TECHNICAL MECHANICS.

BY

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SECOND EDITION, REVISED AND ENLARGED.

SIXTH THOUSAND

NEW YORK:
JOHN WILEY & SONS.
LONDON: CHAPMAN & HALL, LIMITED.
1910.

The Scientific Press
Robert Drummond and Company
New York

PREFACE.

THIS work is not a pure nor an applied mechanics, but a theoretical mechanics for students of engineering, and in accordance with the usage of some German writers, the title "Technical Mechanics" has been given to it.

On the theoretical side, practically all the subjects treated have a direct bearing in engineering problems. Little attention was paid to experimentation, principally because the author's own students come to him after having completed a laboratory course in elementary mechanics. In general, the theory in each chapter has been grouped together and separated from the applications. The principal equations and formulas are set in bold-faced type.

On the applied side, no attempt was made to present fully any one subject, as the analysis of trusses, friction, balancing of rotating systems, etc., for the object in view was the illustration of the use of principles of mechanics and not treatments of roof-trusses, friction, balancing, etc. However, a sufficient number and the proper kind of examples have been included, it is believed, to give to the student a working knowledge of the subject.

In Statics, especially, a distinctive feature is the nearly coextensive use of graphical and algebraic methods. This is due to the author's opinion that it is unwise to present graphical and analytical statics as separate courses, at least to beginners. The treatments of composition and equilibrium of forces are separate. This arrangement is largely due to the author's desire to develop the conditions of equilibrium for the different classes of force systems consecutively and to get all the principles of equilibrium together. The examples on the application of the principles of equilibrium are set off in a separate chapter, number VI. No attempt was made to arrange them in accordance with the orders of Chapters II and V. On the contrary, such an arrangement was avoided, and the entire aim was to emphasize the fact that there is a general method for solving the ordinary problems of Statics proper and that it consists principally in "applying" conditions of equilibrium.

Kinematics is treated mainly as a preliminary to Kinetics, but harmonic motion is discussed more fully than usual in works of this kind.

In Kinetics, D'Alembert's Principle is used considerably. In spite of the fact that it has been described as clumsy and old-fashioned, the author believes that the use of the principle in all but simple cases is decidedly helpful, and besides its use makes graphical methods possible.

As to "mass and weight," what has been called the physicist's usage is followed, that is, mass means quantity of matter and weight the Earth's attraction. The author retains, however, the familiar equation $m=W \div g$, and as holding not only in gravitation systems of units but in all systems in which "unit force gives unit mass unit acceleration." Only such systems are used herein, and then the familiar equation $F=ma$ always holds. This of course requires recognition of special units of mass in gravitation systems. The author has succeeded best with his classes in this matter by doing more than defining these units—he has been using a name for one of them, the "English engineers' unit" (equal to 32.2± pounds). So strongly does he feel that names are helpful in this instance that he has ventured to put them into print. The names herein used are "geepound" and "geekilogram," denoting 32.2± pounds and 9.81± kilograms respectively. They were adopted for their descriptiveness and as better than "matt" and "ert," the only other terms proposed in this connection so far as the author is aware.

For the information of any instructor who may consider using this book as a text, the following modifications and abridgments are pointed out: Chapters III and IV may follow Chapter

VI, Chapter V may be taken simultaneously with Chapter II, and Chapters III, IV, and XV, and parts of all other chapters may be omitted without serious derangement of the course.

In his work the author consulted especially the books of Profs. L. M. Hoskins and John Perry. He is especially indebted to Prof. C. H. Burnside of this University for valuable assistance on Statics.

MADISON, WIS., September, 1903.

PREFACE TO THE SECOND EDITION.

SEVERAL additions and changes have been made; the principal additions are: appendixes E and F, articles 87 and D 15, and an explanation of principal symbols, page 409. All errors detected in the first edition have been eliminated.

September, 1904.

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INTRODUCTION.

1. Nature of the Subject.—Mechanics, broadly defined, is the science which treats of motion. It is a natural science, and not a branch of mathematics as the student is apt to infer from his observation that mathematics is extensively used in the subject. Its foundations consist in a few grand generalizations from experience, such as Newton's Laws of Motion; the superstructure consists in the deduction of the logical consequences of those generalizations.

2. Division of the Subject.—The following divisions may be made:

(1) Mechanics
- Kinematics
- Dynamics
  - Kinetics
  - Statics

Kinematics treats of methods for the description of motion. *Dynamics* deals with the circumstances of motions, such as their causes, etc. *Kinetics* embraces that part of dynamics which deals with variable motion, and *Statics* that part dealing with uniform motion.*

The above classification and order is usually followed in modern works on pure mechanics, and is no doubt the logical one.

(2) Mechanics of
- Rigid Solids
- Non-rigid Solids
- Liquids
- Gases

The general principles are the same for these latter branches, but there are special ones and special methods for each branch;

* In a treatment of mechanics on the plan outlined, it would develop that the principles relating to bodies at rest are included in Statics.

thus it is convenient to treat them separately. The present volume deals with principles common to all these branches, but relates especially to the first one. It is the practice in most American schools to present the mechanics of non-rigid solids, liquids, and gases in separate courses, the first two under the titles of Strength of Materials and Hydraulics respectively, and the last in connection with Thermodynamics.

3. Technical Mechanics.—By this term is meant a presentation of the general principles of mechanics with special reference to their application in the fields of engineering.

Although the order of treatment outlined in the first division of the subject (art. 2) is the logical one, another is followed herein. This is the historical order: Statics, Kinematics, and Kinetics. Statics as presented also occupies a more important part than in the outline above, and is not limited by the preceding definition; it deals especially with principles and applications relating to bodies at rest.

Since the logical order is not followed, several principles appear in the early pages which are not fully explained or justified until later.

4. Two Methods of Analysis.—They are called the *graphical* and the *algebraical*, or analytical.

In the first method, the quantities under consideration are represented by lines and the analysis is wholly by means of geometrical figures. In calculations by this method leading to numerical results the figures are accurately drawn to scale. In the second method the quantities under consideration are represented by symbols, and the analysis is by ordinary algebra. Calculations are carried out by arithmetic.

The graphical method finds its best application in statics, though it is advantageous in the applications of kinematics to mechanisms. Statics is often treated entirely by one of the two methods, such a treatment being known as graphical or analytical statics, as the case may be; in this book both methods are employed. Some discussions and solutions are given by both methods by way of comparison, the others by that method which is the more suitable.

STATICS.

CHAPTER I.

FORCE.

§ I. Preliminary.

5. Force Defined.—Bodies act upon each other in various ways, producing different kinds of results. Those actions which influence the motion of bodies are now to be considered.

Definition.—An action of one body upon another which changes or which exerted alone would change the state of the motion of the body acted upon is called a force.

We say that a force is applied to a body, a force is exerted upon a body, a force acts upon a body, etc. The last expression is faulty and misleading; for, since force is the name for a certain kind of an action, the expression might be rendered thus: an action acts upon a body. Now it is not an action which acts, but some other body. The expression criticised is, however, a current one and is used in this book.

Our first notions about force are founded on our experience with forces exerted by or upon ourselves. From this experience we have learned that a force has *magnitude*, *direction*, and *place of application.*

6. Action and Reaction.—When one body exerts a force upon another, the latter also exerts a force upon the former, and the two forces are equal in magnitude but opposite in direction. This fact is often referred to as the principle of "*action and reaction*," the phrase being an abbreviation of Newton's third law of motion. By "action" is meant one of the forces, and by "reaction" the other.

7. Force at a Distance and Force by Contact.—This is a classification for convenience and is probably not based on fact.

Gravitational, electrical, and magnetic forces have been called actions at a distance, the force between the bodies concerned being exerted without their contact and, as was formerly supposed, without any material connection between them. It is now known that the force between two electrified bodies depends upon the medium in which they are placed, which is proof that the medium has to do with the exertion of the force by one body upon the other. Even gravitation, it is believed, is not a true action at a distance, and "the earth and a stone do not strictly draw each other together, but are pushed together by something which extends from one to the other." But apparently these forces are actions at a distance, and they will be so called.

Pressure of the atmosphere upon any object, the force of a hammer blow upon a nail, etc., are forces by contact. The place of application of a contact force is the surface of contact between the bodies concerned.

8. Distributed and Concentrated Forces.—This is a classification for convenience and is not in accord with fact.

A distributed force is one whose place of application is a surface or a solid. For example, the water pressure on a ship, the place of application being the wetted surface; the attraction of the earth upon a stone, the place of application being the solid defined by the stone; etc. All forces are really distributed forces.

A concentrated force is one of finite magnitude whose place of application is a point. Such a force is imaginary since no actual (finite) force can be applied at a point, but there are actual forces whose place of application is very small and which may be regarded as concentrated forces. The conception is useful in developing principles relating to actual forces. Thus, as just stated, many actual forces are practically concentrated and may be treated as such, and a distributed force is regarded as consisting of a great number of concentrated forces.

The line of action, or action line, of a concentrated force is a line parallel to its direction and containing its point of application. Note the distinctions between the terms action line, direction, and sense. To illustrate, imagine a pull exerted upon a sled by means of a single cord. The action line of the force is the line determined by the taut cord, its direction is upward and to the right 30° with the horizontal, and its sense is upward and to the right, not downward and to the left.

9. *Specification of a Concentrated Force.*—A concentrated force is completely specified by its magnitude, direction, and application point. It is explained later that the effect of such a force applied to a rigid body does not depend upon its application point, but only on its magnitude, action line, and sense, which are therefore called the essential characteristics of a force (as regards rigid bodies).

10. Graphical Representation of a Force. — Since a force has magnitude and direction, it is a vector quantity;* the magnitude and direction of a force may therefore be represented by a vector—the length of the vector representing to some scale the magnitude of the force, and the direction of the vector giving the direction of the force. If the force be concentrated, the vector may also represent the action line, but it is convenient to represent the action line separately. Thus suppose that the irregular outline in fig. 1 represents a body to which a force is applied at *P* horizontally to the right. The vector *AB* represents the magnitude and direction of the force, and a horizontal line, indefinite in length through *P*, its action line.

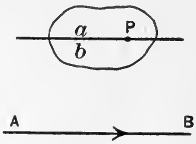


FIG. 1.

If the vector were drawn through *P* it would serve also to locate the action line, but representation by two lines is convenient especially when many forces applied to the same body are under consideration. The part of the entire figure in which the body is represented and the action lines are drawn is called the *space diagram*, and the part in which the vectors are drawn is called the *vector diagram*.

11. Notation.—In the graphical analysis, each vector and the corresponding action line will be marked by the same letters, a capital at each end of the vector and the same small letters, one on each side, at the action line as in fig. 1. Reference to a force in the text will be by capital letters; thus "the force *AB*"

* See Appendix A for brief discussion of vectors.

means one whose magnitude and directicn are represented by AB and whose action line is ab.

In the algebraic analysis, a force will be denoted by a capital letter, which will also stand for the magnitude of the force.

§ II. Measurement of Force; Mass and Weight.

12. Mass.—By mass of a body is meant the quantity of matter in it. There are several independent units of mass in use; only two will be described here.

The "British unit" is the quantity of matter in a certain piece of platinum deposited in the Office of the Exchequer, London; this unit is called a *pound*.

The "French unit" is the quantity of matter in a certain piece of platinum deposited in the Palais des Archives, Paris; this unit is called a *kilogram*.

Copies, more or less exact, of these standards and multiples and submultiples of them are in common use especially for the measurement of quantity of matter in trade.

13. Units of Force.—A *gravitation unit* of force is a force equal to the Earth's attraction on a unit mass.

Gravitation units are not constant with regard to place, for the Earth's attractions on equal masses at different places are, in general, not equal; in fact, the attractions are in the ratio of the values of g in the formula

$$g = 32.0894\,(1 + 0.0052375 \sin^2 l)(1 - 0.0000000957\,e)$$

computed for the two places, l denoting latitude and e elevation above sea level in feet. The extreme variation is between the attractions at a high elevation on the equator and at the pole; this variation is but 0.6 per cent, while for points within the United States the maximum variation is about 0.3 per cent. Ordinarily no account need be taken of the differences in the gravitation units of force employed at different places, for errors introduced into engineering calculations by this variation are practically always negligible.

It is customary to call any gravitation unit of force and the unit mass on which it is based by the same name.* Corresponding to the above-described units of mass we have then

the unit of force called a *pound*, which is a force equal to the Earth's attraction on a pound mass, and

the unit of force called a *kilogram*, which is a force equal to the Earth's attraction on a kilogram mass.

An *absolute unit* of force is one whose value is independent of time or place. Several such units are described later.

14. Measurement of Force.—The lever balance is primarily a force-measuring device, and it is the one commonly used for measuring forces. The force to be measured is applied at one end of a lever or system of levers, and the Earth's attraction on a body of known mass, m pounds say, at the other, the mass being such that the two forces balance. If the ratio between two such forces (determined once for all by the balance-maker) is n, then the magnitude of the force being measured is nm pounds. As the reader knows, the balance-maker provides that the numerical value, nm, may be read off directly. It should be noticed that a lever balance measures forces in terms of a gravitation unit.

By means of lever balances, engineers measure forces applied to materials in testing their strength, and usually also, with a slight indirection, the "brake resistance" when testing engines and other motors.

The spring balance is another force-measuring device. The force to be measured is applied to the spring so as to stretch it merely. The force corresponding to each amount of stretch, within the range of the spring, having been determined by the maker, the magnitude of the force being measured may be inferred from the stretch. As the reader well knows, the maker provides that the numerical value of the force may be read off directly. In theory at least, forces measured with a spring

* Such usage is apt to confuse the beginner, and, when necessary for clearness, we shall add an explanatory word; thus, 'pound mass' or 'pound force.' The word 'second' is used in a closely analogous way. It is the name for two different units, one of angle and the other of time, and the latter is defined with reference to the first. To be clear, we often say second of arc or second of time, as the case may be.

balance are measured in terms of a gravitation unit for the place at which it was graduated. For, consider the theory of the graduation of a spring balance: a body of known mass, a pound say, is suspended from it, and the position of the pointer on the spring is scribed on the scale; the position is marked one pound (force), for it corresponds to a stretch due to a pound force; then other bodies whose masses are multiples or submultiples of the unit mass are successively hung from the balance and the scale is marked correspondingly.

For measuring some forces the spring balance is better adapted than the other. Engineers use it to measure the pressure of the steam in a running engine, the pull of a locomotive, etc.

15. **Weight.**—By weight of a body will be meant the Earth's attraction upon it. Weight as here defined is a force and must be measured in force units. It is customary to express weights in gravitation units.

As before stated, the weight of any body changes slightly with change of its locality. However, if it is expressed in a gravitation unit the numerical value of the weight remains the same, for the relative changes in the weight and the unit are equal. An analogy is the measurement of the length of an iron rod by a standard of the same material at two different times, the temperature having changed during the interval. The length of the rod has changed, but the numerical value as determined by the iron standard remains constant, for the relative changes in the lengths of the rod and standard are the same. If the weight of a body is expressed in an absolute unit, the numerical value will change just as the weight changes if the body be transported.

A lever balance, then, will give the same numerical value for the weight of a body at all places, while a spring balance, if sufficiently sensitive, will show the true variation in its weight.

§ III. Force Systems.

16. Definitions.—Any number of forces collectively considered is called a system of forces or a force system.

The forces of a system are called *coplanar* when their action lines are in the same plane, and *non-coplanar* when they are not in a plane.

The forces of a system are called *concurrent* when their action lines intersect at a point, and *non-concurrent* when they do not so intersect.

The forces of a system are called *parallel* when their action lines are parallel, and *non-parallel* when they are not parallel.

Force systems may be described in accordance with the above definitions, thus: concurrent systems, non-coplanar parallel systems, etc., according as the forces of the system are concurrent, non-coplanar and parallel, etc.

A system of two forces which are equal, parallel, opposite in sense, and have different action lines is called a *couple.*

17. Classification of Force Systems.—A classification may be made in various ways. In Chaps. II and V the treatment is based upon the following classification:

Coplanar	Concurrent	Parallel
		Non-parallel
	Non-concurrent	Parallel
		Non-parallel
Non-coplanar	Concurrent	
	Non-concurrent	Parallel
		Non-parallel

CHAPTER II.

EQUIVALENCE OF FORCE SYSTEMS.

§ I. Preliminary.

18. Definitions.—*Equivalent force systems* are such as may be substituted for each other without change of effect.

The *resultant* of a force system is the simplest equivalent system. The resultant of a system acting upon a rigid body consists of either one or two forces, as will be shown later. It follows from the above definitions that two equivalent systems have the same resultant. The *components* of a force are any forces whose resultant is that force.

Composition of a force system is the process of finding a simpler equivalent system. Finding the resultant of a system is the most important case of composition. *Resolution* of a force system is the process of finding a less simple equivalent system. Finding components of a force is the most important case of resolution.

19. Principle of Transmissibility. — The effect of a force applied to a rigid body is the same for all application points in its action line. This follows from the equations of motion of a rigid body (art. 242), for they are independent of the application points of the applied forces.

This principle may be roughly verified by experiment with the apparatus represented in fig. 2, which consists of a body suspended from two spring balances. The springs are elongated on account of the weight of the body, and if a force, as F, be applied at A, the springs will suffer additional elongations which in a way are a measure of the effect of the applied force. If the application point of F be changed to B

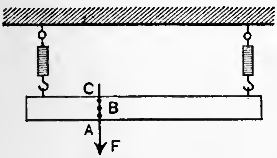


Fig. 2.

or C, the spring readings will not change, hence the effect of F has not changed.

20. Graphical Composition of Two Concurrent Forces.— *The Parallelogram Law.*—If two forces acting upon a rigid body be represented by OA and OB; then their resultant is represented by the diagonal OC of the parallelogram $OABC$ (see fig. 3).

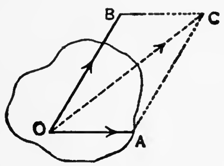


FIG. 3.

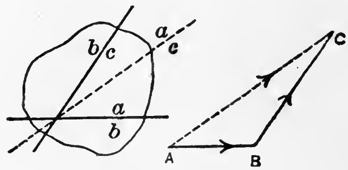


FIG. 4.

The Triangle Law.—If two concurrent forces acting upon a rigid body be represented in magnitude and direction by AB and BC, their resultant is represented in magnitude and direction by the side AC of the triangle ABC (see fig. 4).

The second law is obviously a consequence of the first, and the parallelogram law may be verified experimentally as follows: Set a drawing board in a vertical position and fasten a spring balance and smoothly working pulleys to it somewhat as shown in fig. 5. Next fasten three cords to a small ring, tie their loose ends to the hook of the balance and to two bodies whose weights are known and lay two cords over the pulleys as shown.

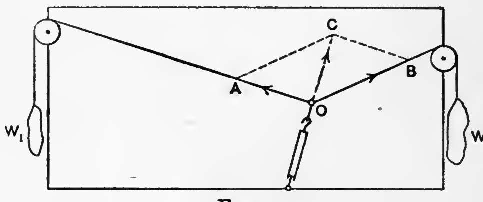


FIG. 5.

The strings now exert three forces on the ring (F_1, F_2, and F_3) equal to the weights (W_1 and W_2) of the suspended bodies and the reading (S) of the balance, respectively. Since F_3 "balances" F_1 and F_2, the resultant of F_1 and F_2 must be equal and directly opposite to F_3. We have only to ascertain now whether a construction according to the parallelogram law gives a resultant equal and opposite to F_3. So lay off on the board OA and OB equal (by some scale) to F_1 and F_2, and complete the parallelogram $OABC$. Then OC should (according to the law) represent the resultant of F_1 and F_2, i.e., it should represent a force equal and opposite to F_3 and it will be found that it does.

EXAMPLES.

1. The large square (fig. 6) represents a board 3×3 feet upon which five forces are applied as shown. Determine completely the resultant of the 4- and 5-lb. forces.

2. Determine completely the resultant of the 7- and 8-lb. forces.

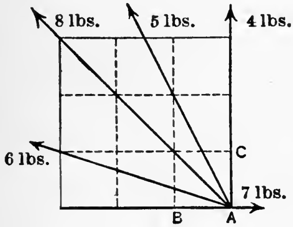


FIG. 6.

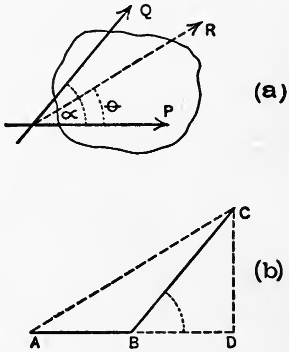


FIG. 7.

21. Algebraic Composition of Two Concurrent Forces.—Let P and Q (fig. 7) be two concurrent forces and α the angle between their action lines. Of course there are two angles here; the one taken is that between the parts of the lines on the sides of their intersection toward which the arrows point.

The lines AB and BC represent the magnitude and direction of P and Q respectively; then AC represents the magnitude and direction of their resultant. The action line, parallel to AC, is marked R Since the angle $CBD = \alpha$,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\overline{AB}\,\overline{BC} \cos \alpha,$$

and

$$\tan CAD = \overline{BC} \sin \alpha / (\overline{AB} + \overline{BC} \cos \alpha).$$

If R denotes the resultant and θ the angle between R and P, then, since AC represents the value of R and the angle CAD equals θ,

$$\mathbf{R^2 = P^2 + Q^2 + 2PQ \cos \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and

$$\mathbf{\tan \theta = Q \sin \alpha / (P + Q \cos \alpha)}. \quad . \quad . \quad . \qquad (2)$$

Special cases: If $\alpha = 90°$, $R = (P^2 + Q^2)^{\frac{1}{2}}$ and $\tan \theta = Q/P$. Describe the resultant if $\alpha = 0°$; if $\alpha = 180°$.

EXAMPLES.

1. Solve ex. 1, art. 20, by the formulas of this article.
2. Compute the resultant of the 6- and 7-lb. forces, fig. 6.
3. Show how the magnitude of the resultant of two forces P and Q changes as α changes from 0° to 180°.

22. Graphical Composition of Three Concurrent Non-coplanar Forces.—*Parallelopiped of Forces.*—If three non-coplanar forces acting upon a rigid body be represented by OA, OB, and OC, their resultant is represented by the diagonal OD of the parallelopiped $OABC$-D. (See fig. 8.)

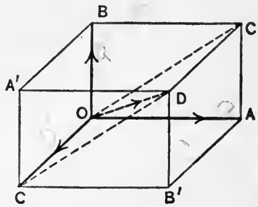


FIG. 8.

Proof: According to the parallelogram law OC' represents the resultant of two of the forces, OD represents the resultant of the third force and OC' and hence of the three given forces.

From an inspection of the figure, it is plain that the magnitude and direction of the resultant of three non-coplanar concurrent forces is given by their vector sum.

23. Algebraic Composition of Three Concurrent Non-Coplanar Forces Whose Action Lines are Mutually at Right Angles.—Denote the three forces and their resultant by P, Q, S, and R, and the acute angles between P and R, Q and R, and S and R, by θ_1, θ_2, and θ_3, respectively. Supposing that P, Q, and S are represented by OA, OB, and OC of fig. 8, it is plain that

$$\mathbf{R^2 = P^2 + Q^2 + S^2}$$

and

$$\mathbf{\cos \theta_1 = P/R, \quad \cos \theta_2 = Q/R, \quad \cos \theta_3 = S/R.}$$

EXAMPLES.

1. Suppose that a force of 9 lbs. acts at A (fig. 6), perpendicularly to the plane of the board and outward. Determine the resultant of the 9-, 4-, and 7-lb. forces.

2. Change the sense of the 9-lb. force and solve ex. 1.

24. Resolution of a Force into Two Concurrent Components.—It may be performed by applying the triangle or parallelogram law inversely. Thus, let it be required to resolve the force AB (fig. 9), applied at P. Draw from A and B two lines which intersect in any point C; then AC and CB represent the magnitudes and directions of components of AB. The action lines of the components must intersect on ab, and their application points must be rigidly connected with P. For, if the two forces AC and CB be compounded, their resultant will be found to be AB.

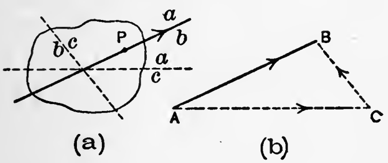


FIG. 9.

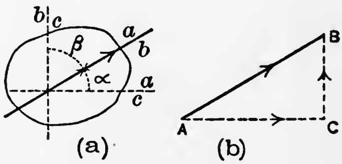


FIG. 10.

The problem just solved is indeterminate, for, C being any point, there are an infinite number of solutions. If conditions are imposed upon the components, the resolution is more or less definite; for example, if in the above it had been specified that the components should be horizontal and vertical, there would be but one answer.

Rectangular Components, or Resolved Parts.—An important case of resolution is that in which the angle between the components is 90°. Each is called a rectangular component, or resolved part, of the force. They can always be readily computed. From fig. 10, it is plain that

$$\left\{\begin{array}{l}\text{the rectangular com-}\\ \text{ponent or resolved}\\ \text{part of a force along}\\ \text{any line}\end{array}\right\} = \left\{\begin{array}{l}\text{the magnitude of}\\ \text{that force}\end{array}\right\} \times \left\{\begin{array}{l}\text{the cosine of the}\\ \text{acute angle be-}\\ \text{tween the force}\\ \text{and that line.}\end{array}\right.$$

If the rectangular components of a force F are parallel to coordinate axes, as x and y, they are called x- and y-components of F respectively, and will be denoted by F_x and F_y. If the acute angles between the force and the x and y axes be denoted by α and β respectively,

$$\mathbf{F}_x = \mathbf{F} \cos \alpha$$

and

$$\mathbf{F}_y = \mathbf{F} \cos \beta = \mathbf{F} \sin \alpha.$$

EXAMPLES.

1. Resolve the 5-lb. force of fig. 6, page 12, into two components whose action lines are parallel to the 4- and 6-lb. forces respectively.

2. Resolve the 5-lb. force of fig. 6 into two components whose action lines are parallel to the 8- and 6-lb. forces.

3. Resolve the 8-lb. force of fig. 6 into two components one of which is horizontal and the other 6.5 lbs. in magnitude.

4. Resolve the 6-lb. force of fig. 6 into two components whose lines of action are horizontal and vertical respectively.

5. Resolve the 4- and 7-lb. forces of fig. 6 into horizontal and vertical components.

25. Resolution of a Force into Three Non-coplanar Forces.—It may be performed by applying the parallelopiped of forces inversely. Thus, let it be required to resolve the force represented by OD (fig. 11). Construct a parallelopiped of which OD is a diagonal. The three edges intersecting at O represent the components of OD; the application points must be rigidly connected to that of the given force.

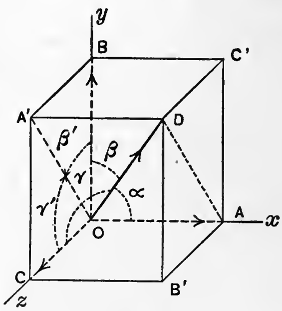


FIG. 11.

The problem just solved is indeterminate, for any number of such parallelopipeds may be thus drawn. If conditions are imposed upon the components, the resolution is more or less definite.

Rectangular Components.—An important case of resolution is that in which the three components are mutually at right angles. Each is a rectangular component, for the two components of *OD* (fig. 11), one of which is either *OA*, *OB*, or *OC*, are at right angles to each other, as *OA* and *OA'*.

If the three components of a force *F* are parallel respectively to coordinate axes *x*, *y*, and *z*, they are called *x*-, *y*-, and *z*-components of *F*, and will be denoted by F_x, F_y, and F_z respectively. If the acute angles between the force *F* and the *x*, *y*, and *z* axes be denoted by α, β, and γ respectively,

$$\mathbf{F}_x = \mathbf{F} \cos \alpha, \quad \mathbf{F}_y = \mathbf{F} \cos \beta, \quad \mathbf{F}_z = \mathbf{F} \cos \gamma.$$

In some instances, it may be more convenient to determine these components as follows: First resolve the given force into two rectangular components one of which is parallel to one of the axes; the other will be parallel to the plane of the other two axes. Then resolve the second component into two forces which are parallel to these two axes. For example, if *OD* (fig. 11) be resolved first along the *x* axis, the first resolution gives *OA* and *OA'*; the resolution of *OA'* gives *OB* and *OC*. Also,

$$OA = \overline{OD} \cos \alpha, \qquad \text{or} \quad F_x = F \cos \alpha;$$

$$OB = \overline{OA}' \cos \beta' = \overline{OD} \sin \alpha \cos \beta', \quad \text{or} \quad F_y = F \sin \alpha \cos \beta';$$

$$OC = \overline{OA}' \cos \gamma' = \overline{OD} \sin \alpha \cos \gamma', \quad \text{or} \quad F_z = F \sin \alpha \cos \gamma'.$$

EXAMPLE.

Resolve each force acting on the cube in fig. 38(*c*) into its *x*, *y*, and *z* components.

26. Moment of a Force with Respect to a Point.—The moment of a force with respect to a point is the product of the magnitude of the force and the perpendicular distance between its action line and the point. The perpendicular distance is called the *arm* of the force with respect to that point, and the point is called an *origin* or a *centre of moments*.

In the following, the moment of a force will usually be denoted by *M*, and the moment of a force with respect to an origin *O* by M_0.

The moment of a force with respect to a point is a measure of its tendency to rotate the body upon which it acts about that point. For, if the body is fixed at that point but free to turn about it in a given plane, any force in that plane will cause it to rotate about the fixed point. The amount of this tendency is proportional to the magnitude of the force and to its arm with respect to that point, and hence to the moment of the force with respect to the point.

The Unit Moment.—From the definition of moment of a force, it follows that the unit moment is the moment of a unit force whose arm is a unit length; hence there are many units of moments. We have no short names for any of them, but they are called a foot-pound, inch-ton, etc., according as the units of length and force are the foot and pound, or inch and ton, etc.

The Sign of a Moment.—Sign, plus or minus, is given the moment of force according as it produces or tends to produce counter-clockwise or clockwise rotation about the origin of moments. In the following, the rotation is supposed to be viewed from the reader's side of the printed page.

27. "Varignon's Theorem."—The algebraic sum of the moments of two concurrent forces with respect to an origin in their plane equals the moment of their resultant with respect to that origin.

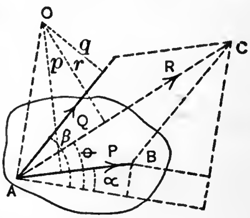


FIG. 12.

Proof: Suppose that the vectors marked P and Q and R (fig. 12) represent two concurrent forces and their resultant respectively. Call the arms of the three forces with respect to O, p, q, and r and the angles between the action lines and a perpendicular to OA, α, β, and θ respectively. From the figure, it is plain that

$$P \cos \alpha + Q \cos \beta = R \cos \theta;$$

therefore $$P\,\overline{OA} \cos \alpha + Q\,\overline{OA} \cos \beta = R\,\overline{OA} \cos \theta,$$

or $$Pp + Qq = Rr. \qquad \text{Q.E.D.}$$

Supply proof when O is between P and R, or Q and R.

According to the theorem, the moment of a force equals the sum of the moments of its x and y components; often it is easier to compute this sum than the moment directly. When one component passes through the origin, the moment of the other equals that of the force.

EXAMPLE.

Compute the moment of the 6-lb. force (fig. 6) about C directly and from its horizontal and vertical components.

28. Moment of a Force with Respect to a Line.—If a force be resolved into components parallel and perpendicular to a given line, the product of the magnitude of the perpendicular component and the distance from its action line to the given line is called the moment of the force with respect to the line. The line is called an *axis of moments*, and the distance referred to above is called the *arm* of the perpendicular component with respect to that axis. Thus, suppose F (Fig. 13) is a force acting upon a body, not shown, and AC and AB are its components parallel and perpendicular respectively to the moment axis OY. Then if YL is the perpendicular between OY and AB, the moment of F with respect to OY is $\overline{AB} \cdot \overline{YL}$, or $F_1\overline{YL}$.

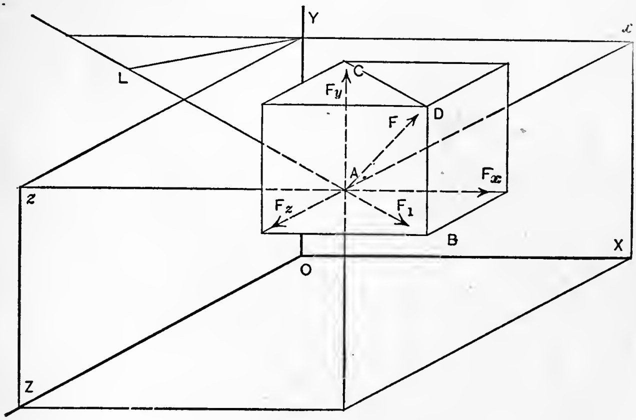


FIG. 13.

The value of the moment does not depend on the point A selected at which the force is resolved, for obviously the value of F_1 does not depend on A, and neither does the value of YL, since YL equals the distance between OY and a plane parallel to it through F, which distance is clearly independent of A.

Some special cases: (1) When the force is parallel to the axis, its moment is zero; (2) when the force intersects the axis, its moment is zero; (3) when the force is perpendicular to the axis, its moment is the product of the force and the distance between it and the axis.

The moment of a force with respect to a line is a measure of its tendency to rotate the body on which it acts about that line. For, if a body is free to turn about the line, OY say, any force, as AD, acting upon it will cause it to rotate. Now the components of AD would produce the same effect upon the body as does AD, but a component parallel to the axis would produce no rotation and therefore the rotation effects of AD and the perpendicular component, AB, are the same. But the tendency of AB to produce rotation is proportional to AB and its arm, hence to their product, or the moment of AD.

The sign of the moment of a force with respect to a line is taken as positive or negative according as the force produces counter-clockwise or clockwise rotation about the axis of moments. The sign will depend upon the side from which the rotation is viewed. If the axis is also a coordinate axis, the rotation is customarily viewed from its positive end.

A second method to compute the moment of a force with respect to an axis: Resolve the force into three rectangular components one of which is parallel to the axis; then compute the moments of the other two components with respect to that axis, and add these moments algebraically; this sum is the moment of the force itself. Thus, by this method, the moment of F about OY is $F_x \cdot \overline{Ax} - F_z \cdot \overline{Az}$. That this method and the first give equivalent results can be shown thus: the moment of the force by the first method $F_1 \cdot \overline{YL}$ is also the moment of F_1 about Y, and the moment by the second method is the algebraic sum of the moments of the components F_x and F_z of F_1 about Y; hence, by Varignon's theorem,

$$F_1 \cdot \overline{YL} = F_x \cdot \overline{Ax} - F_z \cdot \overline{Az}.$$

EXAMPLE.

Compute the moments of each force of fig. 38 (*c*) with respect to the x, y, and z axes, the edges of the cube being 4 ft. long.

29. Couples.—*Definitions.*—Two equal and opposite forces not collinear are called a couple. By *arm* of a couple is meant the distance between the action lines of its forces. The *moment of a couple* with respect to a point in its plane is the algebraic sum of the moments of its forces with respect to that point.

By the definition, the *sign of the moment* of a couple must be the same as the sign of the algebraic sum of the moments of its forces. The sign of the sum is the same for all origins (see proposition below) and can be seen at a glance for an origin between the forces or on the action line of one of them. The *sense* of a couple refers to the sign of its moment. We speak of positive and negative sense or counter-clockwise and clockwise senses. By *aspect* of a couple is meant the aspect * of its plane.

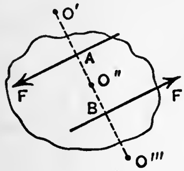


FIG. 14.

Proposition.—The moment of a couple is the same for all origins, and it equals the product of the magnitude of one of the forces of the couple and the arm. Proof: The moments of the couple of fig. 14 with respect to O', O'', and O''' are respectively,

$$-F\cdot\overline{O'A}+F\cdot\overline{O'B}=F\cdot\overline{AB},$$

$$F\cdot\overline{O''A}+F\cdot\overline{O''B}=F\cdot\overline{AB},$$

and

$$F\cdot\overline{O'''A}-F\cdot\overline{O'''B}=F\cdot\overline{AB}.$$

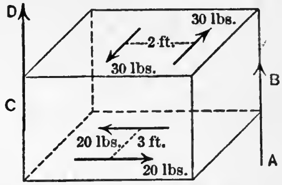
Since O', O'', and O''' represent all possible origins in the plane, the proposition is proved.

30. *Graphic Representation of a Couple.*—The moment of a couple, its aspect, and its sense may be represented by a vector. The length of the vector is made equal, by some scale, to the moment of the couple, it is drawn normal to the plane of the couple, and its arrow is made to correspond with the sense of the couple. This correspondence depends upon some arbitrary rule, such as the following: the arrow on the vector points toward the place from which the rotation of the couple appears counter-clockwise. The vector thus representing a couple will for brevity be called the vector of the couple.

For example, the couple in the upper face of the parallelo-

* Aspect of a plane refers not to its position, but to its direction; it is conveniently specified by the direction of a line normal to it.

piped of fig. 15 is represented by the vector AB or CD. Each vector also represents any other couple whose moment, aspect, and sense are the same as the moment, aspect, and sense of the upper couple; the couple in the lower face is such a one.



Vector scale: 1 in.=100 ft. lbs.

FIG. 15.

It is shown in art. 59 that couples whose moments, aspects, and senses are the same are equivalent. It is therefore consistent to represent by the same vector all couples whose moments, aspects, and senses are the same. Further, it follows from art. 242 that the effect of a couple applied to a rigid body depends only upon its moment, aspect, and sense, which are therefore the essential characteristics of a couple. Hence we may say that a vector completely represents a couple, although it does not give the forces or arm of the couple nor the position of its plane.

31. Resolution of a Force into a Force and a Couple.—*Proposition.*—A force may be resolved into a force acting through any arbitrarily chosen point and a couple.

Proof: Let F (fig. 16a) denote the force to be resolved, and P the chosen point, a distant from F. Imagine two opposite forces equal and parallel to F introduced at P (fig. 16b). Obviously the three forces of this figure are equivalent to the given force, i.e., they are components of it. But the three forces may be grouped into a force at P and a couple.

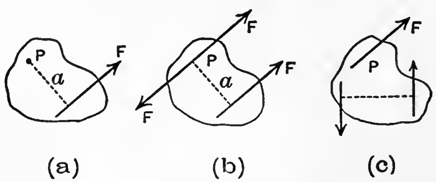


FIG. 16.

Observe that the component force has the same magnitude and direction as the given force, and that the moment of the component couple is the same as that of the given force about the chosen point.

Since couples whose moments, aspects, and senses are the same are equivalent, the force F (fig. 16a) is equivalent to F of Fig. 16c and the couple there represented, provided that its moment equals $F \cdot a$.

EXAMPLE.

Resolve the 8-lb. force (fig. 6) into one acting at B and a couple; into one acting at C and a couple.

§ II. COLLINEAR FORCES.

32. Composition.*—Let F_1, F_2, F_3, etc. (fig. 17), denote the forces to be compounded. From art. 21 it follows that the resultant of those forces of the system having the same sense is equal to their sum, that is,

FIG. 17.

the resultant of F_1, F_3, F_5, etc., or R', $=F_1+F_3+F_5+\ldots$,

and

the resultant of F_2, F_4, F_6, etc., or R'', $=F_2+F_4+F_6+\ldots$.

Also, the resultant of the system, or R, equals the difference between R' and R''. Hence if R be given sign, positive or negative according as it acts right or left,

$$R=R'-R''=(F_1+F_3+F_5+\ldots)-(F_2+F_4+F_6+\ldots)$$
$$=F_1-F_2+F_3-F_4+\ldots$$
$$\mathbf{R}=\Sigma\mathbf{F}.$$

The action line of R is of course the same as that of the given forces.

§ III. COPLANAR CONCURRENT NON-PARALLEL FORCES.

33. Graphical Composition.—Let AB, BC, CD, and DE (fig. 18) be the forces to be compounded. By the triangle law (art. 20), compound AB and BC and replace them by

* In the following articles on composition it is assumed that the force systems are applied to rigid bodies.

their resultant AC; compound AC and CD and replace them by their resultant AD; finally compound AD and DE and replace them by their resultant AE. By successive replacements, the given system has been reduced to a single force, AE, which is therefore the resultant sought. Notice that the lines AC, AD,

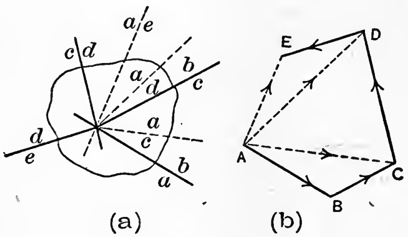


FIG. 18.

ac and *ad* are not necessary in a solution; they are used here only for demonstration purposes.

34. *Force Polygon.*—The figure formed by drawing in succession lines representing the magnitude and direction of the forces of any system is called a force polygon for those forces, or for the system. In fig. 18, $ABCDE$ is a force polygon for the given system. Several force polygons can be drawn for any system, one for each possible order of drawing the lines; thus, for the system compounded in the preceding article twenty-four polygons can be drawn, no two alike.

Proposition.—The algebraic sum of the components of any system of coplanar forces along any line equals the component along that line of a force whose magnitude and direction are represented by the line drawn from the beginning to the end of the polygon for the system.

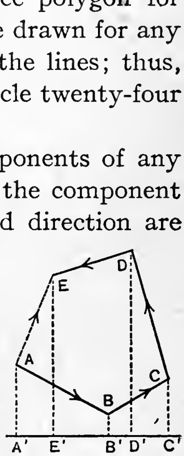


FIG. 19.

Proof: Let $ABCDE$ (fig. 19) be the polygon for the system. The components of the forces along the line $A'C'$ are, in magnitude and direction, $A'B'$, $B'C'$, $C'D'$, $D'E'$, and that of a force whose magnitude and direction are AE is represented by $A'E'$. From the figure, it is plain that $A'E'$ equals the algebraic sum of $A'B'$, $B'C'$, $C'D'$, and $D'E'$.

35. *Rule for Composition.*—Draw a force polygon for the forces; then a line *from* the beginning *to* the end of it. That line represents the magnitude and direction of the resultant; its action line passes through the common point of the action lines of the given forces.

Examining fig. 18, it will be seen that the vector sum of the given forces represents the magnitude and direction of the resultant.

EXAMPLES.

1. Determine the resultant of the forces represented in fig. 6.

2. Solve the preceding example, taking the forces in a different order in the force polygon.

36. Algebraic Composition. — Let fig. 20(a) represent the forces to be compounded and the body to which they are applied. Resolve each force at the origin into its x- and y-components, and replace it by them; the resulting system is represented in

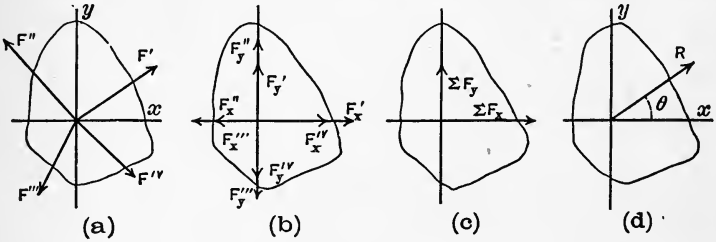


FIG. 20.

fig. 20(b). Next compound all the x-components and replace them by their resultant, ΣF_x, and compound the y-components and replace them by their resultant, ΣF_y; the resulting system is represented in fig. 20(c). Now these three systems are equivalent and have, therefore, the same resultant. If R denotes the resultant and θ its direction angle, from fig. 20(d) it is plain that

$$\mathbf{R} = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2)^{\frac{1}{2}},$$

$$\cos \theta = \Sigma \mathbf{F}_x / \mathbf{R} \quad \text{and} \quad \sin \theta = \Sigma \mathbf{F}_y / \mathbf{R},$$

and the action line of R contains the common point of those of the given forces.

Referring to the figure, or to prop., art. 34, it is plain that the resolved part of R along any line equals the algebraic sum of resolved parts of its components along the same line, that is,

$$\mathbf{R} \cos \theta = \Sigma \mathbf{F}_x.$$

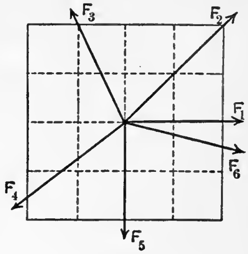


FIG. 21.

EXAMPLES.

1. Solve ex. 1 of the preceding article algebraically.

2. Let F_1, F_2, etc., fig. 21, equal 8, 4, 6, 12, 7, and 5 lbs. respectively, and compute their resultant.

§ IV. COPLANAR NON-CONCURRENT PARALLEL FORCES.

37. Graphical Composition.—Let AB, BC, CD, and DE (fig. 22) be the forces to be compounded. $ABCDE$ is a force

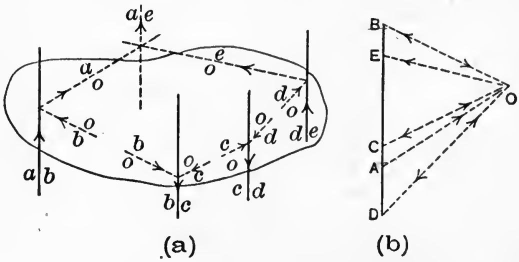


FIG. 22.

polygon for the given forces, and O is any arbitrarily chosen point.

First, resolve AB into AO and OB,
BC " BO " OC,
CD " CO " OD,
and DE " DO " OE.

Next replace each given force by its components. By art. 24, the action lines of AO and OB may intersect anywhere on ab,
" " " " BO " OC " " " " bc,
" " " " CO " OD " " " " cd,
" " " " DO " OE " " " " de.

It is therefore possible to choose the action lines of the components so that those of OB and BO, OC and CO, and OD and DO coincide. Having so taken them, it is plain that OB and BO, OC and CO, OD and DO, balance; therefore the system of components reduces to AO and OE. Finally, the resultant of AO and OE is AE (art. 20), which is also the resultant of the given system.

38. *Funicular Polygon, etc.*—The point O (fig. 22) is called the *pole* of the force polygon. Lines OA, OB, OC, etc., from the pole to the vertexes of the force polygon are called *rays*. Lines oa, ob, oc, etc., are called *strings*, which, considered collectively, are called a *string*, or *funicular polygon*.

If the notation for graphical statics (art. 11) be used, the following rules for drawing the funicular polygon will be found helpful, but the beginner should not depend entirely on them.

(*a*) The two strings intersecting on the action line of any force are parallel to the rays drawn to the ends of the vector corresponding to that force; or, the strings intersecting on mn are om and on.

(*b*) A string joining points in the action lines of two forces is parallel to the ray drawn to the common point of the vectors corresponding to those forces; or, the string joining points on lm and mn is parallel to OM.

39. *Rule for Composition.*—Draw a force and a funicular polygon for the forces; then a line *from* the beginning *to* the end of the force polygon, and a parallel one through the intersection of the first and last strings of the string polygon. The first line represents the magnitude and direction of the resultant, and the second its action line. (The first and last strings are those corresponding to the rays drawn to the beginning and end of the force polygon.)

It is plain from fig. 22 that the vector sum of the given forces represents the magnitude and direction of the resultant.

EXAMPLES.

Fig. 23 represents a board upon which several parallel forces are applied. In each example below determine completely the resultant.

1. The magnitudes of F_1, F_2, etc., are 40, 10, 30, 20, 50, and 15 lbs. respectively. Solve twice, drawing two funicular polygons, starting them at different points.

2. The magnitudes of F_1, F_2, etc., are 20, 10, 30, 30, 10, and 0 lbs. respectively. Solve twice, drawing two funicular polygons using different poles.

40. *The Resultant when the Force Polygon Closes.*—In that case, the beginning and end of the force polygon (A and E,

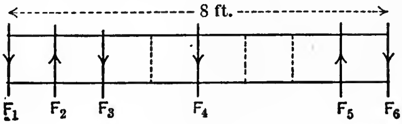


FIG. 23.

fig. 22) coincide; hence the first and last strings are parallel. The forces corresponding to these strings, and to which the given system is equivalent, are equal and opposite (AO and OE); hence they constitute a couple and can not be compounded.

If the strings ao and oe should happen to coincide, then the forces AO and OE would balance; and their resultant, and hence that of the given system, is zero. Therefore

> A funicular polygon for a system whose force polygon closes may be open or closed; if open, the resultant is a couple, and if closed, the resultant is zero.

It is worth noting that in the second case, force and funicular polygons closed, each segment of the funicular polygon is the action line of two forces.

EXAMPLES.

1. The magnitudes of F_1, F_2, etc. (fig. 23), are 20, 55, 0, 15, 10, and 30 lbs. respectively. Determine their resultant.

Solution: The force polygon is $ABCDEF$ (fig. 24), and since it is closed, the resultant is a couple. If we regard A as the beginning of the force polygon, the system compounds into two forces AO and OF whose action lines are ao and of respectively. They constitute the resultant couple; its arm is the dotted line of the space diagram. The arm scales 2.58 ft. and the forces 67.9 lbs., hence the moment of the couple is 175.2 ft.-lbs. The sense is clockwise.

2. Solve the preceding example, taking the forces in a different order in the force polygon.

41. The Principle of Moments.—The algebraic sum of the moments of any number of coplanar parallel forces with respect

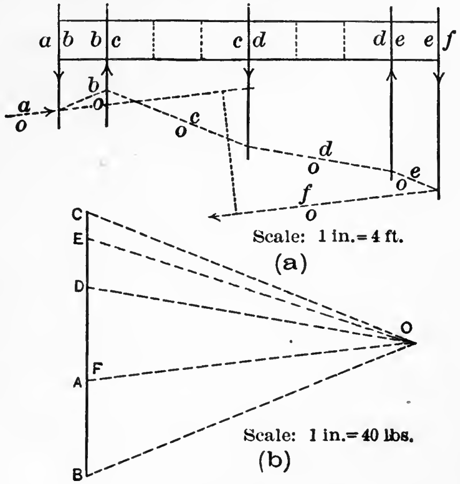


FIG. 24.

to any origin in their plane equals the moment of their resultant with respect to the same origin.

Proof: Let the system be that represented in fig. 22. Remembering

that AO and OB are concurrent components of AB,
" BO " OC " " " " BC, etc.,

and recalling also Varignon's theorem, we may write

moment of AB = moment of AO + moment of OB;
moment of BC = moment of BO + moment of OC;
moment of CD = moment of CO + moment of OD;
moment of DE = moment of DO + moment of OE.

Therefore

Σ(moments of AB, BC, CD, and DE)
$= \Sigma$(moments of AO, OB, BO, OC, CO, OD, DO, and OE).

But the moments of OB and BO, of OC and CO, etc., are equal and opposite, hence

$$\Sigma(\text{moments of } AB, BC, CD, \text{and } DE) = \text{moment of } AO + \text{moment of } OE.$$

If the resultant is a single force,

$$\text{moment of } AO + \text{moment of } OE = \text{moment of } AE;$$

if the resultant is a couple,

$$\text{moment of } AO + OE = \text{moment of the couple (see art. 29).}$$

Therefore, in either case, the sum of the moments of AB, BC, CD, and DE = the moment of their resultant. Proof may readily be extended to a system of more than four forces.

42. **Algebraic Composition.**—I. *If the algebraic sum of the forces is not zero*, the force polygon does not close (art. 34); hence the resultant is a force (art. 37). If F_1, F_2, etc., denote the forces and R their resultant,

$$\mathbf{R} = \Sigma\mathbf{F},$$

and the sense of R is given by the sign of ΣF.

The action line may be determined by means of the principle of moments; thus, if ΣM_0 denotes the algebraic sum of the moments of the forces with respect to any origin O, and a the corresponding arm of their resultant R,

$$\Sigma\mathbf{M}_0 = \mathbf{R}\mathbf{a}, \quad \text{or} \quad \mathbf{a} = \Sigma\mathbf{M}_0/\mathbf{R}.$$

The resultant must act on that side of O which will make the sign of its moment the same as that of ΣM_0.

II. *If the algebraic sum of the forces is zero*, the force polygon closes (art. 34); hence the resultant is a couple (art. 40). According to the principle of moments, the moment of this couple equals the algebraic sum of the moments of the given forces about any point.

EXAMPLES.

1. Solve ex. 1, art. 39, three times, using each time a new origin of moments.

2. Solve ex. 2, art. 39, twice, using different origins of moments.

3. Solve ex. 1, art. 40.

43. *Two Unequal Parallel Forces.*—This is a common case to which the following special methods may be applied. Let P and Q denote the forces, P the larger, and R their resultant.

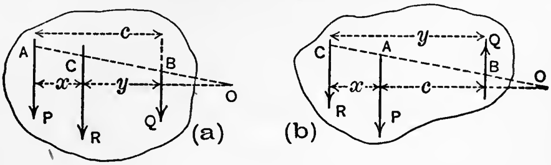


FIG. 25.

When P and Q are alike in sense (see fig. 25a),

$$R = P + Q,$$

the sense of R is the same as that of the given forces,

$$x = Qc/R \quad \text{and} \quad y = Pc/R.$$

When P and Q are unlike in sense (see fig. 25b),

$$R = P - Q,$$

the sense of R is the same as that of the larger force,

$$x = Qc/R \quad \text{and} \quad y = Pc/R.*$$

The student should prove the expressions for x and y and show that R is correctly represented in the figure, i.e., that its action line is between the forces in (a), but beyond the larger force in (b).

The following relations are sometimes convenient:

From either figure $Px = Qy$, therefore

$$P\overline{AC} = Q\overline{BC}, \quad \text{or}$$
$$AC/BC = Q/P;$$

hence the action line of the resultant of two parallel forces divides any secant intersecting their action lines into two seg-

* It is plain from these equations that the smaller R is (P and Q nearly equal) the greater x and y are. As P and Q approach equality, they become more nearly equivalent to a couple; also R approaches zero, and its arm with respect to any origin in the body to which P and Q are applied approaches ∞. *Hence we arrive at the conception that a couple is equivalent to a force of zero magnitude with an infinite arm.*

ments which are inversely proportional to the magnitudes of those forces. Again,

$$P/BC = Q/AC = (P+Q)/(BC+AC) = (P-Q)/(BC-AC),$$

or

$$P/BC = Q/AC = R/AB;$$

hence the forces P, Q, and R are proportional to the distances between the other two.

EXAMPLES.

1. Determine the resultant of F_1 and F_2 of ex. 1, art. 39.
2. Determine the resultant of F_2 and F_3.

§ V. Coplanar Non-concurrent Non-parallel Forces.

44. Graphical Composition.—First method. This consists in compounding two of the forces, then their resultant and a third force, that resultant and a fourth, etc., until the simplest equivalent system has been found. Thus, let AB, BC, CD,

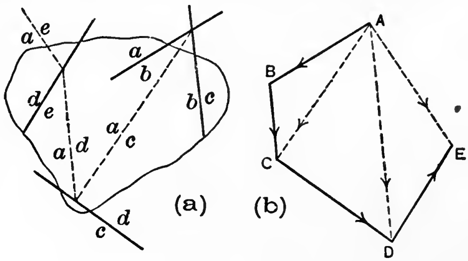


Fig. 26.

and DE (fig. 26) be the forces of such a system. According to art. 20,

the resultant of AB and BC is AC,
" " " AC " CD " AD,
and " " " AD " DE " AE.

Therefore AE is the resultant sought.

It may happen that some of the intersections in the space diagram which are necessary do not fall within convenient limits. This is apt to occur when the action lines of the given forces are nearly parallel. In such cases it is practically impossible to determine the action line of the resultant, and the following method should be employed.

Second method. This is the same as that for the composition of coplanar non-concurrent parallel forces explained in

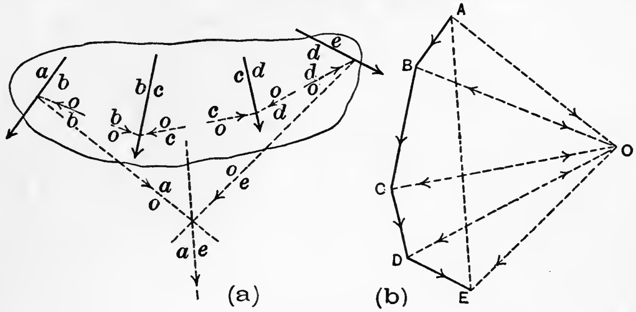


FIG. 27.

arts. 37, 38, and 39, which the student should read in connection with fig. 27.

EXAMPLES.

1. Let F_1, F_2, etc., in fig. 28 equal 8, 4, 6, 7, 12, and 5 lbs. respectively, and determine their resultant.*

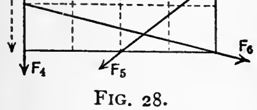


FIG. 28.

2. Solve ex. 1, drawing a second funicular polygon in the same space diagram beginning it at some other point.

3. Solve ex. 1, choosing a new pole but employing the same space diagram.

45. *The Resultant when the Force Polygon Closes.*—Just as in the case of parallel forces, the resultant is a couple. In fact the explanation in art. 40 may be read in connection with fig. 27.

46. The Principle of Moments.—The algebraic sum of the moments of any number of coplanar forces with respect to any origin equals the moment of their resultant with respect to the same origin.

The proof given in art. 41, read in connection with fig. 27, applies to this proposition.

47. Algebraic Composition.—I. *If the algebraic sum of the x and y-components of the forces are not both zero*, the force

* Use scales not less than 1 in. = 2 ft. and 1 in. = 4 lbs.

polygon for the system does not close (art. 34); hence the resultant is a force (art. 44). Since

$$R_x = \Sigma F_x \quad \text{and} \quad R_y = \Sigma F_y,$$

$$\mathbf{R} = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2)^{\frac{1}{2}},$$

$$\sin\theta = \Sigma \mathbf{F}_y / \mathbf{R} \quad \text{and} \quad \cos\theta = \Sigma \mathbf{F}_x / \mathbf{R},$$

θ being the direction angle of R measured from the x axis.

The action line of the resultant may be found by the principle of moments. If ΣM_0 denotes the algebraic sum of the moments of the forces with respect to any origin O, and a the corresponding arm of R,

$$\Sigma \mathbf{M}_0 = \mathbf{R} \cdot \mathbf{a}, \quad \text{or} \quad \mathbf{a} = \Sigma \mathbf{M}_0 / \mathbf{R}.$$

The resultant must act on that side of O which will make the sign of its moment the same as that of ΣM_0.

II. *If the algebraic sums of the x and y-components equal zero*, the force polygon for the system closes (art. 34); hence the resultant is a couple (art. 45). According to the principle of moments, the moment of the couple equals the algebraic sum of the moments of the given forces about any origin.

For a system of couples, ΣF_x and ΣF_y equal zero; hence their resultant is a couple, and its moment equals the algebraic sum of the moments of the given couples.

EXAMPLES.

1. Solve ex. 1, art. 44, algebraically.

Solution: It will be convenient to tabulate the computation.

F	α	a	F_x	F_y	$F.a$
8 lbs.	0	2.00 ft.	+8.00	0	−16.00
4 "	45°	1.41 "	+2.83	+2.83	− 5.64
6 "	63° 25′	1.79 "	−2.68	+5.36	−10.74
7 "	90°	4.00 "	0.00	−7.00	+28.00
12 "	36° 52′	3.20 "	−9.60	−7.20	−38.40
5 "	14° 2′	2.91 "	+4.85	−1.21	+14.55
			+3.40	−7.22	−28.23

In the first column the force magnitudes are recorded, in the second the acute angle between each force and the x axis (taken horizontal), and in the third the arms of the force with respect to the selected origin (centre of the board); then in the fourth, fifth, and sixth columns the computed values of the x-components, the y-components, and the moments. The algebraic sums of these are 3.40 lbs., −7.22 lbs., and −28.23 ft.-lbs. respectively. Hence

(1) $$R=\sqrt{3^2.40+7^2.22}=7.98 \text{ lbs.};$$

(2) the sense of R is downward to the right, the angle with the x-axis being

$$\sin^{-1} 7.22/7.98=64° 47';$$

(3) the action line of R is to the right of the centre of the board a distance $28.23/7.98=3.54$ ft.

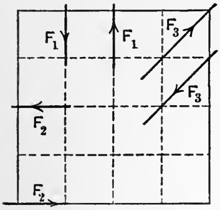


FIG. 29.

2. Solve the preceding example, choosing a different origin of moments.

3. Let F_1, F_2, etc. (fig. 28), be 20, 14.14, 22.36, 15, 25, and 0 lbs. respectively. Compute their resultant.

4. Let F_1, F_2, and F_3 (Fig. 29) be 10, 20, and 30 lbs. respectively. Determine the resultant, the square being 4×4 ft.

48. Reduction of a System to a Force and a Couple.—*Proposition.*—Any system of forces can be compounded into a force acting through any arbitrarily chosen point and a couple.

Proof: * Each force of the given system may be replaced by a force acting through the chosen point and a couple (art. 31). Suppose such a replacement made for each force; the resulting system consists of a concurrent one and a system of couples. But the resultant of the concurrent forces is a single force acting through the chosen point (art. 33 or 36), and the resultant of the couples is a single couple (art. 47).

Computation of the force and the couple. Let F (fig. 30) be one of the forces of the given system and O the chosen point. The components of F are F' (applied at O) and the couple FF''. Observe now that the component of F' along any line is the same in magnitude and sense as that of F along the same line and that the moment of the couple, FF'', is the same as that

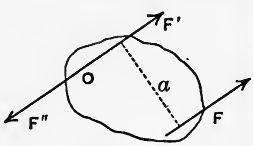


FIG. 30.

* This proof is for a coplanar system, the chosen point being in the plane of the forces. Proof for the general case is given in art. 55.

of F about O. Hence the algebraic sum of the components of all the forces of the concurrent system along any line is the same as that of the components of the given forces along that line, and the sum of the moments of the couples equals the sum of the moments of the given forces about O.

If R_0 denotes the resultant of the concurrent system, and C that of the couples,

$$\mathbf{R}_0 = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2)^{\frac{1}{2}},$$
$$\sin\theta = \Sigma \mathbf{F}_y/\mathbf{R}, \quad \cos\theta = \Sigma \mathbf{F}_x/\mathbf{R}$$

θ being the direction angle of the action line of R, and $\mathbf{C} = \Sigma \mathbf{Fa}$.

EXAMPLE.

Let F_1, F_2, etc. (fig. 28) equal 14, 10, 3, 9, 18, and 11 lbs. Reduce them to a force at the centre and a couple.

§ VI. Non-coplanar Concurrent Forces.

49. Graphical Composition.—Imagine a force polygon $ABC \ldots N$ drawn for the forces to be compounded; it is not of course a plane one. According to the triangle of forces, the resultant of the first two forces, R', is represented in magnitude and direction by AC. Likewise the resultant of R' and the third force, i.e., the resultant of the first three forces, is represented in magnitude and direction by AD, etc. Finally, the resultant of all the given forces is represented by the line AN, joining the beginning and the end of the force polygon.

> Hence the magnitude and direction of the resultant is represented by the vector sum of the given forces, or, otherwise stated, by the line drawn from the beginning to the end of the force polygon for the system.

The action line of the resultant, of course, passes through the common point of the action lines of the given forces.

Since the force polygon is not plane, it is practically necessary to represent it by projections. The line representing the resultant may then be determined by its projections, in the projections of the polygon.

EXAMPLES.

1. Suppose three forces F_1, F_2, and F_3 to act at a point O as shown in fig. 31(a). The horizontal and vertical projections

of O are marked O' and O'' respectively, those of the vector representing F_1 are marked F_1' and F_2'', etc.*

Solution: At any point A we begin to construct the polygon for the forces—$A'B'C'D'$ is its horizontal and $A''B''C''D''$ is its vertical projection. Hence the line AD represents the magni-

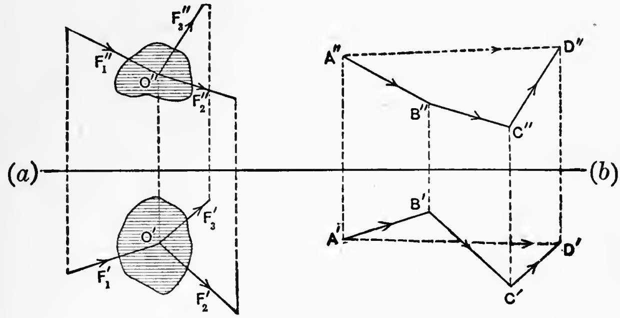


FIG. 31.

tude and direction of the resultant; the action line passes through O.

If the horizontal and vertical projections of the vectors representing the given forces be regarded as force systems, then obviously $A'B'C'D'$ and $A''B''C''D''$ are the force polygons for those systems respectively.

2. Imagine the forces of fig. 38(c) to act in the directions indicated but through the corner diagonally opposite O, and determine their resultant.

50. Algebraic Composition.—Let F', F'', F''', etc., denote the forces to be compounded. At the common point of their lines of action, resolve each force into three rectangular (x, y, and z) components and replace the forces by them. Next, compound separately all the x, y, and z-components and replace them by their resultants, ΣF_x, ΣF_y, and ΣF_z. Now these three systems, the given one, the system of components, and the three forces, ΣF_x, ΣF_y, and ΣF_z, are equivalent; they have, therefore, the same resultant. If R denotes the resultant, and

* All horizontal and vertical projections are marked by primes and double primes respectively.

θ_1, θ_2, and θ_3 the direction angles of its action line, from the third system it is plain that

(1) $$\mathbf{R} = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2 + \overline{\Sigma \mathbf{F}_z}^2)^{\frac{1}{2}},$$

(2) $$\cos \theta_1 = \Sigma \mathbf{F}_x/\mathbf{R}, \quad \cos \theta_2 = \Sigma \mathbf{F}_y/\mathbf{R}, \quad \cos \theta_3 = \Sigma \mathbf{F}_z/\mathbf{R},$$

(3) the action line passes through the common point O.

EXAMPLE.

Solve ex. 2, art. 49, by the method of this article.

§ VII. Non-coplanar Parallel Forces.

51. Graphical Composition.—The vectors F_1 and F_2 (fig. 32) represent two parallel forces, F_1' and F_2' their projections on the xz plane, and F_1'' and F_2'' those on the yz plane. Let R represent the resultant of the two forces, and R' and R'' the projections of the vector R upon the plane xz and yz respectively.

If F_1' and F_2' be regarded as forces, R' represents their resultant for

$$F_1/F_2 = \overline{CB}/\overline{CA} = \overline{C'B'}/\overline{C'A'} = F_1'/F_2',$$

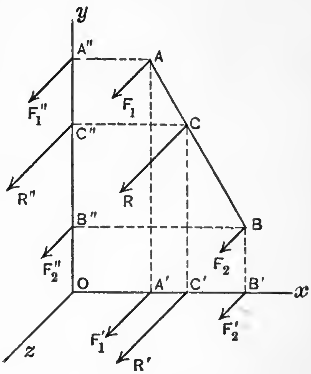


Fig. 32.

i.e., R' divides the line $A'B'$ into segments inversely proportional to F_1' and F_2'; and obviously $R' = F_1' + F_2'$. Similarly, if F_1'' and F_2'' be regarded as forces, R'' represents their resultant. To find the resultant, then, of two parallel forces,

> Project the vectors representing them upon two planes parallel to the forces, and find the resultants of these projections regarded as forces. These resultants are projections of the resultant sought.

This method may obviously be extended to the composition of more than two forces.

EXAMPLE.

Suppose parallel forces of 14, 12, 16, and 8 lbs. applied to a body at points whose x and y coordinates are respectively (2, 4), (3, 5), (4, 2), and (6, 3), all in feet, and suppose that the third force acts in the negative and the others in the positive direction.

Solution: The vectors representing the forces are projected

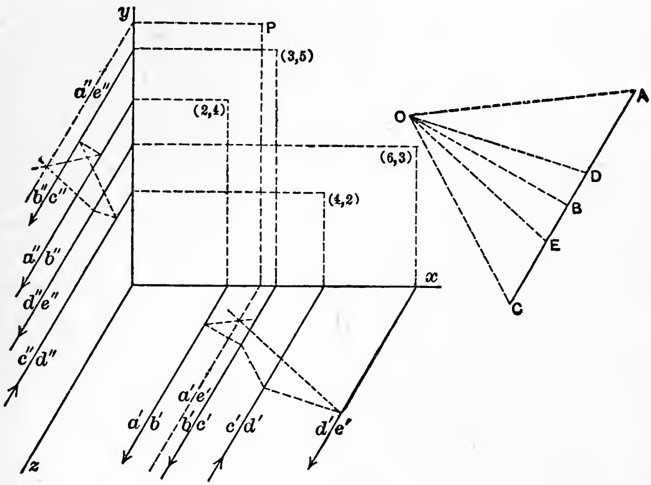


FIG. 33.

on the yz and zx planes; thus $a'b'$ and $a''b''$ (fig. 33) are the projections of the 14-lb. force, $b'c'$ and $b''c''$ those of the 12-lb. force, etc. Each projected system may be compounded by the method of art. 37, one force polygon, as $ABCDE$, sufficing for the two systems. The funicular polygon in the xz plane determines $a'e'$, and that in the yz plane determines $a''e''$. The action line of the resultant sought then passes through the point P in the xy plane. The magnitude and sense of the resultant are given by AE.

