has already been noted that every integral square is the sum of a series of odd numbers. Thus—

$2^2 =$	Sum	of first	two	odd	numbers,	i.e.	$1 + 3$
$3^2 =$	,,	,,	three	,,	,,	,,	$1 + 3 + 5$
$4^2 =$	,,	,,	four	,,	,,	,,	$1 + 3 + 5 + 7$
$n^2 =$	,,	,,	n	,,	,,	,,	$1 + 3 + 5 \ldots 2n - 1$

Children can illustrate this interesting truth for themselves by cutting out a series of gnomons $(2n + 1)$, as in Fig. 65.

These when fitted together form 4^2 or 16, and any square can be treated similarly.

Algebraically it will be noted that, by the addition of $2n + 1$ to n^2 we have obtained the next square, i.e. $n^2 + 2n + 1$ or $(n + 1)^2$.

Perfect Squares. Again children should know from observation that no perfect square can end in 2, 3, 7, or 8. The only possible digits for the units are 0, 1, 4, 5, 6, 9. This may be treated exhaustively by squaring the first ten numbers and noting the results, or may be viewed in the following interesting mathematical form—

"Every square number is either a multiple of five or is one more or one less than a multiple of five." This is easily seen by dividing all numbers into five groups: $5n$, $5n \pm 1$, $5n \pm 2$, and squaring each number. The only possible results are multiples of five or multiples of five + 1, or multiples of five + 4 (i.e. multiples of five − 1).

A simple knowledge of factors suffices for working all problems of the following type—

"By what must I multiply or divide 48 in order to have a perfect square?"

$$\begin{aligned} \text{Now } 48 &= 2 \times 2 \times 2 \times 2 \times 3 \\ &= 2^4 \times 3 \end{aligned}$$

But 2^4 is a square and 3 is not.

Hence, if I add another factor 3, I shall have $2^4 \times 3 \times 3$ or $2^4 \times 3^2$, which is a perfect square, i.e. 144.

Similarly I might *divide* by 3 and obtain 16.

Other factors that could be added are infinite, i.e. 3×2^2, 3×3^2, 3×4^2, 3×5^2, etc., all of which will convert the original 48 into a perfect square.

The Theorem of Pythagoras. Of relations between squares this is the best known and most useful, and of late years it has received considerable attention in primary schools. As usually stated it asserts that in any right-angled triangle the sum of the squares on the two sides containing the right angle is equal to the square on the hypotenuse.

For beginners a rigid mathematical proof of this important truth is out of the question. Some form of empirical or heuristic method of demonstration must be adopted. The careless non-mathematical teacher relies upon some particular case such as the 3 : 4 : 5 triangle, and pretends to "prove" the truth by drawing this example on squared paper; or he will airily announce that the theorem is "proved" by a cursory examination of a right-angled isosceles triangle (having sides $1 : 1 : \sqrt{2}$). This is bad mathematics and worse teaching. This

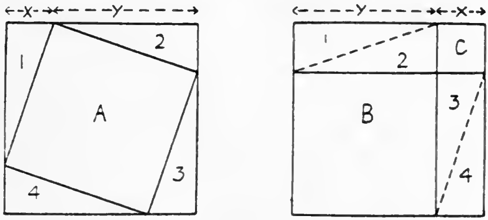


FIG. 66.

remarkable mathematical truth is worthy of better treatment. The teacher owes it to the child's intelligence to perform some sort of dissection of the large square to cover the two smaller squares, or of the small squares to cover the larger square. Numerous solutions of the "dissection" problem are available, and at least once in his life the pupil should have the pleasure of attempting this dissection to demonstrate "Pythagoras," while for proof Fig. 66 for beginners is as obvious as any. It is thought to be possibly the method of Pythagoras himself.

The original large squares are identical. Segments of each side x and y are the same in both. Clearly since the right-angled triangles are all identical in size, square A is equal to square B and C together. But A is the square on the hypotenuse and B and C are the squares on the sides containing the right-angle.

To Convert any Square into Two Smaller Squares. The following method deserves to be better known.

Let ABCD be the given square (Fig. 67). On BC describe any right-angled triangle X, and by a perpendicular from A make another triangle Y identical with X.

Now cut out X and rotate clockwise round C through a right angle. Similarly cut and rotate triangle Y anti-clockwise through a right angle about A. The original square (Fig. 67) has now the appearance of Fig. 68. It is an interesting exercise for beginners to prove that the original large square has been converted into two smaller

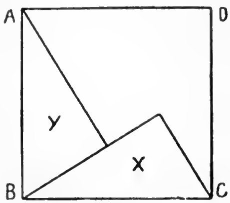


FIG. 67.

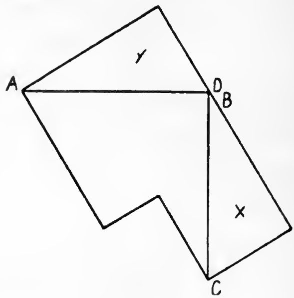


FIG. 68.

squares, and so for any triangle X thus drawn, the square on the hypotenuse is equal to the sum of the squares on the sides.

Nor should we forget the history of this classical problem. The history of Pythagoras himself is vague and legendary, but the problem is undoubtedly much older than the Greeks. It is undisputed that the ancient Egyptians used the truth to orient their temples and tombs, and appointed "rope-slingers" armed with ropes suitably divided in the proportions 3 : 4 : 5 to set off their right angles, the north-south line having been determined astronomically.

Children may with profit repeat this ancient method of setting out a right angle in the open air. Possibilities of error are greater than would be imagined and the exercise is valuable in every sense.

Numerical examples may be set as soon as the principle has been grasped. These may take such forms as: (*a*) Finding the hypotenuse, given the two sides; (*b*) finding one of the sides, given the hypotenuse and the other side. In their anxiety to set examples that "come out" to exact answers, teachers have shown a tendency to restrict their examples to simpler cases such as: 3 : 4 : 5, or 5 : 12 : 13. For the sake of these we give simple numerical methods by which an infinite series of right-angled triangles having integral sides may be found.

(*a*) *Series based on odd numbers.*

$$\text{Basis } n^2 + 2n + 1 = (n + 1)^2.$$

Since n^2 and $(n + 1)^2$ are both squares, all that is necessary is to select an odd *square* number for the odd number $(2n + 1)$.

Thus—

Value of $2n + 1$.	Value of n.	Resultant Identity.	Corresponding Triangle.
9	4	$16 + 9 = 25$	3 : 4 : 5
25	12	$144 + 25 = 169$	5 : 12 : 13
49	24	$576 + 49 = 625$	7 : 24 : 25
81	40	$1600 + 81 = 1681$	9 : 40 : 41

The series may be continued indefinitely.

(*b*) *Series based on even numbers.*

$$\text{Basis } (n^2 - 1)^2 + (2n)^2 = (n^2 + 1)^2.$$

Rule: Take any even number, $(2n)$. Square its half and *subtract* 1, square its half and *add* 1. The results, with $2n$, give the sides of a right-angled triangle.

$2n$.	$n^2 - 1$.	$n^2 + 1$.
4	3	5
6	8	10
8	15	17
10	24	26
12	35	37

This series, too, may be continued indefinitely. These simple methods of finding integral solutions for the indeterminate equation,

$$x^2 + y^2 = z^2,$$

were known to the Greek mathematicians.

The full solution is, of course, given by

$$k\{(m^2 - n^2)^2 + 4m^2n^2\} = k\ (m^2 + n^2)^2,$$

where k, m, and n may have any positive integral value.

Children in senior classes enjoy solving the problem by simple substitution of numbers if it is presented

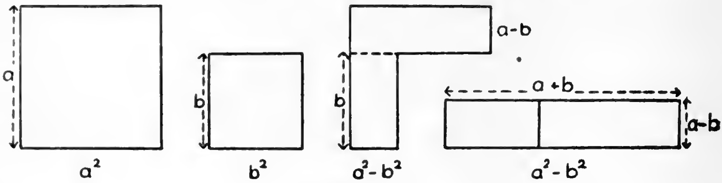


FIG. 69.

in the form of a quest for all pairs of square numbers that, added together, make a third square; while for teachers the methods indicated above provide an easy method of relieving the monotonous repetition of the 3 : 4 : 5 triangle in various guises.

The second problem: Given the length of the hypotenuse and one side to calculate the length of the other side, introduces us to the identity $x^2 - y^2 = (x + y)(x - y)$. This identity is of such endless application in calculation that it now rightly receives considerable attention in schools and is treated by graphical methods similar to those employed to demonstrate $(a + b)^2$, $(a - b)^2$, and $(a + b)(c + d)$.

The diagram (Fig. 69) is self-explanatory and shows how the difference-gnomon may be cut and converted into a rectangle $(a + b)$ long and $(a - b)$ wide. The identity once established should always be used for calculating the difference between two squares.

It is impossible, however, to restrict our problems in squares to *perfect* squares for more than a limited period. Sooner or later the need for a means of finding square roots will arise.

Square Roots. The necessity for finding square roots continually recurs in all forms of practical mathematics. Simple examples will have already been set in connection with factor exercises. Thus the child may find by simple factors the square roots of perfect squares such as 196, 225, 256, 384, 441, 484, etc., though he could hardly be expected to find by factors the square root of such a number as 9409 (97×97).

Again by comparison with known squares he will be able to approximate roughly to the square root required. Thus he will see that $\sqrt{20}$ lies between 4 and 5, since 20 is between 16 and 25. He might also reasonably assume that the root would be very nearly 4·5 (since 20 is approximately half way between 16 and 25), and he would not be far wrong. Such estimation and rough approximation should be encouraged as much as possible. Similarly he should note that $\sqrt{150}$ is not much above 12, and if he tries $12\frac{1}{4}$ he will find that he is very close to the answer ($12\frac{1}{4} \times 12\frac{1}{4} = 144 + 6 + \frac{1}{16}$)

$$= 150\tfrac{1}{16}$$

But the process of finding square roots to any desired degree of accuracy has still to be faced. Every pupil at some stage in his mathematical career learns the "trick." We will work a simple one for reference.

```
      3364 | 58
      25   |
     ------|
 108 | 864 |
     | 864 |
     ======
```

This ancient process, usually most popular with children, is easily explained on the basis of the identity $(a + b)^2 = a^2 + 2ab + b^2$, and may be demonstrated graphically even to children.

We represent the square 3364 graphically on any scale as (A) (Fig. 70). Our first step is to take out the largest possible square (say 50^2), i.e. 2,500 (B). This leaves us with a gnomon (C) containing 864. This gnomon we convert into a rectangle (D).

Clearly 864 has to be split into two factors $(100 + x)$, and x and by trial we find x to be 8.

All cases, integral or fractional, can be explained by similar diagrams. This graphical explanation of the general method is as old as Ptolemy, but has been re-discovered and elaborated with considerable ingenuity by later writers, notably Dr. T. P. Nunn.

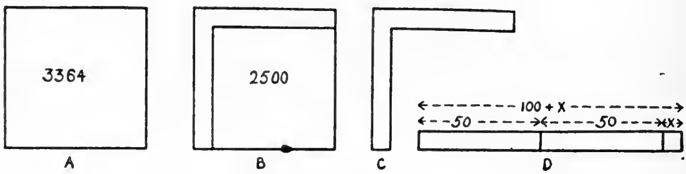


Fig. 70.

The extraction of a square root by this method is a process usually enjoyed by young arithmeticians. The method applies equally to decimals as to whole numbers. It is popular, too, with teachers as it is easily taught and readily assimilated (if not always understood) by an average class. As a practical method for older workers, however, it is gradually being superseded by logarithmic methods and slide-rule calculations.

Geometrical Methods. All simple surd quantities involving "incommensurables" or square roots that do not "come out" may be dealt with by simple geometrical methods involving the use of only a ruler, set square, and compass. The method is the simplest combination of the Theorem of Pythagoras, and of the identity $x^2 - y^2 = (x + y)\ (x - y)$.

Any given number may be expressed as the difference of the squares of two numbers that differ by 1.

Thus—

$$29 = 15^2 - 14^2 \text{ (i.e. } (15 + 14)(15 - 14))$$
$$30 = (15\tfrac{1}{2})^2 - (14\tfrac{1}{2})^2 \text{ (i.e. } (15\tfrac{1}{2} + 14\tfrac{1}{2})(15\tfrac{1}{2} - 14\tfrac{1}{2}))$$

and so for any odd or even number. Combining this fact with Pythagoras' Theorem, we have the following simple method of finding any simple square root graphically.

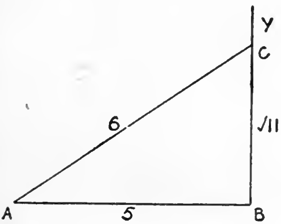


FIG. 71.

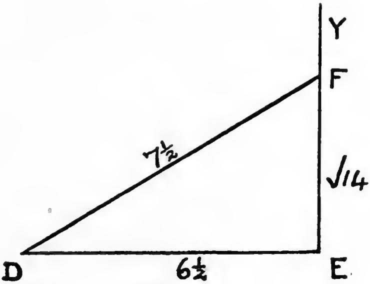


FIG. 72.

Example I. To find $\sqrt{11}$ graphically.

$$11 = 6^2 - 5^2.$$

Set off AB = 5 in any convenient unit (Fig. 71). Draw BY perpendicular to AB. From A as centre cut off AC = 6.

Then BC = $\sqrt{11}$ in the units used.

Example II. To find $\sqrt{14}$ graphically.

$$14 = (7\tfrac{1}{2})^2 - (6\tfrac{1}{2})^2.$$

Set off DE = $6\frac{1}{2}$ in any convenient unit (Fig. 72). Draw EY perpendicular to DE. From D as centre cut off DF = $7\frac{1}{2}$.

Then EF = $\sqrt{14}$ in the units used.

Another simple construction for finding simple surd quantities is based upon "Pythagoras" only (Fig. 73).

Choosing any convenient unit set off AY = 1 at right angles to Ax. On Ax mark off AB = 1. Then set off in succession—

$$AC = YB = \sqrt{2}$$
$$AD = YC = \sqrt{3}$$
$$AE = YD = \sqrt{4}$$
$$AF = YE = \sqrt{5}, \text{ etc., etc.}$$

We may thus simply and quickly construct a scale of surds. The construction is so simple that every child should be allowed to construct such a scale. The

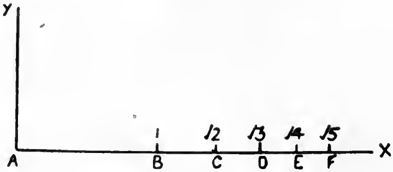


FIG. 73.

accuracy of working may be checked by seeing that $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, etc., coincide exactly with 2, 3, 4, etc., in the scale used.

The same method may be set out alternatively by a process akin to that of vector addition.

The diagram (Fig. 74) explains itself. Successive units a, b, c, d, etc., are set off at right angles. Thus, a is at right angles to 1, b at right angles to $\sqrt{2}$, c at right angles to $\sqrt{3}$, etc. Successive angular points will be seen to be on a spiral, which the ordinary pupil supplied with a large sheet of paper delights to construct.

The "Mean Proportion" Method. The method of finding square roots and "surds" graphically by means of the principle of "mean proportion," is equally fascinating to young learners.

The principle is best introduced with the usual semicircular diagram where it is known that the rectangle $a \times c$ is equal to the square on b (Fig. 75).

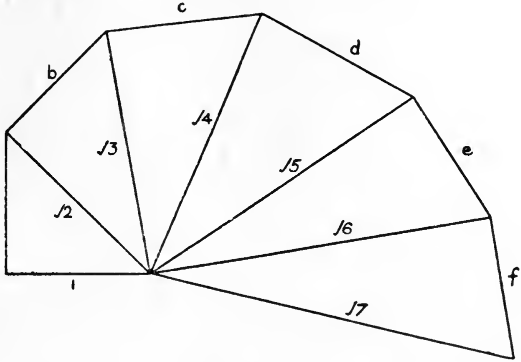


FIG. 74.

This truth may be investigated by actual measurement or may be proved in the orthodox way. For young students unacquainted with similar triangles or

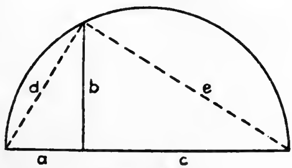


FIG. 75.

rectangle properties of circles, the following simple combination of Pythagoras' Theorem and of the identity $(a + c)^2 = a^2 + 2ac + c^2$ may suffice.

$$(d^2 + e^2) = (a + c)^2 \text{ (Pythagoras')}$$
$$\text{But } d^2 = a^2 + b^2$$
$$\text{and } e^2 = c^2 + b^2$$

$\therefore$ By addition

$$d^2 + e^2 = a^2 + 2b^2 + c^2$$
$$\text{and also } (a + c)^2 = a^2 + 2ac + c^2$$
$$\therefore \text{ Clearly } 2b^2 = 2ac$$
$$\text{i.e. } ac = b^2$$

The application of this principle to square roots is simple.

Example. Find $\sqrt{7}$ graphically.

Choose any unit AB (Fig. 76). Set off BC = 7. On AC as diameter describe a semicircle. Then BD = $\sqrt{7}$. No proof is necessary for the reader.

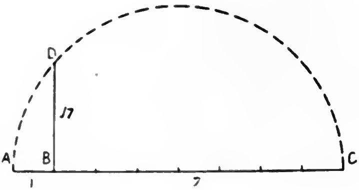


FIG. 76.

The problem might also be solved as follows.

Set off AB = 7: at C (where AC = 1) erect a perpendicular CD. Then AD = $\sqrt{7}$. (Fig. 77.)

We leave the proof to the reader.

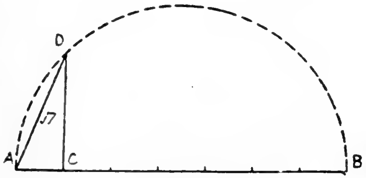


FIG. 77.

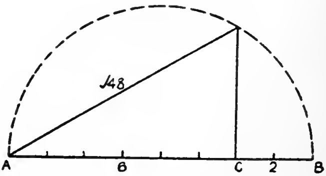


FIG. 78.

Similarly BD = $\sqrt{42}$, and this suggests an easy method for finding the square root of a number that has a pair of convenient factors.

Example. Find $\sqrt{48}$.

Select factors 8 and 6. Describe a semicircle with diameter AB = 8 (Fig. 78). Mark C so that AC = 6. Erect a perpendicular CD at D. Then AD = $\sqrt{48}$.

The uses of this construction are innumerable ; it may be used to construct a square that is any multiple or submultiple or fraction of a given square.

Example I. (Fig. 79.)

Describe a square having three times the area of the square X. Produce AB to C so that BC = 3 AB. Erect a perpendicular BD. Then the square on BD is clearly 3 times the square X, i.e. if X is 1 sq. in., Y is 3 sq. in.

Example II. Describe a square having *half* the area of a given square X (Fig. 80).

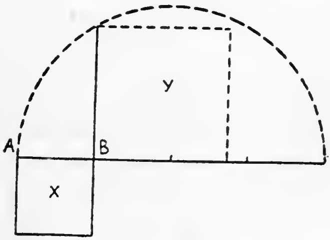


Fig. 79.

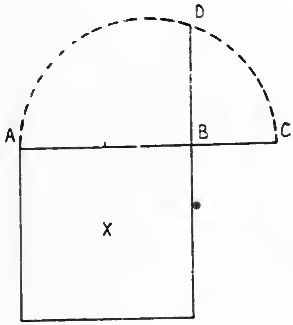


Fig. 80.

Produce AB to C, so that $BC = \frac{AB}{2}$. Describe a semicircle on AC and erect BD perpendicular to it as before. Then the square on BD is equal to half the square X.

This method can be used to solve every problem of the form. " Describe a square having n times the area of a given square " where n is any integer or fraction.

But the arithmetician of the older type having followed us thus far is possibly beginning to protest. " All this is not arithmetic," he may be saying, " Let us have some sums ! " To him we would reply that the subject of squares and square roots in ordinary life is almost entirely a geometrical topic and that the majority of practical applications of squares and square roots occur in such problems as finding sides and diagonals of squares and rectangles and radii and diameters of circles, cylinders, spheres, etc. Since this must be granted we think it only right that the pupil should make some acquaintance with geometrical methods and geometrical connections, instead of relying entirely upon his memory of arithmetical rules of thumb. Let us try to convince the arithmetician by one simple example. Let him ask his older scholars to draw a square having an area of 5 sq. inches.

Proceeding arithmetically, the pupil we presume would first calculate $\sqrt{5}$ to several decimal places, e.g. 2·236 . . . . and then would try to draw a square having a side 2·236 . . . inches long. Let the arithmetician contrast this with the simple geometrical method, whether based on Pythagoras' Theorem or on Mean Proportion. We think he will confess that the geometrical method is simpler and quicker than the arithmetical method and is also *more accurate.*

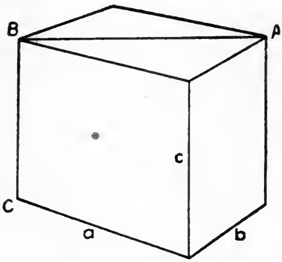


FIG. 81.

To Find the Diagonal of a Rectangular Box. At the risk of being charged with the heinous offence of introducing three-dimensional geometry into the primary school, we venture to suggest the following simple extension of Pythagoras' Theorem to rectangular solids. Our excuse is that we have tried it with boys of 13 in a primary school with success.

The diagonal of a rectangle having sides a and b is $\sqrt{a^2 + b^2}$. This, as we have noted, is taught in most schools.

We will extend this to a simple rectangular box having dimensions a, b, and c (Fig. 81.)

Now the diagonal AB of the top face is clearly $\sqrt{a^2 + b^2}$.

But the triangle ABC is also right-angled,

$$\therefore AC^2 = AB^2 + BC^2$$
$$= a^2 + b^2 + c^2$$

or simply: the distance from any one corner to the opposite corner is $\sqrt{a^2 + b^2 + c^2}$.

Many applications of this statement are ready to hand, such as—

(*a*) Find the longest pencil that could be put in a rectangular box 4 ins. × 5 ins. × 6 ins. (*Ans.*, $\sqrt{4^2 + 5^2 + 6^2}$.)

(*b*) Find the distance from the top corner of a rectangular room to the opposite bottom corner. (*Ans.*, $\sqrt{l^2 + b^2 + h^2}$.)

The mathematical reader will recognize this as a simple arithmetical application of the " solid geometry " theorem $x^2 + y^2 + z^2 = r^2$, or the distance of any point from the origin with reference to three rectangular axes.

It should not be necessary to warn teachers that no reference to " origin," " axes," " dimensions," etc., is advisable in teaching beginners.

CHAPTER XVII

THE TEACHING OF VOLUME

THE teaching of volume and of its measurement in ordinary units is an essential part of all school arithmetic. The idea of volume is in some respects easier for a child to grasp than the mathematical conception of surface or area, and it is certainly capable of much more concrete demonstration. The teacher should be careful not to confine the teaching to word-definitions, tables, and sums, but to strive to build up by practical work a real conception of volume and of the units employed, together with their application to ordinary life.

The idea of volume is closely akin to that of capacity; both give a measure of space, whether of emptiness or of fullness. A teacher with the slightest experience of the child-mind will not *begin* the subject with any attempt at formal definition. We shall preferably begin with a unit of volume, and the simplest will be the cubic inch (or cubic centimetre).

The Cubic Inch. This may be treated synthetically or analytically. Thus, analytically, we may require our learners to examine an inch-cube of wood and to attempt to describe it in precise terms. Nor need we be content with mere formal definition. We shall require them to note that it has length, breadth, and height, that it has six surfaces, eight corners, and twelve edges. They may also note, by actual building up, that cubes of 2 in., 3 in., 4 in. edge, etc., will require 2^3, 3^3, 4^3, etc., inch cubes. The same knowledge may be reached by requiring each child to construct a hollow inch-cube from cardboard. The constructive method takes a little longer time than the purely observational method, but in many respects the time is well spent.

When the inch cube and cubic inch are known as realities, their application to the measurement of rectangular volumes follows easily.

Rectangular Solids. The simple rule, $V = L \times B \times H$ is readily taught by methods analogous to those dealing with simple rectangular area. Thus in the first few instances we may actually build up with inch-cubes some simple rectangular volumes such as that of a brick or a chalk box and simultaneously evolve the simple rule for computing their value from dimensions given. We should not confine the "rule" to its form as given above, but, with an eye to future requirements, should familiarize the pupils with the slightly different forms of volume-measurement such as Base Area × Height, or Length × Cross-Section, and others. Still employing inch-cubes, establish the idea of a *cubic foot* and its connection with the *cubic inch*. The ordinary child realizes readily that a foot-cube requires 12 layers each of 144 inch cubes without actually *handling* the whole 1,728 cubes. The remaining common unit, the *cubic yard*, is generally too cumbersome for concrete illustration in the classroom, nor should such a proceeding be necessary, for at this stage the pupil should be passing from the concrete to the abstract, and should be able to *think* of a yard-cube as necessarily requiring 27 foot cubes. The same fact may, of course, be represented diagrammatically (Fig. 82).

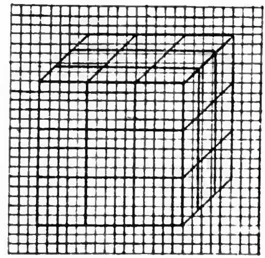


Fig. 82.

The above preliminary exercises will occupy the first few lessons. Following this we have innumerable common applications of volume and volume-measurement even if the work is confined to simple rectangular solids.

Combining practical work and calculation, and not limiting ourselves to text-book examples and sums, we shall apply our new knowledge to the actual measurement and calculation of the volume of rooms, tanks, blocks, boxes and countless other simple volumes. Our actual "sums" are capable of many forms, for in addition to the straightforward

$$\text{(i)} \quad V = L \times B \times H$$

they may be given in the alternative forms—

$$\text{(ii)}\ L = \frac{V}{B \times H}. \quad \text{(iii)}\ B = \frac{V}{L \times H}. \quad \text{(iv)}\ H = \frac{V}{L \times B}.$$

or in the more usual alternative forms of—

(v) V = Length × Cross Section.

(vi) V = Base Area × Height;

and their variations—

$$\text{(vii) Cross Section} = \frac{V}{L}. \quad \text{(viii) Length} = \frac{V}{\text{Cross Section}}.$$

$$\text{(ix) Base Area} = \frac{V}{\text{Height}}. \quad \text{(x) Height} = \frac{V}{\text{Base Area}}.$$

To neglect any of these forms in setting our "sums" is to limit seriously the child's facility in the application of his ideas of volume.

Rooms provide numerous examples. Each child, for instance, should calculate from his own data the number of cubic feet of air space provided for each child in his classroom.

Rectangular Tanks and Cisterns provide ready means for emphasizing the connection between the common units. Thus a child should receive practice in calculating (*a*) the cubic capacity, (*b*) the number of gallons, (*c*) the weight of water, that a tank of given dimensions will contain. The simple connections for this purpose, viz., 1 cu. ft. of water = $6\frac{1}{4}$ galls. and weighs $62\frac{1}{2}$ lbs. approximately, should be constantly in use. Further interesting problems may be set on the rise and fall of the water in a tank according to the amount of water put in or withdrawn.

Rectangular Boxes give endless examples, simple and difficult. Few of us have forgotten our struggles with examples involving both external and internal dimensions of boxes. Perhaps some of these were indeed a little too involved, but many other variations are possible. Thus we may combine surface and volume in the calculation of the area of cardboard, tin, etc., necessary for the construction of a box of given dimensions; while with older scholars and students we may investigate interesting problems in maxima and minima such as finding the dimensions of the box of greatest volume that can be fashioned out of a given sheet of material.

This simultaneous treatment of length, surface and volume should be followed wherever possible. But varieties of examples and applications of volume, even those limited to rectangular volumes, are too numerous to detail in these pages.

It cannot be emphasized too much, that the question is not primarily one of "sums" but of arithmetic really applied to solid objects, and indeed the treatment of any solid object may be both geometrical and arithmetical. The "Suggestions" of the Board of Education (Par. 49, Circular 807) may be quoted in this respect—

"The study of an object from the geometrical and arithmetical points of view will involve some or all of the following four processes—

(i) The measurement of the actual object,
(ii) Its representation by drawings,
(iii) The construction of a model,
(iv) Various calculations involving quantities of material and cost.

The relative value of these four processes will be found to vary according to the progress of the scholar and the nature of the problem; but the construction of a model is likely to lead to clearer conceptions and to yield more interest than a mere examination of the object. One simple example is given to show what may be done with the very commonest materials — an empty match-box may afford many exercises of varying degrees of difficulty, viz., observation of the shape of the sides, the number of edges, the number of corners, and the nature of the angles; measurement of the dimensions of the box; drawing the dissected box to scale; construction of a model of half or double the size of the original; calculation of the size of a sheet of cardboard required to make a box; calculation of the size of a parcel containing a gross of boxes and of the cost, at a given rate, of the cardboard required to make a gross."

Thus will our school arithmetic broaden into mathematics, and our "sums" be based upon actual data and personal experience instead of being obtained from books and figures.

Most of our examples for younger scholars will deal with the volumes of solids and liquids. Gaseous volumes introduce a pressure-factor which is perhaps out of the range of beginners. Yet the humble gas meter should receive some attention. Children should know that gas is frequently measured and sold by the cubic foot; they should learn to read the ordinary three-dial meter; they should be able to calculate from their readings the amount used in any given period, and they may compare the cost of gas, using a slot-meter, with the ordinary quarterly payment.

Again, the idea of volume is innate in the measurement of rainfall. Though possibly the mathematics of the rain gauge and its measuring cylinder may be somewhat too difficult for beginners, yet even beginners are able to work out the implications of a rainfall of 1 inch, and should learn by calculation that 1 inch of rain on a district is equivalent approximately to 100 tons per acre. This is but one of the ways in which simple arithmetic will convince a pupil of the immensity of the natural forces at work around him.

Cubes and Cube Roots. Closely akin to rectangular volume is the subject of cubes and cube roots. The interest here is, of course, mainly mathematical and geometrical, yet children may well make acquaintance even in the primary school with simple facts in connection with cubes treated both arithmetically and geometrically.

The common series of cubes of whole numbers should be recognized in simple cases. Thus the child should know the series 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1725. No formal method for finding cube root should be taught ; in later life this is always found by slide-rule or table. Simple integral cubes, however, may be dealt with by factors and their cube root so discovered. Thus the child may discover by factorizing that

$$2744 = 2 \times 2 \times 2 \times 7 \times 7 \times 7$$
$$\text{or } 2^3 \times 7^3,$$

and that consequently the cube root of 2744 is 14. Again they may be set examples such as : "Find the smallest multiplier that will convert 225 into a perfect cube."

By factors they will discover that $225 = 3^2 \times 5^2$, and that consequently they must supply another 3 and another 5 so that $225 \times 15 = 3^3 \times 5^3$, whence the required multiplier is 15.

Other interesting examples in cubes are such as : "Find the smallest cube that you could build with bricks $9'' \times 4\frac{1}{2}'' \times 3''$ (or chalk-boxes, etc., etc.) all placed the same way. Calculate how many bricks you would want."

Many interesting principles in number-theory arise out of cubes.

Thus every integral cube may be expressed as the difference of two squares, e.g.—

$$1^3 = 1^2 - 0^2$$
$$2^3 = 3^2 - 1^2$$
$$3^3 = 6^2 - 3^2$$
$$4^3 = 10^2 - 6^2$$

Pupils may be asked to continue the series.

[The general term is seen to be

$$n^3 = (\Sigma n)^2 - (\Sigma \overline{n-1})^2$$

and addition of corresponding sides of the series gives an easy method of summing cubes, i.e.

$$\Sigma\,(n^3) = (\Sigma n)^2]$$

So, too, older pupils may note that the simple odd-number series, 1 3 5 7 9 11 13 15 17 19, etc.,

may be grouped so as to show the series of cubes thus—

$$\underbrace{1}_{1^3} \quad \underbrace{3 \quad 5}_{2^3} \quad \underbrace{7 \quad 9 \quad 11}_{3^3} \quad \underbrace{13 \quad 15 \quad 17 \quad 19}_{4^3}$$

Teachers familiar with the summation of series will find here a ready means of finding the sum of the cubes of the first n whole numbers.

If the products of the ordinary multiplication table be arranged in square form as in the diagram (Fig. 83), these also may be grouped so as to show the successive cubes.

If the numbers be cut out by right-angled segments as shown, each segment contains numbers that add together to make a perfect cube, and this simple square of numbers gives us yet another easy method of finding $\Sigma (n^3)$. Mathematical readers interested may supply the method by summing each line successively and finding the grand total [i.e. $(\Sigma n) \times (\Sigma n)$].

1^3	2^3	3^3	4^3	5^3	6^3
1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36

FIG. 83.

In dealing with the Pythagorean Theorem we have noted the infinity of solutions of right-angled triangles with integral sides such as—

$$3^2 + 4^2 = 5^2$$
$$8^2 + 15^2 = 17^2, \text{ etc., etc.}$$

It is interesting to extend this by analogy to the cube, and to find whether three integral cubes will add together to give a fourth cube. We give one simple example. Thus—

$$3^3 + 4^3 + 5^3 = 6^3$$
$$(27 + 64 + 125 = 216)$$

We leave the reader to determine whether there are any others.

But the Pythagorean Theorem may be extended to solids in another direction. We will first consider the cube.

Problem. Find the diagonal of a cube, i.e. the distance from any corner to the opposite corner. (There are clearly four such, viz., AG, BH, CE, DF (Fig. 84).)

By the three-square theorem, if a is the length of the edge of the cube, the diagonal DB is clearly $a \times \sqrt{2}$. Now the triangle DBF is also right-angled, and by a second application of the same theorem we find that DF, the diagonal required, is $a \times \sqrt{3}$.

It is to be feared that this and other simple facts of the geometry of cubes have been somewhat neglected in our schools, both primary and secondary, for, year after year, students of the age of 18 and upwards have come to the author, perfectly familiar with the fact that the diagonal of a square is always: (side) $\times \sqrt{2}$, but quite ignorant of the analogous fact that the diagonal of a cube is (edge) $\times \sqrt{3}$.

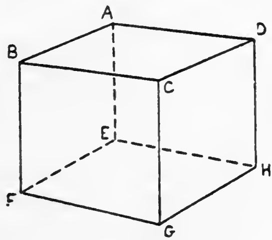


FIG. 84.

Finally, we may introduce even young pupils to the historical problem of constructing a cube having double the volume of a given cube. An approximate numerical solution is easily found, for if we desire to have a cube of volume 2 cubic inches we must clearly have an edge of $\sqrt[3]{2}$ inches. Readers who desire a more exact method will find the whole question discussed in W. W. R. Ball's *History of Mathematics*.

The Volume of Prisms. Following the treatment of rectangular volumes, it is customary to extend the idea of volume to the determination of the volume of other prisms and also of pyramids.

For prisms we base our calculations upon the form—

Volume = Base × Height
or Cross Section × Length

Common Forms are the regular triangular prism and the regular hexagonal prism as shown in the diagram (Fig. 85).

Their volumes in each case are: End - area × Length. The actual construction of these in paper or card-board provides valuable exercises in handwork and mathematics.

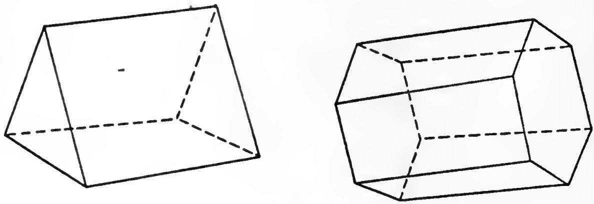


Fig. 85.

The important case of a prism whose end area is a right-angled trapezium deserves to be specially dealt with. Common examples are lean-to sheds, greenhouses, cold frames, swimming baths, etc. Consider the lean-to shed in the diagram (Fig. 86). It has length l, width w, heights a and b.

Now we have already shown that the area of a right-angled trapezium is (base × mean height).

Hence the end area of the shed is clearly $w \times \left(\frac{a + b}{2}\right)$,

and therefore the volume is given by $l \times w \times \left(\frac{a + b}{2}\right)$.

All such volumes are given similarly, viz.,

Length × Width × Mean Height (Mean Depth).

The formula is very easily memorized, or derived if forgotten.

Volume of Pyramids. From Prisms we may readily extend our formulae to deal with simple cases of pyramids.

The volume of a Pyramid is always one-third of the volume of a corresponding Prism of the same base-area and the same height. Hence we may state generally for all Pyramids—

$$\text{Volume} = \tfrac{1}{3} \times (\text{Base Area}) \times (\text{Height}).$$

This fact again can hardly be "proved" to beginners, but may be demonstrated by any of the methods of weighing or measuring indicated in dealing with the

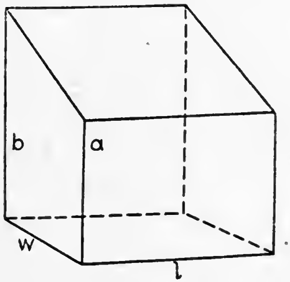


FIG. 86.

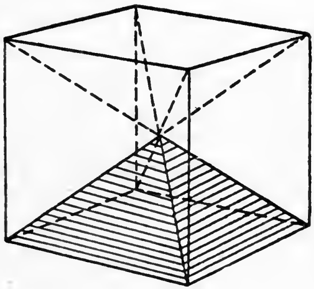


FIG. 87.

Cone, which is of course only a particular case of a Pyramid with a circular, instead of polygonal, base.

It is possible to derive the formula for the volume of a Pyramid from a consideration of one of the six square pyramids into which a cube is divided by its four long diagonals as shown in the diagram (Fig. 87).

The volume of any one of these six is obviously $\dfrac{(\text{Volume of Cube})}{6}$ or $\frac{1}{6}a^3$ where a is the edge of the cube.

But $\frac{1}{6}a^3 = \frac{1}{3}a^2\,\dfrac{a}{2}$ and a^2 is the base of a pyramid and $\dfrac{a}{2}$ its height. Hence, for *this* pyramid the volume is $\frac{1}{3}$ (base $\times$ height). The general truth follows by simple Euclidean reasoning.

CHAPTER XVIII

THE MENSURATION OF THE CIRCLE

ALL primary school arithmetic includes numerical examples on the circumference and area of a circle.

It was formerly the custom to proceed deductively in teaching this subject, and to hand out a few facts such as $\pi = 3\frac{1}{7}$, Circumference $= 2\pi r$, Area $= \pi r^2$, without any attempt to demonstrate these truths. Modern teaching rightly rebels against this rule-of-thumb method of formulae blindly applied to sums, and insists that this subject of circles and circular areas shall be dealt with as experimentally and as practically as possible.

The Circumference of a Circle. The subject is best introduced by an experimental lesson to determine the value of π. For this purpose we shall follow the methods of the physical laboratory and, collecting a sufficiency of circular discs and objects, we shall require our pupil to measure in each case diameter and circumference. We shall tabulate our results in two columns as shown.

Circumference.	Diameter.	Circum. ÷ Diam.

and we shall add a third column $\frac{\text{Circum.}}{\text{Diameter}}$ or π. We must not expect too accurate a value in the third column, but with careful work the average magnitude of $\frac{\text{Circumference}}{\text{Diameter}}$ or π will be sufficiently uniform to enable the pupils to believe that this ratio may have a constant value. It is at this stage that a fairly accurate value such as $3\frac{1}{7}$ may be introduced, though care should be taken to impress even at the start that the value of this ratio cannot be *exactly* expressed since it is known to be incommensurable.

The ordinary formulae may now be easily built up.

$$\text{Since } \frac{\text{Circumference}}{\text{Diameter}} = \text{a Constant} = \pi$$

it follows that Circumference $= \pi d$ or $2\pi r$

$$\text{and that Diameter} = \frac{\text{Circumference}}{\pi}.$$

Applications are endless, but, as always, it is well to combine practical examples with "sums" and occasionally to set the pupils to find the circumference and the radius of a circular tin, jar or pipe, etc.

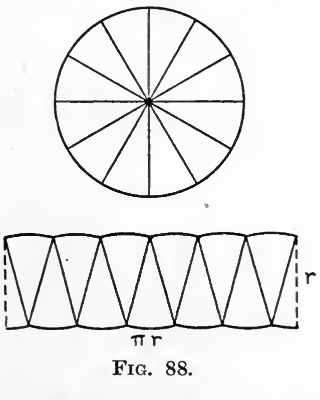


FIG. 88.

The Area of a Circle. Though the actual *proof* of Area $= \pi r^2$ belongs to more advanced mathematics, yet several simple *demonstrations* of this fact are available for teachers.

The demonstration most commonly seen in primary schools is the well-known one shown in Fig. 88, in which the original circle is dissected into an even number of sectors which are re-arranged to form a (so-called) rectangle having an area ($\pi r \times r$) or πr^2. The weakness of the method is that to obtain an *exact* rectangle it would be necessary to have an infinite number of sectors, and to proceed to a limit which is never reached in practice.

The same fact may be realized without troubling to rearrange the sectors in a rectangular form, for since any one triangle drawn in a sector (Fig. 89) has an area equal to $\frac{\text{base} \times \text{height}}{2}$ or $\frac{\text{base}}{2} \times$ height, we may assume this to be true however small the sector be, i.e. however many

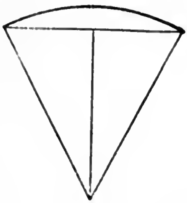


Fig. 89.

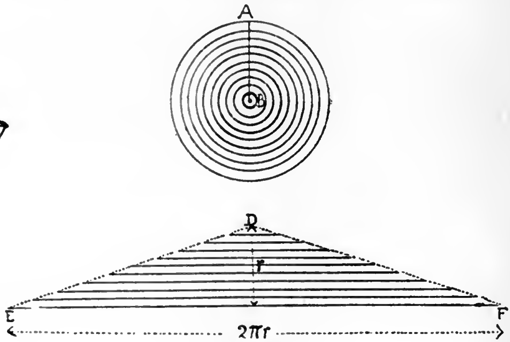


Fig. 90.

sectors are cut from the original circle. But if we assume an *infinite* number, then all the infinitely small bases will form the circumference and the height in every case will be the radius. Hence we have—

$$\begin{aligned}\text{Area of Circle} &= \text{Sum of Areas of Triangles}\\ &= \tfrac{1}{2}\,(\text{Sum of All Bases}) \times \text{Height}\\ &= \tfrac{1}{2}\,(\text{Circumference}) \times \text{Radius}\\ &= \tfrac{1}{2} \times 2\pi r \times r = \pi r^2\end{aligned}$$

This method has the advantage of giving the area of any *sector*, for by similar reasoning the sector may be viewed as an infinite series of triangles and their total area will be $\frac{1}{2} \times$ Arc $\times$ radius.

This fact has an important application in the curved surface of a cone.

An ingenious method for finding the area of a circle based upon the area of an equivalent triangle is that shown in Fig. 90.

The original circle may be assumed to be formed of a series of concentric circles of fine cotton or thread. If the circle be now cut along a radius AB and all the threads straightened out, a triangle DEF will be formed, whose base will be $2\pi r$ and height r, and whose area consequently is—

$$\tfrac{1}{2} \times 2\pi r \times r \text{ or } \pi r^2 \text{ as before.}$$

Experimental methods similar to those employed for finding the circumference of a circle may also be employed. Thus the child may draw a series of circles on squared paper and may find the area of each by the usual method of counting squares. Results should be arranged as in the table below—

Area of Circle.	Area of Square on Radius.	$\dfrac{\text{Area of Circle.}}{\text{Area of Square on Radius.}}$

The third column again should give a reasonably accurate value of π, and thus again we may evolve the "rule"—

$$Area = \pi \times r^2$$

An investigation of the relation between the area of a circle and the area of the square on the radius might also be conducted by cutting out of the same kind of paper or material both circle and square and carefully weighing each, as is indicated in Par. 50, Circular 807 (Board of Education "Suggestions").

Once the rule has been investigated and established, calculations will follow the usual order of—

(*a*) Find area, given radius or diameter.

(*b*) Find radius or diameter, given area.

Examples of type (*b*) will involve usually the extraction of a square root.

Somewhat harder and much less common in actual life is the following type—

(*c*) Find area, given circumference.

The plotting of the areas of a series of circles with gradually increasing radii forms a valuable exercise. Thus the pupils may plot accurately the areas of circles having radii 1″, 2″, 3″, 4″, etc. The graph thus drawn may be used for a variety of problems. Thus we read off from it answers to such problems as—

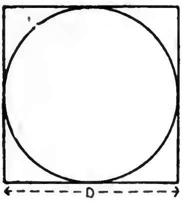


FIG. 90A.

Find the radius of a circle having an area of 1 sq. ft.

The important connection between the area of a circle and the square in its *diameter* (Fig. 90A) is too frequently neglected in schools, though the formula Area $= \frac{\pi d^2}{4}$ or $\cdot 7854\ d^2$ is, in actual workshop mathematics, in much more frequent use than the formula Area $= \pi r^2$. This important relation may be investigated by methods already suggested, such as by drawing and counting squares or by cutting and weighing.

Archimedes gave the result as area $= \frac{11}{14} \times (\text{diameter})^2$, which is clearly based on the approximation $\pi = 3\frac{1}{7}$, and is a convenient form to remember.

In calculating circumferences and areas of circles, the more accurate value of π as $3\cdot 1416 \ldots$ should be introduced as soon as possible. It is essential for really accurate work, and, if logarithms are used, it is manipulated as easily as $\frac{22}{7}$ or $3\frac{1}{7}$.

Parts of Circles. The rules for finding the circumference and area of a circle are easily modified to deal with the simpler parts of circles. Thus it is unnecessary to deal separately with the semicircle and the quadrant. The adaptation of the formulae for these may well be left to each individual pupil as an exercise. The combination of these parts of circles with rectangles gives varied and interesting examples. Thus we may ask our pupils to determine the outside area of a barn with a semi-circular roof, or the distance round a race-course with two semi-circular ends.

The Sector. This is important enough to receive special

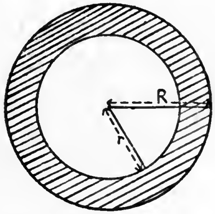


FIG. 91.

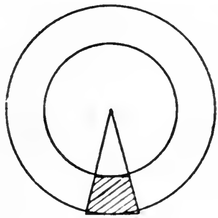


FIG. 92.

treatment. Two simple methods of finding the area of a sector of a circle are here indicated.

(*a*) We may, as already indicated, use the circular arc as the sum of the bases of an infinite series of triangles whence the area of the sector is—

$$\frac{1}{2} \times (\text{Arc}) \times (\text{radius}).$$

(*b*) We may use the angle subtended by the arc at the centre. Thus, if the arc of a sector is 70° the area may be treated as equivalent to $\frac{70}{360}$ or $\frac{7}{36}$ of the whole area of the original circle.

Both methods may easily be taught to children.

The Circular Ring. This is of such common occurrence in calculations dealing with cross sections, especially of pipes, that all children should be familiar with the simple method of calculating its area.

Consider such a ring with outer radius R and inner radius r (Fig. 91). This is usually treated as the difference of two circular areas, viz. $\pi R^2 - \pi r^2$. The simplest algebra transforms this into—

$$\pi \ (R + r) \ (R - r),$$

which is the form most commonly taught.

The following alternative method of treatment based upon the trapezium is in many respects simpler and easier of application. Consider a trapezium drawn as shaded. (Fig. 92). The area of this is, by the ordinary rule—

(Mean of parallel sides) × (Distance between them).

But the whole ring may be imagined as divided up into an infinite number of trapeziums, and their total area (i.e. the area of the ring) will be seen to be—

Mean Circumference × Thickness.

This statement may be expressed in any one of the following formulae—

$$(a) \ 2\pi \ (\text{Mean } r) \ t,$$
$$(b) \ \pi \ (\text{Mean } d) \ t,$$
$$(c) \ \pi \ (r_1 + r_2) \ t,$$

all of which are readily applicable to ordinary problems.

Example. Find the area of a cross section of a pipe with an internal diameter of $1\frac{1}{2}''$ and an external diameter of $1\frac{3}{4}$ inches.

Here the mean diameter is $1\frac{5}{8}''$ and the thickness $\frac{1}{8}''$, hence required area is $(\pi \times 1\frac{5}{8} \times \frac{1}{8})$ sq. in.

The History of π. The history of this symbol is one of the most fascinating topics in the whole history of mathematics. Readers will find an excellent synopsis of it under the title of "The Quadrature of the Circle" in *Mathematical Recreations and Essays*, by W. W. R. Ball (page 293, fifth edition), while in *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young (Longmans, Green and Co.), Professor D. E. Smith has a somewhat more difficult account of "The History and

Transcendence of π." In these brief accounts the reader will learn how for thousands of years a great army of circle-squarers has been at work attempting to find the exact numerical relation between the diameter or radius of a circle and its circumference or area. They will learn, too, that it was not until the eighteenth century that Lambert definitely established the incommensurable nature of π and the consequent impossibility of its exact evaluation; and that in the nineteenth century Lindemann showed that π was transcendental, i.e. could never be the root of a rational Algebraic equation.

Concurrently with the futile attempts of the circle-squarers is a long history of more rational attempts to determine an accurate numerical approximation for π. The attempts have followed two general lines. One group has based its various efforts on the attempt to approximate to π by noting that the circumference of a circle is intermediate in value between the perimeters of inscribed and circumscribed polygons. The methods usually proceed by a continual increase in the number of sides of these polygons. The more modern method is that of determining π as an infinite series for, as Mr. W. W. R. Ball points out, π is a quantity that must of necessity enter into all mathematical analysis and is not confined to circle measurements.

The various values that have been used and propounded for π may well be employed by teachers in their lessons.

The Babylonians and Jews probably used an empirical value 3, but the Egyptians certainly used $\frac{256}{81}$, i.e. $(\frac{16}{9})^2$ or 3·1605. The Greeks devoted considerable attention to the subject, and Archimedes showed that π was less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$, (i.e. that it lay between 3·1428 and 3·1408). Ptolemy gave a sexagesimal value of $\pi = 3° \; 8' \; 30''$ or $3 + \frac{8}{60} + \frac{30}{3600}$, which gives $3{\cdot}141\dot{6}$. The Romans, though they knew of $3\frac{1}{7}$, seem to have preferred $3\frac{1}{8}$ as being handier to manipulate in their duodecimal fractions.

Turning to more Eastern mathematicians, we find Indian mathematicians giving such values as $\frac{49}{16}$, $\sqrt{10}$, and $\frac{62832}{20000}$, all of which were also used by the Arabs. Bhaskara gives $\frac{3927}{1250}$ and $\frac{754}{240}$, both of which are very close to the modern accepted value. The Chinese appear to have had a value $\frac{157}{50}$, which is a convenient but not very accurate form.

The modern evaluations have proceeded in many cases to an extraordinary number of figures, and the history of these calculations has a number of outstanding and picturesque features that appeal to children. Thus they should know of the "lucky guess" of the father of Adrian Metius, a Dutchman of the seventeenth century, who, having proved that π was between $\frac{377}{120}$ and $\frac{333}{106}$, took as his ultimate value $\frac{377+333}{120+106}$ or $\frac{355}{113}$, which is correct to six places. Children appreciate, too, the labours of Ludolph van Ceulen who, at his death in 1610, directed that his value of π calculated to 35 places should be engraved upon his tombstone in the churchyard of St. Peter's, Leyden. Not without reason did the contemporaries of this patient calculator call π "Ludolph's number."

To Draw a Line Equal to the Circumference of a Circle. We give two simple methods by which it is possible to draw a line approximately equal in length to the circumference of a circle.

The first method is given by W. W. R. Ball as one of the simplest.

In the given circle (Fig. 93) inscribe a square. To three times the diameter of the circle add one fifth of the side of the square. The result is the circumference of the circle.

The close approximate correctness of this may be seen by taking d as a unit.

If $d = 1$, then $a = \frac{1}{\sqrt{2}}$, and the line AB $= 1 + 1 + 1 + \frac{1}{5\sqrt{2}}$

$$= 3 + \frac{\sqrt{2}}{10}$$

$$= 3{\cdot}1414 \ldots$$

The second method was found by the writer in use by an employee in the L.N.W.R. locomotive works

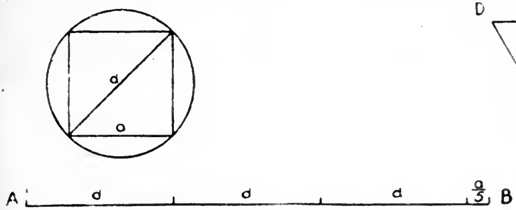


FIG. 93.

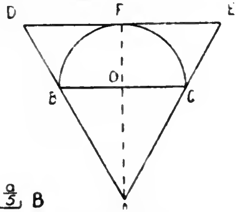


FIG. 94.

at Crewe (though it was probably not original). It was applied to the semicircle.

On the diameter of the semicircle (Fig. 94) an equilateral triangle ABC was described. AB and AC were produced to meet the tangent at D and E, forming a larger equilateral triangle ADE. Then the length DE is approximately the length of the arc of the semicircle.

Assuming a unit radius, we have AO $= \sqrt{3}$, and by simple proportion—

$$\frac{DE}{FA} = \frac{BC}{OA}$$

$$\therefore DE = \frac{BC}{OA} . FA$$

$$= \frac{2}{\sqrt{3}} (1 + \sqrt{3})$$

$$= \frac{2}{3} (3 + \sqrt{3})$$

$$= 3{\cdot}15 \text{ approx.}$$

The Cylinder. It is thought necessary by some teachers to complete the pupil's armoury of mathematical weapons by giving him rules for dealing with such common solids as Cylinders, Cones, and Spheres.

The rules for the cylinder need not concern us long. They are simple and fairly obvious. Since any sheet of paper or thin material may be rolled into a cylinder, it follows that the curved surface of a paper cylinder may, with a single cut parallel to its axis, be developed into a rectangle, as in Fig. 95. The area of this curved surface is then deduced at once as Circumference × Height, or

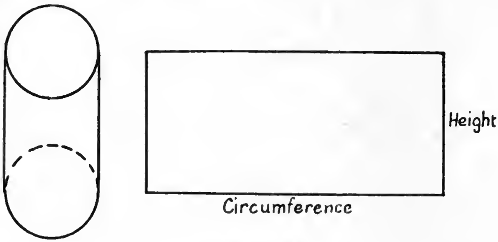


FIG. 95.

in the usual shorthand: Curved surface of cylinder $= (2\pi r) \times (h)$. All children should make this simple model and obtain their own rule.

The volume of a cylinder by analogy with the volume of a prism is also easily developed as Base × Height, or $\pi r^2 h$.

These two formulae for surface and volume are sufficient for dealing with all simple examples.

Again it should be noted, in setting numerical examples, that the *diameter* forms of these formulàe are frequently better for use in actual examples than the radius form.

Using diameters the formulae become—

$$\text{Curved surface of cylinder} = \pi \, . \, d \, . \, h$$

$$\text{Volume of cylinder} = \frac{\pi}{4} d^2 h \text{ or } \cdot 7854 \, d^2 h.$$

These forms are always used in practical mathematics, where it is generally easier to measure the diameter than to measure the radius of a cylinder.

Cylindrical objects such as pipes, wires, tins, chimneys, chimney pots, columns, and a thousand other familiar objects, are so common in every-day life, that examples are never lacking to form the basis of practical and written work. One common example of the cylinder—the ordinary rain-gauge and its accompanying measuring cylinder, will provide a splendid series of numerical examples for upper classes. The actual making of a simple form, including the calibration of a convenient measuring-cylinder, will teach the ordinary "mensuration" of the cylinder in a most attractive form.

Again an excellent revision example may take the form of "A sheet of paper 10″ × 8″ can be rolled into cylinders either 10″ high and 8″ round, or 8″ high and 10″ round. Find which has the greater volume."

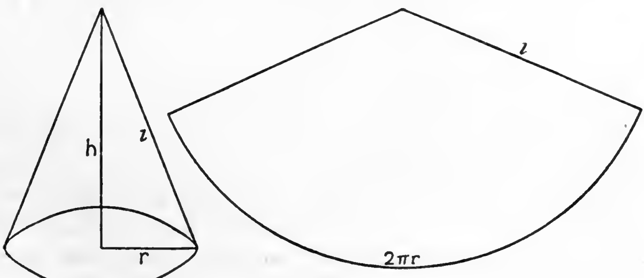


Fig. 96.

The Cone. The formulae for the surface and volume of a cone may be demonstrated by simple methods for beginners though their "proof" is not possible without further mathematical knowledge.

Curved Surface of a Cone. By cutting a paper cone every child can see for itself that the curved surface will flatten out into a sector of a circle (Fig. 96).

Now the arc of this sector can be identified as the circumference of the base of the cone or $2\pi r$, while the radius of the sector is the slant height (l) of the cone.

Hence applying our ordinary rule for a sector, the area of the curved surface of a cone is seen to be—

$$\tfrac{1}{2} \times 2\pi r \times l \text{ or } \pi \,.\, r \,.\, l.$$

The Volume of a Cone. A convenient direct proof of the formula: Volume of Cone $= \frac{1}{3} \times$ (Base area) $\times$ height or $\frac{1}{3}$ Volume of Corresponding Cylinder, is beyond the

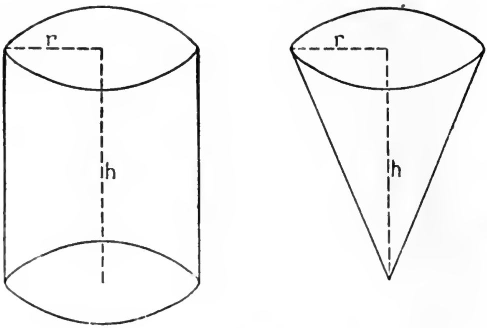


FIG. 97.

range of beginners, hence it is customary to assume this connection and to demonstrate it by filling a hollow cylinder, using as filler a hollow cone of the same height and base radius (Fig. 97). This is a rough-and-ready means of demonstration possible even with the largest class, and will always fix this important relation.

Frustums of Cones. Though these are seldom dealt with in our primary schools, yet the truncated cone or frustum is of such common occurrence in ordinary life that formulae for dealing with this shape are of very great practical use.

Jugs, cans, pails, buckets, milk churns, funnels, flower pots, etc., all show this common shape in one of the two forms (a) or (b) (Fig. 98). Hence, for the sake of teachers of "practical" mathematics, we indicate methods for developing the formulae necessary.

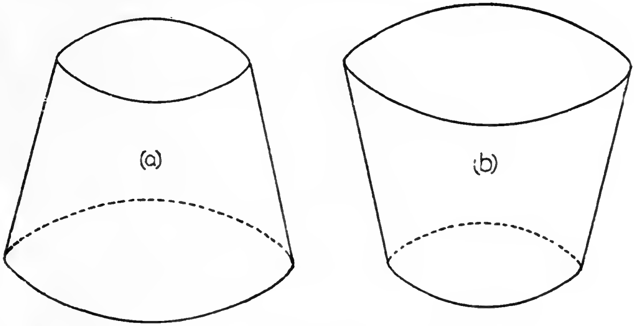


FIG. 98.

Curved Surface of a Frustum. Consider (as in the case of the area of a circular ring) a trapezium drawn on this curved surface (Fig. 99). If the whole curved surface

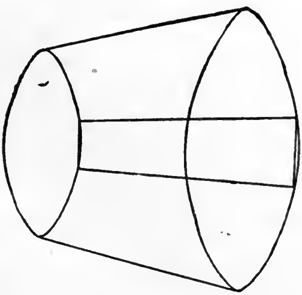


FIG. 99.

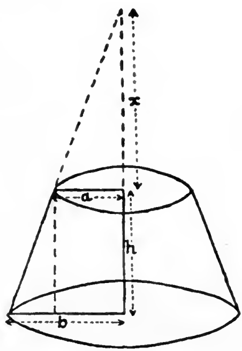


FIG. 100.

be considered to be made up of an infinite series of such trapeziums then the total area may be deduced as (Mean Circumference) × (Slant Height). We leave this as a simple exercise to the reader.

Volume of a Frustum. This is best approached as the difference of two cones.

Let the end radii of the frustum be a and b, and its height h (Fig. 100).

Now by similarity we see at once from the diagram that the height x of the cone cut away may by simple proportion be derived from $\frac{x}{a} = \frac{h}{b - a}$

$$\text{whence } x = \frac{ah}{b - a}.$$

If we add h to x we get the height of the original cone as $\frac{bh}{b - a}$.