52. *The Resultant when the Force Polygon Closes* is a couple. For, obviously, the resultant of all the forces of the system but one is equal and opposite to that one; hence that resultant and the last force constitute the resultant couple. The forces and arm of this couple depend on which force of the system is omitted; accordingly, the system may be reduced

by this method to as many different couples as there are forces in the system, but they are equivalent to each other.

Let the system consist of F_1, F_2, F_3, etc., and let R denote the resultant of F_2, F_3, etc.; then the resultant couple consists of F_1 and R. Imagine the vectors representing the forces to be projected on two planes and, as above, denote the projections on one plane by F_1', F_2', F_3', . . . and R', and those on the other by F_1'', F_2'', F_3'', . . . and R''. Evidently, the resultants of the systems F_1', F_2', F_3', etc., and F_1'', F_2'', F_3'', etc., are couples, and the resultant of the first is F_1', R', and that of the second is F_1'', R''.

It might of course happen that F_1 and R coincide. In that case the resultant of the given system would be zero; and since F_1' and R' and F_1'' and R'' would also coincide, the resultants of the systems F_1', F_2', F_3', etc., and F_1'', F_2'', F_3'', etc., would be zero, and *the funicular polygons for those systems would close* (art. 40).

EXAMPLE.

Solve exs. 1 and 2, art. 54, graphically.

53. The Principle of Moments.—The algebraic sum of the moments of any number of parallel forces with respect to a line equals the moment of their resultant with respect to that line.

Proof: Let F_1 and F_2 (fig. 34) be two forces of the system,

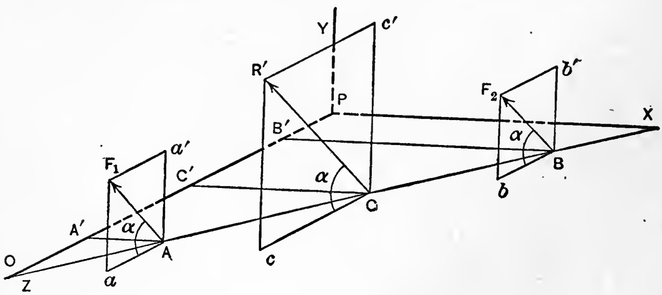


FIG. 34.

R' their resultant, and OP the moment axis. OP is regarded as a z coordinate axis, and the y axis is so taken that the yz plane is parallel to the forces; A, B, and C are the points where F_1, F_2, and R' respectively pierce the xz plane; and

Aa and Aa' represent the components of F_1 parallel and perpendicular respectively to the moment axis, Bb and Bb' those of F_2, and Cc and Cc' those of R'. The moments of the three forces are respectively

$$(F_1 \sin\alpha)\overline{AA'},\quad (F_2 \sin\alpha)\overline{BB'},\quad \text{and}\quad (R' \sin\alpha)\overline{CC'}.$$

Evidently, the perpendicular component of R is the resultant of the perpendicular components of F_1 and F_2. Then, according to the principle of moments for coplanar forces,

$$(R' \sin\alpha)\overline{OC} = (F_1 \sin\alpha)\overline{OA} + (F_2 \sin\alpha)\overline{OB}.$$

This equation multiplied through by sin (POB) becomes

$$(R' \sin\alpha)\overline{CC'} = (F_1 \sin\alpha)\overline{AA'} + (F_2 \sin\alpha)\overline{BB'}.$$

That is, the moment of R' equals the sum of the moments of F_1 and F_2.

If F_1 and F_2 have unlike senses, a slight change in the proof is necessary, which the student can make. The extension of the proof to more than two forces is quite evident.

For the exceptional case in which the resultant is a couple, the proposition still holds if we define the moment of a couple with respect to a line to be the algebraic sum of the moments of its forces with respect to that line.

54. Algebraic Composition.—I. *If the algebraic sum of the forces is not zero*, the force polygon for the system does not close; hence the resultant is a force (art. 51). If F_1, F_2, etc., denote the forces and R their resultant,

$$\mathbf{R} = \Sigma \mathbf{F};$$

the *sense* of R being given by the sign of ΣF.

The action line may be determined by the principle of moments; thus, if ΣM_x and ΣM_y denote the algebraic sums of the moments of the forces with respect to two axes, x and y, which are perpendicular to the forces, and a_x and a_y the corresponding arms of the resultant, R,

$$\Sigma M_x = Ra_x \quad \text{and} \quad \Sigma M_y = Ra_y,$$

whence
$$\mathbf{a}_x = \Sigma \mathbf{M}_x / \mathbf{R}, \quad \mathbf{a}_y = \Sigma \mathbf{M}_y / \mathbf{R}.$$

The resultant must act on such sides of the x and y axis that the signs of its moments are the same as those of ΣM_x and ΣM_y respectively.

II. *If the algebraic sum of the forces is zero*, the force polygon closes; hence the resultant is a couple (art. 52). As explained in the article referred to, the resultant couple is not determinate. A resultant couple can be readily found by computing the resultant of all the forces of the system but one; this resultant force and the omitted force constitute a couple, the resultant of the system.

EXAMPLES.

1. Compute the resultant of five forces of 15, 12, 20, 16, and 21 lbs., the first three acting in the positive z direction and the last two in the negative, the coordinates of the points in which they pierce the xy plane being respectively (2, 3), (4, −2), (2, 4), (3, −1), and (0, 0).

2. Include a force of 10 lbs. acting in the negative z direction at a point whose coordinates are (−8, 10), and compound.

§ VIII. Non-coplanar, Non-concurrent, Non-parallel Forces.

55. The Resultant.—*Proposition I.* A system of non-coplanar, non-concurrent, non-parallel forces may be compounded into two forces, the action line of one being in, and that of the other normal to, any arbitrarily selected plane.

Proof: Let the action lines of the forces of the system be extended till they pierce the selected plane.* This will be referred to as the plane π. At each of these points resolve the corresponding force into two components, one in the plane and the other normal to it. The given system may be regarded as replaced by two, a coplanar system (the components in π), and a parallel system (the components normal to π). In general, each of these systems compounds into a single force, but either

* A force whose action line is parallel to the plane should be replaced by its equivalent, a force in the plane and a couple whose forces are perpendicular to it. (Art. 31.)

or both of them may compound into a couple (arts. 45 and 51). But, as shown in art. 42, a couple may be regarded as "a zero force with an infinite arm," and with this understanding the proposition holds in all cases.

Since the two forces do not usually intersect, it is not possible, in general, to compound them into one force. They may therefore be properly called a resultant of the system. For different planes π we arrive at different pairs of resultant forces, but since they are equivalent to the same system they are equivalent to each other. We will denote the resultant forces in and normal to the plane by R_t and R_n respectively.

Proposition II. A system of non-coplanar, non-concurrent, non-parallel forces may be compounded into a force acting through any arbitrarily selected point and a couple.

Proof: Each force of the given system may be replaced by a force acting through the chosen point and a couple (art. 31). Suppose such a replacement made for each force; the resulting system consists of a concurrent system of forces and a system of couples. But the resultant of the concurrent forces is a single force acting through the chosen point (arts. 49 or 50), and the resultant of the couples is a single couple (art. 60). We will denote this force and couple by R and C respectively.

In general, R and C may be compounded into two non-parallel unequal forces. For, C may be replaced by an equivalent couple one of whose forces intersects R (see art. 59), and that force and R may be compounded into a force; since this last force will not be in the same plane with the other force of the couple these two forces cannot be compounded, and they may properly be called a resultant of the system.

If the plane of C happens to be parallel to R, C and R may be compounded into a single force. For, C may be replaced by an equivalent couple whose plane coincides with R; and, the forces of that couple and R being coplanar, their resultant is a single force. That force is the resultant of the system.

In the following three articles, methods are explained for determining the resultants above discussed.

56. Graphical Composition.—The graphical method can be employed to determine R_t and R_n, the method of art. 44 for R_t and that of art. 51 for R_n. Some elementary principles of descriptive geometry are employed for determining the intersections of the lines of action of the forces of the given system with the plane π. The method will be illustrated by means of an

EXAMPLE.

Let there be three forces in the system, the horizontal projections of the vectors representing them being F_1', F_2', and F_3' (fig. 35), and the vertical projections F_1'', F_2'', and F_3''.

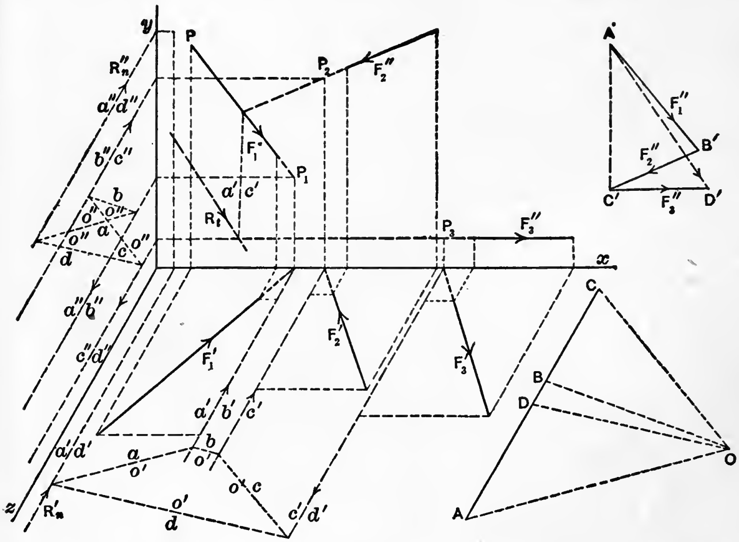


FIG. 35.

The plane π is taken to coincide with the xy plane, the forces piercing it in the points P_1, P_2, and P_3 respectively. The components of the given forces in the plane π are represented by the vectors F_1'', F_2'', and F_3'', while the components normal to that plane act through the points P_1, P_2, and P_3, and their magnitudes are represented by the projections of F_1', F_2', and F_3' upon the z axis.

The resultant of the coplanar system is represented in magnitude and direction by $A'D'$, $A'B'C'D'$ being a force polygon for the system, and its action line is marked R_t.

The resultant of the normal system is represented in magnitude and direction by AD, $ABCD$ being a force polygon for the system. The funicular polygon for the projection of the normal system on the xz plane is oa', ob', oc', and od', and R_n' is the projection of R_n on that plane. The funicular polygon for the projection of the normal system on the yz plane is oa'', ob'', oc'', and od'', and R_n'' is the projection of R_n on that plane. Hence the action line of R_n pierces the plane at P.

57. Principle of Moments.—The moment of the resultant * of any system of forces about a line equals the sum of the moments of the forces.

This follows from the proposition that the sums of the moments of the forces of equivalent systems about any line are the same, which we now prove.

Let l denote the line or axis of moments and imagine any plane containing it as the plane π. Each of the systems may be compounded into a resultant consisting of two forces, one in and one normal to the plane π; since the resultants are identical,† R_t and R_n may denote the forces of each resultant.

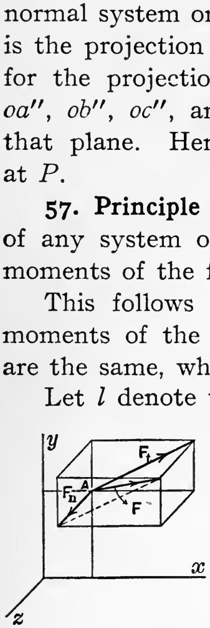


FIG. 36.

Let F (fig. 36) be one of the forces of either system. The x axis is taken along the line l and the y axis in the plane π; then the xy and π

* Here "moment of the resultant" means the algebraic sum of the moments of the forces of the resultant.

† Proof: Suppose that they are not identical, and call the forces of the resultant of one system R_t' and R_n', and those of the other R_t'' and R_n''. Being resultants of equivalent systems, they are equivalent to each other; hence if the forces of one resultant be reversed, the four forces would balance, i.e., their resultant would be zero. Now the resultant of R_t' and R_t'' reversed, is in the plane π, so call it R_t'''; that of R_n' and R_n'' reversed is normal to that plane, so call it R_n'''. But the resultant of R_t''' and R_n''' cannot be zero unless each is zero, and that is impossible unless R_t' and R_t'' and R_n' and R_n'' are identical.

planes coincide. F is shown resolved at A into two components (F_t and F_n) in and normal to the plane π. According to art. 28,

the moment of F about l = the moment of F_n about l;

hence

$$\Sigma(\text{mom. } F) = \Sigma(\text{mom. } F_n).$$

But R_n is the resultant of all the F_n's, and, according to art. 53,

$$\text{mom. } R_n \text{ about } l = \Sigma(\text{mom. } F_n) = \Sigma(\text{mom. } F);$$

hence the sum of the moments of all the forces of either system equals that of R_n, that is, the sums of the moments of the forces of the equivalent systems are equal. Q.E.D.

58. Algebraic Composition. — *Determination of R_t and R_n.* Fig. 37 represents one force, F, of a system and its components

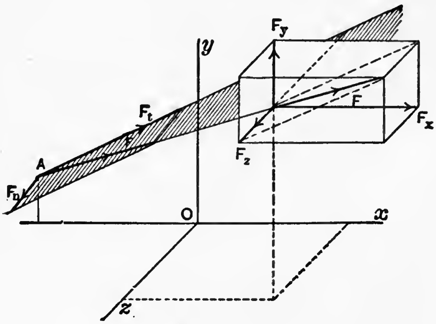


FIG. 37.

F_x, F_y, and F_z, also its components F_t and F_n, the plane π being taken coincident with that of the x and y axes. From the figure, it is plain that F_n equals F_z and that the x- and y-components of F_t are the same in magnitude and direction as those of F. It follows that

(1) $$\mathbf{R}_n = \Sigma \mathbf{F}_z \quad \text{and} \quad \mathbf{R}_t = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2)^{\frac{1}{2}};$$

(2) the sense of R_n is given by the sign of ΣF_z;

(3) the angles which R_t makes with the x and y axes are

$$\cos^{-1} \Sigma \mathbf{F}_x/\mathbf{R}_t \quad \text{and} \quad \cos^{-1} \Sigma \mathbf{F}_y/\mathbf{R}_t.$$

The action lines of R_n and R_t are determined by the principle of moments, thus:

the arm of R_n with respect to any line in the plane π equals (the moment sum of the given forces with respect to that line)$/R_n$;

the arm of R_t with respect to any line normal to π equals (the moment sum of the given forces with respect to that line)$/R_t$.

Determination of R and C. Let F_1, F_2, F_3, etc., denote the forces to be compounded and O (fig. 38*a*) the point through which R is to pass. The components of F_1 are the force F_1' at O and the couple (F_1, F_1''); the components of F_2 are the force F_2' at O and the couple (F_2, F_2''); etc.

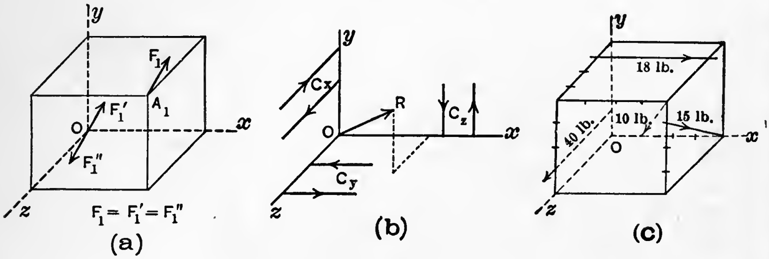


FIG. 38.

It is plain that the components of F_1', F_2', etc., along any line are the same in magnitude and direction as those of F_1, F_2, etc., respectively; hence

$$\mathbf{R} = (\overline{\Sigma \mathbf{F}_x}^2 + \overline{\Sigma \mathbf{F}_y}^2 + \overline{\Sigma \mathbf{F}_z}^2)^{\frac{1}{2}},$$

$$\cos\theta_1 = \Sigma \mathbf{F}_x/\mathbf{R}, \quad \cos\theta_2 = \Sigma \mathbf{F}_y/\mathbf{R}, \quad \cos\theta_3 = \Sigma \mathbf{F}_z/\mathbf{R},$$

ΣF_x, ΣF_y, and ΣF_z denoting the algebraic sums of the x,- y-, and z-components of the *given* forces, and θ_1, θ_2, and θ_3 the direction angles of R (art. 50).

Imagine the couple C resolved into three components whose planes are respectively perpendicular to the x, y, and z axes (art. 61) and denote them by C_x, C_y, and C_z. Now the given system and the system R, C_x, C_y, and C_z (fig. 38*b*), have the same resultant, and hence their moment sums with respect to any axis are equal. But the moment sums of R, C_x, C_y, and C_z, with respect to the x, y, and z axes are C_x, C_y, and C_z. Hence, if ΣM_x, ΣM_y, and ΣM_z denote the moment sums of the given system with respect to the coordinate axes,

$$C_x = \Sigma M_x, \quad C_y = \Sigma M_y, \quad \text{and} \quad C_z = \Sigma M_z.$$

Also, by art. 60, $\mathbf{C} = (\overline{\Sigma \mathbf{M}_x}^2 + \overline{\Sigma \mathbf{M}_y}^2 + \overline{\Sigma \mathbf{M}_z}^2)^{\frac{1}{2}}$ and

$$\cos \phi_1 = \Sigma \mathbf{M}_x/\mathbf{C}, \quad \cos \phi_2 = \Sigma \mathbf{M}_y/\mathbf{C}, \quad \cos \phi_3 = \Sigma \mathbf{M}_z/\mathbf{C},$$

ϕ_1, ϕ_2, and ϕ_3 denoting the direction angles of the vector of C.

EXAMPLES.

1. Compound the four forces of fig. 38 (c) into a force at O and a couple, the edges of the cube being 4 ft. long.

2. Compound the four forces into two whose action lines are in and normal to the xy plane.

§ IX. Theory of Couples.

59. Equivalent Couples.—*Proposition.*—Two couples whose moments, aspects, and senses are the same are equivalent; or, otherwise stated, two couples whose vectors are the same are equivalent.

Proof: I. The planes of the couples coincide. Let (Pp)* and (Qq) (fig. 39) be the two couples, F_1 (identical with Q_1), F_2, F_3 . . . a system of which (Pp) is the resultant. Then the vector sum of F_1, F_2, F_3, . . . F_n is zero, and their moment sum about any point equals Pp.

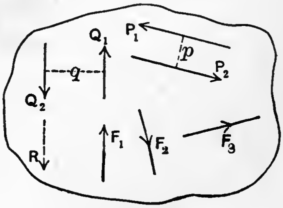


FIG. 39.

It will now be shown that the couple (Qq) is also the resultant of the system F_1, F_2, F_3, . . . by showing that the system can be compounded into two forces which are identical with the forces of the couple. One of those two forces is F_1, and the other one is the resultant of F_2, F_3, . . . which call R. Since the vector sum of all the forces of the sys-

* By "couple (Pp)" is meant one whose forces are P and arm is p. For convenience of designation, the forces of the couple are sometimes marked P_1 and P_2; then P_1 equals P_2.

tem is zero, R is equal and opposite to F_1, i.e., R is the same in magnitude and direction as Q_2. If a denotes the arm of R with respect to an origin *on* Q_1, then

$$Ra = \text{moment sum of } F_2, F_3, \ldots \text{ (for that origin)} = Pp = Qq.$$

Since R equals Q, $a=q$, i.e., R and Q_1 coincide.

Finally, since (Pp) and (Qq) are the resultants of the same system, they are equivalent.

II. The planes of the two couples are parallel. Let (Pp) and (Qq) (fig. 40) be the two couples. According to case I, the couple (Qq) can be replaced by any other in its own plane provided that its moment and sense are the same as those of (Qq). Let (Ss) be such a couple, its forces being parallel to those of (Pp) and one in the y axis. Imagine a system of parallel forces F_1 (identical with S_1), F_2, F_3, . . . the resultant of which is the couple (Pp). The vector sum of the system is zero, and its moment sums with respect to the x and z axes equal the moments of its resultant (Pp) with respect to the same axes, which obviously are zero and Pp respectively.

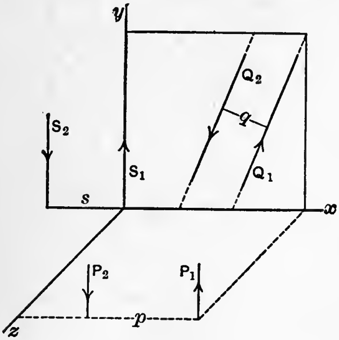


FIG. 40.

It will now be shown that the couple (Ss), and therefore (Qq), is also the resultant of F_1, F_2, F_3, . . . by showing that the system can be compounded into two forces which are identical with the forces of (Ss). One of these two forces is F_1, and the other is the resultant of F_2, F_3, . . . which call R. Since the vector sum of all the forces of the system is zero, R is equal and opposite to F_1, i.e., it is the same in magnitude and direction as S_2. Let a and c denote the arms of R with respect to the x and z axes respectively; then

$$Ra = \text{moment sum of } F_2, F_3, \ldots \text{ about the } x \text{ axis} = 0$$

and

$$Rc = \text{`` `` `` } F_2, F_3, \ldots \text{ `` `` } z \text{ `` } = Pp = Ss.$$

Hence $a=0$ and $c=s$; i.e., R and S_2 coincide.

Finally, since (Pp) and (Qq) are the resultants of the same system, they are equivalent.

60. Composition of Couples.—*Proposition I.* The resultant of any number of couples is a couple.

Proof: Let $AA'B'B$ and $AA''B''B$ (fig. 41) be the planes of

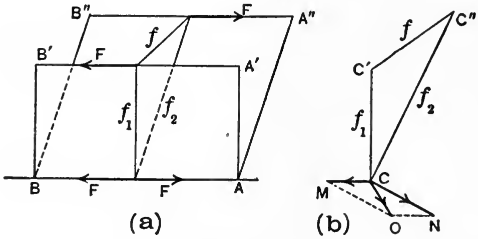


FIG. 41.

two of the couples. Replace the couples by equivalent ones (Ff_1) and (Ff_2) and so that a force of one couple shall "balance" a force of the other; these balancing forces must lie in AB. The four forces are equivalent to two, F in $A'B'$ and F in $A''B''$, which clearly constitute a couple, the resultant of the two given couples. Similarly, the resultant of this resultant couple and the third couple is a couple, etc.

The student should supply a proof for the case in which the planes of the couples are parallel.

Proposition II. The vector of the resultant of any number of couples is the sum of the vectors of those couples.

Proof: Consider first two couples, those compounded above. Let CM and CN (fig. 41) be their vectors, then CO is the sum of CM and CN. (In fig. 41*b*, the planes of the given couples are represented by their traces CC' and CC'' with a plane perpendicular to AB.) It must be shown that CO is the vector of the resultant couple (Ff).

By construction, $\overline{CM}/Ff_1=\overline{CN}/Ff_2$, and the angles CNO and $C'CC''$ are equal; hence the triangles CON and $CC'C''$ are similar, and

$$\overline{CO}/f=\overline{CM}/f_1=\overline{CN}/f_2, \quad \text{or} \quad \overline{CO}/Ff=\overline{CM}/Ff_1=\overline{CN}/Ff_2;$$

that is, the length of the vector CO represents the moment of the resultant couple to the same scale according to which CM and CN represent the moments of the given couples. Since CM is perpendicular to CC', CO is perpendicular to $C'C''$; i.e., the vector CO is normal to the plane of the resultant couple. From an inspection of the figures, it is plain that the arrow on the vector OC and the sense of the resultant couple agree in accordance with the rule of vector representation of couples (see art. 30). The proof is easily extended to more than two couples.

Special Cases.—(*a*) Three couples whose planes are mutually at right angles. Let the three planes be taken as coordinate planes and call the couples whose planes are perpendicular to the x, y, and z axis C_x, C_y, and C_z respectively, C their resultant, and v_x, v_y, v_z, and v their vectors. Then

$$v=(v_x^2+v_y^2+v_z^2)^{\frac{1}{2}}; \quad \text{hence}$$
$$\mathbf{C}=(\mathbf{C}_x^2+\mathbf{C}_y^2+\mathbf{C}_z^2)^{\frac{1}{2}}.$$

Also, if ϕ_1, ϕ_2, and ϕ_3 denote the direction angles of v,

$$\cos\phi_1=v_x/v, \quad \cos\phi_2=v_y/v, \quad \cos\phi_3=v_z/v;$$

hence

$$\cos\phi_1=\mathbf{C}_x/\mathbf{C}, \quad \cos\phi_2=\mathbf{C}_y/\mathbf{C}, \quad \cos\phi_3=\mathbf{C}_z/\mathbf{C}.$$

(*b*) Couples whose aspects are the same. The resultant is a couple whose aspect is the same as that of the given couples and whose moment equals the algebraic sum of the moments of the couples, a result reached in art. 47 for coplanar couples.

61. Resolution of a Couple.—It follows from the preceding article that a couple may be equivalent to two or more couples, which are therefore components of that couple. Also, to resolve a couple we have only to resolve its vector, the component vectors being the vectors of the component couples.

The resolution of a couple into three components whose planes are mutually at right angles is an important special case. Let C be the couple to be resolved and v its vector; and denote the direction angles of the vector by α, β, and γ, the coordinate planes having been taken to coincide with the planes of the desired component couples. Let C_x, C_y, and C_z denote the component couples which are perpendicular to the x, y, and z axes respectively, and v_x, v_y, and v_z the corresponding vectors. Then

$$v_x = v \cos \alpha, \quad v_y = v \cos \beta, \quad v_z = v \cos \gamma,$$

hence

$$\mathbf{C}_x = \mathbf{C} \cos \alpha, \quad \mathbf{C}_y = \mathbf{C} \cos \beta, \quad \mathbf{C}_z = \mathbf{C} \cos \gamma.$$

EXAMPLES.

1. Hold this book, opened 150°, before you and imagine two couples whose moments are 50 and 70 ft.-lbs. to act in the planes of the right- and left-hand covers respectively, their senses being clockwise as viewed by yourself. On the supposition that the book is a rigid body, determine the resultant of the two couples.

2. Imagine two couples whose moments are 40 and 90 ft.-lbs. to act in the front and right-hand side of the parallelopiped of fig. 15, their senses being clockwise and counter-clockwise respectively as viewed by yourself. Determine the resultant of the four couples.

CHAPTER III.

CENTRE OF GRAVITY AND CENTROID.

§ I. Centroid of Parallel Forces.

62. Centroid Defined.—A system of parallel forces having fixed application points possesses an important property which is now to be investigated.

First consider a system of two forces, P and Q, with application points at A and B respectively, fig. 25 (a) or (b). In art. 43 it is shown that $\overline{BC}/\overline{AC}=P/Q$; hence C, determined by the equation, is independent of the angle between AB and the action lines of P and Q. Therefore, if the body upon which the forces act be turned in any manner, the direction of the forces remaining unchanged, their resultant will always pass through the same point of the body. The point C may hence be regarded as the application point of the resultant for all aspects of the body.

Consider next a system of more than two forces.

Let F', F'', F''', etc., denote the forces;

R' the resultant of F' and F'', and C' its application point;

R'' the resultant of R' and F''', and C'' its application point; etc.

If the body be turned, R' always passes through C', and R'', which is also the resultant of F', F'', and F''', always passes through C''. The point C'' may therefore be regarded as the application point of the resultant of F', F'', and F''' for all aspects of the body. Extending this reasoning to the resultant of the first four forces of the system, then to the resultant of the first five, etc., one arrives at the conclusion that

the resultant of any system of parallel forces having definite application points always passes through the same point of the body, or its extension, irrespective of its aspect. This point is called the *centroid* of the system of forces.

63. Determination of the Centroid.—Call the forces F', F'', F''', etc., those of one sense being given the same sign and those of the opposite sense the opposite sign. Let (x', y', z'), (x'', y'', z''), etc., denote the coordinates of their respective application points with respect to a set of rectangular axes which is fixed with reference to the body, and let $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the coordinates of the centroid. Now imagine the body upon which the forces act to be turned so that one of the axes, say that of x, becomes

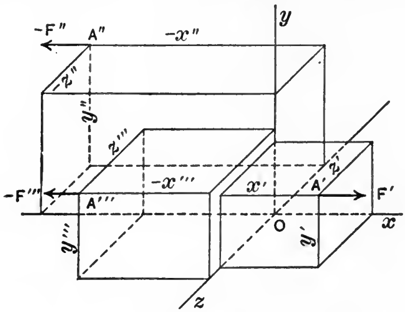


FIG. 42.

parallel to the forces (see fig. 42). Then according to the principle of moments (art. 53), R being equal to ΣF,

$$R\bar{z} = F'z' + F''z'' + F'''z''' + \ldots$$

and

$$R\bar{y} = F'y' + F''y'' + F'''y''' + \ldots$$

Now imagine the body turned until one of the other axes, that of y say, becomes parallel to the forces. The moment equation, with the z as moment axis, is

$$R\bar{x} = F'x' + F''x'' + F'''x''' + \ldots.$$

From these three equations the desired formulas for the coordinates of the centroid follow; thus,

$$\bar{\mathbf{x}} = \Sigma \mathbf{Fx}/\mathbf{R}, \quad \bar{\mathbf{y}} = \Sigma \mathbf{Fy}/\mathbf{R}, \quad \bar{\mathbf{z}} = \Sigma \mathbf{Fz}/\mathbf{R}.$$

In these formulas signs must be given the forces as explained above, and to the coordinates of their application points as customarily. That this is necessary will be seen from an inspection of the moment equations above and the figure.

If the application points of the forces be coplanar, two of the formulas will suffice, provided that two of the coordinate axes be taken in the plane of the application points. The graphic method is easily applied in this case as follows: Imagine the body to be turned so that the action lines of the forces fall into the plane of the application points, their directions remaining unchanged. The force system is then coplanar, and the action line of its resultant may be readily determined graphically (art. 39). Now imagine the body to be turned about an axis which is perpendicular to the plane of the application points through any angle other than 180 or 360 degrees. The forces are still coplanar, and the action line of their resultant may be determined as before. The intersection of the action lines of the two resultants is the centroid of the system.

EXAMPLES.

1. Forces of 10, 20, 15, and 5 lbs. have the same directions; the application points are coplanar and their coordinates in feet are respectively, (5, 3), (2, 4), (1, 5), and (5, 6). Determine the centroid of the forces. *Ans.* $\bar{x}=2.6$ ft.

2. With the forces of the preceding example, include two of 20 and 30 lbs. whose directions are the same as that of the four and whose application points are respectively at (5, 4) and (−6, 3). Determine the centroid of the six forces.

Ans. $\bar{x}=0.5$ ft.

3. Reverse the sense of the thirty-pound force, and determine the centroid of the six.

§ II. CENTRE OF GRAVITY OF A BODY.

64. Definition and General Formulas.—The weights of the particles of a body constitute a system of practically parallel forces having fixed application points. Therefore, these forces have a centroid, that is, the resultant always passes through a certain point fixed with reference to the body no matter how it is turned. It is assumed, of course, that in turning the body, its form and size are not changed.

Definition.—The *centre of gravity of a body* is the centroid of the weights of all its particles.

Centres of gravity, in the following, will be usually specified by means of rectangular coordinates and which will then always be denoted by $\bar{x}$, $\bar{y}$, and $\bar{z}$. General values for these may be deduced as follows: Let (x', y', z'), (x'', y'', z''), etc., denote the coordinates of the particles* of a body; w', w'', etc., their weights, and W the weight of the body. Then, from the formulas for centroid of a system of parallel forces,

$$\bar{x} = \Sigma wx/W, \quad \bar{y} = \Sigma wy/W, \quad \bar{z} = \Sigma wz/W.$$

65. Moment of a Weight with Respect to a Plane.—*Definition.*—The moment of the weight of a body with respect to a plane is the product of the weight and the ordinate of the centre of gravity of the body from that plane. Ordinates on opposite sides of the plane are given opposite signs. A moment, therefore, has the same sign as the corresponding ordinate.

From the definition it follows that if the moment of the weight of the body is zero with respect to a plane, its centre of gravity is in that plane.

Proposition.—The moment of the weight of a body with respect to a plane equals the algebraic sum of the moments of the weights of its parts with respect to the same plane.

Proof: Let W_1, W_2, etc., be the weights of the parts and x_1'. x_1'', etc., be the x-coordinates of particles of the first part, and w_1', w_1'', etc., their weights; x_2', x_2'', etc., be coordinates of particles of the second part, and w_2', w_2'', etc., their weights. etc. Then, according to the formulas above, the x-coordinate of the centre of gravity is

$$\bar{x} = \frac{(w_1'x_1' + w_1''x_1'' + \ . \ . \ .) + (w_2'x_2' + w_2''x_2'' + \ . \ . \ .) + \text{etc.}}{(w_1' + w_1'' + \ . \ . \ .) + (w_2' + w_2'' + \ . \ . \ .) + \text{etc.}}.$$

* By particle is meant a body whose dimensions are vanishingly small or negligible in comparison with other distances involved, which, in this case, are for each particle the distances between it and the coordinate planes.

Now the first parenthesis of the numerator equals $W_1\bar{x}_1$, the second parenthesis equals $W_2\bar{x}_2$, etc., and the denominator equals the weight of the entire body. The equation may therefore be written

$$\mathbf{W\bar{x}=W_1\bar{x}_1+W_2\bar{x}_2+\ \ldots}$$

Since the plane from which the x's are measured may be taken at pleasure, the proposition is proved.

EXAMPLES.

1. Suppose that the heavy lines of fig. 43 represent a bent wire. Determine the coordinates of its centre of gravity.

Ans. $\bar{x}=4.13$ in.

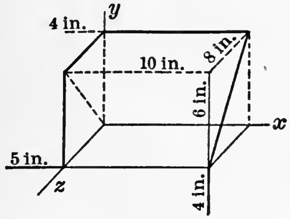


FIG. 43.

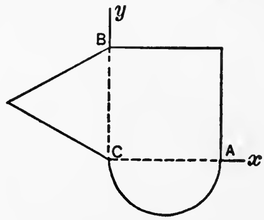


FIG. 44.

2. A piece of tin consists of three parts, square, semicircular, and equilateral triangular (fig. 44). Determine the centre of gravity of the piece. (The distances of the centres of gravity of the semicircular and triangular parts from the base are $4/3\pi$ times the radius and $\frac{1}{3}$ times the altitude respectively.)

3. Imagine the triangular and circular parts described in the preceding example bent forward on the lines BC and AC respectively until their planes are perpendicular to that of the square. Determine the centre of gravity.

4. Suppose that the triangular part of the preceding example is bent backward instead of forward. Write the expressions for the coordinates of the centre of gravity.

5. The weights of four bodies are W_1, W_2, W_3 and W_4, and the distance of the centre of gravity of the first from the plane through the centres of gravity of the other three is h. How far from that plane is the centre of gravity of the four bodies?

6. The weights of three bodies are 12, 18, and 40 lbs.; the

distances between the centres of gravity of the first and second, second and third, third and first, are 10, 14, and 16 inches respectively. How far from the line joining the centres of gravity of the first and second is the centre of gravity of the three?

Ans. 7.91 in.

7. Fig. 45 represents a cylindrical body having a cylindrical hole in the top; a part of the body is cast iron and the remainder (conical) is lead. Determine the centroid. (Cast iron and lead weigh 450 and 711 lbs. per cu. ft. respectively. See art. 80.)

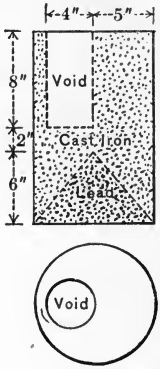


FIG. 45.

66. Centre of Gravity Determined by Integration. — Imagine a body divided into an infinitely large number of parts, i.e., elements. Let dW denote the weight of any element and x, y, z, the coordinates of its centroid. According to the principle of moments,

$$W\bar{x} = \int dW \cdot x, \quad W\bar{y} = \int dW \cdot y, \quad W\bar{z} = \int dW \cdot z. \quad (1)$$

If the body is homogeneous, let w denote its specific weight* and V its volume; then $W = wV$ and $dW = wdV$. Equations (1) reduce to

$$V\bar{x} = \int dV \cdot x, \quad V\bar{y} = \int dV \cdot y, \quad V\bar{z} = \int dV \cdot z. \quad . \quad . \quad (2)$$

These formulas may be employed for determining the centre of gravity of a body which cannot be divided into finite parts whose weights and centres of gravity are known, provided that its form and specific weight, if the body is not homogeneous, are such that the integrations can be performed.

EXAMPLES.

1. Determine the centre of gravity of an octant of a sphere whose specific weight, varying from point to point, is directly proportioned to distance from the centre.

Solution: Let the plane faces of the octant be taken as coordinate planes as shown in fig. 46; denote the radius of the

* By specific weight is meant weight per unit volume.

sphere by r and the specific weight at any point P by w. If k is a proper constant and x, y, and z denote the coordinates of P,

$$w = k(x^2+y^2+z^2)^{\frac{1}{2}} \quad \text{and} \quad dW = k(x^2+y^2+z^2)^{\frac{1}{2}}\, dx\, dy\, dz.$$

Then from (1),

$$W\bar{x} = k\int_0^r \int_0^{(r^2-x^2)^{\frac{1}{2}}} \int_0^{(r^2-x^2-y^2)^{\frac{1}{2}}} (x^2+y^2+z^2)^{\frac{1}{2}}\, dx\, dy\, dz \cdot x;$$

hence $\bar{x} = \frac{2}{5}r$. Evidently, $\bar{x} = \bar{y} = \bar{z}$.

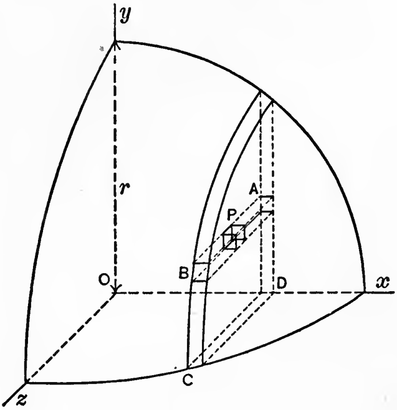


FIG. 46.

2. Determine the centre of gravity of an octant of a homogeneous sphere.

Solutions: Equation (2) may be employed. First, selecting a cubical element as in ex. 1, $dV = dx\, dy\, dz$, and eq. (2) becomes

$$V\bar{x} = \int_0^r \int_0^{(r^2-x^2)^{\frac{1}{2}}} \int_0^{(r^2-x^2-y^2)^{\frac{1}{2}}} dx\, dy\, dz \cdot x;$$

hence $\bar{x} = \frac{3}{8}r$. Evidently, $\bar{x} = \bar{y} = \bar{z}$.

Second, selecting the parallelopiped AB as elementary volume, $dV = dx\, dy(\overline{AB})$. If x and y denote coordinates of B,

$$\overline{AB} = (r^2 - x^2 - y^2)^{\frac{1}{2}},$$

and eq. (2) becomes

$$V\bar{x} = \int_0^r \int_0^{(r^2-x^2)^{\frac{1}{2}}} (r^2-x^2-y^2)^{\frac{1}{2}}\, dx\, dy \cdot x;$$

hence $\bar{x} = \frac{3}{8}r.$

Third, selecting the volume between two planes parallel to the yz plane, dx apart as elementary volume, $dV = \pi\overline{DC}^2 dx/4$. If x denotes the coordinate of C, $\overline{DC}^2 = r^2 - x^2$, and eq. (2) becomes

$$V\bar{x} = \frac{\pi}{4}\int_0^r (r^2 - x^2)dx \cdot x;$$

hence $$\bar{x} = \tfrac{3}{8}r.$$

67. Centre of Gravity Determined Experimentally.—Some bodies are so irregular in shape that their centres of gravity cannot be found by the methods explained above. In such cases, experimental methods may be resorted to.

The Method of Suspension.—The body whose centre of gravity is to be determined is suspended from one point of it and the direction of the suspending cord is then marked in some way on the body. The operation is repeated using another point of suspension. Since the centre of gravity is in each of the lines so fixed in the body, it is at their intersection.

The Method of Balancing.—The body whose centre of gravity is to be determined is balanced upon a straight-edge and the position of the vertical plane containing the edge marked upon the body; then the operation is repeated for two more balancing positions of the body. Since the centre of gravity is in each plane so fixed, it is in the common point of the three.

This method is easily applied to determine the centre of gravity of a thin plate. The plate is balanced in two positions, same face down both times, and the lines of contact of the straight-edge and plate are marked upon the latter. The centre of gravity is midway between the intersection of those lines and a point directly opposite on the other face of the plate.

EXERCISE.

Cut from a sheet of stiff paper the "angle section" described in ex. 4, art. 81, and determine how far the centre of gravity of the paper is from the edges AC and CB.

§ III. CENTROIDS OF SOLIDS, SURFACES, AND LINES.

68. Centroid Defined.—The term centre of gravity, applied to solids,* surfaces, or lines, is inappropriate, for they have no weight. However, the term is much used in that connection. Centroid, a more suitable term, is coming into use. Instead of the terms centroid or centre of gravity of a solid surface or line, centroid or centre of gravity of a volume area or length, respectively, are often employed.

Definitions.—The *centroid of a solid* is that point of it which coincides with the centre of gravity of a homogeneous body which is bounded by the surface of the solid. The *centroid of a surface* is the limiting position of the centre of gravity of a homogeneous thin plate one of whose faces coincides with the surface as its thickness approaches zero. The *centroid of a line* is the limiting position of the centre of gravity of a homogeneous slender rod whose axis coincides with the line as its sectional area approaches zero.

69. Centroid as Mean Point.—*Proposition.*—The ordinate of the centroid of any solid, surface, or line with reference to any plane equals the mean of the ordinates of all the equal elements of the solid, surface, or line. It must be understood that opposite signs are given to ordinates on opposite sides of the plane.

Proof for lines: Let x_1, x_2, x_3, etc., denote the ordinates to the different elementary lengths ds of the line, and n the number of elements (infinite). Then

$$\text{the mean ordinate} = \frac{x_1 + x_2 + x_3 + \ldots}{n},$$

$$\text{but this equals} \quad \frac{(x_1 + x_2 + x_3 + \ldots)ds}{nds} = \frac{\int xds}{l},$$

l denoting the length of the curve. In art. 83 it is shown that the last expression is that for the ordinate of the centroid, hence the proposition is proved.

Proof for solids and surfaces is similar to that just given.

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* Geometrical and not physical solid is meant.

70. Moment of a Volume, Area, or Length.—*Definitions.*—The moment of the volume of a solid (area of a surface or length of line) with respect to a plane is the product of the volume (area or length) and the ordinate of the centroid of the solid (surface or line) from that plane. Ordinates on opposite sides of the plane are given opposite signs; hence a moment has the same sign as the corresponding ordinate.

Proposition.—The moment of the volume of any number of solids with respect to a plane equals the algebraic sum of the moments of the volumes of those solids with respect to the same plane. Similar propositions hold for surfaces and lines.

Proof for volumes: Let V_1, V_2, etc., denote the volume of the solids, $\bar{u}_1$, $\bar{u}_2$, etc., the ordinates of their centroids, and $\bar{u}$ that of the centroid of the collection of solids with respect to any plane. Now the ordinates of the centres of gravity of homogeneous bodies which are bounded by the surfaces of these solids are also $\bar{u}_1$, $\bar{u}_2$, etc., and the ordinate of the centre of gravity of the collection of bodies is also $\bar{u}$. From art. 65, if w denotes the specific weight of the imagined homogeneous bodies,

$$(wV_1+wV_2+\ldots)\bar{u}=wV_1\cdot\bar{u}_1+wV_2\cdot\bar{u}_2+\ldots,$$

or

$$(\mathbf{V}_1+\mathbf{V}_2+\ \ldots\ )\bar{\mathbf{u}}=\mathbf{V}_1\bar{\mathbf{u}}_1+\mathbf{V}_2\bar{\mathbf{u}}_2+\ \ldots \qquad \text{Q.E.D.}$$

Proof for surfaces: Let A_1, A_2, etc., denote the areas of the surfaces, $\bar{u}_1$, $\bar{u}_2$, etc., the ordinates of their centroids, and $\bar{u}$ the ordinate of the centroid of the group. Imagine now as many homogeneous thin plates as there are surfaces, and that one face of each plate coincides with one of the surfaces. Let $\bar{u}_1'$, $\bar{u}_2'$, etc., denote the ordinates of the centres of gravity of the plates, and $\bar{u}'$ that of the centre of gravity of the collection. If w denotes the specific weight of the plates and t their thickness, their weights are approximately A_1tw, A_2tw, etc., the approximation being closer the smaller t is taken. (Of course, for plane plates these expressions are correct for all values of t.) From art. 65,

$$(A_1+A_2+\ldots)tw\bar{u}'=A_1tw\cdot\bar{u}_1'+A_2tw\cdot\bar{u}_2'+\ldots, \text{ approximately;}$$

or

$$(A_1+A_2+\ldots)\bar{u}'=A_1\bar{u}_1'+A_2\bar{u}_2'+\ldots, \text{ approximately.}$$

Now this approximate equality approaches exact equality as t approaches zero, i.e.,

$$\begin{aligned}\lim.[(A_1+A_2+\ldots)\bar{u}'] &= \lim.[A_1\bar{u}_1'+A_2\bar{u}_2'+\ldots] \\ &= \lim.[A_1\bar{u}_1']+\lim.[A_2\bar{u}_2']+\ldots,\end{aligned}$$

or

$$(\mathbf{A}_1+\mathbf{A}_2+\ldots)\bar{\mathbf{u}}=\mathbf{A}_1\bar{\mathbf{u}}_1+\mathbf{A}_2\bar{\mathbf{u}}_2+\ldots \qquad \text{Q.E.D.}$$

Proof for lines is very similar to that for areas.

71. Centroidal Plane. — *Definition.*—Any plane containing the centroid of a line, surface, or solid will be called a centroidal plane of that line, surface, or solid.*

From the definitions of art. 70 it follows that if the moment of a length, area, or volume with respect to a plane is zero, then it is a centroidal plane of the line, surface, or solid.

Proposition.—If the form of a solid (surface or line) is such that it can be divided into parts which may be paired off in such a way that the parts of each pair are equal in volume (area or length), and that the lines joining the centroids of the parts of each pair are bisected by a plane, then that plane is a centroidal one.

Proof: The moments of the volumes (areas or lengths) of the parts constituting a pair with respect to the bisecting plane are equal but of opposite sign; hence the moment of the volume (area or length) of the pair is zero, and the moment of the entire volume (area or length) is zero; therefore that plane is one of zero moment and it contains the centroid of the solid (surface or line).

Evidently the following are centroidal planes:

for a circular arc, any bisecting plane containing a diameter;
for a sector, any bisecting plane containing a diameter;
for a triangle, any plane cutting it in a median;
for a parallelogram, any plane cutting it in a diagonal;
for a triangular pyramid, any plane containing its vertex and median of the base.

* Likewise any straight line containing the centroid of a line, surface, or solid will be called a centroidal axis of the line, surface, or solid.

72. Centroids of Simple Solids and Surfaces.—The centroids of such solids and surfaces are determined in the succeeding articles. The general method consists in finding enough planes or lines containing the centroid to locate its position.

73. *The Centroid of a Triangle* is at the intersection of its medians.

Proof: As before stated, a plane cutting a triangle in a median is a centroidal plane. Since the centroid is in such a plane and in the plane of the triangle, it is in their intersection, i.e., the median line. But there are three such median lines, hence the centroid is at their intersection. Further, OA', OB', and OC' (fig. 47) equal respectively one-third of AA', BB', and CC', from which it follows that

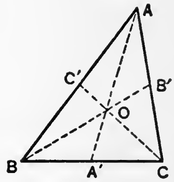

FIG. 47.

the distance of the centroid of a triangle from any side equals one-third the altitude measured from that side.

74. *The Centroid of a Parallelogram* is at the intersection of its diagonals.

Proof: As before stated, a plane cutting the parallelogram in a diagonal is a centroidal plane. Since the centroid is in such a plane and in the plane of the parallelogram, it is in their intersection, the diagonal; hence, etc.

75. *The Centroid of the Surface of a Pyramid* is on the axis* of the surface at a distance from the base equal to one-third of the altitude.

Proof: Imagine the pyramid cut by numerous planes parallel to its base; the part of the surface between any two adjacent planes is a frustum of the surface. Conceive the entire surface as consisting of elementary frustums; the centroid of any one of them and those of the perimeters of its bases approach coincidence. But these perimeters are polygons similar to the perimeter of the base and the relation between any one of them and the intersection of its plane with the

* By axis of the surface of a pyramid (or cone) is meant the line joining its apex with the centroid of the perimeter of its base.

axis is precisely similar to that between the perimeter of the base and its centroid. Hence the centroid of all the polygons and the elementary frustums lie upon the axis, and it follows that the centroid of the surface of the pyramid is on the axis.

The centroids of all the faces of the pyramid lie in a plane distant one-third of the pyramid's altitude from the base. It follows that the plane is a centroidal one, and hence the centroid of the surface is in that plane.

76. *The Centroid of the Surface of a Cone* is on the axis of the surface at a distance from the base equal to one-third the altitude.

Proof: The surface of a cone may be regarded as the limit of the surface of a pyramid, the number of whose faces are increased without limit. It follows that the limiting position of the centroid of the pyramid is the centroid of the surface of the cone, etc.

77. *The Centroid of a Prism with Parallel Bases* is in the axis and midway between the bases.

Proof: Conceive the prism as consisting of elementary laminas whose faces are parallel to the bases. The centroid of any lamina and the centroids of its faces approach coincidence. Obviously, the centroids of the faces are on the axis, hence the centroid of each elementary lamina is also, and it follows that the centroid of the prism is on the axis. Obviously, a plane midway between the bases is a centroidal one, hence the centroid is in that plane.

78. *The Centroid of a Pyramid with a Triangular Base* is on the axis* at a distance from the base equal to one-fourth the altitude (see also next art.).

Proof: As before stated, a plane containing the apex and a median line of the base is a centroidal one. But there are three such planes, and since the centroid is in each, it is in their intersection, the axis. Now each "corner" of the pyramid may be in turn considered as a vertex corresponding to which there

* A line joining the apex of any pyramid or cone with the centroid of the base is the axis.

is an axis, and the centroid being on each axis, is at their intersection.

Two axes, AF and BG, are represented in fig. 48. Since $EF = EB/3$ and $EG = EA/3$, GF is parallel to AB and $GF = AB/3$, and it follows that the triangles OFG and OAB are similar. Hence $OF = OA/3$ and $OF = AF/4$, that is, the centroid O is one-fourth the length of the axis upward from its foot. Also the distance of the centroid from the base equals one-fourth the altitude.

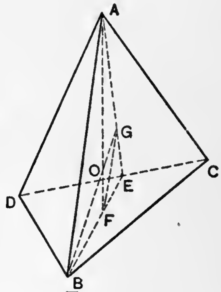

Fig. 48.

79. *The Centroid of Any Pyramid* is on the axis at a distance from the base equal to one-fourth the altitude.

Proof: Conceive the entire pyramid as made up of elementary frustums; the centroid of any one of them and those of its faces approach coincidence. But these faces are surfaces similar to the base, and the relation between any one of them and its intersection with the axis is precisely similar to the relation between the base and its centroid. Hence the centroids of all such surfaces and elementary frustums lie upon the axis, and it follows that the centroid of the pyramid is in the axis.

Conceive the base divided into triangles, and the entire pyramid as consisting of pyramids with these triangles as bases. The centroids of all these component pyramids lie in a plane distant one-fourth of the pyramid's altitude from the base; it follows that the plane is a centroidal one for the entire pyramid, and hence the centroid is in that plane.

80. *The Centroid of a Cone* is in its axis at a distance from the base equal to one-fourth the altitude.

Proof: A cone may be regarded as the limit of a pyramid, the number of whose faces are increased without limit. It follows that the limiting position of the centroid of the pyramid is the centroid of the cone. Hence, etc.

81. Centroids of Solids and Surfaces Consisting of Simple Parts.—The solids and surfaces whose centroids are now to be determined consist of parts whose volumes, or areas, and centroids are known. The solutions are based on the principle of moments, art. 70.

EXAMPLES.

1. Determine the centroid of a trapezoid whose altitude is a and whose minor and major bases are b and B respectively.

Solution: Imagine the trapezoid divided into two triangles. Their areas are $Ba/2$ and $ba/2$, and the distances from the base B to their centroids are $a/3$ and $2a/3$ respectively. The area of the trapezoid is $(B+b)a/2$; and if $\bar{y}$ denotes the distance between its centroid and the base B,

$$\frac{(B+b)a}{2}\bar{y}=\frac{Ba}{2}\frac{a}{3}+\frac{ba}{2}\frac{2}{3}a,$$

or

$$\bar{y}=\frac{2b+B}{3(b+B)}a.$$

The centroid can be determined geometrically in this way: Extend in either direction the major base a distance b, and in the opposite direction the minor base a distance B. Then the line joining the ends of the extensions intersects the line joining the centres of the bases at the centroid of the trapezoid. The student should supply a proof.

2. Determine the centroid of the "tee section" represented in fig. 49. *Ans.* 1.01 in. above base.

3. Determine the centroid of the "channel section" represented in fig. 49. *Ans.* 0.79 in. above base.

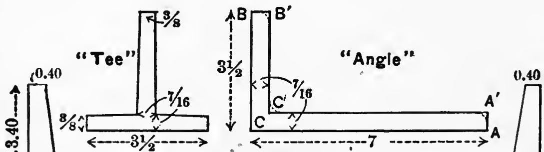

FIG. 49.

4. Determine the centroid of the "angle section" represented in fig. 49, with reference to the sides AC and BC.

5. Locate the centroid of any plane quadrilateral.

Method: Join the centroids of the two triangles into which either diagonal divides it; also join the centroids of the triangles into which the other diagonal divides it. The intersection of those two lines is the centroid sought. Prove.

82. Centroids of Solids and Surfaces Considered as Parts of Other Solids or Surfaces.—The solids and surfaces whose centroids are now to be determined may be conveniently considered as consisting of a simple solid or surface *minus* one or more simple solids or surfaces. It is supposed that the volume or area and centroid of each of the simple solids or surfaces are known. The principle of moments slightly modified is employed thus: Let M_1 be the moment of the volume or area in question, M_2 that of the volume or area of which the first is a part, and M_3 that of the remainder; then

$$M_1 = M_2 - M_3,$$

for, according to the principle of moments, $M_2 = M_1 + M_3$.

EXAMPLES.

1. Determine the centroid of the shaded part of fig. 50.

Solution: The shaded area consists of the large square *minus* the triangle and quadrant. Its value is

$$a^2 - \left(\frac{a^2}{4} + \frac{\pi a^2}{16}\right) = \frac{a^2}{16}(12 - \pi).$$

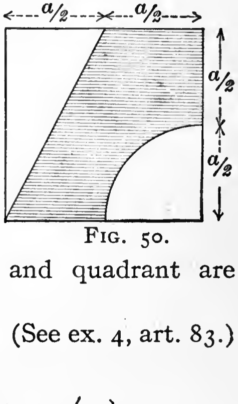

FIG. 50.

Let the base and left side of the square respectively be x and y axes, and $\bar{x}$ and $\bar{y}$ the coordinates of the centroid. The coordinates of the centroid of the triangle and quadrant are respectively

$(a/6,\ 2a/3)$ and $(a - 2a/3\pi,\ 2a/3\pi)$. (See ex. 4, art. 83.)

Hence

$$\frac{a^2}{16}(12 - \pi) \cdot \bar{x} = a^2\frac{a}{2} - \frac{a^2}{4}\frac{a}{6} - \frac{\pi a^2}{16}(a - 2a/3\pi),$$

or

$$\bar{x} = \frac{8 - \pi}{12 - \pi}a.$$

The student should determine $\bar{y}$.

2. Two circles whose diameters are as 2 to 3 are tangent internally. Locate the centroid of the part of the larger circle not included in the smaller.

3. Locate the centroid of a surface bounded by a quarter of a circle and the tangents at its extremities.

Ans. Its distance from either tangent $=0.223r$.

4. Suppose that in fig. 49 three corners of the "angle" are rounded as shown, the radii at A' and B' being $\frac{5}{16}$ in. and that at C' $\frac{1}{2}$ in. Redetermine the centroid.

5. Locate the centroid of a frustum of a cone whose major and minor base radii are R and r respectively and whose altitude is a.

Ans. Distance from larger base is $\frac{a}{4}\frac{R^2+2Rr+3r^2}{R^2+Rr+r^2}$.

6. Locate the centroid of the shaded part of fig. 51.

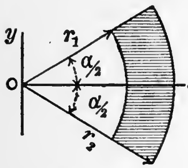

FIG. 51.

Solution: Since any plane cutting the figure in Ox is a centroidal plane, the centroid is on Ox and $\bar{y}=0$. To determine $\bar{x}$, consider the shaded part as consisting of the sector of radius r_2 minus the sector of radius r_1, and write the moment equation with respect to the yz plane. (See ex. 3, art. 83.)

$$Ans.\ \bar{x}=\frac{4}{3}\frac{(r_2^3-r_1^3)}{(r_2^2-r_1^2)\alpha}\sin\frac{\alpha}{2}.$$

7. Locate the centroid of a circular segment.

Ans. Its distance from the centre of the circle equals (base of the segment)^3/(area of the segment)12.

83. Centroids Determined by Integration.—Imagine the solid, surface, or line divided into elementary parts. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the coordinates of the centroid of the solid, surface, or line.

For a solid, let V denote its volume and x, y, z the coordinates of the centroid of any elementary part. Then, according to the principle of moments (art. 70),

$$V\bar{x}=\int dV\cdot x, \quad V\bar{y}=\int dV\cdot y, \quad V\bar{z}=\int dV\cdot z. \quad . \quad . \quad (1)$$

For a surface, let A denote its area, and x, y, z the coordinates of the centroid of any elementary part. According to the principle of moments,

$$A\bar{x}=\int dA\cdot x, \quad A\bar{y}=\int dA\cdot y, \quad A\bar{z}=\int dA\cdot z. \quad . \quad . \quad (2)$$

For a line, let l denote the length, and x, y, z the coordinates of the centroid of any elementary part. According to the principle of moments,

$$\mathbf{l\bar{x}}=\int \mathbf{dl \cdot x}, \quad \mathbf{l\bar{y}}=\int \mathbf{dl \cdot y}, \quad \mathbf{l\bar{z}}=\int \mathbf{dl \cdot z}. \quad . \quad . \quad . \quad (3)$$

The limits of integration in the above formulas, applied in any particular example, must be such that the moments of each element of the solid (surface or line) are included in the summations. The formulas may be used for determining the centroid of a solid (surface or line) provided that the form of it is such that the integrations can be performed. They are to be employed when the solid (surface or line) cannot be divided into parts whose volumes (areas or lengths) and centroids are known.

EXAMPLES.

1. Locate the centroid of a circular arc.

Solution: Evidently the centroid is on OC (fig. 52), i.e., $\bar{y}=0$. To determine $\bar{x}$, eq. (3) is used. Since $dl=rd\theta$, $x=r\cos\theta$, and $l=r\alpha$,

$$r\alpha \cdot \bar{x}=\int_{-\alpha/2}^{\alpha/2} rd\theta \cdot r\cos\theta, \quad \text{or} \quad \bar{x}=\frac{2r}{\alpha}\sin\frac{\alpha}{2}.$$

For a semicircular arc, $\alpha=180°$, and $\bar{x}=2r/\pi=0.637r$.

2. Locate the centroid of a circular arc of 90°.

Ans. Distances from OA and OB equal $2r/\pi$.

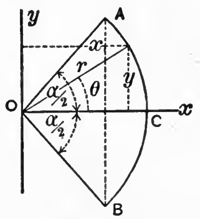

FIG. 52.

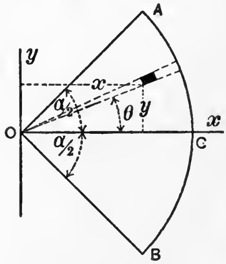

FIG. 53.

3. Locate the centroid of a circular sector.

Solutions: Evidently the centroid is on OC (fig. 53), i.e., $\bar{y}=0$. To determine $\bar{x}$, eq. (2) is used.

(1) If the elementary area be chosen so that $dA=\rho d\theta\, d\rho$, the

x coordinate of the centroid of the element is $\rho \cos \theta$; and since $A=\frac{1}{2}r^2\alpha$,

$$\tfrac{1}{2}r^2\alpha \cdot \bar{x} = \int_0^r \int_{-\alpha/2}^{\alpha/2} \rho^2 \cos \theta d\rho\, d\theta, \quad \text{or} \quad \bar{x} = \frac{4r}{3\alpha} \sin \frac{\alpha}{2}.$$

(2) If the sector is considered as made up of elementary triangles, so that $dA=\frac{1}{2}r^2 d\theta$, the x coordinate of the centroid of any element is $\frac{2}{3}r \cos \theta$. Eq. (2) becomes

$$\tfrac{1}{2}r^2\alpha \cdot \bar{x} = \tfrac{1}{3}r^3 \int_{-\alpha/2}^{\alpha/2} \cos \theta\, d\theta, \quad \text{or} \quad \bar{x} = \frac{4r}{3\alpha} \sin \frac{\alpha}{2}.$$

(3) The student should write the expression for $\bar{x}$, choosing the elementary area so that $dA = dx\, dy$.

For a semicircular area, $\alpha = 180°$, and $\bar{x} = 4r/3\pi = 0.424r$.

4. Locate the centroid of a circular quadrant.

Ans. Distance from either straight side is $4r/3\pi$.

5. Show by the integration method that the centroid of a triangle is one-third the altitude from the base.

(Suggestion: Take the origin of coordinates at the vertex and consider the triangle as consisting of elementary strips parallel to the base.)

6. Locate the centroid of a symmetrical parabolic segment whose altitude is a.

Ans. It is on the axis of the parabola distant $\frac{3}{5}a$ from the vertex.

7. Locate the centroids of the halves into which the axis divides the segment described in the preceding example.

84. *Surfaces of Revolution.*—General formula (2) of the preceding article is used; it will be advantageous to select the elementary area in a certain way, namely, the area described by an elementary part of the generating curve. Let the x axis be taken coincident with the axis of revolution, fig. 54; then the area described by a part of the generating curve whose length is ds is $2\pi y ds$. The coordinates x, y, and z of the centroid of this area are evidently x, o, and o respectively; hence

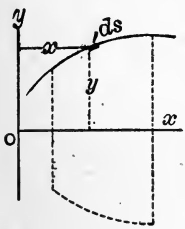

FIG. 54.

$$A\bar{x} = 2\pi \int yds \cdot x, \quad \text{or} \quad \bar{x} = \frac{2\pi}{A}\int xyds, \quad \text{and} \quad \bar{y} = \bar{z} = 0.$$

A stands for the area of the surface of revolution, and

$$A = 2\pi \int y ds.$$

The limits of integration must be assigned so that each element ds is represented in the integrations.

EXAMPLES.

1. Locate the centroid of a segment of a spherical surface.

Solution: The segment may be considered as generated by the revolution of a circular arc, as AC about OC, fig. 52. Since $x = r \cos \theta$, $y = r \sin \theta$, and $ds = r d\theta$,

$$\bar{x} = \frac{r\int_0^{\alpha/2} \cos\theta \sin\theta \, d\theta}{\int_0^{\alpha/2} \sin\theta \, d\theta} = \frac{r}{2}\left(1 + \cos\frac{\alpha}{2}\right).$$

2. Show by the integration method that the centroid of the surface of a cone is one-third of the altitude from the base.

85. *Solids of Revolution.*—General formula (1), art. 83, is used; it will be advantageous to select a certain elementary volume, namely, that generated by an elementary part of the generating plane which is included between two lines perpendicular to the axis of revolution. Thus if, in fig. 55, the generating plane is that bounded by the solid curve, and the x axis is taken coincident with the axis of revolution,

$$dV = \pi(y_2^2 - y_1^2)dx.$$

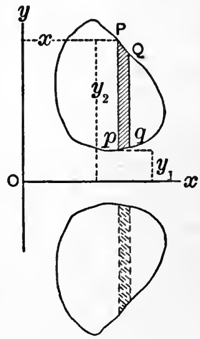

FIG. 55.

Now the centroid of this elementary volume is in the x axis, and its x coordinate is the x in the figure; hence

$$V\bar{x} = \pi \int (y_2^2 - y_1^2)dx \cdot x, \quad \text{or} \quad \bar{x} = \frac{\pi}{V} \int (y_2^2 - y_1^2)x \, dx.$$

V denotes the volume of the solid of revolution, and

$$V = \pi \int (y_2^2 - y_1^2)dx.$$

The limits of integration are to be assigned so that each dV is represented in the integrations.

EXAMPLES.

1. Locate the centroid of a segment of a sphere.

Solution: The segment may be considered as generated by the revolution of one-half of a circular segment, as ACD about OC, fig. 56. Since $x=r\cos\theta$, $dx=-r\sin\theta\,d\theta$. Also $y_2=r\sin\theta$ and $y_1=0$, therefore,

$$\bar{x}=\frac{r\int_0^{\alpha/2}\sin^3\theta\cos\theta\,d\theta}{\int_0^{\alpha/2}\sin^3\theta\,d\theta}=\tfrac{3}{4}r\,\frac{\sin^4\alpha/2}{2-3\cos\alpha/2+\cos^3\alpha/2}.$$

2. Locate the centroid of a paraboloid of revolution whose altitude is a.

Ans. It is in the axis of revolution distant $\tfrac{2}{3}a$ from the apex.

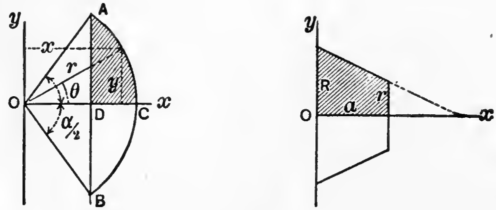

FIG. 56.

3. Locate the centroid of a frustum of a cone.

Suggestion: The frustum may be considered as generated by the revolution of the shaded trapezoid (fig. 56) about Ox.

$$\textit{Ans.}\quad \bar{x}=\frac{a}{4}\,\frac{R^2+2Rr+3r^2}{R^2+Rr+r^2}.$$

86. Theorems of Pappus and Guldinus.—I. The area of the surface of revolution generated by a plane curve revolved about an axis in its plane equals the length of the curve times the circumference of the circle described by its centroid.

Proof: Let A denote the area of the surface, l the length of the curve, and $\bar{y}$ the ordinate of its centroid measured from the axis. Then (see fig. 54)

$$A=2\pi\int y\,ds \quad \text{and, from eq. (3), art. 83,} \quad \bar{y}=\frac{1}{l}\int y\,ds;$$

hence

$$\mathbf{A}=\mathbf{l}\cdot 2\pi\bar{\mathbf{y}}. \qquad (1)$$

II. The volume of a solid of revolution generated by a plane figure revolved about an axis in its plane equals the area of the figure times the circumference of the circle described by its centroid.

Proof: Let V denote the volume of the solid, A the area of the plane figure, and $\bar{y}$ the ordinate of the centroid of A measured from the axis. Then (see fig. 55)

$$V=\pi\int(y_2^2-y_1^2)dx,$$

and, from eq. (2), art. 83,

$$\bar{y}=\frac{1}{A}\int(y_2-y_1)dx\cdot\tfrac{1}{2}(y_2+y_1);$$

hence

$$\mathbf{V}=\mathbf{A}\cdot 2\pi\bar{\mathbf{y}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are available for computing $\bar{y}$ as well as A or V if all other factors in the equations are known. It should be remembered that A has different meanings in (1) and (2),

EXAMPLES.

1. Knowing that the surface of a sphere is $4\pi r^2$, locate the centroid of a semicircular arc.

2. Knowing that the volume of a sphere is $\frac{4}{3}\pi r^3$, locate the centroid of a semicircle.

3. A circle of radius r is revolved about a line in its plane whose distance from the centre is a, a being greater than r. Deduce expressions for the surface area and volume of the solid generated.

4. By eqs. (1) and (2), deduce formulas for the surface area and volume of a cone.

5. Show that the area of the spherical segment generated by revolving the arc AC, fig. 53, about Ox is $2\pi r^2(1-\cos\alpha/2)$.

87. Graphical Determination of the Centroid of a Plane Figure.—I. The figure can be subdivided into parts whose areas and centroids are known. Imagine a uniform lamina shaped like the figure and then determine its centre of gravity by the method outlined in art. 63, p. 54. As an illustration, let it be required to determine the centroid of the shaded part of fig. 57(*a*). Imagining the figure to represent a uniform lamina, first determine the action line of the resultant of

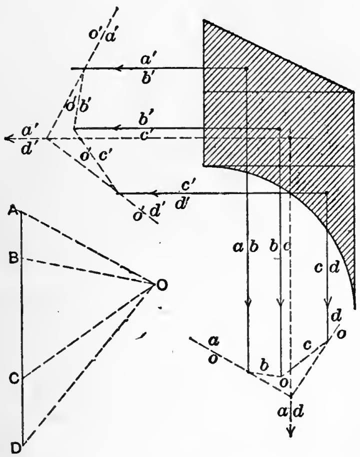

FIG. 57*a*.