Now (Volume of original cone) – (Volume of Cone cut off)

$$= \left(\tfrac{1}{3}\ \pi b^2 \cdot \frac{bh}{b - a}\right) - \left(\tfrac{1}{3}\ \pi a^2 \cdot \frac{ah}{b - a}\right)$$

which reduces to

$$\tfrac{1}{3}\ \pi h \left(\frac{b^3 - a^3}{b - a}\right)$$

and using our knowledge of factors we have the usual form

$$\tfrac{1}{3}\ \pi\ h\ (b^2 + ab + a^2).$$

If, however, we look at this as areas we have

$$\tfrac{1}{3}h\ (\pi b^2 + \pi ab + \pi a^2).$$

which may be expressed simply as

$$\tfrac{1}{3}h\ (S + \sqrt{Ss} + s),$$

where S and s are the end areas. This form is most easily remembered.

The Sphere. The mathematical methods of deriving the formulae for the surface and volume of a sphere are so far beyond a beginner that any attempt at rigid "proof" is useless, and therefore we must adopt the usual method of demonstrating the necessary facts experimentally.

Both surface and volume of a sphere are so intimately connected with the corresponding cylinder that we shall do well to begin our demonstration with the examination of such a cylinder.

Clearly the cylinder which will just enclose a sphere of radius r will have height and diameter each equal to $2r$ (Fig. 101).

The area of the curved surface of this cylinder

$$= 2\pi r \times 2r$$
$$= 4\pi r^2$$

The volume of this cylinder

$$= \pi r^2 \times 2r$$
$$= 2\pi r^3$$

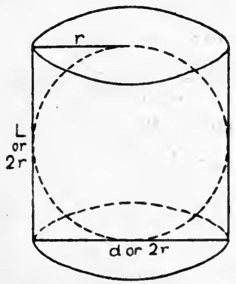


FIG. 101.

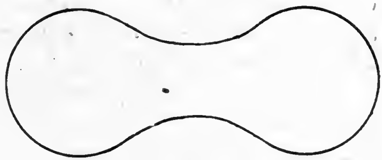


FIG. 102.

Surface of Sphere. This should be remembered as exactly equal to the curved surface of the enclosing cylinder, i.e. $4\pi r^2$.

Proof of this is difficult for learners, but illustration and demonstration are easier. Thus we may mark a square inch on a football bladder (uninflated); on inflating it sufficiently to form a sphere this will be seen to be approximately 4 sq. inches, from which we might expect that the total area of the spherical surface would be four times that of the section through the centre (the Great Circle of Astronomy) or $4 \times \pi r^2$.

Another useful illustration is to strip the covering from an ordinary tennis ball. This is usually in two equal pieces of the form shown in Fig. 102. The area of one of these may be approximately estimated by tracing on squared paper, and comparing with the area of a great circle of the original tennis ball.

A more accurate method would be to procure an accurately-made thin hollow sphere and a disc of the same diameter and made of the same material. The weight of the spherical shell should be approximately four times the weight of the disc. Several makers of physical apparatus supply such spheres and discs.

The rigid mathematician may scoff at these experimental methods, but we are convinced that such have their place in the teaching of beginners and are infinitely preferable to the older methods of so-called teaching by which the pupil was required to memorize what, to him, were of necessity confusing and arbitrary facts.

Volume of a Sphere. This again should be closely connected with the volume of the circumscribing cylinder, and may be remembered as exactly equal to two-thirds of the volume of the cylinder.

The circumscribing cylinder clearly has a volume of $2\pi r^3$ (v.s.).

Hence the volume of the inscribed sphere is $\frac{2}{3}$ of $2\pi r^3$ or $\frac{4}{3}\pi r^3$.

This fact may be demonstrated by using solid spheres and solid cylinders of equal diameters, made in any suitable metal or material.

If these are accurately turned, *three* spheres will be found to balance the weight of *two* cylinders, and the weight of one sphere is then easily deduced.

We may, however, easily deduce the formula for the volume of a sphere from a combination of the formulae for the *surface* of a sphere and the volume of a cone. All that is necessary is to think of the volume of the sphere as composed of an infinite number of cones, each having its base on the surface of the sphere and its apex at the centre (Fig. 103).

Now the volume of any one cone is $\frac{1}{3}$ (base) $\times$ (height), and the sum of all the cone-volumes (or the volume of the sphere) is clearly $\frac{1}{3}$ (Sum of bases) $\times$ (height); but in the limit the sum of the bases is the surface of the sphere and the height is the radius. Hence the volume of the sphere is—

$$\begin{aligned} &\tfrac{1}{3}\ (\text{Surface}) \times (\text{Radius}), \\ \text{but Surface} &= 4\pi r^2 \\ \therefore\ \text{Volume} &= \tfrac{1}{3}\ (4\pi r^2) \times (r) \\ &\text{or } \tfrac{4}{3}\pi r^3 \end{aligned}$$

It is unfortunate that the simple connection of these two formulae belongs to the realm of calculus. Older students with some knowledge of calculus methods should always remember the surface ($4\pi r^2$) as the simple differential of the volume $\frac{4}{3}\pi r^3$, or conversely the volume ($\frac{4}{3}\pi r^3$) as the simple integral of the surface $4\pi r^2$.

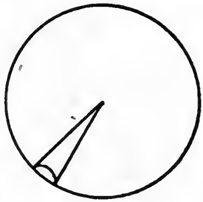


FIG. 103.

A similar connection is, of course, observable between the circumference and area of a circle. Thus the circumference $2\pi r$ is the differential or rate-of-change of the area πr^2.

Area and Volume by Calculus Methods. Proofs of these formulae for the surface and volume of different circular solids such as the cylinder, cone and sphere are most conveniently obtained by calculus methods, and as such are beyond the comprehension of elementary students. They may, however, be obtained so easily by the well-known Theorems of Pappus, that we may be pardoned for digressing slightly to indicate these beautiful methods, in a book intended primarily for teachers of children.

These two theorems deal with the surface and volume formed by a line or area rotating about a fixed axis. In simple form they may be stated as follows—

(*a*) Surface of Solid of Rotation $= 2\pi \bar{x} P$ where P is the length of line rotating and $\bar{x}$ the distance of its centre of gravity from the axis of rotation.

Cylinder. Consider a line of length l rotating round a fixed axis at a distance r (Fig. 104). It clearly sweeps out a cylindrical surface.

The distance $\bar{x}$ is always r.

Hence Curved Surface $2\pi \bar{x} P = 2\pi rl$.

Cone. Consider a line of length l rotating round a fixed axis as in diagram (Fig. 105). The distance $\bar{x}$ of

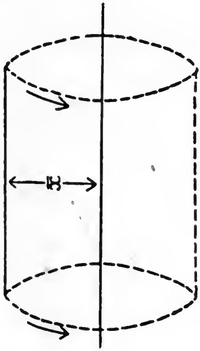


FIG. 104.

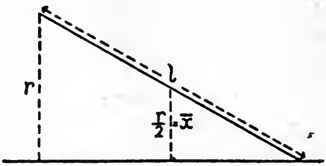


FIG. 105.

the centre of gravity of the line from the axis of rotation is clearly $\frac{r}{2}$.

Hence the curved surface of the cone $(2\pi \bar{x} P)$ is $2\pi \frac{r}{2} . l$ or πrl.

(*b*) Volume of a solid of revolution $= 2\pi \bar{x} A$ where A is the area of the rotating surface and $\bar{x}$ the distance of its centre of gravity from the axis of rotation.

Cylinder. Imagine a rectangle of length h and width r to rotate round one of its sides as a fixed axis (Fig. 106). It clearly sweeps out a cylindrical volume.

The volume of this solid is

$$2\pi \bar{x} . A.$$

But A is clearly $r \times h$ and x, the distance of the centre of gravity of the rectangle from the axis of rotation, is $\frac{r}{2}$.

$$\text{Hence } 2\pi \bar{x} A = 2\pi \cdot \frac{r}{2} \cdot r \times h$$

$$\text{or } \pi r^2 h$$

Cone. Imagine a right-angled triangular surface of sides r and h to swing round a fixed axis in the side h as in Fig. 107. The solid thus swept out is a cone.

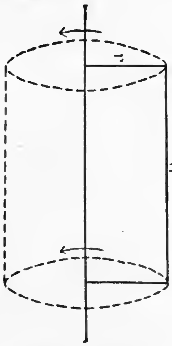


FIG. 106.

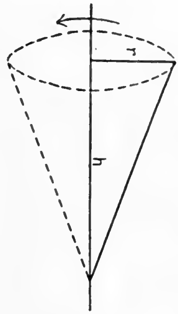


FIG. 107.

Volume of this cone $= 2\pi \bar{x} \, . \, A$.

But A = area of triangle $= \frac{1}{2}(r \times h)$ and $\bar{x}$ is known to be $\frac{r}{3}$.

$$\text{Hence } 2\pi \bar{x} A = 2\pi \cdot \frac{r}{3} \cdot \frac{r \, . \, h}{2}$$

$$= \tfrac{1}{3}\pi r^2 h$$

The proofs of these useful theorems will be found in any elementary text-book of calculus, and the theorems should be familiar to all students, especially those who are reading what is known as Practical Mathematics.

All symmetrical solids may be treated by these methods, and they form the readiest means of calculating the surface and volume of such things as inflated bicycle tubes and other common objects of the " annulus " shape.

CHAPTER XIX

GEOMETRICAL METHODS IN ARITHMETIC

THE heading of this chapter may alarm many teachers. Geometry to them may bring visions of children laboriously learning, and dutifully reproducing, the stilted phraseology of the older editions of Euclid's Elements. We will disarm criticism on this ground by stating that we shall not in the following pages recommend any such rigid and formal treatment of the subject; for the logical and deductive method based upon postulate and axiom is entirely out of place with beginners, a fact now realized by all teachers of mathematics. Yet there are many theorems and simple truths to which even the beginner's attention may well be drawn so that he may learn that such things *are*, even if he is unable for some years to see that these things, from mathematical necessity, *must* be so. The methods that we shall follow and advocate in this chapter will be largely experimental and inductive. (We were tempted to add also heuristic, but remembered in time that *inductive* and *heuristic* are not of necessity synonymous, for the pupil may by heuristic methods be given the joy, not only of discovering the truth but also of discovering the best way in which to apply it. Thus "heurism" or heuristic methods in mathematics may be pressed into service both *inductively* and *deductively*.)

Earlier Teaching. In those other days, not so very long ago when, in our schools, books and material were rare and trained teachers rarer; when all knowledge imparted was learned by rote; when definitions, formal rhymes and rigid statements, all "learned by heart," masqueraded as knowledge; when words took the place of things; the facts of geometry suffered in common with all other subjects of instruction. Thus the infant prodigies of the naive Wilderspin readily exhibited their wonderful geometrical knowledge to the delight of admiring visitors. The following extract from Wilderspin's *Infant Education* (1825 edition) indicates the method then in vogue—

One day some visitors requested I would call out a class of children to be examined, and having so done, I asked the visitors in what they would wish the children to be examined, at the same time stating that they might hear the children examined in Natural History, Scriptural History, Arithmetic, Spelling, Geography, or Geometry. They chose the latter, and I proceeded to examine the children accordingly ; and began with straight lines. Having, as I supposed, continued half an hour in this examination, we were proceeding to enter into particulars respecting triangles ; and having discoursed on the difference between isosceles triangles and scalene triangles, I observed that an acute isosceles triangle had all its angles acute and proceeded to observe that a right angle scalene triangle had all its angles acute. The children immediately began to laugh, for which I was at a loss to account, and told them of the impropriety of laughing at me. One of the children immediately replied " Please, sir, do you know what we were laughing at ? " I replied in the negative. " Then, sir," says the boy, " I will tell you. Please, sir, you have made a blunder." I, thinking I had not, proceeded to defend myself, when the children replied " Please, sir, you convict yourself." I replied, " How so ? " " Why," says the children, " you said a right-angle triangle had one right angle, and that all its angles are acute. If it has one right angle how can all its angles be acute ? " I soon perceived that the children were right and that I was wrong. Here, then, the reader may perceive the fruits of teaching the children to think, inasmuch that children of six years of age and under were able to refute their tutor.

This is a splendid example of how *not* to teach geometry. Here we have a good instance of the fallacy we have noted frequently in this book—the fallacy of mistaking shadow for substance, of teaching words instead of things. And unfortunately, the whole of Wilderspin's teaching of " Geometry " to the " Infant poor from the age of eighteen months to seven years " proceeded on similar lines ; for in the chapter devoted to the teaching of the

"Geometrical Figures," we find that the lessons are a running catechism from which we give but a few selections—

Q. What is this? *A.* A straight line. *Q.* Why do you not call it a crooked line? *A.* Because it is not crooked but straight. . . . *Q.* What does parallel mean? *A.* Parallel means when they are equally distant from each other in every part. . . . *Q.* What does perpendicular mean? *A.* A line up straight like the stems of some tree. . . . *Q.* What does obtuse mean? *A.* When the angle is less sharp than a right angle. . . . *Q.* What do you mean by angles? *A.* The space between two right lines drawn gradually nearer to each other till they meet in a point. . . . *Q.* Can an acute triangle be an equilateral triangle? *A.* Yes, it may be an equilateral, isosceles, or scalene. . . . *Q.* What is the difference between a rhomb and a rectangle? *A.* The sides of a rhomb are equal but the sides of the rectangle are not all equal. . . . *Q.* What do we call these figures that have four sides? *A.* Tetragons. . . . *Q.* Are they called by any other name? *A.* Yes, they are called quadrilaterals or quadrangles. . . . *Q.* What is this? *A.* An ellipse or oval. . . . *Q.* What shape is the top or crown of my hat? *A.* Circular. . . . *Q.* What shape is that part which comes on my forehead and the back of my head? *A.* Oval."

Thus we might follow the futile catechetical method of the writer through Pentagons, Hexagons, Heptagons, Octagons, Nonagons, Decagons, etc., and we learn that "the other polygons are taught the children in rotation in the same simple manner, all tending to please and edify them." In all this we do not doubt the virtuous intentions of the teacher, and we respect the excellent attempts of Wilderspin to educate both the "head" and the "heart" of his infants. We may, however, observe that he makes one heinous omission, common to all such teaching of word-knowledge: he has forgotten that each infant in addition to "head" and "heart" has also "*hands*."

Modern teaching strives to remedy this lip-knowledge and to teach geometrical ideas through the handling, examination, and construction of actual objects.

Again, not so very long ago, drawing in Elementary

Schools included a wonderful subject known as geometrical drawing. In this the pupil learnt a few conventional tricks with ruler and compass, such as bisecting lines and angles, constructing angles of 90°, 45°, 60°, 30°, etc., and drawing triangles, parallels, circles and other figures to given data. The treatment followed always the stereotyped order of the text-books then in vogue, and was as rigid in its practice as the much-maligned Euclid is held to be in his theory. Thus the drawing of perpendiculars from points to lines had to be performed with ruler and compass; the obvious short cut of set square and ruler was not admitted. Each "problem" or "construction" had by repetition to be committed to memory; and little attempt was made to call attention to simple truths and points of theory upon which the work was based. Occasionally the pupil progressed as far as the construction of the regular polygons and of inscribed and circumscribed figures. A few pupils even reached the amazing height of being able to draw simple plan-and-elevation, though this too was stereotyped and frequently reproduced from blackboard, or printed, copy.

Scale-drawing also found a place and a treatment as part of the year's work in drawing. The pupil progressed with the help of "copies" from squared paper to plain paper. Yet always the work was rendered of negligible value and frequently futile by the fact that the copy and the "scale" were always ready-made. The ordinary-life method of making a drawing of an object from actual measurements, and of choosing, on one's own initiative, the most convenient scale to be employed, was seldom encouraged.

Such was the treatment of geometry in the schools of yesterday. Now however ruler, compass, and set square have largely been banished from the drawing lesson in schools. Their place has been more than occupied by the more "arty" mediums of chalk and charcoal, pastels and paint boxes. Thus rendered homeless in a world of subjects isolated in air-tight compartments and time-table pigeonholes, this subject of geometrical drawing pines neglected in many schools, for the drawing-master refuses to recognize it, and the arithmetician resents its intrusion into his "sums." Yet its rightful place is clearly in the mathematical curriculum, and it is one of the objects of this chapter to show how we may cheerfully welcome this cuckoo of elementary school mathematics even though we may be obliged to pull out a few of its more pretentious feathers.

Scale Drawing. It is a sign of advance in our treatment of school mathematics that, especially in schools in industrial areas, this subject now receives a much more extensive and sensible treatment than it did formerly.

Its natural place in ordinary life is in the making of drawings preparatory to practical and constructive work. Thus, in all more advanced work in cardboard and in almost all wood-work, instructors insist upon good working drawings being first prepared. In this natural setting it is possible to teach all the conventions of what is termed *practical* or *mechanical* drawing, and the subject need seldom be isolated from its real purpose. It may, however, be extended far beyond the range of a child's constructive work. Thus we may require a boy to construct to scale, from his own measurements, a front elevation of a cupboard or desk, choosing the scale most convenient for the size of paper used. There is no need to make a fetish of "scales" and the drawing of "scales"; the pupil may be allowed to use the natural scales of "one half actual size," or one-quarter, one-eighth, etc., as well as the more formal 1 in. = 1 ft., etc. In this subject of "practical drawing" applied to real objects, the meaning of scale, elevation, plan, end-elevation, section and other conventional terms may be imparted and acquired under simple and natural conditions, and it is work that boys, in most industrial areas, enjoy thoroughly.

If we apply scale-drawing to larger distances and areas we at once open up a range of interesting geographical applications. A beginning may be made with a simple plan of the school-room or school-hall drawn to scale from actual measurements. In some cases we may extend this work to the school yard or playing field. Again we may, from ordinary field book data, draw to scale a much larger area; and using the simple method of plane-tabling, following the choice of a scale suitable for representing the base line, the rest of the drawing may be said to *draw itself* to scale.

The subject is not exhausted by such examples in practical outdoor geography, for it may still find a place in indoor geography since map-scales provide an endless source both of practical work and of calculations in "scales" if not in scale-drawing. Thus we may estimate distances such as, as the crow flies, railway journeys, steamer-trips, rivers, coast lines; or areas, e.g. urban areas, borough areas, drainage areas or river basins, counties, and even countries from maps drawn accurately to scale.

Finally, we must not omit to mention that *scales* play an important part in all graphic representation. The look of a graph and frequently even its use are largely dependent upon the sensible selection of the most suitable scale. Frequently, too, the anxious teacher spoils the full effect of this useful training in scale-selection by suggesting the best scale or even by insisting upon an arbitrary scale.

Experimental Geometry. The logical order of treatment of the science of geometry proceeds from the ultimate elements of points, lines and angles through surfaces to solids; but this *logical* order is not of necessity the best *teaching* order; for what is too often forgotten by teachers is that what appears to be a simple logical progression is the result of much patient adult labour and analysis. We might almost say that for a child it is necessary to reverse this logical order entirely and to *begin* with the solid—the object—the reality. This must be the method of approach for beginners. The child must gain his geometrical notions from the manipulation of *objects*, not from the memorizing of definitions. We must remember that our object in this work is still two-fold—theory and practice. We still have to impart correct mathematical notions of shape and form, together with appreciation of their simple properties, and also we have to teach practical methods of creating these shapes and forms, which involve carefully graduated instruction in the use of the commoner instruments of mechanical drawing and measurement such as ruler, set square, compass, protractor, etc. To isolate these two aspects into separate divisions of the curriculum is to defeat one great object of our mathematical instruction, which is to give the pupil the means of reducing to manageable and measurable size the complex world of concrete reality around him.

Most of our geometrical teaching will, then, be incidental in the arithmetic lesson. It will begin with the handling of simple objects and will develop through drawing and constructive work in paper, wood, and cardboard. Theory and practice will thus develop side by side, the fundamental ideas of squares, right angles, triangles, etc., growing gradually out of this varied practical arithmetic even if at first their expression is crude and lacks the concise polished phraseology of the practised mathematician. Simple geometrical constructions of angles, parallels, squares, triangles, polygons, etc., will be evolved as the necessity for them arises in the constructive work, and will be remembered far better from the fact that they have been acquired for a definite purpose.

Much of this incidental work in geometry has been already indicated in this book, and the pupil who has thus been taught to arrive at the formulae necessary for numerical calculations in dealing with surfaces and solids (usually termed "mensuration") will have acquired, in passing, much general geometrical knowledge. In this work, objects and instruments should be constantly employed, thus removing reproaches such as that "elementary" children "cannot use a set-square." Nor need our aim be always the establishment of formulae or the working of sums. We may investigate as we proceed the fundamental truths of geometry. In the sections of this book which deal with proportionate division, squares and square roots, proportion and similarity, we see that to divorce the numerical treatment from

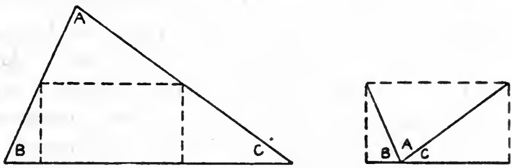


FIG. 108.

the geometrical is impossible, since fundamental truths and formulae are in intimate connection.

Children delight in ruler-and-compass work, and much is possible by these means. Thus the pupil who constructs and graduates a simple clock-face or compass-dial with ruler and compass, will learn much angle-lore, which may be crystallized and applied in the construction and use of a simple semi-circular protractor. The same ruler and compass will enable him to discover, in the construction of triangles, that no triangle can be made unless the sum of two sides is greater than the third side, while he may investigate by a variety of methods the fundamental fact that the three angles of a triangle are equal to two right angles. He may actually measure the three angles and find their sum, or he may by simple paper folding methods show that the three angles together give half-a-revolution. The diagram (Fig. 108) illustrates this convenient method of demonstration.

By folding along the dotted lines the angles are placed in contact and are shown to be together equal to two right angles.

Similarly, by careful drawing, he may discover the important set of "concurrency" theorems in connection with triangles.

Again with ruler, compass and set-square, he may investigate parallels and parallelograms, and will frequently "discover" methods for drawing these, based upon his observations of their properties. So, too, in connection with the mensuration of the circle, investigations need not be limited to the establishing of formulae for circumference and area, but may well extend to tangent, angle, and chord properties of circles. Tangents, together with the useful fact that the angle in a semicircle is a right angle, are especially fascinating to most children.

Thus, in improved ways, we may still teach the old knowledge of geometrical drawing and scale-drawing, the difference between the "old" and the "new" methods of teaching being one of "approach" to the subject rather than of "content."

The scope of experimental geometry is too wide to be indicated in detail in this book. It has, however, been worked out fully for the lower forms of the modern secondary school, and all teachers of the upper classes of the elementary school might well follow the excellent methods of the modern text-books on the subject. There is no need to adopt the old division of "pure" from "practical" geometry. Geometry is neither dull logic, nor a collection of rules-of-thumb and "constructions" for ruler and compass, but is one of the oldest sciences, based on the simplest of obvious truths, with a wide range of application, and the wise teacher will mingle "truth" and application, "construction" and numerical instance, so that his work is neither Euclid nor practical plane and solid geometry, nor mensuration, nor mechanical drawing, but mathematics.

Open-air Work. If philological evidence is correct, geometry or measurement of the earth was originally an intensely practical science capable as much of outdoor application as of indoor investigation, and all teachers must consider the possibilities of these outdoor exercises in mathematics. Of late years the whole subject of "Outdoor Education" and "Open-air schools" has found many enthusiastic followers, both among the "simple life" and "return-to-nature" extremists, and among the more reasonable educationists who recognize chiefly the physical benefits to the weakly, and consequently "backward" child, of a school life spent largely in the open air. In dealing with the possibilities of outdoor mathematics, however, it is necessary to distinguish between work that *may* be done in the open air and work that *must*, from its nature, be performed outside. This distinction will save us from much of the pseudo-outdoor work in mathematics. For example, we may set our children to work "long division" or "stocks and shares" in a shed with the sides knocked out, but we are not there by doing "outdoor mathematics," and a similar criticism may be offered to those methods that insist upon a class "proving" a geometrical theorem by means of a crude drawing in chalk or in sand on the playground or school yard. The work we would include under this heading of outdoor mathematics is of a more genuinely outdoor character. The enthusiastic teacher will find endless possibilities in the subject. The additional apparatus necessary is not over-much. Distance and direction and occasionally time are the fundamentals most frequently to be measured. For this we shall need some convenient form of chain or steel-tape or even a tape-measure or ball of string; we shall also need some simple form of angle-measurer or home-made theodolite; a simple spirit level, plumb line, sighting rule and some form of graduated rod, equivalent to an ordinary surveyor's telescopic rod, will be sufficient for most of our work.

Difficulties are many, but need not dismay us. Accurate work out of doors requires a great deal of time. Some hustling teachers dislike outdoor work on this ground. They know that dozens of sums could be worked in a classroom in the same time that it takes to measure a simple distance or to set off a simple right angle out of doors. Again, crude instruments and inexperienced youthful workers make the limits of possible error very wide. A great deal of patient supervision and checking is necessary. The difference between *knowing* how to do a thing and actually *doing* is never more marked than in outdoor mathematics; a pupil may readily calculate the height of an object when supplied with the numerical data and yet will fail ludicrously when asked to measure the height of some actual object out of doors. The work has a special value on this account alone, for practical accuracy is even more vital to success than the mechanical accuracy of figuring from data supplied.

The work need not be artificial or pedantic. We may attempt many of the common tasks of life. Simple exercises in the actual manipulation of instruments will be found in such ordinary exercises as the accurate marking out of a cricket pitch, football or hockey ground, or tennis court, according to regulation measurements. Again the planning of a garden or the neat arrangement of a vegetable plot is a basis for valuable practical mathematics. Let the sceptic who prefers his classroom and his sums try the simple exercise of setting out an accurate rectangle such as is required for a football pitch. He may return somewhat subdued and chastened, for the difficulties are greater than the unpractised would imagine.

According to the locality of the school we may proceed farther afield and attempt a chain survey of a selected area, returning to the classroom with the data for our actual drawings and calculations. This will enable us to indicate to our pupils numerous little "dodges" and pitfalls of practical surveying with base line and offsets. (It is remarkable to note the number of people who are unable to set out accurately a base line of say 300 yards on a given field or other plane area. We have seen even experienced teachers fail ludicrously to get their base line *straight.*) The simple and beautiful method of the plane-table is more geometrical still, and should be introduced into schools wherever possible.

Levels, levelling, and contouring are subjects that, perhaps, are not quite so geometrical in their nature as those indicated in the preceding paragraph. They are, however, so common in outdoor mathematics of practical life that we may well discuss them here. The actual setting of levels and levelling pegs as practised by builders may well be tried. Again it may be necessary to set pegs to indicate a gentle rise or fall in the laying of a field or garden and the method of doing this should be practised. With a simple form of telescopic staff we may measure actual rises and falls of ground, and may even, in some cases, teach the surveyor's method of "foresight and backsight" in determining differences of level, and the extreme limit of our work in this connection will perhaps be the mapping and contouring of a selected area.

The subject of sun and shadow is an excellent topic for outdoor geometry and mathematics. Observations of the shadow of a vertical post will teach much concerning the sun's inclination at different times of the day or year, and may even be made the basis for an attempt to construct and graduate a simple form of horizontal or vertical sun-dial.

Lastly, we shall bring geometrical methods to our aid in the calculation of "heights" as well as of distances and areas. Here various methods are possible, from the geometrical method of drawing to scale to the purely arithmetical use of numerical trigonometry. Beginners should learn both methods and use the one as a check to the other.

To conclude, we have tried to emphasize in the chapter the fact that the treatment of geometry for beginners, especially for the beginner who is not likely to become a mathematical specialist, the only reasonable treatment is not mainly logical and deductive but largely experimental, practical, and applied.

CHAPTER XX

THE TEACHING OF PROPORTION

So far in this book we have dealt almost entirely with *processes* in arithmetic, i.e. Addition, Subtraction, Multiplication, and Division in Numbers, Quantities, Fractions or Decimals. In the subject of Proportion we at once advance the learner to a higher mathematical plane. The Four Rules in any kind of number or quantity may with patience be mastered by any normal child, and may be accurately, if mechanically, applied by anyone gifted with an average memory for tables and rules. Here assuredly practice will " make perfect," but in *proportion* we have a mathematical conception of simple beauty, of enormous importance, and yet of great difficulty to children. The idea of one quantity depending mathematically upon another is one that grows but slowly. It is questionable perhaps whether some people, even adults, ever have a clear conception of two connected variables. Whatever in our pedantry we may call the subject—ratio, proportion, or variation—the fundamental idea is one that in all branches of mathematics—arithmetic, algebra, geometry, trigonometry—proves a stumbling block. Many can master rules and processes, but few indeed can really think accurately in ratios and comparisons.

The result has been that the subject has received very mechanical treatment in schools. Instead of teaching what is perhaps the most comprehensive mathematical idea of all, covering a multitude of types of sums, we have reduced proportion to the mechanical application of a particular trick to a particular type of sum. Generations of scholars have in the past been taught the " Rule of

Three " by the method of dots : :: : as a mechanical trick which they never understood, and frequently applied absurdly ; and they have classed it in their minds as a new type sum equal in value to L.C.M. or H.C.F. Generations of modern scholars are taught the *method of unity* equally mechanically, and use it with a minimum of intelligence and discrimination as an infallible nostrum to be applied to the solution of an " if " sum. Where such mechanical teaching is in vogue the pupils apply their " dots " or their " Unitary Method " with clumsy conscientiousness to such examples as the following—

" If 1,000 men build a ship in 400 days how many men would build it in $\frac{1}{2}$ a day ? "

" If a train takes 30 mins. on a journey carrying 100 people, how long will it take carrying 200 people ? ".

If we grasp the absurdity of a blind application of " rules " to such problems as these we are approaching the heart of the matter. Proportion is an *idea* to be grasped before any figures or calculations are made. If the real connection and the *kind* of connection between the quantities is not seen at the outset, figuring is mechanical and frequently ludicrous. And the error is more fundamental than any error in calculation—it is an error in judgment or reasoning. It may be vexatious if our memory plays us unkind tricks, so that we write, e.g. 7×8 as 54, but the defect is much graver if we go wrong in our comparative arithmetic and assume the wrong kind of numerical connection between quantities, or even assume proportion where none exists. It is not our mechanical automatic self that is at fault ; it is our real, intelligent, rational self. To be illogical is a graver crime than to be merely inaccurate in the narrow sense. Thus if an answer was 960 we might overlook an answer 959 due to bad *calculation*, but could not excuse an answer 240 due to incorrect *thinking*.

Methods of Working. It has been usual for many years to begin the subject with *simple* proportion ; to subdivide this into *direct* and *inverse* proportion, and to follow these with cases of *compound* proportion.

In actual setting-out we have a choice of methods—

(*a*) The traditional *dot* method depending upon *extremes* and *means*

$$: \quad :: \quad :$$

(*b*) The method of unity or unitary method.

(*c*) The so-called fractional method.

(*d*) The method of the simple fractional equation involving one unknown, i.e.

$$\frac{x}{10} = \frac{20}{50}$$

Children taught any of these four methods soon attain a considerable amount of skill and accuracy with straightforward examples.

Method (*a*). *The "Rule of Three."* This method (*a*) was formerly first favourite, and teachers are still found who habitually use it. It is now falling generally into disuse, as being too rigid, mechanical, and unintelligent. Children learnt to apply it with phenomenal accuracy but seldom understood the reason for the process. If asked to solve a simple problem such as: "If 5 books cost 7s. 6d. what would 10 cost ?" they would plunge industriously into the maze of—

5 : 10 :: 7s. 6d. : x (i),
or x : 7s. 6d. :: 10 : 5 (ii),
or even 5 : 7s. 6d. :: 10 : x (iii),

thereby raising the ire of the pedant who declaimed on the impossibility of comparing 5 (books) with $7\frac{1}{2}$ (shillings). Yet it is obvious that according to the "trick" this so-called incorrect statement (iii) gives the correct answer as readily as (i) or (ii). The fact, which is not always realized by teachers, is that what we really compare are *numbers*, not things, and accordingly any one of the three statements given above will serve if the quantities are treated solely as *numbers*.

(*b*) *The Method of Unity*, or Unitary Method. This is perhaps the most popular and widespread method employed with young scholars. Upholders of the method claim for it many advantages. They maintain that it is reasonable, based on common sense, easily taught and capable of simple explanation. Against these undoubted advantages we have to set the following defects—

(1) It is hardly an "adult" method; for to proceed always through the "unit," and to write out the three lines of statement, is frequently cumbersome and often unnecessary. Actually, the unit line is a convenient prop for children and insistence upon its use may easily destroy their real ability to think readily in comparisons.

(2) It lends itself easily to unintelligent routine use, thus resulting in absurd statements. We have seen a child write down rubbish such as the following—

£5 12s. 6d. represents the wages of 5 men

£1 " " " $\frac{5}{\text{£5 12s. 6d.}}$ men

The absurdity of such statements as $\frac{5}{\text{£5 12s. 6d.}}$ men needs no comment. Yet such are frequently overlooked by teachers whose sole aim at any cost seems to be mechanical accuracy.

(*c*) *The "Fractional" Method*. This method for dealing with the simpler cases of proportion is becoming more and more common in schools.

The essence of the method is that the unit line of the previous method is discarded and the final statement of the answer in correct form is obtained in one step, calculation only being necessary to simplify the answer.

A graded series of examples will explain the method.

(1) *Multiples and Submultiples.*

(*a*) If 12 eggs cost 3s. 9d., what will 24 cost? 36? 48? etc.

Ans. *Twice* as much, *three* times as much, etc.

(*b*) If 5 yds. cost 7s. 6d., how many yards could I buy for 15s. ?

Ans. *Twice* as many.

(*c*) If 12 eggs cost 3s. 9d., what would 6 cost ? 4 ? 3 ? 2 ?

Ans. *Half* as much, *one-third* as much, etc.

(2) *Extension* of the idea to numbers and quantities that are not simple multiples or submultiples.

(*a*) If 12 eggs cost 3s. 9d. what would 9 cost ?

Ans. $\frac{9}{12}$ or $\frac{3}{4}$ of 3s. 9d.

(*b*) If 5 yds. cost 7s. 6d. what would 4 yds. cost ?

Ans. $\frac{4}{5}$ of 7s. 6d.

(*c*) If 20 pigs cost £36 what would 16 cost ?

Ans. $\frac{16}{20}$ of £36 or $\frac{4}{5}$ of £36.

(*d*) If 12 yds. of carpet cost £3 10s., what would 18 yds. cost ?

Ans. $\frac{18}{12}$ or $\frac{3}{2}$ of £3 10s.

This second stage is the crux of the method. Many children who can reason readily with multiples and submultiples break down when faced with examples that do not involve simple multiples. They must be taught to reason always in the same simple steps—

(*a*) Is my answer number or money or quantity, etc. ?

(*b*) Is it more or less ?

(*c*) In what proportion is it more or less ?

Thus in the example : Some iron bars counted 30 to the ton. What would 85 such bars weigh ?

Answer to (*a*) above is "tons," etc.

" " (*b*) " " "more."

" " (*c*) " " "$\frac{85}{30}$ as much."

Therefore my answer is $\frac{85}{30}$ of 1 ton or $2\frac{5}{6}$ tons.

(3) *Extension* of the idea to fractional quantities involving complex ratios.

This is the hardest of all for beginners. Thus even in such a simple example as—

"$7\frac{1}{4}$ yds. cost 19s. $11\frac{1}{4}$d. ; what would $4\frac{3}{4}$ yds. cost ?"

it is questionable if the average child could reasonably argue *in one step* that the cost would be $\frac{4\frac{3}{4}}{7\frac{1}{4}}$ of 19s. $11\frac{1}{4}$d.

Thus it would appear necessary to compromise between the Method of Unity and the Fractional Method, and not to require children to adhere slavishly to the former nor to apply mechanically the latter, but to settle each example on its merits and never to use the unit line when it can obviously be dispensed with. To attempt to teach the one or the other as the *rule* to be *always* followed is pedantic and cramps the development of the child.

(*d*) *The Equational Method.* This method of setting down the results of our reasoning in comparative arithmetic is simple yet mathematical, and should be taught to all older scholars.

Taking the example: "A map scale was 5 miles to the inch. What length of line represented a distance of 42 miles?" this might be set out as—

$$\frac{x}{1} = \frac{42}{5}$$

So, too, the example discussed above: "If $7\frac{1}{4}$ yds. cost 19s. $11\frac{1}{4}$d., what would $4\frac{3}{4}$ yds. cost?" might be set out as $\frac{x}{19\text{s. } 11\frac{1}{4}\text{d.}} = \frac{4\frac{3}{4}}{7\frac{1}{4}}$ hence $x = \frac{4\frac{3}{4}}{7\frac{1}{4}} \times$ 19s. $11\frac{1}{4}$d.

This method has all the merits of the older "dot" method or "rule of three" with none of its defects. It is readily taught; it is mathematically sound; the selection of the two fractions to be equated involves excellent practice in comparative thinking, while the process of solution is sensible, and at the same time speedy and accurate.

To conclude this review of methods of working proportion we will summarize our recommendations. Avoid any method that is ill-understood by the average child even though you may be able to apply it readily yourself. Do not make a fetish of any one particular method, but rather give every child's mathematical common sense free scope. If he can "jump" to the answer, let him; but otherwise let him proceed in his own way and at his own pace. This may be slower than the daily grind at a mechanical mill whose sole merit is that it turns out the right answer, but in the end it will be worth the trouble. People who can *think* accurately in comparisons are of more value to the world at large than those who can patiently and obediently turn the handle of the answer-mill when the data are supplied.

Direct Proportion. The two fundamental ideas of simple proportion, namely, direct and inverse proportion, may be studied quite apart from particular sums; and if we can make sure that our pupils have a clear conception of the functional interdependence of two quantities, the actual particular cases or sums will present but little difficulty.

Direct proportion or the functional idea of two quantities increasing or decreasing together. Simple cases abound. A few only need be mentioned—

(*a*) Numbers and Costs.

(*b*) Distance and Time (at uniform speed).

(*c*) Mileage and Fares (on railways).

(*d*) Length and Weight (pipes and rods of uniform cross section).

(*e*) Circumferences and Radii (circles and solids of circular cross section).

(*f*) Wages and Hours (piece-work).

The intimate connection of the numerical conception with the geometrical treatment is seen in that all the above lend themselves readily to graphic representation, and all result in the simple *straight line* graph (using rectangular co-ordinates and the ordinary squared paper). Children may well be set to draw these simple graphs. Thus we may devote a lesson or two to the drawing of such graphs as "Eggs at 5 for 1s." The class will soon determine sufficient connected values to give the necessary line (Fig. 109). Once constructed to a suitable scale, it may be used for countless oral examples, e.g.—

(*a*) What shall I pay for 15 eggs ?

(*b*) How many eggs can I buy for 2s. 6d. ? etc.

All such answers may be read off from the graph, and the *use* of these graphs in this way is as important as their construction.

It follows that every sum in simple proportion has a graphic solution or illustration of the above "straight

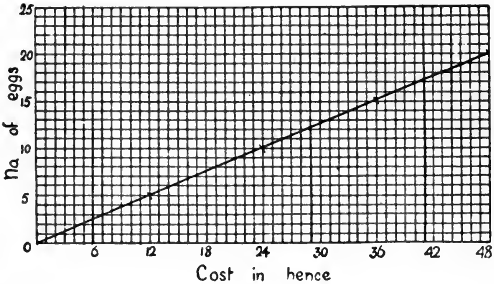


Fig. 109.

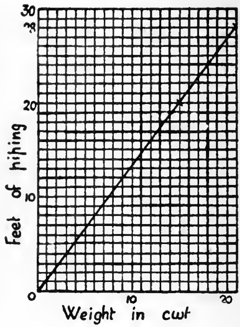


Fig. 110.

line" form. Thus in the following (Fig. 110): "Twenty feet of iron piping weighed 15 cwt. Find what 28 ft. of the same piping would weigh," we may work graphically by the following steps—

(*a*) Join the point (20, 15) to (0.0).

(*b*) Read off the cwts. corresponding to 28 ft. on this line.

But at this stage it is not particular answers to particular sums that are required, but rather the fundamental notion of one quantity depending on another in simple mathematical proportion.

Nothing would appear easier at first sight than to collect and multiply instances of simple direct proportion. Yet, in actual life, we find that actual instances are not quite so common as we imagined. Consider for a minute the simple example of sunshine and shadow. Here we seem to have a perfect example in nature of simple proportion. Clearly the length of the shadow is proportionate to the vertical height of the obstacle. We might equally well claim that it also depends upon the time of day, and upon the latitude of the place of observation, and upon the season. Thus the shadow-length is a complexity resulting from various factors, and if we set to children a "sum" about a church 80 ft. high casting a shadow of 100 ft. and requiring them to calculate the length of the flagstaff which casts a 40 ft. shadow, we do so only on a tacit understanding that it is an instantaneous view we assume of the same spot at the same instant of time.

Even the favourite example of costs and quantities is frequently fallacious. For example, a child will dutifully supply the answer "Ten times as much" to such an example as: "If 5 cwt. cost £27 10s., what would 50 cwt. cost?" Yet we know that in real life the larger quantity would probably be sold at a lower rate, so that the simple direct proportion only holds within very narrow limits.

The familiar "men and work" sums are even more glaring examples of unwarranted assumptions of strict proportionality. The ever-green "porter" example will serve for illustration.

"Two porters load a truck in eight minutes. How long should 12 porters take?"

In a recent book we have seen the following table set out—

	2 men take	8 mins.
	1 man takes	16 ,,
(a very weak man)	$\frac{1}{2}$ man takes	32 ,,

We may of course extend this kind of reasoning, thus—

(a boy)	$\frac{1}{4}$ man might take	64 mins.
(an infant)	$\frac{1}{8}$,, ,, ,,	128 ,,

The absurdity is clear, for all the time we assume irreproachable working automatons who never cease a moment, and we ignore altogether the fact that the packages to be loaded may be altogether beyond the powers of the "very weak man."

At the other end of the table we have equally amazing results. Clearly if we want to speed up we increase our men thus—

4	men	would	take	4	mins.
40	,,	,,	,,	·4	,,
400	,,	,,	,,	·04	,,

This would be quick work, though we shudder at the scene round the one truck when 400 men got to work.

But we have said enough to convince a reader that the assumption of strict proportionality is not always justified and is often absurd. Yet the traditional text-book abounds in sums capable of solution only upon the assumption of simple proportionality which a moment's reflection would show to be non-existent.

Let us give one more example. A man papered a room in $3\frac{1}{2}$ hours. What fraction of the room had he done at the end of the first hour? The thoughtless will answer promptly $\frac{2}{7}$, but the thoughtful will reply that answer is hardly possible, for the first hour might be occupied in mixing paste and stripping walls, and the hardened cynical householder will add that the average job-man will not have progressed far beyond lighting his pipe and sending his boy for some missing tool.

It is then essential that a properly trained pupil shall be able not only to work examples where strict simple proportionality may be admitted, but shall also be able to detect at once cases where the rule of three breaks down. We will return to this subject later in the chapter.

Having broken the ground and having given the beginner some idea, by means of preliminary and graphical illustrations, of what simple proportion implies, we are ready now to deal with the tradition sums included in this category. Here the range of examples is so vast that only by a most careful selection and grading can we hope to cover the field with anything approaching completeness.

We suggest the following as a convenient division. The method of solution is left to the teacher, but we assume that it will be a mixture of fractional and unity methods without a rigid adherence to either.

(*a*) *Multiples.*

(i) If 3 lbs. cost 5s. give the cost of 6 lbs., 9 lbs., 12 lbs., etc.

Ans. *Twice* as much, *three times* as much, etc.

(ii) If the fare for 12 miles is 1s. 6d., what is it for 24 miles, 36 miles, 48 miles, etc. ?

Ans. *Twice* as much, etc.

(iii) If 10s. 6d. buys 3 shirts, how many can I buy for £1 1s., £1 11s. 6d., £2 2s., etc. ?

Ans. 6, 9, 12, etc.

(*b*) *Submultiples.*

(i) Towels were 16s. 6d. a dozen. What was the cost of 6, 4, 3, 2, 1 ?

Ans. *Half* as much, *one third* as much, etc.

(ii) A man walked at the rate of 5 miles an hour. How far did he walk in 30 min., 20 mins., 15 mins., 10 mins., etc. ?

Ans. $2\frac{1}{2}$ m., $1\frac{2}{3}$ m., $1\frac{1}{4}$ m., $\frac{5}{6}$ m., etc.

General Type.

(i) *Money answers.*

If 7 books cost 17s. 6d., what would 5 cost at the same rate ?

(ii) *Number answers* (converse of (i)).

If £18 15s. provides 500 dinners, how many dinners would £16 10s. provide ?

(iii) *Introducing fractional quantities.*

(*a*) If $3\frac{1}{2}$ yds. cost 5s. 10d., what would $5\frac{1}{2}$ yds. cost at the same rate ?

(*b*) $\frac{3}{5}$ of a ship was worth £7,500. What was the value of a $\frac{3}{8}$ share ?

(iv) *Involving quantities all of the same kind.*

(*a*) A steeple 120 ft. high cast a shadow 140 ft. long. What would be the length of the shadow of a tree 42 ft. high at the same time and place ?

(*b*) At 2s. 4d. a lb. a keg of butter was worth £4 4s. What would it be worth at 2s. 6d. a lb. ?

(*c*) On a debt of £29 10s. a bankrupt paid £11 1s. 3d. What would he pay on £50 at this rate ?

This section (iv) includes numerous money transactions such as rates, taxes, simple interest, percentage, all of which may be treated by proportion methods.

Simple Inverse Proportion. This subject is usually taught immediately after and sometimes almost concurrently with the subject of simple direct proportion.

Following our previous treatment, we shall first see that pupils grasp the fundamental idea of inverse variation, and then we shall set examples where this inverse proportion has to be detected.

The Fundamental Idea. Traditional text-books have given us a surfeit of examples involving men-and-work, hay-eating horses, besieged garrisons, and shipwrecked mariners. Simpler and more practical examples are ever ready to hand—if we will look for them.

The least mathematical of our readers will recognize that $xy = k$ represents all simple cases of one quantity x varying inversely as y. This constant k can be in any kind of units.

Constant Length. Example. 240 miles. Tabulate times at different rates, e.g.—

	Rate.	*Time.*
240	120 m.p.h.	2 hrs.
	60 ,,	4 ,,
	40 ,,	6 ,,
	30 ,,	8 ,,
	20 ,,	12 ,,

These may be plotted graphically and the ordinary inverse curve obtained and used.

Constant Area. Rectangle of 16 sq. ft.
Tabulate pairs of sides, e.g.—

One side.	*Other side.*
4	4
8	2
16	1
32	½

etc.

Constant Volume (introducing time).
(*a*) Tank of any volume *V*.
Suppose 2 taps fill it in 10 mins.
Tabulating taps and time we have—

1 tap . .	20 mins.
2 taps . .	10 ,,
4 ,, . .	5 ,,
8 ,, . .	2½ ,,

This assumes that delivery from each tap is uniform and equal.

(*b*) Ditch or wall, etc. (of constant volume, introducing men and work).
Suppose 10 men take 12 days.
Tabulating men and time—

30 men . .	4 days
20 ,, . .	6 ,,
10 ,, . .	12 ,,
5 men . .	24 days
4 ,, . .	30 ,,
3 ,, . .	40 ,,
2 ,, . .	60 ,,

This assumes men of equal and automatic working capacity.

Constant Sum of Money. Suppose sufficient to buy 8 yds. at 3s. 6d. a yd.

Vary the cost of 1 yard and tabulate yards and price per yard.

At	3s. 6d.	a yard I can buy	8 yds.
,,	7s.	,, ,, ,, ,,	4 ,,
,,	14s.	,, ,, ,, ,,	2 ,,
,,	28s.	,, ,, ,, ,,	1 ,,
,,	1s. 9d.	,, ,, ,, ,,	16 ,,
,,	10½d.	,, ,, ,, ,,	32 ,,

Constant Weight (i.e. hay, provisions, etc.).

Suppose constant weight enough for 200 men for 20 days.

Vary the number of men and tabulate results—

200 men	. .	20 days
100 ,,	. .	40 ,,
50 ,,	. .	80 ,,
25 ,,	. .	160 ,,
5 ,,	. .	800 ,,
1 ,,	. .	4,000 ,,
400 ,,	. .	10 ,,
800 ,,	. .	5 ,,
4,000 ,,	. .	1 ,,

This, of course, assumes the most rigid rationing system.

But we need not continue our examples. Our object is to try to convince teachers that the idea of inverse proportion is best taught not from single isolated instances such as occur in the usual set exercises, but from a full discussion of real inverse types such as are indicated above. We should not be satisfied until the pupil unaided, readily recognizes cases where the inverse law holds.

We have already noted that the examples given in the usual text-books tend to be somewhat traditional, conventional and artificial. It is not difficult to find real and simple examples. The following are but a few—

(*a*) At 40 miles an hour a train took $3\frac{3}{4}$ hours. How long would it have taken at 50 miles an hour ?

Ans. $\frac{4}{5}$ of $3\frac{3}{4}$ hrs.

(*b*) 39 boxes of fish had 2 doz. in a box. How many boxes would be necessary if packed 3 doz. in a box ?

Ans. $\frac{2}{3}$ of 39.

(*c*) A wheel of 7 ft. diameter turned 250 times along a road. How many turns would a wheel of 5 ft. diameter make in the same distance ?

Ans. $\frac{7}{5}$ of 250.

(*d*) Carriage on 80 tons for 40 miles cost £11. How far should 60 tons be carried at the same rate for the same money ?

Ans. $\frac{80}{60}$ (or $\frac{4}{3}$) of 40 miles.

(*e*) A 16-knot boat took 12 days on a voyage. How long should an 18-knot boat have taken?

Ans. $\frac{16}{18}$ (or $\frac{8}{9}$) of 12 days.

(*f*) One shilling's worth of gas keeps 6 burners alight for 5 hours. For how long should it keep 4 burners alight?

Ans. $\frac{6}{4}$ (or $\frac{3}{2}$) of 5 hrs.

(*g*) Gears and Turns: The large chain wheel of a bicycle had 42 teeth, and the small back cog wheel had 18 teeth. If the chain wheel turned 120 times, how many turns did the small cog wheel make? ("Fixed" wheel assumed.)

Ans. $\frac{42}{18}$ (or $2\frac{1}{3}$) of 120.

It is the duty of every teacher to see that his examples are as real and practical as possible, and that the answers are within the range of possibility. Thus examples like the following can be "worked" accurately enough as inverse proportion, but the answers would have no real significance.

"1,000 men built a battleship in 400 days. How many men would be necessary to build it in 1 min.?"

The actual method followed in examples of inverse proportion will depend upon particular teachers and schools. The answers to the examples given above show that the fractional method is the readiest.

With both direct and inverse proportion treated in this way the same questions should be asked and answered—

(*a*) Is the answer more or less?

(*b*) In what proportion is it more or less?

Beginners, in inverse as in direct proportion, may find it necessary to proceed through the unit line, but more experienced students should dispense with this step.

Compound Proportion. Examples of the following type have long been known as compound proportion—

(*a*) 45 men could build a wall in 30 days, working 12 hours a day. How many days of 8 hours each would 60 men take?

(*b*) The carriage on 35 tons for 45 miles is £31 5s. How far, at this rate, could 63 tons be carried for £27 10s. ?

If children can readily apply the fractional method, these present no new difficulties at all but are simply combinations of a series of simple proportions, sometimes direct, sometimes inverse.

Thus we may set out (*a*)—

45 men working 12 hours a day take 30 days.
60 ,, ,, 8 ,, ,, ,, ?

We argue thus—

Original number of days = 30 days.

(*a*) More men ∴ less days, i.e. $\frac{45}{60}$ of 30 days.

(*b*) Less hours per day, ∴ more days, i.e. $\frac{12}{8}$ of $\frac{45}{60}$ of 30 days.

Children taught according to this method can deal with this type of example with astonishing speed and accuracy.

So-called compound examples may frequently be reduced to simple examples ; thus in (*a*) above we might use simply men and hours, and in (*b*) above we might work in ton-miles and costs, but using the fractional method of solution such reduction is unnecessary.

All such examples may, of course, be worked by continued application of the method of unity, but this usually involves much needless repetition of lines.

Compounding Ratios. The art of compounding ratios is not known or practised in schools nearly as much as it should be. An example will show its use.

At a concert seats were of three kinds, and the prices of tickets were in the proportion 4 : 2 : 1. Numbers were in the proportion 1 : 2 : 3. In what proportion were the takings ?

Prices of Tickets	4	:	2	:	1
Numbers sold	1	:	2	:	3
Receipts are	(4 × 1)	:	(2 × 2)	:	(3 × 1)
i.e.	4	:	4	:	3

The original ratios are simply compounded in turn.

This may be reversed. Thus in a similar example suppose—

Prices of Tickets	5 : 2 : 1
Receipts	2 : 3 : 4

Find the proportionate numbers of people.

Here the proportion of numbers (by division) is—

Numbers are	$\frac{2}{5} : \frac{3}{2} : 4$
i.e.	4 : 5 : 40

The principle has endless applications. We give a few more.

Radii of three cylinders are	1 : 2 : 3
Heights ,, ,, ,,	3 : 2 : 1

Compare volumes.

Now volumes depend upon radii *squared* as well as heights, hence we write down—

$(\text{Radii})^2$ are	1 : 4 : 9
Heights ,,	3 : 2 : 1
∴ Volumes ,,	3 : 8 : 9

(*b*) Two cylinders have volumes in the ratio 3 : 4. Their heights are 3 : 1. Compare their diameters.

Volumes are	3 : 4
Heights ,,	3 : 1
∴ By division $(\text{Radii})^2$,,	1 : 4
∴ Radii ,,	1 : 2
and Diameters ,,	1 : 2

This ability to compound and "de-compound" ratios is essential to rapid intelligent work in this branch of mathematics.

Cases where the "Rule of Three" or Simple Proportion does not hold.

Work of an intelligent nature in proportion must be characterized by ability to detect at sight cases where simple proportion or the method of unity breaks down. Examples are innumerable. It is a sound plan to intersperse these with the orthodox examples. They should not of course be treated in bulk as a new type of sum. In that well-known book *Easy Mathematics*, Sir Oliver

Lodge advocates the use of examples such as these with beginners in the teaching of proportion. We give here a small selection. Every teacher will be able to add to the list.

Cases where no Proportion at all exists, including well-known catches and comic examples—

(*a*) Two men on a cliff can see 10 miles out to sea. How far could 4 men see ?

(*b*) The thickness of a halfpenny is ·05 inch. What is the thickness of a penny ?

(*c*) Four eggs were cooked in $3\frac{1}{2}$ mins. How long should two eggs take ?

(*d*) A hen laid 24 eggs in a month. How many did it lay in a year ?

(*e*) A shower wetted 2 people through in 10 mins. How long would it take to wet 4 people ?

(*f*) A cockerel woke a family of 4 people. How many cockerels would be necessary to wake 12 people ?

(*i*) A man fished for 6 hours. He caught 3 in the first hour. How many did he catch altogether ? (Similarly for cricketers and runs, footballers and goals, etc.)

(*j*) With a run of 30 yards a boy cleared a high jump of 4 ft. 3 ins. How high would he have jumped with a run of 40 yds.?

Cases where Proportion exists but is not simple.

(*a*) A man dug a ditch 3 ft. deep in 2 hours. How long would he take to dig one 12 ft. deep ?

(*b*) At 12 years a boy weighed 6 stone, what would he weigh at 18 years ? (Similarly for heights, etc.)

(*c*) A box half full of tea weighed 40 lbs. What did it weigh when full ?

(*d*) Six men in 6 days earned £6 by piece-work. What would 8 men earn in 8 days at the same rate ?

(*e*) A train travelled at 40 m.p.h. What would be its speed if it burnt twice as much coal per hour ? (Similarly for motors, ships, etc. The connection between speed and fuel-consumption is never simple.)

Cases where Proportion exists but only within strict limits.

(*a*) A camel can carry a load of 1 ton for 100 miles. How far could it carry 30 tons?

(*b*) A weight of 1 lb. stretched a wire ·02 inch. How much would a weight of 1 ton stretch it?

(*c*) 1,000 men build a factory in 100 days. How many men could build it in 1 day?

(*d*) Walking for 8 hours, a man walked $4\frac{1}{2}$ miles in the first hour. How far did he walk altogether?

Testing Quantities for Proportionality. All our examples so far have been based upon simple proportion, stated or implied. Equally interesting and perhaps even more fascinating is the converse process of testing for proportionality or determining laws. Here we are confronted with a series of corresponding values, whose connection has to be represented in a law expressed as simply as possible.

This work is an excellent feature of the modern books on practical mathematics. Most of the simpler cases are forms of simple proportion expressible in the law

$$y = ax + b,$$

where x and y represent the varying quantities. The work is of such real practical application that it might well find a place in *all* schools and not be restricted to evening and technical schools. The work is simple and interesting.

Let x and y represent connected values. If *simple proportion* holds, then the plotting of x and y will yield a *straight line.* If *inverse proportion* holds, then the plotting of x and y will yield a hyperbola, or, more simply, if inverse proportion is suspected, the plotting of x and $\frac{1}{y}$ or y and $\frac{1}{x}$ should give a straight line. Similarly if one quantity depends upon the square of the other, then x and y, if plotted, will yield a parabola, or more simply x and y^2 (or else y and x^2), if plotted, will yield a straight line. As soon as the line is drawn the constants a and b

may be found. For simple examples we refer the reader to any of the many modern books on practical mathematics. The work is a pleasant change from the eternal round of sums.

Proportionate Division. This common application of proportion usually receives special attention in schools. The problem is presented in various forms in the ordinary sums. Thus we may have the following and other varieties—

1. Divide a quantity (number, sum of money, length, weight, etc.) into two parts proportional to 2 : 3.

2. Divide a quantity into two parts so that one part is $\frac{2}{3}$ of the other part.

3. Divide a quantity into two parts so that one part is half as much again as the other part.

4. Divide a sum of money between two people so that for every 2s. received by one the other receives 3s.

These represent the simple type. In each case we have to divide in the ratio 2 : 3. A line (Fig. 111) serves as the most ready illustration—

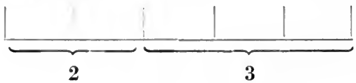


Fig. 111.

Clearly if the segments are 2 : 3, the corresponding fractions are $\frac{2}{5}$ and $\frac{3}{5}$ or $\frac{2}{2+3}$ and $\frac{3}{2+3}$, whence the ordinary rule. The teaching of proportionate division seldom gives much trouble, and the average pupil likes the type of example. He should be taught to recognize the problem in any of its above forms.

We may, of course, extend and complicate the type of example to any required extent. Thus, we may divide in the proportion 1 : 2 : 3, or 1 : 3 : 5 : 7. Again we may vary it by dividing in the proportion $\frac{1}{2} : \frac{1}{3} : \frac{1}{4}$. (These should be converted to integers, i.e. 6 : 4 : 3.)

Or we may require a sum of money to be divided into *equal numbers* of shillings and sixpences (which is hardly proportionate division) or into twice as many shillings as sixpences, and so on. Further, we may divide between A, B, and C so that A : B is 1 : 2 and B : C is 1 : 2.

But we need not continue our varieties. They are well known to all arithmeticians and provide many ingenious and puzzling examples. Teachers should beware of their fascination—their importance in the ordinary school syllabus is very limited. Frequently, however, a hoary or evergreen example may be brightened and refurbished with the aid of a little proportionate division. Thus, in the case of our old friend: "A does a piece of work in 6 days and B in 8 days. How long should they take working together;" we may add, "and if £7 were paid for the work how should A and B divide it?"

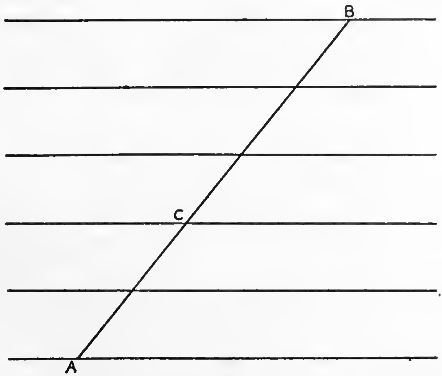


FIG. 112.

Graphical Work. The subject of proportionate division in arithmetic is so closely connected with the geometrical conception of the subject that the two should be taught side by side. The child should be familiar with the ordinary method of dividing a line in any given proportion. In this connection the machine-drawn parallels on foolscap paper or exercise book paper are of very great use. Let a line AB be drawn across five foolscap spaces (Fig. 112). Then clearly the intermediate lines divide it into five equal parts, and C, for example, divides AB in the ratio 2 : 3, and similarly for other points. The amount of drawing thus saved and the extra accuracy obtained is very great. The method is, of course, limited by the length of AB and the number of parallels available.