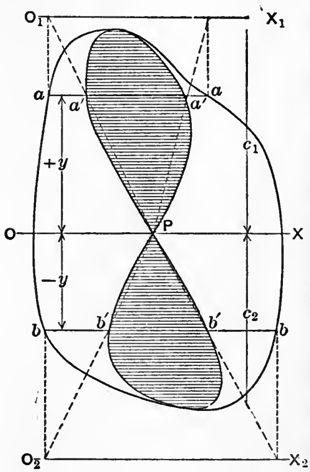

FIG. 57*b*.

the weights of the three component parts of the lamina acting through their respective centres of gravity; these weights are proportional to the areas of the component parts of the figure. First draw a force polygon for the forces, as *ABCD*, then a funicular polygon, as *oa*, *ob*, *oc*, *od*, and extend the first and last strings *oa* and *od*; then a line through their intersection parallel to the forces is the action line sought, *ad*. Next, instead of turning the lamina about as suggested in art. 63, imagine the direction of gravity to be changed so that *a′b′*, *b′c′*, and *c′d′* will represent the weights of the parts of the lamina; then find the action line of the resultant of these weights. A second force polygon is unnecessary, for the second funicular polygon, *o′a′*, *o′b′*, *o′c′*, *o′d′*, can be constructed from the first force polygon, *o′a′* being perpendicular to *OA*, *o′b′* perpendicular to *OB*, etc. The intersection of *o′a′* and *o′d′* fixes *a′d′*, which is the action line sought, and the intersection of *ad* and *a′d′* is the centroid of the entire figure.

In this case, the graphical method is not so convenient as the algebraic, but in the following case it is generally better.

II. The figure cannot be divided into simple parts whose areas and centroids are known.—The method requires the use of a planimeter or other device for determining the area of an irregular figure. Let *abba* (fig. 57*b*) be the figure whose centroid is to be determined, OX any convenient reference axis, and P any point on it. (1) Draw O_1X_1 and O_2X_2 at any convenient distance m from OX. (2) Draw a line *aa* parallel to the x axis through the figure and mark its intersections with the perimeter *a*. (3) Project *aa* on O_1X_1. (4) Join the ends of the projection with P and mark the intersections of *aa* with the joining lines a'. (5) Repeat the construction for other lines like *aa*, as *bb*, thus locating points b'. (6) Draw a smooth curve through all points, as $a'b'c'$, etc. (7) Measure the area of the two loops * (shaded in the figure) and that of the given figure. (8) Finally, if A denotes the area of the given figure, A_1' and A_2' the areas of the loops on the positive and negative sides of the x axis, and $\bar{y}$ the ordinate of the centroid, then

$$\bar{y}=m(A_1'-A_2')/A.$$

Proof: The moment of the area of the given figure with respect to the x axis is

$$\bar{y}A=\int ydA=\int_{-c_2}^{+c_1} ywdy=\int_{-c_2}^{0} ywdy+\int_{0}^{+c_1} ywdy,$$

w denoting any width of the figure, as *aa* or *bb*. If w' denotes the width of the loop corresponding to w, i.e., $a'a'$ or $b'b'$, then, from similar triangles in the figure, it follows that

$$\pm y/w'=m/w, \quad \text{or} \quad yw=\pm mw',$$

the positive or negative sign to be used according as w and w' refer to widths above or below the x axis. Substituting this value of yw in the last expression for the moment, we have

$$\bar{y}A=m\left[-\int_{-c_2}^{0} w'dy+\int_{0}^{c_1} w'dy\right].$$

Now the first integral equals A_2' and the second A_1'; hence, etc.

* If the x axis does not cut the given figure, there will be only one loop.

CHAPTER IV.

ATTRACTION AND STRESS.*

§ I. Gravitation.

Every body in the universe attracts every other body, the attraction in each case depending upon the masses of the two bodies and their distance apart. This attraction is called gravity, gravitation, and gravitational attraction.

88. Law of Gravitation.—Every particle attracts every other particle with a force which is proportional to the product of the masses of the two particles directly and to the square of their distance apart inversely.

The law may be stated algebraically thus: let F denote the attraction, m' and m'' the masses, and r the distance; then

$$\mathbf{F} \propto \frac{\mathbf{m'm''}}{\mathbf{r}^2}, \quad \text{or} \quad \mathbf{F} = \mathbf{k}\frac{\mathbf{m'm''}}{\mathbf{r}^2},$$

k being a constant whose value depends, as explained below, upon the units used to express force, mass, and distance.

Gravitation Constant.—It is shown in art. 97 that two homogeneous spheres attract each other as though the mass of each were concentrated at its centre, i.e., the attraction is the same as that between two particles placed at the centres of the spheres, the mass of each being equal to that of the corresponding sphere. Hence the formula for the attraction between particles applies to that between homogeneous spheres, r denoting

* The principles of mechanics employed in this chapter are mainly those relating to composition of forces, and most of the chapter might have been distributed as problems on composition in Chap. II. However, as it all relates to but two subjects, it is convenient to treat the matter together.

the distance between their centres; and the attraction between two spheres of unit mass, their centres being unit distance apart, is $k\frac{1 \cdot 1}{1^2}=k$.

Therefore, the numerical value of k equals the attraction between two homogeneous spheres of unit mass whose centres are unit distance apart.

The numerical value of the gravitation constant, k, hence depends upon the units employed for expressing force, mass, and distance. If the pound, pound, and foot respectively be employed, $k=3.31\times10^{-11}$, and for C.G.S. units (see art. 232) $k=6.65\times10^{-8}$. It is possible to choose the units of force, mass, and length so as to make $k=1$; two of them may be chosen arbitrarily, but such a third unit is an uncommon one, and therefore k will be retained in the formula.

A determination of the gravitation constant involves the measurement of the attraction between known masses at known distance apart. If F denotes the measured force, m' and m'' the masses of two homogeneous attracting spheres, and r the distance between their centres, then

$$k=Fr^2/m'm''.$$

EXAMPLE.

In an early determination of the gravitation constant, the attracting spheres were of lead, two and twelve inches in diameter. When their centres were nine inches apart, the attraction was measured. What was its value in pounds?

89. Density.—By density of a body is meant its mass per unit volume. We will denote it by δ.

In a homogeneous body, the mass of a unit volume is the same no matter where, in the body, the volume is selected; hence the density is constant, and its value is found by dividing the mass of the body by its volume, i.e., if m and V denote mass and volume of the body respectively,

$$\delta=\frac{\mathrm{m}}{\mathrm{V}}.$$

In a heterogeneous body, the density is variable and the expression m/V gives the average density; and if Δm denotes the mass of a portion whose volume is ΔV,

$$\frac{\Delta m}{\Delta V} = \text{average density of the portion.}$$

If the small portion, as its volume approaches zero, always includes a point P, then

$$\lim. \frac{\Delta m}{\Delta V} = \text{density at } P,$$

or, if δ denotes density at P,

$$\delta = \frac{dm}{dV}.$$

90. Attraction at a Point or "Strength of Field."—By attraction at a point due to any body is meant the attraction which it would exert upon a particle of unit mass placed at the point or upon a homogeneous sphere of unit mass whose centre is at the point. This is also known as strength of field, the term field being a contraction of "field of force," which means the region at all points of which there is an attraction.

91. Attractions in Some Simple Cases.—To compute the attraction at a point it is necessary to compound the gravitational forces exerted by all the particles of the attracting body upon a particle of unit mass placed at the point. This system of forces is concurrent, and may be collinear, coplanar or non-coplanar; hence to determine the resultant of the system the methods of §§ 2, 3, or 6, Chap. II, may be used.

92. *Attraction at a Point on the Produced Axis of a Straight Slender Rod.*—Let AB (fig. 58) be the rod and P the point. If

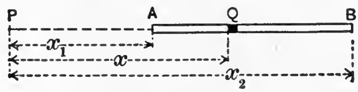

FIG. 58.

a denotes the area of the cross-section of the rod, and dm the mass of an elementary volume such as that represented at Q, then $dm = \delta a dx$. Such an element may be considered as a particle, for its dimensions are negligible in comparison with x; hence the attraction due to it at P is $k\delta a dx/x^2$. Denoting the attraction of the whole rod by F, then

$$F=k\delta a\int_{x_1}^{x_2}\frac{dx}{x^2}=k\delta a\left(\frac{1}{x_1}-\frac{1}{x_2}\right)=k\frac{m}{l}\left(\frac{1}{x_1}-\frac{1}{x_2}\right),$$

m and l denoting mass and length of the rod respectively.

93. *Attraction at any Point Due to a Straight Slender Rod.*—Let AB (fig. 59) be the rod and P the point c distant from the rod. If a denotes the cross-sectional area of the rod and dm the mass of an element such as that represented at Q, then $dm = \delta a dy$. Such an element may be considered as a particle, since its dimensions are negligible compared with r; hence the attraction due to it at P is $k\delta a dy/r^2$.

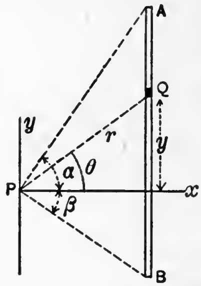

FIG. 59.

The x- and y-components of this attraction are respectively

$$\frac{k\delta a dy}{r^2}\cos\theta \quad \text{and} \quad \frac{k\delta a dy}{r^2}\sin\theta.$$

It is plain from the figure that $r=c/\cos\theta$, and $y=c\tan\theta$; hence $dy=cd\theta/\cos^2\theta$ and the expressions above can be written

$$\frac{k\delta a}{c}\cos\theta\, d\theta \quad \text{and} \quad \frac{k\delta a}{c}\sin\theta\, d\theta.$$

If F_x and F_y denote the x- and y-components of the attraction at P,

$$F_x=\frac{k\delta a}{c}\int_{-\beta}^{\alpha}\cos\theta\, d\theta=\frac{k\delta a}{c}(\sin\beta+\sin\alpha),$$

and

$$F_y=\frac{k\delta a}{c}\int_{-\beta}^{\alpha}\sin\theta\, d\theta=\frac{k\delta a}{c}(\cos\beta-\cos\alpha).$$

If F denote the magnitude of the resultant attraction and ϕ its angle with the x axis,

$$F=(F_x^2+F_y^2)^{\frac{1}{2}}=\frac{2k\delta a}{c}\sin\frac{\alpha+\beta}{2},$$

$$\tan\phi=F_y/F_x=\tan\frac{\alpha-\beta}{2}, \quad \text{or} \quad \phi=(\alpha-\beta)/2.$$

94. *Attraction Due to a Circular Ring at a Point on its Axis.* —Let the ring be that represented in fig. 60 and P the point b distant from its centre. Imagine the ring resolved into elements as that represented at Q, and call the mass of one dm. Such an element may be treated as a particle, for its dimensions are negligible compared with PQ. The attraction at P due to any element is $kdm/(r^2+b^2)$. It is plain from the symmetry that the algebraic sums of the y- and z-components of the attractions of all the elements are zero, hence the attraction at P due to the ring equals the sum of the x-components. Let ϕ denote the angle CPQ, then

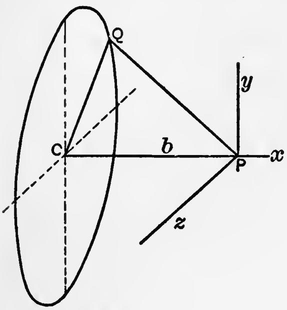

FIG. 60.

the x-component of the attraction of an element is

$$\frac{kdm}{r^2+b^2}\cos\phi,$$

and if F denotes the resultant attraction,

$$F=\frac{k}{r^2+b^2}\cos\phi\int dm=\frac{km}{r^2+b^2}\cos\phi,$$

m denoting the mass of the ring.

95. *Attraction Due to a Thin Circular Plate at any Point on its Axis.* —Let AB (fig. 61) be the plate, a its radius, t its thickness, and P the point. Imagine the plate as consisting of rings, or hollow cylinders, whose inner and outer radii are r and $r+dr$ respectively. The mass of such a ring is $\delta 2\pi r dr\cdot t$, and the attraction of the ring at P equals

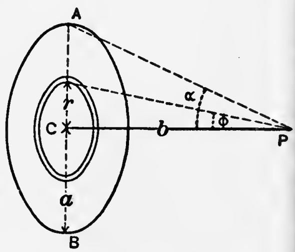

FIG. 61.

$$k\frac{\delta 2\pi r dr\cdot t}{r^2+b^2}\cos\phi$$

(see art. 94). Let F denote the attraction of the whole plate, then

$$F=2\pi k\delta t\int_0^a\frac{r\cos\phi}{r^2+b^2}dr.$$

Since $r=b\tan\phi$, $dr=b\,d\phi/\cos^2\phi$, and

$$F=2\pi k\delta t\int_0^\alpha \sin\phi\,d\phi=2\pi k\delta t(1-\cos\alpha)=2\pi k\frac{m}{A}(1-\cos\alpha).$$

The expression $2\pi(1-\cos\phi)$ is the value of the solid angle* subtended by the plate at P; hence, if ω denote that angle,

$$F=k\frac{m}{A}\omega.$$

For a point very near the plate α is nearly 90°, and

$$F=k2\pi\frac{m}{A},\text{ approximately.}$$

96. *Attraction Due to a Spherical Shell at any Point.*—Let ABC (fig. 62) be the shell, represented by a diametral section,

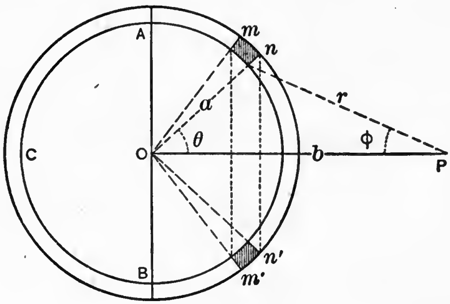

FIG. 62.

and P the point. Imagine the shell resolved into elementary rings which are cut out from the shell by cones whose common apex and axis are O and OP respectively. Two of the ring surfaces then are conical and two are zones of the outer and inner sur-

* The student is reminded that a solid angle is measured by the ratio of the area of that part of a sphere, whose centre is at the apex of the angle, which is included within the surface bounding the angle to the square of the radius of the sphere. Thus if for the solid angle subtended by the plate at P a sphere of radius R be used, the cone bounding the angle cuts from the sphere a segment whose area is $2\pi R^2(1-\cos\alpha)$, see ex. 5, art. 86; hence the solid angle equals $2\pi(1-\cos\alpha)$.

faces of the shell. Let dA denote the area of the inner zone, and t the thickness of the shell; then the mass of the ring is $\delta dA \cdot t$ and

$$\text{the attraction of the ring at } P = \frac{k\delta dA \cdot t}{r^2} \cos \phi,$$

(see art. 94). Let F denote the attraction of the shell, then

$$F = k\delta t \int \frac{\cos \phi}{r^2} dA,$$

the limits being chosen so that all rings of the shell are included.

To integrate, we will express dA and $\cos \phi$ in terms of r. From the figure,

$$dA = 2\pi a \sin \theta \cdot a d\theta = 2\pi a^2 \sin \theta \, d\theta,$$

and $\qquad r^2 = a^2 + b^2 - 2ab \cos \theta, \quad$ or $\quad r\,dr = ab \sin \theta \, d\theta;$

hence

$$dA = 2\pi \frac{a}{b} r \, dr.$$

Since $a^2 = r^2 + b^2 - 2rb \cos \phi$,

$$\cos \phi = (r^2 + b^2 - a^2)/2br.$$

Finally,

$$F = \frac{\pi k\delta at}{b^2} \int \frac{r^2 + b^2 - a^2}{r^2} dr.$$

The integral equals $\displaystyle\int dr + (b^2 - a^2) \int \frac{dr}{r^2} = \frac{r^2 + a^2 - b^2}{r}.$

The limits of the integration depend on whether the point P is within or without the shell.

(a) When the point is external,

$$F = \frac{\pi k\delta at}{b^2} \left[\frac{r^2 + a^2 - b^2}{r} \right]_{b-a}^{b+a} = \frac{k\delta 4\pi a^2 t}{b^2} = \frac{km}{b^2},$$

m denoting the mass of the shell. The final expression shows that the attraction at a point outside of a homogeneous shell is the same as though its mass were concentrated at its centre.

When the point is on the surface, $b = a$ and $F = k 4\pi \dfrac{m}{A}$, A denoting the area of the surface of the shell.

(*b*) When the point is internal,

$$F = \frac{\pi k\delta a t}{b^2}\left[\frac{r^2 + a^2 - b^2}{r}\right]_{a-b}^{a+b} = 0.$$

97. *Attraction at a Point Due to a Sphere.*—If the sphere may be resolved into homogeneous shells, the results of the preceding article may be employed.

For exterior points, the attraction due to each shell is the same as though its mass were concentrated at the centre and hence the attraction due to the sphere is the same as though its mass were concentrated at the centre. If m denotes the mass of the sphere and b the distance of any exterior point from the centre, the attraction at that point is km/b^2, i.e., the attraction varies inversely as the square of the distance from the centre.

For interior points, the attraction due to each shell which includes the point is zero, and the attractions of all shells to which the point is external are the same as though their masses were concentrated at the centre of the sphere; hence only the part of the sphere which is nearer the centre than the point is, exerts an attraction at that point, and it is the same as though the mass of that part were concentrated at the centre. If m' denote the mass of that part and b the distance of the point from the centre, the attraction at that point is km'/b^2.

If the sphere is homogeneous, $m' = \delta\frac{4}{3}\pi b^3$; hence the attraction at the point is $k\frac{4}{3}\pi\delta b$, i.e.. it varies directly as the distance from the centre.

§ II. Electric and Magnetic Attractions.

Electrified bodies attract or repel each other, and so do magnetized ones. A study of these forces does not fall within the scope of this book, but their laws are so similar to that of gravitation that several important propositions relating to electric and magnetic attractions can be readily deduced from the results derived on gravitation.

98. Laws of Electro-Static and Magnetic Forces.—(1) Bodies similarly electrified or magnetized repel, and those dissimilarly electrified or magnetized attract, each other.

This law points out a difference between electric or magnetic force and gravitation; the latter is always attractive.

(2) The force, attraction or repulsion, between two bodies electrified or magnetized is proportional directly to the product of the quantities of electricity or magnetism and inversely to the square of the distance between them.

In order that the distance between the bodies may be definite their dimensions must be negligible compared with the distance between any two points of the bodies.*

The second law may be stated algebraically; thus, let F denote the force, q' and q'' the quantities, and r the distance, then

$$F \propto \frac{q'q''}{r^2}, \quad \text{or} \quad F = k\frac{q'q''}{r^2}. \qquad (1)$$

k being a constant corresponding to the gravitation constant. It is customary to employ units of force, quantity, and distance which make k equal to unity; then

$$\mathbf{F} = \frac{\mathbf{q'q''}}{\mathbf{r^2}}. \qquad (2)$$

99. Strength of Field.—By strength of field at any point due to any quantity of electricity or magnetism is meant the force which that quantity would exert upon a unit quantity of electricity or magnetism concentrated at that point. It is assumed that the unit quantity would not affect the field.

100. Analogy between Electrical or Magnetic and Gravitational Attractions.—Comparing the equation of art. 87 and (1) of art. 98, it is seen that they are perfectly similar; hence the strength of field due to any distribution of electricity or magnetism may be found by analogy from the gravitational attraction due to a body the distribution of whose mass is similar to that of the electricity or magnetism.

101. Strengths of Field Due to Some Simple Distributions of Electricity and Magnetism.—I. *Electrified Circular Plate.*

* These statements of the laws are somewhat loose, but they serve the present purpose better than the usual forms. The reader is assumed to have at least an elementary knowledge of the phenomena of these forces.

—The electricity resides at the surface, and is nearly uniformly distributed, being more "dense" at the edges than elsewhere on the surface. If we regard it uniformly distributed, the expression for the strength of field at any point on the axis of the plate due to the electricity on either face may be deduced from the expression for the gravitational attraction due to a homogeneous thin plate at any point of its axis, which is $k\frac{m}{A}\omega$ (art. 95) For strength of field due to the electricity on one side, we make $k=1$, and substitute q for m, q denoting the quantity of electricity on the side considered. If F denotes the strength of field, then

$$F=\frac{q}{A}\omega.$$

Now q/A is the quantity of electricity per unit area, and is called the surface density; if it be denoted by ρ,

$$F=\rho\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a point at the centre of the face, $\omega=2\pi$, and

$$F=2\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the plate be very thin, so that its thickness is negligible compared to the distance between plate and the point of the axis considered, the strength due to electricity on both sides is called strength due to the plate, and its value is $q\omega/A$, q denoting quantity on both faces. For such thin plates q/A, the quantity of electricity on both sides per unit area of one side, is called surface density. With this meaning of q/A, or ρ, equations (1) and (2) may be used to compute strengths due to thin plates.

II. *Electrified Sphere.*—The electricity resides at the surface and is uniformly distributed over it; hence the expression for the strength of field due to an electrified sphere may be deduced from that for the gravitational attraction due to a homogeneous spherical shell.

(*a*) For external points, the expression is km/b^2 (art. 96). For strength due to an electrified sphere we make $k=1$, and substitute q for m, q denoting the quantity of electricity. Then if F denote strength,

$$F=\frac{q}{b^2}=\frac{4\pi r^2}{b^2}\frac{q}{A},$$

A denoting area of the surface of the sphere and r its radius. Now q/A is the surface density; if it be denoted by ρ,

$$F=\frac{4\pi r^2}{b^2}\rho. \quad \text{. . . . .} \quad (3)$$

For a point at the surface $b=r$, and

$$F=4\pi\rho. \quad \text{. . . . .} \quad (4)$$

(*b*) For internal points, the gravitational attraction due to a shell is zero, and hence the strength of field due to an electrified sphere at an internal point is also zero.

III. *Magnetic Shell.*—This is a magnetized plate, the magnetism on one side being "north-seeking" and that on the other side "south-seeking." The expression for strength of field due to the magnetism on one side of a circular plate at a point on its axis may be deduced from the expression for the gravitational attraction due to a homogeneous thin circular plate. That expression is $k\frac{m}{A}\omega$ (art. 95). For strength of field due to the magnetism on one side we make $k=1$, and substitute q for m, q denoting the quantity of magnetism. Then if F denote strength,

$$F=\frac{q}{A}\omega.$$

Now q/A is the quantity of magnetism on one side per unit area, and is called the surface density of magnetism; if it is denoted by ρ,

$$F=\rho\omega. \quad \text{. . . . .} \quad (5)$$

For a point at the surface of the shell $\omega=2\pi$, and

$$F=2\pi\rho. \quad \text{. . . . .} \quad (6)$$

§ III. Stress.

102. Stress Defined.—The term stress is variously defined. Some writers mean by it the forces which any two bodies or two parts of a body exert upon each other, it being then a term which refers to any "action" and its "reaction" (art. 6). For example, such writers designate as a stress the forces which the earth and sun exert upon each other, the forces which the upper and lower halves of a monument exert upon each other, etc.

Most engineers, however, use the term in a narrower sense, meaning by it the force which one part of a body exerts on an adjacent part at the surface of contact of the parts. Such engineers designate as stresses the force which the upper or lower half of a monument exerts upon the other half, the force which either half of a stretched string exerts on the other half, etc.

We will use the term in the engineer's sense slightly extended and define it thus: *Stress is any force whose place of application is a surface.* The force may be exerted between parts of one body or between two different bodies which are in contact, a part or all of the contact surface being the place of application of the force.

103. Units for Expressing Stress.—Since a stress is a force, it must be expressed in force units, the pound force, the kilogram force, etc.

104. Classification of Stresses.—If the parts of a stress on all equal *small* portions of its place of application are equal, the stress is said to be *uniform as to distribution;* if otherwise, *non-uniform.* If the action lines of the resultant forces on all the small portions are parallel, the stress is said to be *uniform as to direction;* if otherwise, *non-uniform.*

If the action lines of the resultants are all normal or tangential to the surface of application, the stress is called *simple;* if otherwise, *complex.* Stresses are, in general, complex, but it is possible to describe any complex stress in terms of simple stresses; only such are discussed herein.

Simple stresses may be classified into *normal* and *tangen-*

tial stresses according as the action lines of all the forces on the small portions of the place of application are normal or tangential to the surface to which the stress is applied. Normal stresses are subdivided into *pressures*, or *compressions*, and *tensions* according as the force, or stress, acts toward or away from the place of application. A tangential stress is also called a *shear*.

The classification may be presented thus:

Simple stresses	normal...	pressure
		tension
	tangential..	shear

105. Description of a Simple Stress.—This requires a statement as to its kind (pressure, tension, or shear), and as to the manner of its distribution. The distribution is described by a statement, for each point of the place of application, of the value of the

106. Intensity of Stress.—By intensity of stress, or stress intensity, at any point of the place of application of the stress is meant the stress per unit area at that point.

If the stress is uniform as to distribution, then the stress per unit area is the same at all points, and its value is found by dividing the stress by the area of its place of application. If F, A, and p denote the stress, area, and intensity respectively,

$$p=F/A. \qquad (1)$$

If the stress is non-uniform as to distribution, then the stress per unit area is different at different points, and the expression F/A gives the value of the average intensity. Also if ΔA denotes the area of any part of the place of application and ΔF is the value of the stress applied to that part, $\Delta F/\Delta A$ is the value of the average intensity of that part, ΔF, of the whole stress. Now if ΔA, as it approaches zero, always includes a point P, then $\Delta F/\Delta A$ approaches, in general, a finite limit, and the value of that limit is the intensity of stress at P, or

$$p=\frac{dF}{dA}. \qquad (2)$$

The *unit of intensity of stress* depends on the unit used for expressing F and A. If, for example, the pound and square inch are used for these respectively, then the pound per square inch is the corresponding unit for stress intensity.

[Note: What is herein called intensity of stress is very commonly called by engineers "unit stress." Strictly, unit stress is the general name for the units employed for expressing stresses, the pound, kilogram, etc., and the engineer's usage is avoided in this book as being confusing to the student. Once thoroughly familiar with the quantities involved, he may safely adopt the engineer's term.]

107. Graphical Representation of a Simple Stress.*—

(*a*) *Normal Stress.*—Let $abcd$, in the xy plane (fig. 63), be the place of application of the stress, and imagine ordinates erected at all points of it proportional to the intensities of stress at the points; thus, if p denote the intensity at P, z the ordinate, and k any constant (the scale number),

$$p = kz, \quad \text{or} \quad z = p/k.$$

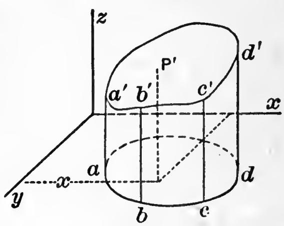

FIG. 63.

Then the volume of the solid defined by those ordinates represents the stress, for its altitude at any point represents the intensity there; and the volume represents the value of stress, as can be shown, thus:

the stress, or $F = \int p dA = k \int z dA$,

and the volume, or $V = \int z dA$;

hence

$$F = kV, \quad \text{or} \quad V = F/k.$$

If the stress is partly tensile and partly pressural, the ordinates corresponding to tensions and pressures are drawn from the plane in opposite directions. Then p is regarded as posi-

* In this and the following articles it is assumed that the place of application of the stress is plane, but some of the results are not restricted to such cases.

tive for one kind and negative for the other kind of stress. (See fig. 64.)

(*b*) *Tangential Stress.*—A tangential stress which is uniform as to direction may also be represented by the method described above.

108. Centre of Stress.—Any stress which is uniform as to direction can be conceived as a system of parallel forces having definite application points. Thus imagine the place of application of such a stress divided into elementary parts, and let dA denote the area of any element and p the intensity of stress at that element; then the stress or force on that element is pdA. Now all such forces as pdA make up a system of parallel forces and their application points are definite points of the place of application of the stress.

Since a system of parallel forces having definite application points has a centroid (art. 62), therefore a stress which is uniform as to direction has a centroid, or a centre as it is more commonly called.

The centre of stress is in the plane of the place of application, and it is where the action line of the resultant of the elementary forces, pdA, pierces that plane. Formulas for the position of the centre of stress in the plane may be deduced from those for the centroid of a system of parallel forces (art. 63). Thus, let fig. 64 represent a stress and x_c and y_c the coordinates of the centre of the stress. Also, let x and y denote the coordinates of the application point of the force pdA on any elementary area dA. Then

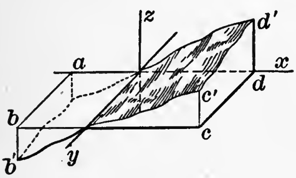

FIG. 64.

$$x_c = \frac{\int pdA \cdot x}{F} \quad \text{and} \quad y_c = \frac{\int pdA \cdot y}{F},$$

F denoting the value of the stress, or $\int pdA$.

Proposition.—A line drawn through the centroid of the solid representing a stress and perpendicular to its place of application passes through the centre of the stress.

Proof: According to the preceding article, the equations above can be rewritten thus:

$$x_c=\frac{\int zdA\cdot x}{V} \quad \text{and} \quad y_c=\frac{\int zdA\cdot y}{V};$$

hence

$$x_c=\frac{\int dV\cdot x}{V} \quad \text{and} \quad y_c=\frac{\int dV\cdot y}{V}.$$

Now $\frac{\int dV\cdot x}{V}$ and $\frac{\int dV\cdot y}{V}$ are the expressions for the coordinates of the centroid of the solid whose volume is V (art. 83); hence the corresponding coordinates (x and y) of the centre of stress and the centroid of the solid representing the stress are equal. It follows that the line joining those two points is parallel to the z axis, hence, etc.

109. A Uniformly Varying Normal Stress.—By this is meant one whose intensity at any point is proportional to the distance of that point from some straight line in the plane of the place of application of the stress. The straight line is called a *neutral axis*, or *zero line*. A familiar example of such a stress is the pressure of a liquid which is at rest upon an immersed flat surface which is not horizontal, and an important case is the "fibre stress" on any cross-section of a moderately loaded beam.

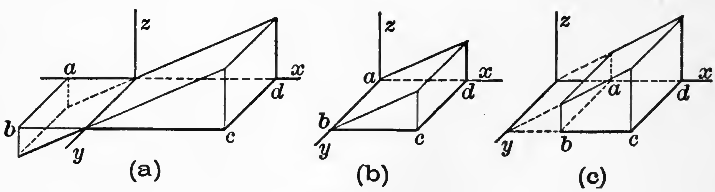

Fig. 65.

Fig. 65 represents three such stresses; the places of application are *abcd*, and the zero lines are coincident with the y axes. In (a) the stress is part tension, part pressure; in (b)

and (c) it is all of one kind. In any case, the law of variation can be expressed thus:

$$p = ax,$$

p being the intensity at a point whose distance from the zero line is x and a a constant. When the stress consists of a tension and a pressure, then p in the equation must be regarded as having sign, its sign being the same as that of x.

When $x=1$, $p=a$; hence the numerical value of a equals the value of p at points at unit distance from the zero line.

The value of the stress.—The stress on an elementary area is

$$dF = pdA = axdA,$$

hence

$$F = a\int xdA. \qquad (1)$$

Since $\int xdA = \bar{x}A$, A being the area of the place of application of the stress and x the x coordinate of its centroid,

$$F = a\bar{x}A. \qquad (2)$$

Let $\bar{p}$ denote the intensity of stress at the centroid of the place of application, then

$$\bar{p} = a\bar{x} \quad \text{and} \quad F = \bar{p}A. \qquad (3)$$

Proposition.—The intensity of a uniformly varying stress at the centroid of its place of application and the average intensity of the stress are equal.

Proof: From equation (3), $\bar{p} = F/A$; since F/A is the value of the average intensity (art. 106), the proposition is proved by that equation.

110. *Centre of a Uniformly Varying Stress.*—Substituting values of p and F for a uniformly varying stress in the first expressions for the x_c and y_c in art. 108, we have

$$x_c = \frac{a\int (xdA)x}{aA\bar{x}} = \frac{\int x^2 dA}{A\bar{x}},$$

$$y_c = \frac{a\int (xdA)y}{aA\bar{x}} = \frac{\int xydA}{A\bar{x}}.$$

$A\bar{x}$ is the moment of the area of the place of application of the stress with respect to the zero line. The integrals in the expressions for x_c and y_c are of great importance and have special names; they are discussed in App. D, where their values for many forms of surfaces are deduced.

EXAMPLES.

1. Where is the centre of a stress whose place of application is a rectangle, the intensity at any point being proportional to its distance from one side?

2. Where is the centre of a stress whose place of application is a semicircle, the intensity at any point being proportional to its distance from the diameter?

111. *Moment of a Uniformly Varying Normal Stress.*—The stress on an element of area dA whose ordinate is x, is $axdA$, and its moments with respect to the x and y axes are respectively

$$(axdA)y \quad \text{and} \quad (axdA)yx.$$

If M_x and M_y denote the moments of the entire stress about the x and y axes respectively,

$$M_x = a\int xydA$$

and

$$M_y = a\int x^2dA.$$

112. *Position of the Neutral Axis when the Stress is Part Tensile and Part Pressural and the Two Parts are Equal.*—The algebraic sum of the elementary stresses is zero; hence

$$a\bar{x}A = 0, \quad \text{or} \quad \bar{x} = 0,$$

i.e., the zero line contains the centroid of the surface of application of the stress.

In a beam, as ordinarily loaded, there is on each cross-section a stress of the kind described in the title to this article, and the result above deduced is of great practical importance. It will bear another deduction; it differs but slightly from that given above. Let A' and A'' be the areas of the parts of the place of application of the stress which sustains tension and pressure respectively, and $\bar{x}'$ and $\bar{x}''$ the distances (not coordinates) of their centroids from the zero line. Then

the value of the tension is $a\bar{x}'A'$;
" " " " pressure is $a\bar{x}''A''$ (art. 109).

Since these are equal, $\bar{x}'A' = \bar{x}''A''$, that is, the moments of the two parts of the entire area A about the yz plane are equal; hence that plane contains the centroid of the whole area (art. 71). But the centroid is also in the xy plane, hence it is in the y axis, or the neutral axis.

CHAPTER V.

GENERAL PRINCIPLES OF EQUILIBRIUM.

§ I. Preliminary.

113. Definitions.—An *external force* is one exerted on a body by some other body. An *internal force* is one exerted on a part of a body by some other part. The same force may be classed either as external or internal, depending upon the point of view. For example, consider a block resting upon a table; the two bodies exert forces, or pressures upon each other. If the block and table are considered as one body, then both of these forces are internal, but if they are considered as two separate bodies, then each of the forces is an external one. Again, the upper and lower halves of the block exert forces upon each other, and with reference to the whole block considered as one body each is an internal force; but with reference to either half considered as a body, the force exerted upon that half is an external one.

A body is said to be in *equilibrium* if (1) it is at rest or (2) its state of motion is unchanging. The first is the important case in this connection.

The system of external forces applied to a body which is in equilibrium is also said to be in equilibrium.

114. General Condition of Equilibrium for a System of Forces Applied to a Rigid Body.—If the system of external forces applied to a rigid body is in equilibrium, its resultant is nil. For the system, being in equilibrium, produces no change of motion, and therefore its resultant would produce none; hence the resultant must be nil.

Conversely, if the resultant of all the external forces applied to a rigid body is nil, that system of forces is in equilibrium.

For the resultant, being nil, would produce no motion, therefore its equivalent, the system, produces none; hence it is in equilibrium.

The condition fulfilled by a system in equilibrium and the condition to insure the equilibrium of a system are one and the same, namely, *the resultant is nil;* this is called, therefore, the general condition of equilibrium.

115. Equilibrium of a System of Forces Applied to a Non-rigid Body.—The condition stated in the preceding article for the equilibrium of forces applied to a rigid body is, as shown, both necessary and sufficient. For the equilibrium of a deformable body it is necessary that the resultant of the external applied forces be nil, but it is not sufficient. This may be explained by illustration:

Consider the water in a cup; the system of external forces applied to it consists of its weight, the pressures exerted by the cup, and the air pressure on top. Now if this same system of forces could be applied to the water when frozen, it would certainly be in equilibrium and its resultant would be nil; hence the resultant of the external system on the body of water is nil. But any forces applied to the body of ice which with the weight have a zero resultant will maintain its equilibrium; thus, a single vertical force equal to the weight and acting through the centre of gravity will answer. The same force applied to the body of water will not of course maintain its equilibrium, although the resultant of the two applied forces is nil. Other conditions than a vanishing resultant therefore are necessary. In this case, pressure must be exerted on the entire surface of the body of water except the top, and in a certain manner.

Again, consider two similar boards fastened together with one nail as in fig. 66(*a*) and two fastened with several nails as in fig. 66(*b*), and suppose their planes vertical. It can be proved experimentally and otherwise that the equilibrium of the first two boards can be maintained by two forces such as R' and R''. Clearly, the same two forces would maintain the equilibrium of the second two boards; hence the resultant of the system R', R'', W', and W'' is zero. But any forces which together with W' and W'' have a zero resultant would maintain the equilibrium of the boards in fig. 66(*b*); thus, a single vertical force, as R, equal to W' and W'' will answer. The same force applied to the first two boards will not of course maintain their equilibrium, although the resultant of the applied

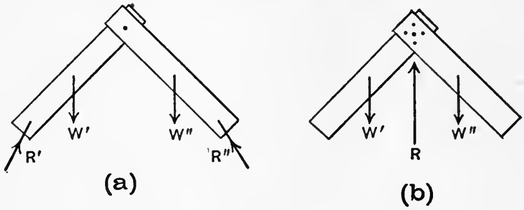

FIG. 66.

forces is zero. Therefore, other conditions than a vanishing resultant are necessary.

Summing up, if the system of external forces applied to a deformable body is in equilibrium, the resultant is nil; but the converse is not necessarily true.

§ II. THE CONDITIONS OF EQUILIBRIUM FOR THE VARIOUS CLASSES OF FORCE SYSTEMS.

116. Collinear Forces.—The *algebraic condition* of equilibrium is,

$$\Sigma \mathbf{F} = 0,$$

i.e., the algebraic sum of the forces equals zero.

The *graphical condition* of equilibrium is that the force polygon for the forces closes, or, the vector sum of the forces equals zero.

For if ΣF equals zero or the force polygon for the forces closes, the resultant is nil.

117. Coplanar Concurrent Non-Parallel Forces.—The *algebraic conditions* of equilibrium may be expressed in various ways:

(*a*) $$\Sigma \mathbf{F}_x = 0, \quad \Sigma \mathbf{F}_y = 0; \quad \ldots \ldots \quad (1)$$

i.e., the algebraic sum of the resolved parts of the forces along two lines, x and y, equals zero.

(*b*) $$\Sigma \mathbf{F}_x = 0, \quad \Sigma \mathbf{M} = 0; \quad \ldots \ldots \quad (2)$$

i.e., the algebraic sum of the resolved parts of the forces along a line x equals zero and the moment sum for the forces with respect to a point equals zero. (The direction

x is not to be perpendicular to the line joining the common intersection of the action lines of the forces with the origin of moments.)

(c) $$\Sigma \mathbf{M}_a = 0, \quad \Sigma \mathbf{M}_b = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

i.e., the moment sums of the forces for each of two origins equal zero. (The origins and the common intersection of the action lines of the forces are not to be collinear.)

For, in either case the resultant is zero, as may be thus explained:

(a) According to art. 36, the resultant R of the system, if there is one, is given by
$R = (\overline{\Sigma F_x}^2 + \overline{\Sigma F_y}^2)^{\frac{1}{2}}$; hence, if ΣF_x and ΣF_y equal zero, R equals zero.

(b) If ΣM_a is zero, then the resultant, if there is one, must pass through a as well as through the common point of the action lines of the forces, O. If the angle between Oa and the x axis be α, $R_x = R \cos \alpha = \Sigma F_x$; and since $\Sigma F_x = 0$ and α is not $90°$, R must be zero.

(c) As before, the resultant, if there is one, must pass through O and a; but if ΣM_b is also zero, and O, a, and b are not collinear, R must be zero.

The *graphical condition* of equilibrium is that the force polygon for the forces closes, i.e., the vector sum of the forces equals zero. For, if the polygon closes, the resultant is zero, see art. 33.

118. *Special Condition of Equilibrium for Three Forces.*—The following form of the algebraic conditions is often more convenient of application than the conditions given in the preceding article.

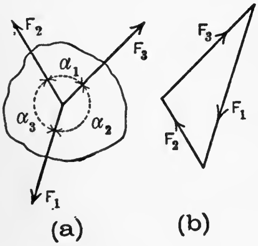

FIG. 67.

$$\mathbf{F}_1/\sin \alpha_1 = \mathbf{F}_2/\sin \alpha_2 = \mathbf{F}_3/\sin \alpha_3;$$

i.e., each force is proportional to the sine of the angle between the other two (see fig. 67). For, from the force polygon for the forces, which closes (fig. 67*b*),

$$F_1/\sin(180 - \alpha_1) = F_2/\sin(180 - \alpha_2) = F_3/\sin(180 - \alpha_3), \text{ or}$$
$$F_1/\sin \alpha_1 = F_2/\sin \alpha_2 = F_3/\sin \alpha_3.$$

119. Coplanar Non-Concurrent Parallel Forces.—The *algebraic conditions* of equilibrium may be expressed in several ways:

(*a*) $$\Sigma \mathbf{F}=0, \quad \Sigma \mathbf{M}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i.e., the sum of the forces and the sum of the moments of the forces each equal zero.

(*b*) $$\Sigma \mathbf{M}_a=0, \quad \Sigma \mathbf{M}_b=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e., the moment sums for the system for each of two origins equal zero. (The line joining the origins is not to be parallel to the forces.)

For in either case the resultant is zero, as may be thus explained: In art. 42, it is shown that the resultant of such a system is either a force or a couple, and if a couple, its moment equals the algebraic sum of the moments of the given forces about any point.

(*a*) If ΣF equals zero, the resultant is not a force, and if ΣM equals zero, the resultant is not a couple; hence the resultant vanishes.

(*b*) If ΣM_a equals zero, the resultant is not a couple, and the resultant force, R, (if there is one,) must pass through a. But if ΣM_b equals zero, R must equal zero, or pass through b; but R is parallel to the forces of the system and cannot therefore pass through a and b. Hence R equals zero, i.e., the resultant vanishes.

The *graphical conditions* of equilibrium are (1) the force polygon and (2) the funicular polygon for the forces must close. For, if the force polygon closes, the resultant is not a force, and if the funicular polygon closes, it is not a couple (see art. 40); hence the resultant vanishes.

120. Coplanar Non-Concurrent Non-Parallel Forces. — The *algebraic conditions* of equilibrium are

(*a*) $$\Sigma \mathbf{F}_x=0, \quad \Sigma \mathbf{F}_y=0, \quad \Sigma \mathbf{M}=0; \quad . \quad . \quad . \quad . \quad (1)$$

i.e., the algebraic sums of the resolved parts of all the forces along each of two lines and the moment sum of the forces with respect to any origin equal zero.

(*b*) $$\Sigma \mathbf{F}_x = 0, \quad \Sigma \mathbf{M}_a = 0, \quad \Sigma \mathbf{M}_b = 0; \quad . \quad . \quad . \quad . \quad (2)$$

i.e., the algebraic sum of the resolved parts of all the forces along any line and the moment sums for the system with respect to two origins equal zero. (The direction of resolution is not to be perpendicular to the line joining the two moment origins.)

(*c*) $$\Sigma \mathbf{M}_a = 0, \quad \Sigma \mathbf{M}_b = 0, \quad \Sigma \mathbf{M}_c = 0; \quad . \quad . \quad . \quad (3)$$

i.e., the moment sums for the system with respect to three origins equal zero. (The three moment origins are not to be collinear.)

For, in either case the resultant is nil, which may be proved as follows: It is shown in art. 47 that the resultant of a system of this kind is a force or a couple; if a force, its magnitude equals $(\overline{\Sigma F_x}^2 + \overline{\Sigma F_y}^2)^{\frac{1}{2}}$, and if a couple, its moment equals ΣM.

(*a*) If ΣF_x and ΣF_y equal zero, the resultant is not a force, and if ΣM equals zero, the resultant is not a couple; hence the resultant vanishes.

(*b*) If $\Sigma M = 0$, the resultant is not a couple, and the resultant force, R, (if there is one) must pass through a and b in order to make ΣM_a and ΣM_b equal to zero. If now $\Sigma F_x (= R_x)$ equals zero, R must equal zero.

(*c*) If $\Sigma M = 0$, the resultant if there is one, is not a couple, but a force. If ΣM_a, ΣM_b, and ΣM_c equal zero, this resultant force must equal zero, or else pass through a, b, and c. Since the latter case is impossible, the resultant vanishes.

The *graphical conditions* of equilibrium are (1) the force polygon and (2) the funicular polygon must close. For, if the force polygon closes, the resultant is not a force, and if the funicular polygon closes, it is not a couple; hence the resultant vanishes (see art. 45).

121. *Condition of Equilibrium for Four Forces.*—The resultant of either pair balances that of the other pair. This special condition may often be advantageously employed in graphical solutions of problems.

122. Non-Coplanar Concurrent Forces.—The *algebraic conditions* of equilibrium are:

$$\Sigma \mathbf{F}_x = 0, \quad \Sigma \mathbf{F}_y = 0, \quad \Sigma \mathbf{F}_z = 0;$$

i.e., the algebraic sums of the resolved parts of all the forces along three directions, not coplanar, equal zero. For, in art. 50 it is shown that the resultant R of this kind of a system is given by $R = (\overline{\Sigma F_x}^2 + \overline{\Sigma F_y}^2 + \overline{\Sigma F_z}^2)^{\frac{1}{2}}$; hence if the conditions be fulfilled, R must equal zero.

The *graphical conditions* of equilibrium are that the force polygons for two projections of the vectors representing the given forces, regarded as force systems, must close. For, the closing lines of such polygons are the projections of the resultant of the given forces, and if the force polygons close, the projections vanish and the resultant is zero (see art. 49).

123. Non-Coplanar Non-Concurrent Parallel Forces.—The *algebraic conditions* of equilibrium are :

$$\Sigma \mathbf{F} = 0, \quad \Sigma \mathbf{M}_x = 0, \quad \Sigma \mathbf{M}_y = 0;$$

i.e., the algebraic sum of the forces and their moment sums with respect to two axes equal zero. (Neither of the axes is to be parallel to the forces.) For, in art. 54 it is shown that the resultant, if there is one, is a force or a couple. If ΣF is zero, the resultant is not a force, and if ΣM_x and ΣM_y equal zero, the resultant is not a couple; hence the resultant vanishes.

The *graphical conditions* of equilibrium are (1) the force polygon for the forces and (2) the funicular polygons for the projections of the vectors representing the given forces, regarded as forces, on two planes must close. For, the resultant of the system, if there is one, is a force or a couple; if the force polygon closes, the resultant is not a force, and if the funicular polygons close, it is not a couple; hence the resultant vanishes (see art. 52).

124. Non-Coplanar Non-Concurrent Non-Parallel Forces.—The *algebraic conditions* of equilibrium are:

$$\Sigma F_x=0, \quad \Sigma F_y=0, \quad \Sigma F_z=0, \quad \Sigma M_x=0, \quad \Sigma M_y=0, \quad \Sigma M_z=0;$$

i.e., the algebraic sums of the resolved parts of the forces along three lines and the moment sums for the system with respect to three axes equal zero. (The three lines must not be coplanar, nor the three axes.)

For, in art. 55 it is shown that the resultant of such a system is a force and a couple, and in art. 58 that the magnitude of the force is $(\overline{\Sigma F_x}^2+\overline{\Sigma F_y}^2+\overline{\Sigma F_z}^2)^{\frac{1}{2}}$, and the moment of the couple is $(\overline{\Sigma M_x}^2+\overline{\Sigma M_y}^2+\overline{\Sigma M_z}^2)^{\frac{1}{2}}$; hence, if the conditions are fulfilled, the resultant vanishes.

The *graphical conditions* may be expressed thus: If the system be resolved into two component systems, one coplanar and one parallel, the forces of the latter being perpendicular to the plane of the former, then the following polygons must close:

the force polygon for the coplanar system,
" funicular polygon for the coplanar system,
" force polygon for the non-coplanar system,
" funicular polygons for the projections of the vectors representing the parallel system on each of two non-parallel planes.

For, if these conditions are fulfilled, the resultants of the component systems are zero (see arts. 45 and 52), and hence the resultant of the system itself vanishes.

125. Special Condition of Equilibrium and Summary.—

Proposition.—If three forces are in equilibrium, they must be coplanar and concurrent or parallel.

Proof: The resultant of the first two forces must be a single force, since it must balance a force, the third one. Since the two forces and their resultant are coplanar and the resultant and the third force are collinear, the three given forces are coplanar. If the first two forces are parallel, their resultant is parallel to them, and hence the third force is also; if the first two forces intersect, their resultant passes through their intersection, and hence the third force does also.

Algebraic Conditions of Equilibrium.

Coplanar Systems:

Collinear. $\Sigma F = 0$.

Concurrent Non-parallel. . . . . $\Sigma F_x = \Sigma F_y = 0$,
$\Sigma F_x = \Sigma M = 0$, or
$\Sigma M_a = \Sigma M_b = 0$.

Non-concurrent Parallel. . . . . $\Sigma F = \Sigma M = 0$, or
$\Sigma M_a = \Sigma M_b = 0$.

Non-concurrent Non-parallel $\Sigma F_x = \Sigma F_y = \Sigma M = 0$,
$\Sigma F_x = \Sigma M_a = \Sigma M_b = 0$, or
$\Sigma M_a = \Sigma M_b = \Sigma M_c = 0$.

Non-coplanar Systems:

Concurrent. . . . . $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.

Non-concurrent Parallel. . . . . $\Sigma F = \Sigma M_x = \Sigma M_y = 0$.

Non-concurrent Non-parallel. $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$ and
$\Sigma M_x = \Sigma M_y = \Sigma M_z = 0$.

Graphical Conditions of Equilibrium.

Coplanar Systems:

Concurrent. . . . . . The force polygon closes.

Non-concurrent. . . The force and funicular polygons close.

Non-coplanar Systems:

Concurrent. . . . . . The polygons for the projections of the vectors representing the system on any two non-parallel planes close.

Non-concurrent Parallel. . . . . . . The force polygon closes and the funicular polygons for the projections of the vectors representing the system on two non-parallel planes close.

Non-concurrent Non-parallel. . . . If the system be resolved into component systems, one coplanar and one parallel, the forces of the latter being normal to the plane of the former, then the appropriate conditions stated above apply to each component system.

CHAPTER VI.

APPLICATIONS OF THE PRINCIPLES OF EQUILIBRIUM.

§ I. Preliminary.

126. Nature of the Problems.—The problems involved in the application of the principles of equilibrium are usually of this kind: a system of forces is in equilibrium and some of them are partly or wholly unknown; it is required to determine the unknown elements. The required elements may be the magnitudes, directions, or action lines of forces.

127. General Method of Solution.—These problems can be solved by two methods, the algebraic and the graphical. The *algebraic method* is to write the appropriate equations of equilibrium for the kind of a force system under consideration, and then to solve them for the unknown quantities. This process is called applying the algebraic conditions of equilibrium. The *graphical method* is to apply the appropriate graphical conditions of equilibrium for the kind of a force system under consideration. How these conditions are applied is explained in the solution of some of the following examples.

Often the force system in equilibrium of which the partly or wholly unknown forces are a part, and to which the conditions of equilibrium are to be applied, is not specified. In such cases the system may be recognized by directing one's attention to a body which is in equilibrium and with reference to which some or all the unknown forces are external. *All the external forces applied to that body constitute a system in equilibrium.*

In all but the simplest of the following examples the student is strongly urged to make a sketch representing the body considered and the external forces exerted upon it before applying the conditions of equilibrium. He will be aided in his enumeration of the external forces if he will represent first the actions through distance * exerted upon the body by other bodies and then count the number of contacts between the body under consideration and other bodies; at each place of contact a force may be exerted upon the body (art. 7).

§ II. Flexible Cords.

128. Definitions.—A perfectly flexible cord is one which may be bent without resistance. Such a cord is ideal, but some cords are practically perfectly flexible; such, for brevity, will be called flexible, other cords being called stiff. A heavy flexible cord if unsupported cannot be stretched straight, but will sag more or less, depending upon the applied pulls. The lighter the cord, the less the sag; and, if the weight of the cord be small compared with the pulls, it will be practically straight. In the following, cords will be assumed without weight except where otherwise stated.

129. Tension in a Cord.—The phrase tension in a cord refers to the forces which two parts of a taut cord exert upon each other. The forces are equal and opposite (see art. 6). By *magnitude of the tension* is meant the magnitude of either force.

To illustrate, suppose that AB (fig. 68) is a flexible cord without weight subjected to equal pulls at its ends, and imagine a plane of separation at any place, C, between the ends of the cord. Since the part AC is in equilibrium, a force acts upon it at its right end equal and opposite to P'; this force is exerted by the part BC. Similarly, a force acts on BC at its left end equal and opposite to P''; this force is exerted by the part AC. These two equal and opposite forces at C hold the parts AC and BC together. The magnitude of the tension is P' (or P'') no matter where C is taken.

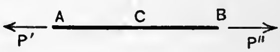

Fig. 68.

If the pulls P' and P'' are unequal, the cord will not be in equilibrium, and the magnitude of the tension varies with the section C.

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* The only actions through distance considered in this chapter are the weights of bodies.

EXAMPLES.

1. A body weighing 100 lbs. is suspended by a single cord which is deflected from the vertical by a horizontal force of 20 lbs. applied to the body. How great are the deflection and the tension in the cord?

Solutions: (1) Algebraic. Considering the forces acting upon the body, it is seen that they are three in number, namely, its weight, the horizontal force, and the pull of the string. The conditions of equilibrium, with axes as in fig. 69, are (art. 125)

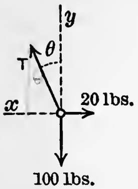


FIG. 69.

$$\Sigma F_x = 20 - T \sin \theta = 0;$$

$$\Sigma F_y = -100 + T \cos \theta = 0.$$

Solving these it is found that

$$T = 102 \text{ lbs.} \quad \text{and} \quad \theta = 11° \, 19'.$$

(2) Graphical. The condition of equilibrium is that the force polygon for the three forces must close (art. 125). To construct the polygon, the wholly known forces are represented first; thus AB and BC (fig. 70) represent the weight and the horizontal pull respectively, scale 1 in. = 100 lbs. The closing side CA then represents the magnitude and direction of the pull of the string exerted upon the body.

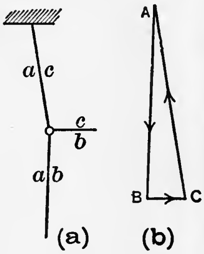


FIG. 70.

2. In the preceding example, how great a horizontal force would be required to deflect the cord 30°, and how great would the corresponding tension be?

3. The two ends of a cord are fastened to hooks in the same horizontal line, and at the middle a second cord is knotted which sustains a freely hanging body weighing 100 lbs. The distance between the hooks being such that the first cord makes angles of 20° with the horizontal, determine the tension in each half of the first cord. *Ans.* 146 lbs.

4. Call the angle in the preceding example α and the weight of the body W. Deduce an expression for the tension.

5. Suppose that the second cord in ex. 3 is knotted to the first at such a point that the inclinations are 10 and 50 degrees. Determine the tensions, employing the special condition of equilibrium of art. 118. *Ans.* 74.2 lbs.

6. Fig. 71 represents an upright frame within which is arranged a network of cords; the connection *EC* consists of a spring balance. Suppose that the balance, wrong end up, reads 10 lbs. Determine the tension in *AF*. *Ans.* $26\frac{2}{3}$ lbs.

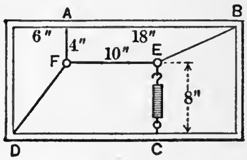


FIG. 71.

7. A given bar is to be supported in a certain position by two strings fastened to its ends. What condition must the directions of the strings satisfy? (See prop., art. 125.)

8. A uniform bar weighing 110 lbs. is supported in a horizontal position by strings fastened to its ends. If one is inclined 30° to the bar, what is the inclination of the other and what are the tensions?

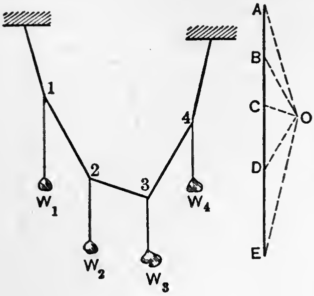
9. A uniform plate triangular in shape is suspended in a horizontal position by vertical cords, one being fastened to each corner. Show that each cord sustains one third the weight of the plate.

10. Three hooks, *A*, *B*, and *C*, in a ceiling lie on a circle of 3 ft. radius, and the arcs *AB*, *BC*, and *CA* are equal. A small ring is suspended at a point 4 ft. below the centre of the circle by means of three cords fastened to the hooks. If a body weighing 100 lbs. is suspended from the ring, how great is the tension in each cord? *Ans.* $41\frac{2}{3}$ lbs.

11. Suppose that in the preceding example, the arc *AB* is a quadrant and *AC* and *BC* are equal. Determine the tensions.

12. It is required to determine the weights necessary to hold a cord in the position shown in fig. 72(*a*).

Solution: The lines radiating from *O* (fig. 72*b*) are drawn parallel to the segments of the cord, and *AE* is any vertical line. If the weights W_1, W_2, W_3, and W_4 are proportional to *AB*, *BC*, *CD*, and *DE* respectively, they will hold the cord in the given position. For, imagine all the knots except the first connected by vertical cords to a fixed support below, and a body hung from the first knot; the cord is then in the required position. According to *some* scale, AB represents W_1, and according to the same scale BC, CD, and DE represent the tensions in the imaginary vertical cords at the second, third, and fourth knots respectively. But these tensions can be supplied by bodies suspended from the knots.



(a) (b)

FIG. 72.

130. Position Assumed by a Cord Sustaining Loads.—Let fig. 73 represent the nth knot on the cord, the load suspended from which call W_n and the tensions in the cord segments on the left and right T_n and T_{n+1} respectively. Since the three forces represented are in equilibrium,

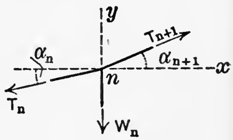


FIG. 73.

$$\Sigma F_x = T_{n+1} \text{ co } \alpha_{n+1} - T_n \cos \alpha_n = 0;$$
$$\Sigma F_y = T_{n+1} \sin \alpha_{n+1} - T_n \sin \alpha_n - W_n = 0.$$

From the first equation, it follows that the horizontal components of the tensions in the cord segments are equal; that is, if H denotes the horizontal component in any segment,

$$H = T_n \cos \alpha_n = T_{n+1} \cos \alpha_{n+1} = \text{etc.} \quad . \quad . \quad . \quad (1)$$

This equation combined with the second one above gives

$$\tan \alpha_{n+1} = \tan \alpha_n + W_n/H. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By means of this equation the direction of each segment may be computed if, in addition to all the weights, the tension in and inclination of one segment or the inclinations of two adjacent segments are known.

EXAMPLES.

1. Three bodies weighing 100, 120, and 200 lbs. are suspended in the order named from three knots in a cord which is so supported that the second segment is horizontal and its

tension is 160 lbs. Determine the directions of the other segments. *Ans.* $\alpha_1 = 32°$.

2. Suppose that the supports in the preceding example are such that the middle knot is the lowest and that the segments adjacent to that knot are inclined 30° above the horizontal. Determine the inclinations of the other segments and all the tensions.

3. Solve the preceding examples graphically.

131. *The Loads are Equal and Uniformly Spaced Horizontally.*—The knots are on a parabola; proof follows. Let $O, 1, 2 \ldots n$ (fig. 74) be knots on the cord from which the loads, W, are suspended, and let x and y axes be taken as in the figure. Denote the angles which $O1$, 12, 23, etc., make with the axes by $\alpha_1, \alpha_2, \alpha_3 \ldots$ etc., the horizontal distance between consecutive loads by s, and the coordinates of the nth knot by x and y. Then

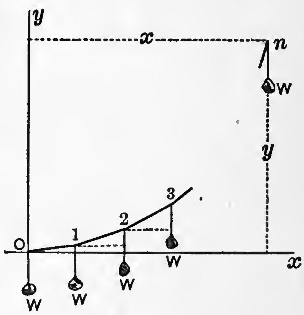


Fig. 74.

$$x = ns; \quad \ldots \ldots \ldots \quad (1)$$
$$y = s\,(\tan\alpha_1 + \tan\alpha_2 + \ldots).$$

From eq. (2), art. 130,

$$\tan\alpha_2 = \tan\alpha_1 + W/H;$$
$$\tan\alpha_3 = \tan\alpha_2 + W/H = \tan\alpha_1 + 2W/H.$$

Similarly,

$$\tan\alpha_4 = \tan\alpha_1 + 3W/H, \text{ etc.};$$

hence

$$y = s[n\tan\alpha_1 + \tfrac{1}{2}n(n-1)W/H]. \quad \ldots \ldots \quad (2)$$

Combining (1) and (2) and eliminating n, we get

$$x^2 + \frac{2\tan\alpha_1 H - W}{W}sx = \frac{2Hs}{W}y, \quad \ldots \ldots \quad (3)$$

which is the equation of a parabola.

If the loads are so closely spaced as to constitute a practically continuous load, the cord is practically parabolic. The equation of the parabola referred to vertical and horizontal axes though the lowest point on it may be derived from eq. (3), or independently as follows:

Let O (fig. 75) be the lowest point on the cord, Q any other point, and let w denote the load per unit horizontal distance; then OQ sustains a load wx. The tension at O call H, and that at Q, T. The three forces applied to OQ are concurrent, and their equilibrium equations are

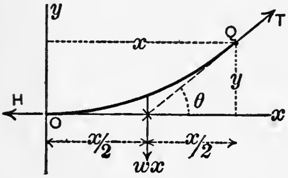


FIG. 75.

$$\Sigma F_x = -H + T\cos\theta = 0; \quad \ldots \quad (1)$$
$$\Sigma F_y = -wx + T\sin\theta = 0. \quad \ldots \quad (2)$$

Combining these with $\tan\theta = y \div x/2$, we get

$$x^2 = \frac{2H}{w}y, \quad \ldots \quad (3)$$

which is the equation sought.

If the points of suspension are at the same level, then (see fig. 76 and eq. (3)),

$$d = wa^2/2H. \quad \ldots \quad (4)$$

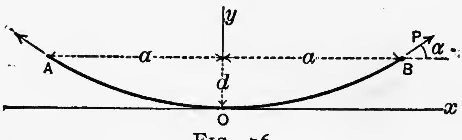


FIG. 76.

EXAMPLES.

1. A cord is supported at two points on the same level 30 ft. apart, and its lowest point is 8 ft. below the level of the supports. If the load is 20 lbs. per ft., what are the tensions at the supports and at the lowest point? *Ans.* $H = 281\frac{1}{4}$ lbs.

2. Suppose that one support is 3 ft. higher than the other and that the lowest point of the cord is 8 ft. lower than the lower support. What is the greatest tension in the cord?

132. Position Assumed by a Heavy Flexible Cord Suspended from Two Points.—The curve assumed by such a cord if uniform in weight (and such only are discussed below) is called a *common catenary.* Let AB (fig. 77) represent such a cord, C its lowest point, Q any other point, s the length CQ, H and T

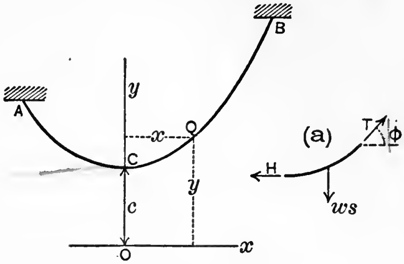


FIG. 77.

the tensions at C and Q respectively, and w the weight of the cord per unit length. Then the forces on the portion CQ are H, T, and ws acting as shown in fig. 77(*a*); hence

$$\Sigma F_x = -H + T\cos\phi = 0;$$
$$\Sigma F_y = -ws + T\sin\phi = 0.$$

From these, by division, we get

$$\tan\phi = ws/H.$$

For convenience, let c denote a length of the cord whose weight equals the tension at C, then $H/w = c$; and since $\tan\phi = dy/dx$,

$$\frac{dy}{dx} = \frac{s}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integration of equation (1) leads to the equation of the catenary. Differentiating, we get

$$d\left(\frac{dy}{dx}\right) = \frac{ds}{c}.$$

Since $\quad (ds)^2 = (dy)^2 + (dx)^2, \quad ds = dx\sqrt{1 + (dy/dx)^2};$

hence
$$d\left(\frac{dy}{dx}\right) = \frac{dx}{c}\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

or
$$d\left(\frac{dy}{dx}\right) \div \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{dx}{c}.$$

Integrating, we get

$$\log_e\left(\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) = \frac{x}{c} + C'.$$

Now $dy/dx = 0$, where $x = 0$ (see fig.), hence $C' = 0$.

Solving the equation for dy/dx, we find that

$$\frac{dy}{dx}=\tfrac{1}{2}(e^{x/c}-e^{-x/c}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

e being the base of the Naperian system of logarithms.
Integrating the last equation, we get

$$y=\frac{c}{2}(e^{x/c}+e^{-x/c})+C''.$$

Since $y = c$ when $x=0$ (see fig.), the constant C'' is zero; hence, the equation of the catenary referred to the axes of the figure is

$$y=\frac{c}{2}(e^{x/c}+e^{-x/c}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Several interesting relations may be deduced as follows: From (1) and (2)

$$s=\frac{c}{2}(e^{x/c}-e^{-x/c}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining (3) and (4), we find that

$$y^2=s^2+c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also, $\quad y+s=ce^{x/c}, \quad$ or $\quad x=c\log_e\frac{y+s}{c}. \quad . \quad . \quad . \quad . \quad . \quad (6)$

From the equilibrium equations,

$$T^2=w^2(s^2+c^2);$$

hence $\quad T=wy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$

EXAMPLES.

1. A measuring-tape 400 ft. long weighing 0.005 lbs. per ft. is suspended from two points, A and B, the supporting pulls there being respectively 1.6 and 2.0 lbs. Compute the horizontal and vertical distances between A and B.

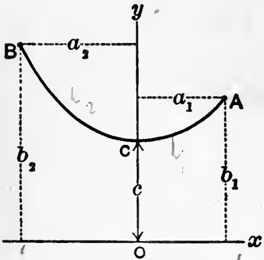


FIG. 78.

Solution: Let T_1 and T_2 denote the pulls at A and B, and l_1 and l_2 the lengths AC and BC respectively (fig. 78). From eq. (7),

$$T_1=wb_1, \quad \text{or} \quad b_1=1.6\div 0.005=320 \text{ ft.},$$

and $T_2 = wb_2$, or $b_2 = 2.0 \div 0.005 = 400$ ft.;

hence
$$b_2 - b_1 = 80 \text{ ft.}$$

From eq. (6),
$$a_1 = c \log_e \frac{b_1 + l_1}{c},$$

and
$$-a_2 = c \log_e \frac{b_2 - l_2}{c}.$$

To determine a_1 and a_2 values of l_1, l_2, and c are needed.

From eq. (5),
$$b_1^2 = l_1^2 + c^2 = 102\,400$$

and
$$b_2^2 = l_2^2 + c^2 = 160\,000;$$

also,
$$l_1 + l_2 = 400.$$

Solution of these three equations shows that

$$l_1 = 128, \quad l_2 = 272, \quad \text{and} \quad c = 293+ \text{ ft.},$$

which values if substituted in the expressions for a_1 and a_2 give

$$a_1 = 122.7 \quad \text{and} \quad a_2 = 242.7 \text{ ft.};$$

hence
$$a_1 + a_2 = 365.4 \text{ ft.}$$

2. What is the tension at the lowest point of the tape in ex. 1?

3. A tape whose length is 100 ft. weighs 0.005 lbs. per ft. and is subjected to end pulls of 10 lbs., the ends being at the same level. Compute the distance between the ends and the "sag."

Ans. Distance = 99.996 ft.
Sag = 0.625 ft.