Similarity. We conclude this chapter on Proportion with a discussion of the subject of Similarity and the possibility of its simple treatment numerically and geometrically in schools.

Similarity has not yet achieved its rightful position in

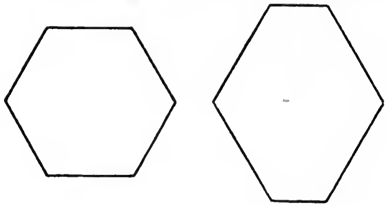


Fig. 113.

elementary mathematics. The principle is of such remarkable mathematical simplicity and beauty, and has such an extraordinary wide range of application, that we plead with all teachers to emphasize its importance on every possible occasion.

The treatment we here sketch out is free, experimental and heuristic, and perhaps unorthodox enough to offend the rigid mathematician, but we do not apologise. In this case the importance of the end justifies the means.

The Idea. Similarity applies to *surfaces* (and solids) rather than *lines*. Two conditions only are requisite for similarity—

(*a*) The same shape (equal corresponding angles).

(*b*) Proportionate dimensions.

Both conditions are generally necessary. Children may investigate two figures (Fig. 113), which are equiangular but not similar, since the dimensions are not proportionate.

More interesting and more difficult is the case of two rectangles (Fig. 114), arranged with a border of equal width all round. Most children say at once that these are similar only to find by measurement that the ratio $\frac{\text{width}}{\text{length}}$ is *not* the same in both, hence they are *not* similar.

The same conditions are necessary for curvilinear figures. All circles are similar, but two ellipses arranged as in Fig. 115 (the ordinary mirror frame with moulding

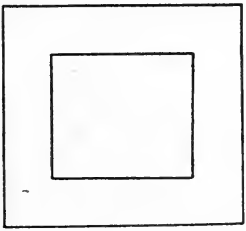


FIG. 114.

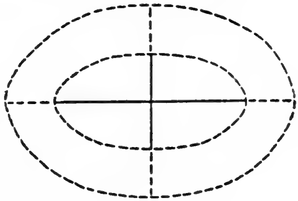


FIG. 115.

of equal width all round) are *not* similar for their dimensions, i.e. major axis and minor axis, are not proportionate.

Similar Triangles. Beginners soon grasp that with triangles (and *only* with triangles) one of the original conditions of similarity is sufficient, i.e. if triangles are equiangular they also have proportionate dimensions, and so are similar, and conversely, if triangles have proportionate dimensions, they are also similar.

This they may discover heuristically by using two rays only from a pole and cutting these rays with a series of parallel lines (Fig. 116).

Actual measurement will convince them that the sides of the triangles PAB, PCD, PEF, etc., are proportionate.

We may use a sun ray and shadow for the same purpose

(Fig. 117). Actual measurement will show the equality of the ratio $\frac{h}{s}$ in every case using the same ray. The same principle is the basis of all simple proportion, illustrated graphically as straight-lined proportion.

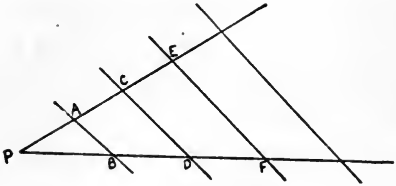


FIG. 116.

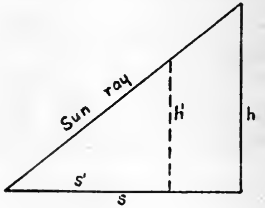


FIG. 117.

This idea of proportionate dimensions needs to be very firmly fixed. Children will willingly admit that all *squares* are similar, but if told that the *sides* of two squares A and B (Fig. 118) are 2 : 3, they will not always admit that the *diagonals* are also in the ratio 2 : 3.

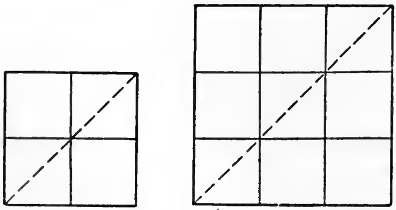


FIG. 118.

Again, according to Pythagoras, they know that a square with a side 1 has a diagonal $\sqrt{2}$, and by similarity they deduce that in *every* square $\frac{\text{side}}{\text{diagonal}}$ is 1 : $\sqrt{2}$. Applying this to calculations, they readily calculate the *diagonal* for any given length of side, as for a side of 6 ins. the diagonal is $6\sqrt{2}$. Many children, however (and some children of a larger growth), find it difficult to proceed from diagonal to side, i.e. to reverse the above operation. They cannot calculate the side if the *diagonal* is 6 ins. Yet we simply *divide* by $\sqrt{2}$ instead of *multiplying* by $\sqrt{2}$, i.e. if diagonal is 6 ins. the side is $\frac{6 \text{ ins.}}{\sqrt{2}}$ or $3\sqrt{2}$ ins. The main idea of similarity is not really grasped until such operations can be performed almost automatically.

Again, it is essential that the pupil should realize the fact that not only are *sides* proportion in similar figures, but *any* corresponding lines maintain the same proportion.

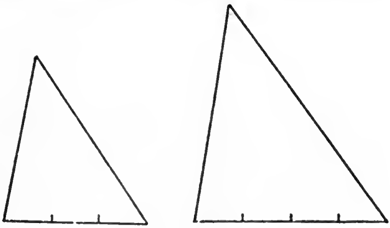


FIG. 119.

Thus if two triangles (Fig. 119) are similar and have sides 3 : 4, it follows that *heights* are 3 : 4; medians are 3 : 4, radii of the inscribed circle are 3 : 4, circumscribing radii are 3 : 4, etc., etc. Indeed, any two lines correspondingly situated one in each triangle are in the same proportion 3 : 4.

The same is true in circles. Children do not always see the extent to which proportionate dimensions holds good; if told that the radii of two circles are 2 : 3, they will admit after a pause that the *diameters* are also 2 : 3, but are not always prepared to admit that *circumferences* are also 2 : 3.

Enlarging and Reducing. The simple process of enlarging or reducing figures while preserving their similarity may well find a place in all schools. It follows directly from similar triangles.

To enlarge any figure such as X (Fig. 120), all that is necessary is to select a convenient pole (P) and to draw rays through the angular points. Then lines drawn

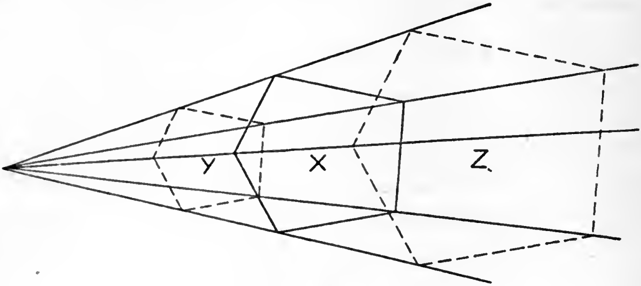


FIG. 120.

parallel to the sides of the original figure X will give larger or smaller replicas as required (e.g. Y and Z).

No extensive geometrical knowledge is necessary to

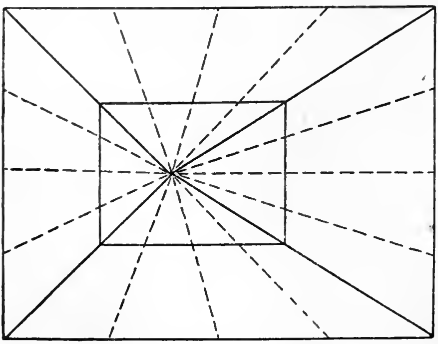


FIG. 121.

draw these. A fair skill with ruler and set square alone is required. Children may experiment with the pole in different positions within and without the original. The method may well be applied to the enlargement of a map (Fig. 121). Thus, selecting a pole anywhere, preferably within the original, we may draw rays to fix any convenient larger rectangle, and finally with other rays (shown dotted) we may fix the position of the lines of latitude and longitude. The method is simple, speedy, and accurate.

The Areas of Similar Figures. The precise statement of Euclid that the areas of similar plane figures are in "the duplicate ratio of homologous sides" may be interpreted simply for our purpose as "areas of similar figures are proportional to squares on corresponding dimensions."

Squares provide the best starting point for showing this comprehensive truth. The child may compile his own table, drawing figures if necessary.

Length of side	1	2	3	4	5	6	7	8	9	10
Area of square	1	4	9	16	25	36	49	64	81	100

These same figures will form the basis of the useful square, and square root, graph ($y = x^2$) for positive integers. This graph should be drawn, examined, and used. The term *parabola* need not of course be mentioned.

Circles may be similarly investigated—

Radius	1	2	3	4	5	6	7	8	9	10
Area	π	4π	9π	16π	25π	36π	49π	64π	81π	100π

As before a graph may be drawn giving corresponding areas and radii for any circle.

The learner should be able to state readily that if radii (or diameters) are 2 : 3 then areas of corresponding circles are 4 : 9. Exercises of the following type are also useful: Two semicircles stand on diameters in the proportion 3 : 4. Compare their areas. (*Ans.* 9 : 16.)

Triangles for beginners need a little more careful investigation. A learner who admits the truth in the case of squares and circles may hesitate to admit that the areas of similar *triangles* are proportionate to squares on corresponding dimensions. Thus if two similar triangles have their basis in the ratio 2 : 3, the pupil does not aways see at sight that their areas must be 4 : 9. We may investigate as follows.

Right-angled Triangles by Calculation. Draw a number of similar right-angled triangles (Fig. 122). (The sun-and-shadow idea is the readiest.) Tabulate (*a*) Bases,

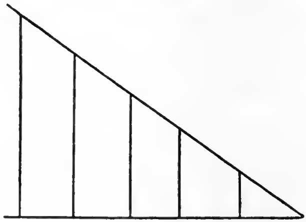


FIG. 122.

(*b*) Areas ($\frac{1}{2}$ Base $\times$ Height). (*b*) Will give a simple series of squares, and the plotting of (*a*) and (*b*) will give the simple parabolic curve of squares. Equilateral triangles may be similarly investigated and a curve may be drawn giving corresponding areas for equilateral triangles of different bases.

The truth may be illustrated for *any* triangles by a simple drawing exercise (Fig. 123) where the dimensions are in simple proportions such as 1 : 2 : 3. Thus—

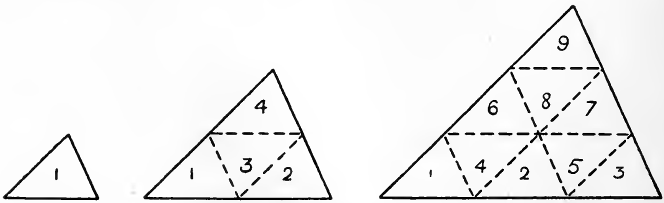


FIG. 123.

Here the bases are 1 : 2 : 3.

By drawing parallels and, if necessary, by cutting along the parallels we demonstrate that where bases are

1 : 2 : 3 the areas of the triangles are 1 : 4 : 9, as shown in the diagram.

We need not stop at triangles, squares, and circles. The principle is an all-pervading one. For *spherical* surfaces the same truth holds. These vary as the squares on their radii, so that if two spheres have radii 1 : 2 their spherical surfaces are 1 : 4. The principle is all-important in physics, for since spherical surfaces round any given centre increase as the square of their radii it follows that the *intensity* of the energy received from a radiating centre, whether heat, light, sound, electrical, or magnetical, will vary *inversely* as the square of the distance from the centre. Thus this universal *inverse square* law is but one obvious deduction from the principle controlling the areas of similar figures.

The Volumes of Similar Solids. Extending our conception, we shall now investigate the truth that *volumes* of similar solids are proportionate to *cubes* of corresponding dimensions.

For learners cubes form the best introduction, as they may build up the cubes of 1″, 2″, 3″, 4″ edge, etc., and discover that the numbers of inch cubes required as 1, 8, 27, 64, etc. From cubes they may proceed to any parallelopipeds, such as bricks or chalk boxes, and find that to make a brick twice as broad, twice as thick, and twice as long would require *eight* ordinary bricks, and so they can deduce the numbers necessary for dimensions three times, four times, etc., the original brick.

The truth may be applied to spheres. Thus, spheres with radii 1 : 2 will have volumes 1 : 8.

The general truth that the volumes of any similar solids vary as *cubes* of corresponding dimensions is a little harder to grasp, but is in constant use in calculation. The non-mathematical reader may pause before he believes that an exact replica of a pint bottle or pint pot made with every linear dimension doubled would actually contain a gallon (eight pints), nor would he be prepared to admit that a small replica of a statue to the scale 1 : 5, i.e. $\frac{1}{5}$ the original height, etc., would only weigh $\frac{1}{125}$ of the original statue. Yet such is the case if our simple similarity law holds.

But here, interesting as the subject of similarity is, we must conclude with the hope that we have said sufficient to cause teachers to give the subject the attention that its importance and applicability demand.

CHAPTER XXI

PERCENTAGE

" The necessity for reducing fractional amounts of a quantity to the same denominator in order to compare their values should lead children to appreciate the advantage in choosing 100 as the standard denominator for the purpose of expressing a fraction as a percentage. The term " per cent " is in everyday use, and the need for calculating percentages is frequent. Children should therefore be trained to express one length sum of money, etc., as the percentage of another length, sum of money, etc., and to evaluate a given percentage of a given quantity."—(Circular 807, par. 43, " Suggestions.")

In this chapter we propose to indicate a somewhat wider treatment of this important subject than that which usually obtains, hoping thereby to remove the reproach that teachers tend to restrict " percentage " to fictitious examples in " Profit and Loss," and to ignore its many more important applications.

The Meaning of Percentage. Percentage may be defined mathematically as the standardizing of ratios for purposes of comparison on the convenient basis of 100. The subject is therefore closely allied to both fractions and decimals. The latter is an important point which we shall develop, and has been too frequently forgotten by teachers, as seen in the tendency to treat all percentages as vulgar fractions rather than decimals.

The ratio or fraction idea of percentage serves as the best beginning. Selecting a real case, we may introduce the subject with some such example as: In a box of oranges 1 out of every 10 was bad. How many was this per hundred (or what per cent) were bad ?

The square-inch showing hundredths forms a very convenient means of illustrating this and all simple percentages.

Thus in the square inch in the diagram (Fig. 124) we may mark the top square in each vertical column as the bad one, showing clearly 10 per hundred or 10 per cent. Similarly we may treat other simple ratios or fractions. Ratios are preferable to fractions especially if we express the ratios in the concrete form of 1 or more *out of* a number. We may deal with 1 out of 5, 1 out of 20, 1 out of 25, 1 out of 4, 1 out of 2, and require all these to be standardized as *out of* 100 or as percentages.

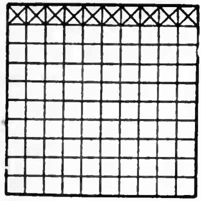


FIG. 124.

As a second stage we should teach the manipulation of percentages *as* percentages and not as fractions. Extending our original example, we might ask: "If 10 per cent of the oranges were bad, what per cent were good?"

Other simple examples involving the addition and subtraction of percentages should be taken, e.g.—

(1) 16% of a class were absent. What per cent were present?

(2) 55% of a mixed class were boys. What per cent were girls?

(3) In a regiment there were 24% casualties. What per cent escaped unhurt?

(4) On a train 60% travelled 3rd class, 25% travelled 2nd. What per cent travelled 1st? (15%.)

All these may be readily illustrated with the square inch. It is advisable that such examples be taken at a very early stage to accustom the child to *think* in percentages and to correct the vicious tendency to treat all percentages as disguised fractions.

Ratio to Percentage. Following the preliminary work outlined above, we may next treat the general case of converting any ratio to a corresponding percentage.

Simple aliquot parts of 100 and their combinations present no difficulty, and children will readily convert

$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{25}, \frac{1}{50}$ to percentages, and should receive oral practice in these until they are firmly fixed as percentages.

Percentages based on these simple ones may next be taken, such as: $\frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{3}{20}, \frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{17}{20}$, etc.

Finally we may take any ratio, i.e. $\frac{5}{7}$ or $\frac{8}{9}$, and express it as a percentage.

The method is of course akin to working with fractions, i.e. $\frac{8}{9} = \frac{\frac{8}{9}}{1} = \frac{\frac{8}{9} \times 100}{100} = \frac{88\frac{8}{9}}{100} = 88\frac{8}{9}\%$. This is usually abbreviated to the simple evaluation of $\frac{8}{9} \times 100\% = 88\frac{8}{9}\%$.

Concurrently with this we should give ample practice in the converse process of converting any percentage to its corresponding simple ratio or fraction. Oral work should make the children familiar with the simpler cases, such as: $50\% = \frac{1}{2}$, $20\% = \frac{1}{5}$, $25\% = \frac{1}{4}$, $10\% = \frac{1}{10}$, etc., while for the harder cases the ordinary method may be followed, e.g. $85\% = \frac{85}{100}$ or $\frac{17}{20}$ and $24\% = \frac{24}{100}$ or $\frac{6}{25}$.

At the same time the equally important connection between percentages and decimals should be developed. This is even simpler—

$$25\% = \cdot 25, \qquad 35\% = \cdot 35$$
$$12\tfrac{1}{2}\% = \cdot 125, \qquad 3\tfrac{3}{4}\% = \cdot 0375$$
$$4\tfrac{1}{6}\% = \cdot 041\dot{6}, \qquad 2\tfrac{1}{2}\% = \cdot 025$$

Similarly any decimal may be converted to a percentage by moving the decimal point two places to the right. Thus—

$$\cdot 163 = 16\cdot 3\%, \quad \cdot 255 = 25\cdot 5\%$$
$$\cdot 456 = 45\cdot 6\%, \quad \cdot 0455 = 4\cdot 55\%$$

These simple transformations of decimals to percentages and percentages to decimals are of endless application in calculation. In this decimal work *vulgar* fractions need never be mentioned.

Calculating Percentages. Every good teacher will have introduced this in the preliminary work outlined above. The direct form is as follows—

(*a*) Find 10%, 20%, 25%, 5%, etc.	of a given number.
(*b*) Find 15%, 30%, 45%, etc.	,, ,, sum of money.
(*c*) Find 37%, 83%, 92½%, etc.	,, ,, quantity.

No rigid fractional rule should be insisted upon and practice methods should be used wherever possible. Thus, in finding 35% of 3 tons 15 cwts—

Method (*a*) $= \frac{35}{100}$ or $\frac{7}{20}$ of 3 tons 15 cwt.

	T.	cwts.	qrs.
	3	15	–
25% = ¼		18	3
10% = $\frac{1}{10}$		7	2
	1	6	1

is as rapid and as clear as method (*b*), i.e. →

	T.	cwts.	qrs.
	3	15	–
			7
2\0	26	5	–
	1	6	1

The practice method is especially useful in that application of percentage known as simple interest, e.g. Find the interest on £287 10s. for 1 year at 3½%. (This is equivalent to: Find 3½% of £287 10s.)

	£	*s.*	*d.*	
	287	10	–	= 100%
	28	15	–	= 10%
2½% = ¼	7	3	9	= 2½%
1% = $\frac{1}{10}$	2	17	6	= 1%
	£10	1	3	

All direct calculation of simple interest lends itself to this simple and obvious method.

The converse process of that outlined above, viz., the evaluation of a given percentage of a quantity, is a little harder for beginners.

Examples usually take some such form as : 15% of my money is 2s. 6d. How much have I ?

(*a*) Such may be laboriously treated by "unity," thus—

$$15\% = 2\text{s. } 6\text{d.}$$

$$1\% = \frac{2\text{s. } 6\text{d.}}{15}$$

$$100\% = \frac{2\text{s. } 6\text{d.} \times 100}{15}$$

(*b*) Or by fractions—

$$\frac{15}{100} = 2\text{s. } 6\text{d.} \quad \therefore \text{ whole } = \frac{100}{15} \text{ of } 2\text{s. } 6\text{d.}$$

(*c*) Or by decimals—

$$\cdot 15 = 30 \text{ pence}$$

$$\text{whole} = \frac{30}{\cdot 15} = 200 \text{ pence} = 16\text{s. } 8\text{d.}$$

(*d*) Or by any common-sense method, e.g.—

$$15\% = 2\text{s. } 6\text{d.}, \quad \therefore 5\% = 10\text{d.},$$
$$\text{and } 100\% = 200\text{d.} = 16\text{s. } 8\text{d.}$$

The decimal method (*c*) above deserves very careful attention. It is the simplest in nearly every case, provided that the pupil is well drilled in decimal manipulations.

Increase and Decrease per cent. This important aspect may well receive definite attention as part of the subject of evaluating a given percentage of a quantity.

Thus we should find time for plenty of examples of the following form—

(*a*) Increase (or decrease) 12500 by 10%, 20%, etc.

(*b*) Increase (or decrease) £37 10s. by 15%, $7\frac{1}{2}$%, etc.

(*c*) Increase (or decrease) $12\frac{1}{2}$ miles by 8%, 12%, etc.

These again should be worked by practice methods. Fractional methods should be used only as a last resort. So to increase £37 10s. by 15% of itself we may work as follows—

	£	s.	d.		
	37	10	–	=	100%
$10\% = \frac{1}{10}$	3	15	–	=	10%
$5\% = \frac{1}{2}$	1	17	6	=	5%
	£43	2	6		*Ans.*

The same problem might appear disguised as "Find 115% of £37 10s.," which only the clumsy calculator would treat by the laborious method of evaluating £$\frac{115}{100}$ of $37\frac{1}{2}$, when the required result is so readily reached by the simple adding on of parts by the practice method shown above.

We will indicate a further real application. "A consignment of cheese originally 15 tons underwent 15% shrinkage in weight while in store. What was its final weight? We work thus—

	15 tons	
10%	1·5 tons	∴ Final weight = $12\frac{3}{4}$ tons
5%	·75 „	or 12 tons 15 cwt.

Total shrinkage: 2·25 „ or $2\frac{1}{4}$ tons

We might of course have evaluated either—

(*a*) $\frac{17}{20}$ of 15 tons, or (*b*) ·85 of 15 tons,

in order to obtain our answer.

Compound Increase or Decrease per cent. This subject yields readily to simple arithmetical treatment. Consider the examples—

(*a*) The population of a village originally 5,000 increases every year by 10% of what it was at the beginning of the year. Find the population after 3 years.

This is clearly solved by successive additions of $\frac{1}{10}$, i.e.—

5,000	original population
500	
5,500	after 1 year
550	
6,050	after 2 years
605	
6,655	after 3 years

(*b*) A machinery plant originally costing £10,000 is held to depreciate each year by 10% of its value at the beginning of the year. After how many years will it be valued at less than half its cost ?

£10000	original value
1000	
9000	after 1 year
900	
8100	after 2 years
810	
7290	after 3 years
729	
6561	after 4 years
656·1	
5904·9	after 5 years
590·49	
5314·41	after 6 years
531·441	
4782·969	after 7 years

Similarly we may treat all such examples. The method is easy and interesting, and forms a splendid groundwork for Compound Interest.

Discounts, Dividends, Commissions. These commercial aspects of the general problem of evaluating a given percentage of a given quantity must always receive attention. No set formal methods need be taught, but as far as possible the work should be reduced to simple mental calculation.

Thus a child should know that a discount, dividend, or commission of 5% is equivalent in money to 1s. in the £. Similarly $2\frac{1}{2}$% is equivalent to 6d. in the £.

From these a multitude of others follow, many simple cases of which should be noted, e.g. $3\frac{3}{4}\% = 9$d. in the £; $7\frac{1}{2}\% =$ 1s. 6d. in the £.

Where odd shillings and pence are included it is perhaps better to work by ordinary "practice" methods.

To find to the nearest penny the dividend on £389 10s. 6d. at 6%, we would work as follows—

	£	*s.*	*d.*	
	389	10	6	
10%	38	19	0·6	
$5\% = \frac{1}{2}$	19	9	6·3	
$1\% = \frac{1}{5}$	3	17	10·8	
	£23	7	5·1	*Ans.* £23 7s. 5d.

The "10%" line is useful for lower sub-multiples.

Ordinary traders' discounts should not be neglected, though these are seldom given (in retail transactions) as percentages. "1s. in the £" or "1d. in the shilling" are more usual forms. It will, however, add an extra touch of reality to the "Bills" and "Accounts" if the answers occasionally have to be "discounted" at some given rate "for cash," and the rate in the case of larger amounts may be given as a percentage.

It is possible, of course, to treat all examples in discounts, dividends, and commissions by the method of unity, but such rigidity is pedantic and seldom necessary.

Thus we might have worked our previous example by the following method—

On £100	. the dividend is	£6
,, £1	,, ,,	$\frac{£6}{100}$
,, £389 10s. 6d.	,, ,,	$£\frac{6 \times £389\ 10s.\ 6d.}{100}$

$$= £\frac{2{,}337\ \ 3s.}{100}$$

$$= £\frac{2{,}337{\cdot}15}{100} = £23{\cdot}3715$$

$$= £23\ \ 7s.\ 5d.\ \text{nearly.}$$

This method should be compared with the practice method already shown. For intelligent common-sense working in arithmetic it is essential that rigid methods for solving set types of examples should be emphasized as little as possible. Children should be encouraged to treat every example by the neatest available method and should not rely upon one particular method for one particular class of example.

One Quantity as a Percentage of Another. In the preceding pages we have dealt pretty fully with the calculation of a given percentage of a given quantity, and have indicated some of its commoner applications. We have now to deal with the alternate process, viz., expressing one quantity as a percentage of another similar quantity.

For subject matter we have much ready to hand in class-room and school. Thus we may express—

(*a*) The attendance of a class on any one day as a percentage of the possible, i.e. (43 out of 50 = 86%).

(*b*) The attendances of the school per week as a percentage of the possible.

(*c*) The numbers of boys and girls in a mixed school as percentages of the total.

(*d*) The number of marks gained by one candidate as a percentage of the possible number of marks.

(*e*) The percentage of "passes" and "failures" in an examination.

(*f*) The percentage of seeds that germinate (in gardening experiments).

All these and many similar involve the manipulation of numbers only, and form a convenient starting point. Any convenient method should be allowed; the unity method should seldom be necessary, except in fractional form, i.e. 23 out of 40 should be viewed as ($\frac{23}{40}$ of 100)% *directly* without any intermediate unity stage.

From numbers we should pass to examples in money and compound quantities generally, dealing with such methods as expressing (*a*) 16s. 6d. as the percentage of £1; (*b*) 13s. 4½d. as the percentage of 15s.; (*c*) 3 cwt. 2 qrs. 14 lbs. as the percentage of 1 ton.

Here a variety of methods are open to us. We may employ practice, decimal, fractional, or unity methods, incidentally employing any convenient short cut that offers itself.

These examples are worked out in a variety of ways to enable comparisons to be made as to the most convenient method.

Example (*a*). *Practice.*

£	*s.*	*d.*	
1	–	–	100%
	10	–	50
	5	–	25
	1	–	5
		6	2·5
	16	6	82·5

{ Simple
Rapid
Direct

Decimal.

16s. 6d. = £·825 = 82·5%

Very simple in this case but not in every case.

Fractional.

$$\frac{16\text{s. } 6\text{d.}}{\pounds 1} = \frac{33}{40} = \left(\frac{33}{40} \times 100\right)\% = \frac{3300}{40} = 82{\cdot}5\%$$

A method frequently clumsy.

Unity.

Out of £1 we have 16s. 6d.
" £100 " 16s. 6d. × 100
= £82 10s. = 82·5%

Example (*b*). *Practice.*

s.	d.	
15	–	100%
10	–	66·666 . . . ($\frac{2}{3}$)
3	4	22·222 . . . ($\frac{1}{3}$ of $\frac{2}{3}$)
	$\frac{1}{2}$	·277 . . . ($\frac{1}{80}$ of 3s. 4d.)
		89·166 . . .

Fractional.

$$\frac{13\text{s. } 4\frac{1}{2}\text{d.}}{15\text{s.}} = \frac{107}{120} = \left(\frac{107}{120} \times 100\right)\% = \frac{1070}{12} = 89\tfrac{1}{6}\%$$

Unity.

Out of 15 shillings we have 13s. 4½d.

" 1 shilling " $\frac{13\text{s. } 4\frac{1}{2}\text{d.}}{15}$

" 100 shillings " $\frac{13\text{s. } 4\frac{1}{2}\text{d.} \times 100}{15\text{s.}}$

Or, using decimals—

$$\frac{13{\cdot}375 \times 100}{15}$$
$$= \frac{1337{\cdot}5}{15}$$
$$= 89{\cdot}166 \ldots$$
$$= 89\tfrac{1}{6}\%$$

Example (*c*). 3 cwts. 2 qrs. 14 lbs. as a percentage of 1 ton.

Practice.

1 ton	100%
2 cwts.	10%
1 cwts.	5%
2 qrs.	2·5%
14 lbs.	·625%
	18·125%

Fractional and Decimal.

$$\frac{3 \text{ cwts. } 2 \text{ qrs. } 14 \text{ lbs.}}{1 \text{ ton}} = \frac{3{\cdot}625}{20} = {\cdot}18225$$
$$= 18{\cdot}125\%$$

Unity.

Out of 1 ton we have 3 cwts. 2 qrs. 14 lbs.
" 100 " " " 3 cwts. 2 qrs. 14 lbs. × 100
= 18 tons 2 cwts. 2 qrs.
or 18·125%

Clearly, since decimals and percentages are so closely akin, whatever methods are used for decimalizing may also be used for expressing one quantity as the percentage of another.

Profit and Loss. We are convinced that, in the average Arithmetical syllabus, far too much attention is devoted to this type of sum. Percentage has all too frequently been taught as if intended merely for the solution of a *gain* or *loss per cent.* The reason is not far to seek: this type of example is concise and clear cut, and thus dear to the heart of the teacher who still places sums in the forefront of elementary mathematics, and who is still more concerned with "four sums right" every lesson than with the fuller development of the child's mathematical intelligence. The result has been that the original straightforward calculation of a gain or loss per cent in a commercial transaction has degenerated into side-ways and *cul-de-sacs* full of highly ingenious puzzles in percentages, but often very remote from the world of real life. This type of example need not concern us very greatly.

We have three elements in every transaction (regarded from one person's point of view), viz., cost price, selling price, gain or loss. Most of our profit and loss examples express the third element as a percentage of the *cost*—an assumption perfectly correct in theory but, unfortunately, not always followed in the commercial world, where in many cases a seller bases his gains or losses *not* on his costs but on his *takings*. With this warning, however, we will proceed to indicate the usual types.