4. A tape is supported at two points on the same level, its length = $2l$, its weight per unit length = w and the end pulls = P. If $2a$ and d denote distance between supports and the sag respectively, show that

$$a = (P^2/w^2 - l^2)^{\frac{1}{2}} \log_e \frac{P/w + l}{(P^2/w^2 - l^2)^{\frac{1}{2}}},$$

$$d = P/w - (P^2/w^2 - l^2)^{\frac{1}{2}}.$$

5. Suppose that a chain 200 ft. long is suspended from two points on the same level and 50 ft. apart. Find the sag.

Solution: This example can be solved by trial only; thus, referring to fig. 77, it is seen that for B, $x=25$ and $s=100$ ft.; hence eq. (4) becomes

$$\frac{100}{c}=\tfrac{1}{2}(e^{25/c}-e^{-25/c}).$$

It will be found that $c=7.65$ will nearly satisfy this equation.* Then eq. (5) for B becomes

$$y^2=100^2+7^2.65, \quad \text{or} \quad y=100.28;$$

hence the sag is

$$100.28-7.65, \quad \text{or} \quad 92.63 \text{ ft.}$$

6. Determine the length of a chain which sags 20 ft. if suspended from two points on the same level and 80 ft. apart.

Solution: This example can be solved by trial only; thus, referring to fig. 77, it is seen that for B, $x=40$ and $y=c+20$, hence eq. (3) becomes

$$c+20=\frac{c}{2}(e^{40/c}+e^{-40/c}).$$

It will be found that $c=42.7$ nearly satisfies this equation; hence eq. (5) for B becomes

$$(42.7+20)^2=s^2+42^2.7,$$

or $s=45.9$. The length of the chain is therefore 91.8 ft.

133. *Approximate Equations.*—If the points of suspension are at the same level and the sag is small, the length of the cord and distance between supports are nearly equal, and the numbered equations below are close approximations. They may be deduced as follows:

Since the slope of the cord is everywhere small, the load per horizontal unit length is nearly uniform; hence the catenary

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* The determination of c in exs. 5 and 6 is much facilitated by use of a table of hyperbolic functions, which gives values of $\tfrac{1}{2}(e^{x/c}-e^{-x/c})$, or sinh x/c, and $\tfrac{1}{2}(e^{x/c}+e^{-x/c})$, or cosh x/c, for different values of x/c.

is very nearly parabolic and the equations deduced in art. 131 must be nearly correct for a flat catenary.

From eq. (4) of that article,

$$d = wa^2/2H. \quad \ldots\ldots\ldots \quad (1)$$

In works on calculus it is shown that the length of a parabolic arc, as OB (fig. 76), is given by

$$l = \frac{H}{w}\left[\tan\alpha\sqrt{1+\tan^2\alpha} + \log_e(\tan\alpha + \sqrt{1+\tan^2\alpha})\right],$$

or, developed,

$$l = \frac{H}{w}\left(\tan\alpha + \frac{\tan^3\alpha}{6} + \ldots\right).$$

Since, in a parabola, $\tan\alpha = wa/H$, approximately,

$$l = a(1 + w^2a^2/6P^2), \quad \ldots\ldots\ldots \quad (2)$$

and

$$a = l(1 - w^2l^2/6P^2). \quad \ldots\ldots\ldots \quad (3)$$

EXAMPLES.

1. Solve ex. 3, art. 132, by the approximate equations of this article. *Ans.* (1) gives for sag 0.625 ft.
(3) " " distance 99.99 "

2. Hard copper wire weighs $3.85A$ lbs. per foot, A being the area of its cross-section in square inches. If such a wire be suspended between two points at the same level 200 ft. apart, and the maximum pull on it be limited to 20000 lbs. per sq. in., what is the proper length of the wire and the sag?

Ans. Sag = 0.96 ft.

3. The rope of a rope drive when not running is observed to sag 1 ft., and the distance between the centres of the wheels is 70 ft. What is the maximum tension in the rope?

§ III. TACKLE.

134. The Pulley.—A wheel with a flat-faced or grooved rim which can rotate about its axis is often used as a "power transmission" device; when so used it is called a *pulley*. It may rotate freely about an axle or shaft, and is then called a *loose pulley;* or it may be fastened to a shaft which turns in bearings with the pulley, when it is called a *fast pulley.*

A combination of one or more grooved pulleys and a rope or chain used for raising heavy bodies, is called a tackle. The pulley or pulleys rotating on the same axle together with the frame supporting the axle is called a *block.* A pulley is called *movable* or *fixed* according as the block of which it is a part does or does not move while the load is raised or lowered.

135. *Tension in a Cord on a Pulley.*—Fig. 80 represents a single pulley, loose or fast, and fixed or movable, about a part of whose rim a flexible cord or belt is tightly wrapped. We wish to find the relation between the tensions in the cord or belt on opposite sides of the pulley on the supposition that the rubbing surfaces at the axle are "smooth."

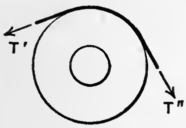


FIG. 80.

If the rubbing surfaces at the axle are smooth, then the pressure, P, of bearings on the axle (fast pulley) or the pressure of the axle on the pulley (loose) is such that its line of action passes through the axis of the pulley (see art. 138). Then of the three forces T', T'', and P, the only ones having a moment about the axis are T' and T'', and if the cord or belt is flexible their arms are equal; hence $T'=T''$. That is, *the tensions in a cord or belt on opposite sides of a pulley which the cord or belt encircles are equal if the cord or belt is flexible and the rubbing surfaces at the axle are smooth.*

In the following examples this relation is made use of, the cords being assumed flexible and the rubbing surfaces smooth. This assumption is far from the truth in actual tackle, and it should be remembered that the results obtained below are for ideal cases. Axle friction and rigidity of cordage are discussed later.

EXAMPLES.

1. In the arrangements shown in fig. 81 (a) and (b), regard F and α as given and compute W and θ.

2. Compute the pressure upon the axle (fig. 82), α and W being given.

3. What is the relation between F and W in the tackle

shown in fig. 83, neglecting the weight of the parts? (Consider the equilibrium of the part below the dotted line.)

4. What is the relation between F and W in the tackles shown in figs. 84 and 85, the

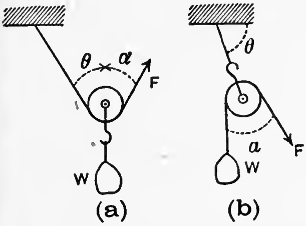


FIG. 81.

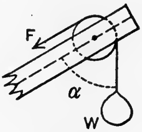


FIG. 82.

FIG. 85.

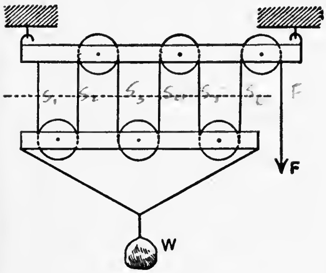


FIG. 83.

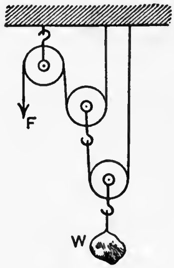


FIG. 84.

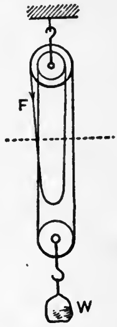


FIG. 87.

FIG. 86.

weight of the parts being neglected? What is the tension in the cord supporting the tackle of fig. 85?

Ans. For fig. 85, $W=6F$.

5. What is the relation between F and W in the tackle shown in fig. 86? What are the pulls on the hooks at A and B?

6. Fig. 87 represents, in principle, a Weston differential tackle. The wheels in the upper block are fastened together, are of unequal diameters, and their grooves are made so that the hoisting chain will not slip in them. Determine the relation between F and W, and the pull at the upper hook.

Solution: Consider the equilibrium of the part of the tackle above the dotted line. The external forces upon it are four in number if its weight be neglected, namely, the upward pull at the hook, the pull F, the tension in the chain on the left of the smaller wheel ($W/2$), and the tension in the chain to the right of the larger wheel ($W/2$). One condition of equilibrium for these forces is that their moment sum with respect to the axis of the wheels is zero; hence if r' and r'' denote radii of smaller and larger pulleys,

$$Fr''+Wr'/2-Wr''/2=0,$$

or

$$W=\frac{2r''}{r''-r'}F.$$

7. Suppose that in fig. 81(b), α is zero and that a man whose weight is W' sits upon the load and exerts a pull F. How great must the pull be to raise man and load? Is it greater or less than W?

§ IV. SMOOTH SUPPORTS.

136. Definitions.—It is known from experience that to slide one body over another even at constant speed requires the application of more or less force; also that a moderate "sliding force" may not cause a body to move. It is inferred that the second body exerts a force upon the first which is opposed to the sliding.

More definitely, let fig. 88 represent two bodies whose surface of contact is a horizontal plane, the upper one being subjected to a horizontal force P. The lower body exerts upon the upper a force such as R, the horizontal component of which, R_x, is the resistance to sliding. If the upper body is in equilibrium, $R_x=P$. Now it is known that the smoother the surfaces of contact, the smaller is the force required to cause sliding and hence the smaller the resistance to sliding. We are thus led to the conception of a

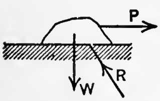


FIG. 88.

Perfectly smooth surface, which may be defined as one which offers no resistance to the sliding of a body upon it. Such a surface is ideal, but there are surfaces which are nearly perfectly smooth.

For simplicity, we will assume the surfaces of contact in some of the following examples as perfectly smooth. For brevity, they will be called smooth surfaces, while those whose resistance to sliding is to be taken into account will be called rough.*

137. Reaction of a Smooth Surface.—A perfectly smooth **plane** surface can exert a force only along a normal to it. For, by definition, such a surface offers no resistance to the sliding of a body over it, that is, the reaction which the surface offers has no component along the surface and hence that reaction must be directed along the normal. If the surface is not a plane one, then the resistance exerted by each elementary part of the surface is directed along the normal to that part.

EXAMPLES.

1. A body weighing 100 lbs. rests upon a smooth plane inclined at an angle of 30° with the horizon, and is prevented from slipping by a cord fastened to it which leads off up the incline and over a smooth pulley at the top and supports a body which hangs freely from that end. Determine the weight of the suspended body and the resistance of the plane.

Solutions: Consider the forces applied to the body upon the inclined plane; they are three in number, namely, its weight, the pull of the cord, and the resistance of the incline. The system is coplanar and concurrent; and the pulley being smooth, the pull of the string equals the weight of the suspended body.

(1) Algebraical. Let W denote the weight of the suspended body and R the resistance of the plane. The forces acting upon the body are represented in fig. 89. For such a system there are two equilibrium equations (art. 125), and if the x and y axes be taken horizontal and vertical respectively,

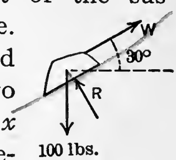


FIG. 89.

$$\Sigma F_x = W \cos 30° - R \cos 60° = 0;$$
$$\Sigma F_y = W \sin 30° + R \sin 60° - 100 = 0.$$

* Rough surfaces are discussed in § VIII.

These equations determine W and R. (The student should select the coordinate axes parallel and normal to the incline, write the equilibrium equations and compare the length of their solution with that of those above.)

(2) Graphical. The condition of equilibrium is that the polygon for the three forces must close (art. 125). To construct the polygon, the wholly known force (the weight of the body) is represented first; thus, AB (fig. 90) is drawn vertically and one inch long. Next, a line from A (or B) parallel to the incline, and one from B (or A) parallel to the normal to the plane are drawn, thus determining C. Then BC and CA represent the magnitudes and directions of the forces sought.

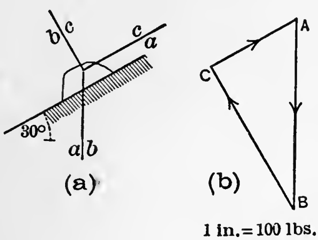


FIG. 90.

2. Solve ex. (1), supposing that the pulley is so placed that the cord leads off horizontally from the body upon the inclined plane. *Ans.* $W = 57.7$ lbs.

3. Solve ex. (1), supposing that the pulley is so placed that the cord leads off from the body upon the inclined plane at an angle of 50° with a horizontal.

4. Two planes which are inclined at angles α and β with a horizontal plane intersect in a horizontal line, and a cylinder whose weight is W rests between and upon them. Compute the pressure on each plane.

5. A bar 24 in. long rests in a smooth hemispherical bowl 30 in. in diameter. The centre of gravity of the bar is 10 in. from the lower end. Determine the position of equilibrium.

Ans. Inclination of bar to horizontal is 12° 32′.

[Suggestion: The three forces maintaining the equilibrium of the bar are coplanar and concurrent. The action lines of two, the pressures of the bowl on the bar, intersect at the centre of the bowl and therefore the centre of gravity of the bar must be vertically below the centre of the bowl. Make a sketch showing this relation, mark the known lengths, and solve trigonometrically for the inclination of the bar.]

6. A uniform bar whose length is 40 in. is supported by a

smooth vertical wall and a smooth peg whose axis is horizontal, parallel to the wall, and 15 in. therefrom. Determine the position of equilibrium. *Ans.* Angle with vertical is 65° 18′.

138. Pin Joint or Hinge.—In some of the following examples reference is made to bodies joined by means of a pin hinge. Such a joint may consist of a cylindrical pin and two cylindrical holes, one in each of the bodies joined, into which the pin is inserted allowing relative rotation about the pin. Still more simply, the joint may consist of a pin which is rigidly fastened to one body and inserted into a hole in the other, allowing rotation.

If the cylindrical surfaces of the pin and hole are smooth, the forces exerted upon these surfaces act normally, and their resultant, therefore, passes through the axis of the pin.

EXAMPLES.

1. The body *ABC* (fig. 91) is supported by a smooth hinge at *A* and a smooth surface at *B*, and a horizontal force *F* is applied at *C*. Compute the reactions of the supports, neglecting the weight of the body.

Ans. Reaction at *B* is Fa/l.

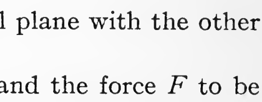


FIG. 91.

2. Suppose that the support of the bar at *B* in ex. 1 is also a smooth pin hinge in the same horizontal plane with the other one. Determine the supporting forces.

Solution: The reactions at *M* and *N* and the force *F* to be in equilibrium can not be parallel, hence they must be concurrent (art. 125); also their force polygon must close. Any reactions fulfilling these two conditions will balance *F*. Many pairs of reactions can satisfy the conditions—for example, *BC* and *CA*, or *BC′* and *C′A* (fig. 92); hence the problem is not statically determinate.

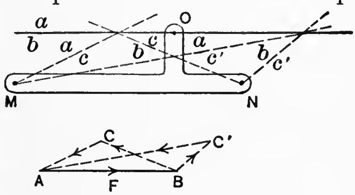


FIG. 92.

The vertical components of the reactions are determinate, which may readily be proved thus: Call the reactions at *M* and N, R' and R'' respectively, and imagine them resolved horizontally and vertically. The system F, R_x', R_y', R_x'', R_y'' is coplanar non-concurrent non-parallel, and there are three algebraic conditions of equilibrium (art. 125). They are

$$\Sigma F_x = F + R_x' + R_x'' = 0;$$
$$\Sigma F_y = R_y' + R_y'' = 0;$$
$$\Sigma M_M = R_y''l - Fa = 0.$$

From the last two equations it follows that

$$R_y'' = Fa/l \quad \text{and} \quad R_y' = -Fa/l.$$

The indeterminateness, therefore, is really with the horizontal components; their arithmetic sum or difference equals F, as may be seen from the first equation or from the force polygon in fig. 92.

3. A uniform bar weighing 200 lbs. and 10 ft. long leans against a smooth vertical wall, and its lower end is fastened to a floor by a smooth pin hinge whose axis is horizontal and parallel to the wall. The distance of the hinge from the wall being 8 ft., determine the forces supporting the bar, solving graphically.

Ans. Force on upper end is horizontal and equals $133\frac{1}{3}$ lbs.

§ V. Three Typical Problems on Coplanar Non-concurrent Forces.

139. Problem I.—*The forces of a parallel system in equilibrium are completely known except two whose action lines only are known. It is required to determine completely these two forces.*

Algebraic Solution.—There are two conditions of equilibrium for this system (art. 125), namely,

$$\Sigma F = 0 \quad \text{and} \quad \Sigma M = 0,$$

or

$$\Sigma M_a = 0 \quad \text{and} \quad \Sigma M_b = 0.^*$$

Either set of equations furnishes the solution. It is advantageous to select moment origins on the lines of action of the

* Whenever a force whose sense is unknown is to be entered into a resolution or moment equation, a sense should be assumed for that force and adhered to in the solution of the equation. The correct sense is indicated by the sign of the computed value of that force. *It is as assumed or opposite according as the sign is positive or negative.*

unknown forces. Then each moment equation will contain but one unknown quantity and may be readily solved.

Illustration: Let it be required to determine the magnitudes and directions of the two forces F' and F'' (fig. 93), all the forces there represented being in equilibrium, F_1, F_2, and F_3 being 700, 300, and 500 lbs., and a, b, c, and d 1, 3, 5, and 2 ft. respectively.

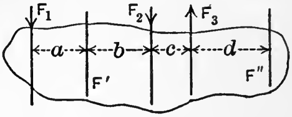


FIG. 93.

If F' and F'' are assumed to act upward and the moment origin is taken on F'',

$$\Sigma F = -700 + F' - 300 + 500 + F'' = 0;$$
$$\Sigma M = +700 \cdot 11 - F' \cdot 10 + 300 \cdot 7 - 500 \cdot 2 = 0.$$

From the second equation, $F' = 880$ lbs., and this value substituted in the first gives $F'' = -380$ lbs. The minus sign means that F'' does not act up as assumed, but down.

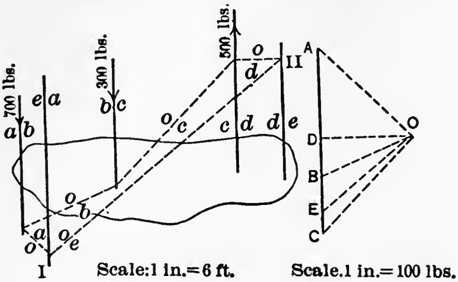
Employing the second set of equilibrium equations, and selecting moment origins on F' and F'' respectively,

$$\Sigma M = 700 \cdot 1 \quad - 300 \cdot 3 + 500 \cdot 8 + F'' \cdot 10 = 0;$$
$$\Sigma M = 700 \cdot 11 - F' \cdot 10 + 300 \cdot 7 - 500 \cdot 2 = 0.$$

These solved give the same values as those found above.

Graphical Solution.—The conditions of equilibrium are that the force and funicular polygons close (art. 125). In the process of constructing them the unknown quantities will be determined.

Illustration: Let the data be the same as that in the preceding illustration. First the force polygon should be drawn as far as possible; thus, AB, BC, and CD (fig. 94) representing F_1, F_2, and F_3 respectively. If either one of the unknown forces be lettered DE, the other must be EA because the polygon must close. It remains to locate the point E; this is done by means of the funicular polygon. Before beginning to draw the polygon, recall what the strings represent (see arts. 37 and 38). If the polygon be begun on ab, strings oa and ob must be drawn from that point; then oc is drawn from where ob intersects bc, and od from where oc intersects cd. Now oa is the action line of one of the components of EA, and should be extended to ca, thus determining a point I in the action line of the other component, EO. Also, od is the action line of one of the com-



(a) (b)

FIG. 94.

ponents of DE, and should be extended to de, thus determining a point II in the action line of the other component, OE. Now, if the funicular polygon is to be closed, the action lines of EO and OE must coincide (see art. 40); hence there is but one string oe, and it passes through points I and II. The ray corresponding to oe may now be drawn, thus determining E.

EXAMPLES.

1. Reverse F_1 and F_3 (fig. 93) and solve algebraically the example used for an illustration above.

2. Solve the preceding example graphically.

140. Problem II.—*The forces of a non-parallel system in equilibrium are completely known except two. Of these, the action line of one and a point in that of the other are known. It is required to determine completely these two.*

Algebraic Solution.—There are three conditions of equilibrium for a system of this kind (art. 125), namely,

$$\Sigma F_x=0, \quad \Sigma F_y=0, \quad \Sigma M=0;$$
$$\Sigma F_x=0, \quad \Sigma M_a=0, \quad \Sigma M_b=0;$$
or,
$$\Sigma M_a=0, \quad \Sigma M_b=0, \quad \Sigma M_c=0.$$

Either set furnishes the solution, the three unknowns being the magnitude (and sense) of one force, the magnitude (and sense) and inclination of the other.

Illustration: Suppose a horizontal beam which is supported at one end by a pin hinge and at the other by an inclined cord sustains given loads, and that it is required to determine the tension in the cord and the pin reaction.

Let fig. 05 represent the beam, loading, notation etc.; the

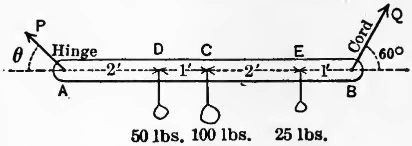


FIG. 95.

100-lb. force may be the weight of the beam. The unknowns are denoted by Q, P, and θ.* Employing the first set of equilibrium equations and selecting the x and y axes horizontal and vertical respectively, and a moment origin at A,

$$\Sigma F_x = -P \cos \theta + Q \cos 60° = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma F_y = P \sin \theta + Q \sin 60° - 50 - 100 - 25 = 0; \quad . \quad . \quad (2)$$

$$\Sigma M_A = Q \cdot 6 \sin 60° - 50 \cdot 2 - 100 \cdot 3 - 25 \cdot 5 = 0. \quad . \quad . \quad (3)$$

These equations furnish the solution. Observe that by selecting A as moment origin, two unknowns, P and θ, were eliminated from the moment equation.

Usually it is advantageous to replace the force whose magnitude and direction are unknown by two rectangular components acting at the known point in the action line of the force. Then the unknowns, instead of P and θ of the illustration, would be P_x and P_y, and when these latter become known, P and θ are easily computed. Such substitution transforms the problem to Problem III (art. 141). The example employed above as illustration is solved in art. 141 on this plan.

Graphical Solutions.—(1) The conditions of equilibrium are that the force and funicular polygons close (art. 125). In the process of constructing the two polygons the unknowns will be determined.

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* In this example it is evident that the senses of P and Q are as represented and that $\theta < 90°$. When the senses of P and Q and "quadrant" of θ are not evident, they should be sketched as they appear to be, and then the equilibrium equations should be written. The correct senses may be inferred as explained in the foot-note, p. 120, and the quadrant of θ will appear from the solution.

Illustration: Let the data be the same as in the preceding illustration. First, the force polygon is drawn as far as possible; thus, AB, BC, and CD (fig. 96), representing the 50, 100, and 25 lb. forces, and then an indefinite line parallel to the cord from D (or A). The fourth force, tension, then is to be lettered DE, and the remaining one must be marked EA, since the polygon is to be closed. It remains to determine E; this is done by means of the funicular polygon.

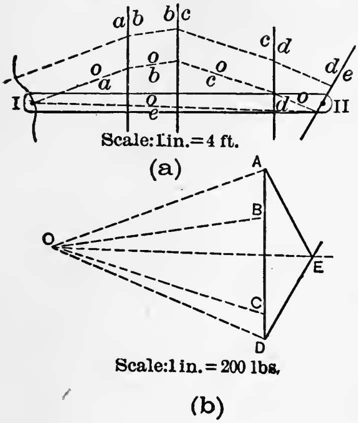


FIG. 96.

If the polygon be begun at any point of ab, bc, cd, or de, it can not be completed, as is plain from an examination of the unlettered funicular polygon which was begun from a point of ab taken at random. The difficulty is in determining the desired intersection of the string corresponding to OA and the unknown action line of EA; this point corresponds to point I of fig. 94. If the funicular polygon be begun at the *given* point of the unknown action line, the difficulty is avoided; thus, oa is drawn through that point, and then ob, oc, and od. Now I and II respectively are points in the action lines of the forces EO and OE, and if the funicular polygon is to be closed, the lines of action of EO and OE must coincide (art. 45); hence there is but one string oe, and it passes through points I and II. The ray corresponding to OE may now be drawn, and thus the point E is determined.

(2) Find the resultant of the wholly known forces, and consider the system as consisting of that force and the two unknown ones. Then, making use of the fact that the three forces are parallel or concurrent (see art. 125), determine the unknowns.*

* If the resultant of the known forces is a couple, the two unknowns must also constitute a couple. How might one determine graphically the forces of the latter couple?

Illustration: Let the data be the same as that of the preceding illustration. The resultant of AB, BC, and CD (fig.

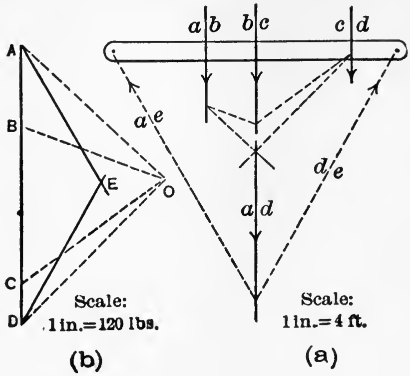


FIG. 97.

97) is found to be AD. Extending ad and de we find their intersection which is one point in the action line of the other unknown force; another point in its action line being given, the centre of the pin, the line is known. The action lines, ad, de, and ea, of three forces in equilibrium and the magnitude and sense of one, AD, are now known. To determine the remaining elements, we have only to draw the force triangle for the three forces, $ADEA$. Then DE represents the magnitude and sense of the pull of the chord, and EA the magnitude and sense of the pin pressure.

EXAMPLE.

Suppose that the bar represented in fig. 98 is supported by a smooth surface at A (then the reaction there must be horizontal) and by a pin joint at B. Determine the supporting forces graphically.

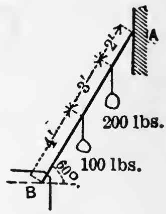


FIG. 98.

141. Problem III. — *The forces of a non-parallel system in equilibrium are completely known except three whose lines of action only are known. It is required to determine completely these three.**

Algebraic Solution.—There are three conditions of equilibrium for a system of this kind, as in the preceding

* If the three unknown forces are concurrent or parallel, the problem is indeterminate.

article. Either set of equations determines the unknowns, the magnitude (and sense) of three forces.

Illustration: Let the data be the same as that of the illustration of the preceding article, but imagine P replaced by its components P_x and P_y as in fig. 99.

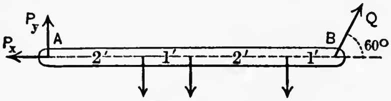


FIG. 99.

(*a*) Employing the first set of equilibrium equations,

$$\Sigma F_x = -P_x + Q \cos 60° = 0; \quad \ldots \quad (1)$$
$$\Sigma F_y = P_y + Q \sin 60° - 50 - 100 - 25 = 0; \quad \ldots \quad (2)$$
$$\Sigma M_A = Q \cdot 6 \sin 60° - 50 \cdot 2 - 100 \cdot 3 - 25 \cdot 5 = 0. \quad \ldots \quad (3)$$

Equation (3) is just like eq. (3) of the preceding article. The value of Q determined from (3), substituted in (1) and (2) leaves them with two unknowns, P_x and P_y. Now it is considerably easier to solve (1) and (2) for P_x and P_y than to solve (1) and (2) of the last article for P and θ, as may be seen by trial. The reason lies in the fact that both $\sin \theta$ and $\cos \theta$ appear in those equations, and one really has to solve three equations, the third one being $\sin^2\theta + \cos^2\theta = 1$.

(*b*) Employing the second set of equilibrium equations,

$$\Sigma F_x = -P_x + Q \cos 60° = 0; \quad \ldots \quad (4)$$
$$\Sigma M_A = Q \cdot 6 \sin 60° - 50 \cdot 2 - 100 \cdot 3 - 25 \cdot 5 = 0; \quad \ldots \quad (5)$$
$$\Sigma M_B = -P_y \cdot 6 + 50 \cdot 4 + 100 \cdot 3 + 25 \cdot 1 = 0. \quad \ldots \quad (6)$$

Equations (5) and (6) contain but one unknown each; no elimination therefore is necessary to obtain P_y and Q. The value of Q substituted in (4) leaves but one unknown in that equation.

(*c*) The third set of equilibrium equations may be written for the forces of this example so that but one unknown will appear in each equation, and then no elimination is necessary in solving the equations. The student should so write them.

Graphical Solutions—(1) If one imagines any two of the unknown forces replaced by their resultant, the problem is transformed to Prob. II (art. 140). Thus, let F', F'', and F'''

denote the three unknowns, and R the resultant of any two of them, F'' and F''' say; then in the transformed system there are two partially unknown forces F' and R. The action line of F' is known, also one point in that of R, the intersection of the action lines of F'' and F'''. Therefore, by the method of Prob. II, F' and R may be determined, and then R may be resolved into two components whose action lines are those of F'' and F''' respectively; these components are F'' and F'''.

Illustration: It is required to determine the supporting forces, "reactions," on the overhanging framework of fig. 100 when loaded as shown. The supports are such that the reaction above is directed along the rafter, and reactions below are horizontal and vertical. Imagine the lower two reactions compounded and call the resultant R; its line of action is unknown, but it passes through the lower point of support. Then the system in equilibrium consists of R, the upper reaction, and the loads. The force polygon for the known forces is $ABCDE$, and since the direction of the upper reaction is known, the polygon can be continued by a line from E parallel to the upper reaction. The as yet unknown end of that line is to be marked F, and then the force R will be FA, since the force polygon is to be closed.

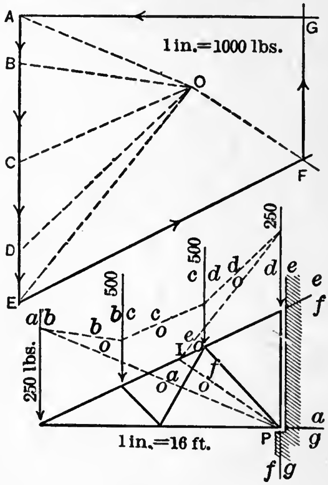


FIG. 100.

The strings which intersect on af are oa and of (art. 38), and since one point on af, P, is known and also OA, the string oa may be drawn. Continuing the construction, ob is drawn next, then oc, od, and oe. Now I is a point in of and P is another; hence of may be drawn and then the corresponding ray. That ray determines F, and then EF and FA represent the upper reaction and R respectively, and the horizontal and vertical components of FA (FG and GA) represent the supporting forces at P.

(2) Find the resultant of the wholly known forces; then the system in equilibrium may be regarded as consisting of that resultant and the three partially unknown forces. The special condition of equilibrium for four such forces is that the resultant of either pair balances that of the other pair (art. 121).

Illustration: Let the data be the same as that of the preceding illustration, and let F denote the upper reaction and H and V the lower ones (fig. 101). The resultant of the loads is a force of 1500 lbs.; its action line is ab. Let R' denote the resultant of the pair F, 1500, and R'' that of the pair H, V. Now R' passes through Q and R'' through P, and since they are collinear, PQ is the action line of each. Since the three forces F, 1500, and R'' are in equilibrium, and since their action lines and the magnitude and sense of one are known, their force polygon can be drawn; it is $ABCA$, BC representing the magnitude and direction of F. Since the four forces, 1500, F, H, and V, are in equilibrium, their polygon must close; hence draw lines from A and C parallel to H and V determining D. Then $ABCDA$ is the force polygon sought, and CD and DA represent the magnitudes and directions of V and H respectively.

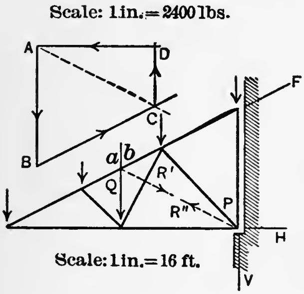


FIG. 101.

EXAMPLES.

1. Suppose the truss represented in fig. 102 to be supported

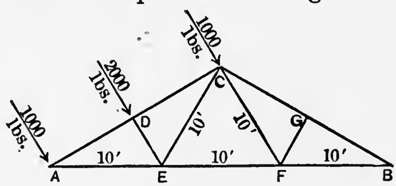


FIG. 102.

on a smooth surface at A and to be pinned to the wall at B. Determine the reactions at A and B due to the loads, by both methods.

2. Determine the reactions on the crane in fig. 103 due to the load and weights of members by both methods. (The

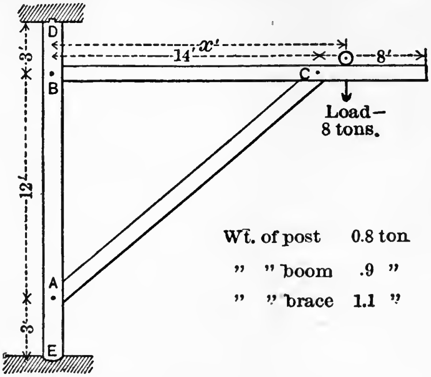


FIG. 103.

supporting surfaces are such that the upper reaction is horizontal and below one is horizontal and one vertical.)

§ VI. JOINTED FRAMES.

142. Definitions.—A *plane framework* is one all members of which are parallel to a plane; only such frames are considered in this section. A *jointed frame* is one the members of which are fastened by pin-joints, the pin being perpendicular to the plane of the frame. It is assumed in the following that the pins are smooth.

143. The Pin Pressures.—The simplest kind of a member is one which is straight and is joined to others at its ends (fig. 104). Such a member, if sustaining no load, is subjected to

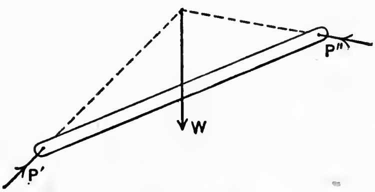


FIG. 104.

three forces, its weight and the pin pressures, W, P', and P'' respectively. *Unless the member is vertical, the action lines of* *the pin pressures do not coincide with the axis of the member;* for, assuming that they do, it is plain that P' and P'' cannot balance W. If the weight of the member is neglected, the two forces P' and P'' act along the axis and are equal; but, *in general*, a pin pressure is not directed along the axis of the member on which it acts.

144. General Direction for Solving Examples.—The student should read again art. 127. In the following examples it will not always be evident to the beginner which body and which force system to consider for determining any particular force. Often several may be selected in the manner mentioned in the article referred to, and it may be that there is but little choice between them; but, as a general rule, the body should, if possible, be so selected that the number of unknown elements in the external system of forces acting upon it shall not exceed the number of conditions of equilibrium for that system.

EXAMPLES.

1. Determine the forces upon each member of the crane in fig. 103, neglecting their weights and taking x equal to 16 ft.

Solutions: (1) Algebraic. Fig. 105 represents the whole crane, its members and groups of them, and the corresponding *external* forces. There are four external forces applied to the crane, namely, the load, and the reactions X, Y, and H (fig. 105*a*). The brace (fig. 105*b*) is subjected to two forces only; hence they are collinear, and each acts along AC. The boom (fig. 105*c*) is subjected to three forces,—one at C, one at B, and the load; the one at C is the reaction corresponding to the pressure on the upper end of the brace, and hence is collinear with that pressure. The force at B is unknown in direction, and is therefore represented by its components B_x and B_y. The post (fig. 105*d*) is subjected to five forces,—X, Y, H, one at A, and one at B; the latter two are reactions corresponding to the forces upon the left ends of the boom and brace respectively.

An examination of these systems reveals several orders of procedure. In general, it is advisable to determine the reactions on the entire frame first, if possible; here it is possible, for the system applied to the entire crane is coplanar non-concurrent non-parallel with three unknowns. Their determination is left for the student; the values for X, Y, and H are 7.11, 8, and 7.11 tons respectively. System (c) might be

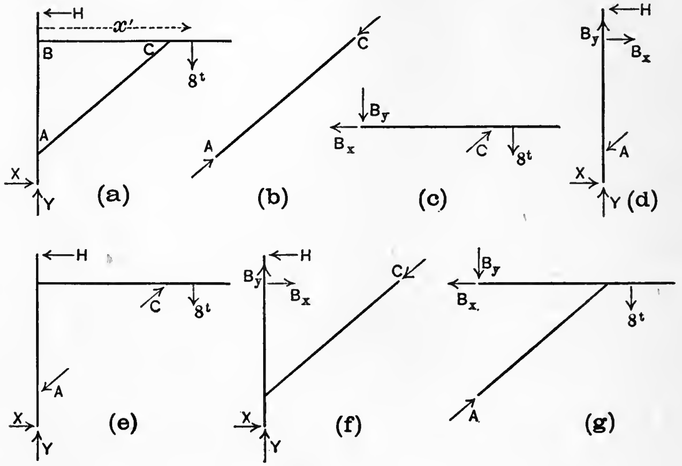


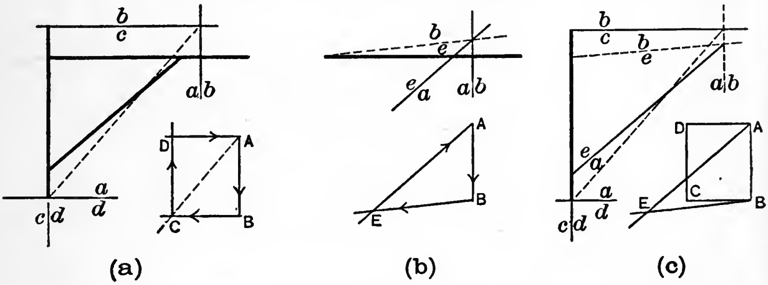
FIG. 105.

solved next, for it is coplanar non-concurrent non-parallel with three unknowns. The student should prove that B_x, B_y, and C equal respectively 10.67, 1.14, and 14.05 tons. From system (b) it is plain that A equals C. Finally, $B = 10.73$, and the inclination of B with the horizontal is $\tan^{-1} B_y/B_x = 6° 7'$. All the unknowns have now been determined.

The student should examine the remaining force systems represented in fig. 105, and determine other orders of procedure for determining the unknowns.

(2) Graphical. The system of external forces applied to the crane may be solved like Prob. III (art. 141). The force polygon for the four forces is $ABCDA$ (fig. 106a), AB representing the load. The forces applied to the boom, being three in number, are concurrent; hence their action lines are known. Since the magnitude of one force is known, their force polygon can be drawn, thus determining the magnitudes of the other two. The polygon is $ABEA$ (fig. 106b), AB representing the load. All the unknowns have now been determined.

Instead of solving the force system applied to the boom, we might have solved that applied to the mast. There are five forces in that system,—three wholly known (*BC*, *CD*, and *DA*), the action line of one, and a point in that of the last. The resultant of the three wholly known forces is *BA* (fig. 106*c*);



Scales, of space diagram 1 in.= 20 ft.
" vector " 1 in.= 20 tons.
FIG. 106.

and if the three known forces be imagined replaced by their resultant, then the system consists of three forces, and it must be concurrent. The action line of the last force is now known since it passes through the action lines of the result-

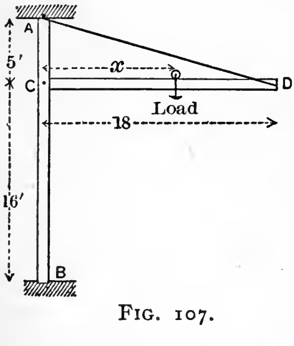


FIG. 107.

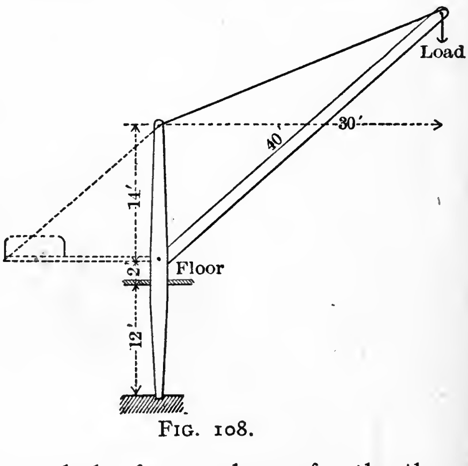


FIG. 108.

ant and the fourth force, and the force polygon for the three forces may be drawn, thus determining the remaining unknowns. The polygon is $BAEB$, AE and EB representing the fourth and fifth forces.*

2. The weights of the post and jib of the crane represented in fig. 107 are 1 and ¾ ton respectively; that of the tie may be neglected. Take the load as 8 tons and x as 10 feet, and compute all the forces on each member by both methods.

3. The weights of the post and jib represented in fig. 108 are 1200 and 1500 lbs. respectively; that of the tie may be neglected. For a load of 4 tons compute all the forces upon each piece by both methods. Disregard counterweight, shown dotted.

4. Solve the preceding example on the supposition that the crane has a counterbalance whose weight is 10 tons, its centre of gravity being 9 ft. from the axis of the post.

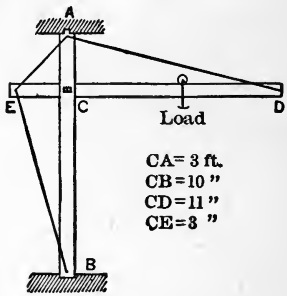


FIG. 109.

5. Determine all the forces upon each member of the crane represented in fig. 109 due to a load of 3 tons 7 ft. from C. It is impossible to analyze this crane by the preceding principles. To make it possible, suppose that the hole in the mast at C is slotted as shown (then the pressure there is vertical), and that there is no member AE.

6. Fig. 110a represents a type of hydraulic crane. The plunger works inside a hollow mast and, pressing against the

* The solution of ex. 1 suggests these general directions for "analyzing" cranes:

(1) Make a sketch of the entire crane, and represent as far as possible all the external forces acting upon it.

(2) Apply the proper conditions of equilibrium (art. 125) to such external forces and determine as many of the unknown elements as possible.

(3) Make sketches of members or collections of members of the crane, and represent as far as possible all the external forces acting upon them. (In this connection, bear in mind art. 143 and the "law of action and reaction.")

(4) Inspect the force systems of the sketches, noting especially the unknown forces in each. Careful inspection will suggest how to determine the unknowns.

bottom of the boom, raises and lowers the load and boom together. Compute the forces upon each piece due to a load of 10 tons, x feet from the axis of the plunger when the pin A is y feet above the floor.

Solution: Consider first the entire crane (fig. 110*b*), except post and plunger. The external forces applied to it consist of the load, the post pressure upon the upper roller, R_1, the post pressures on the lower rollers, R_2 each, and the plunger pressure P. The three unknowns may be determined, for the sys-

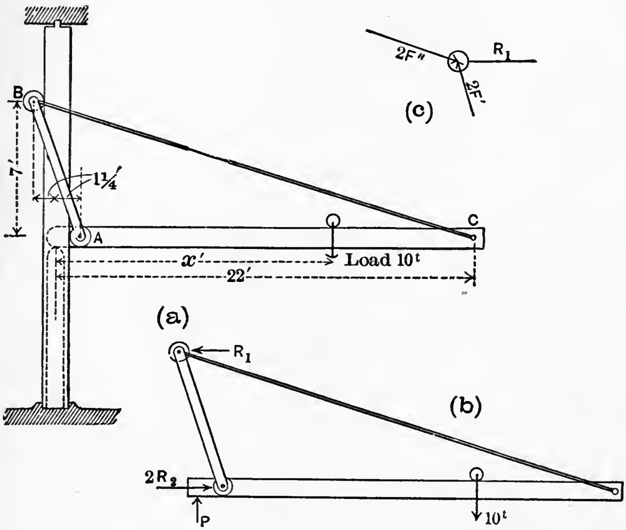


FIG. 110.

tem is coplanar non-concurrent non-parallel. The solution gives

$$R_1 = 1.43x, \quad R_2 = 0.71x, \quad \text{and} \quad P = 10 \text{ tons}.$$

To continue, we may consider next the pin and roller at B. The external forces upon this body are five in number, the pressure on the roller $1.43x$, the forces exerted on the pin by the two members AB and the two BC. The four pin pressures are directed along the axes of corresponding members (fig. 110*c*). Solution of the equations of equilibrium of this system gives

$$F' = 0.26x \quad \text{and} \quad F'' = 0.84x \text{ tons}.$$

Upon the pin at A there are applied five forces, two by the members AB, two by the rollers, and one by the boom; their magnitudes are respectively $0.26x$, $0.71x$, and $1.67x$. The student should prove the values given.

7. Include the weights of the members in ex. 1 and solve. (Notice that the forces applied to the ends of the brace are not directed along its axis. It will be convenient in an algebraic solution to replace each unknown force whose direction is unknown by its horizontal and vertical components.)

8. The frame of fig. 111(a) rests upon smooth surfaces at A and B. Determine the forces (pin pressures) upon each member.

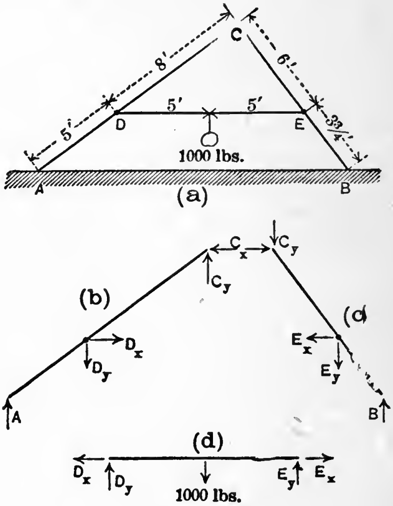


FIG. 111.

Solution: From a consideration of the external forces on the collection of bars it follows readily that the forces at A and B are 446 and 554 lbs. respectively. Fig. 111 (b), (c), and (d) represents the external system on each member, each pin pressure being replaced by its horizontal and vertical components. No unknown of the system (b) or (c) can be determined from the equilibrium equations of the system, but the forces D_y and E_y may be computed from the equations for system (d). Making use of these values in (a) and (b), the remaining unknowns can be determined. The student should make the determination.

9. The bars of the frame of fig. 112(a) are uniform, AC, BC, and AB weighing 150, 100, and 200 lbs. respectively. Compute the forces upon each bar.

Solution: The forces applied to each member are represented in fig. 112 (b), (c), and (d), each force whose direction is unknown being represented by its horizontal and vertical components. The senses of the vertical components at C are not obvious, so they are assumed (see foot-note, p. 120).

From a consideration of the external system on the entire frame, R_1 and R_2 are readily found to be 229 and 221 lbs. respectively. From the system (d), A_y and B_y are found to be 129 and 121 lbs. respectively. Supplying the value of A_y in system (b) or B_y in system (c), the remaining unknowns may be computed.

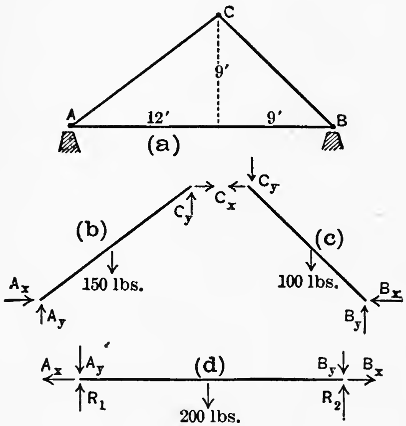


FIG. 112.

Instead of following the order above, one might write the equilibrium equations for systems (b) and (c) and solve them for the six unknowns, and lastly determine R_1 and R_2.

§ VII. JOINTED FRAMES (CONTINUED).

145. Kind of Frames Considered.—The jointed frames considered in this section differ from those of the preceding section in construction and in loading. It is assumed that

(a) each member connects only two joints,
(b) each load is applied so that its action line passes through the axis of a joint.

146. "Force or Stress in a Member."—Fig. 113(a) represents a member of such a frame as described in the preceding article, under two loads L' and L''. The pin pressures are denoted by P' and P'', and the weight of the member is neglected or not considered. Let the resultants of the forces at the left and right ends be denoted by R' and R'' respectively; the action line of each passes through the centre of the corresponding hole. Since R' and R'' balance, they must be collinear, and their action lines must coincide with the axis of the member.

Now any two parts of the member, as M and N, exert forces on each other, and *the lines of action of those forces coincide*

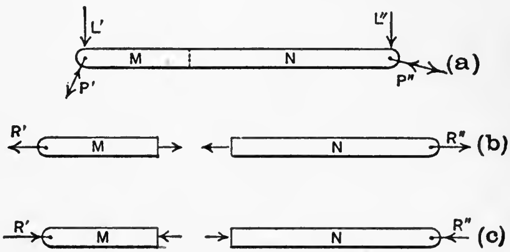


FIG. 113.

with the axis of the member. For, the force which M exerts on N balances R'', therefore its action line must coincide with that of R''; also the force which N exerts on M balances R', therefore its action line must coincide with that of R'.

Observe carefully the relations in fig. 113 (b) and (c).

In (b), the forces at the section between M and N are pulls, or the stress is tensile (art. 104), and the member is stretched by R' and R''.

In (c), the forces at the section between M and N are pushes, or the stress is pressural (art. 104), and the member is compressed by R' and R''.

By force or stress in a member is meant either of the forces which a part of it exerts upon the other. The examples in this section relate to the determination of the forces in the members of jointed frames.

147. Method for Determining the Force or Stress in a Member.

(1) Determine the reactions on the frame, or truss.

(2) Imagine the truss separated into two distinct parts* so

* Such division of a truss is also described as "passing a section," the term section referring to the imaginary surface of separation.

that the member under consideration is one of the members separated, and so that the unknown elements in the system of external forces applied to one part of the truss does not exceed the number of algebraic conditions of equilibrium for that system.

(3) Apply the appropriate condition or conditions of equilibrium necessary to determine the desired force.

The student should bear in mind that the system of external forces with reference to any part of a truss consists of the loads and reactions applied to that part *and* the forces which the

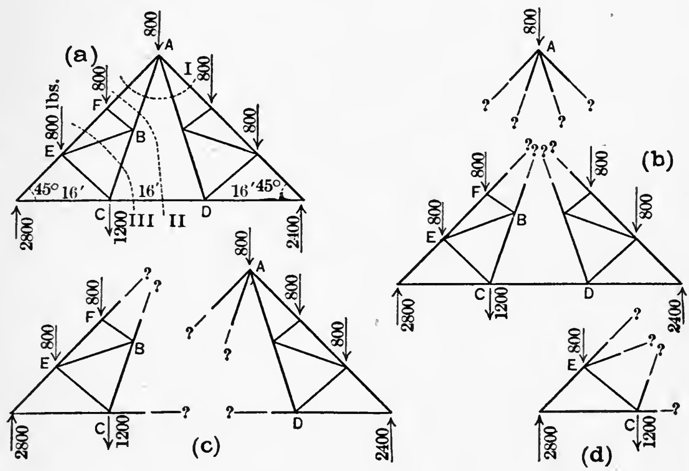


FIG. 114.

other part exerts upon it. These latter forces are exerted upon the "cut" ends of the members belonging to the part considered and are exerted along their axes.

Illustration 1.—It is required to determine the force in the member AB of the truss of fig. 114(*a*).

The reactions are supposed to have been determined. If the section cutting AB be passed as at *I*, the external system on either the upper or the lower part of truss (fig. 114*b*) includes four unknown forces, those in the four members cut. Now the former system is concurrent and since it has only two conditions of equilibrium the unknowns cannot be determined from that system. The system on the lower part is non-concurrent, and as it has only three conditions the unknowns cannot be determined from it.

If the section be passed as at *II*, the system on each part of the truss contains only three unknown forces (fig. 114*c*); and since each system is non-concurrent, there are three conditions of equilibrium, and the force desired can be obtained from either system.

Illustration 2.—It is required to determine the stress in member *BC*, fig. 114(*a*).

No matter where the section cutting *CB* is made, the external system on either part will contain more unknown forces than the number of conditions of equilibrium for the system, and the desired stress cannot be so directly determined. Thus, if the section is made as at *III*, the system on the left part is as represented in fig. 114(*d*), and there are four unknowns, the forces in the four cut members. If now the force in *CD*, *BE*, or *EF* can be determined, its value may be supplied in fig. 114(*d*) and the system can then be solved for the desired force, for there are three conditions of equilibrium and but three unknowns. The force in *CD* can be determined from the system of fig. 114(*c*).

When the stress in each member of a truss is required, a certain order of determining them, depending on the case in hand, is more convenient than others; but in all cases, the sections are made according to direction (2) as stated above. It does not fall within the scope of this book to explain fully the most convenient orders of procedure for the different cases. A good general method is to make the sections so that the external system on one of the parts of the truss shall be simple, containing few unknowns and easy to solve. This matter is partially illustrated in the solution of the first of the following examples and in arts. 148–151.

EXAMPLES.

1. Determine the force in each member of the frame of fig. 115(*a*) due to the load of 1000 lbs

Solution: Imagine the truss divided into two distinct parts as shown in fig. 115 (*b*) and (*c*). The external forces upon the left part are 400 lbs., F_1' and F_2'; upon the right part 600 lbs., 1000 lbs., F_1'' and F_2''.* Each is a system in equilibrium, and the unknowns may be determined from the equilibrium equations for either system.

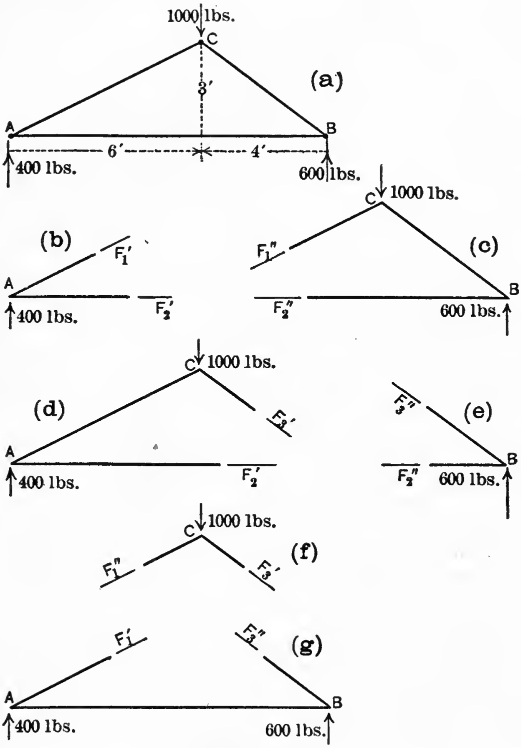


FIG. 115.

To determine the force in the member *BC*, the truss should be imagined as separated into two parts so that *BC* is one of the members cut, for example as in fig. 115 (*d*) and (*e*). The external forces on the left part are 400 lbs., 1000 lbs., F_2' and F_3'; upon the right part 600 lbs., F_2'' and F_3''. Each is a system in equilibrium, and the equilibrium equations for either determine the remaining unknown F_3.

* $F_1' = F_1''$, $F_2' = F_2''$, etc.

In fig. 115 (*f*) and (*g*), there are represented the external systems on the parts of the truss made by cutting members *AC* and *BC*. The equilibrium equations for either determine F_1 and F_3.

Of course, consideration of all the systems represented is not necessary for a solution. They are referred to here merely to show that the solution may be made in several different ways. One of these ways is by means of the systems of fig. 115 (*e*) and (*f*), which may be carried out thus:

Equilibrium equations for system (*e*) are

$$\Sigma F_x = F_3'' \cos 36° 52' - F_2'' = 0;$$
$$\Sigma F_y = -F_3'' \sin 36° 52' + 600 = 0;*$$

hence

$$F_3'' = 1000 \quad \text{and} \quad F_2'' = 800 \text{ lbs.}$$

Having determined F_3'', F_3' is known; it is a push, that is, it acts upward (fig. 115*f*) and its value is 1000 lbs. Only one unknown remains in system (*f*), and the following equilibrium equation suffices for its determination:

$$\Sigma F_x = F_1'' \cos 26° 34' - 1000 \cos 36° 52' = 0;$$

hence $$F_1'' = 894.5 \text{ lbs.}$$

Supposing the senses of the unknown forces in the two systems just considered not apparent, and following the suggestion of the foot-note, the first two equations above become

$$-F_3'' \cos 36° 52' - F_2'' = 0;$$
$$F_3'' \sin 36° 52' + 600 = 0;$$

from which $F_3'' = -1000$ and $F_2'' = +800$ lbs.

---

* In simple trusses the kind of stress in any member is apparent. For example, in Fig. 115(*a*), *AC* and *BC* are in compression and *AB* in tension; then F_1 and F_3 are pushes and F_2 is a pull. When the senses of the forces are not apparent, we may follow the suggestion in the foot-note, p. 120, but *it is convenient to always assume the force to be a pull. Then, according to the foot-note, the force is actually a pull or push (and the member is in tension or compression) according as its computed value is positive or negative.*

Interpreting these signs in accordance with the foot-note, F_3'' is a push and F_2'' a pull, i.e., BC is in compression and AB in tension—results agreeing with the first solution.

2. Determine the force in each member of the truss of fig. 116. *Ans.* AF, 1600 lbs. tension.

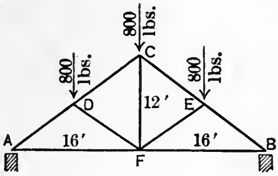


FIG. 116.

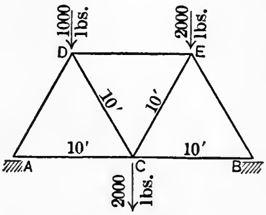


FIG. 117.

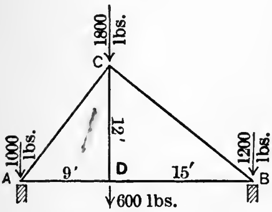


FIG. 118.

3. Determine the force in each member of the truss of fig. 117.

Ans. AD, 2600 lbs. compression;
AC, 1300 lbs. tension.

4. Determine the force in each member of the truss of fig. 118.

5. Determine the force in each member of the truss of fig. 102, it being supported as there described.

148. Graphical Method for "Analyzing Trusses."—Graphical methods are especially well adapted for solving problems like the preceding. As in the algebraic method, the truss is imagined separated into two parts and then the attention is directed to the external forces acting upon either part. Graphical instead of algebraical conditions of equilibrium are then applied to the system of forces to determine the unknowns. In making the imaginary separations of the truss, care should be taken to cut not more than three members, the forces in which are unknown.* It is *advantageous* to make the separation so that not more than two such members are cut. If that be done, a single force polygon will determine the two unknowns, while if three be cut, a force polygon and a funicular polygon, or the equivalent, are necessary to determine the unknowns.

* These members must not meet at the same joint.

149. *Notation.*—The notation described in art. 11 when applied in the graphical analysis of trusses can be advantageously systemized as follows. Each triangular space in the truss diagram is marked by a small letter, also the space between consecutive action lines of the loads and reactions (see fig. 119) Then the two letters on opposite sides of any line serve to designate that line, and the same large letters are used to designate the magnitude of the corresponding force. This scheme of notation is a great help in graphical analyses of trusses.

Illustration.—Determine the force in each member of the truss of fig. 119.

Solution: Evidently the reactions each equal one-half the load, or 2000 lbs. Imagine the truss separated into two parts, as

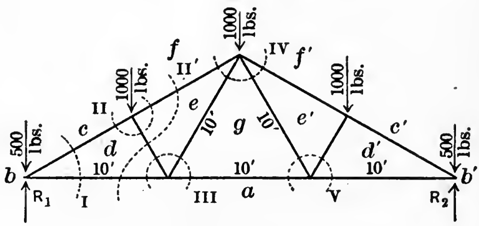
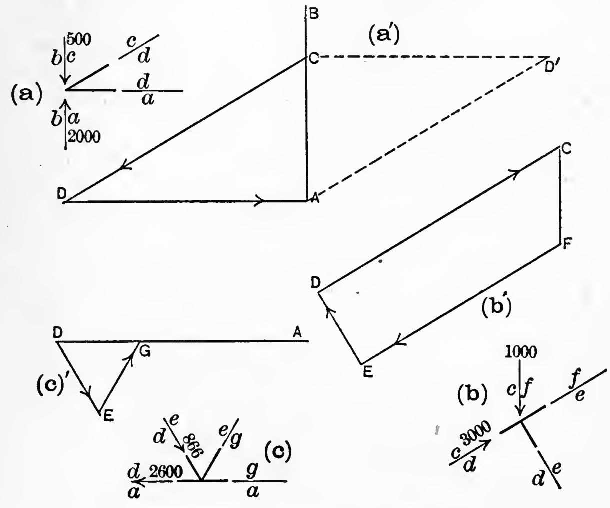


FIG. 119.

by the arc *I*. The external forces upon the left part are represented as far as known in fig. 120(*a*); since they are in equilibrium, their polygon closes, and in constructing it, the unknowns will be determined. Beginning with the knowns, *AB* is drawn to represent 2000 lbs., *BC* to represent 500 lbs.; and then a line from *A* (or *C*) parallel to the action line of one of the unknowns and a line from *C* (or *A*) parallel to the other are drawn. The last two lines determine *D* (or *D′*), and the closed polygon is *ABCDA* (or *ABCD′A*); hence the forces in the members *cd* and *ad* are represented by *CD* and *DA* (3000 and 2600 lbs.) respectively. From the force polygon, it is seen that *CD* is a push, and *DA* is a pull; hence the members *cd* and *ad* are in compression and tension respectively.

We may next imagine the truss separated into two parts as by *II* or *II′* (fig. 119); in either case, there are but two unknown forces in the external system applied to either part. If we choose the part within *II*, we have the simpler system to deal with; the forces of it are represented in fig. 120(*b*) as far as known. The force polygon may be drawn thus: *DC* to represent 3000 lbs., *CF* to represent 1000 lbs., a line from *F* parallel to one of the unknowns and one from *D* parallel to the other. The last two lines determine *E*, and the force polygon



Scale of force polygons: 1 in. = 2000 lbs.

FIG. 120.

is *DCFED*; hence the forces in the members *de* and *ef* are represented by *ED* and *FE* (866 and 2500 lbs.) respectively. Both members are in compression.

We next imagine the truss separated into parts as by *III*. Choosing the part within *III*, we have a simple system to deal with; the forces of it are represented as far as known in fig. 120(*c*). Their force polygon may be drawn thus: *AD* to represent 2600 lbs., *DE* to represent 866 lbs., a line from *E* parallel to one of the unknowns, and a line from *A* parallel to the other. The last two lines determine *G*, and the force polygon is *ADEGA*; hence the forces in the members *eg* and *ag* are represented by

EG and *GA* (866 and 1732 lbs.) respectively. Each member is in tension.

On account of the symmetry of the truss and loading, the forces in the remaining members are now known.

150. *Polygon for a Joint.*—In drawing the force polygon for all the external forces on the part of a truss included within a small circle struck from a joint, it will be advantageous to represent the forces in the order in which they occur about the joint.

A force polygon so drawn will be called a *polygon for the joint;* and for brevity, if the order taken is clockwise, the polygon will be called a *clockwise polygon*, and if counter-clockwise it will be called a *counter-clockwise polygon.* *ABCDA* (fig. 120*a*′) is a clockwise polygon for joint *I* of fig. 119; *ABCD′A* is a force polygon for the "forces at joint *I*," but it is not a polygon for the joint, because the forces are not represented in the polygon in the order in which the forces occur about the joint.

The student should draw the counter-clockwise polygon for the joint and compare with *ABCDA*.

151. *Stress Diagrams.*—If the polygons for all the joints of a truss are drawn separately as in the illustration in art. 149, then the stress in each member will have been represented twice. It is possible to combine the polygons so that it will not be necessary to represent the stress in any member more than once, thus reducing the number of lines to be drawn. Such a combination of force polygons is called a *stress diagram.*

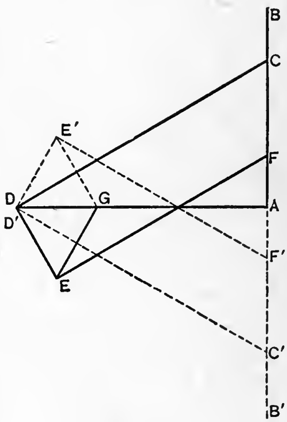


FIG. 121.

Figure 121 is a stress diagram for the truss of fig. 119 loaded as there shown. Comparing the part consisting of solid lines with figs. 120 (*a*′), (*b*′), and (*c*′), it is seen to be a combination of the latter three figures. It will also be observed that the polygons are all clockwise polygons, but counter-clockwise polygons could be combined into a stress diagram.

To construct a stress diagram for a truss under given loads:

(1) Letter the truss diagram as directed in art. 149.

(2) Determine the reactions.

(3) Construct a force polygon for all the external forces applied to the truss (loads and reactions), representing them in the order in which their application points occur about the truss, clockwise or counter-clockwise.*

(4) On the sides of that polygon, construct the polygons for all the joints. They must be clockwise or counter-clockwise ones according as the polygon for the loads and reactions was drawn clockwise or counter-clockwise. (The first polygon drawn must be for a joint at which but two members are fastened; the joints at the supports are usually such. Next that joint is considered (and its polygon is drawn) at which not more than two stresses are unknown, that is, of all the members fastened at that joint the forces in not more than two are unknown. Then the next joint at which not more than two stresses are unknown is considered; etc.†)

These directions are illustrated in the following solutions.

EXAMPLES.

1. Solve ex. 4 of art. 147 by the graphical method.

Solution: Supposing the reactions to have been determined, we draw the force polygon for the loads and reactions *ABCDEFA* (fig. 122*b*); it is a clockwise polygon. We may begin by drawing the clockwise polygon for joint *I* or *II*; for the former it is *FABGF*.‡ Member *bg* is therefore in compression

---

* The part of that polygon representing the loads is called a *load line*.

† In some trusses, after the polygons for a few joints are drawn, there remains no joint at which there are but two unknown stresses; fig. 123 represents such a one. The solution of ex. 5 explains several ways of procedure in such cases.

‡ The student is urged to make sketches of the bodies (parts of truss) upon which the forces, whose polygons are being drawn, act. A force acting upon the "cut" end of a member and toward the joint is a push, and the stress in the member is compressive; if it acts away from the joint, it is a pull and the stress is tensile.

and *gf* in tension. Next we may draw the clockwise polygon for joint *II*, *III*, or *IV*; for the first it is *CDEHC*. Member *ch*

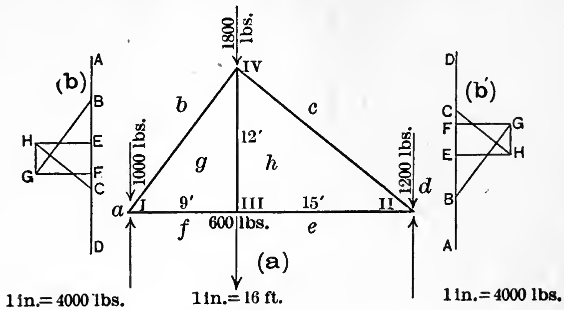


FIG. 122.

is in compression and *eh* in tension. For joint *III*, the polygon is *HEFGH* and member *gh* is in tension. If the work has been correctly and accurately done, the line *GH* is parallel to *gh*.

2. Solve ex. 2 of art. 147 graphically.

3. Solve ex. 3 of art. 147 graphically.

4. Solve ex. 5 of art. 147 graphically.

5. Analyze the truss of fig. 123 under the loads shown.

Solution: Evidently each reaction equals one-half the whole load. *ABCDEE′D′C′B′A′FA* is a clockwise polygon for the loads and reactions. The polygon for joint 1 may be drawn first; it is *FABGF*. Next, that for joint 2 may be drawn; it is *GBCHG*. Then that for joint 3 may be drawn; it is *FGHIF*. The polygons for joints 1′, 2′, and 3′ are *FA′B′G′F*, *G′B′C′H′G′*, and *FG′H′I′F* respectively.

No joint remains at which there are but two unknown forces, and no more polygons can be drawn. If in any way the number of unknown forces at a joint can be reduced to two, the polygon for that joint can be drawn and the stress diagram can be completed. There are several ways of making that reduction. For example, if the force in *ij*, *jm*, or *mf* were known, the polygon for joint 4 could be drawn, then that for 5, 6, 7, and 8.

The force in *mf* may be determined by "passing a section" as at *I* and solving the external system on either part of the truss for the desired force. The system consists of the loads and reaction on that part and the forces in the members cut.

The system may of course be solved graphically or algebraically, but in this truss the algebraic solution is much the simpler. A moment equation for either system with joint 8 as

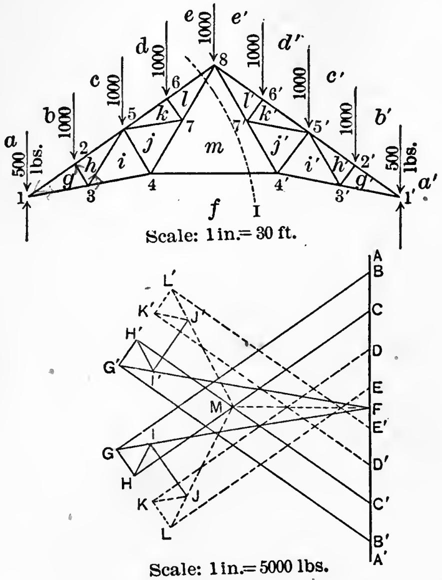


FIG. 123.

origin furnishes the value of the force in *mf* readily; the value is 3420 lbs. and the stress is tensile.

This force in *fm* may now be represented in the proper place in the stress diagram determining *M*, and then the polygon for joint 4 can be drawn; it is *MFIJM*. The student should pick out the polygons for the remaining joints and determine the kind of stress in each member.

There are other ways of meeting the difficulty presented in this form of truss, but that here given is the most general and can be applied readily to other forms.

6. Analyze the truss represented in fig. 114(*a*) under the loads shown.

§ VIII. Rough Supports; Friction.

152. **Definitions.**—It is a fact of experience that when one body slides or tends to slide over another, the sliding of the first is opposed or resisted by the second. Thus, suppose that fig. 124 represents a block which slides or tends to slide over another body towards the right; the second body exerts some such force as R upon the block.

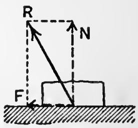


Fig. 124.

The force which one body exerts upon another which slides or tends to slide over the first is called the *total resistance;* it will be denoted by R. The component of the total resistance along the (plane) surface of contact is called *sliding resistance*, or more commonly *friction;* the component normal to the surface is called *normal pressure*. (If the surface of contact of the two bodies is not plane, the force exerted at each elementary part of the surface is the total resistance applied to that element, and its components in and normal to the element are the friction and the normal pressure applied to the element.)

Friction is called *kinetic* or *static* according as sliding does or does not take place. Static friction only is here considered (kinetic friction is discussed later).

Suppose that the block represented in fig. 125 weighs 10 lbs.,

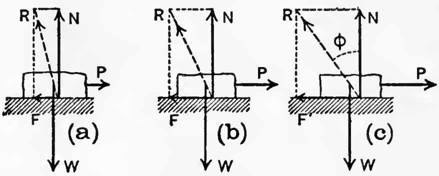


Fig. 125.

that it is subjected to a horizontal pull, P, and the rubbing surfaces are such that P must exceed 6 lbs. to start the block. Fig. 125 (a), (b), and (c) represent the forces acting upon the block when P, as it increases, reaches values of 2, 4, and 6 lbs. respectively. Since the block is at rest, the friction at the three stages equals 2, 4, and 6 lbs. and in all stages N equals 10 lbs. When P reaches 6+ lbs. the block will move and the kinetic friction would be something less than 6 lbs., as has been discovered experimentally. For any two bodies then, the friction may have many values depending on whether slipping occurs or not, and if not, on how great the tendency to slipping is.

The friction corresponding to impending motion is called *limiting friction;* it will be denoted by F'. Evidently limiting friction is the *maximum value* of the friction corresponding to any given normal pressure, see fig. 125 (a), (b), and (c). Limiting friction has been studied experimentally and many important results have been thus deduced.

153. *The Coefficient of Static Friction* for two rubbing surfaces is the ratio of any normal pressure between the surfaces and the corresponding limiting friction; if it is denoted by f,

$$f=F'/N, \quad \text{or} \quad F'=fN.$$

154. *The Angle of Friction* for two rubbing surfaces is the angle between the directions of the normal pressure and the total resistance when motion is impending. Denoting it by ϕ (see fig. 125c),

$$\tan\phi=F'/N; \quad \text{hence} \quad \tan\phi=f.$$

155. *Angle of Repose.*—If a block be placed upon an inclined plane, the inclination at which slipping would be impending is called the *angle of repose* for the two rubbing surfaces; it will be denoted by α. From fig. 126 (representing a body on an incline, the angle being that of repose), and the equations of equilibrium for the forces,

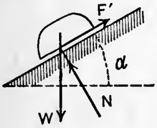


FIG. 126.

$$F'=W\sin\alpha \quad \text{and} \quad N=W\cos\alpha;$$

hence $$\tan\alpha=F'/N.$$

Since $$F'/N=f=\tan\phi, \quad \alpha=\phi, \quad \text{and} \quad \tan\alpha=f;$$

that is, the angle of repose for two surfaces equals their angle of friction, and the tangent of the angle of repose equals the coefficient of friction.

156. Laws of Friction.—The following laws relate to "solid friction" (friction between solids) and are based entirely on experiment.

1. The coefficient of friction for two surfaces depends upon the nature of the surfaces. Thus the coefficient varies with the materials, with the smoothness of the surfaces, and with the lubricant, if any is used.

2. The coefficient of friction is independent of the normal pressure between the two surfaces and of the extent of the contact. This law is not exactly true; especially for such low pressures at which a considerable part of the sliding resistance is due to adhesion, and for high pressures which result in a change in the character of the surfaces; also when lubrication is excessive, for then the friction is mixed, being neither "solid" nor "fluid."

157. Determination and Values of the Coefficient. — The coefficient of friction for two surfaces may be determined by measuring their angle of repose (see fig. 126). The tangent of the angle is the coefficient sought. Or, the two bodies may be placed as in fig. 125, and then measuring the pull P necessary to start the block, F' is known since F' equals that pull. Also, $N=W$; hence $f=P/W$.

In one of these two ways many determinations have been made, the values of the coefficient for a few materials ranging as follows:

Wood on wood, soaped	0.22 – 0.44
" " " dry	0.30 – 0.70
Metal on metal, dry	0.15 – 0.24
" " " as in polished and well-lubricated bearings	0.05 – 0.08
Wood on metal, dry	0.60
Hemp rope on wood	0.50 – 0.80
Sole-leather on wood or cast iron, as in packings, dry	0.40 – 0.60
Leather belting on pulleys	0.25 – 1.00
Stone on stone, as in arches	0.40 – 0.60

EXAMPLES.

1. If the block represented in fig. 127 weighs 100 lbs., θ is 10°, and the coefficient of friction is 0.2, how great must P be to start the block?

FIG. 127.

Solution: Suppose motion to be impending, then $F' = 0.2N$; and since the block is at rest, $P \cos 10° - F' = 0$ and $P \sin 10° + N - 100 = 0$. Solution of these three equations gives $P = 19.62$ lbs.

A force slightly greater than this will start the block.

2. Solve ex. 1 if the sense of P is reversed. *Ans.* 21.05 lbs.

3. If P in ex. 2 is 15 lbs., how great is the friction?

4. What is the least value of P acting as shown in fig. 127 that will start the block? What is the corresponding value of θ? *Ans.* $\theta = 11° 18'.6$.

5. If the block represented in fig. 128 weighs 100 lbs., $\beta = 25°$, $f = \frac{1}{3}$, and $\theta = 0$, how great must P be to start the block? *Ans.* 72.46 + lbs.

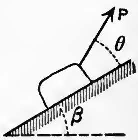


FIG. 128.

6. To prevent slipping, how great must P be?

7. If P is 20 lbs., determine the friction. *Ans.* 22.26 lbs.

8. If P is 50 lbs., determine the friction.

9. Take $\theta = 10°$ in ex. 5, and solve. *Ans.* 69.51 lbs.

10. Show that P (fig. 128) to start the body up is a minimum if $\theta = \tan^{-1} f$.

11. Two bodies connected by a cord are placed upon a plane inclined 12° to the horizontal, the string being taut but without initial tension and inclined 12° with the level base. If the bodies weigh 10 and 15 lbs. and the corresponding coefficients of friction are $\frac{1}{4}$ and $\frac{1}{3}$, determine the frictions.

12. In the preceding example, change the coefficient $\frac{1}{4}$ to $\frac{1}{6}$ and suppose the lower body to be the lighter one. Solve and also determine the tension.

13. A bar weighing 100 lbs. rests upon two end supports at the same level. Suppose a force is applied to it so that the action line passes through the points of support. If the coefficients of friction for the rubbing surfaces be 0.2 and 0.25, how

great must the force be to move the bar? What can you say about the frictions when the force is 18 lbs. and when it is 20 lbs.?

14. How great must P (fig. 129) be to start the wedge against the force Q?

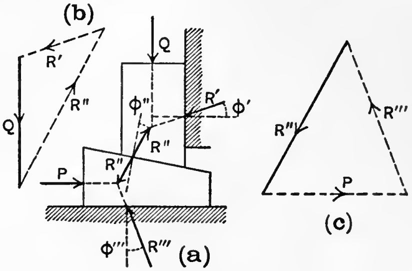


FIG. 129.

Solution: When the wedge is about to slip, the directions of the resistances at the rubbing surfaces are known, for the inclination of each to the normal to the surface on which it acts equals the corresponding angle of friction. Of the three forces, Q, R', and R'', applied upon the upper body, the direction of all and the magnitude of one, Q, are known. Their force triangle determines the magnitude and sense of R' and R'' (see fig. 129b). Having determined R'', P and R''' may be determined by means of the force triangle for those forces (see fig. 129c).

15. How great must P be to prevent the wedge from slipping out, the wedge angle being 25°, and all ϕ's 10° ?

16. Show that the wedge will not slip out if its angle is less than $\phi''+\phi'''$ when $P=0$.

17. Fig. 130 represents a jack-screw. How great a couple whose plane is horizontal must be applied to the screw to "overcome" Q?

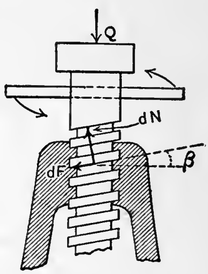


FIG. 130.

Solution: At each point of the lower surfaces of the thread on the screw, the nut exerts a pressure whose normal and tangential components call dN and dF respectively. When the tendency of the screw is to rise, dF acts downward. Let C denote the moment of the couple, β the pitch angle, and r the average arm of the frictions and normal pressures with respect to the screw axis. From the conditions of equilibrium (see art. 125),

$$\Sigma F_y = -Q - \Sigma(dF \sin\beta) + \Sigma(dN \cos\beta) = 0;$$
$$\Sigma M_y = C - \Sigma(dF \cos\beta \cdot r) - \Sigma(dN \sin\beta \cdot r) = 0;$$

and, when slipping is impending, $dF = f dN$.
These three equations make

$$C = Qr(\sin\beta + f\cos\beta)/(\cos\beta - f\sin\beta).$$

18. Show that to cause the screw to descend,

$$C = Qr(f\cos\beta - \sin\beta)/(f\sin\beta + \cos\beta).$$

19. Show that if $\beta > \phi$, the screw will descend under the action of Q alone.

20. Fig. 131 represents a lever supported in a triangular bearing. How great a force, P, is required to "overcome" Q?

Solution: When slipping is about to occur, the reactions at A and B act in the directions indicated. Since the action lines of P, Q, R', and R'' and the magnitude of Q are known, the magnitude of P (and of R' and R'') may be determined (see art. 141).

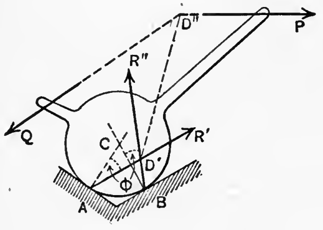


FIG. 131.

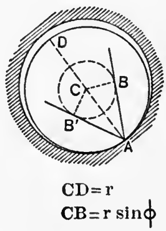


FIG. 132.

158. Friction Circle.—Fig. 132 represents a journal and its bearing (also a pin joint), the fit being loose so that the contact is along a line practically. Let r denote the radius of the journal and ϕ the angle of friction for the rubbing surfaces. Then a circle concentric with a cross-section of the journal whose radius equals $r \sin\phi$ is called the *friction circle* for the journal and bearing. It is useful in solving certain problems involving the friction of a loose-fitting journal and bearing, pin joint, etc.

Proposition.—When slipping is about to occur at a loose bearing, the action line of the resistance offered by the bearing is tangent to the friction circle.

Proof: The resistance is applied at the line of contact, represented by A, and it makes an angle ϕ with the normal AC (art. 154). If AB is tangent to the circle, then

$$\sin BAC = r \sin \phi / r = \sin \phi, \quad \text{or} \quad BAC = \phi;$$

hence the tangent line and the action line of the resistance coincide.

Remembering that the tangential component of the resistance (the friction) opposes the tendency to slip, the student will have no difficulty to tell which one of the two tangent lines which may be drawn is the action line of the resistance.

EXAMPLES.

1. Fig. 133 represents a lever supported in a loose cylindrical bearing. How great a force P is required to "overcome" Q?

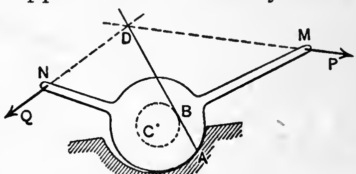


FIG. 133.

Solution: There being three forces applied to the lever, P, Q, and the resistance of the bearing R, their action lines intersect in a point (art. 125); hence R passes through D. Since R is also tangent to the friction circle, its action line is determined, and the system P, Q, R may be solved for the unknown magnitudes.

2. Let MCN (fig. 133) equal 90°, CM 2 ft., CN 6 in., the angles CND and CMD 60° and 90° respectively, radius of axle 2 in., the coefficient of friction $\frac{1}{2}$, and Q 1000 lbs. Determine P and the friction.

3. Determine the largest force P which Q will overcome, data as in ex. 2.

4. When the lever is straight ($NCM = 180°$) and P and Q act at right angles to NCM, show that

$$P(\overline{CM} \mp r \sin \phi) = Q(\overline{CN} \pm r \sin \phi),$$

according as the impending motion is with P or Q.

159. Cone of Friction.—Let P (fig. 134) denote the resultant of all the forces applied to the body represented, not including the resistance of the supporting surface. Then the cone whose apex is at C, whose axis is the normal through C, and whose apex angle equals twice the angle of friction is called a *cone of friction*. In the figure the cone is represented by ACB.

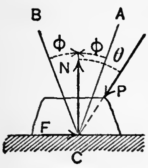


Fig. 134.

Proposition.—If the action line of the resultant of all the forces applied to a body not including the resistance of the support falls within the cone of friction, sliding will not occur; if without, it will occur.

Proof: The force causing or tending to cause sliding is the horizontal component of P (fig. 134), equal to $P \sin \theta$. Since the maximum sliding resistance, or friction, is

$$F' = \tan \phi \cdot N = \tan \phi \cdot P \cos \theta,$$

$$(\text{sliding force})/F' = \tan \theta / \tan \phi.$$ Therefore

if $\theta > \phi$ (P falls without the cone),

the sliding force > limiting friction;

if $\theta < \phi$ (P falls within the cone),

the sliding force < limiting friction.

EXAMPLES.

1. In fig. 128 suppose that θ and $\beta = 30°$, $P = 50$ lbs., $f = \frac{1}{4}$, and the body on the incline weighs 100 lbs. Determine whether the body will slide and the value of the friction graphically.

2. Let P in the preceding example be 10 lbs., and solve.

3. A prismatic block of wood is sawed into two parts so that the cut is inclined at an angle θ with the ends. If the two parts are laid together matching and end pushes are applied along the axis, for which values of θ will slipping not occur?

4. Fig. 135 represents a slider which may slide in the guides A and B; ϕ' and ϕ'' are the angles of friction for the rubbing surfaces respectively. Show that any horizontal force applied above C cannot move the slider.

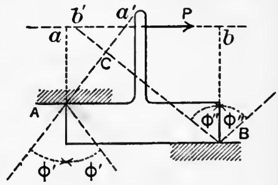


Fig. 135.

Solution: Imagine slipping about to occur at B; then the resistance there acts along the line BC. To keep the slider at rest, the resistance at A must act through b', and since the line Ab' is within the cone at A, such resistance is possible, and equilibrium will be preserved.

5. A boy climbs a ladder which rests on a horizontal floor and against a vertical wall. Determine graphically how far he can ascend without causing the ladder to slip. Suppose that the length of the ladder is 20 ft., its centre of gravity is 8 ft. from the foot, its inclination is 45°, its weight and that of the boy are equal, and the coefficients of friction for the surfaces at the top and bottom are 0.4 and 0.6 respectively.

Ans. About 19 ft.

6. A uniform ladder rests with one end against a rough horizontal, the other against an equally rough vertical plane. Determine the least coefficient of friction that will allow the ladder to rest in all positions.

Ans. 1.

7. Fig. 136 represents a bar AB resting in a horizontal position upon two inclined planes, ϕ' and ϕ'' being the angles of friction at A and B respectively. Show that if the weight of the bar is neglected, any body suspended from a point between m and n will not cause it to slide, but that if suspended beyond m or n it will cause sliding.

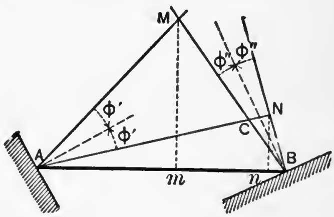


FIG. 136.

8. The front of a drawer is 4 ft., and its sides 10 in. long. If the angle of friction for the rubbing surfaces at the sides is 22°, and the drawer handles are 3 ft. apart, show that the drawer cannot be opened by a forward pull at one handle.

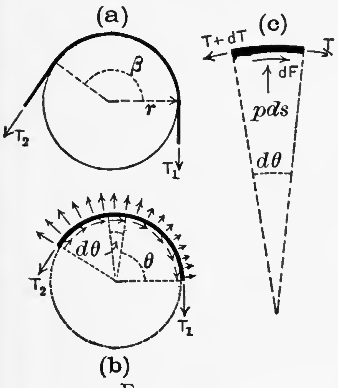


FIG. 137.

160. Belt Friction.—Fig. 137(a) represents a cylinder about a part of which a belt (or cord) is wrapped. If the cylinder is not smooth, the pulls T_1 and T_2 may be quite unequal without causing slipping of the belt, as may easily be verified by trial.

The forces acting upon the part of the belt in contact with the cylinder consist of the tensions T_1 and T_2, the normal pressure and the friction (see fig. 137*b*). Let p denote the normal pressure per unit length of arc; then the normal pressure on any part whose length is ds (fig. 137*c*) is pds. The friction on that part may be called dF and the tensions T and $T+dT$. Since the portion is in equilibrium,

$$pds = 2T \sin\frac{d\theta}{2} = Td\theta;$$

hence $$p = T/r; \quad \ldots \ldots \quad (1)$$

that is, the normal pressure per unit length at any point of the contact equals the belt tension there divided by the radius of the cylinder.

When slipping is impending, $dF = f \cdot pds$, and since $dF = dT$,

$$dT = f\frac{T}{r}ds, \quad \text{or} \quad \frac{dT}{T} = f\frac{ds}{r} = fd\theta.$$

Integration gives $$\Big[\log_e T\Big]_{T_1}^{T_2} = f\Big[\theta\Big]_0^{\beta} \quad \text{(see fig. 137}a\text{);}$$

hence $$\log_e T_2 - \log_e T_1 = f\beta, \quad \ldots \ldots \quad (2)$$

or $$T_2 = T_1 e^{f\beta}. \quad \ldots \ldots \quad (3)$$

The angle β must be expressed in radians; e is the base of the Naperian system of logarithms, 2.718. The formulas apply also when β is greater than 2π, that is, when the cord more than encircles the cylinder.

For a given value of T_1, T_2 increases very rapidly with β as shown by fig. 138 which represents the locus of equation (3). T_2 and β are the variables and e, f, and T_1 constants, f being taken as $\frac{1}{6}$ and $T_1 = OA$. To the scale $OA = T_1$, OB represents T_2 when $\beta = AOB$.

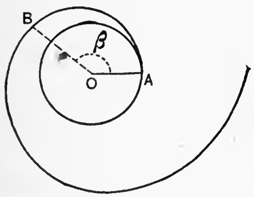


FIG. 138.

EXAMPLES.

1. Compute the ratio between T_1 and T_2, when f is $\frac{1}{2}$ and the cord is wrapped twice around the cylinder.

2. Plot in fig. 138 the locus of equation (3) when f is $\frac{1}{4}$

§ IX. Forces in Space and Miscellaneous.

161. Examples Involving Non-Parallel Non-Coplanar Forces.—The principles for solving such examples are stated in art. 125. Sometimes in simple cases the example can be resolved into others the forces in which are coplanar. Such separation is usually a simplification; ex. 3 is an illustration.

EXAMPLES.

1. Determine the relation between P and W of the windlass represented in fig. 139 and the reactions of the bearings in terms

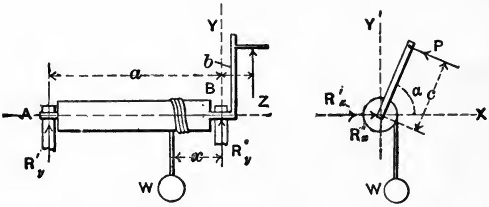


FIG. 139.

of P. Take P always at right angles to the crank as shown, and neglect friction.

Solution: The forces acting on the windlass are P, W, and the reactions of the bearings, A and B; the weight of the windlass is neglected. Let R' and R'' denote the reactions at A and B respectively, and α the angle which the crank makes with the x axis; then

$$\Sigma F_x = -P \sin\alpha + R'_x + R''_x = 0,$$
$$\Sigma F_y = P\cos\alpha + R'_y + R''_y - W = 0,$$
$$\Sigma M_x = -P\cos\alpha \cdot b - Wx + R'_y a = 0,$$
$$\Sigma M_y = -P\sin\alpha \cdot b - R'_x a = 0,$$
$$\Sigma M_z = Pc - Wr = 0,$$

r being the distance from the axis of the rope to that of the windlass.

The z-resolution equation vanishes since none of the forces have z components. Solving the equations, we find that

$W = Pc/r,$
$R'_x = -P \sin\alpha \cdot b/a, \qquad R'_y = P(b\cos\alpha + cx/r)/a,$
$R''_x = \quad P\sin\alpha(1 + b/a), \quad R''_y = P(1 - x/a)c/r - P\cos\alpha(1 + b/a).$

2. Fig. 140(*a*) represents a derrick consisting of a post (AB), boom (AC), two "stiff legs" (BD and BE), and hoisting cables which are not shown in detail. Determine the forces on the parts of the derrick due to the load W, assuming for simplicity that B and C are merely connected by a cable, that the load is

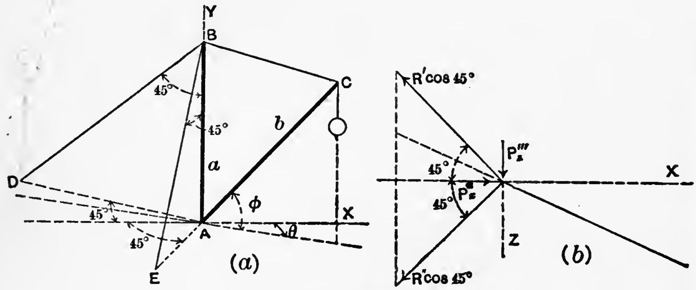


FIG. 140.

suspended from C, and that the boom is connected to the post at its lower end practically.

Solution: The external forces on the whole derrick are W and the reactions at A, D, and E. The directions of these reactions will depend on the nature of the supports at A, D, and E; we will assume that these are such that the reaction at A is a single force acting through A, and that those at D and E act along DB and EB respectively.

We first resolve this system into two component systems, one coplanar (its plane being that of the ground) and one parallel (its forces being vertical). Let the reactions at D and E be called R' and R'' respectively; then the components of the R' are

R' cos 45° acting in the line AD,
R' sin 45° " vertically at D,

and the components of R'' are

R'' cos 45° acting in the line AE,
R'' sin 45° " vertically at E.

Let the reaction at A be called R''', and imagine it resolved into x, y, and z components at A. The coplanar component system consists of forces as shown in fig. 140(b), and the parallel component system consists of four vertical forces, $R' \sin 45°$ at D, $R'' \sin 45°$ at E, R_y''' at A, and W at C (all not shown).

Each of the component systems being in equilibrium, we may write the appropriate equations of equilibrium for each; thus for the first (concurrent)

$$\Sigma F_x = -R' \cos 45° \cos 45° - R'' \cos 45° \cos 45° + R_x''' = 0,$$
$$\Sigma F_z = -R' \cos 45° \sin 45° + R'' \cos 45° \sin 45° + R_z''' = 0;$$

and for the second

$$\Sigma F_y = R_y''' - W - R' \sin 45° - R'' \sin 45° = 0,$$
$$\Sigma M_x = Wb \cos \phi \sin \theta - R' \sin 45° \cdot a \cos 45° + R'' \sin 45° \cdot a \cos 45° = 0,$$
$$\Sigma M_z = -Wb \cos \phi \cos \theta + R' \sin 45° \cdot a \cos 45° + R'' \sin 45° \cdot a \cos 45° = 0.$$

These equations determine the five unknown forces R', R'', R_x''', R_y''', and R_z'''. The solution is left to the student.

The tension in the cable BC and the pressure between the post and the boom may be determined from a consideration of the forces acting on the boom. As these forces are coplanar their determination is left to the student.

3. Fig. 141(a) represents a "shear-legs crane." It consists of two posts, CD and CE, hinged together at the top and hinged

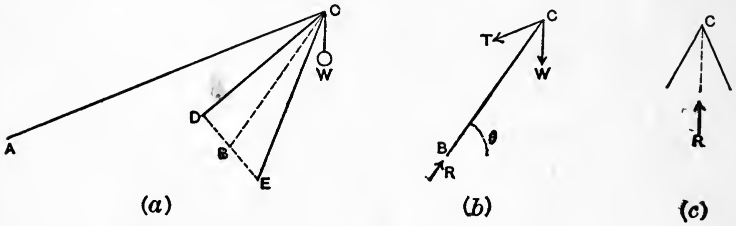


FIG. 141.

at their bases so that they can rotate about the line joining D and E. The stay AC may be a cable or a "stiff leg"; if stiff, AC is constant in length and the load is swung in or out (the posts rotating about DE) by moving the lower end of A along the track AB. Determine the forces on the parts due to the load W.

Solution: Imagine that the two posts are replaced by a single one CB, and determine the tension in the stay and the compression (R) in CB from a consideration of the forces applied (see fig. 141*b*). Then resolve this compression R into two components whose action lines coincide with the axes of the posts (see fig. 141*c*).

4. In fig. 141 take $AC=50$ feet, $BD=BE=10$ feet, $\theta=60°$, $W=10$ tons, and solve the preceding example graphically.

5. Fig. 142 represents a small dipper dredge, the side elevation (*a*) representing a position for filling the dipper, and the

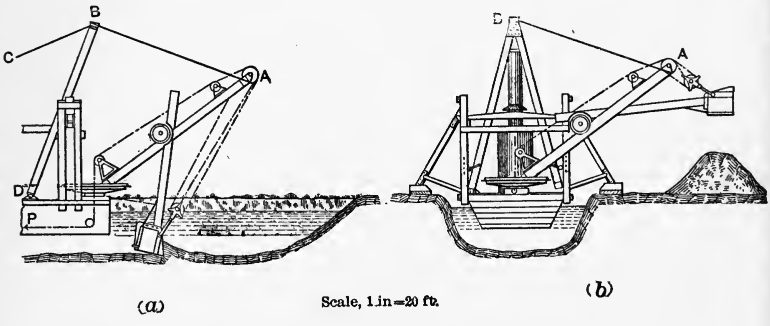


FIG. 142.

front elevation (*b*) a position for emptying. The boom swings about a vertical axis at its lower end, a ratchet on the lower side of the dipper handle engages a pinion on the boom by means of which the effective length of the handle can be changed, and the two back stays BC are fastened at points 10 feet apart, 36 feet to the left of D. In (*a*) assume that the dipper is stuck, the pull in the chain being 5000 lbs., and in (*b*) that the dipper load is 1000 lbs. Neglect the weights of all parts and determine in each case the tensions in the stays AB and BC, the compressions in the posts of the "A frame," the reaction at the pivot at the base of the boom, and the pressure on the ratchet. Solve graphically.

6. Fig. 143(*a*) represents a giant wharf-crane. The structure consists of a rigid framework, T-shaped, and a rigid tripod.

The stem of the T stands within the tripod and rests against the tripod head and on rollers at the base. Determine the reactions at the base of the tripod legs, as far as possible, due to a load P of 150 tons at the position shown in the figure. When the cross-piece of the T is represented in plan by Aa, the short arm being on the side A, what are the reactions?

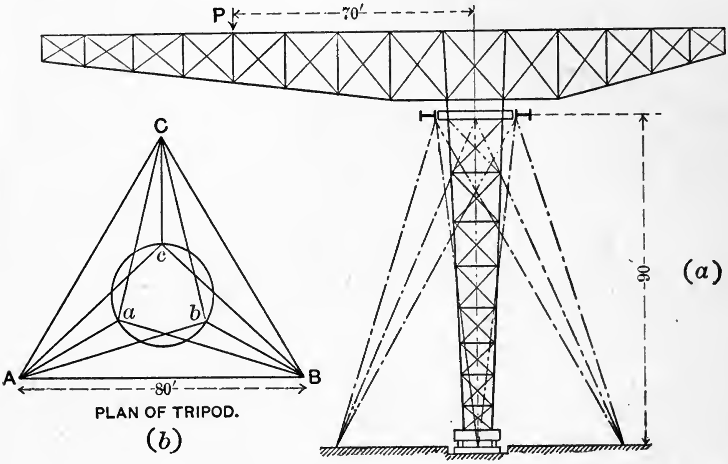


FIG. 143.

7. The crane of the preceding example is revolved by means of a circular rack on the tripod head and two pinions at opposite ends of a diameter of the rack. It is estimated that when the crane is being turned against a maximum resistance (due to friction, inertia, and wind-pressure), the reaction on the rack is a horizontal couple of 112,000 foot-pounds. The diameter of the rack being 22 feet compute the reactions at the base of the tripod legs due to the reaction mentioned.

162. Miscellaneous.—The principles for the solution of the following examples are given in art. 125. Similar examples have been worked in the first part of this chapter, and the student should have no difficulty in solving the following set.

EXAMPLES.

1. Fig. 144(a) represents a platform scale which consists of a frame FF with five knife-edges, one at K_1, two at K_2, and

two at K_3. The platform rests on three levers which bear upon the four knife-edges at K_2 and K_3. The short levers are also supported by the long one at S, and the long one is connected by

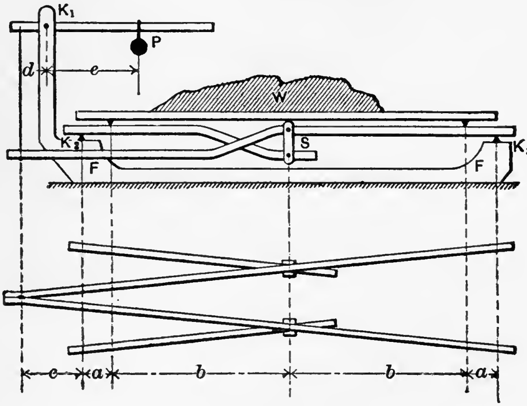


FIG. 144(*a*).

a vertical rod to the "scale-beam" PK_1. Determine the relation between the weight of the poise (P) and that of the body (W) on the platform, and show that it is independent of the position of the body.

2. Assume that the steam-pressure P just balances the load W on the hoisting-engine (fig. 144*b*). If $P = 1000$ lbs. and the angle BCA is 60°, compute the compression in the connecting-rod, the pressure against the cross-head guide, the tangential component of the crank-pin pressure, and W.

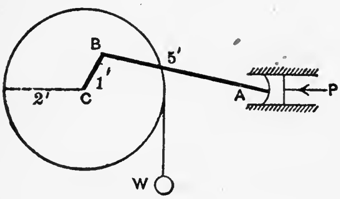


FIG 144(*b*).

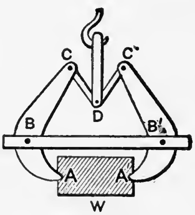


FIG. 144(*c*).

3. Fig. 144(*c*) represents a "crab hook." The lengths AB and BC are 12 and 21 in. respectively, the angle ABC is 100°, $CD = 12''$, and $BB' = 3$ ft. Determine the stresses in CD and

BB' if $W = 1000$ lbs. and $AA' = 18''$, the weight of the parts being neglected.

4. Fig. 144(d) represents a simple elevator-car. What are the pressures of the wheels on the rails due to the load W?

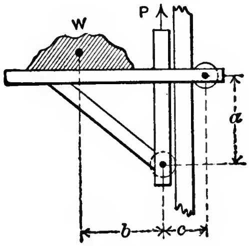


FIG. 144(d).

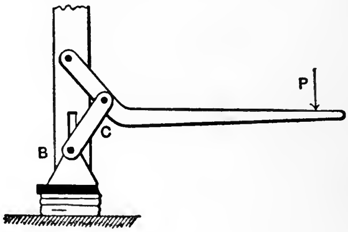


FIG. 144(e).

5. Fig. 144(e) represents a hand-press, consisting of a lever ACP and two short links BC pinned to a cross-head at their lower ends; $AC = BC = 15''$. Compute the pressure on the bale in the press and the pin-pressures when $AB = 2$ ft., and the arm of P with respect to A is 5 ft., P being 100 lbs.

KINEMATICS.

CHAPTER VII.

RECTILINEAR MOTION OF A PARTICLE.

§ I. Velocity and Acceleration.

163. Specification of Position.—The position of a point in a given line can be specified by a single quantity, namely, the abscissa of the point with respect to any other point in the line assumed as "origin." Thus if the abscissas of points to the right of the origin O (fig. 145) be given the plus sign and those of points to the left the minus sign, then P' is specified by the abscissa $\frac{2}{3}$ inch and P'' by $-\frac{1}{2}$ inch. Position abscissas will be denoted by x.

Fig. 145.

164. Space-Time Curves.—A rectilinear motion of a point can be well represented by means of a line called the space-time curve for the motion. This is a line the ordinate and abscissa to any point of which represent the position abscissa and corresponding value of the time respectively. To construct this curve plot corresponding values of x and the time along vertical and horizontal axes as shown in fig. 146, and join all such plotted points; the connecting line is the space-time curve.

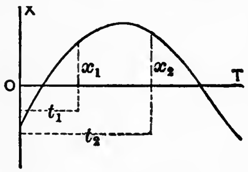


Fig. 146.

Evidently the space-time curve for a motion gives the position of the moving point at each instant, and it is therefore a complete record of the motion.

165. Displacement.—If x_1 denotes the abscissa of a moving point at the instant t_1, and x_2 that at a later instant t_2, then the displacement for the interval t_2-t_1 is defined as x_2-x_1. Evidently a displacement (x_2-x_1) may be positive or negative; hence, two displacements must agree in sign as well as in magnitude to be equal.

166. Kinds of Rectilinear Motion.—If the displacements of a point in equal intervals of time (large or small) are equal, the motion is called *uniform;* and if the displacements are not equal, the motion is called *non-uniform*. Non-uniform motions are further classified as explained in art. 172.

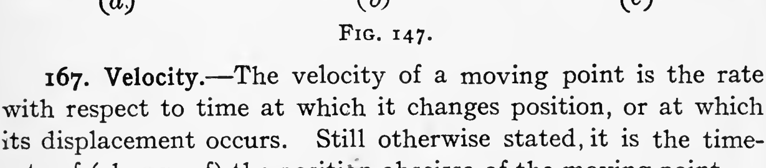
The space-time curve for a uniform motion is obviously an inclined straight line, and the space-time curve for a non-uniform motion is a curved line.

QUESTIONS.

1. What is the difference between the motions represented in fig. 147 (*a*) and (*b*)?

2. What can you tell of the motion whose space-time curve is that in fig. 147(*c*)?

3. Are horizontal and vertical space-time curves possible?



(*a*) (*b*) (*c*)

FIG. 147.

167. Velocity.—The velocity of a moving point is the rate with respect to time at which it changes position, or at which its displacement occurs. Still otherwise stated, it is the time-rate of (change of) the position abscissa of the moving point.

Let x and t denote position abscissa and time respectively; then, as shown in works on calculus (and in Appendix B), the time-rate of x is dx/dt; hence, if v denotes velocity,

$$\mathbf{v = dx/dt.} \qquad (1)$$

This gives the value of v at any instant t; its value at a particular instant equals the value of dx/dt for that instant.

If the motion is uniform, x changes uniformly, and the time-rate of x is $\Delta x/\Delta t$, Δx denoting the displacement which occurs in any interval Δt. Hence

$$\mathbf{v} = \Delta \mathbf{x}/\Delta \mathbf{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and plainly the velocity is constant.

Since $\Delta x/\Delta t$ is the average time-rate of the displacement, equation (2) gives also the average velocity when applied to non-uniform motions.

Since the space-time curve is the "locus" or "graph" of the equation between x and t, dx/dt is the general expression for the slope or gradient of that curve. Hence the velocity corresponding to any point on a space-time curve is represented by the slope of the curve at that point. We say "is represented by" instead of equals, because, while the velocity at a certain instant is definite, the slope depends on the scale used in plotting the space-time curve. The slopes must therefore be interpreted by scale or be computed in a certain way, as explained in ex. 1, art. 169.

168. *Unit Velocity.*—The expressions for velocity in eqs. (1) and (2) of the preceding article imply a certain unit of velocity, namely, the velocity of a point moving uniformly and so that it describes unit distance in unit time. Specific units of velocity are one foot-per-second, one mile-per-hour, etc. There are no short names for these units except for the nautical mile-per-hour, which is called a knot.

The term *per* is conveniently replaced by the solidus, /; foot-per-second, mile-per-hour, etc., are abbreviated thus: ft./sec., mi./hr., etc.*

169. *Sign of a Velocity.*—The expressions dx/dt and $\Delta x/\Delta t$ may be positive or negative; therefore v must be regarded as having the same sign as that of dx/dt or $\Delta x/\Delta t$ (see eqs. (1) and (2), art. 167). When the point is moving in the positive direction, dx/dt and $\Delta x/\Delta t$ are positive, and when it is moving in the negative direction they are negative (see fig. 147); hence

the sign of the velocity of a moving point at any instant is the same as that of the direction in which it is then moving.

* For dimensions of a unit velocity see Appendix C.

EXAMPLES.

1. What is the velocity when $t=2$ secs. in the motion whose space-time curve is shown in fig. 148?

Solution: We find first the point P of the curve corresponding to $t=2$ secs., and then draw a tangent to the curve at that point. Next we take any point Q in the tangent line, and from it draw a perpendicular to the horizontal through P. Then we measure by scale the lines QR and PR, and take their ratio as measured. We find that $QR=35$ ft. and $PR=4$ secs.; hence

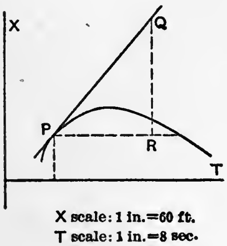


FIG. 148.

$$v=35/4=8\tfrac{3}{4} \text{ ft.-per-sec.}$$

2. A point moves so that $x=ct^3$, c being a constant. Show that its velocity at any time t is $3ct^2$.

3. Let c in the preceding ex. be 10, x being in ft. and t in secs. When $t=3$ secs., where in its path is the moving point, and what is its velocity? *Ans.* $v=270$ ft./sec.

4. A point moves so that $x=100t$, x and t being in ft. and secs. respectively. What is its velocity?

Solution: Here x varies uniformly; hence $v=\Delta x/\Delta t$. From the law of the motion, $\Delta x=100\Delta t$, or $v=100$ ft./sec. (Can the value of v be deduced from eq. (1), art. 167?)

5. A body falls in a vacuum according to the relation $x=16.1t^2$, x and t being in ft. and secs. respectively. What is the formula for the velocity?

6. Draw a space-time curve for the motion of a falling body.

7. A point moves so that $x=10t-t^2$, x and t being in ft. and secs. respectively. What is its velocity when $t=6$ secs.?

8. A point moves so that $x=c\cos(kt)$, c and k being constants. Deduce an expression for its velocity at any time t. Also let $c=2$, $k=3$, and x, t, and kt be in ft., secs., and radians respectively; compute the velocity when $t=4$ secs.

9. A sprint of 100 yards being accomplished in 10 secs., what was the sprinter's average velocity in ft.-per-sec.? In mi.-per-hr.?

10. In a certain motion $v=3t^2$, v and t being expressed in ft.-per-sec. and secs. respectively. Determine the displacement in the interval from the second to the fourth sec.

Solution: Since $v=3t^2=dx/dt$,

$$dx=3t^2\,dt, \quad \text{or} \quad x=t^3+C,$$

C being a constant of integration whose value depends on the mode of reckoning x and t, not specified.* Let x_2 and x_4 denote the values of x when $t=2$ and 4 secs. respectively; then

$$x_4=4^3+C=64+C, \quad \text{and} \quad x_2=2^3+C=8+C.$$

Hence $$x_4-x_2=64-8=56 \text{ ft.}$$

Instead of introducing a constant of integration we might integrate between limits; thus, from $dx=3t^2\,dt$,

$$\int_{x_2}^{x_4} dx=3\int_2^4 t^2\,dt, \quad \text{or} \quad x_4-x_2=3\left[\frac{t^3}{3}\right]_2^4=56 \text{ ft.}$$

11. In a certain motion $v=3t^2+4$, v and t being expressed in ft.-per-sec. and secs. respectively. If the moving particle is 6 ft. to the right of the origin at the instant from which t is reckoned, determine the position at any time t and draw the space-time curve for the motion.

170. Velocity-Time Curve.—The way in which the velocity of a moving point changes with respect to time can be represented graphically by a line called the velocity-time curve for the motion. This is a line the ordinate and abscissa of any point of which represent the velocity and the corresponding value of the time respectively.

To construct this curve plot corresponding values of velocity (v) and time (t) along vertical and horizontal axes respect-

* *Note on the Determination of Constants of Integration.*—The student is reminded that to determine a constant of integration he has only to substitute for the variables in an equation containing the constant any simultaneous values of them and then solve for the constant. Thus in the case above, suppose it had been stated that x is measured from the place occupied by the moving particle at the instant from which t is reckoned; then when t was zero x was also zero, i.e., simultaneous values of x and t are $x=0$ and $t=0$. These substituted in the equation containing C make it

$$0=0^3+C; \quad \text{hence} \quad C=0.$$

ively, as shown in fig. 149, and join all such plotted points. The connecting line is the velocity-time curve.

171. Velocity Increment.—If v_1 denotes the velocity of a point at an instant t_1, and v_2 that at a later instant t_2, then $v_2 - v_1$ (not $v_1 - v_2$) is the velocity increment for the interval $t_2 - t_1$. Evidently a velocity increment may be positive or negative; hence, two velocity increments must agree in sign as well as in magnitude to be equal.

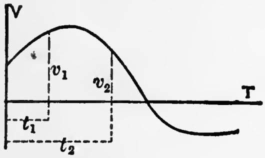
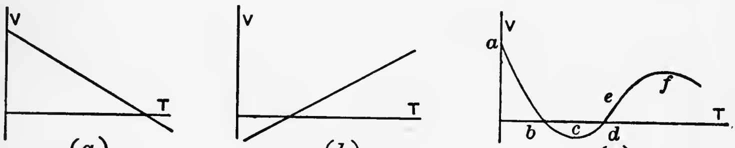


FIG. 149.

172. Kinds of Non-Uniform Motion.—A non-uniform motion whose velocity increments for equal intervals (large or small) are equal is called *uniformly varying;* one whose velocity increments are unequal is called *non-uniformly varying.*

Evidently the velocity-time curve for a uniformly varying



(*a*) (*b*) (*c*)

FIG. 150.

motion is a straight line (fig. 150 *a* and *b*) and that for a non-uniformly varying one is a curved line (fig. 150*c*).

QUESTIONS.

1. What is the difference in the motions whose velocity-time curves are shown in fig. 150 (*a*) and (*b*)?

2. What can you say of the motion whose velocity-time curve is shown in fig. 150(*c*)?

3. Are horizontal or vertical velocity-time curves possible?

173. Acceleration.—By acceleration of a moving point is meant the rate at which its velocity changes with respect to time, or simply the time-rate of (change of) its velocity.

Let v and t denote velocity and time respectively; then, as shown in works on calculus (and in Appendix B), the time-rate of v is dv/dt; hence if a denotes acceleration,

$$\mathbf{a = dv/dt = d^2x/dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equation (1) gives the value of a at any instant t; its value at a particular instant equals the value of dv/dt or d^2x/dt^2 for that instant.

If the motion is uniformly varying, v changes uniformly and the time-rate of v is $\Delta v/\Delta t$, Δv denoting the velocity increment for any interval Δt. Hence in this case

$$\mathbf{a}=\Delta \mathbf{v}/\Delta \mathbf{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and plainly the acceleration is constant.

Since $\Delta v/\Delta t$ is the average time-rate of the velocity, equation (2) gives also the average acceleration when applied to non-uniform motions non-uniformly varying.

Since the velocity-time curve is the locus or graph of the equation between v and t, dv/dt is the general expression for the slope or gradient of that curve. Hence the acceleration corresponding to any point on a velocity-time curve is represented by the slope of the curve at that point. The slopes must be interpreted by a scale, or be computed in a certain way as explained in ex. 1, art. 175.

174. *Unit Acceleration.*—The expressions for acceleration in eqs. (1) and (2) art. 173 imply a certain unit of acceleration, namely, the acceleration of a point whose velocity varies uniformly and so that it changes by a unit in each unit time. Specific units of acceleration are

one knot-per-hour (one nautical mile-per-hour-per-hour)
one foot-per-second-per-second, etc.

Abbreviating the term *per* as before, the above-named units are written thus: knot/hr. (mi./hr./hr.), ft./sec./sec.; or still more briefly, mi./hr^2, ft./$sec.^2$ *

175. *Sign of an Acceleration.*—The expressions dv/dt and $\Delta v/\Delta t$ may be positive or negative; therefore a must be regarded as having the same sign as that of dv/dt or $\Delta v/\Delta t$ (see eqs. (1) and (2), art. 173). Now when the velocity increases algebraically, dv/dt and $\Delta v/\Delta t$ are positive, and when it decreases algebraically, dv/dt and $\Delta v/\Delta t$ are negative (see fig. 150c); hence

* For dimensions of a unit acceleration see Appendix C.

the sign of the acceleration of a point at any instant is positive or negative according as the velocity is then increasing or decreasing (algebraically).

EXAMPLES.

1. What is the acceleration when $t=3$ secs. in the motion whose velocity-time curve is represented in fig. 151?

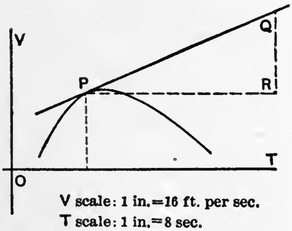


FIG. 151.

Solution: First we find the point P on the curve corresponding to $t=3$ secs., and then draw a tangent to the curve at that point. Next we take any point Q on the tangent and draw from it a perpendicular to the horizontal line through P. Then we measure by scale the lines QR and PR and take their ratio as so measured. We find that $QR=7$ ft.-per-sec. and $PR=8$ secs.; hence

$$a=7/8=0.875 \text{ ft.-per-sec.-per-sec.}$$

2. In a certain experiment on "getting up speed" of electric trains the speeds recorded at 5-minute intervals were as follows: 0, 19, 30, 35, 38, and 40.5 mi.-per-hr. Draw the velocity-time curve for the motion and determine the accelerations at the beginning and end of the period.

3. The motion of a point being according to the equation $x=ct^3$, c being a constant, show that the acceleration equals $6ct$.

4. Let c in the preceding example be 10, x being in ft. and t in secs. What is the value of the acceleration when $t=3$ secs.?

5. A point moves so that $v=50t$, v and t being in ft.-per-sec. and secs. respectively. What is its acceleration?

Solution: Since v varies uniformly, $a=\Delta v/\Delta t$. From the law of the motion $\Delta v=50\Delta t$; hence $a=50$ ft.-per-sec.^{2} (Can the value of a be deduced from eq. (1), art. 173?)

6. A body falls in a vacuum according to the law $x=16.1t^2$, x and t being in feet and seconds respectively. What is the value of the acceleration?

7. Draw a velocity-time curve for a body falling in vacuum, and compare with the curve drawn in your solution of ex. 6, art. 169.

8. A point moves so that $x=10t-t^2$, x and t being in ft. and secs. respectively. What is its acceleration when $t=6$ secs.?

9. A point moves so that $x=c\cos(kt)$, c and k being constants. Deduce an expression for its acceleration at any time t. Also let $c=2$, $k=3$, and x, t, and kt be in ft., secs., and radians respectively; compute the acceleration when $t=4$ secs.

10. The "acceleration due to gravity" is about 32.2 ft.-per-sec.-per-sec. Express the same in yard-minute units.

11. The law of a motion is $a=10t$ (ft.-sec. units), and the velocity is 10 ft.-per-sec. when t is 4 secs. Determine the velocity at any instant.

Solution: Since $a=dv/dt=10t$, $dv=10t\,dt$,

and
$$v=10\int t\,dt=5t^2+C.$$

Now $v=10$ and $t=4$ being simultaneous values of v and t, they satisfy the last equation; hence

$$10=5\times 4^2+C, \quad \text{or} \quad C=-70,$$
and
$$v=5t^2-70.$$

12. If the law of a motion is $a=10t+5$ (ft.-sec. units) and x, v, and t are simultaneously zero, determine the values of v and x at any time.

176. "Acceleration-Time" and Other Curves.—If simultaneous values of the acceleration and the time of a motion be plotted on two rectangular axes, then all such plotted points determine a curve called the acceleration-time curve for the motion. Evidently it is a graphical representation of the way in which the acceleration varies with the time.

Other curves descriptive of a motion can be drawn. Thus values of the velocity (v) and the position-abscissa (x) of a moving point if plotted make a "velocity-space curve," and values of a and x if plotted make an "acceleration-space curve."

§ II. IMPORTANT SPECIAL MOTIONS.

177. Uniform Motion.—The velocity is constant, and therefore the acceleration is zero and the displacement (x_2-x_1) in any interval (t_2-t_1) given by

$$(x_2-x_1)=v(t_2-t_1),$$

v denoting the velocity.

178. Uniformly Accelerated Motion.—The acceleration is constant, and the velocity increment (v_2-v_1) in any interval (t_2-t_1) is given by

$$(v_2-v_1)=a(t_2-t_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a denoting the acceleration. According to this equation the velocity varies uniformly; hence the average velocity for the interval (t_2-t_1) is $\frac{1}{2}(v_2+v_1)$, and the displacement (x_2-x_1) in the interval is given by

$$(x_2-x_1)=\tfrac{1}{2}(v_2+v_1)(t_2-t_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let x_0, v_0, and o be simultaneous values of x, v, and t. Since

$$a=dv/dt, \quad dv=a\,dt, \quad \text{or} \quad v=at+C_1.$$

Substituting simultaneous values of v and t in the last equation, we find that $C_1=v_0$; hence

$$v=at+v_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $v=dx/dt$, $dx=at\,dt+v_0\,dt$; and hence

$$x=\tfrac{1}{2}at^2+v_0t+C_2.$$

Substituting simultaneous values of x and t in the last equation, we find that $C_2=x_0$; hence

$$x=\tfrac{1}{2}at^2+v_0t+x_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

EXAMPLES.

A body moving near the earth under the influence of its attraction would have a constant acceleration were it not for air resistance. In the following examples neglect this resistance and denote the acceleration by g and the distance (measured positively downward) of the moving body from the starting-point by x.

1. If a body falls from rest, show that

$$v = gt, \quad x = \tfrac{1}{2}gt^2, \quad \text{and} \quad v^2 = 2gx.$$

2. If a body is projected down with a velocity v_0, show that

$$v = gt + v_0, \quad x = \tfrac{1}{2}gt^2 + v_0 t, \quad \text{and} \quad v^2 = 2gx + v_0^2.$$

3. If a body is projected up with a velocity v_0, show that

$$v = gt - v_0, \quad x = \tfrac{1}{2}gt^2 - v_0 t, \quad \text{and} \quad v^2 = 2gx + v_0^2.$$

179. Simple Harmonic Motion.—This may be defined as a rectilinear motion in which the acceleration of the moving point is proportional to its distance from an origin in the path and is always directed from the point to the origin. It may also be defined thus: If a point travels in a circle describing equal distances in all equal intervals of time, then the motion of the projection of the point on any diameter of the circle is a simple harmonic one. We will choose the latter definition and show in the sequel that it is in accord with the former.

Imagine P (fig. 152*a*) to start from P_0 and move uniformly in the circle in the counter-clockwise direction. The motion of the projection of P on O_1Y is harmonic and will now be discussed.

Let ε denote the angle XO_1P_0, the *lead* (*lag* if negative);
$(\theta + \varepsilon)$ " " " XO_1P, the *phase angle* at time t;
y " " ordinate O_1V, the *displacement;*
T " " time of one revolution of P, the *period;*
n " " number of revolutions per unit time, the *frequency;*
r " " radius of the circle, the *amplitude;*
v " " velocity of the projection (V) of P;
t " " time elapsed after starting.

If θ is expressed in radians, it is plain that

$$\theta = \frac{2\pi}{T}t = 2\pi nt = \omega t,$$

ω being an abbreviation for $2\pi/T$ and $2\pi n$; ω may be defined also as the angle described by O_1P per unit time. From the figure it will be seen that

$$y = r \sin(\omega t + \varepsilon), \quad \ldots \ldots \quad (1)$$

and since $v = dy/dt$,

$$\left.\begin{aligned} v &= \omega r \cos(\omega t + \varepsilon) = \omega r \sin(\omega t + \overline{\varepsilon + \pi/2}) \\ &= \omega x = \omega(r^2 - y^2)^{\frac{1}{2}}. \end{aligned}\right\} \quad \ldots \quad (2)$$

Since $a = dv/dt$,

$$\left.\begin{aligned} a &= -\omega^2 r \sin(\omega t + \varepsilon) = \omega^2 r \sin(\omega t + \overline{\varepsilon + \pi}) \\ &= -\omega^2 y. \end{aligned}\right\} \quad \ldots \quad (3)$$

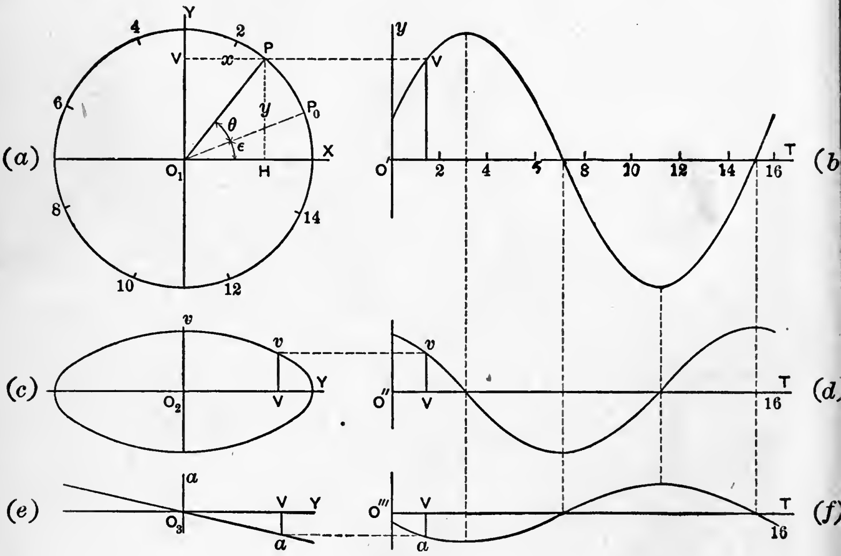


FIG. 152.

Equation (1) is represented in fig. 152(*b*). The curve is a sinusoid, and since the ordinates and abscissas denote "displacement" and time respectively, it is a displacement-time, or a space-time curve. The curve may be constructed as follows: Having drawn the "circle of reference" (fig. 152*a*) and having fixed the initial position of P, divide the circumference into a number of equal parts (16 is convenient) beginning with P_0, and number that point 0 and the others successively in the direction of the motion of P, 1, 2, etc., 16 coinciding with 0. Then on the t axis (fig. 152b) lay off any convenient length to represent the period, divide it in 16 equal parts, and erect ordinates at the points of division. Number these beginning with $O'y$, 0, 1, 2, 3, etc., the last being 16. Then project points 0, 1, 2, etc., of the circumference of the circle upon the corresponding ordinates. The curve through the projections is the displacement-time curve.

Equations (2) are represented in fig. 152 (c) and (d), (c) representing $v=\omega x$ and (d) the other one. The equation $v=\omega x$ shows that the velocity at any displacement y equals ω times the corresponding value of x. Hence to construct the curve in (c) lay off O_2V equal to O_1V in (a) and make Vv equal to $\omega\overline{O_1H}$, laying it off up or down according as O_1H (x) is positive or negative. Repeat this construction for several positions of P and thus determine the curve.

The curve in (d) is the velocity-time curve for the harmonic motion and may be constructed in various ways: for instance, lay off $O''16$ to represent the period, divide it into sixteen equal parts, and number the points of division beginning with O'', 0, 1, 2, etc. At these points erect values of the velocity which may be obtained from (c) by obvious methods.

It is worth noting that the equation $v=\omega r \sin(\omega t+\overline{\varepsilon+\pi/2})$ is very similar to $y=r\sin(\omega t+\varepsilon)$, v, ωr, and $\varepsilon+\pi/2$ in the first replacing y, r, and ε in the second. Hence the variation in v is simply harmonic and can be represented by a harmonic motion whose period, phase, and amplitude are respectively equal to, 90° ahead of, and ω times the period, phase, and amplitude of the given motion.

Since (b) and (d) are y-t and v-t curves respectively and $v=dy/dt$, any ordinate in the latter figure should be equal (by scale) to the slope of the tangent to the curve in the former at the corresponding point. Thus the ordinate Vv in (d) should equal the slope of the tangent at V in (b).

Equations (3) are represented in fig. 152 (e) and (f), (e) representing $a=-\omega^2y$ and (f) the other one. Equation $a=-\omega^2y$ shows that the acceleration is proportional to the displacement (y) and that it is always opposite to the displacement in sign, i.e., it is always directed from the moving point toward the middle of the path. Hence the two definitions of simple harmonic motion previously given agree.

To construct the curve in (*e*), lay off O_3V equal to y and make Va equal to $\omega^2 y$, laying it off down or up according as y is positive or negative. Repeat this construction for a number of positions of P and thus determine the line O_3a.

The curve in (*f*) is the acceleration-time curve for the harmonic motion and may be constructed in various ways: For instance, lay off $O'''16$ to represent the period, divide it into 16 equal parts, and number the points of division, beginning with O''', 0, 1, 2, etc. At these points draw ordinates to represent the corresponding values of the acceleration which may be obtained from (*e*) by obvious methods.

The equation $a = \omega^2 r \sin(\omega t + \overline{\varepsilon + \pi})$ is the same in form as $y = r \sin(\omega t + \varepsilon)$, a, $\omega^2 r$, and $(\varepsilon + \pi)$ in the first replacing y, r, and ε in the second. Hence the variation of a is analogous to that of y; in fact the variation in a is simply harmonic and it can be represented by a harmonic motion whose period, phase, and amplitude are respectively equal to, 180° ahead of, and ω^2 times the period, phase, and amplitude of the given motion.

Since (*d*) and (*f*) are *v-t* and *a-t* curves respectively and $a = dv/dt$, any ordinate in the latter figure equals (by scale) the slope of the tangent to the curve in the former at the corresponding point. Thus the ordinate Va in (*f*) equals the slope of the tangent at v in (*d*).

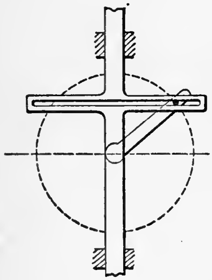


FIG. 153.

180. *Mechanism for Producing a Simple Harmonic Motion.*—The mechanism represented in fig. 153 consists of a crank and a slotted slider, the pin of the crank fitting the slot of the slider. As the crank is rotated the slider moves up or down, and plainly if the crank rotates uniformly every point of the slider executes a simple harmonic motion.

The space-time curve for the motion of the slotted slider can be automatically drawn as follows: Fasten a pencil to the slider so that it will mark on a plane surface as the slider moves, and then cause a sheet of paper to move uniformly over the surface in a direction at right angles to that of the motion of the pencil; the line traced on the paper is the space-time curve. If the mechanism for moving the paper is connected with the crank-shaft, then the curve traced by the pencil is a space-time curve for a harmonic motion whether the mechanism is driven uniformly or not.

181. Motion of the Piston of a Steam-engine.—Let OP (fig. 154) represent the crank and the connecting-rod of a steam-engine slider-crank mechanism, and suppose that the crank turns uniformly.

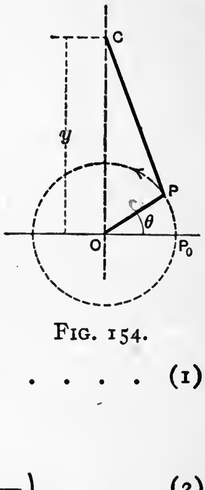


FIG. 154.

Let n denote number of revolutions of the crank per unit time;
c " length of crank;
r " " " connecting-rod;
y " the distance of C from O;
t " " time required to describe the angle P_0OP.

Then $\theta = 2\pi nt = \omega t$, ω being an abbreviation for $2\pi n$, and

$$y = (r^2 - c^2 \cos^2 \omega t)^{\frac{1}{2}} + c \sin \omega t. \quad . \quad . \quad . \quad . \quad (1)$$

Hence the velocity of the piston is

$$\frac{dy}{dt} = c\omega\left(\cos \omega t + \frac{\sin 2\omega t}{2(r^2/c^2 - \cos^2 \omega t)^{\frac{1}{2}}}\right), \quad . \quad . \quad . \quad (2)$$

and its acceleration is

$$\frac{d^2y}{dt^2} = -c\omega^2\left(\sin \omega t + \frac{\cos^4 \omega t - (r^2/c^2) \cos 2\omega t}{(r^2/c^2 - \cos^2 \omega t)^{\frac{3}{2}}}\right). \quad . \quad (3)$$

Close approximate values of the velocity and acceleration can be found as follows: Since approximately

$$r\left(1 - \frac{c^2}{r^2} \cos^2 \omega t\right)^{\frac{1}{2}} = r\left(1 - \frac{c^2}{2r^2} \cos^2 \omega t\right),$$

$$y = \left(r - \frac{c^2}{4r}\right) - \frac{c^2}{4r} \sin (2\omega t + \pi/2) + c \sin \omega t, \text{ very nearly.} \quad (4)$$

Hence, approximately, the velocity of the piston is

$$\frac{dy}{dt}=c\omega[\cos \omega t-(c/2r) \cos (2\omega t+\pi/2)], \quad . \quad . \quad . \quad (5)$$

and the acceleration is

$$\frac{d^2y}{dt^2}=-c\omega^2 [\sin \omega t-(c/r) \sin (2\omega t+\pi/2)]. \quad . \quad . \quad (6)$$

If the connecting-rod were infinitely long, c/r would be zero and the second term of eq. (3) would vanish; hence the motion of the piston would be simply harmonic. The smaller the ratio c/r the more nearly is the motion of an actual mechanism simply harmonic.

EXAMPLE.

Take r/c equal to four and compute the values of the acceleration in terms of $c\omega^2$ from equations (3) and (6), when $\theta=0°$, $\pm 30°$, $\pm 60°$, and $\pm 90°$ (see fig. 154). Plot these values on the same base, draw the a-θ curves and compare. Draw also on the same base the curve $a=-c\omega^2 \sin \omega t$.

CHAPTER VIII.

CURVILINEAR MOTION.

§ I. VELOCITY AND ACCELERATION.

182. Specification of Position.—It is usually convenient herein to specify the position of a point in space by Cartesian coordinates, but for the purpose of this section it is more convenient to specify position by means of a vector. The vector drawn from a fixed origin of reference to the point to be located is called the *position-vector* of the point. For, if the vector is known the position of the point with reference to the origin is also known. Thus, the direction of the vector OP (fig. 155) fixes the direction of P from the origin O, the length of the vector fixes the distance of P from O, and thus the position of P is determined.

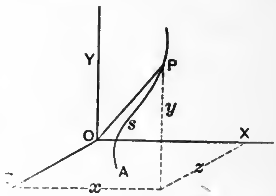


FIG. 155.

The position of a point in a given line can be specified by a single quantity: Thus if A is an origin of reference in the line AP, the abscissa s measured from A along the line fixes P, it being understood that s is positive for points on one side of A and negative for those on the other.

183. Space-Time Curve.—A curvilinear motion can be represented in part by means of a line, the space-time curve for the motion. The ordinate and abscissa of each point of it represent respectively simultaneous values of the position abscissa s of the moving point and of the time t. It is analogous to the space-time curve for a rectilinear motion (art. 164) and is similarly constructed.

This curve must not be confounded with the path; the latter may be tortuous, but the former is always a plane curve.

184. Displacement.—The displacement of a point during an interval in which it moves from A to B is defined to be the vector AB. The term therefore does not refer to the path actually described between A and B, but simply to the straight path between A and B. Observe that if O is any point of reference, vectorially

$$AB = OB - OA.$$

Two displacements must agree in direction as well as in magnitude in order to be equal.

185. Kinds of Motion.—The definitions under this title given in art. 166 refer to rectilinear motion; those following are general.

If the displacements of a moving point in equal intervals of time (large or small) are equal, its motion is called uniform; and if unequal, the motion is called non-uniform.

In order that all displacements may be equal, the path must be straight; hence a uniform motion as here defined must be rectilinear.

186. Velocity.—By velocity of a point is meant the rate with respect to time at which it changes position or at which its displacement occurs. Still otherwise stated, velocity is the time-rate (of change) of its position-vector.

Let P (fig. 156) be a point moving in the path APB, and O and O' points of reference; then OP is the position-vector of P, and s its position-abscissa. Now in general the rate of the vector OP, i.e., the velocity, changes with the time as shown in Appendix B, and when the moving point is at P the rate is a vector whose direction is that of the tangent at P, whose magnitude equals the time-rate of (change of) s.

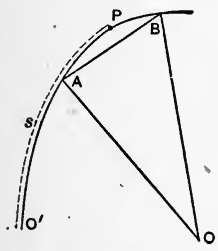


FIG. 156.

If the *magnitude* of the velocity* be denoted by v, then

$$v = ds/dt; \quad \dots\dots \quad (1)$$

and *if s varies uniformly*, ds/dt is constant and

$$v = \Delta s/\Delta t. \quad \dots\dots \quad (2)$$

* The word "speed" has been set apart by several recent writers to denote magnitude of a velocity. Such usage is very convenient and will be followed herein.

Observe that our definitions lead to the result that the *velocity* in a curvilinear motion cannot be constant, for although its magnitude might be constant its direction continually changes. This result is in accord with the definitions of uniform and non-uniform motion (art. 185).

EXAMPLE.

The point P (fig. 157) describes the circle in such a way that $s=2t^3$, s and t being in ft. and secs. respectively. (*a*) Deduce an expression for the speed at any time t. (*b*) What are the magnitude and direction of the velocity when $t=2$ secs.?

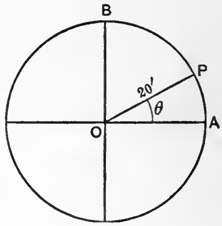


FIG. 157.

Solution: (*a*) Since the speed (v) equals ds/dt, from the equation of the motion, $v=6t^2$.

187. Speed-Time Curve and Hodograph.—The way in which the speed of a moving point varies can be represented by a line called the speed-time curve for the motion. The ordinate and abscissa of each point of it represent simultaneous values of the speed and the time respectively. It is analogous to the velocity-time curve for a rectilinear motion (art. 170) and is similarly constructed.

If from any point vectors be drawn which represent the velocities of a moving point and the free ends of all such vectors

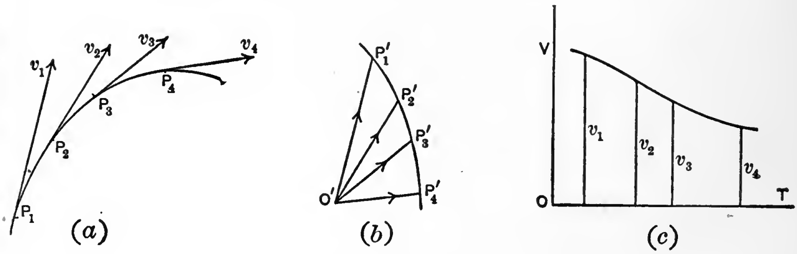


FIG. 158.

be joined, the joining line is called the *hodograph* for the motion. Thus let P_1, P_4 (fig. 158) be the path of a moving point, and let its velocities at the points P_1, P_2, etc., be as represented by

P_1v_1, P_2v_2, etc. The hodograph is $P_1'P_4'$, $O'P_1'$ being equal and parallel to P_1v_1, $O'P_2'$ parallel and equal to P_2v_2, etc.

A hodograph may be plane or tortuous, but a speed-time curve is always plane. The latter curve corresponding to fig. 158 (*a*) and (*b*) is shown at (*c*).

EXAMPLES.

1. Suppose that P (fig. 157) describes the circle with a constant speed of 10 ft.-per-sec. Draw a hodograph and a speed-time curve for the motion from A to B.

2. Draw a hodograph and a speed-time curve for a motion from A to B (fig. 157) according to the law stated in the ex. of art. 186, determining at least four points on the hodograph.