These are—

(*a*) Given the C.P. and the S.P.—find the gain or loss per cent.

This is the straightforward calculation.

(*b*) Given the C.P. and the gain or loss per cent, find the S.P.

This is almost as direct as (*a*).

(*c*) Given the S.P. and the gain or loss per cent, find the C.P. Children find this type most difficult, and it therefore requires careful teaching.

All three types may, of course, be dealt with by the mechanical application of the method of unity. Most, however, yield readily to the fractional method. Thus we have seen numerous conscientious but unenterprising teachers work an example such as: "A farmer sold a horse which cost him £27 at a gain of 10%. Find what he received," by droning through the ritual of—

S.P. of £100 is £110

" £1 " $£\frac{110}{100}$

" £27 " $£\frac{110}{100} \times 27$

instead of taking the obvious "short cut" of adding on $\frac{1}{10}$, i.e.,

£27 + £2 14s. = £29 14s.

This addition (or subtraction) of simple parts is but one other example of the practice method, and should be employed wherever possible.

Children may well be drilled in the simpler cases, e.g. What fraction of the cost price is the selling price—

(i)	When sold at	50%	gain.	*Ans.*	$\frac{3}{2}$
(ii)	,, ,, ,,	25%	,,	,,	$\frac{5}{4}$
(iii)	,, ,, ,,	10%	loss.	,,	$\frac{9}{10}$
(iv)	,, ,, ,,	15%	,,	,,	$\frac{17}{20}$

etc., etc.

This will save endless repetition of unnecessary "unity" statements.

Type (*c*), i.e. proceeding from the S.P. to the C.P. or "backwards" also may be treated fractionally.

Example. A motor cycle was sold for £44, a gain of 10 per cent on its cost. Find its cost.

The unity method would give us—

Cost Price of £110 is £100

,, ,, £1 ,, $£\frac{100}{110}$

,, ,, £44 ,, $£\frac{100}{110} \times £44$

whereas we might have reached the same by the simple deduction of $\frac{1}{11}$, i.e. £44 − £4, or £40. If necessary we might "justify" ourselves more fully thus—

110%	£44	
10%	4	($\frac{1}{11}$ of 110%)
100%	£40	

For a *loss* we may work similarly.

Example. A lathe was sold for £270 at a loss of 25 per cent on its cost. Find its cost.

Here we may simply add on $\frac{1}{3}$, i.e. £90 and give our answer as £360, or more fully express ourselves—

75%	£270
25%	90 ($\frac{1}{3}$ of £270)
100%	£360

The fractional method makes plain to the beginner the old puzzle that to resell to the original owner at the same percentage is not necessarily to have your money returned.

E.g.: A sells an article to B at 10% gain. B resells it to A at 10% loss, but B loses. Why?

Clearly because adding $\frac{1}{10}$ to A's cost is not the same as deducting $\frac{1}{10}$ from B's cost. To correct our $\frac{1}{10}$ addition we must deduct $\frac{1}{11}$.

In this connection we may treat similarly that familiar type, so perplexing to children, of the shopkeeper who *marks* at a certain percentage above cost, but *discounts* at a certain percentage for cash.

Example. A shopkeeper marks his goods 25% above cost but allows 10% discount for cash. What percentage of profit does he really make?

Cost . . .	100
Add $\frac{1}{4}$ . .	25
Selling . .	125
Deduct $\frac{1}{10}$ . .	12·5
Discount Price .	112·5

or $12\frac{1}{2}$% actual profit.

Excellent drill in the straightforward types may be compactly arranged in three columns, thus—

	C.P.	S.P.	Gain or Loss %.
(*a*)	x	x	?
(*b*)	x	?	x
(*c*)	?	x	x

This arrangement saves much blackboard space and unnecessary wording, where practice in calculation alone is desired.

Statistics. The application of percentage to money should not be allowed to obscure other equally important uses of this convenient conception.

Percentage is almost universally used as a convenient means of handling and comparing statistics of all kinds. Thus we find it used in dealing with populations, whether increase or decrease or actual percentage composition. Similarly it crystallizes facts of births, marriages, deaths, on the basis of total population. Here frequently a very small fractional percentage is avoided by the simple artifice of " per thousand " instead of " per cent." Thus the non-mathematical can grasp 2 per 1,000, where they would fail to realize ·2 per cent. In geography, again, applications of percentage are innumerable. We may apply percentage to help us to realize any quantitative facts, exports, imports, shipping, etc. Indeed, economic geography generally has a strong percentage basis. The modern books on practical geography provide plenty of data for these percentage calculations.

Percentage Composition in Chemistry, etc. This subject, so closely allied to proportional division, again has endless applications, though from the older type of arithmetical text-book it would appear that gunpowder is the only substance whose percentage composition is worthy of estimate. It may well be applied to all alloys, mixtures, and compounds in connection with chemistry, manures, soils, milk, etc., etc., while even for girls the hygiene lessons provide plenty of scope for percentages examples in food analyses and relative food values.

Percentage of Error. Lastly, to the practical mathematician, *percentage of error* has an important significance, though perhaps this subject is beyond the scope of arithmetic for beginners.

With the possible exception of percentage of error, all the above and countless other topics readily lend themselves to actual examples in percentage, and for the more extensive inclusion of these in our school syllabuses we would gladly sacrifice a good many of the mythical transactions of the phantom traders A and B.

CHAPTER XXII

INTEREST AND INVESTMENTS

(*a*) " It is now agreed that in the past too many of the sums set in Elementary Schools consisted of money operations, often of a difficult nature, connected with the business of the shopkeeper or the clerk or even of the banker and the stockbroker, and to some extent this objection to the arithmetic commonly taught still obtains."[1]

(*b*) " . . . Exercises on interest and investments if taken at all should illustrate lessons on the simple economic aspect of these subjects, e.g. it is useful for the children to know that the rate of interest should depend upon the nature of the security and that high interest often betokens an unsafe investment. The rate of interest allowed on Post Office and other savings bank deposits should be thoroughly understood. Examples should be given involving " usurer's interest " charged on small loans for short periods such as a week or a fortnight in order that the uneconomic nature of such borrowings may be appreciated."[1]

We think we hear the old-time arithmetician snort and scoff at this view. Why should he interrupt his pleasant series of sums in interest or stocks and shares to deal out dull platitudes on the morality of the whole business of usury? Of what use would his warnings be? No knowledge of the " uneconomic nature " of some transactions will ever make some people thrifty, nor will a dissertation on the obvious pitfalls of investment ever prevent a fool and his money from being parted. He is there (he will say) to teach *mathematics* and not prudence to his scholars—fools and rogues there will always be. . . .

And many will secretly sympathize with him and will object to the addition of mathematics to the weapons of the didactic, garrulous improver of mankind.

Yet there is strong necessity in our present arithmetical curriculum for the striking of a normal balance, and this

[1] Circular 807, " Suggestions."

is particularly the case in the subject of interest. Without hesitation we say that far too much time is spent in the average school on the subject of interest, especially in the working of endless examples remote from the child's experience, and remote also from life itself. Here again the teacher has been unthinkingly led astray by the text-book maker, until he thoughtlessly attaches as much importance to artificial and unpractical bookish types such as: "At what rate per cent?" etc.; "In what time?" etc.; "What principal will amount to," etc.; as he does to the straightforward calculation of simple interest itself, which is the only type of calculation commonly required in the financial world. It is to this type accordingly that we shall devote most of our attention in this chapter—noting, however, the other possible types of example for the sake of those who look for mathematical completeness.

The Fundamental Idea. Interest is merely a particular application of percentage with the addition of a simple *time* factor.

It has been customary for many years now to begin the subject by constant reference to topics which the children readily grasp, such as the interest allowed on Post Office Savings Bank deposits. The rate of interest here, $2\frac{1}{2}$ per cent per annum, or 6d. in the £ per year, or $\frac{1}{2}$d. per £1 per month, is easily understood and applied especially under the P.O. Regulation of paying only on *complete* pounds deposited for *complete* months. Thus our first exercises might be based on a *very* simple bank-book embracing as far as possible deposits in complete pounds and not being complicated by "withdrawals." With this practical introduction there will be little need for prosy talking and moralizing on the meaning of interest.

The older method of beginning the subject, one by which the beginner was plunged at once into the most general case, and wrote down parrot-like and obediently :

"Interest on £100 for — years," etc., has little to recommend it.

It is possible, however, that of late years, especially since 1914, the attraction of the Post Office Savings Bank has been obscured by the more productive War Savings Certificates which unfortunately, with their compound basis of interest and awkward percentage of £5 4s. 7d. per cent, do not lend themselves to simple interest calculations, though they may well form the basis of the arithmetic of investments and the buying of simple shares.

As for the pawnbroker and moneylender, the arithmetical possibilities of these are rather limited, and we are almost inclined to leave these topics for the moral instruction lesson, though we think we know a certain type of teacher of both sexes who would refuse to have them even there, acting no doubt on the truly psycho-therapeutic (and somewhat ostrich-like) principle of refusing to recognize the existence of the unpleasant. Yet a little knowledge is not always a dangerous thing, and children, especially those of the poorer classes, have a right to the knowledge that pawnbroking and money-lending are highly remunerative to the principals, and highly unpopular with the customers, no doubt from the point of view that there is something immoral in enriching oneself out of other people's misfortunes. In this same connection we might also work out the actual rate of interest charged in some of the less reputable hire systems and deferred-payment purchases. Yet here, again, caution is necessary, for this convenient method of payment is recognized by firms of the highest repute and has always been the basis of the ordinary "building society" transaction. Hence, we must distinguish between the use and abuse of the system. It is a system highly convenient to wage-earners and salaried people, but has been unscrupulously exploited among the poor and the ignorant who have been tempted to buy furniture, etc., which they did not need, at prices the enormity of which they did not realize. But here again we are apt to confuse the issue. We cannot hope by arithmetical examples alone to inculcate thrift and prudence, nor to curb the desire for possession which is innate in nearly every human being.

The child may work out the actual rate per cent per annum of some such apparent simplicity as " 1d. in the 1s. per month," but whether he is impressed or frightened by the actual 100 per cent per annum we do not know. He might even secretly determine to *lend* a shilling on these advantageous terms. There is indeed something elusive about all such calculations, for in most of such transactions the actual financial needs *of the moment* outweigh all other considerations. Similarly, we may introduce our pupils to the legal charges of the pawnbroker, but here again there is not scope for more than a few incidental examples. Nor, in the time of personal economic stress which might necessitate a discreet visit to one of these establishments, should we be concerned with the rate of charge. Our most intense anxiety would be the " bird in hand."

We must then regretfully return from these highly interesting by-ways of Interest to the more serious work of calculating the interest on a given sum of money, at a given rate per cent, though even here we cannot refrain from one more shock to those who consider it pedagogically important, with the remark that those whose daily business concerns this subject of interest never trust their own calculation—*they use a ready reckoner* or *interest table*.

The Calculation of Simple Interest. The controversy still rages round the respective merits of Unity and Formula. We are still asked—those of us whose work takes us daily into different schools—which method we prefer. Our answer: that we use neither Unity nor Formula if we can help it, is not always popular. Our examples, however, as worked below must justify our contention.

Example. Find the interest on £116 6s. 8d. for $4\frac{1}{2}$ years at $2\frac{1}{2}$ per cent.

(*a*) *Unity*—

Preliminary Calculation : $£4\frac{1}{2} \times 2\frac{1}{2} = £11\frac{1}{4}$.

Interest on £100 for given time $= £11\frac{1}{4}$

" " £1 " " $= £\frac{11\frac{1}{4}}{100}$

" " £116 6s. 8d. " " $= £\frac{11\frac{1}{4} \times 116\frac{1}{3}}{100}$

$$= £\frac{45 \times 349}{4 \times 3 \times 100}$$

$$= £\frac{1{,}047}{80}$$

$$= £13 \quad 1s. \quad 9d.$$

(*b*) *Formula*—

$$I = £\frac{P \times T \times R}{100}$$

$$= £\frac{116\frac{1}{3} \times 4\frac{1}{2} \times 2\frac{1}{2}}{100}$$

$$= £\frac{349}{\cancel{3}} \times \frac{\overset{3}{\cancel{9}}}{2} \times \frac{\cancel{5}}{2} \times \frac{1}{\underset{20}{\cancel{100}}}$$

$$= £\frac{1{,}047}{86}$$

$$= £13 \quad 1s. \quad 9d.$$

(*c*) *Practice method*—

Total Interest $= 11\frac{1}{4}\% \ (2\frac{1}{2} \times 4\frac{1}{2})$

	£	*s*	*d.*
100% =	116	6	8
10% = $\frac{1}{10}$	11	12	8
$1\frac{1}{4}\%$ = $\frac{1}{8}$	1	9	1
	£13	1	9

This method is simple, obvious, and easy. It can be applied to all ordinary examples requiring the calculation of interest or amount.

A few more examples are worked to illustrate the method.

Example. Find the interest on (or amount of) £413 13s. for 5 years at $3\frac{1}{3}\%$.

$$3\tfrac{1}{3}\% \times 5 = 16\tfrac{2}{3}\%$$

Percentage to be found $= 16\frac{2}{3}\%$

Here we will work both in £ s. d. and in decimals.

(a)

	£	*s.*	*d.*
100% =	413	13	0
$10\% = \frac{1}{10}$	41	7	3·6
$5\% = \frac{1}{2}$	20	18	7·8
$1\frac{2}{3}\% = \frac{1}{3}$	6	17	10·6
	£482	11	10

(b)

	£
	413·65
$10\% = \frac{1}{10}$	41·365
$5\% = \frac{1}{2}$	20·6825
$1\frac{2}{3}\% = \frac{1}{3}$	6·894166
	482·5916
	£482 11s. 10d.

Example. Find to the nearest penny the interest on £1,953 for $1\frac{1}{3}$ years at $2\frac{3}{4}\%$.

$$1\tfrac{1}{3} \times 2\tfrac{3}{4} = 3\tfrac{2}{3}.$$

Required percentage is therefore $3\frac{2}{3}\%$.

	£	*s.*	*d.*
100%	1,953	–	–
10%	195	6	–
$3\frac{1}{3}\%$	65	2	0
$\frac{1}{3}\%$	6	10	2·4
	£71	12	2·4

Teachers are earnestly advised to adopt this method of practice in all straightforward calculations of interest or amounts.

It may be objected that the method cannot be readily applied to *times* including odd numbers of days. To them we would reply that all such can be reduced to simple division by 73, and that this again can be performed to a surprising degree of approximate accuracy by the well-known "third, tenth, and tenth" rule which is but another example of a useful practice method. If any reader should possibly be unfamiliar with this old rule he will find it explained in any ordinary complete text-book of arithmetic.

But some teachers will persist in pursuing the subject in all its implications, and for these we will discuss the other types. Since we have such varied terms as interest, principal, amount, rate, time, we may clearly have at least five different varieties of examples, requiring the finding of any one of these. The older generation of arithmeticians, in their eagerness for "rules," would have us learn a new rule for each type, but teachers as a whole are more and more inclined to treat these by common-sense methods or by formula (if they teach them at all).

It may be said at once that all such examples as: "What principal will amount to, etc."; "at what rate per cent, etc."; "in what time, etc.," have a mathematical rather than a practical everyday value and may well be omitted in the average school curriculum.

The Formula for Finding Simple Interest. Teachers who love mathematical completeness and are partial to precise, rigid, never-failing methods are very fond of the "formula" method. This may be readily established from definition as—

$$I = \frac{P \times T \times R}{100} \quad . \quad . \quad . \quad . \quad . \quad \text{i}$$

From this, by the process known to modern teachers of Algebra as "changing the subject of a formula," we deduce by simplest algebra three others—

$$P = \frac{I}{T \times R} \times 100 \quad . \quad . \quad . \quad . \quad . \quad \text{ii}$$

$$T = \frac{I}{P \times R} \times 100 \quad . \quad . \quad . \quad . \quad . \quad \text{iii}$$

$$R = \frac{I}{P \times T} \times 100 \quad . \quad . \quad . \quad . \quad . \quad \text{iv}$$

Armed with these four formulae the pupil can deal easily with all ordinary types of interest sums. The only one to which the formula does not apply directly is the type : "What sum of money will amount to, etc." This is best dealt with by ordinary proportionate methods on the basis of £100.

But for the average child the Formula method is most mechanical. It is applied without intelligence and gives accordingly an answer uninteresting and mostly unintelligible to the pupil. To those who would argue that it is mathematical we would reply that it is distinctly a method for older scholars and adults. It is not suitable for beginners.

Additional Methods. The following methods are occasionally used.

(*a*) Decimal Method (as for C.I.).

Example. Find the simple interest on £215 for 1 year at $3\frac{3}{4}$%.

£	215	
	6·45	(Multiplying by 3 and moving two places [to right.)
	1·6125	($\frac{1}{4}$ of previous line.)
	8·0625	

Ans. £8 1s. 3d.

(*b*) All-decimal method based on the interest on £1.

Example. Find the interest on (or amount of) £232 10s. for 3 years at $4\frac{1}{2}$%.

$$4\tfrac{1}{2}\% \times 3 = 13\tfrac{1}{2}\%$$

∴ £1 becomes £1·135.

Hence £232·5 becomes £(232·5) × (1·135) or gains £(232·5) × (·135) interest.

The necessary multiplication of two decimals is now performed by the ordinary method.

Compound Interest.

"Amongst the branches of arithmetic which by general consent are being laid aside may be mentioned 'stocks,' especially 'stocks with brokerage,' 'foreign exchange,' 'true discount,' 'partnerships,' and all compound interest except that simple form of it required in connection with savings bank deposits." (Circular 807, "Suggestions.")

This may be interpreted as an invitation to teach the principle underlying compound interest by means of simple examples.

In the previous chapter we have prepared the way for this subject in our discussion of compound increase or decrease per cent. All we have to do is to extend the principle to money and to work out a few representative examples. The actual meaning of compound interest is best conveyed to beginners by means of simple examples worked on common-sense lines.

Where the number of years are few and the percentage simple, it is usual to work by decimals or by a combination of decimals and practice.

Thus: "Find the C.I. on £430 for 3 years at 5%."

	£	
	430	. . First Principal
$5\% = \frac{1}{20}$	21·5	
	451·5	. . Second Principal
	22·575	
	474·075	. . Third Principal
	23·70375	
	497·77875	. . End of Three Years
	430	. . First Principal
Interest =	£67·77875	= £67 15s. 7d. approx.

Example (2). Find the C.I. on £825 for 2 years at $2\frac{3}{4}\%$.

	£	
	825	First Principal
$2\frac{1}{2}\% = \frac{1}{40}$	20·625	
$\frac{1}{4}\% = \frac{1}{10}$	2·0625	
	847·6875	Second Principal (end of one year)
	21·19218	
	2·11921	
	870·99890	End of Two Years
	825	
Interest =	45·9989	
=	£46 nearly.	

A few examples worked on some such elementary methods as those given above will probably suffice for beginners.

Older children, especially those who are studying simple geometric progression, may well look upon compound interest from this point of view and so evolve for themselves the ordinary formula.

It is well to derive this formula as inductively as possible, starting with some simple percentage, e.g. 5 and an original amount of £1. Successive increases may then be written down: $(1{\cdot}05)$, $(1{\cdot}05)^2$, $(1{\cdot}05)^3$, $(1{\cdot}05)^4$, etc.; hence, principal or amount after n years in this case will be $(1{\cdot}05)^n$ and the corresponding interest will be $(1{\cdot}05)^n - 1$.

From discussion of a few of these we may deduce the ordinary formula for any example. Good practice in algebraic and logarithmic calculation may then be obtained in finding either interest, principal, amount, time or rate.

With simple cases the formula method is not more rapid than the more elementary method, but is very useful in "inverse" problems, such as: "What principal will amount to, etc., etc., at C.I.?" These do not yield readily to elementary arithmetical methods.

Yet we cannot thus shortly dismiss this important subject for what is known as the "Compound Interest Law" is of such wide significance and application that it merits a little more definite attention. For the sake of the teachers of older boys in Central Schools and Evening Schools (of a more definitely technical character) we indicate briefly the simple practical treatment of this law.

The Compound Interest Law. The amount of interest due on a compound basis is clearly dependent upon the amount after the last payment. (We are assuming equal intervals of payment, e.g. 1 year.) Thus the interest each year increases in regular geometrical progression. The simplest case will illustrate.

Suppose 5% Compound Interest.

Principals	100	105	110·25	115·7625	etc.
Interests	5	5·25	5·5125	5·788125	etc.

Both of these represent geometric series whose constant ratio is $1\frac{1}{20}$.

But we may extend this payment of interest in our imagination to payment every half year, then every month, then every day, then every *instant*. Finally, we shall reach the fundamental law of increase which covers so many actual cases in nature.

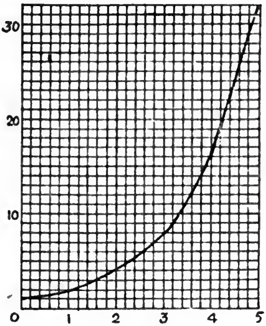


Fig. 125.

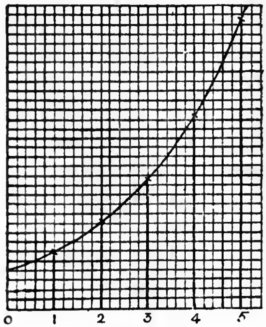


Fig. 126.

Since the quantity is now changing every *instant* we are clearly dealing with a *continuous* quantity changing *continually* (not in periodic jumps), and changing in such a way that its rate of change at any instant is proportional to the quantity *at that instant.*

This is a difficult conception for beginners. Graphical work may make it plainer.

Consider a very simple example—

$$y = 2^x$$

Corresponding values are—

x	0	1	2	3	4	5	etc.
y	0	2	4	8	16	32	etc.

These may be graphed and the resulting curve examined

(Fig. 125). It will be noted that the successive ordinates form a geometrical progression.

Clearly any compound interest series will give a curve of similar properties. Thus we might plot 1, 1·05, $(1{\cdot}05)^2$ $(1{\cdot}05)^3$ $(1{\cdot}05)^4$ . . . and construct a curve to show graphically the interest due at these equal intervals.

We may approach this curve by an even simpler method by marking a series of equidistant ordinates,

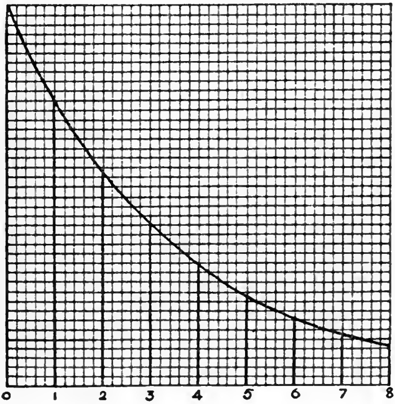


FIG. 127.

each of which is a definite multiple or fraction of the preceding one.

Thus in our example (*a*) (Fig. 126), ordinate (1) is $1\frac{1}{2}$ times ordinate (0), ordinate (2) is $1\frac{1}{2}$ times ordinate (1), etc.

Similarly in example (*b*) (Fig. 127), ordinate (1) is $\frac{3}{4}$ of ordinate (0); ordinate (2) is $\frac{3}{4}$ of ordinate (1), etc.

These and similar examples will give the general properties of the curve.

Finally with the help of tables the pupil may plot the general case $y = e^x$ and after this $y = e^{-x}$. He will find that $y = e^x$ and $y = e^{-x}$ are symmetrical about the axis of y. (In case of difficulty he may plot on the same axis $y = 2^x$ and $y = 2^{-x}$ or $y = (\frac{1}{2})^x$.)

Having plotted both on the same axes we may even be introduced to the simplest catenary (plotted by halving the sum of corresponding ordinates). All this is possible with a class of youths having a minimum of mathematical knowledge.

Investments. The subject of interest which we have just discussed at length can never be divorced entirely from the world of finance. It is essentially a vital part of the whole subject of investment, which to-day in some form or other figures in every person's life. It follows then that it is our duty as teachers to give our pupils some insight into this complex financial world which, in unseen yet powerful ways, controls us all.

Yet in the past *stocks* were merely another type of sum with new tricks to be learnt, and seldom was there any attempt to connect the subject with everyday life. We do not wonder then that in Elementary Schools "amongst the branches of arithmetic which by general consent are being laid aside may be mentioned 'stocks' especially 'stocks with brokerage'" (Circular 807, "Suggestions"), for we recall the stilted and artificial types of sums that we waded through under these headings. From the older text-books it would appear that every financial transaction was based upon the "£100 share," especially "Consols," and that brokerage was always *one-eighth per cent.* On these assumptions we bought and sold stock, calculating our gain and loss; we estimated our percentage returns; our differences of incomes, etc., by more or less mechanical juggling of figures remote from any references to practical life.

To-day if we would make the subject live for our pupils it is necessary to remember that *consols* and *railways* are not the only forms of investment; that the £1 share, the £5 share, the £10 share figure much more frequently in the investments of ordinary people than does the £100 share; and that the principle of shares and companies has spread throughout the industrial and commercial world.

Many investments will have been noted in dealing with interest such as the Post Office Savings Bank, the more modern War Bonds and War Savings Certificates, shares in co-operative societies, shares in municipal undertakings, corporation loans, etc. All that is necessary is to develop the idea of financial co-operation and the fluctuation in the value of "shares." An excellent lesson may be given in a senior class by the formation of a mock company with imitation share certificates. With regard to actual arithmetical examples, it is well to begin with the buying and selling of some simple shares, e.g. Gas Company £10 shares, etc. No rigid rules need be taught. Each question can be treated by ordinary common-sense, frequently by practice methods. Thus the example: "Find the cost of 25 £10 shares in a gas company at £8 11s. 6d. each, does not require any different working from the example: "Find the cost of 25 *pigs* at £8 11s. 6d. each." Following such examples involving the buying and selling of shares, may come examples involving the calculation of the income or dividends from shares. Here, again, methods already taught such as simple multiplication or practice will suffice; and later the hardest and perhaps the most important examples will deal with actual percentage returns. Thus a child usually finds it difficult at first to realize that a dividend of 15 per cent on shares is not always as much as it looks, since, if the shares stand at three times their face value, i.e. if a £1 share costs £3, the actual return is only 5 per cent. Beyond this stage we do not think it necessary to take the child. Intricate questions in the transference of big investments and the consequent change of income seem out of place with pupils who are hardly of an age to grasp the fundamental idea.

Some teachers have discarded the whole subject of stocks from their arithmetical syllabuses. We feel that this is perhaps hardly fair to the average child, who has a right to a "general knowledge" acquaintance with the subject of investment; yet undoubtedly the more intricate ramifications of this difficult subject may well be left to the more commercial type of schools for older pupils.

CHAPTER XXIII

ALGEBRAIC METHODS IN ARITHMETIC

THE possibilities of algebraic methods in arithmetic are still largely unrealized by teachers, and consequently, to a great extent, this promising field is still unexplored. Here and there teachers are found who have made timid and tentative efforts to introduce the methods and 'shorthand' of algebra into the arithmetic of their pupils, but on the whole the subject is feared and shunned.

The reason for the neglected position of this most important mathematical tool is fairly clear. Algebra to the ordinary teacher is a *subject* rather than a *method*. Its beginnings (for most of them) consisted in the well-worn pathway of humdrum hypothetical substitutions of the type "if $a = 1$, $b = 2$, $c = 3$, find the value of, etc.," followed by hosts of mechanical manipulations in the four rules and in L.C.M., H.C.F., brackets, fractions, and factors—all work which at the time seemed aimless, dull, stupid, and consequently unattractive. Teachers with memories of courses such as these rightly hesitate to use valuable school-time in wading through the shallows and deeps of such purposeless exercises. Though this is perhaps the real reason at the back of the teacher's mind, he is not always keen, in his professional pride, to admit as much, for this would apparently decry much of his own superior learning. Consequently he seeks refuge in other excuses and apologies. Algebra, he will insist, is too hard for the average scholar even of the higher classes in the elementary school, especially where scholarships have skimmed the cream of the youthful intellects and have left him but very ordinary material in the remaining scholars.

It is to teachers of this kind that we address this chapter, and first, we will try to correct their fundamental misconception. It is not algebra in the traditional sense which we would ask them to attempt; for we do not wish them to enter upon the fruitless and impossible task of teaching the manipulation and analysis of abstract generalities to classes of beginners. What we ask is that they shall attempt the much humbler task of introducing certain of the simpler algebraic methods and transformations, into their ordinary *arithmetic*. The rigid "purist" would perhaps refuse to call this constant reference to numbers and quantities—(nearly always *positive* quantities)—algebra. As in his pure geometry, so in his algebra, he has no place for units, numbers and concrete quantities. But the fact remains that in this *literal* arithmetic we have an additional mathematical weapon of the highest value for every kind of school.

One of the most important books of recent years on the subject of mathematical teaching is *The Teaching of Algebra*, by Prof. T. P. Nunn, in connection with which the author has also published *Algebraic Exercises*, Part I and Part II. No teacher of older scholars, whether in elementary or secondary schools, should be without these remarkable books. Here they will find all the mechanical useless lumber of *introductory algebra* ruthlessly cast aside. They will look in vain for the time-honoured exercises, but in their place they will find novel, sensible, *purposeful* exercises and a new treatment of old methods and topics inspiring in its freshness and vigour. Much of what is here suggested is due to the contents of Prof. Nunn's books and such will be obvious to all who have read them. The present writer's debt is a large one, and is fully and gratefully acknowledged.

Making Formulae. It is well to introduce algebraic methods, not with a pompous, noisy, pedantic flourish, but quietly, unobtrusively, incidentally, as opportunity serves. A beginning may be made in what is termed *generalized* arithmetic. To take a simple case: we fell that our teaching of area has not succeeded if we have not reached the general statement *in words* that

$$\text{Area} = \text{Length} \times \text{Breadth}.$$

The harder *algebraic* step is to express this in the *literal* form

$$A = l.b. \text{ (or } l \times b, \text{ or simply } lb).$$

This should be our line of progress: through *verbal* statements to *symbolical* statements: through *words* to *letters-and-symbols*.

Incidentally, too, we may thus teach the simple notation of *shorthand* of algebra. The symbols of operation $+$, $-$, $\times$, $\div$ will be familiar from ordinary arithmetic. Very little extra is required except that $\times$ is still further shortened so that $l \times b$ may be written as $l.b.$ or lb. Similarly $a \times a$ might be written aa but is shortened still more to the index form a^2. Though brackets with their intricacies of sign-changing may easily be avoided, and may well be omitted, we can hardly dispense with a simple bracket accompanied by a multiplying factor such as $2\,(l + b)$, which should be recognizable by scholars as the perimeter of a rectangle where l and b represent the length and breadth. Nor is this multiplication notation really new, since teachers for decades have used in mensuration such forms as: circumference $= 2\pi r$, and area $= \pi r^2$, without stopping to explain in detail that $2\pi r$ means $2 \times \pi \times r$ or that πr^2 means $\pi \times r \times r$. The extension of this algebraic shorthand to all simple mensuration cases is desirable and easy. Thus we should require the statement, interpretation and application of such simple forms as—

$$A = \frac{bh}{2} \text{ (Triangle)}$$

$$A = b\left(\frac{h_1 + h_2}{2}\right) \text{ (Right-angled trapezium)}$$

$$A = 2\pi\left(\frac{r_1 + r_2}{2}\right) t. \quad \text{(Circular ring)}$$

and numerous others.

In every case the pupil should be encouraged to express his general statement (in words) in the more convenient symbol or literal form.

To what extent this same symbolic form will be used for expressing other general rules will depend upon the age of the pupil. Yet the extension from the mensuration rules to other rules should be made wherever possible. For example, they may express totals in various ways, i.e.—

C (Total Cost) $= np$ (n = number; p = price of each)
M (Total Miles) $= mh$ (m = miles per hr.; h = number of hours)

Similarly, all reduction steps lend themselves readily to symbolism, e.g.—

$$
\begin{aligned}
p \text{ pounds} &= 20p \text{ shillings} \\
s \text{ shillings} &= \frac{s}{20} \text{ pounds} \\
y \text{ yards} &= 3y \text{ feet} \\
&= 36y \text{ inches} \\
c \text{ cwt.} &= 4c \text{ quarters} \\
&= 112c \text{ lbs.}
\end{aligned}
$$

etc., etc., etc.

Volumes extend the practice to more complex work such as—

$$
\begin{aligned}
V &= lbh \text{ (rectangular)} \\
V &= lb\left(\frac{h_1 + h_2}{2}\right) \text{ (lean-to shed, etc.)} \\
V &= \tfrac{4}{3}\pi r^3 \text{ (sphere)} \\
V &= Ah \text{ (prism, cylinder)} \\
V &= \tfrac{1}{3} Ah \text{ (pyramid, cone)}
\end{aligned}
$$

In many schools simple interest is taught on a basis of the formula $I = \dfrac{PTR}{100}$.

But these are all well known. What we would insist on is that pupils be accustomed to the *making* of formulae and not merely to the application of formulae ready-made. Thus they may be asked to find a formula for the cost of a telegram under Post Office Regulations (1s. for 12 words or less; ½d. for each additional word).

We might work inductively thus, tabulating each step—

Number of Words over 12.	*Cost in pence.*
0	12
1	$12 + \frac{1}{2}$
2	$12 + 2\cdot\frac{1}{2}$
3	$12 + 3\cdot\frac{1}{2}$
n	$12 + \frac{n}{2}$

whence required formula is—

$$c \text{ (cost)} = 12 + \frac{n}{2}$$

where c is the cost in pence and n the number of words over 12.

Similarly, we may set our pupils to discover the symbolic statement of such simple domestic rules as: "One for each person and one for the pot" (in making tea), "Allow $\frac{1}{4}$ hour for each lb. and 20 mins. over" (cooking meat).

This making of formulae is algebraic work of the most valuable kind. It is a true application of the educational maxims: "Proceed from particulars to generals," or "from concrete to abstract." In every case it is possible to proceed inductively by the examination of a series of numerical instances arranged in order. With practice the pupil may be able to give the general statement from the examination of a single numerical instance.

Harder exercises will involve pure symbolism without reference to numerical examples. The boy who can state the total number of words in a book having p pages, l lines per page, and an average of w words per line, as plw without having recourse to *number* instances has progressed a fair way. Similarly we can hardly expect beginners to state rapidly that the cost of x articles (where y are known to cost z) is $\frac{xz}{y}$.

Changing the Subject. Once the general statement, expressed in literal form, is constructed and understood, important applications follow. Particularly important is the cultivating of ready ability to *change the subject* of a formula. This phrase *changing the subject* is due to Professor T. P. Nunn, and aptly expresses a very old difficulty known to all teachers of arithmetic and algebra.

A simple case will show all that is implied. For a triangle—

$$A = \tfrac{1}{2}bh \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From this we may derive two others—

$$h = \frac{2A}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$b = \frac{2A}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In (*b*) we have "changed the subject" of the formula, i.e. the left-hand member, from A to h, and similarly in (*c*) the "subject" has become b.

Practice in this transformation of formulae is essential for rapid numerical calculation. Examples are endless —every formula of necessity provides suitable practice. Yet all too often a neglect of this kind of work leads to stereotyped work and types of sums mechanically taught. Thus many schools teach methodically and carefully

$$I = \frac{PTR}{100}$$

but neglect the three derived formulae having as subjects P, t, r, i.e.—

$$P = \frac{100\ I}{TR},\ T = \frac{100\ I}{PR},\ R = \frac{100\ I}{PT}$$

Similarly, formulae relating to the circle should be known in all their common forms, such as: $C = \pi d = 2\pi r$ whence $d = \dfrac{C}{\pi}$ and $r = \dfrac{C}{2\pi}$; $A = \pi r^2 = \dfrac{\pi d^2}{4}$ whence $d = \sqrt{\dfrac{4A}{\pi}}$ and $r = \sqrt{\dfrac{A}{\pi}}$.

The work however need not be confined to ordinary mensuration formulae. The range of examples is infinite, and we would refer every teacher to Prof. Nunn's *Algebraic Exercises*,[1] Part I, Section I.

Using Formulae. Concurrently with the making of formulae the beginner should have ample practice in *using* formulae. This does not mean that the work need be confined to the mechanical substitution of numerical data for one particular case included in a general formula.

The nature of functional dependence cannot be grasped from a single instance. What is required is the orderly evaluation of the formula according to a series of values of the variable. Thus in the simple formula, $s = \frac{1}{2}gt^2$, or $s = 16{\cdot}t^2$, which represents the distance travelled by a body falling freely from rest, we may evaluate the distance corresponding to values of $t = 1, 2, 3, 4, 5$, etc. This type of work forms a natural start-point for smooth-curved graphs and illustrates in the best possible way the idea of one quantity varying with another. If the real nature of this has been grasped, a definite mathematical advance has been made. It is by this definite numerical treatment that the implication of a formula is best revealed to a beginner. Numerous useful rules may be explored in this way. Thus the well-known rule for determining the approximate distance of the visible horizon at sea is $D^2 = \frac{3}{2}h$, where D is the distance in miles and h is the height in ft. of the observer above the sea level. This is also expressed as $D = 1{\cdot}22\sqrt{h}$.

Here, again, we may ask beginners to tabulate a series of corresponding values of h and D, and to draw a graph sufficiently smooth to enable intermediate values to be read off by what is termed *interpolation*.

Identities. The use of algebraic methods in numerical calculations is nowhere better seen than in the application of some of the simpler identities. There is at present in

[1] Published by Longmans, 1s. 6d. net.

elementary schools a tendency to restrict this work to well-known forms such as—

$$a^2 - b^2 = (a + b)(a - b)$$
$$(a + b)^2 = a^2 + 2ab + b^2$$

It has already been shown in this book how formulae such as these may be demonstrated graphically by simple areas. They may also, as algebraic statements, be verified and explored numerically. But much more may be accomplished in the direction of encouraging neat algebraic methods of calculation and arrangement of figures. This is the place and perhaps the only place of *factors* and *brackets* in the mathematics of the primary school. Every child should be encouraged to make use of such identities as $a (b \pm c \pm d) = ab \pm ac \pm ad$, so that in a calculation such as $59 \times 12 + 33 \times 12$, he works $(59 + 33)\ 12$, thus performing one multiplication instead of two. The saving of labour in manipulation is even more noticeable in the case of differences such as $21 \times 12 - 33 \times 6$, which may be thrown into the easy form $6\ (42 - 33)$ or 6×9.

This should be the keynote of our work in factors and brackets. They should be used wherever possible for throwing calculations into easier form for calculation.

Miscellaneous work in area gives ample opportunities for algebraic arrangement and methods. In determining an area such as that in the diagram (Fig. 128), we clearly have to compute (1) ad, (2) bd, (3) two pieces cd, and (4) a centre piece d^2.

Expressing the total as $ad + bd + 2cd + d^2$ before calculating we throw the total into the form

$$d\ (a + b + 2c + d).$$

If only we could persuade our scholars to arrange every calculation in the most convenient form for computation *before beginning* to work, the saving of time, ink, and paper would be very great, and the gain in accuracy would be equally valuable. In involved calculations it is safe to say that the great majority of mistakes arise from bad arrangement rather than from incorrect manipulation of numbers. We adapt one more example from Prof. Nunn's *Exercises* to show the advantage of arrangement. The four pointed star is inscribed in a square of 12-inch side, and we require to calculate its area (Fig. 129).

Analysis indicates the method, viz., to deduct from

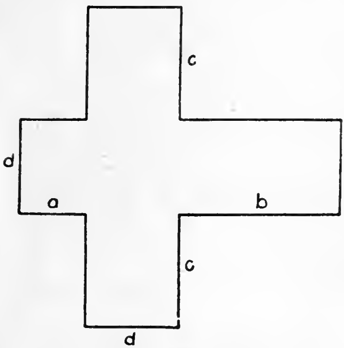


FIG. 128.

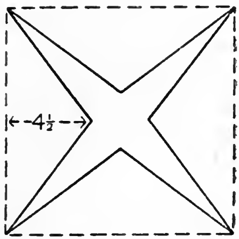


FIG. 129.

the whole area the sum of the areas of four equal triangles each $\frac{12 \times 4\frac{1}{2}}{2}$. Before calculating, however, we proceed a little further with our analysis and note that what we require to evaluate is—

$$12^2 - \frac{4 \cdot 12 \cdot 4\frac{1}{2}}{2}$$
$$\text{or } 12^2 - 2 \cdot 12 \cdot 4\frac{1}{2}$$
$$\text{or } 12\ (12 - 9)$$
$$\text{or } 12 \times 3 = 36$$

The gain by using algebraic methods is obvious.

We conclude this section with the following advice to teachers. Use simple factors and brackets wherever possible in calculations, and insist on every calculation being arranged in the most convenient form before any working is allowed. In this way you will develop valuable mathematical power and at the same time gain vastly in orderliness and accuracy.

Simple Equations. Finally, " when the initial difficulty of using letters to represent numbers has been overcome, many problems can be simplified by using the methods of the simple equation." Children generally love solving a simple equation but are not always so willing or able to *form* the equation before solving. The so-called " rules " of an equation are capable of being deduced from one broad general statement. " Treat both sides in the same way." Here the analogy of the " Balance " is useful. The equality or balance of the equation is undisturbed if we perform identical operations on *both sides* of the equation. The following example worked in full shows all common operations and the ready method of demonstrating them—

$$4x - 3 = \tfrac{3}{2}x + 7$$

Solution in full—

(*a*) Double both sides,

$8x - 6 = 3x + 14$

(*b*) Add 6 to both sides,

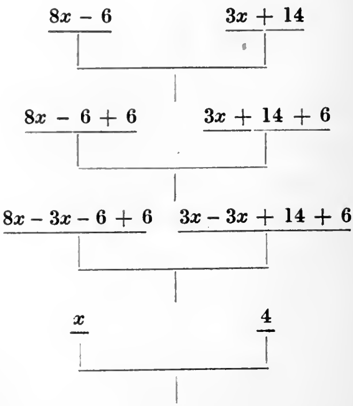
$8x = 3x + 20.$

(*c*) Subtract $3x$ from both sides,

$5x = 20.$

(*d*) Divide both sides by 5,

$x = 4.$



It should be noted, too, that the order in which the various operations are performed is immaterial to the correctness of the answer.

Numerous simple number puzzles which will give plenty of practice in the solution of equations may be framed by any teacher. We may even encourage children to invent examples for one another. But the importance of the subject goes far beyond mere mathematical amusement. It represents a method to be applied wherever possible to aid our numerical calculations. We indicate a few of the ordinary numerical problems where the methods of the simple equation may usefully be employed.

Unequal Division. Example. Divide 1s. between two boys so that one has 3d. more than the other.

Shares : p pence and $p + 3$ pence $\begin{pmatrix}\text{or } p - 3 \\ p\end{pmatrix}$

Equation : $2p + 3 = 12$ (or $2p - 3 = 12$)

Proportionate Division. Divide one guinea between two people in the proportion 3 : 4.

Shares s shillings
and $\frac{3}{4}\,s$,,
Equation $s + \frac{3}{4}s = 21$
or $7s. = 84$

"*Work, Pipes,*" *etc.* Two pipes fill a cistern—one in 10 minutes, the other in 12 minutes. How long will they take together ?

Suppose m minutes to be the time,

$$\text{Then } \frac{m}{10} + \frac{m}{12} = 1 \text{ whence } m = 5\tfrac{5}{11} \text{ mins.}$$

(The actual reasoning here is perhaps beyond beginners.)

"*Clocks.*"—We have already indicated the algebraic method of dealing with these traditional examples. We add one more.

At what time between 4 and 5 are the two hands of a clock in a straight line ?

Let t be the time,

$$\text{Then } t - 20 - \frac{t}{12} = 30$$

$$\text{or } \frac{11}{12}t = 50$$

A diagram will make this clear.

Finally, we should note that the process already discussed as "changing the subject of a formula" is based entirely upon the rules controlling the manipulation of the simple equation, for every formula is of course an equation between interdependent quantities, any one of which is expressible in terms of the other.

In concluding this chapter on the possibilities of algebra in the elementary school, we will summarize our advice to teachers in as concise a form as possible.

(*a*) Before attempting the subject read carefully some modern text-books and "method" books.

(*b*) Teach algebraic *methods* as applied to arithmetic and not the elementary algebra of a generation ago.

(*c*) Proceed through simple formulae to simple equations.

(*d*) Remember that the *construction* of formulae is as important as their *use*.

(*e*) Teach brackets only with a view to shortening calculation and insist that no arithmetical calculation shall be performed unless it is first expressed in its simplest algebraical form.

CHAPTER XXIV

GRAPHS IN ARITHMETIC

THE concluding paragraph of Circular 807 issued by the Board of Education, "Suggestions for the Teaching of Arithmetic," for teachers in Elementary Schools, is brief enough to be quoted in full—

> "Of late years attention has been given to the drawing of 'graphs.' These geometrical illustrations are of two kinds. In the first place there are the charts which form a convenient means of recording quantities which fluctuate in an irregular way, e.g. the attendance of the school or the attendance of the class-room. In the second place, there are the curves which express in a diagrammatical form the relation between two connected quantities. Thus it is easy to construct a diagram which may be used as a ready-reckoner, e.g. one which will give the square or square root of any number within a given range or the relation between the side and area of similar triangles, or the relation between 'cost price' and 'selling price' for any particular percentage of profit. The interpretation and use of such curves is as important as their construction. Both kinds of graphs may from time to time be drawn by children in the upper part of an Elementary School.
>
> "It is suggested that in dealing with graph-drawing the Elementary School teacher should refrain from all attempts to give ideas as to the 'equation' of a curve or to teach analytical geometry in some disguised form."

All teachers whether of elementary or secondary schools will find a much fuller treatment of the subject in Circular 884, issued by the Board of Education, on *The Place and Use of Graphs in Mathematical Teaching*, one of the *Memoranda on Teaching and Organization in Secondary Schools.*

Many teachers will recall the appearance of "Graphs" in mathematical syllabuses and examination papers. Older teachers are still to be found who resent the intrusion of this "new fad," and would prefer the cut-and-dried manipulations of symbols which filled the older text-books of algebra. Still more out of place did the subject of graphs seem in arithmetic where the object was always "sums," each sum presenting its own intrinsic difficulties and data isolated from all the rest. Converts to graphical methods and to the use of squared paper generally, were made but slowly, in spite of the valiant efforts of pioneers such as the late Professor J. Perry, whose *Six Lectures to Working Men*, published about twenty years ago, contained many valuable hints to teachers on graphical methods of working in mathematics and on the possibilities of squared paper. Publishers and text-book makers as usual were soon hot on the scent of this new "examination" subject. Their first efforts were not very successful. New editions of old text-books appeared with a few pages on *graphs* inserted here and there. The compilers could not escape from the baneful influence of the isolated example, problem, or sum, and a graph was usually presented not as an illustration of a function or of inter-dependence of changing quantities, but as one more means of finding an answer to some concocted example. Thus teachers and scholars laboured to *solve graphically* equations, problems, and sums that usually lent themselves to much more rapid and accurate solution by ordinary methods. It is not to be wondered that both teacher and pupil, proceeding on these lines, occasionally felt that much of the belauded graphical work in mathematics was largely a futile waste of time and effort.

The newer text-books present the subject in a truer perspective. The point of view is radically altered. To realize the full import of a graph and of graphic methods we must enter the realms of philosophy and take a broader and deeper view of the whole range of mathematical thought. No longer shall we in our mathematics amuse ourselves, after the fashion of the schoolmen of the middle ages, in propounding to one another ingenious little problems and puzzles for arithmetical or algebraic solution. Rather shall we view the universe as a unity amid all its diversity; where phenomena are analysed into their inter-dependent elements; where change is ceaseless and yet orderly; where rise and fall, increase and decrease, growth and decay eternally proceed. We may (and indeed we must under the limitations of our intellect) make mentally a cross-section of this never-ceasing stream of change and thus secure a sufficiency of data to concoct our sums, but this static view is largely an illusion for, even as we work our example, the reality it seeks to represent is changing; and the ultimate value of all mathematics is this: that in it we have man's attempt to reduce this eternal flux of things to order and law within the measure of his intelligence and the means of his expression. Only from this dynamical view of the universe can the meaning of function and functional variation be grasped, and this idea of functionality is the only sound foundation for the teaching of graphs.

Discontinuous and Continuous Quantities. It is, accordingly, with this ultimate object in view that all our graphical work should proceed even from its beginnings in the elementary school. The extract quoted at the opening of the chapter would appear at first sight to be too dogmatic in its division of graphs into "kinds," for if, in the view outlined above, every graph represents functional dependence, then all graphs are ultimately of the same kind.

Yet a definite distinction is possible between graphs that exhibit discrete or "discontinuous" quantities and graphs that show "continuous" quantities. From the latter, values other than those observed and tabulated may be deduced by the process of interpolation, while the former are not graphs in the narrower sense but merely convenient illustrations to exhibit certain facts for comparative purposes. But even in this division the difference is frequently more apparent than real, as will appear from a simple example.

It is customary in many schools to keep a chart or graph of classroom temperatures. If these temperatures are read, for example, at 11 a.m. each day and the results "graphed," we have an excellent example of the type of graph where no "interpolation" is possible, for it would clearly be absurd to attempt to deduce graphically the temperature at 11 p.m. on Thursday from the tabulated temperatures at 11 a.m. Yet the quantities thus graphed may be made more continuous by plotting temperatures read every *hour* instead of every *day*. With the temperatures at 11 a.m. and 12 a.m. for each of a number of days we might with comparative safety deduce from the curve

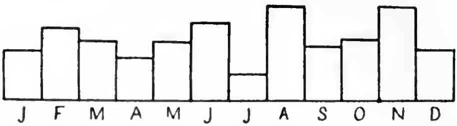


FIG. 130.

the probable temperature at 11.30. A barograph such as is provided by every self-recording barometer illustrates the extreme case of these irregular curves. No simple mathematical law may be found to be the basis of this curve, and yet we are able to read off the actual pressure as recorded at any moment.

Rainfall curves indicating rainfall month by month may, on the other hand, be taken as typical of the curve of discontinuous quantities, for interpolation is clearly impossible, the data from its nature cannot give, when graphically displayed, any more information than the original figures contained. For this reason the line graph for illustrating the fluctuations of such quantities is clearly of no use for numerical deductions. Consequently many mathematicians refuse to employ the line graph in such cases, preferring to use discontinuous areas as in the diagram (Fig. 130). The method has distinct advantages.

In all cases of discontinuous quantities thus displayed graphically, the graph, though further interpolation is impossible, nevertheless has its uses, for it forms a ready and striking means of comparison and of detecting maximum and minimum values; and for this reason alone children should receive considerable practice in thus displaying a series of quantities.

Introduction of Graphs. We shall, then, do well in our schools to begin with the graphic representation of statistics, whether fluctuating regularly or irregularly. Numerous examples are ready to hand. "Graphs" may be made of attendance, of marks obtained, of temperature readings, of barometer readings, of average monthly temperatures, of monthly rainfall. The work is simple. No negative quantities need be employed. The horizontal scale hardly exists; all that is necessary is that the vertical measurements shall be made at equal horizontal distances.

From these *discontinuous* graphs we may pass in easy stages to graphs of statistics that appear to follow some sort of law. Thus, if we plot the average height or weight of boys or girls at different ages we may reasonably deduce the average height or weight at intermediate ages, even though the curve is hardly regular in the mathematical sense or *smooth* as the modern books phrase it. Numerous simple cases can be given where interpolation is possible; we show graphically the times of sunrise or sunset on different days, and from the curve we may reasonably deduce the time for any day not specified. Every graph should be systematically explored from this point of view: Can we deduce *more* results than those upon which the graph is based? In this way the full implications of every graph may be revealed to the child, and the fact is again emphasized, in this connection, that the *use* of a graph is as important as its construction.

Regular Curve Graphs. Following these "statistical" and more or less irregular graphs we may proceed to the more regular curves. Numerous examples have already been given of straight-line graphs. We may require our scholars to construct and use ready reckoners for converting inches to centimetres, pints to litres, pounds to kilograms, miles to kilometres. The method of constructing these has already been indicated fully.

These are of very great use for quick approximate conversions. They should be carefully drawn on a large scale by each pupil and then be made the basis of numerous oral examples. All cases of proportion lend themselves readily to graphic representation especially simple or straight-line proportion. Thus we may easily exhibit graphically the line representing eggs at 5 a shilling, or distance travelled at 12 miles an hour. These graphs, however, are so obvious that children frequently see no use whatever in them, and prefer to calculate rather than to read off answers to sums set on these lines of simple proportion. So, too, we may draw straight-line graphs for showing interest (simple and compound) and time, or interest and rate, or interest and principal. Again we may show any given percentage or profit on various sums of money within the limits of the graph ; or we may show the relation between marked price and discount price. All these curves may be grouped as ready reckoners.

Akin to these are graphs showing uniform rates, such as occur in the time-honoured examples of work, pipes, races, trains, etc. To spend too much time on such is inadvisable. It is of course possible to make graphical solutions to sums of this type, but to do this is seldom necessary, for answers are more easily obtained by ordinary methods. Again, most of these examples set for graphical solution have a very artificially uniform basis. The type is well known : A and B start from places 30 miles apart, A at 4 m.p.h. and B at $4\frac{1}{2}$ m.p.h. When and where do they meet ? This according to the data may be exhibited neatly and graphically by the intersection of two straight-line graphs, but less artificial and more distinctly human would be the plotting from actual data of the distances walked by a man in successive hours. These will show generally that a natural motion is seldom uniform but tends to slacken gradually whether from fatigue or friction.

In general, it may be stated that to use graphs to solve isolated "sums" of the above type is to lose sight of their main import and of their chief benefits to scholars.

Examples of smooth curves other than straight lines, which may be drawn by all young scholars have been already noted, such as the inverse proportion curve or simple rectangular hyperbola, and the simple curve of squares, the parabola. Instances of the latter are common: Circle areas and their radii; areas of similar triangles and their bases; distance and time for falling bodies. The data for each curve may be calculated or supplied according to the nature of the problem. In every case the curve when drawn may be used for further deductions and numerous oral examples.

Graphical Introduction to the Calculus. It is now recognized that the natural introduction to the calculus and calculus methods is a graphic one. At once let it be said that we have no idea of asking teachers of younger scholars to introduce such a subject into their classes. We do not even desire them to introduce that term now so glibly used, viz., *gradient*. But we are sure that the idea of faster or slower rates of change corresponding to the greater or less "steepness" of graphs is readily grasped even by beginners, and teachers accordingly would do well to direct their pupils' attention continually to this fundamental and essential notion. Those teachers whose work lies mainly with older scholars will, therefore, develop the idea of rate-of-change of a varying quantity as expressible in terms of the steepness or gradient of the curve at any point which in turn is measured by the slope of the tangent to the curve at that point. With such preliminary work much of the initial difficulty of the meaning of a *differential coefficient* will disappear.

But with the plotting of "statistics," irregular curves, and smooth curves so far indicated we have by no means exhausted the possibilities of graphs even in the primary school.

Connection between Graph and Formula. It will be noticed that so far we have made no mention of "variables" x and y. This is intentional, for these symbols represent a degree of generalization to which primary children cannot be expected to attain, and to attempt to interpret or explain a curve in terms of an x y formula or, as is more usually the practice, to exhibit graphically the implications of some such formula, is work that may easily become mechanical and meaningless. Yet the connection between graphs and corresponding formulae is so fundamental that they cannot well be treated apart, though the formulae can usually be expressed in more striking symbolism than the x and y of Cartesian and algebraic geometry.

This connection between formula and graph may be viewed from two aspects. In the first place we look upon the formula, whether empirical or graphical, as the *end* of our graphical investigation and, on the other hand, we may accept any formula as data whose implications are best displayed through the medium of the graph. The first aspect is clearly seen in the plotting of experimental results, and in the attempt to detect from the resulting graph the underlying law. Exercises of this type are properly based upon actual practical work. Thus the pupil may collect for himself data in connection with the length of the shadow of a vertical pole at equal intervals of time throughout a day. Elementary physical experiments are pre-eminently the source of these results of experiment and observation. Thus we may tabulate and graph results for the pendulum; (length and time of swing); the extension of a rubber cord; (extension and weight attached); a body rolling down an incline; (distance) and time.

Numerous other instances will occur to the teacher of elementary physics, all of which may be made the basis of useful exercises in graphs. It is hardly to be expected that many students at first will be able to derive unaided a formula connecting a set of observed results of two connected quantities. Consequently we frequently set our pupils to verify some given law in an experiment but this verification is a lame evasion of the difficulty. In practical mathematics of recent years it has been fashionable to state the problem in the following form: "The series of values given are supposed to follow the law—(the law is then given). Test whether this is so and find the probable values of the constants." Actually we should set our pupils the harder task of finding the relation between s and t, or p and v, or any two connected quantities.

Most of the simpler laws being reducible graphically to the straight-line form, it follows that if this type of work is pursued the straight line and its equation $y = ax + b$, must be thoroughly investigated, though here again we are trespassing beyond the confines of the ordinary elementary school, for with younger children it is largely a waste of time to talk of the equation of a line or curve, especially in terms of x and y.

The second method of approach, which may be termed deductive, deserves some notice, though it may easily become mere mechanical and blind substitution. Here the problem is to investigate graphically all the implications of a formula. We may tell the pupil that the expenses of a household per week—partly constant and partly depending upon the number of people in the house—may be expressed in pounds by—

$$\frac{3}{4}n + 1 \text{ or } \frac{3n + 4}{4}$$

when n is the number of people, and ask him to exhibit this graphically. He will probably proceed by tabulation as follows—

n	1	2	3	4	5	
Total Cost	$1\frac{3}{4}$	$2\frac{1}{2}$	$3\frac{1}{4}$	4	$4\frac{3}{4}$	etc.

Plotting these on rectangular axes he will discover that such a formula has a straight-line graph. Other and more difficult formulae may similarly be investigated.

By this means the idea of a *function* as displayed in a series of numerical values will gradually form in the pupil's mind. That this idea of a function is still far from receiving just and ample treatment even with older scholars is shown by the author's experience with students in a training college. The following two questions are of a converse nature—

(*a*) Find the *n*th term of the series 3, 6, 9, 12, etc.

(*b*) Find the series whose *n*th term is $\frac{3n-2}{4}$.

Presenting these to students, the writer has found that though average students find little difficulty with (*a*) they are frequently at a loss with (*b*). The functional idea, i.e. that n may be *anything* in integers, seldom occurs to them, and they wrestle with formulae such as—

$$a + (n - 1)d = \frac{3n-2}{4}$$

when a simple substitution of $n = 1, 2, 3$, etc., would give them the required result.

Graphical Solution of Equations. It is in connection with this deductive and graphical treatment of formulae that a place may be found for the simpler cases of graphical solution of equations. Thus, if a pupil has shown graphically the curve of a function such as $n^2 - 3n$, he may easily be led to deduce from the graph values for n that will give $n^2 - 3n = 4$ or $n^2 - 3n = 0$. Similarly he may investigate simultaneous values for intersecting curves. All such work may easily be overdone, and it should be impressed that graphical solutions are seldom resorted to in practice except with equations that do not yield readily to the ordinary analytical methods of solution.

Maxima and Minima. More important, because of much wider practical application, than the graphical solution of equations is the graphical determination of simple cases of maxima and minima. All ordinary cases yield readily to graphic treatment. For instance, we may easily determine by a graph the greatest rectangular area that may be enclosed with 12 inches of wire or string for such an area must be expressible in the form

$$A = s\,(6 - s),$$

or $6s - s^2$ where s represents one side;

and since the limits of s are 0 and 6, we may show graphically A for values of s within this range, and thus determine at sight the maximum value.

Area Under a Graph. Finally in connection with graphs, regular or irregular, we must not omit to give ample practice in determining the area *under* a graph. This, another fundamental idea of the calculus, may well be treated in the first place by some such elementary rule as the mid-ordinate rule to which we have already alluded. Older and more proficient scholars may be introduced to the more accurate rules of Simpson. It is to be regretted that in elementary schools this useful work in what is really graphic integration should receive such scanty attention. If continuation education is to be *continuous* as well as simply *continued*, teachers must endeavour to bridge the mathematical gaps which too often exist between primary and other education, especially in respect of the concocted artificial examples which always "come out" in convenient whole numbers and the more practical cases which do not "come out" but require careful and patient approximate evaluation.