188. Acceleration.—By acceleration of a point is meant the time-rate (of change) of its velocity.

In deducing the expression for this rate it must be remembered that velocity is a vector quantity. According to Appendix B the rate is a vector, and if $A'P'$ (fig. 159) is a hodograph and $O'P'$ represents the velocity at any instant t, then the acceleration at that instant is a vector

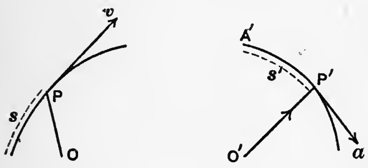


FIG. 159.

whose direction is that of the tangent to the hodograph at P',
whose magnitude is the time-rate of s',

s' being the distance of P' from any fixed point on the hodograph as origin.*

If a be used to denote the *magnitude* of the acceleration,

$$a = ds'/dt; \quad \ldots \ldots \quad (1)$$

and *if a varies uniformly*,

$$a = \Delta s'/\Delta t. \quad \ldots \ldots \quad (2)$$

Observe carefully that the tangents to the hodograph and to the path at corresponding points P and P' are not parallel or in

* As explained in the appendix referred to, the fixed point A' from which s' is measured is to be taken so that P' moves in the positive s' direction and the arrow on the tangent (giving the sense of the vector) points in the same direction.

any way related as to direction, but the vector representing the acceleration always points from P toward the side of the tangent on which the path lies.

EXAMPLE.

1. What is the acceleration of a point describing a circle of radius r with constant speed v?

Solution: Let AB (fig. 160) be the circular path and P the position of the moving point at any instant. When P is at A

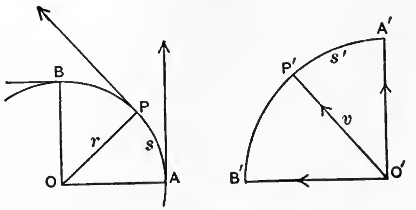


FIG. 160.

the velocity is represented by the vector $O'A'$, when at P by $O'P'$, when at B by $O'B'$, etc. Hence the hodograph is the circular arc $A'P'B'$, its radius being v. Now the angles POA and $P'O'A'$ are equal, and since the former equals s/r and the latter s'/v,

$$s/r = s'/v, \quad \text{or} \quad s' = sv/r.$$

Hence $$ds'/dt = (ds/dt)v/r = v^2/r.$$ Since $ds'/dt = a$,

$$a = v^2/r.$$

The direction of the acceleration of P is that of the tangent to the hodograph at P'. Since this tangent is parallel to the normal to the path at P,

> the acceleration of a point describing a circle with constant speed is directed along the radius of the circle drawn to the moving point, and its value equals the speed squared divided by the length of the radius.

§ II. Resolutions of Velocities and Accelerations.

189. Components and Resultant of Velocities or Accelerations Defined.—Velocities and accelerations are, as defined in arts. 186 and 188, vector quantities; they may therefore be resolved and compounded. The velocity (or acceleration) represented by the sum of the vectors representing any number of velocities (or accelerations) is called the resultant of those velocities (or accelerations). The velocities (or accelerations) represented by the components of a vector representing any velocity (or acceleration) are called the components of that velocity (or acceleration).

190. Axial Components of a Velocity.—These components are parallel to three rectangular axes (x, y, and z), and will be denoted by v_x, v_y, and v_z respectively.

Let v denote a velocity, its direction angles with the x, y, and z axes being α, β, and γ respectively. Then the x, y, and z components equal respectively $v \cos \alpha$, $v \cos \beta$, and $v \cos \gamma$. Since $v = ds/dt$, $\cos \alpha = dx/ds$, $\cos \beta = dy/ds$, and $\cos \gamma = dz/ds$,

$$\mathbf{v}_x = \mathbf{dx/dt}, \quad \mathbf{v}_y = \mathbf{dy/dt}, \quad \mathbf{v}_z = \mathbf{dz/dt}. \quad . \quad . \quad . \quad (1)$$

These equations show that the x, y, and z velocity components of a moving point are respectively the time-rates at which its distances from the yz, zx, and xy planes change, and that they equal respectively the velocities of the projections of the moving point on the axes x, y, and z.

191. Tangential and Normal Components of Velocity.—These components are parallel to the tangent and normal to the path at the point where the velocity is resolved. They will be denoted by v_t and v_n respectively. Since the velocity at any point in the path is directed along the tangent to the path at that point,

$$\mathbf{v}_t = \mathbf{v} \quad \text{and} \quad \mathbf{v}_n = \mathbf{0}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

192. Axial Components of Acceleration.—These components are parallel to three coordinate axes (x, y, and z), and will be denoted by a_x, a_y, and a_z respectively. For simplicity, the deduction of the expressions for the components is limited to the case of plane paths with the coordinate axes in the plane of the path, but the discussion can be extended to the general case (resolution of the acceleration of a point along three axes, the path being tortuous).

Let PQ (fig. 161) be the path of a moving point and $P'Q'$ the hodograph of the motion, P and P' being any two corresponding

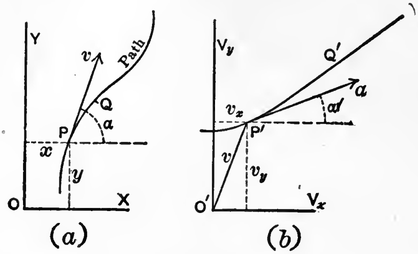


FIG. 161.

points on the path and hodograph respectively. Also let v and α denote the magnitude of the velocity at P and its angle with the x axis respectively. Then the *polar* coordinates of P' are v and α, and its rectangular coordinates are $v \cos \alpha$ and $v \sin \alpha$, or v_x and v_y; hence the coordinate axes in fig. 161(*b*) are marked $O'V_x$ and $O'V_y$ instead of OX and OY.

As shown in art. 188, the acceleration at P is directed along the tangent to the hodograph at P'; hence $P'a$ may represent the acceleration. Now if a, a_x, and a_y denote the acceleration and its x and y components respectively, it follows from the figure that

$$a_x = a \cos \alpha' \quad \text{and} \quad a_y = a \sin \alpha';$$

and from the analogy between the two parts of the figure,

$$\cos \alpha' = dv_x/ds' \quad \text{and} \quad \sin \alpha' = dv_y/ds',$$

ds' being the length of the infinitesimal arc on the hodograph at P'. Also, from art. 188, $a = ds'/dt$; hence

$$a_x = (ds'/dt)(dv_x/ds') = dv_x/dt,$$

and $$a_y = (ds'/dt)(dv_y/ds') = dv_y/dt.$$

In the general case (resolution into three components) it can be shown that the z component of the acceleration (a_z) equals dv_z/dt; hence

$$\left.\begin{aligned} \mathbf{a}_x &= \mathbf{dv}_x/\mathbf{dt} = \mathbf{d^2x/dt^2}; \\ \mathbf{a}_y &= \mathbf{dv}_y/\mathbf{dt} = \mathbf{d^2y/dt^2}; \\ \mathbf{a}_z &= \mathbf{dv}_z/\mathbf{dt} = \mathbf{d^2z/dt^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equations state that the x, y, and z components of the acceleration of a moving point P equal the time-rates of the x, y, and z components of its velocity; also that they equal the accelerations of the projections of the point on the x, y, and z axes respectively.

193. Tangential and Normal Components of Acceleration.— These are components whose directions are parallel to the tangent and normal to the path at the point where the acceleration is resolved; they will be denoted by a_t and a_n respectively. For simplicity, the deduction below is limited to the case of a plane path, but it might be extended to the general case. Let a denote the acceleration of a moving point when it arrives at P (fig. 162). Since the values of the tangential and normal components of the acceleration are independent of the axes of reference,* these may be chosen parallel to the tangent and normal at P. This is done solely for simplicity in the deduction of the values of a_t and a_n. *With axes thus chosen*, it is plain that the tangential and x components and the normal and y components are equal, or

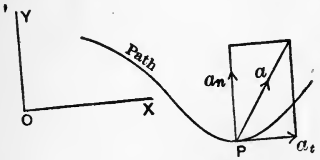


FIG. 162.

$$a_t = a_x = dv_x/dt, \quad a_n = a_y = d^2y/dt^2.$$

Also $d^2y/dt^2 = (d^2y/dx^2)(dx^2/dt^2) = (d^2y/dx^2)v_x^2$; and d^2y/dx^2 equals the curvature of the path at P. Denoting the radius of curvature at P by ρ, and since $v_x = v_t = v$,

$$\mathbf{a}_t = \mathbf{dv/dt} = \mathbf{d^2s/dt^2}, \quad \text{and} \quad \mathbf{a}_n = \mathbf{v}^2/\rho. \quad . \quad . \quad . \quad (1)$$

* This independence may be explained thus: From fig. 161 it is plain that $a_t = a \cos (\alpha - \alpha')$ and $a_n = a \sin (\alpha - \alpha')$. Now both a and $(\alpha - \alpha')$ are independent of the coordinate axes and hence a_t and a_n are also.

These equations state that the tangential and normal accelerations at P (any point in the path) equal respectively the time-rate (of change) of the speed at P and the square of the speed divided by the radius of curvature at P.

If the moving point travels in a circle of radius r and with a constant speed v, then $dv/dt=0$ and

$$\mathbf{a}=\mathbf{a}_n=\mathbf{v}^2/\mathbf{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLES.

1. A point P (fig. 163a) starts from a point X and moves in a circle of 20 ft. radius and so that the distance described (in feet) equals twice the cube of the time (in seconds) after starting. Compute the value of v, v_t, v_n, v_x, v_y, a, a_t, a_n, a_x, and a_y, when $t=2$ secs.

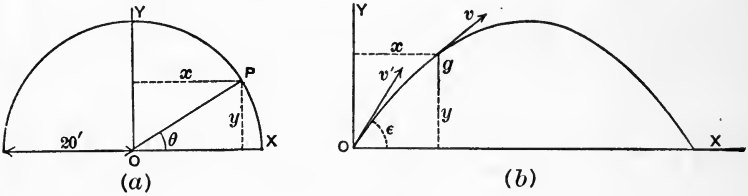


FIG. 163.

Solution: From the equation of motion, $s=2t^3$;

$$\therefore ds/dt=6t^2=v, \quad d^2s/dt^2=12t=a_t; \quad \text{also,} \quad a_n=36t^4/20.$$

Hence, when $t=2$ secs.,

$$v=24 \text{ ft./sec.}, \qquad v_t=24 \text{ ft./sec.}, \quad v_n=0,$$
$$a_t=24 \text{ ft./sec.}^2, \quad \text{and} \quad a_n=28.8 \text{ ft./sec.}^2.$$

Since $x=20\cos\theta$ and $y=20\sin\theta$,

$$dx/dt=-20\sin\theta\, d\theta/dt=v_x, \quad dy/dt=20\cos\theta\, d\theta/dt=v_y,$$
$$d^2x/dt^2=-20\sin\theta\, d^2\theta/dt^2-20(d\theta/dt)^2\cos\theta=a_x,$$

and

$$d^2y/dt^2=20\cos\theta\, d^2\theta/dt^2-20(d\theta/dt)^2\sin\theta=a_y.$$

Since $\theta=s/20=0.1t^3$, $d\theta/dt=0.3t^2$, and $d^2\theta/dt^2=0.6t$. By means of these values and the expressions for v_x, v_y, a_x, and a_y, we find that when $t=2$ secs.,

$$v_x = -17.21 \quad \text{and} \quad v_y = 16.7 \text{ ft./sec.},$$
$$a_x = -37.2 \quad \text{and} \quad a_y = -3.9 \text{ ft./sec.}^2.$$

The determination of the magnitude and direction of a is left to the student; he should also plot to scale, in the figure, all the computed values.

2. Discuss the motion of the centre of gravity of a projectile neglecting the resistance of the air if the velocity and angle of projection are v' and ε respectively (see fig. 163*b*).

Solution: It is shown in art. 240 that the acceleration of the centre of gravity at each instant is just like that of a freely falling body, i.e., equal to g and vertically downward; hence

$$a_x = 0 \quad \text{and} \quad a_y = -g. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since $d^2x/dt^2 = 0$, $v_x = C_1$ and $x = C_1t + C_2$, C_1 and C_2 being constants of integration. Since v_x is constant during the entire motion, it is equal to its initial value, i.e., $C_1 = v' \cos \varepsilon$, and since $x = 0$ when $t = 0$, $C_2 = 0$; hence

$$v_x = v' \cos \varepsilon \quad \text{and} \quad x = v' \cos \varepsilon \cdot t. \quad . \quad . \quad . \quad . \quad (2)$$

Since $d^2y/dt^2 = -g$, $v_y = -gt + C_3$ and $y = -\frac{1}{2}gt^2 + C_3t + C_4$, C_3 and C_4 being constants of integration determinable like C_1 and C_2 from "initial conditions." These are $y = 0$ and $v_y = v' \sin \varepsilon$ when $t = 0$; hence $C_3 = v' \sin \varepsilon$ and $C_4 = 0$, and therefore

$$v_y = -gt + v' \sin \varepsilon \quad \text{and} \quad y = -\tfrac{1}{2}gt^2 + v' \sin \varepsilon \cdot t. \quad . \quad (3)$$

3. Show that the path is parabolic and that its equation is

$$y = x \tan \varepsilon - gx^2/2(v' \cos \varepsilon)^2.$$

4. Deduce expressions for the range and greatest height reached, and show what values of ε make them a maximum.

§ III. RELATIVITY OF MOTION.

194. Path, Displacement, Velocity, and Acceleration of a Point Relative to a Body.—Position of a point can be specified only by means of a set of reference axes or some other equivalent base. When we speak of the position of a point with respect to a body we would specify that position relative to some reference base in the body; thus to specify the position of a bug on a drawing-board, say, we might describe it as being on the upper surface of the board, two feet above the lower edge and three feet from the left edge. The fact that specification of position is necessarily relative makes the path, displacement, velocity, and acceleration of a moving point also relative, as can be shown clearly by illustration.

Imagine a drawing-board with a sheet of paper lying upon it, and a bug upon the paper. Suppose that the paper is slid over the board and that the bug walks about on the paper and punches holes rapidly through the paper into the board. The successions of holes in the paper and board mark out his paths relative to the paper and board respectively. Evidently these paths would, in general, be very dissimilar.

Let P_1 and B_1 denote the holes punched into the paper and board at any instant, and P_2 and B_2 those punched at a later instant. Then the vectors P_1B_1 and P_2B_2 are the displacements of the bug relative to the paper and the board respectively for the interval of time. Evidently these two displacements would, in general, be very dissimilar. The rates at which the displacements relative to the paper and the board occur are the velocities of the bug relative to the paper and the board respectively. Since these displacements for *any* interval of time are, in general, unlike, the velocities at any instant are also unlike. Furthermore, the changes or increments in the velocities for any interval are unlike also.

The rates at which the velocities relative to the paper and the board change are the accelerations of the bug relative to the paper and the board respectively. Since the changes in the velocities for *any* interval of time are, in general, unlike, the accelerations at any instant are also unlike.

195. Path, Displacement, Velocity, and Acceleration of a Point Relative to Another Point.—By path, displacement, velocity, and acceleration of a point relative to a second point is meant the path, displacement, velocity, and acceleration of the point relative to a set of axes having fixed directions through the second point.

If a bug walks about on a sheet of paper whose edges are kept fixed in direction, then its motions relative to the paper and any point of the same are alike; but if the edges are not kept fixed in direction, then those motions are unlike. Thus imagine two sheets of paper pivoted together at some point O, that they are slid over a flat surface in any way except that the edges of the lower sheet remain fixed in direction, and that a punching bug walks about on the upper sheet. The holes punched into the lower sheet mark the path of the bug relative to that sheet and to any point of the sheet. The holes punched in the upper sheet mark the path relative to that sheet; this path would in general be unlike the first, which is the path of the bug relative to the point O of the upper sheet.

196. Relation between the Velocities and the Accelerations of (a) Two Points and (b) Three Points.—(*a*) *Proposition.*—The velocities and the accelerations of two points relative to each other are equal and opposite.

Proof for uniplanar motion of the points: Imagine two sheets of paper being shifted about on a flat surface, their edges remaining fixed in direction. Also imagine punching bugs fixed at the middles of the sheets and call the one on the upper sheet A and the other B; then the successions of holes punched by A and B (into the lower and upper sheets respectively) mark the paths of A relative to B and B relative to A. Let (*a*) and (*b*) (fig. 164) represent the positions of the sheets at the beginning and end of any motion; then A_1 and B_1 are the first positions of the bugs A and B, and A_2 and B_2 are their last positions. If $A_2A_1B_1B'$ and $B_2B_1A_1A'$ are parallelograms, then B' and A' are the final positions of the first holes punched into the upper and lower sheets respectively, and hence the displacements of A relative to B and B relative to A are the vectors $A'A_2$ and $B'B_2$. Since the sides A_2B' and B_2A' of the quadrilateral $A_2B'B_2A'$ are equal and parallel, the mentioned vectors (and displacements) are equal; plainly they are opposite.

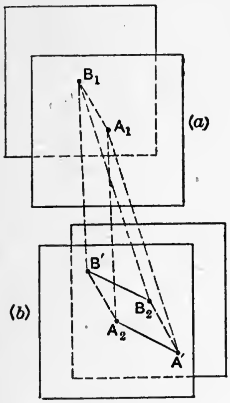


FIG. 164.

It follows that the displacements per unit time, and hence the velocities, of the two points are equal and opposite. The velocities being at each instant equal and opposite, it follows that the changes in those velocities for any interval and the velocity changes per unit time (hence the accelerations also) must be equal and opposite.

(*b*) *Proposition.*—The velocity (and acceleration) of A relative to B plus that of B relative to C equals that of A relative to C.

Proof for uniplanar motion of the three points: Imagine a sheet of paper slid about on a drawing-board which is also slid about on a flat surface, the edges of the paper and board being fixed in direction. Imagine also a punching bug walking about on the paper; let it be A, a corner of the paper B, and a corner of the board C. Suppose that at the beginning of some interval the board, paper, and bug are at C_1c_1, B_1b_1, and A_1 (fig. 165) respectively, and at the end of that interval they are at C_2c_2, B_2b_2, and A_2. If $C_2C_1B_1B'$, $B'B_1A_1A'$, and $B_2B'A'A''$ are parallelograms, it follows respectively that $B'B_2$ is the displacement of B relative to C; that A' is the final position of the first hole punched in the board, and hence the displacement of A relative to C is $A'A_2$; and that A'' is the final position of the first hole punched in the paper, and hence $A''A_2$ is the displacement of A relative to B. It is plain from the figure that vectorially $A''A_2+B'B_2=A'A_2$, i.e., the displacement of A relative to B plus that of B relative to C equals that of A relative to C.

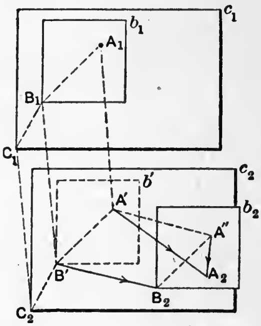


FIG. 165.

It follows that the displacement per unit time, and hence the velocities, are similarly related. The velocities at each instant being thus related, it follows that the changes in those velocities in any interval and the velocity changes per unit time (hence the accelerations also) are so related.

197. Meaning of Composition of Motions.—According to the preceding article the displacement of the bug (of the first illustration of art. 194) with respect to a point C on the board may be found by compounding its displacement with respect to a point B on the paper and that of B with respect to C. Hence we say that *the motion* of a point A with respect to a point C may be regarded as consisting of the motion of A with respect to B and that of B with respect to C.

EXAMPLES.

1. A man rows a boat in a stream whose velocity is 3 miles per hour, so that his velocity relative to a floating chip is 5 mi.-per-hr. "straight across" the stream. Determine his absolute velocity.*

2. Two locomotives run at speeds of 40 and 50 mi.-per-hr. on two different tracks, that of the first locomotive being east and west and the other northeast and southwest. If both locomotives run eastward, what is the velocity of each relative to the other?

§ IV. Composition of Simple Harmonic Motions.

198. Mechanism for Compounding Simple Harmonic Motions.—Fig. 166(a) represents two cranks with their slotted sliders, the crank-shaft of one turning in a bearing fixed upon the slider of the second. If both cranks be turned uniformly, the lower

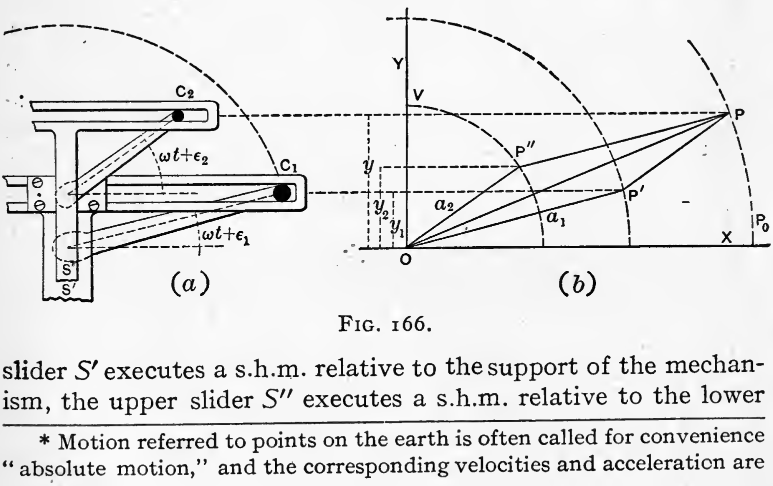


FIG. 166.

slider S' executes a s.h.m. relative to the support of the mechanism, the upper slider S'' executes a s.h.m. relative to the lower

* Motion referred to points on the earth is often called for convenience "absolute motion," and the corresponding velocities and acceleration are also called absolute.

slider, and its motion relative to the fixed support is compounded of these two harmonic motions. In theory at least, a third crank and slider could be mounted on the second slider, a fourth crank and slider on the third slider, etc. Then if all cranks were turned uniformly, the motion of the last slider would be compounded of the simple harmonic motions executed by the several sliders relative to the supports of their cranks.

When the sliders are in line the s.h.m.'s are described as "along the same line" or "collinear," and when the sliders are inclined to each other the s.h.m.'s may be described as "oblique" or "non-parallel."

199. Composition of Two Collinear Simple Harmonic Motions of the Same Period.*—Let fig. 166 represent the position of the mechanism at any time t of the motion. We will consider the motion of the projection of C_2 on OY, i.e., V.

From O draw vectors OP' and OP'' to represent the cranks as shown, and complete the parallelogram $OP'P''P$; then OP is the sum of the vectors representing the cranks. It is plain from the figure that V is also the projection of P on the line OY. Since the periods of the two s.h.m.'s are the same, the angle $P'OP''$ remains constant during the motion; hence OP is constant in length and turns uniformly, i.e., describes a circle at a uniform rate. Therefore the motion of V is simply harmonic.

The amplitude of the motion of V is OP, the phase is XOP, the period is the same as that of the given s.h.m.'s. The epoch can be found from the figure as follows: Turn the parallelogram back to its position when $t=0$; let P_0 be the corresponding position of P, then XOP_0 is the desired epoch.

Values of the amplitude and epoch can be computed from the figure; thus let

a_1	denote	the	amplitude	of	the	first	s.h.m.,
a_2	"	"	"	"	"	second	"
ε_1	"	"	epoch	"	"	first	"
ε_2	"	"	"	"	"	second	"
y_1	"	"	displacement	"	"	first	"
y_2	"	"	"	"	"	second	"

* Only motions of the same period are compounded herein.

and $\qquad 2\pi/\omega$ their common period;

then $\qquad y_1 = a_1 \sin(\omega t + \varepsilon_1)$ and $y_2 = a_2 \sin(\omega t + \varepsilon_2)$.

Since the motion of V is simply harmonic, period equal to $2\pi/\omega$, its displacement is given by

$$y = a \sin(\omega t + \varepsilon),$$

a and ε being the amplitude and epoch respectively. It can be shown from the figure that

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\varepsilon_2 - \varepsilon_1),$$

and $\qquad \tan \varepsilon = (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2)/(a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2)$.

200. Resolution of a Simple Harmonic Motion into Two Components Collinear with it.—Let OP (fig. 166) represent the crank of any s.h.m. in the line OY, and let OP' and OP'' be the cranks of two others in the same line. According to the preceding article the resultant of the second two gives the first and the latter are therefore called components of the first.

Obviously a s.h.m. may be resolved into many pairs of components, for many parallelograms, as $OP'P''P$, can be drawn on the same diagonal OP, and the sides OP' and OP'' of each represent the cranks of a pair of components.

Special Case.—Resolution into two components which differ 90° in phase, the epoch of one being zero. Let OP (fig. 167) represent the crank of the s.h.m. to be resolved when $t=0$; then XOP is the epoch of that motion and OP' and OP'' are the cranks of the component motions when $t=0$; the epoch of one component (crank OP') is zero and that of the other is 90°. Also if $OP=a$ and $XOP=\varepsilon$, the equation of the given s.h.m. is

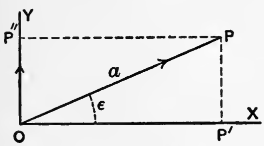


FIG. 167.

$$y = a \sin(\omega t + \varepsilon),$$

and those of the first and second components are

$$y' = a \cos \varepsilon \cdot \sin \omega t,$$

and $$y'' = a \sin \varepsilon(\sin \omega t + 90°) = a \sin \varepsilon \cdot \cos \omega t.$$

201. Composition of Many Collinear Simple Harmonic Motions of Equal Periods.—It follows from art. 199 that the resultant of three or more collinear s.h.m.'s of equal period is simply harmonic and of that period. For the first two can be replaced by a single s.h.m., and in turn this one and the third can be replaced by a single s.h.m., etc.

To obtain the "crank" of the resultant motion, add the vectors representing the several cranks in their positions at any instant; the sum represents the crank of the resultant motion in its corresponding position. Thus, let OP', $P'P''$, $P''P'''$, etc. (fig. 168), represent the cranks in their simultaneous positions;

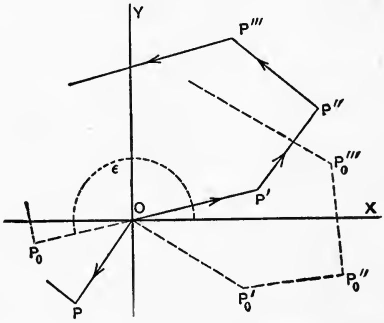


FIG. 168.

then OP represents the length and direction of the crank of the resultant motion at the same instant.

The epoch can be determined by turning the polygon about O until P' falls into its position when t was zero. Let P_0 be the corresponding position of P; then XOP_0 is the epoch sought.

EXAMPLES.

1. Compound two collinear s.h.m.'s whose amplitudes and periods are equal, their phases differing (*a*) by 90°, (*b*) by 180°, (*c*) by 270°.

2. Resolve the s.h.m. whose equation is $y=4 \sin (2\pi t+30°)$ into two collinear with it, their phase difference being 45° and the epoch of one being zero.

3. Compound three collinear s.h.m.'s whose amplitudes and periods are equal, their phases differing by 120°.

202. Composition of Two Simple Harmonic Motions in Lines at Right Angles.—Imagine the two sliders of fig. 166 turned at right angles as represented in fig. 169; then if the absolute motion of the lower one is simply harmonic and the motion of the upper one relative to the lower is also simply harmonic, the absolute motion of the upper is compounded of two s.h.m.'s at right angles.

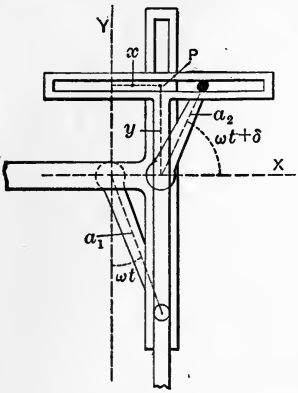


FIG. 169.

For simplicity, we choose the origin of time so that the epoch of the first s.h.m. is zero and consider the motion of a point P at the middle of the slot of the second slider. With notation and axes as in the figure, the x and y coordinates of P are

$$x=a_1 \sin \omega t \quad \text{and} \quad y=a_2 \sin (\omega t+\delta). \quad . \quad . \quad . \quad (1)$$

The equation of the path of P is found from these by eliminating t; thus we find that

$$\frac{x^2}{a_1^2}-\frac{2 \cos \delta}{a_1 a_2} xy+\frac{y^2}{a_2^2}=\sin^2 \delta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This represents an ellipse and hence the motion has been called "elliptic harmonic motion."

The path of a point whose motion is the resultant of two s.h.m.'s in lines at right angles (or inclined) can be traced graphically by plotting on the two lines simultaneous values of the displacements due to the component motions and then fixing the position of the point from the displacements. Thus let the circles of fig. 170 be the circles of reference of the two component s.h.m.'s, and let $Y'Oa_0$ be the epoch of the horizontal and XOb_0 that of the vertical component motion; then the displacements in the two motions when $t=0$ are respectively Ox_0 and Oy_0, and the position of the point describing the elliptic motion is c_0. The constructions for the other points on the ellipse should be obvious from the figure.

Special Cases.—(1) If the phases of the component motions are the same or differ by 180°, $\sin \delta = 0$; hence according to eq. (2) the path is a straight line. The resultant motion is a s.h.m., its period equals that of the given motions, its phase is

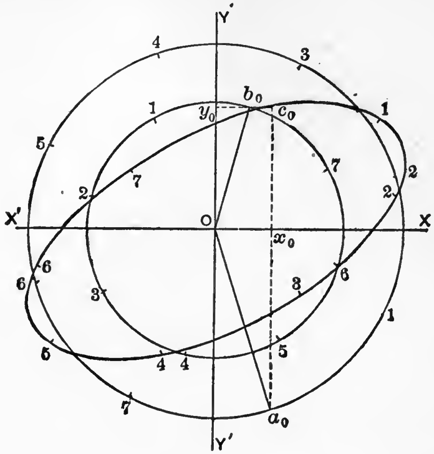


FIG. 170.

the same as that of one or both of the given motions, and its amplitude equals $(a_1^2 + a_2^2)^{\frac{1}{2}}$. Prove.

(2) If the amplitudes of the component motions are equal and they differ in phase by 90°, $a_1 = a_2$, $\cos \delta = 0$, and $\sin \delta = 1$; hence according to eq. (2) the path is a circle whose radius equals $a_1 = a_2$. The circular motion is "uniform," of a period equal to that of the component motions. Prove.

203. Resolution of a S.H.M. into Two Components at Right Angles to Each Other.—Let OZ (fig. 171) be the path of the motion to be resolved, O being the centre, and let the displacement, amplitude, period, and epoch be z, a, $2\pi/\omega$, and ε respectively; then

$$z = a \sin (\omega t + \varepsilon).$$

The x and y components of this displacement are (see the figure),

$$x = (a \cos \alpha) \sin (\omega t + \varepsilon)$$

and $$y = (a \sin \alpha) \sin (\omega t + \varepsilon);$$

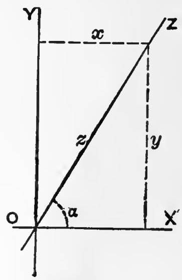


FIG. 171.

hence the periods and epochs of the components are the same as of the given motion, and their amplitudes are respectively $a \cos \alpha$ and $a \sin \alpha$.

204. Composition of More than Two S.H.M.'s Not Collinear.—According to the preceding article each s.h.m. may be resolved and replaced by two components along two axes x and y. According to art. 201 all the components in the x axis can be compounded into a single s.h.m., and those in the y axis also. According to art. 202 these two s.h.m.'s compound, in general, into an elliptic harmonic motion.

Special Case.—Three s.h.m.'s in lines inclined 120° to each other of equal amplitudes and periods but differing 120° in phase, compound into a uniform circular motion.

Proof: Let $z'z''$ and z''' denote simultaneous displacements in the component motions and a and $2\pi/\omega$ their common amplitude and period; then

$$z' = a \sin \omega t,$$
$$z'' = a \sin (\omega t + 120°),$$
$$z''' = a \sin (\omega t + 240°).$$

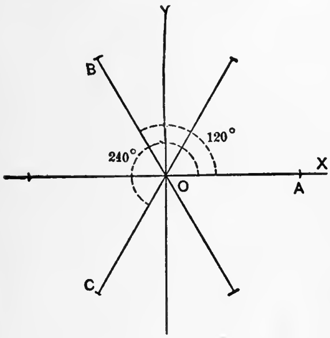


FIG. 172.

Let OA, OB, and OC (fig. 172) be the paths of the component motions, O being the centre of each, x', x'', and x''' the x components of the displacements, and y', y'', and y''' their y components. Then, according to the preceding article,

$$x' = a \sin (\omega t), \qquad y' = 0,$$
$$x'' = a \cos 120° \cdot \sin (\omega t + 120°), \quad y'' = a \sin 120° \cdot \sin (\omega t + 120°),$$
$$x''' = a \cos 240° \cdot \sin (\omega t + 240°), \quad y''' = a \sin 240° \cdot \sin (\omega t + 240°);$$

hence the sums of the x and y components (which call x and y) are

$$x = \tfrac{3}{2}a \sin \omega t, \quad \text{and} \quad y = \tfrac{3}{2}a \cos \omega t = \tfrac{3}{2} \sin (\omega t + 90°).$$

These two equations show that the given motions are equivalent to two s.h.m.'s in the coördinate axes, equal as to amplitudes and periods but differing 90° in phase. According to special case (2), art. 202, these two motions are equivalent to a uniform motion in a circle of radius $\tfrac{3}{2}a$, its period being equal to that of the given motions.

CHAPTER IX.

MOTION OF A RIGID BODY.

§ I. Translation.

205. Translation Defined.—A translation is a motion in which each straight line of the moving body remains fixed in direction. Thus the motions of the coupling-rods of a locomotive which runs on a straight track are translations; also the motions of those on a locomotive which runs on a transfer-table.

206. Motions of all Points of a Body in Translation are Alike.—Let A and B be any two points of a body having a translatory motion, A' and B' their positions at a given instant, and A'' and B'' those at another instant. By definition, the lines $A'B'$ and $A''B''$ are parallel, and, since they are also equal in length, the figure $A'B'A''B''$ is a parallelogram and $A'A''$ and $B'B''$ are equal and parallel. Hence the displacements of all points of the moving body for the same interval of time (long or short) are equal in magnitude and the same in direction. It follows that at each instant the velocities, and hence the accelerations, of all points of the moving body are exactly alike.

207. Velocity and Acceleration of the Body.—By velocity and acceleration of a body having a translatory motion is meant the velocity and the acceleration respectively of any one of its points.

§ II. Rotation.

208. Rotation Defined.—A rotation is a motion in which one line of the moving body or of its extension remains fixed. The fixed line is called the *axis* of the rotation.

Obviously all points of the moving body must describe circles whose centres lie on the axis unless the axis cuts the body; in this case all points of the body on that line are at rest, the others describing circles. The planes of the circles are perpendicular to the axis, and any plane perpendicular to the axis may be called the *plane of rotation.* All points of the body on any line parallel to the axis move alike; hence the motion of the projection of the line on the plane of rotation represents that of all the points, and the motion of the body itself is represented by the motion of its projection.

209. Angular Displacement.—By angular displacement of a rotating body during any time interval is meant the angle described during that interval by any line of the body perpendicular to the axis. Obviously all such lines describe equal angles in the same interval, and we select a line which cuts the axis. For convenience, angular displacements are given sign—positive if during the interval the body has turned counter-clockwise, and negative if clockwise.

Let the irregular outline (fig. 173) represent a rotating body, the plane of rotation being that of the paper, and O the intersection of the axis with that plane. Let P be any point and θ the angle XOP, OX being any fixed line of reference. As customarily, θ is regarded as positive or negative according as OX when turned about O toward OP moves counter-clockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any rotation, then the

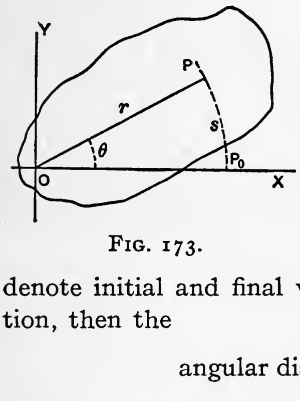


FIG. 173.

$$\text{angular displacement} = \theta_2 - \theta_1 = \Delta\theta.$$

210. Angular Velocity.—The angular velocity of a rotating body is the time-rate at which its angular displacement occurs; or, otherwise stated, it is the time-rate at which any line of the body perpendicular to the axis describes angle.

The time-rate at which OP (fig. 173) describes angle or the time-rate (of change) of θ is, as shown in works on calculus and in Appendix B, $d\theta/dt$. Hence if ω denotes angular velocity,

$$\omega = d\theta/dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the body turns uniformly, θ is a uniform variable, and its time-rate is $\Delta\theta/\Delta t$, $\Delta\theta$ denoting the angular displacement occurring in the interval Δt. Hence

$$\omega = \Delta\theta/\Delta t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the angular velocity is constant.

211. *Units of Angular Velocity.*—The formulas of the preceding article imply as unit an angular velocity corresponding to a unit angular displacement in each unit time, the velocity being constant. There are several such units; thus, one revolution-per-second, one degree-per-hour, one radian-per-second, etc. The last is the one usually used herein.*

212. *Sign of Angular Velocity.*—An angular velocity must be regarded as having sign, the same as that of $d\theta/dt$ (and of $\Delta\theta/\Delta t$ if the angular velocity is constant). Now $d\theta/dt$ and $\Delta\theta/\Delta t$ are positive or negative according as θ increases or decreases algebraically; hence

> The angular velocity of a rotating body at any instant is positive or negative according as it is turning in the counter-clockwise or clockwise direction at that time.

213. Angular Acceleration.—The angular acceleration of a rotating body is the time-rate (of change) of its angular velocity.

If, as in the preceding, ω denotes the angular velocity, then the general expression for the time-rate of the angular velocity is $d\omega/dt$; hence if α denotes the angular acceleration,

$$\alpha = d\omega/dt = d^2\theta/dt^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the angular velocity changes uniformly, its time-rate is $\Delta\omega/\Delta t$, $\Delta\omega$ denoting the increment in the velocity for any interval Δt; hence

$$\alpha = \Delta\omega/\Delta t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the angular acceleration is constant.

214. *Units of Angular Acceleration.*—The formulas of the preceding article imply as unit an angular acceleration corresponding to a unit angular velocity change in each unit time, the angular acceleration being constant. One revolution-per-second-

* For dimensions of a unit angular velocity, see Appendix C.

per-second, one radian-per-second-per-second, etc., are such units.*

215. *Sign of Angular Acceleration.*—An angular acceleration must be regarded as having sign—the same as that of $d\omega/dt$ (and of $\Delta\omega/\Delta t$ if the angular velocity changes uniformly). Now $d\omega/dt$ and $\Delta\omega/\Delta t$ are positive or negative according as ω increases or decreases algebraically; hence

> An angular acceleration is positive or negative according as the angular velocity is increasing or decreasing (algebraically).

216. Velocity and Acceleration of Any Point of a Rotating Body.—Let P (fig. 173) be any point of the rotating body there represented, let r denote its distance from the axis and s the length of the arc P_0P. Then *if θ is expressed in radians*, $s=r\theta$; hence

$$ds/dt=r\,d\theta/dt \quad \text{and} \quad d^2s/dt^2=r\,d^2\theta/dt^2.$$

Now ds/dt is the velocity of P (art. 186) and d^2s/dt^2 is its tangential acceleration (art. 193); hence, if as heretofore the velocity, the acceleration, and its tangential and normal components be denoted by v, a, a_t, and a_n respectively,

$$\mathbf{v}=\mathbf{r}\omega, \qquad (1)$$

$$\mathbf{a}_t=\mathbf{r}\alpha, \quad \mathbf{a}_n=\mathbf{r}\omega^2, \qquad (2)$$

and

$$\mathbf{a}=\mathbf{r}\sqrt{\alpha^2+\omega^4}; \qquad (3)$$

ω and α denoting respectively the angular velocity and acceleration of the body. These equations show that the velocity and acceleration of a point in a rotating body are proportional to its distance from the axis.

EXAMPLES.

1. Write in the proper place below the signs of the angular velocity and acceleration of a body which rotates as follows:

(*a*) Clockwise,

(1) when "getting up speed," sign of ω is . . ., of α . . .;
(2) " "slowing down," " " " " . . ., " " . . .;

* For dimensions of an angular acceleration, see Appendix C.

(*b*) Counter-clockwise,

(1) when "getting up speed," sign of ω is . . ., of α . . . ;
(2) " "slowing down," " " " " . . ., " " . . . .

2. Express an angular velocity of n rev.-per-sec. in rad.-per-sec.

3. If the angular velocity of a wheel changes from 100 to 120 rev.-per-min. in one-half a min., what is its average angular acceleration?

4. A wheel is set rotating in such a manner that the number of turns made after starting equals the square of the time (in minutes) after starting. Deduce expressions for the angular velocity and the acceleration at any time.

Solution: The law of the motion, if θ and t denote the number of turns and the time, is $\theta = t^2$; hence

$$d\theta/dt = 2t = \omega \text{ (turns-per-min.)},$$

and $$d^2\theta/dt^2 = 2 = \alpha \text{ (turns-per-min.-per-min.)}.$$

5. Compute the velocity and the acceleration of a point on the rim of a wheel whose diameter is six feet, if its angular velocity is 4 rev.-per-sec.

Solution: $\omega = 4 \times 6.283 = 25.13$ rad./sec.
According to eq. (1), $v = 3\omega = 75.4$ ft./sec., and
" " eq. (2), $a_n = v^2/3 = 1895$ ft./sec.^{2}

Since the angular velocity is constant, the angular acceleration is zero; hence, see eqs. (2) and (3), $a_t = 0$ and $a = a_n$. The direction of a is from the point on the rim to the centre of the wheel.

§ III. Plane Motion.

217. Plane Motion Defined.—A plane motion is one in which each point of the moving body remains at a constant distance from a fixed plane. The fixed plane (or any plane parallel to it) is called *the plane of the motion.*

The wheel of a car running on a straight track has plane motion; so also has a book sliding about on a table. A translation may or may not be a plane motion (see illustrations, art. 205), but a rotation is always a plane motion.

As in a rotation, all points on any line of the moving body which is perpendicular to the plane of the motion move alike, hence the motion of the projection of the line on the plane of the motion represents that of all the points, and the motion of the body itself is completely represented by that of its projection on the plane of the motion.

218. Angular Displacement.—By angular displacement of a body whose motion is plane is meant (as in rotations) the angle described by any line of the body which is parallel to the plane of the motion. Obviously all such lines describe equal angles in the same time interval. As in rotations also, displacements are regarded as positive or negative according as they are due to counter-clockwise or clockwise turning of the body.

Let the irregular outline (fig. 174) represent the projection of the moving body on the plane of the motion, AB a fixed line of the projection, OX a fixed reference line, and let θ denote the angle XOA, it being regarded as positive or negative according as OX, when turned about O *toward* AB, turns counter-clockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any motion of the body, then the

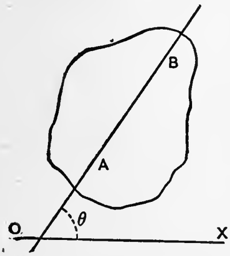


FIG. 174.

$$\text{angular displacement} = \theta_2 - \theta_1 = \Delta\theta.$$

219. Angular Velocity and Angular Acceleration.—If a body has a plane motion, its angular velocity is the time-rate at which its displacement occurs, and its angular acceleration is the time-rate at which its angular velocity changes.

These definitions are precisely similar to those of the angular velocity and acceleration of a rotation (arts. 210 and 213); hence the expressions, units, and rules of signs given in those articles and the following hold for any plane motion. Rewriting the expressions,

$$\omega = d\theta/dt, \quad \text{and} \quad \alpha = d\omega/dt = d^2\theta/dt^2,$$

ω and α denoting angular velocity and acceleration of the moving body respectively.

EXAMPLES.

1. Determine the angular velocity and acceleration of the connecting-rod of a steam-engine, assuming the angular velocity of the crank to be constant.

Solution: Let OP and CP (fig. 175) be the crank and connecting-rod respectively, their lengths being c and r, and let the

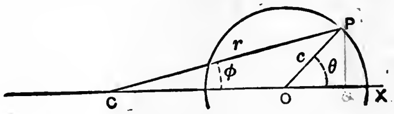


FIG. 175.

angles XOP and XCP be denoted by θ and ϕ (measured from the horizontal line and counter-clockwise positive); then for all positions

$$r \sin \phi = c \sin \theta, \quad \text{or} \quad \phi = \sin^{-1}\left(\frac{c}{r} \sin \theta\right).$$

Let ω and α denote the angular velocity and acceleration of the rod respectively, and ω_c the angular velocity of the crank. Then, since

$$\omega = d\phi/dt, \quad \alpha = d^2\phi/dt^2, \quad \omega_c = d\theta/dt, \quad \text{and} \quad d^2\theta/dt^2 = 0,$$

$$\omega = \frac{\cos \theta}{(r^2/c^2 - \sin^2 \theta)^{\frac{1}{2}}} \omega_c \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\alpha = -\frac{(r^2/c^2 - 1) \sin \theta}{(r^2/c^2 - \sin^2 \theta)^{\frac{3}{2}}} \omega_c^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. Take r/c equal to 4 and compute the values of the coefficients of ω_c and ω_c^2 in eqs. (1) and (2) of the preceding solution, when $\theta = 0°$, $30°$, $60°$, $90°$, $120°$, $150°$, and $180°$. Plot those values, thus showing how ω and α vary during a stroke.

220. Velocity and Acceleration of any Point of the Body.— Let P and P' (fig. 176) be two points of the moving body and O a point without, the three being in a plane parallel to that of the motion. According to art. 196 the velocity of P relative to O equals the vector sum of the velocities of P relative to P' and P' relative to O. Let the last velocity be v' directed as shown, and let r denote the distance PP' and ω the angular velocity of the body at the instant considered. Since the motion of P relative to P' is circular, the velocity of P relative to P' equals $r\omega$ (art. 216), and its direction is perpendicular to PP' as shown. The velocity of P relative to O then is the vector sum of v' and $r\omega$.

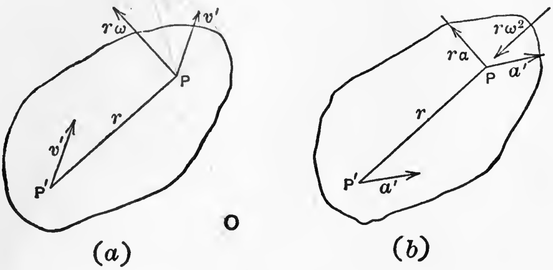


FIG. 176.

Similarly, the acceleration of P relative to O equals the vector sum of the accelerations of P relative to P' and P' relative to O. Let the last acceleration be a' directed as shown, and let α denote the angular acceleration of the body at the instant considered. The path of P relative to P' being a circle, the tangential and normal components of the acceleration of P relative to P' are respectively $r\alpha$ and $r\omega^2$ (art. 216) directed as shown, and the acceleration of P relative to O is the vector sum of a', $r\alpha$, and $r\omega^2$.

Proposition.—The components of the velocities of any two points of a body having a plane motion * along the line joining them are equal and agree in sense.

Proof: Let P and P' (fig. 176*a*) be two points in the plane of the motion. Since the velocity of P (v) is the resultant of $r\omega$ and v' and $r\omega$ is perpendicular to PP', the component of v along PP' is the same as that of v' along that line; but v' is the velocity of P', hence, etc.

It will be noticed that the proof is not general, the points being in the plane of the motion; this is the case to which the proposition is applied later.

* Really true in any motion of a rigid body.

EXAMPLES.

1. A wheel rolls uniformly and makes one turn every second. What is its angular velocity?

Solution: Any line of the body parallel to the plane of the wheel describes an angle of 360° each second; hence the angular velocity is 360 deg.-per-sec.

2. Determine the velocity of any point on the rim of a wheel of radius r whose angular velocity is ω rad.-per-unit time.

Solution: We use the foregoing principles, choosing the centre of the wheel as P'. Let θ denote the angle described by the radius PP' after any origin of time; then the distance (s) travelled by P' in that time is given by $s = r\theta$. Hence $ds/dt = r\, d\theta/dt$, or the velocity of P' at any instant equals r times the angular velocity of the wheel at that instant (see fig. 177). Relative to P', the selected point P on the rim describes a circle and the velocity of that point relative to P' is $r\omega$ (art. 216), its direction being that of the tangent to the circle at P. Hence the absolute velocity of P (v) is represented by the diagonal Pp of the parallelogram on the two velocities $r\omega$.

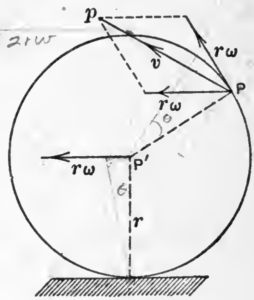


FIG. 177.

3. From the result of the preceding solution, determine the velocities of the highest and lowest points on the rim of the wheel.

221. Plane Motion Regarded as a Combined Translation and Rotation.—Imagine the velocity of each point, P_1, P_2, etc., of the moving body to be resolved into two components, one of which is the same as the velocity of any particular point P' of the body (fig. 178a). It follows from the preceding article that the other component is perpendicular to the line joining the point with P' and is equal to the product of the length of the line and the angular velocity of the body.

Also imagine the acceleration of each point resolved into two components, one of which is the same as the acceleration of P' (fig. 178b). It follows from the preceding article that the other component can be resolved into two components directed along and perpendicular to the line joining the point with P', they being equal to the products of the length of the line and the square of the angular velocity and angular acceleration respectively.

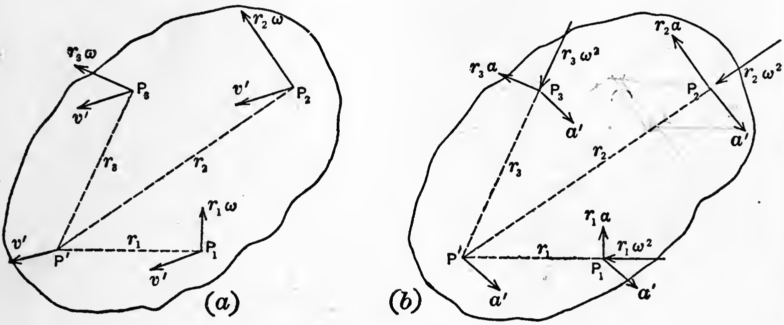


FIG. 178.

Now if the points of the body had the first sets of component velocities and accelerations only, the motion of the body would be a translation, the velocity and acceleration of which would be like those of the chosen point P'. And if the particles had the second sets of component velocities and accelerations only, the motion of the body would be a rotation about the line through the chosen point and perpendicular to the plane of the motion, the angular velocity and acceleration of the rotation equaling respectively the angular velocity and acceleration of the actual motion. The two motions are hence regarded as components of the actual motion.

222. Instantaneous Axis (of no Velocity).—*Proposition.*—If a body has a plane motion which is not translatory, then at each instant there is a line in it or in its extension all points of which have no velocity.

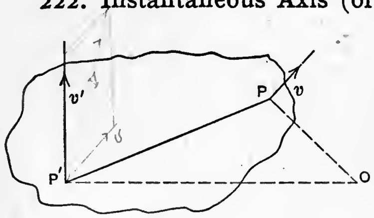


FIG. 179.

Proof: Let P and P' (fig. 179) be any two points in the plane of the motion, and draw lines PO and $P'O$ perpendicular to the directions of the velocities of

P and P' respectively. The velocities of points on PO and $P'O$ have no components along those lines respectively (prop., art. 220), and since O is on both lines its velocity has no components along PO and $P'O$, and hence that velocity must be zero. All points on the line through O perpendicular to the plane of the motion have the same velocity as that of O, i.e., zero.

If the motion is translatory, the velocities of P and P' have the same direction and the point O is "at infinity," but no points of a finite extension of the body have a zero velocity.

Definitions.—The line of a moving body, all points of which have at a certain instant no velocity, is called the *instantaneous axis* of the motion at that instant. The intersection of the instantaneous axis with the plane of the motion is called the *instantaneous centre.*

In general the instantaneous centre moves about in the body and in space. Its path in the body (i.e., the path relative to axes fixed in the body) is called *body centrode*, and its path in space (i.e., that relative to axes fixed in space, or in the earth) is called *space centrode.*

It follows from the solution of exs. 2 and 3, art. 220, that the velocity of the lowest point of a rolling wheel is zero; hence that point is the instantaneous centre. The body centrode is the circumference of the wheel, and the space centrode is the line on the plane surface along which the wheel rolls.

223. Instantaneous Rotation.—Let P and Q denote two points of a body in a plane of motion, the latter being such that at some instant during the motion it is an instantaneous centre. At all times the motion of P relative to Q is circular and its velocity relative to Q at any instant equals the product of the distance between P and Q (r) and the angular velocity of the body at that instant.

Consider now the state of the motion when Q is the instantaneous centre, calling the angular velocity of the body at that instant ω. Since the absolute velocity of Q is zero, the absolute velocity of P (v) equals its velocity relative to Q, or

$$v = r\omega. \qquad (1)$$

Hence the absolute velocity of any point of a moving body at any instant equals the product of the distance of the point from the instantaneous axis and the angular velocity of the body at that instant.

Since in a rotation about a fixed axis the velocity of any point of the body is also proportional to its distance from the axis of rotation, the state of a plane motion at any instant is described as an instantaneous rotation about the instantaneous axis.

EXAMPLES.

1. Show how to find the instantaneous centre of the connecting-rod of a steam-engine in any position.

Solution: The directions of the velocities of the ends of the rod are known for all positions; that of C (fig. 180) is OC and that of P is the same as that of the tangent to the crank-pin circle at P. Hence to find the instantaneous centre, draw perpendiculars to the directions of these velocities at C and P respectively; their intersection is the instantaneous centre of the rod for the position represented.

2. Where is the instantaneous centre when the crank is horizontal? When vertical? What can you say about the state of the motion of the rod in these cases?

3. Show how to find the angular velocity of the connecting-

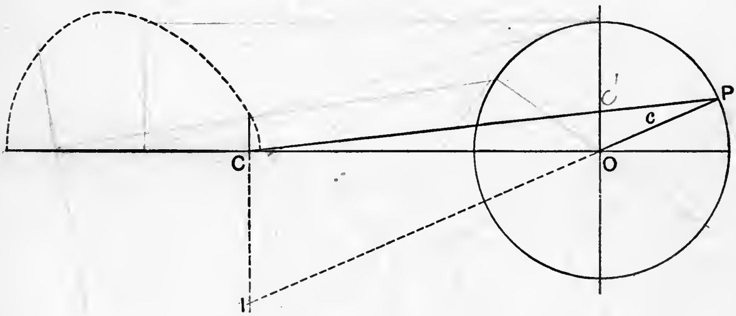


FIG. 180.

rod and the velocity of the cross-head C (fig. 180) in terms of the velocity of the crank-pin P for any position.

Solution: Let c denote the length of the crank;
v_1, " " velocity of the crank-pin;
v_2, " " " " " cross-head;
ω, " " angular velocity of the connecting rod.

According to eq. (1), $\omega = v_1/\overline{IP}$ and

$$v_2 = \overline{IC} \cdot \omega = (\overline{IC}/\overline{IP})v_1.$$

If it is desired to draw a velocity-space curve for the motion of the cross-head (as the dotted curve of the figure) when the crank-pin velocity is constant, the following simple constructions may be employed: Draw a vertical diameter of the crank-pin circle and mark its intersection with the connecting-rod (extended if necessary) C'; then to the same scale by which OP represents the crank-pin velocity OC' represents the cross-head velocity. For, from the figure,

$$\overline{IC}/\overline{IP} = \overline{OC'}/c, \quad \text{hence} \quad v_2/v_1 = \overline{OC'}/c.$$

4. A sheet of paper is caused to slide on a draughting-board so that two points (P and Q) of the paper move along two lines (OX and OY) on the board. Show how to find the instantaneous centre of the motion for any position of the paper.

5. The velocity of one point of the paper of the preceding example being given, show how to find the velocity of any other point.

EXERCISE.

Take a sheet of paper and move it as described in ex. 4, the lines OX and OY being taken at right angles to each other, and determine the instantaneous centre for several positions of the paper. As each instantaneous centre is determined, mark it by pricking a hole through the paper and into the board, and join the holes in the paper and those in the board by smooth curves. These curves are the body and space centrodes respectively.

Now cut the paper along the centrode and replace it on the board in one of its positions (P and Q falling on OX and OY respectively). Then move the paper so that the body centrode *rolls* on the space centrode. If carefully done (a template fitted to the space centrode helps to get the rolling motion), it will be noticed that P and Q move along OX and OY; hence the actual motion is a rolling of the body centrode over the space centrode. It can be shown that any plane motion is equivalent to such a rolling.

KINETICS.

CHAPTER X.

MOTION OF A PARTICLE (RESUMED) AND OF A SYSTEM OF PARTICLES.

§ I. Mass and Mass-Centre.

224. **Quantity of Matter.**—It must be confessed that the usual definition of mass (art. 12) needs explanation. The question at once arises, How shall matter be measured? If this is answered, then the meaning of "quantity of matter" is clear.

"As long as we have to do with bodies of the same exact kind, there is no difficulty in understanding how the quantity of matter is to be measured. If equal quantities of the substance produce equal effects of any kind, we may employ these effects as measures of the quantity of the substance.

"For instance, if we are dealing with sulphuric acid of uniform strength, we may estimate the quantity of a given portion of it in several ways. We may weigh it, we may pour it into a graduated vessel and so measure its volume, or we may ascertain how much of a standard solution of potash it will neutralize.

"We might use the same methods to estimate a quantity of nitric acid if we were dealing only with nitric acid; but if we wished to compare a quantity of nitric acid with a quantity of sulphuric acid we should obtain different results by weighing, by measuring, and by testing with an alkaline solution." *

Now these methods are not equally appropriate, and indeed that of titration cannot be applied to all bodies, that of measuring would lead us to say that the amount of gas in a tight rubber bag could be changed by simply squeezing (changing the volume

* Quoted from Maxwell's "Matter and Motion."

of the bag), which is absurd. It will be shown that the method of weighing is an appropriate one.

Any appropriate method must be based on a common property of matter. Now all matter is inert, i.e., force must be applied to any body to change its velocity, and this property (inertia) is the basis of the fundamental method of determining quantity of matter. Not only is this method (explained in detail later) employed in mechanics, but also sometimes in ordinary affairs. Thus, suppose that we wish to ascertain whether a barrel lying upon a floor is full or empty; we push it and conclude that it is full or empty according as a large or small force is required to roll it, i.e., to "overcome the inertia."

Along this line we can determine the relative amounts of matter in two bodies if we have a means of measuring forces. Thus, calling the two bodies A and B, place each upon a light and easy-running carriage (fig. 181) and connect these by means

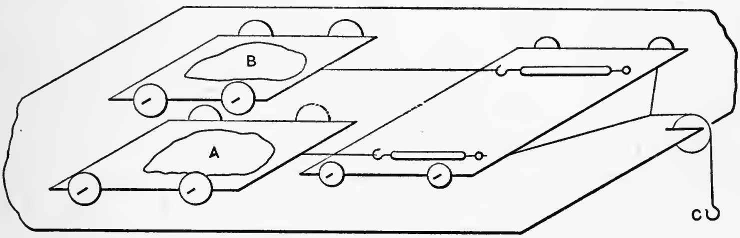


FIG. 181.

of cords to spring-balances which rest upon and are fastened to a third carriage as shown. If this last carriage is pulled to the right, the others follow and the spring-balances measure the pulls required to move the smaller carriages and their loads. Clearly it will be in accord with the crude test applied to the barrels to say that the quantities of matter in A and B (neglecting that in the carriages) are as the forces applied to them, as read from the spring-balances. It will also be in accord with a comparison of quantities of matter in bodies of the same kind by the method of volumes; for suppose that A and B are of the same kind and that the volume of A is n times that of B, we would say at once that there is n times as much matter in A as in B. The same ratio would be arrived at by the inertia method, i.e., from the readings of the spring-balances.

We now make definite our notions about quantity of matter as just expressed by means of the following

Definition.—The quantities of matter in bodies are proportional to the forces required to give them equal accelerations.

If we should adopt as a standard, or unit, the amount of matter in any particular body, we could determine the quantity in any other body in terms of this unit (ideally at least) by placing it and the standard body on the two smaller carriages and then measuring the forces required to give them the equal accelerations. If the force on the body in question is n times that on the standard, the quantity of matter in the body would be n.

Of course this scheme of measurement is not put forth as a practical one, but rather as a help to understand the meaning of mass. The practical method of measuring quantity of matter is by weighing, which is (as explained in art. 226) precisely similar in principle to that just described.

225. **Mass.**—*Definition* (repeated from art. 12).—By mass of a body is meant the quantity of matter in it. The word is merely an abbreviation for "quantity of matter."

Units of Mass.—There are many units of mass in use; they may be grouped into two classes:

(*a*) Absolute Units; so called to express the fact that their magnitudes are independent of locality. Two of these, the pound and the kilogram, are described in art. 12. Their relation is as follows:

$$1 \text{ pound} = 0.4536 \text{ kilograms},$$

or

$$1 \text{ kilogram} = 2.205 \text{ pounds}.$$

(*b*) Gravitation Units; so called because their magnitudes vary with locality precisely as acceleration due to gravity varies. Two of these units are described in art. 233.

226. **Practical Determination of Mass.**—As is well known, bodies falling at the same place in vacuum move with equal accelerations, i.e., the forces of gravity upon bodies (their weights) at the same place accelerate them equally. It follows from this fact and the definitions of arts. 224 and 225 that

the masses of bodies are proportional
to their weights at the same place.

Hence to determine the mass of a body, i.e., to determine the ratio of its mass to that of a standard (pound for example), we determine the ratio of the weights of the body and the standard. If this latter ratio is n, then the former is n also, and the mass of the body is n times that of the standard (or n pounds).

227. Moment of Mass.—The product of the mass of a particle and its ordinate with respect to any plane is called the moment of the mass of the particle with respect to that plane. An ordinate is regarded as positive or negative according as the particle is on one side of the plane or the other; hence a moment has the same sign as the corresponding ordinate.

By moment of the mass of a system of particles is meant the algebraic sum of the moments of the masses of its particles. Thus, let the particles of a system be referred to a set of rectangular axes, and denote the coordinates of the particles by (x_1, y_1, z_1), (x_2, y_2, z_2), etc., and their masses by m_1, m_2, etc. Then the moments of the mass of the system with respect to the y-z, z-x, and x-y planes are respectively

$$m_1x_1 + m_2x_2 + \ldots = \Sigma mx;$$
$$m_1y_1 + m_2y_2 + \ldots = \Sigma my;$$
$$m_1z_1 + m_2z_2 + \ldots = \Sigma mz.$$

228. Mass-Centre Defined.—It is obvious that the mass of a system may be multiplied by some distance (positive or negative) such that the product equals the moment of the mass with respect to a given plane. Thus, let M be the mass of a system and $\bar{x}$, $\bar{y}$, and $\bar{z}$ such multipliers that

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M\bar{z} = \Sigma mz. \quad . \quad . \quad . \quad (1)$$

The point whose coordinates are $\bar{x}$, $\bar{y}$, and $\bar{z}$ is called the mass-centre of the system. The formulas for the coordinates of the mass-centre are therefore

$$\bar{x} = \frac{\Sigma mx}{M}, \quad \bar{y} = \frac{\Sigma my}{M}, \quad \bar{z} = \frac{\Sigma mz}{M}. \quad . \quad . \quad . \quad . \quad (2)$$

229. Relation Between Mass-Centre and Centre of Gravity.—Since the masses of the particles of a system are proportional to their weights, we may substitute for the mass terms in the equations (2) the corresponding weights, i.e.,

$$\bar{x}=\frac{\Sigma wx}{W}, \quad \bar{y}=\frac{\Sigma wy}{W}, \quad \bar{z}=\frac{\Sigma wz}{W},$$

w denoting the weight of any particle whose coordinates are x, y, and z, and W the weight of the system. But these values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ are identical with those for the x, y, and z coordinates of the centre of gravity of the system (see art. 64); hence the mass-centre and centre of gravity of a system of particles (as herein defined) are coincident.

§ II. Motion of a Particle.

230. Laws of Motion.—(1) When no force is exerted upon a particle it remains at rest or continues to move uniformly in a straight line.

(2) When a force is exerted upon a particle it is accelerated; the direction of the acceleration is the same as that of the force, and its magnitude is proportional to the force directly and to the mass of the particle inversely.

(3) When one particle exerts a force upon another the latter exerts one on the former, and the two forces are equal, collinear, and opposite.*

These laws are inductions from observation and experiment, made not of course on particles, but on bodies of ordinary size. We do not attempt a full discussion of the experience leading to the laws, but limit ourselves to a brief statement. Really the best evidence of the correctness of the laws are the many agreements noted between observed results and those calculated from the laws, and the fact that the laws are not known to be in disagreement with any phenomenon.

(1) We know of no body which is free from the influence of

* These are essentially Newton's Laws of Motion. The form of statement here given, however, differs from that in which they were originally announced (1687).

all others, i.e., a body not acted upon by force, and so no direct observation or experiment leads to this law. But all have perhaps noticed that a small body, if projected along a sheet of ice, moves in a straight path and continues to move for a considerable period; also that the smoother the ice the longer will the body move, i.e., the smaller is its retardation or the more nearly is its motion uniform. The retardation is rightly ascribed to the frictional resistance offered by the ice; and it is a fair inference that if that resistance were zero, the retardation would also be zero and the motion would be uniform.

(2) The second law may be roughly verified by the means of the apparatus represented in fig. 181. Thus place a body whose mass is known on one of the small carriages and another on the hook C. Then let the system move, measure the acceleration of A and the pull recorded by the spring-balance. If this is repeated with other bodies on C, it will be found that the accelerations are roughly proportional to the pulls. Also place bodies of unequal mass successively on A and cause the carriages to roll so that the spring-balance reading is the same for the different bodies, measuring the acceleration of each motion. It will be found that the accelerations are roughly inversely proportional to the masses of the bodies. The lack of exact proportionality may properly be ascribed to the neglect of friction and the mass of the carriage A, and to unavoidable experimental errors involved in such an apparatus.

(3) It is well known that a magnet exerts a force on a piece of soft iron in its proximity; that the iron also exerts a force on the magnet, may be proven as follows: Place the magnet and piece of iron in small vessels; then float them so loaded in water. If they are not too far apart they will be observed to move toward each other; hence not only does the magnet attract the soft iron, but the iron attracts (exerts a force) on the magnet.

If two spring-balances be laid on a table, the hook of one engaging the hook of the other, and then they be pulled apart by their rings, each will register the pull exerted upon it by the other. If the balances are accurate, it will be found that the amounts registered are equal; hence the forces exerted by the hooks on each other are equal. $F_1 = F_2$

231. Quantitative Expression of the Second Law of Motion.—Let m' and m'' denote the masses of two particles and a' and a'' the accelerations given them by two forces F' and F'' respectively. The second law of motion asserts that

$$a':a''::F'/m':F''/m'', \quad \text{or} \quad F'/m'a'=F''/m''a''.$$

It follows from this equation that the ratio of the force acting on a particle to the product of the mass and the acceleration of the particle produced by the force is constant, i.e., in the form of an equation,

$$\mathbf{F/ma=k}, \quad \text{or} \quad \mathbf{F=k \cdot ma}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k being the value of the constant ratio.

The numerical value of k depends upon the units used for expressing value of force, mass, and acceleration. It is convenient, but not necessary, so to choose these units that k becomes 1. A system of units so chosen is called *a kinetic system.*

232. Kinetic System of Units.—It follows from the last equation that in any kinetic system the unit force acting upon the unit mass produces unit acceleration (i.e., unit velocity in unit time). Evidently any three of the units involved in a kinetic system (units of force, mass, length, and time) may be chosen arbitrarily, but the fourth must be such as to satisfy the requirement just stated. The unit of force or of mass is the one selected as the derived or fourth unit. There are two classes of kinetic systems.

(*a*) *Absolute Kinetic Systems.*—Units of length, mass, and time are arbitrarily selected, the unit of force being derived.

Centimetre-Gram-Second (c.g.s.) System. — The units of length, mass, and time are the centimetre, the gram (one-thousandth of a kilogram, see art. 12), and the second respectively. The corresponding unit of force, i.e., the force which acting on a gram mass for one second gives it a velocity of one centimetre per second, is called *a dyne.* This is the system now universally used in scientific work and literature.

Foot-Pound-Second (f.p.s.) System.—The units of length, mass, and time are the foot, the pound (see art. 12), and the second respectively. The corresponding unit of force, i.e., the force which acting on a pound mass for one second gives it a velocity of one foot per second, is called a *poundal*. This system has never met with favor; accordingly, it is not used herein, but is mentioned and explained because of the relations which it bears to other important systems.

(*b*) *Gravitation Kinetic Systems* (also called Engineers' Systems).—Units of length, force, and time are arbitrarily selected and the unit of mass is derived. The units of force in these systems are weights of absolute units of mass, i.e., they depend upon the force of gravity and are gravitation units (see art. 13) The units of length and time being independent of place, the units of mass in these systems must vary just as the units of force vary, and are hence properly called gravitation units of mass.

Foot-Pound (force) -Second System.—The units of length, force, and time are the foot, the pound (see art. 13), and the second respectively. The corresponding unit of mass, i.e., one in which a pound force would produce in one second a velocity of one foot-per-second, is called herein a *geepound*.*

Metre-Kilogram (force) -Second System.—The units of length, force, and time are the metre, the kilogram (see art. 13), and the second respectively. The corresponding unit of mass, i.e., one in which a kilogram force would produce in one second a velocity of one metre-per-second, is called herein a *geekilogram*.*

233. Relations Between Force Units and Between Mass Units.—The *dyne* and the *poundal* might ideally at least be "preserved" as follows: Place one gram (or one pound) on one of the small carriages of fig. 181, and then make the large carriage move to the right with an acceleration of one cm.-per-sec.-per-sec. (or one ft.-per-sec.-per-sec.) and note the reading of the spring-balance. Assuming the mass of the small carriage compared to

* This is a new term, and the student should remember that fact, for it is not now and perhaps never will be current. The unit is usually called "engineers' unit of mass," an appellation which is on a par with "wood-choppers' unit of volume" (the cord). Other terms have been proposed, but they have never been adopted. The author is confident that names for gravitation units of mass are convenient *and that their use tends to clearness.*

that of its load, and its frictional resistance compared with the balance reading to be negligible, the force causing the stretch of the spring is one dyne (or one poundal).

Instead of employing the foregoing method, we make use of measurements made on the acceleration due to gravity, thus comparing the dyne and poundal with standards or units of weight. A force equal to the weight of a gram (or a pound) acting on one gram (or one pound), as in a falling body, produces an acceleration of approximately 981 cm.-per-sec.-per-sec. (or 32.2 ft.-per-sec.-per-sec.); hence a force equal to 1/981 of the weight of a gram (or 1/32.2 of the weight of a pound) would produce in one gram (or one pound) an acceleration of one cm.-per-sec.-per-sec. (or one ft.-per-sec.-per-sec.). Now by definition the forces producing in one gram and in one pound accelerations of one cm.-per-sec.-per-sec. and one ft.-per-sec.-per-sec. are the dyne and poundal respectively; therefore

$$\text{one dyne} = 1/981 \pm \text{ gram (force)},$$
$$\text{one poundal} = 1/32.2 \pm \text{ pound (force)}.*$$

The "*geepound*" and the "*geekilogram*" might also be determined by means of the apparatus represented in fig. 181. Thus adjust a load on the smaller carriage so that the spring-balance will read one pound (or one kilogram) when the larger carriage is drawn to the right with an acceleration of one ft.-per-sec.-per-sec. (or one m.-per-sec.-per-sec.). If the mass and frictional resistance of the small carriage are negligible, the mass of the load is one geepound (or one geekilogram).

A better determination of these units of mass can be made by means of experiments on the acceleration due to gravity. A one-pound (or a one-kilogram) force produces in a one-pound (or a one-kilogram) mass an acceleration of approximately 32.2 ft.-per-sec.-per-sec. (or 9.81 m.-per-sec.-per-sec.); hence a one-pound (or a one-kilogram) force would produce in 32.2 pounds (or 9.81 kilograms) mass an acceleration of 1 ft.-per-sec. per-sec. (or one m.-per-sec.-per-sec.). Now by definition, two masses which under the action of forces of one pound and one kilogram

* One gram and one pound (force) being the weight of one gram and one pound (mass) respectively.

receive accelerations of one ft.-per-sec.-per-sec. and one m.-per-sec.-per-sec. are the geepound and the geekilogram respectively. Therefore

$$1 \text{ geepound} = 32.2 \pm \text{ pounds},$$
$$1 \text{ geekilogram} = 9.81 \pm \text{ kilograms}.$$

234. Relation Between the Mass and the Weight of a Body.—This relation is implicitly given in the preceding article; we will now state it definitely. Let W denote the weight of a body, m its mass, and g its acceleration due to gravity, *the three quantities being expressed in units of any one kinetic system.* When this body falls, its acceleration (g) is due to its weight (W); hence the equation of the motion is (see art. 231)

$$W = mg, \quad \text{or} \quad m = W/g,$$

and this gives the relation between the weight and the mass of any body when they (W, m, and g) are expressed in units of any one kinetic system. Thus for any body

$$W \text{ (in dynes)} = m \text{ (in grams)} \times 981 \pm,$$
$$W \text{ (in poundals)} = m \text{ (in pounds)} \times 32.2 \pm,$$
$$m \text{ (in geepounds)} = W \text{ (in pounds)} \div 32.2 \pm,$$
$$m \text{ (in geekilograms)} = W \text{ (in kilograms)} \div 9.81 \pm$$

The weight and mass of a body are also numerically the same if expressed in certain units, more definitely if the unit weight is the weight of the unit mass (see art. 15). Thus the weight of a barrel of flour is 196 pounds and its mass is also 196 pounds.

EXAMPLE.

Express the mass of a cubic foot of water in pounds, geepounds, kilograms, and geekilograms.

235. Acceleration of a Particle Acted Upon by Several Forces.—The acceleration of a particle acted upon by several forces may be determined in several ways:

(*a*) By the methods of "Statics" determine the resultant of the forces and then compute the acceleration due to this resultant. This acceleration is identical with that due to the actual forces applied to the particle, for by definition the resultant of any number of forces is equivalent to them in producing motion.

(*b*) Compute the acceleration of the particle due to each force acting alone and add these accelerations vectorially. The vector sum represents the actual acceleration due to the combined action of the forces. The correctness of this method may be proved from the first; we give the proof for the case of two forces: Let m be the mass of the particle, F_1 and F_2 the applied forces, a the acceleration due to their combined action, R their resultant, and a_1 and a_2 the accelerations due to F_1 and F_2 acting singly. Let AO and BO (fig. 182) represent F_1 and F_2 respectively; then the diagonal of the parallelogram $OABC$ represents R (art. 20), and the continuation Oc of that diagonal represents a if $Oc = R/m$ (art. 231). Also, Oa and Ob represent a_1 and a_2 respectively if

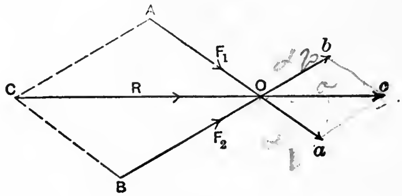


FIG. 182.

$$Oa = F_1/m \quad \text{and} \quad Ob = F_2/m.$$

It remains to show that Oc is the vector sum of Oa and Ob. To do this, join a and c and b and c, and show that $Oabc$ is a parallelogram.

236. Equations of Motion of a Particle.—Let m, a, and R denote the mass and acceleration of a particle and the resultant of the forces applied to it respectively. Then, as explained in the preceding article,

$$\mathbf{R} = \mathbf{ma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and a and R have the same direction.

Let F', F'', etc., denote the forces applied to the particle and α, β, and γ the angles which R and a make with a set of coordinate axes x, y, and z. Also let ΣF_x, ΣF_y, and ΣF_z denote the algebraic sums of the x, y, and z components of F', F'', etc., and a_x, a_y, and a_z the x, y, and z components of a. Now

$R \cos \alpha = ma \cos \alpha,\quad R \cos \beta = ma \cos \beta,\;$ and $\; R \cos \gamma = ma \cos \gamma;$

and since $R \cos \alpha = \Sigma F_x,\quad R \cos \beta = \Sigma F_y,\quad R \cos \gamma = \Sigma F_z,$

and $\qquad a \cos \alpha = a_x,\qquad a \cos \beta = a_y,\qquad a \cos \gamma = a_z,$

$$\Sigma \mathbf{F}_x = \mathbf{m}\mathbf{a}_x,\quad \Sigma \mathbf{F}_y = \mathbf{m}\mathbf{a}_y,\quad \Sigma \mathbf{F}_z = \mathbf{m}\mathbf{a}_z. \quad . \quad . \quad . \quad (2)$$

Let ϕ denote the angle between the action line of R and the tangent to the path of the particle. Then

$$R \cos \phi = ma \cos \phi \quad \text{and} \quad R \sin \phi = ma \sin \phi,$$

or $$\Sigma \mathbf{F}_t = \mathbf{m}\mathbf{a}_t \quad \text{and} \quad \Sigma \mathbf{F}_n = \mathbf{m}\mathbf{a}_n. \quad . \quad . \quad . \quad . \quad (3)$$

ΣF_t and ΣF_n denoting the algebraic sums of the tangential and normal components of the forces, and a_t and a_n the tangential and normal acceleration of the particle.

§ III. MOTION OF A SYSTEM OF PARTICLES.

237. Definitions.—Any number of particles collectively considered are called a *system of particles*. If the distances between the particles remain invariable, the system is called a *rigid one* or a *rigid body*.

Among the forces exerted upon the system of particles, some may be exerted by particles not belonging to the system. Such forces have already been named *external forces* (art. 113), and all such forces the *external system*. A force exerted by a particle upon another of the same system has been named an *internal force* (art. 113), and all such forces the *internal system*.

According to the third law of motion (art. 230), if one particle of a system exerts a force upon another, the second also exerts one upon the first, and these two forces are equal, opposite, and collinear. Hence the internal forces of a system of particles occur in pairs, the forces of each pair being equal, opposite, and collinear.

By *effective force* for a particle of a system is meant the resultant of all the forces acting on that particle. The effective forces for all the particles of a system are called the *effective system*. If m and a denote the mass and acceleration of any particle of a system, then (art. 236) for that particle the

$$\text{effective force} = ma.$$

238. D'Alembert's Principle.—Since the effective system consists of the external and the internal forces acting upon the particles of a system, the resultant of the effective forces is identical with that of the external and the internal forces. Now the resultant of the internal forces is zero since they occur in pairs, the forces of each being equal, opposite, and collinear; hence

Proposition I.—For any system of particles the resultants of the effective and the external forces are identical. Obviously this proposition is equivalent to

Proposition II.—For any system of particles the effective forces reversed * and the external forces together are in equilibrium. This proposition is known as D'Alembert's Principle, after him who first announced it (1742) for rigid bodies.

These propositions are not fundamental, being deducible from the laws of motion (art. 230), but they express an important relation in convenient forms.

239. Component of an Effective System Along Any Line.—Let the line be taken as an x axis, and as before let m_1, m_2, etc., denote the masses of the particles and a_x', a_x'', etc., the x components of their accelerations. Then the algebraic sum of the x components of the effective forces for the particles is

$$m_1a_x' + m_2a_x'' + \ldots$$

This sum equals the product of the mass of the system (Σm) and the x component of the acceleration of its mass-centre ($\bar{a}_x$). For, according to art. 228,

$$m_1x_1 + m_2x_2 + \ldots = \Sigma m \cdot \bar{x}, \quad \ldots \ldots \quad (1)$$

hence $$m_1dx_1/dt + m_2dx_2/dt + \ldots = \Sigma m \cdot d\bar{x}/dt, \quad \ldots \quad (2)$$

and $$m_1d^2x_1/dt^2 + m_2d^2x_2/dt^2 + \ldots = \Sigma m \cdot d^2\bar{x}/dt^2, \quad \ldots \quad (3)$$

or $$m_1a_x' + m_2a_x'' + \ldots = \Sigma m \cdot \bar{a}_x. \quad \ldots \ldots \quad (4)$$

240. Motion of the Mass-Centre of any System of Particles.—According to Prop. I, art. 238, the algebraic sums of the com-

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* Often called the "reversed effective system."

ponents of the external and effective forces for any system of particles along any line are equal. Hence, if ΣF_x, ΣF_y, and ΣF_z denote the sums of the components of the *external forces* along an x, y, and z axis respectively,

$$\Sigma \mathbf{F}_x = \Sigma \mathbf{m} \cdot \bar{\mathbf{a}}_x, \quad \Sigma \mathbf{F}_y = \Sigma \mathbf{m} \cdot \bar{\mathbf{a}}_y, \quad \Sigma \mathbf{F}_z = \Sigma \mathbf{m} \cdot \bar{\mathbf{a}}_z. \quad . \quad (1)$$

These equations show that the acceleration of the mass-centre depends only on the values of the components of the external forces and that

> The acceleration of the mass-centre of a system is just like that of a particle, whose mass equals that of the system, acted on by forces equal and parallel to the external forces which are applied to the system.

EXAMPLES.

1. Show that if the air resistance were zero the acceleration of the mass-centre of any body thrown into the air in any way would equal g and be directed vertically downward.

Solution: Let W denote the weight of the body, and refer the motion to a set of axes fixed in the earth, the y axis being vertical, its positive end being up. Then since the only force acting on the body during the motion is its weight,

$$\Sigma F_x = 0 = (W/g)\bar{a}_x, \quad \Sigma F_y = -W = (W/g)\bar{a}_y, \quad \Sigma F_z = 0 = (W/g)\bar{a}_z,$$

or $\bar{a}_x = \bar{a}_z = 0$, and $\bar{a}_y = -g$; hence $\bar{a} = -g$.

2. The mass-centre of a "stationary" steam-engine when running is in general not a fixed point. Show that the reaction of its supports is not constant in amount.

Solution: Let m and W denote the mass and weight respectively of the engine, and $\bar{a}$ the acceleration of its mass-centre. Also let R_x, R_y, and R_z denote the x, y, and z components of the reaction of the supports, the y axis being taken vertical. Then eqs. (1) become

$$R_x = m\bar{a}_x, \quad R_y = +W + m\bar{a}_y, \quad R_z = m\bar{a}_z.$$

3. What is the greatest acceleration which a locomotive can give a train?

Solution: When the locomotive is pulling, the drivers tend to slip on the rails and the latter therefore exert on the former frictional forces directed forward. (These may be regarded as the forces which directly make the train move.) The rails exert on the other wheels of the train forces directed backward; call the sum of their horizontal components R', and the resistance of the air R''. If F denotes the sum of the frictional forces, m the mass of the train, and a its acceleration,

$$F-R'-R''=ma, \quad \text{or} \quad a=(F-R'-R'')/m.$$

F is maximum when the drivers are about to slip, R'' depends on the velocity, and is least when the velocity is zero, and R' does probably not depend much on the velocity and is practically independent of the acceleration. Hence the acceleration is greatest at low speeds if the drivers are about to slip.

4. What can you say of the forces which a travelling crane, moving with an acceleration, exerts on its track?

241. Moment of the Effective System About Any Axis.—Let the moment axis be taken as an x coordinate axis, and call the mass of any particle m, its acceleration a, its coordinates x, y, and z. The effective force for the particle is ma, and its x, y, and z components are ma_x, ma_y, and ma_z respectively, and the moment of the force about the x axis equals the sum of the moments of its components (Prop. I, art. 28), i.e.,

$$ma_z \cdot y - ma_y \cdot z.$$

The moment of the entire system about the x axis equals the sum of all such expressions as the above, or

$$\Sigma(ma_z y - ma_y z).$$

242. "Angular Motion" of a System of Particles.—According to Prop. I, art. 238, the moment sums of the external and effective force systems about any axis are equal. Hence if ΣM_x, ΣM_y, and ΣM_z denote the moment sums of the *external forces* about the x, y, and z coordinate axes respectively,

$$\Sigma M_x = \Sigma(ma_z y - ma_y z),$$
$$\Sigma M_y = \Sigma(ma_x z - ma_z x),$$
$$\Sigma M_z = \Sigma(ma_y x - ma_x y).$$

It can be shown that these and eqs. (1) art. 240 completely determine the effect of a force acting upon a rigid body. Since the components and the moments of a force do not depend on its application point, the equations also do not; hence the effect of a force on the motion of a rigid body does not depend on its application point. This is called the "principle of transmissibility."

CHAPTER XI.

TRANSLATION OF A RIGID BODY (RESUMED).

§ I. General Principles.

243. Equations of Motion.—Let m denote the mass of a body, a its acceleration, and a_x, a_y, and a_z the x, y, and z components of a. Then, from the equations of motion of the mass-centre of any system having any motion (art. 240),

$$\Sigma \mathbf{F}_x = \mathbf{m}\mathbf{a}_x, \quad \Sigma \mathbf{F}_y = \mathbf{m}\mathbf{a}_y, \quad \Sigma \mathbf{F}_z = \mathbf{m}\mathbf{a}_z. \quad . \quad . \quad . \quad (1)$$

ΣF_x, ΣF_y, and ΣF_z denoting the algebraic sums of the x, y, and z components of the external forces. It is advantageous to select the coordinate axes in a certain way in special cases; thus if the motion is a plane one, two axes should be taken in the plane of the motion, for then the equations reduce to two in number; and if the motion is rectilinear, one of the axes should be taken parallel to the direction of the motion, for then the number of equations reduces to one.

It is shown in the next article that the resultant of the effective forces for the particles of a translating body is a single force acting through the mass-centre in the direction of the acceleration, its magnitude being equal to the product of the mass of the body and its acceleration. According to D'Alembert's Principle, the resultants of the external and effective systems are identical; hence if the resultant of the *external* forces is denoted by R,

$$\mathbf{R} = \mathbf{m}\mathbf{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and it acts through the mass-centre in the direction of the acceleration.

244. Resultant of the Effective System.—In a translation, the accelerations of all the particles of the moving body at each instant are alike in magnitude and in direction. Hence the effective system consists of forces having the same direction, and they are proportional to the masses of the corresponding particles; therefore the resultant of the effective system is a single force.

Obviously the direction of the resultant at each instant is the same as that of the acceleration. The magnitude equals the sum of the separate effective forces, i.e.,

$$(dm)_1a_1+(dm)_2a_2+\ \dots\ =a\int dm=ma,$$

$(dm)_1$, $(dm)_2$, etc., denoting the masses of the particles. The action line of the resultant passes through the mass-centre, as can be shown thus: The effective forces constitute a system of parallel forces with fixed application points; hence they have a centre or centroid (art. 62). Let x, y, and z denote the coordinates of any particle with reference to a set of fixed axes, x_0, y_0, and z_0 the coordinates of the centroid of the effective forces, and $\bar{x}, \bar{y}$, and $\bar{z}$ the coordinates of the mass-centre. Then (see art. 63)

$$x_0=\frac{\int(dm\cdot a)x}{ma}=\frac{\int dm\cdot x}{m}=\bar{x}\quad \text{(see art. 228).}$$

Similarly it can be shown that $y_0=\bar{y}$ and $z_0=\bar{z}$; hence the centroid of the effective forces coincides with the mass-centre, and the resultant of the effective forces passes through the mass-centre as stated.

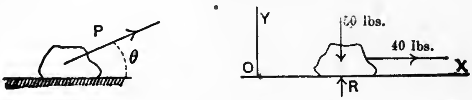
§ II. APPLICATIONS.

245. General Method of Procedure.—In the following problems the forces applied to a body are wholly or partially given and it is required to determine the acceleration, or else the acceleration is given and one or more forces are required. Such problems are solved by writing the equations of motion (or as many as necessary) and then solving them for the unknowns.

If it is required to determine the motion completely, i.e., to compute the acceleration, velocity, and position at each instant of the motion, the acceleration is found first as just indicated and then the velocity and position may be found by methods explained in Chapters VII and VIII.

BXAMPLES.

1. If the body represented in fig. 183(*a*) weighs 50 lbs., the pull P is 40 lbs., the angle θ is 0, and the supporting surface is



(*a*) (*b*)

FIG. 183.

smooth, compute the acceleration and the reaction of the support.

Solution: Let R denote the reaction of the plane; then the external forces acting on the body are as represented in fig. 183(*b*). The mass of the body is $50/32.2 = 1.553$ geepounds. Hence

$$\Sigma F_x = 40 = 1.553 a_x,$$
$$\Sigma F_y = R - 50 = 1.553 a_y,$$
$$\Sigma F_z = 0 = 1.553 a_z.$$

From the third equation $a_z = 0$, and from the first $a_x = 25.76$ ft./sec.^{2} Obviously $a_y = 0$, hence the acceleration of the body is 25.76 ft./sec.^{2} in the plus x direction. Since a_y equals zero, the second equation shows that $R = 50$ lbs.

2. Suppose that at a certain instant the mass-centre of the body of ex. 1 is at the origin, its velocity being 30 ft.-per-sec. in the plus x direction. Compute the velocity and position of the mass-centre two seconds later.

3. Suppose that at a certain instant the mass-centre of the body of ex. 1 is at the origin and that the velocity of the body at that instant is 30 ft.-per-sec. in the plus z direction. Determine the velocity and the position of the mass-centre two seconds later.

4. Solve ex. 1, supposing that θ equals 20°.

Ans. $a = 24.2$ ft./sec./sec.

5. Suppose that $P = 40$ lbs. and $\theta = 0$ (fig. 183), that the supporting surface is rough, the frictional resistance being 10 lbs.,

and that at a certain instant the body is at rest. Determine the subsequent motion.

6. Suppose that at a certain instant the body of ex. 5 is moving in the minus x direction with a velocity of 30 ft.-per-sec. Determine the subsequent motion.

7. Suppose that the inclined plane (fig. 184a) is smooth, that the body upon it weighs 50 lbs., and that P equals 40 lbs. Determine the acceleration and the reaction of the plane.

Solution: Let R denote the reaction of the plane; then the external forces acting on the body are as shown in fig. 184(b). Taking coordinate axes as shown, the mass being 1.553 geepounds,

$$\Sigma F_x = -40 + 50 \cos 60° = 1.553 a_x,$$
$$\Sigma F_y = R - 50 \sin 60° = 1.553 a_y,$$
$$\Sigma F_z = 0 = 1.553 a_z.$$

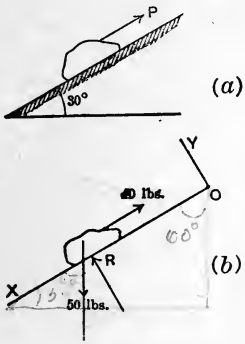


FIG. 184.

From the first and third respectively,

$$a_x = -9.66, \quad a_z = 0, \quad \text{and obviously} \quad a_y = 0;$$

hence $a = 9.66$ ft./sec.² in the negative x direction.

From the second equation it follows that $R = 43.3$ lbs.

8. Suppose that the plane in fig. 184(a) is rough and that the frictional resistance is 10 lbs. If at a given instant the velocity of the body is zero, determine the subsequent motion.

9. Suppose that at the instant mentioned in the preceding example the velocity is 40 ft.-per-sec. down the plane. Determine the subsequent motion.

10. Fig. 185(a) represents an open box on a horizontal surface, the box containing a stone and subjected to a force P. Let the weights of the box and stone be 90 and 70 lbs. respectively, P 100 lbs., and suppose that the supporting surfaces are smooth. Compute the pressure between the stone and the rear end of the box.

Solution: The external forces applied to box and stone together consist of the pull, the reaction of the horizontal support and the weights (see fig. 185b). The masses of the two bodies are 70/32.2 and 90/32.2, or 2.17 and 2.80 geepounds respectively. Evidently the acceleration a is in the direction of the 100-lb. force and its value is given by

$$100 = 4.97a, \quad \text{or} \quad a = 20.1 \text{ ft./sec./sec.}$$

The external forces acting on the box and stone respectively are shown in (c) and (d), P' and R' denoting pressures between the stone and the end and bottom of the box respectively. Knowing the acceleration of each body, the forces can be determined from the equations of motion of either; thus for the stone,

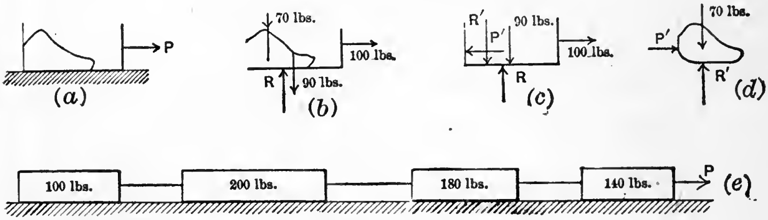


FIG. 185.

$$P' = 2.17 \times 20.1 = 43.62 \text{ lbs.}$$

11. Four bodies are connected by cords as shown in fig. 185(e) and are pulled along on a horizontal plane by a force P of 200 lbs. Supposing the frictional resistance on each body to equal one-fourth its weight, compute the tension in each cord.

12. In an elevator there are two boxes weighing 610 and 1000 lbs. respectively, the lighter being on top of the other. What are the pressures on the bottoms of the boxes if (a) the elevator is started up with an acceleration of 4 ft./sec./sec.? (b) If started down with the same acceleration?

Ans. (a) 1810 lbs. on the bottom of the lower box.

13. If the elevator of the preceding example weighs 1600 lbs., compute the tensions in the hoisting cable in the two cases mentioned.

14. Fig. 186(a) represents two bodies, A and B, suspended from the ends of a cord which passes over a "smooth pulley" C. Let the weights of A and B be W_1 and W_2 (W_2 being the greater).

Assume that the cord and pulley are practically without mass and that the axle friction is zero; then the tension is the same at all sections of the cord. Compute the acceleration of the bodies and the tension.

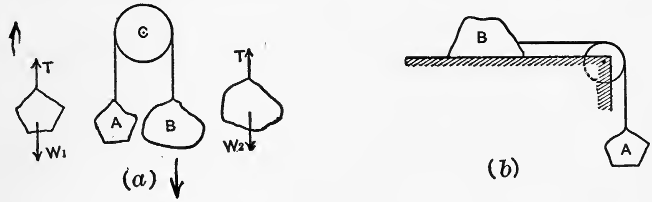


FIG. 186.

Solution: Let T denote the tension; then the external forces applied to each body are as represented. Assuming the string to be inextensible, the accelerations of the two bodies are equal; we will denote them by a. Evidently the accelerations of A and B are respectively upward and downward; hence the resultant forces on A and B act up and down respectively, i.e., $W_1 < T < W_2$. The masses of A and B being W_1/g and W_2/g respectively, the equations of motion for these bodies are

$$T - W_1 = \frac{W_1}{g}a \quad \text{and} \quad W_2 - T = \frac{W_2}{g}a.$$

Solving these equations for T and a, we find that

$$a = \frac{W_2 - W_1}{W_1 + W_2}g \quad \text{and} \quad T = \frac{2W_1W_2}{W_1 + W_2}.$$

15. Fig. 186(b) represents two bodies, A and B, connected by a cord passing over a smooth pulley, one hanging freely and the other supported by a horizontal surface. Let the weights of A and B be W_1 and W_2 respectively, and assume the surface to be smooth and that the tensions at different sections of the cord are equal. Compute the acceleration of the bodies and the tension.

$$\textit{Ans.}\ a = \frac{W_1}{W_1 + W_2}g, \quad T = \frac{W_1W_2}{W_1 + W_2}.$$

16. Solve the preceding ex., supposing that the $W_1 = 64$ lbs., $W_2 = 96$ lbs., and that the horizontal surface is so rough that its frictional resistance on the body is 19.2 lbs. (Take $g = 32$.)

T = 46.08# a = 8.96 sec²

17. Suppose that a stone weighing 100 lbs. is placed on a rough horizontal board, the coefficient of static friction for the stone and board being 1/4. If the board is moved horizontally with an acceleration of 4 ft./sec./sec., will the stone remain on the board?

18. Suppose that the stone of the preceding example is square in cross-section, 1 by 1 ft., and 8 ft. high, and that it stands so that two of its sides are parallel to the direction of the motion. How large an acceleration of the board will cause the stone to tip, supposing that the friction between the board and stone to be large enough to prevent slipping.

246. Kinetic Reactions.— When a body rests upon a horizontal support, the reaction between the support and body equals the weight, but if the body is in motion, the reaction in general does not equal the weight. Thus, as seen in ex. 12, art. 245, when an elevator moves with an acceleration a, the pressure which it exerts upon a body and its floor equals $W \pm ma$ (W, m, and a denoting the weight, mass, and acceleration of the body respectively), the plus or minus sign being used according as the acceleration is up or down. The reactions of the cranks on a coupling-rod of a locomotive when it is at rest each equal one-half the weight of the rod, but when it is in motion they have a different value. The difference depends in part on the acceleration as is shown below.

The components of a force acting on a body which depend upon acceleration are said to be "due to acceleration" and "due to inertia," and such are often called *inertia forces*. Sometimes it is desirable to determine the components of a reaction which are dependent and independent of acceleration. To distinguish them we shall call the former and latter *kinetic and static reactions* respectively. No new principles are necessary for the computation of these.

EXAMPLES.

1. Suppose that the mass of the slider represented in fig. 187 is m, the length of the crank c, and the number of revolutions per unit time n. Compute the crank-pin pressure at several

points of the stroke and plot curves showing how it varies with the displacement and the time.

Solution: Neglecting friction the only horizontal force on the slider is the crank-pin pressure. Hence, denoting that pressure by Q and the acceleration of the slider by a,

$$Q=ma,$$

the positive direction being taken the same for Q and a. Since $a=-4\pi^2n^2c \sin\theta=-4\pi^2n^2x$ (see eq. 3, art. 179),

$$Q=-m4\pi^2n^2c \sin\theta=-m4\pi^2n^2x;$$

and the maximum value of Q is given by

$$Q_m=-m4\pi^2n^2c.$$

Fig. 187 (*b*) and (*c*) shows graphically how Q varies with the displacement and the time.

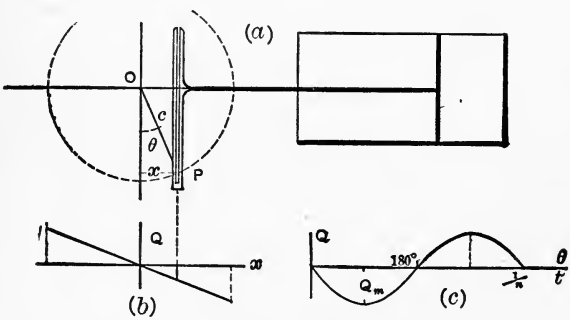


FIG. 187.

2. Assume that the slider of the preceding example is vertical, that $m=180$ lbs., $c=10$ in., $n=120$ rev.-per-min. Draw curves similar to those of fig. 187, showing the static and kinetic components of the entire pin reaction.

3. Assume that the slider of fig. 187 is driven by steam pressure P, its value being known at each point of the stroke. Assuming that the crank turns uniformly, determine the crank-pin pressure for any position.

Solution: (1) Algebraic. With notation as in ex. 1,

$$P+Q=ma \quad \text{or} \quad Q=ma-P,$$

a being equal to $-4\pi^2n^2c \sin\theta$ or $-4\pi^2n^2x$, as in ex. 1.

(2) Graphical and employing D'Alembert's Principle. Since P, Q, and the reversed effective force for the slider are at all times in equilibrium (art. 238), Q is equal and opposite to the resultant of P and the reversed effective force.

Let $a'a''$ (fig. 188) represent the length of the stroke, and the

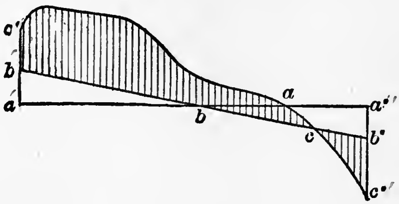


FIG. 188.

ordinates from $a'a''$ to the curve $c'c''$ the values of the steam pressure, P. From a' to a the pressure is forward and from a to a'' backward. The effective force for the slider at each instant equals the product of its mass and acceleration; it varies therefore with the acceleration, and the ordinates from $a'a''$ to the straight line $b'b''$ may represent the effective forces. During the first part of a stroke the acceleration is forward (in the direction of the motion) and in the second part backward; hence the reversed effective force in the first part of the stroke is backward and in the second forward.

The directions of the steam pressure and reversed effective force from a' to b are opposite, from b to a the same, and from a to a'' opposite again. Hence the resultants of the two forces from a' to b and from a to a'' are represented by the differences between the ordinates to the two curves, and from b to a by their sum, i.e., by the ordinates of the shaded area. Obviously from a' to c the direction of the horizontal pin pressure is opposite to the direction of the motion and from c to a'' in that direction.

4. Compute the kinetic reactions on a coupling or side rod of a locomotive running at a constant speed.

Solution: Let the notation be as follows:

r, radius of the crank-pin circles,	v, speed of the locomotive,
R, " " " drivers,	m, mass of the rod.

Since (by supposition) the acceleration of the locomotive is zero, the absolute acceleration of the rod is the same as its acceleration relative to the locomotive (see Prop., art. 196). Now the motion of each point of the rod is just like that of the centre of a crank-pin, and relative to the locomotive that motion is circular, the speed being rv/R. When a point travels in a circle with a constant speed, its acceleration equals the speed squared divided by the radius of the circle and its direction is along the radius toward the centre (art. 193). Hence the acceleration of the rod equals rv^2/R^2, and its direction at each instant is parallel to the crank.

The effective force therefore equals mrv^2/R^2, and if the mass-centre is at the middle of the rod each kinetic reaction equals $mrv^2/2R^2$, its direction being from the corresponding crank-pin to the centre of the wheel.

The total crank-pin pressure depends also on the weight of the rod and on the train resistance which is being "overcome."

5. Assume that the weight of one side rod is 275 lbs., crank radius 1 ft., wheel diameter $5\frac{1}{2}$ ft., and the speed 60 mi.-per-hr. Compute the crank-pin reactions due to acceleration and weight of the rod when it is at its lowest, highest, and middle positions.

247. Vibrations.—A study of vibrations furnishes applications of equations of motion in which the force is variable. As illustrations, we choose vibrations caused by open coil-springs in various circumstances. If such a spring hangs vertically from one end and supports a body at the other, and the body is displaced vertically from its position of rest and then released, it will oscillate or vibrate up and down, the vibration enduring for a time and then ceasing. This is called a *natural vibration*, and in general a natural vibration is one executed by a body or system which has been displaced or distorted, then released and left to itself.

If the support of the coil-spring is not fixed, but has a periodic up and down motion, the motions of the spring and suspended body are called *forced vibrations*. In general if a body or system executes a natural vibration and is then subjected to a periodic influence on its motion, the resulting vibration is called a forced one.

The dying out of a vibration is ascribed to forces of the nature of friction; thus in the preceding illustrations the forces are the resistance of the air upon the moving bodies and the internal or molecular friction in the spring. This effect (dying out) is known as *damping*, and vibrations in which damping occurs are called *damped vibrations*. For simplicity we will assume that there are springs without internal friction, and hence that such a spring and a suspended body might execute an undamped (or "simple") vibration.

If an open coil-spring is elongated or compressed (not excessively), the elongating or compressing force is proportional to the elongation or compression of the spring. Strictly, this is true only when the act of elongation or compression is slow; we will assume it to be true for a vibrating spring. No important error results if the mass of the suspended body is considerable as compared with that of the spring. Then if e and e' denote the elongations or compressions of a spring caused by forces T and T' respectively,

$$T/T'=e/e' \quad \text{or} \quad T=(T'/e')e.$$

248. *Undamped Natural Vibration.*—We use the following notation (see fig. 189):

l = natural length of the spring;
W = weight of the suspended body;
m = mass " " " "
e' = elongation caused by W;
$k = W/e'$;
y = displacement of the body from its position of rest;
a = acceleration " " " at the displacement y;
T = force exerted upon the body by the spring at the displacement y.

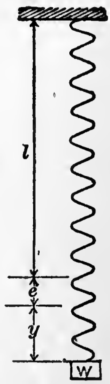


FIG. 189.

We choose the downward direction as positive for forces and displacements.

The elongation (or contraction) of the spring for all positions of the body is $(y+e')$; the spring is elongated or compressed according as $(y+e')$ is positive or negative. Also

$$T=-(W/e')(y+e')=-(W/e)y-W,$$

and the resultant force on the body in any position is

$$W+T=-(W/e')y=-ky. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence the equation of motion is

$$-ky=ma, \quad \text{or} \quad a=-(k/m)y, \quad . \quad . \quad . \quad . \quad (2)$$

i.e., the acceleration is proportional to y (displacement) and is always directed toward a fixed point (from which y is measured) in the path; therefore the motion is simply harmonic (see art. 179) and its period is $2\pi/\sqrt{k/m}$.

EXAMPLE.

Suppose that $W=5$ lbs., $e'=3$ in., and that the body is released from a position 10 in. below that of rest. Describe the motion.

249. *Damped Natural Vibration.*—The laws of the damping forces are various, depending upon the circumstances of the motion. If the vibrating system is like that represented in fig. 189, the suspended body being a thin vertical plate immersed in a viscous liquid, then (neglecting the internal friction in the spring) the damping force consists of the fluid friction at the sides of the plate. If the velocity of the plate is small, this friction is approximately proportional to the velocity. We choose this law of damping because the corresponding motion is analogous to an important electrical phenomenon.

Let v and v' denote any two velocities of the immersed plate and F and F' the corresponding frictions; then

$$F/F'=v/v' \quad \text{or} \quad F=(F'/v')v=cv,$$

c being an abbreviation for F'/v'. If F be regarded as positive or negative according as it acts down or up on the plate, then because $v=dy/dt$ (see the preceding article), $F=cdy/dt$, and the resulting force on the plate is

$$W+T+F=-ky-cdy/dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of motion of the plate becomes, instead of eq. (2) art. 248,

$$\left.\begin{aligned} -ky-c\frac{dy}{dt}&=m\frac{d^2y}{dt^2} \\ \text{or} \qquad \frac{d^2y}{dt^2}+\frac{c}{m}\frac{dy}{dt}+\frac{k}{m}y&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For brevity, let $p=c/m$ and $q=\sqrt{k/m}$; then the solution of eq. (2) gives the following:

(*a*) If $\frac{1}{4}p^2<q^2$,

$$y=A\epsilon^{-\frac{1}{2}pt}\sin(\sqrt{q^2-p^2/4}\,t+\epsilon), \quad . \quad . \quad . \quad . \quad (3)$$

ϵ being the "Naperian base" and A and ϵ constants of integration depending upon "initial conditions." Thus let $t=0$ and $dy/dt=v_0$ when $y=0$, then substituting these in eq. (3) and in the expression for dy/dt, we find that

$$\epsilon=0 \quad \text{and} \quad A=v_0/\sqrt{q^2-p^2/4}.$$

(*b*) If $\frac{1}{4}p^2>q^2$,

$$y=A\epsilon^{-\alpha t}+B\epsilon^{-\beta t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

α and β being abbreviations for $-p/2\pm\sqrt{p^2/4-q^2}$ respectively and A and B constants of integration depending on initial conditions. If as under (*a*), $t=0$ and $dy/dt=v_0$ when $y=0$, then substituting these in eq. (4) and in the expression for dy/dt, we find that

$$A=v_0/(\beta-\alpha) \quad \text{and} \quad B=v_0/(\alpha-\beta).$$

EXAMPLE.

Let $q=4$ rad.-per-sec. and $v_0=16$ ft.-per-sec. Plot on the same axes,

(1) Equation (4), p being 9 rad.-per-sec.
(2) Equation (3), p being 2 rad.-per-sec.
(3) Equation (3), p being zero.

250. *Undamped Forced Vibration.*—Imagine the support of the spring in fig. 189 to oscillate up and down and call its displacement from the position shown x, regarding x as positive or negative, according as the displacement is down or up. The elongation of the spring at any instant is not $y+\epsilon'$ as in art. 248, but $y+e'-x$; hence

$$T=-(W/e')(y+e'-x),$$

and the resultant force on the suspended body is

$$W+T=-k(y-x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation of motion is

$$\left.\begin{array}{ll} & -k(y-x)=m\, d^2y/dt^2 \\ \text{or} & d^2y/dt^2+(k/m)(y-x)=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now let the motion of the support be simply harmonic, its amplitude, period, and epoch being A, $2\pi/\omega$, and 0 respectively; then eq. (2) becomes

$$\frac{d^2y}{dt^2}+q^2y=q^2A\sin\omega t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

q being an abbreviation for $\sqrt{k/m}$, as in art. 249. The solution of the last equation is

$$y=\frac{A}{1-\omega^2/q^2}\sin\omega t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This shows that the forced vibration of the suspended body is simply harmonic, its period and epoch being the same as of the motion of the support, and its amplitude $1/(1-\omega^2/q^2)$ times that of the support. Notice that ω/q is the ratio of the frequency of the motion of the support to that of the natural vibration of the spring.

EXAMPLE.

Let $A=1$, and plot a curve showing how the amplitude of the forced vibration varies for different values of ω/q, between $1/10$ and 10.

251. *Damped Forced Vibration.*—Imagine that the suspended body of the preceding article is a vertical thin plate immersed in a viscous liquid. Then in addition to the forces acting on the body as described in the preceding article, there is the frictional resistance $F=-c\,dy/dt$ (see art. 249). Hence the equation of motion is

$$\left.\begin{array}{ll} & \dfrac{d^2y}{dt^2}+\dfrac{c}{m}\dfrac{dy}{dt}+\dfrac{k}{m}\dfrac{dy}{dt}=\dfrac{k}{m}A\sin\omega t, \\ \text{or} & \dfrac{d^2y}{dt^2}+p\dfrac{dy}{dt}+q^2y=q^2A\sin\omega t, \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

p and q, being abbreviations as in art. 249. The solution of this equation is

$$y=\frac{q^2\sin\varepsilon}{\omega p}A\sin(\omega t-\varepsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ε being an abbreviation defined by

$$\tan \varepsilon = \omega p(q^2 - \omega^2).$$

This equation shows that the motion of the suspended body is simply harmonic, its period and amplitude being respectively the same as and $(q^2 \sin \varepsilon)/\omega p$ times that of the motion of the support and lagging behind the latter an amount equal to ε.

EXAMPLE.

Assume that $q = 10$ rad.-per-sec. and that ω varies from 7 to 13 rad.-per-sec. Draw curves showing (*a*) how the lag ε varies with ω for values of p equal to 2, 1, 1/2, and 1/10, and (*b*) how the amplitude varies with ω for the four values of p just given.

252. Kinetic Friction.—*Definitions.* The friction between two bodies which move relative to each other is called kinetic friction, and the ratio of the kinetic friction to the normal pressure between the bodies is called their *coefficient of kinetic friction.* If f denotes the coefficient and F and N the friction and normal pressure respectively,

$$f = F/N \quad \text{or} \quad F = fN. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Laws of Friction for Dry Surfaces.—(1) The coefficient depends on the nature of the rubbing surfaces.

(2) The coefficient is approximately independent of the intensity of the normal pressure. Strictly it falls off slightly as the intensity increases up to a point when "seizing" is about to occur; then it increases rapidly.

(3) The coefficient decreases as the velocity increases—not directly, but rapidly as the velocity increases from 0 to 0+ and then less rapidly with increasing velocity.

(4) The coefficient decreases with the lapse of time during the motion.

These laws are qualitative; their reduction to a quantitative form has not been effected, but in some cases we know quite accurately how the coefficient varies, especially with velocity. Thus in certain experiments on the braking of railway trains, the following values of the coefficient for the brake-shoe (cast iron) and the wheel (steel) were determined:

Velocity in mi.-per-hr. .	17	21	27	31	37	47
Coefficient.	0.16	0.15	0.13	0.11	0.10	0.08

These coefficients were taken five seconds after the brakes were set; fifteen seconds after setting, the coefficients were about 0.04 less than those given above.

EXAMPLES.

1. Fig. 190(*a*) represents a body resting on a plank, which in turn rests on rollers. If f' is the coefficient of static friction for

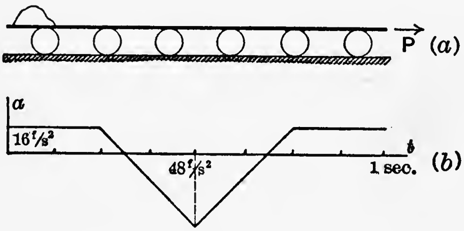


FIG. 190.

the body and plank, how large an acceleration can be given to the plank without causing slipping between it and the body?

Ans. $f'g$.

2. Suppose that the body is slipping on the plank, and let f'' denote the coefficient of kinetic friction assuming it to be independent of velocity. How long (t) and how far (s) will the body slide in coming to rest from a velocity v under the influence of friction?

Ans. $t=v/f''g$ and $s=v^2/2f''g$.

3. Suppose that the plank is caused to oscillate on the rollers by a force P, so that the acceleration-time curve for the motion is represented by fig. 190(*b*). Draw the velocity-time and space-time curves for the motion of the plank; also for the body, taking $f'=0.5$, $f''=0.2$, and $g=32$ ft./sec.^{2} Where on the plank is the body at the end of one second?

Ans. It has slid forward on the plank $2\frac{7}{24}$ ft.

CHAPTER XII.

ROTATION (RESUMED).

§ I. Second Moments of Mass (Moment of Inertia, etc.).

253. Occurrence of Second Moments.—In a study of the rotation of a body, certain quantities are met with which are expressed by integrals of the kind and form $\int dm \cdot u^2$ and $\int dm \cdot uv$, m denoting mass and u and v distances. Such quantities have been called "second moments of mass," the term being in line with "first moment of mass," which is applied to quantities expressed by integrals like $\int dm \cdot x$ (see arts. 227 and 228). We distinguish between second moments of mass employing special names for the kinds.

254. Moment of Inertia.—The moment of inertia of a body with respect to any axis is the sum of the products obtained by multiplying each elementary mass of the body by the square of its distance from the axis. The axis will often be called "*inertia-axis*" to distinguish it from other axes, coordinate, geometrical, etc.

[Euler introduced the term "moment of inertia," and he explained its appropriateness somewhat as follows ("Thoria Motus Corporum Solidorum," p. 167): The choice of the name, moment of inertia (Ger. *trägheitsmoment*) is based on analogies in the equations of motion for translations and rotations. In a translation, the acceleration is proportional directly to the "accelerating force" and inversely to the mass, or "inertia," of the moving body, and in a rotation the acceleration (angular) is proportional directly to the *moment* of the accelerating force and inversely to a quantity, $\int r^2 dm$, depending on the mass, or inertia; this quantity, to complete a similarity, we may call moment of inertia. Then we have—

for translations, acceleration = (force) / (inertia, or mass);
for rotations, acceleration = (moment of force) / (moment of inertia)].

Expression for Moment of Inertia.—Let I * denote the moment of inertia of a body with respect to any axis, dm the mass of any elementary portion all points of which are equidistant from the axis, and ρ that distance. Then the definition states that

$$\mathbf{I}=\int \mathbf{dm}\cdot\rho^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the limits of integration being such that all the elementary parts of the body are included in the integration.

If the body is homogeneous, δ denoting its density and dV the volume of the elementary mass, $dm=\delta dV$; hence

$$\mathbf{I}=\delta\int \mathbf{dV}\cdot\rho^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Units of Moment of Inertia.—From eq. (1) it is plain that the units involved in a moment of inertia are those of mass and length, and hence the unit of moment of inertia will depend upon the units of mass and length employed. No names are in use for the different units of moment of inertia, but each unit is described by stating the corresponding units of mass and length. Thus a moment of inertia computed by using the pound and foot is said to be expressed in a pound-foot unit. The unit corresponding to the geepound and foot may be called the engineers' unit of moment of inertia or the geepound-foot unit.†

255. Radius of Gyration.—Since a moment of inertia is one dimension in mass and two in length, it can be expressed as the product of a mass and a length squared; it is sometimes convenient to so express it.

Definition.—The radius of gyration of a body with respect to an axis is such a length whose square multiplied by the mass of the body equals the moment of inertia of the body with respect to that axis. That is, if k and I denote the radius of gyration and moment of inertia of the body with respect to any axis and m its mass,

$$\mathbf{k^2m=I} \quad \text{or} \quad \mathbf{k=\sqrt{I/m}}.$$

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* A subscript affixed to the symbol refers to the inertia-axis; thus I_x stands for moment of inertia with respect to an x axis.

† For dimensions of a unit moment of inertia, see Appendix C.

The square of the radius of gyration of a homogeneous body with respect to any axis is the mean of the squares of the distances of all the equal elementary parts of the body from that axis. For let ρ_1, ρ_2, etc., be the distances from the elements, dm, to the axis, and let n denote their number (infinite). Then the mean of the squares is

$$(\rho_1^2+\rho_2^2+\ldots)/n=(\rho_1^2dm+\rho_2^2dm+\ldots)/n\,dm=I/m.$$

But I/m is the square of the radius of gyration, hence, etc.

EXAMPLES.

1. Show that the moment of inertia and radius of gyration of a *homogeneous right circular cylinder* with respect to its geometric axis are respectively

$$\tfrac{1}{2}mr^2 \quad \text{and} \quad r\sqrt{1/2},$$

m denoting its mass and r the radius of its base.

Solution: Let a denote the altitude and imagine the cylinder to consist of elementary prisms parallel to the axis and extending from base to base. If dA denotes the area of the cross-section of any element, then $dV=adA$, and if ρ denotes the distance of the element from the axis,

$$I=\delta\int adA\cdot\rho^2.$$

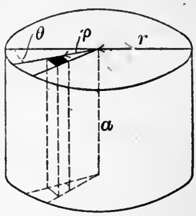


FIG. 191.

For convenience, select the prisms as indicated in fig. 191; then $dA=\rho d\rho\cdot d\theta$, and

$$I=a\delta\int_0^r\int_0^{2\pi}\rho^3d\rho\cdot d\theta=a\delta\tfrac{1}{2}\pi r^4=\tfrac{1}{2}mr^2.$$

Since $k^2=I/m$, $k=r\sqrt{1/2}$.

2. Show that the moment of inertia and radius of gyration of a *homogeneous parallelopiped* with respect to one of its geometrical axes are respectively

$$m(a^2+b^2)/12 \quad \text{and} \quad \sqrt{(a^2+b^2)/12},$$

m being the mass and a and b the lengths of the edges which are perpendicular to the inertia-axis.

3. Show that the moment of inertia and radius of gyration of a *homogeneous sphere* with respect to a diameter are respectively

$$\tfrac{2}{5}mr^2 \quad \text{and} \quad r\sqrt{2/5},$$

m being its mass and r its radius.

Solution: Let fig. 192 represent a diametrical section of the sphere and OY the inertia-axis. Imagine the sphere to consist of elementary laminas perpendicular to OY; then the mass of the lamina is given by

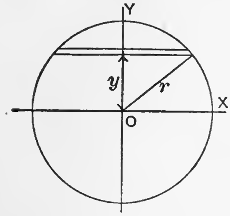


FIG. 192.

$$dm = \delta\pi(r^2 - y^2)dy.$$

Now according to ex. 1, the moment of inertia of the lamina with respect to its geometrical axis (OY) is $\tfrac{1}{2}dm(r^2 - y^2)$, or $\tfrac{1}{2}\delta\pi(r^2 - y^2)^2dy$. Therefore the moment of inertia of all the laminas, or of the sphere, is the sum of all such moments, i.e.,

$$I = \tfrac{1}{2}\delta\pi\int_{-r}^{+r}(r^2 - y^2)^2dy = \tfrac{8}{15}\delta\pi r^5 = \tfrac{2}{5}mr^2.$$

4. Show that the moment of inertia and radius of gyration of a homogeneous *right circular cone* with respect to its geometrical axis are respectively

$$\tfrac{3}{10}mr^2 \quad \text{and} \quad r\sqrt{3/10},$$

m being its mass and r the radius of the base.

5. Show that the moment of inertia of a circular lamina with respect to a diameter of either base is approximately $\tfrac{1}{4}mr^2$, m and r being its mass and the radius of the bases respectively.

Solution: Let t denote the thickness of the lamina. Imagine it to consist of elementary prisms whose bases coincide with those of the lamina, their cross-sections being as shown in fig. 193. The volume of each elementary prism is given by $t\rho d\theta \cdot d\rho$. Now all parts of each elementary prism are not equally distant from the inertia-axis, but they are nearly so, except in the case of prisms near the axis. These prisms, however, contribute little to the mo-

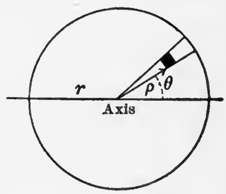


FIG. 193.

ment of inertia of the lamina and the error made in assuming the parts of such near prisms as equally distant from the axis is small. For any prism the distance is $\rho \sin \theta$; therefore approximately *

$$I = t\delta \int_0^{2\pi} \int_0^r \sin^2\theta \, d\theta \cdot \rho^3 d\rho = \tfrac{1}{4}\pi r^4 t\delta.$$

Hence, etc.

6. Show that the radius of gyration of a thin elliptic plate with respect to either of its axes is $\frac{1}{2}a$, $2a$ being the length of the other axis.

256. Relations Between Moments of Inertia and Between Radii of Gyration with Respect to Parallel Axes.—*Proposition.*—The moment of inertia (I) with respect to any axis equals the moment of inertia ($\bar{I}$) with respect to a parallel central axis plus the product of the mass (m) and the square of the distance (d) between the axes, that is,

$$\mathbf{I} = \mathbf{\bar{I}} + \mathbf{md^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Proof: Let fig. 194 represent a section of the body perpendicular to the inertia-axis. Let O and C be the points where that axis and the parallel central axis respectively pierce the section. Then (see the figure)

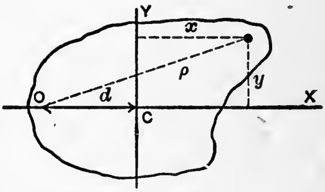


FIG. 194.

$$I = \int dm \cdot \rho^2, \text{ and since } \rho^2 = y^2 + (x+d)^2,$$

$$I = \int dm(y^2 + x^2 + 2xd + d^2)$$

$$= \int dm(x^2 + y^2) + 2d\int dm \cdot x + d^2 \int dm.$$

Now $\int dm(x^2+y^2) = \bar{I}$; $\int dm \cdot x = m\bar{x} = 0$; and $\int dm = m$; hence, etc.

Corollary: Dividing the sides of eq. (1) by m we get

$$I/m = \bar{I}/m + d^2,$$

or

$$\mathbf{k^2 = \bar{k}^2 + d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* The approximation is the closer the less the thickness.

k denoting radius of gyration of a body with respect to any axis, $\bar{k}$ that with respect to a parallel central axis, and d the distance between the axes.

Equations (1) and (2) show that the moment of inertia and radius of gyration of a body with respect to a central axis are less than for any other parallel axis, and that approximately, when $\bar{k}$ is small compared to d, $k=d$ and $I=md.^2$

EXAMPLES.

1. Determine the moment of inertia of a homogeneous parallelopiped, the lengths of its edges being a, b, and c and its mass m, with respect to the third edge. *Ans.* $m(a^2+b^2)/3$.

2. Determine the radius of gyration of a homogeneous sphere with respect to a line whose distance from the centre is x.

3. Determine the radius of gyration of a right circular cylinder with respect to a line parallel to its axis distant 10 ins. therefrom, the radius of the base being 4 ins.

4. Compute the radius of gyration of a rod 24 ins. long and 1×1 in. in cross-section with respect to a line parallel to a 1-in. edge and 12 ins. from the centre of the rod.

5. Show that the moment of inertia of a *right circular cylinder* with respect to a central axis parallel to the bases is $m(r^2/4+a^2/12)$, m denoting mass, a altitude, and r radius of the base.

Solution: Imagine the cylinder to consist of elementary circular laminas. Call the distance of any one of them from the inertia-axis x, then its thickness is dx and its volume is πr^2dx. According to ex. 5, art. 255, the moment of inertia of a lamina with respect to its central axis which is parallel to the inertia-axis is $\frac{1}{4}dmr^2$, and according to eq. (1), its moment of inertia with respect to the inertia-axis of the cylinder is

$$\tfrac{1}{4}dm\cdot r^2+dm\cdot x^2=\tfrac{1}{4}\delta\pi r^4dx+\delta\pi r^2x^2dx.$$

Therefore the moment of inertia of all the laminas, or of the cylinder, is given by (if a denotes the altitude),

$$I=2\delta\pi r^2\left(\tfrac{1}{4}r^2\int_0^{a/2}dx+\int_0^{a/2}x^2dx\right)$$

$$=2\delta\pi r^2(\tfrac{1}{8}r^2a+\tfrac{1}{24}a^3);\text{ etc.}$$

6. Show that the moment of inertia of a *right elliptic cylinder* with respect to a central axis parallel to either axis of the base is $m(a^2/4+h^2/12)$, $2a$ being the length of the other axis of the base and m and h the mass and altitude of the cylinder respectively.

7. Show that the moment of inertia and radius of gyration of a *hollow right circular cylinder* with respect to its geometrical axis are respectively

$$\tfrac{1}{2}m(r_1^2+r_2^2) \quad \text{and} \quad \sqrt{(r_1^2+r_2^2)/2},$$

m being its mass and r_1 and r_2 the inner and outer radii of its bases.

257. Composite Bodies.—We refer now to bodies which can be divided into simple component parts; thus, a flywheel consists of hub, spokes, and rim whose forms are usually simple. The moment of inertia of such a body with respect to any axis can be computed by adding the moments of inertia of the component parts *taken with respect to that same axis.*

EXAMPLE.

Compute the moment of inertia and radius of gyration of the cast-iron flywheel (weight 450 lbs.-per-cu. ft.) represented in fig. 195, the spokes being four in number and elliptic in cross-section.

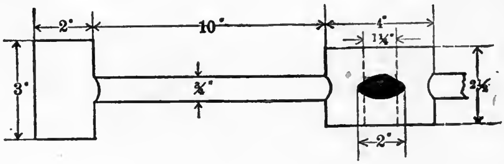


FIG. 195.

258. Experimental Determination of Moment of Inertia.—There are a number of methods; we give two.

Pendulum Methods.—(1) Suspend the body from an axis coinciding with or parallel to the inertia-axis. Let it oscillate like a pendulum noting the "time" of an oscillation (T), and determine the distance (a) from the axis of suspension to the centre of gravity of the body. Then substitute these values in the equation

$$T=\pi\sqrt{k^2/ag},^* \quad \text{or} \quad k=(T/\pi)\sqrt{g/a}$$

* This is the formula for the time of oscillation of a pendulum (see art. 267).

(g denoting the acceleration due to gravity), and solve for k. This is the radius of gyration of the body with respect to the axis of suspension, and from which the methods of art. 256 the desired moment of inertia can be computed.

(2) From the same axis about which the suspended body swings, suspend by means of a cord a body whose dimensions are small compared with the length of the cord. Adjust the length of the cord so that the times of oscillation of the two suspended bodies are equal. Then measure the distance (l) from the axis of suspension to the centre of gravity of the small body and solve for k in the equation

$$k^2/a=l \quad \text{or} \quad k=\sqrt{al},*$$

k and a having the same meanings as in (1).

Torsion-Balance Method.—There are many variations of the method here given. The balance for the present purpose may be arranged as follows: Suspend an elastic wire vertically, making the connection between the wire and support rigid, and fasten a flat plate in a horizontal position rigidly to the lower end of the wire—the form of the plate to be such that its moment of inertia with respect to the wire can be computed.

Place the body whose moment of inertia is desired and a second body on the plate so that they "balance," i.e., leave the plate horizontal. The first body is to be so placed also that the wire is parallel to the axis with respect to the inertia-axis and the form of the second body is to be such that its moment of inertia with respect to the wire can be computed. Now, cause the loaded balance to oscillate and note the time (T) of one oscillation; then remove the two bodies and cause the empty balance to oscillate, noting the time (T_1).

Let I_1, I_2, and I denote the moments of inertia with respect to the wire of the plate, of the second body, and of the one whose moment of inertia is desired respectively; then, as shown in art. 268,

$$T:T_1::\sqrt{I_1+I_2+I}:\sqrt{I_1},$$

or

$$I=I_1T^2/T_1^2-(I_1+I_2).$$

* Proved in art. 267.

259. Product of Inertia.—*Definition.*—The product of inertia of a body with respect to two coordinate planes is the sum of the products obtained by multiplying each elementary mass of the body by its ordinates from those planes.

Expression for Product of Inertia.—Let J denote the product of inertia of a body with respect to any two planes, as the yz and zx coordinate planes, dm the mass of any "third order" elementary volume, and x, y, and z its three coordinates; then the definition states that

$$J = \int dm \cdot xy,$$

the limits of integration being so assigned that all the elementary parts of the body are represented in the integration.

Since x and y have signs, the product $dm \cdot xy$ may be positive or negative, and hence a product of inertia may be (unlike a moment of inertia) negative or zero.

260. Principal Axes.—It can be proved that through any point of a body there are three mutually rectangular axes with respect to two of which the moments of inertia are greater and less than with respect to any other axis through the point. The three axes are called the principal axes of the body at that point.

The condition that a line may be a principal axis of a body at some point of its length is that $\int dm \cdot xz$ and $\int dm \cdot yz$ equal zero, the line being regarded as a z axis and the point as origin. (Proof must be omitted.) It follows from the foregoing that

(1) An axis of symmetry of a homogeneous body is a principal axis at every point of it.

(2) Any line perpendicular to a plane of symmetry of a homogeneous body is a principal axis at the point where it pierces that plane.

§ II. General Principles.

261. The Effective Forces.—Each particle of a rotating body revolves in a circle whose plane is perpendicular to the axis of rotation; therefore the acceleration of, and hence the effective force for, each particle has no component along the axis.

Let the irregular outline (fig. 196) represent a rotating body, the axis of rotation being perpendicular to the plane of the figure at O, and P any particle; also let

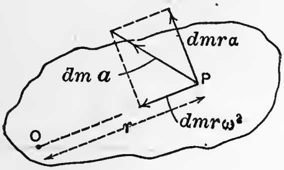


FIG. 196.

dm denote the mass of P;
r its distance from the axis;
a " acceleration;
α the angular acceleration of the body;
ω its angular velocity.

Then the effective force for P is $dm \cdot a$, its direction being the same as that of a and the tangential and radial components of the force are respectively (since $a_t = r\alpha$ and $a_n = r\omega^2$, art. 216)

$$\mathbf{dm \cdot r}\alpha \quad \text{and} \quad \mathbf{dm \cdot r}\omega^2.$$

262. Moment of the Effective System.—Of the two components of the effective force for any particle, only the tangential one has a moment about the axis; hence the sum of the moments of the effective forces is

$$\int (dm \cdot r\alpha)r = \alpha \int dm \cdot r^2 = I\alpha,$$

I being the moment of inertia of the body with respect to the axis of rotation.

263. Equations of Motion.—According to D'Alembert's principle, the sums of the moments of the external and effective forces are equal. Hence if ΣM denotes the sum of the moments of the external forces about the axis for any instant,

$$\Sigma \mathbf{M} = \mathbf{I}\alpha, \quad \ldots \ldots \ldots \quad (1)$$

α being the angular acceleration of the body at that instant. This is the equation of motion of the body.

As shown in art. 239 the sum of the components of the effective forces along any line equals the product of the mass of the body and the component of the acceleration of the mass-centre along that line. We wish this sum for three lines, the tangent, the normal to the path of the mass-centre at the mass-centre, and the axis. These sums are respectively

$$m\bar{a}_t = m\bar{r}\alpha, \quad m\bar{a}_n = m\bar{r}\omega^2, \text{ and } \quad 0,$$

wherein m denotes the mass of the body;

$\bar{a}_t$ the tangential acceleration of the mass-centre;

$\bar{a}_n$ " normal acceleration of the mass-centre;

$\bar{r}$ " distance from the mass-centre to the axis.

If ΣT, ΣN, and ΣA denote the algebraic sums of the components of the *external* forces along the tangent, normal and axis respectively, then according to D'Alembert's Principle,

$$\Sigma \mathbf{T} = \mathbf{m}\bar{\mathbf{r}}\alpha, \quad \Sigma \mathbf{N} = \mathbf{m}\bar{\mathbf{r}}\omega^2, \quad \Sigma \mathbf{A} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

These equations may be called the equations of motion of the mass-centre, since they involve terms depending on its motion. However, they are useful not to determine motion but especially to determine forces when the motion is known.

264. Resultant of the Effective System.—It is assumed in this article that the rotating body is homogeneous and that it has a plane of symmetry perpendicular to the axis of rotation. Then the axis of rotation is a principal axis of the body where it pierces the plane of symmetry (art. 260).

Imagine the body divided into elementary rods parallel to the axis of rotation and then one of these rods into elementary portions of equal length. These portions have at any instant the same acceleration, and hence the effective forces for them are equal and have the same direction. It follows that the resultant of these effective forces is a single force whose action line is in the plane of symmetry. The effective forces for all the rods therefore constitute a coplanar system, its plane being the plane of symmetry. The resultant of a coplanar system of forces is in general a force, and in special cases a couple (arts. 44 and 45); we proceed to determine these.

Let fig. 197(a) represent the plane of symmetry of a body perpendicular to the axis of rotation, O the intersection of the axis and the plane, and C the mass-centre. Also in addition to the preceding notation let

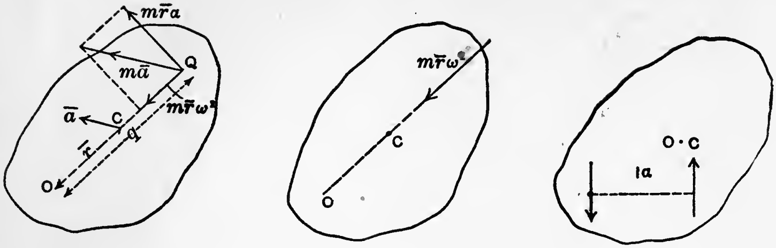
k be the radius of gyration of the body with respect to the axis;

$\bar{a}$ the acceleration of the mass-centre;

R denote the resultant effective force;

R_t its tangential component ($\perp$ to OC);

R_n " normal component ($\parallel$ to OC);



(a) (b) (c)

FIG. 197.

According to art. 263 the sums of the components of the effective forces along and perpendicular to OC equal respectively $m\bar{a}_n = m\bar{r}\omega^2$ and $m\bar{a}_t = m\bar{r}\alpha$. Therefore the components of the resultant effective force and the force itself are given by

$$\mathbf{R}_t = \mathbf{m}\bar{\mathbf{r}}\alpha, \quad \mathbf{R}_n = \mathbf{m}\bar{\mathbf{r}}\omega^2, \quad \mathbf{R} = \mathbf{m}\bar{\mathbf{a}},$$

and the direction of R is the same as that of the acceleration of the mass-centre ($\bar{a}$).

The action line of the resultant cuts the line OC at a point Q whose distance from O is $k^2/\bar{r}$. This may be proved as follows: Imagine the resultant force to be resolved into its two components ($m\bar{r}\alpha$ and $m\bar{r}\omega^2$) at Q. The moment of the force about O equals $(m\bar{r}\alpha)q$. But the moment of the force equals the sum of the moments of its components, and this was shown in art. 262 to be $I\alpha$; hence $m\bar{r}\alpha q = I\alpha$, or

$$\mathbf{q} = \mathbf{k}^2/\bar{\mathbf{r}}.$$

Three Special Cases.—(*a*) The angular velocity is constant, $\alpha = 0$. The resultant effective force equals $m\bar{r}\omega^2$, acts toward the axis and through the mass-centre (see fig. 197*b*).

(*b*) The axis of rotation contains the mass-centre, $\bar{r} = 0$. The resultant force equals zero, but $q = \infty$, i.e., the resultant is a couple and its moment is $I\alpha$ (see fig. 197*c*).

(*c*) The axis contains the mass-centre and the angular acceleration is zero. The resultant vanishes completely.*

* It can be shown that the resultant of the effective system is a force or a couple whenever the axis of rotation is a principal axis of the rotat-

265. *Centripetal and Centrifugal Force.*—When the resultant of the effective system is a single force, that of the external system is also a single force, and these resultants are identical; hence if C denotes the component of the latter resultant along the line joining the centres of rotation and mass,

$$\mathbf{C} = \mathbf{m}\bar{\mathbf{r}}\omega^2 = \mathbf{m}\bar{\mathbf{v}}^2/\bar{\mathbf{r}}.$$

The component C of the external forces acting on the rotating body is called *centripetal force* and the reaction corresponding to it is called *centrifugal force.* Observe that the first is exerted *on* the rotating body by other bodies and the second *by* the rotating body on the others, and that the centripetal force acts from the mass-centre towards the axis of rotation and the centrifugal in the opposite direction.

§ III. Applications.

266. Determination of the Motion.—If all the external forces which have a moment about the axis of rotation are given for any instant during the motion, the angular acceleration of the body at that instant can be computed from the equation of motion (art. 263). If the acceleration can be thus determined for each instant and the initial conditions of the motion (i.e., the angular velocity and the position of the body for any one instant) are also given, the motion can be completely determined.

EXAMPLES.

1. A body whose moment of inertia is I is made to rotate about an axis through the mass-centre by a constant force P applied to a cord wrapped about a cylindrical portion of the

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ing body at some point of the axis, and that, in this case, all the results of art. 264 hold if fig. 197 represents a section of the rotating body perpendicular to the axis of rotation, O the point where the axis is principal, and C the projection of the mass-centre on the section.

body, as shown in fig. 198. Determine the motion, neglecting axle friction and supposing that the angular velocity and the position of the body at a certain instant are known.

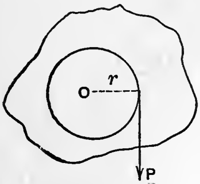

FIG. 198.

Solution: The external forces on the body are P, its weight, and the reactions of the supports on the axle. Of these only P has a moment about the axis, hence the equation of motion becomes

$$Pr = I\alpha \quad \text{or} \quad \alpha = Pr/I. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation shows that the angular acceleration is constant, and for a given "turning moment" (Pr) applied to different bodies, their angular accelerations are inversely proportional to their moments of inertia with respect to the axes of rotation.