Graphical Tracing of Loci. Thus far in this chapter nearly all the graphs have been referred to two rectangular axes. There remains another form of graphic work that always has a fascination for beginners and that is the graphic tracing of loci under given conditions. This subject is usually treated as belonging to the realms of pure geometry, yet numerous curves lend themselves to a purely graphic if non-algebraic treatment. Most children at some time or other learn to describe an ellipse by means of pencil and string. The other curves such as parabola and hyperbola may similarly be drawn by the application of some simple property. By defining the parabola as the path of a point moving so that its distance from a fixed point is always equal to its distance from a fixed straight line, we may easily construct a parabola by using the ordinary ruled parallels of a sheet of foolscap paper.

Thus, in the diagram (Fig. 131)—

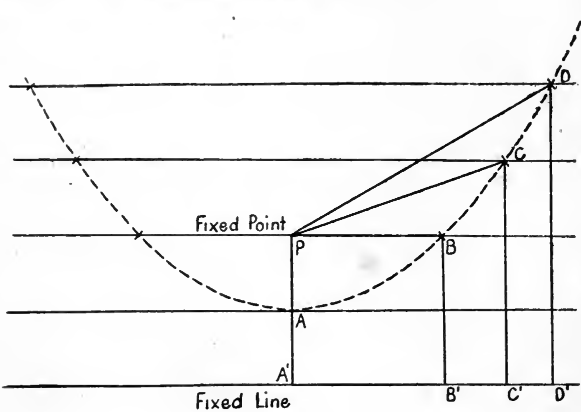


Fig. 131.

$$PA = AA'$$
$$PB = BB'$$
$$PC = CC'$$
$$PD = DD', \text{ etc.}$$

The method, requiring only compass or divider, is easily grasped by children.

But in addition to such well-known curves other lesser known curves may be traced such as the cycloid or path described by a point on the rim of a rolling wheel. More difficult would be the path of a point on the circumference of one circle rolling round the circumference of another circle, the two circumferences always being in contact. Again numerous examples in curves of pursuit may be set as suggested by Prof. Nunn in such an example as the dog on the common running to over-take his master on a road some distance away. Thus AB (Fig. 132) indicates the direction of the man and A his position when first seen by the dog at C. The resultant curve of the dog's motion is a most interesting example of approximate graphic methods. The reader will find the method of solution indicated in Prof. Nunn's *The Teaching of Algebra*. Numerous other examples will occur to the teacher such as the curve formed by reflection from the side of a teacup on the surface of the tea.

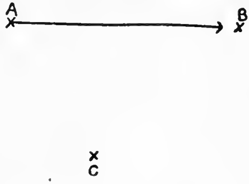


FIG. 132.

Treatment of Graphs. We have now indicated fairly comprehensively in this chapter the main lines of the development of graphic methods which we consider possible and advisable in schools. In conclusion we would again emphasize that the graph is best treated as an additional and striking means of showing the connection or law underlying a series of observed results, or alternately as a means of showing clearly the implications of a formula. If treated narrowly as a new form of sum—a new means of obtaining an answer—the work soon loses its zest. It is the pictorial, illustrative and visual side that should be emphasized, and from this aspect it is essential that the information conveyed by a graph should be as self-evident as possible. For the sake of the younger teachers we will indicate a few of the points which experience has shown to be essential to good graphs.

(1) Whatever the graph, the units chosen should be *as large as the paper will conveniently allow.* This involves a careful choice of *scales*, and the same scale for horizontal and vertical quantities is seldom possible. Inattention to this point results in the production of absurdly small and cramped graphs, usually in one corner of the paper.

(2) Each axis should be carefully *numbered* and *named.* If the numbering is plainly shown it is unnecessary to make statements such as: "Scale: One square = ," etc.

(3) The *points* of the graph should be indicated by some such simple convention, as + or ⊙. The following three examples (Fig. 133) will indicate our meaning—

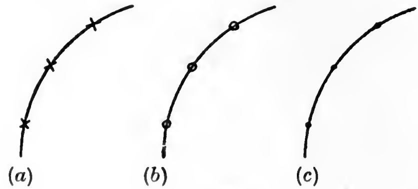


FIG. 133.

(*a*) and (*b*) show suitable methods of indicating points, but remembering the Euclidean definition of a point, (*c*) should be discouraged.

(4) The curve should be drawn *in pencil* preferably, and should be as smooth as the data permits.

(5) Any results obtained by interpolation should be indicated by dotted lines parallel to the axes.

(6) The completed graph may well be finished off by some title of the form: "Graph showing the connection between — and —."

CHAPTER XXV

THE TEACHER'S BOOKSHELF—SOME HINTS

In this concluding chapter we propose to indicate a few of the more important books on the subject of the *History and Method of Arithmetic* and *Elementary Mathematics* generally, which may well form the nucleus of the special library of the mathematical teacher-enthusiast.

Considerations of space have compelled us to limit the list to books and publications in English (including several American publications). It should be understood, however, that numerous books of outstanding interest and importance in this subject have been published in French and German. Some of the general books mentioned in the lists below contain extensive bibliographies of both English and foreign reference works.

Class Books. For elementary schools the field of choice is so extensive that no complete list can be given. Practically every educational publisher has produced one or more sets of class-books (and accompanying teacher's books) for use in the "standards" of the Elementary School. Of late years the standard of excellence reached by these series has been high and there is little to choose between the best. Two of the most recent series may be mentioned: *Pitman's Common Sense Arithmetics* (in eight books), compiled by the present writer; and *The Cambridge Elementary Arithmetics*, compiled by J. H. Webster (Cambridge University Press).

Text-books. Assuming that our readers are familiar with the host of well-known text-books we do not propose to attempt to give anything approaching a complete list, but will content ourselves with indicating a few books which in our opinion have broken through the thick crust of convention and tradition in the selection and arrangement of Arithmetical topics—

An Arithmetic for Preparatory Schools, by Trevor Dennis, M.A. (G. Bell & Sons, 4s. 6d.). This book is an advance on anything we have yet seen. The contents are based upon the syllabus issued by the Curriculum Committee of the Headmasters Conference. Though intended for candidates for Public Schools the book is of such catholic interest and so well produced that it might well be found in every school in the country.

A Modern Arithmetic with Graphic and Practical Exercises, by H. S. Jones, M.A. (Macmillan & Co.). This book, although published a decade since, is still in advance of most of its contemporaries. It is particularly valuable to teachers for its excellent practical exercises.

The Winchester Arithmetic, by Godfrey & Bell (Cambridge University Press). A book of modern and progressive character, particularly valuable to teachers for its varied miscellaneous examples.

Arithmetic, by G. W. Palmer (Macmillan & Co.). This consists entirely of examples and problems. The contents are extremely well grouped and graded. It will prove a splendid source-book for the teacher who is eager for "fresh" examples.

Algebra. Here, perhaps, the teacher of some years' experience in Elementary Schools is apt to be somewhat out of touch with the text-books and class books that have appeared since his own college days. For such teachers we accordingly indicate a few of the more recent books.

Exercises in Algebra, by T. P. Nunn (Longmans). This book has already been referred to repeatedly in the chapter on Algebra in this book. Every teacher of older scholars in every type of school should possess at least

Part I of these exercises with the corresponding Teachers' Book—*The Teaching of Algebra.*

Elementary Algebra, by Godfrey & Siddons (Cambridge University Press). This, though hardly marked by the individuality of Professor Nunn's work, is nevertheless in every way an admirable elementary text-book.

Elementary Algebra, by Carson and Smith (Ginn & Co.), Parts I and II. In these admirably produced little volumes we have the happiest combination of enlightened English and American effort.

GEOMETRY. Teachers who are only able to find time to deal with this subject in incidental and practical fashion will find all the inspiration they need in the practical and experimental sections of the following works—

Elementary Geometry, by Godfrey and Siddons (Cambridge University Press).

Plane Geometry, by Carson and Smith (Ginn & Co.).

PRACTICAL MATHEMATICS. Much modern reform in mathematical teaching is due to the excellent and unwearied efforts of the late Prof. Perry. Teachers should possess the famous 1901 British Association *Report of a Discussion on the Teaching of Mathematics*, and his *Six Lectures to Working Men on Practical Mathematics* (Wymans).

Every teacher should be familiar, too, with some of the more recent text-books on *Practical Mathematics*. These books approach the whole subject of mathematics from a new angle, and most of them still retain the vigour and freshness imparted to the subject by Prof. Perry, the pioneer of this pragmatic method in mathematics. Indicating only those that we have tried personally, we may mention—

(*a*) *An Introduction to Practical Mathematics*, by F. M. Saxelby (Longmans).

(*b*) *A Course in Practical Mathematics*, by F. M. Saxelby (Longmans).

The larger work (*b*) is valuable specially for its copious examples.

(*a*) *A First Book of Practical Mathematics*, by Usherwood and Trimble (Macmillan & Co.).

(*d*) *Practical Mathematics for Technical Students.* Parts I and II, by Usherwood and Trimble (Macmillan & Co.).

Part II of (*a*) is perhaps the most comprehensive of recent books on the subject. The more advanced of these books are of course beyond the ability of young beginners, but we recommend them to teachers on the sound principle that the more familiar a teacher is with a subject generally, the better teacher of that subject he is likely to be.

The Teaching of Mathematics. Most of the older manuals of teaching method that are arranged in a "subject" classification contain a chapter on the *Teaching of Arithmetic*, and we assume that our readers are familiar with the contents of these.

Of late years the revival of interest in the teaching of mathematics generally has caused the appearance of books devoted specially to the subject. In Arithmetic we may note—

(*a*) *The Teaching of Arithmetic*, by M. Punnett (Longmans).

(*b*) *The Art of Teaching Arithmetic*, by J. B. Thomson (Longmans).

(*c*) *The Teaching of Primary Arithmetic*, by H. Suzzallo (Riverside Press, N.Y.).

Miss Punnett's book hardly develops the subject beyond the lower standards of the Elementary School, but contains a sound treatment of the fundamentals. Miss Thomson's book contains a good chapter on the teaching of proportion by the "Fractional" method.

Two well-known American books: (*a*) *The Teaching of Elementary Mathematics*, by D. E. Smith (Macmillan & Co.); (*b*) *The Teaching of Mathematics in the Elementary and Secondary School*, by J. W. A. Young (Longmans

& Co.), each contain chapters on Arithmetic, though their complete contents should be known by every teacher.

In the *method* of Algebra the outstanding work of recent years is Prof. T. P. Nunn's *The Teaching of Algebra* (Longmans), which should be in constant use with the companion volumes *Exercises in Algebra*. The full effect of these books has not yet been felt in schools—revolutions can never be hurried. In a generation perhaps, or less, examiners, teachers and pupils will realize their importance and their value.

Of great interest to both teachers and students are books such as: *Easy Mathematics*, by Sir Oliver Lodge (Macmillan & Co.); *Calculus Made Easy*, by Sylvanus P. Thomson (Macmillan & Co.); yet though such books give teachers ample food for thought, they seem to suffer from the fallacy that what is easy and simple to their illustrious authors is thereby easy and simple to every beginner. Lodge's book more especially, in its frequently didactic manner, seems to imply that all the difficulties of pupils are due to their teachers—an implication which is ungenerous because so often untrue.

THE HISTORY OF MATHEMATICS. Here the field is vast indeed, but, fortunately for the busy teacher, a brilliant worker in the person of W. W. R. Ball has narrowed it down to one volume, *A Short Account of the History of Mathematics* (Macmillan & Co.). This work is too well known to need comment. It should be found on every mathematical teacher's book-shelf.

An excellent American book is *A History of Elementary Mathematics*, by F. Cajori (The Macmillan Co.). To these two comprehensive works we may add *Greek Geometry from Thales to Euclid*, by G. J. Allman (Longmans).

There is perhaps room for a modern popular account of the history of arithmetic and its methods in form and price which would appeal to the average teacher in the Elementary School.

General. Of late years a number of books has appeared dealing very thoroughly with the logical and philosophical foundations of mathematics and mathematical reasoning. The average teacher need not fear the contents of many of these books. He will find them neither difficult nor dull.

Two little books of marvellous cheapness are *The Nature of Mathematics*, by P. E. B. Jourdain (Jacks: The People's Books), and *Introduction to Mathematics*, by A. N. Whitehead (Williams & Norgate: Home University Library).

Carson: *Essays on Mathematical Education* (Ginn & Co.) contains some good things, though the essay on the teaching of arithmetic is meagre.

Benchara Branford's *A Study of Mathematical Education* (Clarendon Press) is an inspiring and remarkable book.

Somewhat harder reading for the non-specialist are excellent books such as: *Monographs on Modern Mathematical Topics*, edited by J. W. A. Young (Longmans); *Lectures on the Fundamental Concepts of Algebra and Geometry*, by J. W. A. Young (The Macmillan Co.).

Finally in our list of general books we include what to many teachers will prove the most popular books of all. These are books on the Amusements of Mathematics. The standard work is of course *Mathematical Recreations and Essays*, by W. W. R. Ball (Macmillan & Co.).

This extremely comprehensive volume contains sufficient good things to occupy a life-time of leisure.

More popular in style are the works of that versatile puzzle propounder known to all magazine readers, viz., Mr. H. E. Dudeney. Very cheap and very good are his *Mathematical Amusements* (Cassell & Co.), and *Canterbury Puzzles*.

Of more recent origin (1919) we may mention *Number Stories of Long Ago*, by David Eugene Smith (Ginn & Co.), and *Number Puzzles*, by the same author (Ginn & Co.).

Board of Education Publications. These publications are the cheapest of all and are invaluable to a teacher.

All the following may be obtained direct from Wyman & Sons, Ltd., Fetter Lane, London, E.C., or through any bookseller.

(*a*) *Suggestions for the Teaching of Arithmetic,* Circular 807.

(*b*) *The Place and Use of Graphs in Mathematical Teaching*, Circular 884.

(*c*) *Geometry*, Circular 85.

(*d*) *Special Reports on Educational Subjects*: (i) "The Teaching of Arithmetic in London Elementary Schools"; (ii) "The Teaching of Arithmetic in English Elementary Schools."

APPENDIX

LIST OF ALL-PRACTICAL EXERCISES IN USE AT WERRINGTON INDUSTRIAL SCHOOL, STOKE-ON-TRENT

THE appended list of All-practical Exercises is printed by the courtesy of Mr. J. D. Johnstone, Superintendent of the Werrington Industrial School, Stoke-on-Trent. All the exercises are actually in use there substantially in the form introduced by Mr. Johnstone

JUNIORS.

Lengths.

EXERCISES.	APPARATUS REQUIRED
1. How much longer is your foot than that of the boy next to you? If you draw a line on the floor 12 of your feet long, and he drew a line 12 of his feet long, how much longer would your line be than his? Do it and see if you are right.	*Tape, Ruler.*
2. What is the distance round this card? Measure the shortest distance between the opposite corners. Put 4 cards together to make a rectangle four times the area of this one. How far is it round this rectangle?	*Card, Ruler.*
3. Find as carefully as you can the distance all round this T-square.	*T-Square.*
4. Work with another boy and take each other's measurements: head, neck, chest ("up and down"), waist, thigh, knee, calf, ankle, "muscle," and wrist.	*Tape.*
5. Measure the diameter of this ball, pipe, jam jar, cylinder.	*Sliding Callipers.*
6. Here is a match-box, carefully open it out, and with this cardboard make one similar to it.	*Match-box, Cardboard, Scissors or Knife.*
7. Measure the length and breadth of the platform (or top of cupboard, desk, floor, etc.), and find the distance all round it.	*Tape or Ruler.*
8. How much farther is it round the top of the table than round the top of the stool?	*Tape.*

App. Required

Tape. 9. What height are you? How much higher are the blackboard and the door?

Book. 10. If a fly walked from a corner of your book right round the edge till it came to the corner at which it started, and then walked to the opposite corner, how far has it gone?

11. Estimate the lengths of the cupboard, floor, a desk, the heights of the door, the window, the board, your teacher. Write your answers down. Now measure them and compare your results.

String, Book. 12. Here is a book. Estimate how much string you would need to tie it up, allowing 3 inches for the knot. Now do it and see if you were right.

13. How far can you span? Measure the length of the desk by spanning. Measure it next with a ruler. Were you nearly right? Which has the larger span, your right hand or your left? How much larger?

14. Take ten ordinary steps from a chalk line. How far have you gone? How long was each step, supposing each was the same length? Remember how long your step is.

Map Pole or Lath. 15. How high do you think the room is? Get a long stick and measure it.

16. I want to put a skirting board or picture rail round the room. How long will it be?

17. Take all the measurements you want to make a "ruler" drawing of the door (window, picture frame).

Paper, Thread, Scissors. 18. Draw a triangle, an oblong, a circle, a square. Cut them out and find the distance all round (use thread for the circle).

Atlas. 19. Obtain a map of England. Measure the distances from (*a*) London to Stoke, (*b*) Carlisle to Newcastle, (*c*) Bristol to Hull.

String. 20. Here is a piece of string. Tie six knots in it. Measure it now. How much string does a knot take up?

String. 21. Here is a piece of string. Make a square with it. How long is one side? Make a triangle with three equal sides. How long is one side?

22. How long would 20 pencils be if placed end to end?

23. Cut off $\frac{5}{8}$ of this piece of string.

App. Required

24. Measure the distance across the floor. Now measure it with your feet. How long is your foot? *String* 1′ 9″.

25. Measure the distance from your elbow to the tip of your middle finger. Remember it. When you stand with "arms sideways stretched," how far is it from the tip of one middle finger to the tip of the other?

26. Measure the distance round a bicycle wheel (or barrow wheel or hoop). How many times would it revolve in going across the yard? *Wheel.*

Areas.

EXERCISES.

1. Find the area (the table top, cupboard, desk, stool, etc.) in square inches.

2. Find the area of the floor in square feet (measure to nearest foot).

3. Find the area of the floor in square yards (wall, yard, garden plot, etc.). Measure to nearest yard.

4. Draw an oblong 8 inches by 5 inches. How many square inches are there? Show them. Shade 16 of them. What part of the whole oblong are these?

5. Cut out a square of side 5½ inches. How many whole square inches are there? Add up the parts of square inches left. What is the area of the whole square? *Paper, Scissors.*

5. A draughts board is square and has 8 rows, with 8 squares in each. Make one with this piece of cardboard. Start in a corner and shade alternate squares. What part of the whole have you shaded? *Cardboard, Knife or Scissors.*

7. How many whole square inches are there on this card? If you put a sixpenny insurance stamp on each, what would they cost?

8. Draw a square 9 square inches in area and a rectangle 9 square inches in area. Measure the distance all round them. Which is the greater?

9. Cut out of this paper an oblong piece 5 inches by 3 inches. How many square inches are left? *Paper, say 6″ by 5″.*

10. How many pieces of cardboard this size would just cover this big piece? *Card, 2″ by 1½″. Card 8″ by 12″.*

11. If it cost 1½d. a square foot to paint both sides of the door, what would be the total cost?

App. Required

12. How many square inches does this tile cover? How many tiles like it would cover this floor? (hearth, table).

Miscellaneous.

EXERCISES.

3s. $7\frac{1}{2}$d. in Cardboard Money.

1. If this money is $\frac{1}{4}$ of what I have altogether, how much is $\frac{1}{3}$ of my money?

Watch or Clock.

2. What time is it by this watch? How many minutes have you been in school? How many until you go home?

1 or 2 or 5 or 10 Pennies. Scales, etc.

3. What is the weight of ten pennies? How much would a sovereign's worth of pennies weigh?

Books, Scales, etc.

4. Weigh 4 reading books altogether, and then calculate the weight of one. Try it.

5. Pick out six boys. Ask them their age (years and month). Find the total age. Divide by six. Your answer should give their average age. What is it?

6. Use the cardboard money. Put in 4 separate heaps these amounts: 2s. $7\frac{1}{2}$d., £1 15s. 6d., 7s. $10\frac{1}{4}$d., 18s. $5\frac{1}{2}$d. How much altogether?

Letter (say) $1\frac{1}{2}$ oz. Scales, etc. Postal Information.

7. (*a*) How much would this letter cost in postage?

(*b*) Wrap your jersey in a parcel. Address it to your mother. How much would it cost to send away?

Envelopes with Cardboard Money.

8. How much money is in each of these 3 envelopes? How much altogether? How much would you want to make £10?

Book.

9. If you have read $\frac{1}{8}$ (?) of the pages of this book, how many have you not read?

10. Write down the name of each article of clothing that you have on. Estimate their costs, and find the total.

11. How many panes of glass are there in the schoolroom? If each cost $10\frac{1}{2}$d. (?) what would be the total cost?

12. How much would it cost to supply new ink-wells for this room at $1\frac{1}{2}$d. each?

Pint Measure, Water, Scales, etc.

13. Find out the weight of a pint of water. What will (*a*) a quart, (*b*) a gallon, weigh? Remember this.

	App. Required
14. Weigh a brick. Take its measurements. Soak it in a bucket of water for a quarter of an hour. Weigh it again. What difference is there?	*Brick.*
15. Cut a cubic inch of plasticine. Weigh it. Mould a rectangular piece 4 ins. by 2 ins. by 1 in. Weigh it. Divide your result by 8. What is the weight of one cubic inch? Is this answer the same as your first? If not, why not?	*Plasticene, Scales, etc.*
16. Model a plasticene ball. Now model one twice as heavy.	*Plasticene, Scales, etc.*
17. What do 16 new pencils weigh? Find the weight of one.	*Pencils, Scales, etc.*
18. How many ounces of soil does this plant pot hold?	*Scales, Soil, Plant Pot.*
19. Sweets are sold at 2 ozs. for 3d. Suppose pebbles are sweets and weigh me sixpenny worth.	*Pebbles, Scales, etc.*
20. Weigh out 1 lb. of stones on the table, divide them into two equal groups. See if they balance.	*Pebbles, Scales, etc.*
21. What weight of water will this bottle hold? How many gills is that (1 gill = 5 ounces)?	*Bottle, Water.*
22. Here are 4 pieces of wood—oak, pine . . . . all the same size. Arrange them in order of weight, the lightest first.	*Four rectangular Pieces of Wood, say 6″ by 4″ by 2″. Scales, etc.*
23. How many pebbles do you think would just weigh ½ lb.? Test your answer by weighing. Were you right?	*Pebbles, Scales, etc.*
24. Suppose sand is flour. Flour is 2s. 11d. a stone. Weigh me four lbs. How much would it cost?	*Sand, Scales, etc.*

SENIORS.

Lengths.

PROBLEMS.

1. With this piece of cord form successively a square, a rectangle, an equilateral triangle and a circle. Draw each one as you form it. Calculate area of each, and arrange in order.	*Piece of Cord 11″ long, Drawing Material.*
2. Take another boy's anthropometrical measurements—head, neck, chest, biceps, wrist,	*School Weighing*

App. Required	
Machine, Tape Measure.	waist, calf, height, weight. Let him take yours. Make a table of results and compare.
Tape or Ruler.	3. Measure the adjacent sides of a table (desk, floor, hall, or book, etc.), and calculate the length of a diagonal. Check your answer by actual measurement.
Bundle of Cards.	4. (*a*) How thick is this card? (*b*) A bundle of cards like this is an inch thick. How many cards are there?
Piece of Cord 15″ long, with 6 Knots in it.	5. (*a*) Here is a piece of cord with several knots in it. Find the length of string taken up by the knots. (*b*) After unravelling the knots tie another knot dividing the string in the ratio of 1 to 2.
Piece 2′ 4″ *long.*	6. (*a*) I cut off $\frac{7}{80}$ of a ball of string. Here it is. How many yards were in the ball?
200 *yds. Reel.*	(*b*) Cut from this reel of thread $2\frac{1}{2}$ yards. What percentage of the whole have you cut off?
Atlas.	(*c*) Use this piece to measure on your atlas the lengths of the (*a*) Severn, (*b*) Danube, (*c*) Volga.
Books, String, Brown Paper, Weights, etc. Postal Information.	7. Make a parcel of three reading books. Estimate the length of string needed to tie it up (allow for the knot). Tie it up. Weigh, and find out cost of sending by parcel post. Unpack and check your estimate of the length of string required.
Ordnance Map.	8. Find the distance from school to —— ? (*a*) As the crow flies. (*b*) By road.
Wheel, Tape, Long Tape.	9. How many revolutions would my bicycle (or barrow) wheel make in going across the playfield?
Halfpenny, Thread, Ruler.	10. (*a*) Measure the diameter of a halfpenny. Find the circumference by rolling it along a ruler, or with thread. Divide the latter by the former.
Outside or Sliding Callipers, Jam Jar.	(*b*) Measure the outside diameter of this jam jar. Measure its circumference. Divide the latter by the former. Do you notice anything peculiar in the answers to (*a*) and (*b*)?
Threaded Bolt, Ruler.	11. (*a*) Find the distance between the threads of this screw. (*b*) What is the approximate length of the thread?

	App. Required
12. Find the height of *that* wall, (*a*) by counting the bricks; and (*b*) by using an angle measurer.	*Some form of Angle Measurer, Protractor, Drawing Material, Long Tape.*
13. Find the height of *that* tree, (*a*) by measuring its shadow and comparing it with the shadow of a stick; (*b*) by using an angle measurer.	*Stick, Long Tape, Angle Measurer, Drawing Material, Protractor.*
14. What would it cost to floor *this* room with boards 6 inches wide at 6d. a foot?	*Long Tape.*
15. Take 50 paces and measure the distance covered. Do this three times and find the average. Now calculate the average length of your step. Measure the length of the schoolroom by pacing it. Check your result by a scale drawing of the schoolroom.	*Long Tape, Plan of School.*
16. From the weather chart find the average daily rainfall for the last month. Draw a diagram to show the variation.	*Weather Chart, Squared Paper.*

Areas.

PROBLEMS.

1. How many blocks (or tiles) like *this* would be needed to cover the entrance hall?	*Wooden Block or Tile.*
2. Here are four pieces of paper. Cut out a rectangle, a square, a triangle, and a circle, each 9 sq. inches in area.	*Drawing Paper, Scissors.*
3. Here is a piece of paper. What is the area of the largest circular piece you can cut out from it? Cut out this circular piece.	*Piece of Paper (say) 7″ by 3½″. Scissors.*
4. (*a*) What area of paper was used in making this book? (*b*) Calculate roughly how many words there are in the book.	*Book.*

App. Required	
Tape, Long Stick.	5. What would it cost to paper *that* wall with paper 1 ft. 9 ins. wide, costing 3s. a roll of 12 yards ?
Paper, Scissors.	6. Cut out of paper a square, a rectangle, an equilateral triangle, each with a perimeter of 9 ins., and find their areas.
Match-box, Cardboard and Knife, Binding.	7. Here is a box (match-box or box made of cardboard). What area of material is needed to make one like it ? Make one.
Any Geometrical Model, small size.	8. What area of paper would be needed to exactly cover this ? Cover it.
Picture Frame, Washer, Piece of Piping.	9. Find the area— (*a*) Of wood in this picture frame. (*b*) Metal in this washer (one side only). (*c*) The cross section of this pipe.
Long Tape.	10. What area of carpet would be needed for this floor, leaving a one foot margin all round ? Find its cost at 7s. 6d. per sq. yard.
Tape.	11. Measure diameter of garden roller (or cylinder, rolling pin, etc.). Calculate its circumference. What area would it cover in one revolution ?
Tape.	12. Find the area of a circular flower bed.
Ruler.	13. What would it cost to reglaze the windows in this room at 7½d. per sq. ft. ?
Paper, Scissors.	14. Cut out— (*a*) A parallelogram. (*b*) A circle. (*c*) A hexagon, each with a perimeter of 11 ins. Calculate their areas.
Paper, Scissors.	15. Here is a piece of paper. Cut out 4 circular pieces each 3½ ins. in diameter. What area is left ?
Piece of Cardboard of any Shape.	16. Here is a piece of cardboard. Find out what you can about it.

Volumes.

PROBLEMS.	App. Required
1. What volume of timber is in that door ? (cupboard or other).	*Ruler.*
2. What weight of water will this bucket hold ?	*Bucket, Pint or Quart, Weights.*
3. Find out the weight of a cubic inch of these different kind of wood (or metal).	*Rectangular Pieces of Oak, Pine, etc.*
4. Here is a stone ; find its volume.	*Stone, Graduated Cylinder.*
5. How many books like the ones shown you will this box hold ? Test your answer. Were you right ?	*Book, Box, Ruler.*
6. Arrange these numbered boxes in order of weight, first by estimating, then by weighing.	*Six Match-boxes containing different quantities of Plasticene.*
7. What weight of water would half fill the swimming bath ? (or cylindrical or rectangular cistern).	*Tape.*
8. What weight of water would this bottle (jug, mug, basin) hold ?	*Bottle or Jug, etc. Scales, etc.*
9. Without weighing find the weight of this piece of piping.	*Piping, Callipers, Table of Relative Densities.*
10. Find the volume of this room. How many boys has it to accommodate ? How many cubic feet each ?	*Tape.*
11. Weigh this brick. Soak it in water for ten minutes. Weigh it again. If there is any change, find the percentage increase in weight.	*Brick, Scales, Water.*
12. What would 6 gallons, 3 quarts, 1 pint of water weigh ?	*Scales, etc. Pint Measure.*
13. Suppose the store (or other rectangular floor, or piece of ground) were a lawn, and ½ inch of rain fell on it. What would be the weight of the water ?	*Tape.*
14. Find out all you can about this model (area, volume, weight).	*Geometrical Model.*
15. If the number 1 represents the weight of a halfpenny, what number would represent the weight of a penny ?	*Halfpenny, Penny, Scales, etc.*

App. Required	
Milk Tin or Urn, Pint, Callipers, Tape.	16. How many boys could get a pint of tea from this tin if it were full? Do it by measuring only. Check by trial—with water.
Brick, Ruler, Scales, etc.	17. What is the weight of a cubic foot of brickwork?
Scales. A rectangular Vessel. Tin Box, say, that will hold the material: volume about 8 cu. ins.	18. Find out the weight of a cubic foot (*a*) of water, (*b*) sand, (*c*) soil.
Six Letters weighing ½ oz., 1 oz., 1½ oz., 2 oz., 2¾ oz., 3 oz., 3¾ oz. Postal Information.	19. Find the postage on these letters.
Scales, Paper, Postal Information.	20. Wrap your (1) boots, (2) coat, into parcels. Weigh them. What would the postage be?
Box of Drawing Pins, Scales, etc.	21. Here is a box of drawing pins (paper fasteners). Find the weight of one.
School Weighing Machine.	22. Guess the weight of six of your friends. Weigh them and find the average weight.

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Printed in Bath, England, by Sir Isaac Pitman & Sons, Ltd.
(1822w)