Let the position of the body be specified by means of the angle θ which a fixed line of it makes with a fixed reference line as in art. 209, and suppose that when P begins to act the angle θ is zero and the angular velocity (ω) is also zero. Since $\alpha = d\omega/dt$,

$$d\omega = (Pr/I)dt \quad \text{or} \quad \omega = (Pr/I)t + C_1.$$

If time is reckoned from the instant when P begins to act, $\omega = 0$, when $t = 0$. Substituting these values in the last equation we get

$$0 = 0 + C_1, \quad \text{or} \quad C_1 = 0,$$

and hence $$\omega = (Pr/I)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $\omega = d\theta/dt$,

$$d\theta = (Pr/I)tdt, \quad \text{or} \quad \theta = \tfrac{1}{2}(Pr/I)t^2 + C_2.$$

Now $\theta = 0$ when $t = 0$, and these values substituted in the last equation give

$$0 = 0 + C_2, \quad \text{or} \quad C_2 = 0,$$

and hence $$\theta = \tfrac{1}{2}(Pr/I)t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. Suppose that the body in ex. 1 is a right circular cylinder of cast iron (weight 450 lbs.-per-cu.-ft.) 4 in. thick and 2 ft. in diameter, and that P is applied at the rim, its value being 6 lbs. Determine the acceleration. *Ans.* 0.82 rad./sec.²

3. Suppose that at a certain instant the wheel of ex. 2 is rotating at 20 rev.-per-sec. in the counter-clockwise direction.

What is its velocity 10 secs. later, P acting upon the wheel during that time?

4. Solve ex. 1, supposing that a body whose weight is W is suspended from the cord wrapped around the drum.

Solution: Let T represent the tension in the cord. This is the force replacing P of ex. 1, and the equation of motion for the rotating body is

$$Tr = I\alpha. \qquad (1)$$

This contains two unknowns T and α; to determine them we write next the equation of motion for the suspended body. The external forces on it are T and W, and as its acceleration is downward the resultant force on it is down, i.e., W is larger than T; hence

$$W - T = (W/g)a, \qquad (2)$$

a being the acceleration of the suspended body. These two equations contain three unknowns, T, α, and a, but we have the following additional relation:

$$a = r\alpha. \qquad (3)$$

From these three equations we find that

$$a = \frac{Wg}{W + mgk^2/r^2} \quad \text{and} \quad \alpha = \frac{Wg/r}{W + mgk^2/r^2},$$

m denoting the mass of the rotating body and k its radius of gyration with respect to the axis of rotation. The equations also determine the value of the tension; thus

$$T = W - W^2/(W + mgk^2/r^2).$$

5. Suppose that a wheel rotates about a horizontal axis "out of centre" as represented in fig. 199, and that the only external forces on the wheel are its own weight and the reaction of the supports (no friction). The angular velocity when C is directly to the right of O being given as ω_0 counter-clockwise, determine the angular velocity of the wheel in any position.

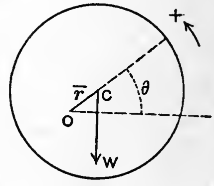

FIG. 199.

Solution: The equation of motion is

$$-W\bar{r}\cos\theta = I\alpha = Id^2\theta/dt^2,$$

and integration of it gives

$$-W\bar{r}\sin\theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2,$$

from which ω can be computed for any value of θ.

6. Solve ex. 14 art. 245 taking into account the mass of the pulley (then the tensions on opposite sides of the pulley are unequal), its radius of gyration with respect to the axis of rotation being k, its radius r, and its weight W.

$$Ans.\ \alpha = \frac{(W_2 - W_1)gr}{Wk^2 + (W_1 + W_2)r^2}$$

7. Solve ex. 15 art. 245 taking into account the mass of the pulley, its radius of gyration with respect to the axis being k, its radius r, and its weight W.

8. Solve ex. 16 art. 245 supposing that $k = \sqrt{7/9}$ ft., $r = 1\frac{1}{4}$ ft., $W = 144$ lbs., and $g = 32$.

9. Fig. 200 represents a tub floating upside down. Two cords are wrapped about the tub in opposite directions and lead off in parallel directions over pulleys as shown, and sustain bodies W and W. Discuss the motion of the tub under the influence of the suspended bodies and the fluid frictional resistance which assume to be proportional to the velocity.

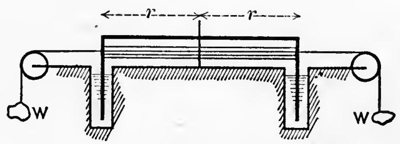

FIG. 200.

Solution: Let θ denote the angular distance described by the tub in any time t after starting, ω the angular velocity, and α the angular acceleration. Let F denote the frictional resistance at any instant; then since it varies as the velocity, $F = c\omega$, c being a constant depending on the liquid, diameter of tub and extent of the wetted surface. Let T denote the tensions in the cords, evidently the same at any instant, but not constant in time. Also let I denote the moment of inertia of the tub with respect to the axis of rotation, $2r$ its diameter, m and W the mass and weight of the suspended bodies, and a their acceleration.

The equations of motion for each suspended body and the tub are

$$W-T=ma \quad \text{and} \quad 2Tr-2Fr=I\alpha.$$

Combining these with $a=r\alpha$, we get

$$(I+2mr^2)\alpha+2rc\omega=2Wr,$$

and abbreviating, we have as the equation of motion

$$A\frac{d\omega}{dt}+B\omega=C. \quad \ldots \ldots \quad (1)$$

The first integration of this gives

$$\omega=\frac{C}{B}\left(1-\varepsilon^{-\frac{B}{A}t}\right), \quad \ldots \ldots \quad (2)$$

and the next

$$\theta=\frac{C}{B}\left(t+\frac{A}{B}\varepsilon^{-\frac{B}{A}t}-\frac{A}{B}\right). \quad \ldots \ldots \quad (3)$$

10. Plot a curve showing how the angular velocity changes, taking A, B, and C as 2, 5, and 10 respectively.

11. Suppose that there are no cords and suspended weights and that the tub is given an angular velocity ω_0. Discuss the motion of the tub under the influence of fluid friction.

Ans. $\omega=\omega_0\varepsilon^{-bt}$ (b being equal to $2rc/I$).

267. *Pendulums.*—A body which rotates about a horizontal axis under the influence of its weight and the reaction of the support is called a *compound or physical pendulum.* Let fig. 201 represent a section of such a pendulum perpendicular to the "axis of suspension" and through the mass-centre C. Let O be the intersection of the axis of suspension and the section, and let

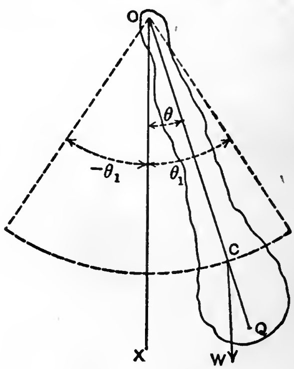

FIG. 201.

a denote the distance OC;

k the radius of gyration with respect to the axis of suspension;

T " time of one oscillation (from one extreme position to the other);

θ the angle XOC;
θ_1 " maximum value of θ;
W " weight of the pendulum;
m its mass.

We regard the counter-clockwise direction as positive and assume that the support is frictionless, or has no moment about the axis of suspension. Then the equation of motion becomes

$$-Wa \sin \theta = mk^2\alpha = mk^2 d^2\theta/dt^2,$$

or

$$d^2\theta/dt^2 = -(ag/k^2) \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The complete integration of eq. (1) is expressible by an infinite series, but it is not here given because we wish the value in a special case which admits of a simple approximate integration.

We will assume that the amplitude of the oscillations (θ_1) is so small that practically $\sin \theta = \theta$; then eq. (1) becomes

$$d^2\theta/dt^2 = -(ag/k^2)\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the first integration of which gives

$$(d\theta/dt)^2 = -(ag/k^2)\theta^2 + C_1.$$

Now when $\theta = \theta_1$, $d\theta/dt = 0$; therefore substituting these values in the last equation we find that

$$0 = -(ag/k^2)\theta_1^2 + C_1, \quad \text{or} \quad C_1 = (ag/k^2)\theta_1^2,$$

and

$$(d\theta/dt) = \pm(ag/k^2)^{\frac{1}{2}}(\theta_1^2 - \theta^2)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The plus or minus sign is to be used according as $d\theta/dt$ is positive or negative, i.e., according as the pendulum is swinging in the positive or negative direction. The integration of eq. (3) gives

$$\sin^{-1}(\theta/\theta_1) = \pm(ag/k^2)^{\frac{1}{2}}t + C_2.$$

Now if we reckon time from the instant when the pendulum passes through its lowest position, i.e., $t = 0$ when $\theta = 0$, the last equation becomes for these values

$$\sin^{-1}(0) = 0 + C_2; \text{ hence } C_2 = 0.$$

If when $t = 0$ the pendulum is moving in the positive direction,

$$\left.\begin{aligned} &\sin^{-1}(\theta/\theta_1) = (ag/k^2)^{\frac{1}{2}}t \\ \text{or} \quad &\theta = \theta_1 \sin(\sqrt{ag/k^2}\, t) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation is analogous to that for a simple harmonic motion (see art. 179), and hence the motion of a pendulum is often called "a simple harmonic oscillation."

The time of an oscillation can be found from eq. (4). Let t_1 and t_2 denote the values of t when the pendulum is in its highest positions on the right and left respectively. Then when $t=t_1$, $\theta=\theta_1$, and when $t=t_2$, it can be shown that $\theta=-\theta_1$, hence

$$t_1=\sqrt{k^2/ag}\,\sin^{-1} 1=\tfrac{1}{2}\pi\sqrt{k^2/ag},$$

and

$$t_2=\sqrt{k^2/ag}\,\sin^{-1}(-1)=\tfrac{3}{2}\pi\sqrt{k^2/ag},$$

or

$$T=\pi\sqrt{k^2/ag}.^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This expression for T being independent of θ, shows that the time of oscillation of a pendulum is the same for all values of θ_1, provided that it is so small that $\sin\theta_1$ practically equals θ_1.

The point Q (in OC) whose distance from O equals k^2/a is called the *centre of oscillation* and the line through it parallel to the axis of suspension is called the *axis of oscillation*. The following is a simple geometrical construction for locating the centre of oscillation: Let O (fig. 202) be the centre of suspension and C the mass-centre; then $OC=a$. Let $\bar{k}$ denote the radius of gyration of the pendulum with respect to a central axis parallel to the axis of suspension and lay off CK equal to $\bar{k}$; join O and K and draw KQ perpendicular to OK; then Q is the centre of oscillation. For

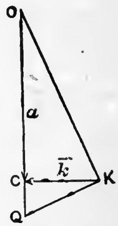

FIG. 202.

$$\overline{CK}^2=OC\cdot\overline{CQ}, \quad \text{or} \quad \overline{CQ}=\bar{k}^2/a,$$

hence $\overline{OQ}=a+\bar{k}^2/a=(a^2+\bar{k}^2)/a=k^2/a$

(see eq. (2), art. 256).

**Experimental Proof that the Masses of Bodies are Proportional to their Weights at the Same Place.*—The expression for the time of oscillation of a pendulum was deduced on the assumption that mass is proportional to weight, for we substituted g for W/m in deducing eq. 1. If this substitution is not made eq. (5) becomes $T=\pi k\sqrt{m/aW}$.

Now take two pendulums, each consisting of a sphere suspended by means of a light cord, the lengths of the cords being the same; also, to make the air resistances the same, the spheres should be equal in size. Next, compare their times of oscillation; *it will be found that they are equal.* Call this time T', and the weights and masses of the pendulums

By length of a compound pendulum for any given axis of suspension is meant the distance from that axis to the centre of oscillation. It is shown in the next paragraph that this is the length of an equivalent "simple pendulum."

A simple or mathematical pendulum is an ideal one consisting of a particle suspended by means of a massless cord. Evidently this is a special form of the physical pendulum, and the preceding discussion applies. Let l equal the length of the cord, then for the simple pendulum

$$k=a=l, \quad \text{or} \quad k^2/a=l,$$

hence

$$T=\pi\sqrt{l/g}. \qquad (6)$$

EXAMPLES.

1. A "seconds pendulum" (one whose time of oscillation is one second) is found to be 39.12 in. long at a certain place. What is the value of g at that place?

2. Show that the "axes of suspension and oscillation are interchangeable," i.e., that the times of oscillation are the same whether a pendulum oscillates about O or about Q.

3. Show that on any line OCQ there are four points or axes about which the pendulum will oscillate with the same time T.

4. According to the law of gravitation (art. 87) the attraction of the earth and hence the acceleration due to it varies inversely as the square of the distance from the earth's centre. Or if g_1 and g_2 denote the accelerations at two points whose distances from the centre are r and $r+e$ respectively,

$$g_1/g_2=(r+e)^2/r^2.$$

Show that if T_1 and T_2 are the times of oscillation of a pendulum at the two places respectively, approximately

$$T_1/T_2=1-e/r, \quad \text{and} \quad e=r(1-T_1/T_2).$$

W_1 and W_2, and m_1 and m_2, respectively. Evidently k has the same value for the two pendulums; also a. Hence the equation above becomes for the two pendulums

$$T=\pi k\sqrt{m_1/aW_1} \quad \text{and} \quad T=\pi k\sqrt{m_2/aW_2}$$

or

$$m_1/m_2=W_1/W_2.$$

This is practically the method employed by Newton; he used hollow wooden spheres containing gold, silver, lead, glass, sand, common salt, wood, water, and wheat.

268. *Torsion Balance.*—When a torsion balance (see art. 258) is displaced through a small angle (the "pan" being rotated about the wire), the moment of the couple required to produce the displacement is proportional to the displacement. Thus if M and M' denote two values of the displacing couple and θ and θ' the corresponding displacements

$$M/M' = \theta/\theta', \quad \text{or} \quad M = C\theta,$$

C being an abbreviation for M'/θ'.

The wire exerts upon the pan in any displaced position a couple whose moment is equal but opposite in sign to that of the displacing couple.

Imagine the pan displaced an amount θ_1, and then released; it will oscillate under the influence of the couple which the wire exerts upon it. If I denote the moment of inertia of the pan (and its contents if any) with respect to the axis of the wire, then the equation of the motion is

$$-C\theta = I\alpha, \quad \text{or} \quad d^2\theta/dt^2 = (-C/I)\theta. \quad . \quad . \quad . \quad . \quad (1)$$

This is analogous to eq. (1) art. 267, therefore its solution is left to the student. He should find that the time of oscillation (T) is given by

$$T = \pi\sqrt{I/C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

269. Conical Pendulum.—A conical pendulum consists of a body suspended from a fixed point by a cord and so that it can be made to rotate about the vertical axis through the fixed point (see fig. 203). The motion might be caused and maintained by means of a vertical board rotating about the axis and pressing laterally against the suspended body. We wish to determine the relation between the angular velocity (ω) of the body when constant and the "height" (h) of the pendulum.

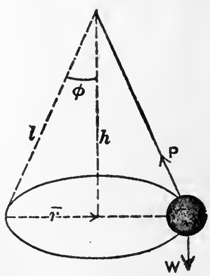

FIG. 203.

Let P denote the tension in the string;
W the weight of the body;
m its mass;
R the pressure of the board against the body;
n the revolutions per unit time.

Then neglecting air resistance, eqs. (2) art. 263, become

$$\Sigma T = R = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma N = P \sin \phi = m\bar{r}\omega^2 = ml \sin \phi \cdot \omega^2. \quad . \quad . \quad . \quad (2)$$

$$\Sigma A = P \cos \phi - W = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $h = g/\omega^2 = g/4\pi^2 n^2$, and $\cos \phi = g/l\omega^2 = g/l4\pi^2 n^2$.*

EXAMPLE.

Suppose that the weight of the rotating body (fig. 203) is 10 lbs., that it makes 100 rev.-per-min. and l is 15 in. What are the values of h and P?

270. Weighted Conical Pendulum Governor.—This consists of three heavy bodies, A, B, and C (fig. 204), connected by light links as shown, the whole system being supported at D and revolving about a vertical axis AD. We wish to determine the height (h) for a given angular velocity (ω). Let P_1 denote the force exerted on B by BD;

P_2 the force exerted on B by BA:
W " weights of B and C;
W_1 " weight of A.

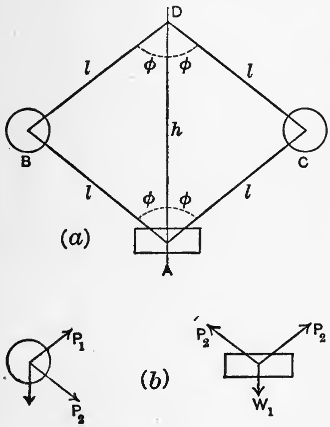

FIG. 204.

Then the forces exerted upon A and B are as shown in fig. 204(b). Eqs. (2) art. 263 for A become

$$\Sigma A = 2P_2 \cos \phi - W_1 = 0, \quad (1)$$

and for B,

$$\Sigma N = P_1 \sin \phi + P_2 \sin \phi = (W/g)l \sin \phi \cdot \omega^2, \quad . \quad . \quad (2)$$

$$\Sigma A = P_1 \cos \phi - P_2 \cos \phi - W = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $$h = 2(W + W_1)g/W\omega^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

EXAMPLE.

Let $W_1 = 10$ and $W = 8$ lbs., and draw a curve showing how h varies with the number of rev.-per-min.

---

* For any deflected position of the cord, $\cos \phi < 1$; hence $\omega > \sqrt{g/l}$ and $n > \sqrt{g/l}/2\pi$. If $\omega < \sqrt{g/l}$, or $n < \sqrt{g/l}/2\pi$, the pendulum will not remain in *any* deflected position however small. The time of one rotation at the critical speed, $n = \sqrt{g/l}/2\pi$, is the same as the time of one complete vibration of the bob as a simple pendulum (see page 268).

271. Kinetic Reactions.—*Definition* (repeated from art. 246).—By the kinetic reactions upon any body is meant such components of the forces acting upon it which depend upon its acceleration. The determination of kinetic reactions of rotating bodies is illustrated in the solution of some of the following

EXAMPLES.

1. A cubical box, into which a sphere just fits without pressure, is made to rotate about a vertical axis, as shown in fig. 205. Determine the kinetic reactions on the sphere when the angular velocity and acceleration are ω and α respectively.

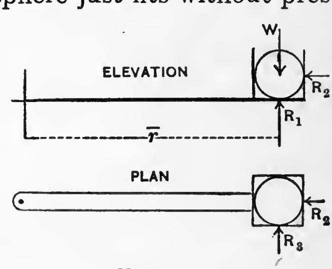

FIG. 205.

Solution: Let R_1 denote the pressure of the bottom of the box, R_2 that of the outer side, and R_3 the third one. Evidently the latter acts as shown if the acceleration is counter-clockwise. Equations (2) art. 263 become, if W and m denote the weight and mass of the sphere respectively, and $\bar{v}$ the velocity of the mass-centre,

$$\Sigma T = R_3 = m\bar{r}\alpha,$$
$$\Sigma N = R_2 = m\bar{r}\omega^2 = m\bar{v}^2/\bar{r},$$
$$\Sigma A = R_1 - W = 0, \quad \text{or} \quad R_1 = W.$$

These show that R_2 and R_3 are entirely kinetic and that R_1 is static.

2. Let $\bar{r} = 15$ ins., $W = 10$ lbs., and suppose that the angular velocity increases every second by 2 rev.-per-sec. Determine the kinetic reactions when the angular velocity is 10 rev.-per-min.

3. A body rests upon the floor of a car which moves in a horizontal circular curve of radius r with a constant speed v. Determine (1) the kinetic reaction on the body and (2) the direction of the resultant pressure of the body on the floor.

Ans. (2) Inclination to the vertical, $\tan^{-1}(v^2/rg)$.

4. A body is suspended by means of a cord from the ceiling of a car which moves in a horizontal circular curve of radius r with a constant speed v. Determine the direction of the sustaining cord and the tension in it.

5. Fig. 206 represents a car on a tilted track. Suppose that the track is a horizontal circular curve of radius r and that the car moves with a constant speed v. Determine the kinetic reaction on the car and the angle of tilt which makes the resultant of the flange pressures * zero.

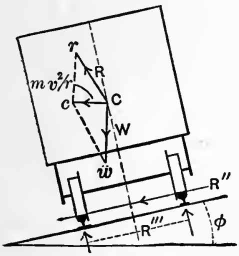

FIG. 206.

Solution: Imagine each wheel pressure resolved into three components, one parallel to the rails, one parallel to the ties, and one perpendicular to the first two. Call the sums of these components R', R'', and R''' respectively, and the resultant of R'' and R''' R; also let P' and P'' denote the pulls at the front and rear of the car respectively, which assume to be practically parallel.

Since the velocity of the car is constant, $\alpha = 0$ and (see eqs. (2) art. 263),

$$\Sigma T = P' - P'' - R' = 0,$$
$$\Sigma N = R'' \cos\phi + R''' \sin\phi = mv^2/r,$$
$$\Sigma A = R''' \cos\phi - R'' \sin\phi - W = 0.$$

These equations show that $R' = P' - P''$ and that $R = (m^2v^4/r^2 + W^2)^{\frac{1}{2}}$.

If R'' (the sum of the flange pressures) equals zero

$$\Sigma N = R''' \sin\phi = mv^2/r \quad \text{and} \quad \Sigma A = R''' \cos\phi - W = 0;$$

hence, combining $\tan\phi = v^2/gr.$†

* By flange pressure is here meant the component of the pressure on a wheel parallel to a tie of the track.

† This relation makes the *sum* of the flange pressures, but not each one necessarily, equal to zero. It has been discovered experimentally that if the wheels are coupled together in fours as usual, the front outer wheel always experiences a flange pressure.

These results can be reached graphically a little more simply as follows: Since R', P', and P'' are in equilibrium, R and W are equivalent to the effective force for the car, i.e., the resultant of R and W is identical with the resultant effective force. So draw from C a line Cc to represent the resultant effective force $mr\omega^2 = mv^2/r$, a line Cw to represent W, and complete the parallelogram $Cwcr$. Then Cr represents R, and from the figure it is seen that R equals the value given in the foregoing. In order that R may have no component along the tie, i.e., $R''=0$, the angle Crc must equal ϕ, or

$$\tan \phi = \overline{Cc}/\overline{cr} = (mv^2/r)/W.$$

272. *Weight of a Body as Influenced by the Earth's Rotation.*—Let fig. 207 represent a meridional section of the earth, ON being the polar axis. Imagine a body resting on the surface at A or suspended by means of a cord. The forces acting upon the body are two in number, the attraction of the earth (P), and the reaction of the support or the pull of the cord (Q). P is directed somewhat as shown; let its magnitude be represented by AB. Q is not collinear with P (except at the equator or pole) because the resultant of P and Q must be directed the same as the resultant effective force for the body; the direction of this is AD (the radius of the path of A).

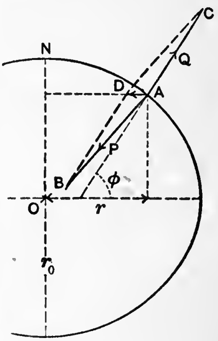

FIG. 207.

Let R denote the resultant effective force (also the resultant of P and Q), m the mass of the body, ω the angular velocity of the earth, and r the radius of the path of A; then

$$R = mr\omega^2,$$

and if AD represents R, the side AC of the parallelogram drawn on ABD represents Q. It follows from the figure that

$$P^2 = Q^2 + R^2 + 2QR \cos \phi,$$

or,

$$Q = P\sqrt{1 - \sin^2 \phi (R/P)^2} - R \cos \phi. \quad . \quad . \quad . \quad (1)$$

It is shown in ex. 1 below that R/P is less than $1/289$; hence, approximately

$$Q=P-R\cos\phi=P-mr\omega^2\cos\phi. \quad . \quad . \quad . \quad . \quad (2)$$

Q or its opposite is the force which we actually measure by spring- or beam-balance and call the weight of the body. It may be called "apparent weight" to distinguish it from the attraction P, or "real weight." Eq. (2) shows that the apparent weight is always less than the real, and that the difference depends on $r\cos\phi$.

Notice that ϕ is the latitude at A, since it is the angle made by the plumb line at A with the equatorial plane. Since R ($=mr\omega^2$) is *equal* to the centrifugal force of the body, the relationship in eq. (2) is sometimes expressed thus: "The weight of a body is diminished by the product of its centrifugal force and the cosine of the latitude."

EXAMPLES.

1. Show that at the equator the difference between the real and apparent weights of a body is about $1/289$ of the apparent weight.

Solution: From eq. (1), since $\sin\phi=0$ at the equator

$$(P-Q)/Q=R/Q=mr_0\omega^2/Q,$$

r_0 denoting the equatorial radius. Now $m/Q=1/g$, g denoting the acceleration due to gravity at the equator as measured experimentally, i.e., g is also "apparent"; hence

$$(P-Q)/Q=r_0\omega^2/g.$$

Now $\omega=2\pi/t$, where t denotes the time of one revolution of the earth; and since $t=86{,}164$ sec., $r_0=20{,}920{,}000$ ft., and $g=32.09$ ft./sec.^{2}, $r_0\omega^2/g=0.003467=1/289$.

2. Show that if the earth rotated 17 times as fast as it does, then the apparent weight of a body at the equator would be practically zero.

273. *Centrifugal Hoop Tension.*—Imagine a hoop to lie upon a horizontal table which rotates about a vertical axis through the centre of the hoop. The tension which exists at each cross section of the hoop is called "centrifugal hoop tension"; we now deduce an expression for it.

Let fig. 208 represent one half of the hoop. The forces acting on this half consist of its weight, the reaction of the table, and the forces exerted by the other half, i.e., the hoop tensions at the two sections. Since the first two forces balance each other, the resultant of the remaining two must equal the resultant effective force. Hence if m denotes the mass of one half of the hoop, $\bar{r}$ the distance from the axis to its mass-centre, ω its angular velocity, and P the hoop tension,

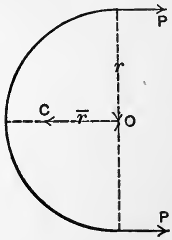

FIG. 208.

$$2P = m\bar{r}\omega^2, \quad \text{or} \quad P = \tfrac{1}{2}m\bar{r}\omega^2.$$

EXAMPLE.

Let r denote the radius of the hoop and w its specific weight. Regard the tension at a section as uniform (practically true when the thickness is small compared to the radius), and show that the intensity of the hoop tension equals $wr^2\omega^2/g$.

274. Hinge Reactions.—Rotating bodies often turn (a) about a fixed shaft or (b) with a shaft in fixed bearings. The force exerted by the shaft on the body in the first case and those exerted by the bearings on the shaft in the second will be called hinge reactions. Determination of hinge reactions in the following is limited to cases in which the rotating body has a plane of symmetry perpendicular to the axis. Then the resultant of the effective system for the body consists of a single force (see art. 264).

Case I. Rotation about a Fixed Shaft.—We assume that the applied forces are such that the hinge reaction is equivalent to a single force which call R. Let fig. 209 represent a section of the rotating body through the mass-centre (C) and perpendicular to the axis (O). Imagine R resolved into three components, one parallel to OC* one parallel to the axis, and one perpendicular to OC and the axis, and denote them by R_n, R_a (not shown), and R_t respectively.

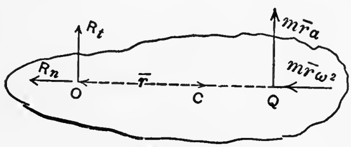

FIG. 209.

* If the mass-centre is in the axis the direction of OC may be taken any way perpendicular to the axis.

Let ΣF_n, ΣF_a, and ΣF_t denote the algebraic sums of the components of all applied forces (*i.e., all the external forces except the hinge reaction*), parallel to R_n, R_a, and R_t respectively. The components of the resultant effective forces parallel to these same directions are $m\bar{r}\omega^2$, 0, and $m\bar{r}\alpha$, acting as shown in fig. 209.

Since the external forces and the reversed effective forces are in equilibrium (art. 238),

$$R_t + \Sigma F_t = m\bar{r}\alpha, \quad \text{or} \quad R_t = m\bar{r}\alpha - \Sigma F_t,$$
$$R_n + \Sigma F_n = m\bar{r}\omega^2, \quad \text{or} \quad R_n = m\bar{r}\omega^2 - \Sigma F_n,$$
$$R_a + \Sigma F_a = 0,^* \quad \text{or} \quad R_a = -\Sigma F_a.$$

These equations show that R_a has no kinetic component, and that the kinetic components of R_t and R_n equal zero if $\bar{r}=0$, i.e., if the mass-centre is in the axis and if, as was assumed at the outset, the body has a plane of symmetry perpendicular to the axis.

Case II. The Body Rotates about a Shaft in Bearings.—We assume that there are two bearings whose reactions call R' and R''. Let A and B (fig. 210) be the bearings, and the parallelogram the plane of symmetry of the body.

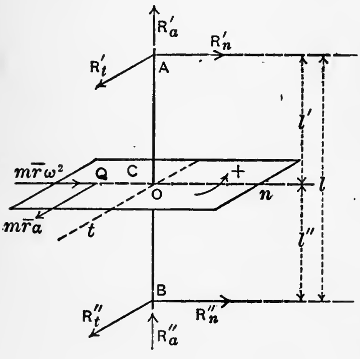

FIG. 210.

Imagine R' and R'' (like R, Case I) resolved into three components, and extend the notation of that case to the present one. Also let ΣM_t and ΣM_n denote the moment sums of the applied forces (*all the external forces not including* R' and R'') with respect to the lines marked Ot and On respectively. Then as all the external forces and the reversed resultant effective force are in equilibrium,

$$R_t' + R_t'' + \Sigma F_t = m\bar{r}\alpha,$$
$$R_n' + R_n'' + \Sigma F_n = m\bar{r}\omega^2,$$
$$R_a' + R_a'' + \Sigma F_a = 0,$$
$$R_n''l'' - R_n'l' + \Sigma M_t = 0,$$
$$R_t'l' - R_t''l'' + \Sigma M_n = 0.$$

* These follow also from eqs. (2), art. 263.

From these equations we find that

$$R_t'l=(m\bar{r}\alpha-\Sigma F_t)l''-\Sigma M_n,$$
$$R_n'l=(m\bar{r}\omega^2-\Sigma F_n)l''+\Sigma M_t,$$
$$R_t''l=(m\bar{r}\alpha-\Sigma F_t)l'+\Sigma M_n,$$
$$R_n''l=(m\bar{r}\omega^2-\Sigma F_n)l'-\Sigma M_t,$$
$$R_a'+R_a''=-\Sigma F_a.$$

These equations show that the kinetic components of the hinge reactions are zero if $\bar{r}=0$, i.e., if the mass-centre is in the axis, and if, as was assumed at the outset, the rotating body has a plane of symmetry perpendicular to the axis of rotation. An axis of a body for which the kinetic components of the hinge reactions are zero is called a "free axis." It can be shown that the three central principal **axes** of *any* body are free axes.

EXAMPLES.*

1. Suppose that in ex. 5 art. 266, $W=100$ lbs., $I=10$ (gee-pound-foot units), $\bar{r}=1/2$ ft., and $\omega_0=4$ rad.-per-sec. Compute the hinge reaction when C is directly to the right of O.

Solution: According to the solution of ex. 5, the angular acceleration in the position under consideration is -5 rad.-per-sec.-per-sec. Hence (see fig. 211)

$$R_t-100=-(100/32.2)\cdot\tfrac{1}{2}\cdot 5, \text{ or } R_t=92.2 \text{ lbs.}$$
$$R_n=(100/32.2)\cdot\tfrac{1}{2}\cdot 16=24.8 \text{ lbs.}$$

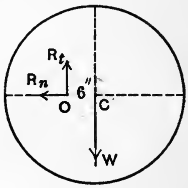

FIG. 211.

2. Determine the hinge reactions in ex. 1, when C is vertically above O, below O, and to the left of O.

3. Suppose that the wheel of the preceding example revolves about a vertical axis with a constant angular velocity of 4 rad.-per-sec. Determine the hinge reaction in any position of the wheel, AO and OB (see fig. 210) being 5 inches.

* The student is advised not to use the foregoing formulas, but to proceed as follows: (1) Determine the resultant effective force for the rotating body in the position under consideration, remembering that the normal component acts from the mass-centre toward the axis and the tangential component in the direction of the tangential component of the acceleration of the mass-centre. (2) Write the conditions of equilibrium for the force system consisting of that resultant reversed and all the external forces (including the hinge reactions). (3) Solve.

4. Determine the hinge reaction on the rotating body described in ex. 2, art. 266.

5. Imagine the rotating body of fig. 210 to be a parallelopiped whose weight is 90 lbs., and whose radius of gyration with respect to a central axis parallel to the axis of rotation is 7.84 in., that $l' = l'' = 18$ in., and $OC = 1$ in. Suppose that the body is rotated by means of a couple applied to a thin disk (mass negligible), as shown in fig. 212. Determine the hinge reactions when the motion is about to begin ($\omega = 0$).

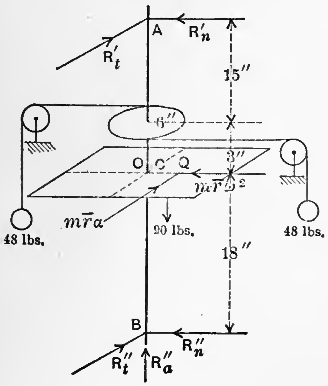

FIG. 212.

Solution: The masses of the suspended and rotating bodies are respectively

$$48/32.2 = 1.49 \quad \text{and} \quad 90/32.2 = 2.79 \text{ geepounds.}$$

The moment of inertia of the rotating body about the axis of rotation is

$$2.79\ (7.84/12)^2 + 2.79\ (1/12)^2 = 1.21 \text{ (geepound-foot units).}$$

Hence the equations of motion of the suspended and rotating bodies are respectively, T denoting tension in the cord,

$$48 - T = 1.49a, \quad T \cdot 1 = 1.21\alpha;$$

and since here $a = \alpha/2$, we find from the equations that $\alpha = 24.5$ rad.-per-sec.-per-sec.

The components of the resultant effective force are

$$m\bar{r}\alpha = 2.79 \times \tfrac{1}{12} \times 24.5 = 5.68 \quad \text{and} \quad 0,$$

and act as shown in the figure. Since the reversed resultant effective force and the external forces are in equilibrium, we next write as many conditions of equilibrium for that system as are necessary to determine the unknowns. Thus, for the action line of R_t'' as moment axis,

$$R_n' \cdot 3 - 90 \cdot 1/12 = 0, \quad \text{or} \quad R_n' = 2.5 \text{ lbs.,}$$

for the action line of R_n'' as moment axis,

$$R_t' \cdot 3 - 5.68 \cdot 1\tfrac{1}{2} = 0 \quad \text{or} \quad R_t' = 2.84 \text{ lbs.};$$

and since $R_t' + R_t'' = 5.68$ and $R_n' + R_n'' = 0$,

$$R_t'' = 2.84 \quad \text{and} \quad R_n'' = -2.5 \text{ lbs.}$$

From the "axis resolution equation"

$$R_a'' - 90 = 0, \quad \text{or} \quad R_a'' = 90 \text{ lbs.}$$

6. Determine the hinge reactions when the body has rotated through 90°, 180°, 270°, and 360°, and record your results in a tabular form as follows:

	a	ω	$m\bar{r}a$	$m\bar{r}\omega^2$	R_t'	R_n'	R_t''	R_n''	R_a''
0°	24.5	0	5.68	0	2.84	2.5	2.84	−2.5	90
90°	24.5								

7. Suppose that in ex. 5 there is no couple applied and that the body rotates with a constant speed of 100 rev.-per-min. Determine the hinge reactions.

8. Suppose that in ex. 7 there is a second parallelopiped just like the first attached to the shaft, also 1 in. "out of centre," and so that the two mass-centres are on opposite sides of the shaft, the second one being 3 in. vertically above the first. When they are rotating at 100 rev.-per-min., determine the hinge reactions.

275. Balancing of Rotating Bodies.—As illustrated by exs. 5–8 of the preceding article, when one or more bodies rotate with a shaft, the hinge reactions have in general kinetic components, i.e., the hinge reactions depend in part at least upon the motion. These kinetic components continually change direction so that the bearings are subjected to injurious influences to prevent which it is sometimes desirable to "balance" the rotating system.

Balancing consists in arranging the rotating bodies on the shaft so that the effective forces for the rotating system are in equilibrium; then the kinetic components of the hinge reactions vanish, and the hinge reactions depend only on external forces.

In the following we consider only rotation at constant speed ($\alpha=0$), and rotating systems consisting of one or more bodies which have planes of symmetry perpendicular to the axis. Then the resultant effective force for each rotating body is a force whose action line is in that plane and passes through the mass-centre and the axis, and whose magnitude is $m\bar{r}\omega^2$ (m denoting mass of the body, $\bar{r}$ the distance of its mass-centre from the axis, and ω the angular velocity of the system).

Since the centrifugal force which each rotating body exerts upon the shaft is equal and opposite to the resultant effective force for the body, balance is effected if the centrifugal forces (or the reversed effective forces) for all the rotating bodies are in equilibrium. This is the usual view of balancing and we shall follow it.

For convenience we will write r instead of $\bar{r}$ and will often say "body" instead of "mass-centre of a body." We will also denote both a body and its mass by m.

276. *Balancing Bodies whose Mass-centres rotate in the Same Plane.*—Let m_1, m_2, and m_3 (fig. 213) represent three bodies whose common centre of rotation is O. Their centrifugal forces equal respectively

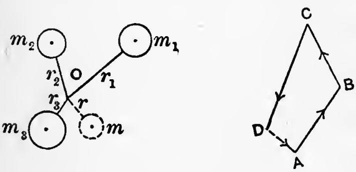

FIG. 213.

$$m_1r_1\omega^2, \quad m_2r_2\omega^2, \quad m_3r_3\omega^2,$$

their directions being from O to the mass-centre of the rotating body in each case.

Let AB, BC, and CD represent the magnitudes and directions of the three centrifugal forces. Then a fourth force acting through O, which would close the force polygon $ABCD$, would balance the three centrifugal forces. This fourth or balancing force can be supplied by adding to the system of rotating bodies a fourth one whose centre of rotation is O, whose mass-centre is in the direction DA from O, and whose distance from the axis (r) and mass (m) are such that

$$mr\omega^2=\overline{DA} \text{ (by scale)}, \quad \text{or} \quad mr=\overline{DA}/\omega^2.$$

It will be noticed that ω is a common term in the forces represented by the lines of the force polygon. Hence, if the polygon for the centrifugal forces closes, it will also close for forces parallel to them of magnitudes m_1r_1, m_2r_2, etc.; so that if AB, BC, and CD are drawn parallel to the first three centrifugal forces respectively but equal (by scale) to m_1r_1, m_2r_2, m_3r_3, then DA represents mr to that same scale.

277. *Balancing Bodies whose Mass-centres Rotate in Different Planes.*—In general such bodies cannot be balanced by adding to the rotating system a single body. It will now be shown that any rotating system can be balanced by adding *two* bodies which rotate in *any* selected planes.

A Single Rotating Body.—Let m denote the mass of the given body, r the distance from its mass-centre to the axis, ω the angular velocity, and O (fig. 214*a*) its centre of rotation.

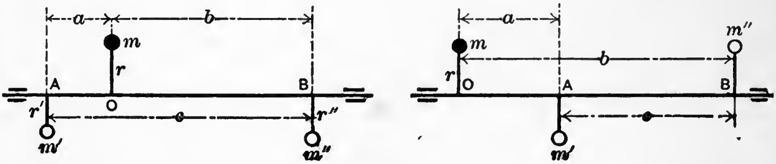

FIG. 214*a*.

Also let A and B be the selected centres of rotation of the two balancing bodies, m' and m'' their masses, and r' and r'' the distances from their mass-centres to the axis as shown.

The centrifugal forces of m, m', and m'' are $mr\omega^2$, $m'r'\omega^2$, and $m''r''\omega^2$ respectively. In order that they may be in equilibrium,

(1) The plane of the mass-centres of the three bodies must contain the axis of rotation, and

(2) The algebraic sum of the moments of the three forces about any point in their plane must equal zero; hence

$$m'r' = mrb/c \quad \text{and} \quad m''r'' = mra/c.$$

(Notice that it is not necessary to use units of a kinetic system in these equations; any unit of length and unit of mass may be used. Generally the foot and pound, or inch and pound, will be most convenient.)

From these two equations we can compute $m'r'$ and $m''r''$ in terms of the given quantities, m, r, a, b, and c; we may arbitrarily select either of the factors in $m'r'$ and $m''r''$ and then compute the remaining two.

Two cases may be distinguished: (*a*) the balancing bodies are on opposite sides of the given body; (*b*) they are on the same side of it. Condition (2) requires in each case that the middle one of the three bodies (m, m', and m'') shall be alone upon one side of the axis and the outer ones together on the opposite side of the axis.

Any Number of Rotating Bodies.—Let m_1, m_2, m_3, etc., denote the masses of the given rotating bodies, and let the centres of rotation of the balancing bodies be denoted by A and B. By the method just explained, determine two bodies, m_1' and m_1'' rotating about A and B respectively which will balance m_1; likewise two bodies m_2' and m_2'' rotating about A and B respectively which will balance m_2, etc. Then the given bodies together with m_1', m_2', . . . m_1'', m_2'', . . . would be "in balance."

Next compound the centrifugal forces of m_1', m_2', . . . and determine a single body m' whose centrifugal force is equivalent to that resultant; likewise compound the centrifugal forces of m_1'', m_2'', . . . and determine a single body m'' whose centrifugal force is equivalent to that resultant. Then m' and m'' are the two bodies sought.

EXAMPLES.

1. In the left-hand part of fig. 214(*a*), let $m=20$ lbs., $r=15$, $a=6$, and $b=14$ in. Determine the balancing bodies m' and m''.

Solution: According to the equation under (2), page 281,

$$m'r'=(20\cdot 15\cdot 14)/20=210 \text{ lb.-in.}$$

and

$$m''r''=(20\cdot 15\cdot 6)/20=\ 90 \text{ lb.-in.*}$$

If r' and r'' are taken as 6 and 8 in. respectively, then

$$m'=210/6=35 \quad \text{and} \quad m''=90/8=11.25 \text{ lbs.}$$

* It should be noticed that products like mr are moments of mass, see arts. 227 and 228.

2. Solve the preceding example, but let the data refer to the right-hand part of fig. 214 (a).

3. Let m_1, m_2, and m_3 (fig. 214b) be three bodies on the

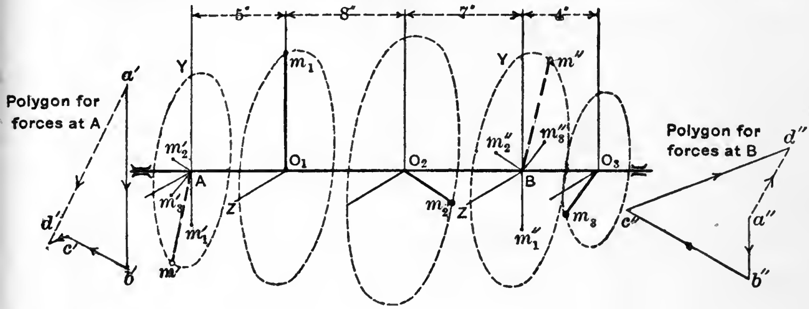

FIG. 214b.

shaft AB, and determine two bodies rotating about A and B which will balance them.

Solution: The data are arranged in the first five columns of the adjoining tabulation, θ denoting the angles which the "radii" of the different rotating bodies make with the vertical plane through the axis of rotation, and a and b having the same meanings as in fig. 214 (a). The quantities in columns 6 to 12 were determined by the methods explained in the foregoing text.

1	2	3	4	5	6	7	8	9	10	11	12
m	r	θ	a	b	mr	mra	mrb	$m'r'$	θ'	$m''r''$	θ''
10	6	0	5	15	60	300	900	45	180	15	180
8	7	240	13	7	56	728	392	19.6	60	36.4	60
9	4	110	24	4	36	864	144	7.2	110	43.2	290

The centrifugal forces of the imagined balancing bodies rotating about A are $45\omega^2$, $19.6\omega^2$, and $7.2\omega^2$; $a'b'c'd'$ is their force polygon, and hence $a'd'$ represents their resultant. The angle which $a'd'$ makes with the vertical is 147° 40′, and the mass-moment represented by $a'd'$ is 44.5 lb.-in.; hence a single body of 8.9 lbs. fixed to the shaft with a radius of 5 in. in the position m' is one of the required balancing bodies.

The centrifugal forces of the imagined balancing bodies rotating about B are $15\omega^2$, $36.4\omega^2$, and $43.2\omega^2$; $a''b''c''d''$ is their force polygon, and hence $a''d''$ represent their resultant. The angle which $a''d''$ makes with the vertical is $328° 40'$, and the mass-moment represented by $a''d''$ is 20.4 lb.-in.; hence a single body of 3.4 lbs. fixed to the shaft with a radius of 6 ins. in the position m'' is the other one of the required balancing bodies.

4. Move A of fig. 214 (b) to a point midway between O_1 and O_2, and move B to a point 4 in. to the right of O_3; then solve the preceding example.

278. Pivot and Journal Friction.—*Flat Pivot.*—Let fig. 215(a) represent the flat end of a shaft which is pressed or "thrust" against a flat bearing and rotates. Let r denote the radius of the end, N the thrust or normal pressure, and f the coefficient of friction for the rubbing surfaces.

Regarding the normal pressure as uniform, its value per unit area is $N/\pi r^2$, and the normal pressure on any elementary area as that indicated in fig. 215(a) is $(N/\pi r^2)\rho d\theta d\rho$ while the friction on that area is f times that elementary pressure. The moment of the elementary frictional force about the axis of

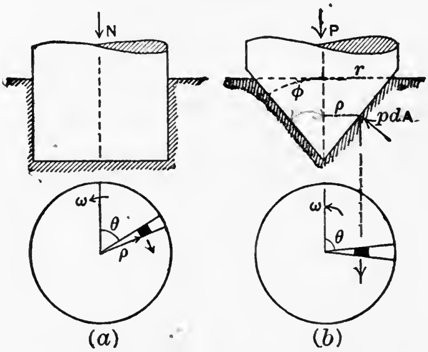

FIG. 215.

the shaft is ρ times the force, and the sum of all such moments is

$$f(N/\pi r^2)\int_0^r \int_0^{2\pi} \rho^2 d\rho d\theta = fN\tfrac{2}{3}r.$$

The resultant friction is not a force but a couple, and hence we may regard the actual frictional resistance as a couple whose forces equal fN and whose arm is $\frac{2}{3}r$.

Conical Pivot.—Let fig. 215(b) represent a rotating conical pivot which is pressed into its step by a thrust P directed along the axis. Let r denote the radius of the step, ϕ the angle shown, and p the normal pressure per unit area regarded as constant.

The normal pressure on an elementary area dA of the bearing is pdA and its vertical component is $(pdA)\sin\phi$. Since the friction has no vertical component the sum of all the vertical components of the normal pressure must equal P, hence

$$P = p\sin\phi\int dA = p\sin\phi\cdot A = p\pi r^2, \quad \text{or} \quad p = P/\pi r^2.$$

Note that the normal pressure per unit area is independent of ϕ.

The frictional force on each element of area dA is $f(P/\pi r^2)dA$, and its moment with respect to the axis is ρ times the force (see the figure); hence the entire frictional moment equals

$$f(P/\pi r^2)\int dA\cdot\rho.$$

For simplicity take dA of such form that its horizontal projection equals $\rho d\rho d\theta$ (see the figure), i.e., that $(dA)\sin\phi = \rho d\rho d\theta$. Then the above expression can be written

$$f(P/\pi r^2)\csc\phi\int_0^r\int^{2\pi}\rho^2 d\rho d\theta = f\frac{P}{\sin\phi}\frac{2}{3}r.$$

Journal Friction.—In general it is not known how the normal pressure (and hence the friction) varies over the surface of a journal. It is customary to compute the "axle" or "journal friction" from

$$F = f'R,$$

in which F denotes the value of a single resistance applied to the surface of the journal whose moment about the axis is the same as that of the actual frictional resistance, R the resultant pressure between journal and bearing, and f' a coefficient of journal friction.

The coefficient f' is determined from $f' = F/R$, R and F having been experimentally measured. The values of f' and of f (for pivots) depend on circumstances, as described below. They range from about 0.004 in the most favorable cases to about 0.08 in ordinary lubrication.

*Friction of Lubricated Surfaces.**—"The laws which appear to express the behavior of well-lubricated surfaces are almost the reverse of those of dry surfaces." Thus the frictional resistance

(1) is almost independent of the pressure with bath lubrication . . . ;

(2) varies directly as the speed for low pressures . . . ;

(3) depends more upon the temperature than on any other condition . . . ;

(4) with flooded bearings, depends but slightly upon the nature of the material of which the surfaces are composed...;

(5) of rest is enormously greater than the friction of motion...;

(6) is least at first, and rapidly increases with the time after the two surfaces are brought together. . . .

EXAMPLES.

✓ 1. Show that the value of the frictional moment on a hollow flat pivot is $\frac{2}{3}fN(r_2^3-r_1^3)/(r_2^2-r_1^2)$, r_1 and r_2 denoting the inner and outer radii respectively of the pivot.

2. Deduce an expression for the frictional moment on a pivot formed of a frustrum of a cone, there being no pressure on the lower base of the frustrum. Use notation of fig. 215(*b*) and call the radius of the step r_2 and that of the end of the pivot r_1.

* Abridged and quoted from Goodman's "Mechanics Applied to Engineering." Chap. VII of that work is an extensive discussion of the subject of Friction.

CHAPTER XIII.

ANY PLANE MOTION OF A RIGID BODY (RESUMED).

§ I. General Principles.

279. The Effective Forces.—Let fig. 216 be a section of the moving body parallel to the plane of the motion, and P and P' any two points of the body in that section.

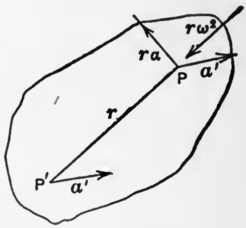

Fig. 216.

Let a' denote the acceleration of P';
α the angular acceleration of the body;
ω its angular velocity;
r the distance $P'P$;
dm " mass of the particle at P.

According to art. 220, the acceleration of P can be resolved into three components as shown in the figure, one being the same as the acceleration of P' and the other two being components of the acceleration of P relative to P'. Therefore the effective force for the particle at P can be resolved into three components whose directions are the same as those of the three component accelerations, their values being

$$\mathbf{dm \cdot a'}, \quad \mathbf{dm \cdot r\alpha}, \quad \text{and} \quad \mathbf{dm \cdot r\omega^2}.$$

In accordance with art. 221, all components like $dm \cdot a'$ may be called the translational effective forces and all those like $dm \cdot r\alpha$ and $dm \cdot r\omega^2$ may be called the rotational effective forces.

280. Moment of the Effective System.—The moment axis is taken perpendicular to the plane of the motion and through any point of the body. Let P' (fig. 217) be the point and $P'X$ and $P'Y$ be two axes parallel to the plane of the motion and fixed in direction. Now the moment of the effective force for the particle at P equals the sum of the moments of its components; the moment of $dm \cdot r\omega^2$ is zero, that of $dm \cdot r\alpha$ is $(dm \cdot r\alpha)r$, and that of $dm \cdot a'$ equals the sum of the moments of its x and y components ($dm \cdot a_x'$ and $dm \cdot a_y'$) or $-(dm \cdot a_x')y$ and $(dm \cdot a_y')x$ respectively. Hence the moment of the effective force equals

$$(dm \cdot r\alpha)r + (dm \cdot a_y')x - (dm \cdot a_x')y,$$

and the sum of all such moments (for all the particles of the body) is

$$\alpha \int dm \cdot r^2 + a_y' \int dm \cdot x - a_x' \int dm \cdot y,$$

or $$I\alpha + ma_y'\bar{x} - ma_x'\bar{y},$$

I being the moment of inertia of the body with respect to the moment axis, $\bar{x}$ and $\bar{y}$ the coordinates of the mass-centre at the instant for which the moment is computed.

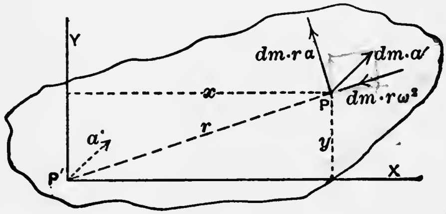

FIG. 217.

This expression for moment simplifies considerably for two special moment axes, as follows:

(*a*) If the moment axis contains the mass-centre, $\bar{x}$ and $\bar{y}$ are zero, and the expression for moment reduces to $\bar{I}\alpha$, $\bar{I}$ denoting the moment of inertia with respect to that axis.

(*b*) If the moment axis coincides with the instantaneous axis of no acceleration (the line all points of which have at the instant no acceleration), a_x' and a_y' are zero, and the expression for moment reduces to $I\alpha$, I denoting the moment of inertia of the body with respect to that axis.

281. Equations of Motion.—Let the motion of the mass-centre be referred to a set of fixed axes, x, y, and z, the last being perpendicular to the plane of the motion; also let

$\bar{a}_x$ denote the x acceleration of the mass-centre;
$\bar{a}_y$ the y acceleration of the mass-centre;
ΣF_x " sum of the x components of the external forces;
ΣF_y " " " " y " " " " "
$\Sigma \bar{M}$ " " " " moments " " " " about an axis through the mass-centre and perpendicular to the plane of the motion.

As shown in art. 240

$$\Sigma \mathbf{F}_x = \mathbf{m}\bar{\mathbf{a}}_x, \quad \Sigma \mathbf{F}_y = \mathbf{m}\bar{\mathbf{a}}_y; \quad \text{also} \quad \Sigma \bar{\mathbf{M}} = \bar{\mathbf{I}}\alpha, \quad . \quad . \quad (1)$$

since, according to D'Alembert's principle, the sums of the moments of the external and effective forces about any line are equal.

282. Resultant of the Effective System in Important Special Cases.—It is assumed in this article that the body is homogeneous and has a plane of symmetry which is parallel to the plane of the motion. Imagine the body divided into elementary rods perpendicular to the plane of the motion. As explained in art. 264, the effective force for each of these is one whose action line is in the plane of symmetry; hence the effective forces for all the rods constitute a coplaner system, its plane being the plane of symmetry.

We proceed to determine the resultants of the effective forces corresponding to translational and rotational components of the motion. In general, the resultant of each set is a single force. Let fig. 218(a) represent the section of symmetry of the body, C the mass-centre and P' any other point of the section. In addition to the notation of the foregoing articles, let

$\bar{r}$ denote the distance from C to P';
$\bar{a}'$ the acceleration of C relative to P', i.e., its acceleration in the rotational component;
k the radius of gyration of the body with respect to the axis through P' perpendicular to the plane of the motion.

(a) Regarding the motion as resolved into a rotation about the axis through P' and the corresponding translation.—According to art. 244, the resultant of the translational effective forces equals ma' and acts in the direction of a' through the mass-centre as shown. According to art. 262, the resultant of the rotational effective forces equals $m\bar{a}'$ and acts in the direction of $\bar{a}'$ through a point Q in the line $P'C$ such that $P'Q$ equals $k^2/\bar{r}$. This last force may be resolved into two components $m\bar{r}\alpha$ and $m\bar{r}\omega^2$, as represented, fig. 218(*a*).

(*b*) Regarding the motion as resolved into a rotation about the axis through the mass-centre and the corresponding translation (P' coincides with C).—The resultant of the translational effective forces is a force equal to $m\bar{a}$ and acts in the direction of $\bar{a}$ through the mass-centre as shown in fig. 218(*b*). According to art. 262, the resultant of the rotational forces is a couple whose moment is $\bar{I}\alpha$.

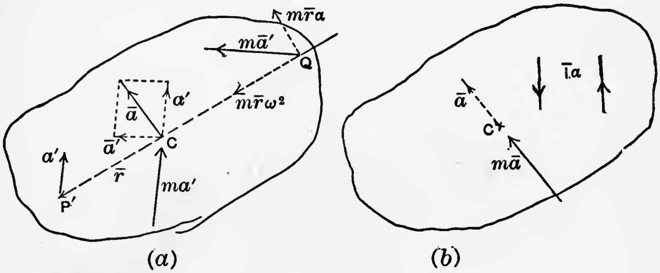

FIG. 218.

If the acceleration of the mass-centre and the angular acceleration of the body are both zero, the resultant of the effective forces vanishes.

§ II. APPLICATIONS.

283. Determination of the Motion.—The general method consists in writing the equations of motion for the case in hand and deducing from them the value of the angular acceleration of the body and that of the acceleration of a point of it. The method is further explained in the solution of some of the following

EXAMPLES.

1. A homogeneous cylinder rolls without slipping down an inclined plane. Determine the motion.

Solution: Let W denote the weight of the cylinder, and m its mass; also let F and N denote the components of the reaction of the plane along and perpendicular to its surface (fig. 219).*

The external forces acting on the cylinder are F, N, and W; hence with coordinate axes as shown, eqs. (1), art. 281 become

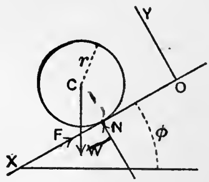

FIG. 219.

$$\Sigma F_x = +W \sin\phi - F = m\bar{a}_x, \quad \Sigma F_y = N - W\cos\phi = m\bar{a}_y,$$

$$\Sigma \overline{M} = Fr = \bar{I}\alpha.$$

Evidently $\bar{a}_y = 0$, hence $\bar{a} = \bar{a}_x$. Now $\bar{v} = r\omega$ (see ex. 2, art. 220); hence $\bar{a} = r\alpha$. This equation and the first and third above determine $\bar{a}$ and α. We find that (since $\bar{I} = \frac{1}{2}mr^2$)

$$\alpha = \tfrac{2}{3}(g\sin\phi)/r,$$
$$\bar{a} = \tfrac{2}{3}g\sin\phi.$$

We may also determine the reaction of the plane; from the first equation

$$F = \tfrac{1}{3}W\sin\phi \quad \text{and from the second,} \quad N = W\cos\phi.$$

2. Show that there will be slipping between the cylinder and plane if $\frac{1}{3}\tan\phi$ is greater than the coefficient of static friction.

3. If in ex. 1 $\phi = 30°$ and the coefficients of static and kinetic friction are respectively 0.25 and 0.2, determine the angular acceleration of the cylinder and the linear acceleration of its mass-centre.

4. A homogeneous cylinder rests on a smooth horizontal plane, and a horizontal force P is applied by means of a string wrapped about it (see fig. 220). Determine the angular acceleration of the cylinder and the acceleration of its mass-centre.

---

* We assume in this and the following examples on rolling that the rolling body and the surface on which it rolls do not distort each other, thus leaving a point or line of contact; then there is no "rolling resistance" (art. 285).

Solution: The external forces acting on the cylinder are P, weight (W), and the resistance (N) of the surface. Equations (1) of art. 281 become (see fig. 220),

$$\Sigma F_x = P = m\bar{a}_x, \quad \Sigma F_y = 0 = m\bar{a}_y, \quad \Sigma \overline{M} = Pr = \bar{I}\alpha.$$

From the last equation, $\alpha = Pr/\bar{I}$, and from the first $\bar{a}_x = P/m$. Since $\bar{a}_y = 0$, $\bar{a} = \bar{a}_x = P/m$.

5. Solve ex. 4, supposing that the plane is rough, its coefficient of kinetic friction being f.

6. Fig. 221 represents a wheel rolling on a horizontal surface. If the distance of its centre of gravity from the centre is c, and its radius of gyration with respect to an axis through its centre and perpendicular to the plane of the motion is k, show that

$$-gc \sin\theta = rc \sin\theta \cdot \omega^2 + (r^2 - 2rc\cos\theta + k^2)\alpha.$$

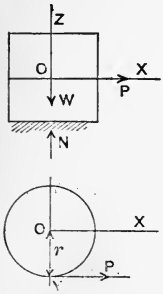

FIG. 220.

284. Kinetic Reactions.—In the following examples the motion of the body under consideration is given and the unknown quantities are forces. These are determined by use of the equations of motion, but sometimes other forms of the equations are more convenient. These other forms are obtained, like those of art. 281, by D'Alembert's principle, i.e., by expressing algebraically the fact that the external forces acting on a body are equivalent to the resultant effective force for it. This equivalence can be expressed in many ways; thus, to deduce the equations of art. 281, we wrote two resolution equations and one moment equation with moment-axis through the mass-centre and perpendicular to the plane of the motion. But we may write one resolution and two moment equations or three moment equations. The first example below is solved from two sets of equations, thus illustrating different forms of the equations of motion.

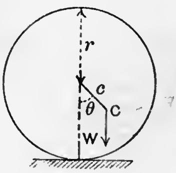

FIG. 221.

EXAMPLES.

1. A wheel whose mass-centre is not in its geometrical axis is rolled along a horizontal roadway at a constant speed by means of a horizontal force (P) applied at its centre. Determine the value of P and the reaction of the roadway for any position of the wheel.

Solutions: (1) By use of the equations of art. 281. Let fig. 222 represent the wheel, P' being its centre and C its mass-centre, and let F and N denote the horizontal and vertical components of the reaction of the roadway. The external forces acting on the wheel are W, P, F, and N; hence eqs. (1), art. 281, become

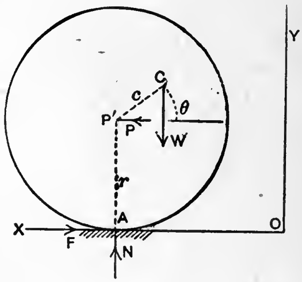

FIG. 222.

$$\Sigma F_x = P - F = m\bar{a}_x, \quad . \quad (1)$$
$$\Sigma F_y = N - W = m\bar{a}_y, \quad . \quad (2)$$
$$\Sigma \overline{M} = F(r + c \sin\theta) - Nc \cos\theta - Pc \sin\theta = 0. \quad . \quad . \quad . \quad (3)$$

Now the acceleration of C ($\bar{a}$) equals the vector sum of the acceleration of P' and that of C relative to P' (art. 196). Since the wheel rolls uniformly, the acceleration of P' is zero and that of C relative to P' is $c\omega^2$ in the direction CP'; hence

$$\bar{a} = c\omega^2 \text{ directed along } CP'.$$

Also $\bar{a}_x = c\omega^2 \cos\theta$ and $\bar{a}_y = -c\omega^2 \sin\theta$,

and eqs. (1), (2), and (3) may be written

$$F = P - mc\omega^2 \cos\theta, \quad N = W - mc\omega^2 \sin\theta, \text{ and}$$

$$(P - mc\omega^2 \cos\theta)(r + c \sin\theta) - (W - mc\omega^2 \sin\theta)c \cos\theta - Pc \sin\theta = 0.$$

From these we find that

$$P = (Wc/r + mc\omega^2) \cos\theta,$$
$$F = W(c/r) \cos\theta,$$
and
$$N = W - mc\omega^2 \sin\theta.$$

(2) The three components of the resultant effective force as described in art. 282, case (a), are ma', $m\bar{r}\alpha$, and $m\bar{r}\omega^2$. P'

(fig. 218a) being chosen at the centre of the wheel, a' is zero; and since α is zero, the resultant effective force is $m\bar{r}\omega^2 = mc\omega^2$ directed from C to P'. Now by D'Alembert's principle F, N, P, and W are equivalent to $mc\omega^2$, or they are in equilibrium with $mc\omega^2$ reversed. Hence

$$\Sigma M_A = Pr - Wc\cos\theta - mc\omega^2 r\cos\theta = 0,$$
$$\Sigma M_{P'} = Fr - Wc\cos\theta = 0,$$
$$\Sigma F_y = N - W + mc\omega^2 \sin\theta = 0.$$

From these (without elimination) we get the same values for P, F, and N as were found in (1).

The *kinetic reaction* (on the wheel) is vertical and equals $mc\omega^2 \sin\theta$; it acts downwards for values of θ between 0° and 180° (C is above P') and upwards for values of θ between 180° and 360° (C is below P'). Otherwise stated, when C is above P', N is less than W, and when C is below P', N is greater than W. The kinetic component of the reaction N is called "hammer blow" in locomotive parlance.

2. Show that the wheel of ex. 1 lifts from the roadway in a certain position if the speed of the centre of the wheel is greater than $r\sqrt{g/c}$.

3. In which position of C is N a maximum, and what is that value if the velocity is slightly less than $r\sqrt{g/c}$.

4. A steam-engine connecting-rod with no piston-rod attached is drawn by the crank. It is required to determine the kinetic reactions at the ends of the rod in any position.

Solution: Let AB and $P'B$ (fig. 223) be the crank and rod respectively, C being the mass-centre of the latter. Let $\bar{r}$ denote the length CP', a the acceleration of P', m the mass of the rod, α and ω its angular acceleration and velocity respectively, and k the radius of gyration of the rod with respect to a line perpendicular to the plane of the motion at P'.

We first determine the resultant effective force (R) for the rod. Its three components as given by art. 282 are

$$ma, \quad m\bar{r}\alpha, \quad \text{and} \quad m\bar{r}\omega^2,$$

acting as shown in the figure, the distance of Q from P' being

$k^2/\bar{r}$. Supposing these to have been computed, R can be readily found graphically (the construction is not shown).

Let V and P denote the reactions on the rod at P' and B respectively; the "guide-rods" being supposed frictionless, V

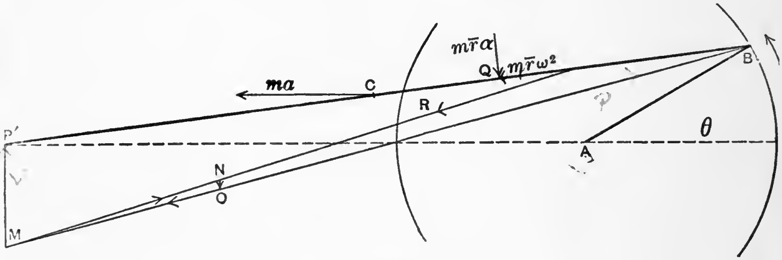

FIG. 223.

is vertical. Since V, P, and R reversed are in equilibrium, their action lines intersect in a point. So produce R to intersect V and join that intersection with B; this line is the action line of P. To determine the values of V and P, we draw a force triangle for V, P, and R reversed. Thus, lay off MN to represent the magnitude of R, and from N draw a vertical line to intersect the action line of P, marking that point O; NO and OM respectively represent the magnitudes and directions of V and P.

5. Let r and c denote the lengths of the connecting-rod and crank respectively, and ω_0 the angular velocity of the crank assumed constant. Take $r/c=4$, $\bar{r}=r/2$, $k^2=r^2/3$, and compute the values of V and P when θ (see fig. 223) equals 0°, 30°, 60°, 90°, 120°, 150°, and 180°. Also plot the values of V at the corresponding positions of P' and represent each P, scaling it from the corresponding position of B; then draw smooth curves, joining the ends of the vectors representing V and P thus drawn.

(For a method of computing a, α, and ω, see art. 181 and ex. 1, art. 219. Notice that θ of fig. 154 equals $\theta-90°$ of figs. 175 and 223.)

285. Rolling Resistance.—Let fig. 224 represent a wheel or cylinder rolling upon a horizontal surface or roadway at constant speed. Let W denote its weight, P, applied as shown, the force required to maintain the constant speed, and R the resultant reaction of the roadway. Since the acceleration of the mass-centre and the angular acceleration of the wheel are zero, the resultant effective force for the wheel is zero (art. 282); hence the external forces W, P, and R are in equilibrium, and R must act through the centre and be inclined as shown.

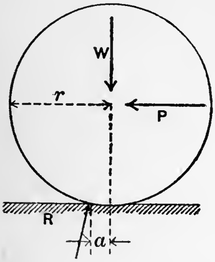

FIG. 224.

The horizontal component of the reaction of the roadway is called "rolling resistance," also "rolling friction." The three forces being in equilibrium, the horizontal component of R equals P, and hence an expression for P is also a value of the rolling resistance. If a denotes the distance from the vertical through the centre to the intersection of the action line of R and the circumference of the wheel, and if a is small (as it is except on soft roadways), then approximately

$$Pr = Wa, \quad \text{or} \quad P = (a/r)W.$$

From this equation the rolling resistance in a given case can be computed if a is known.

Like the coefficients of friction, a is determined from experiment. Thus, suppose that the force P for a given wheel and roadway has been measured; that value and the weight and radius of the wheel substituted in either preceding equation determine a for the case in hand. The distance a is sometimes called the "coefficient of rolling resistance." Observe that it is not an abstract number like the coefficients of friction, but a length.

Practically no general facts or laws are known concerning the coefficient of rolling resistance. The coefficient is usually regarded as independent of the weight (if moderate so that no permanent deformation of roller or roadway occurs), but there is disagreement as to the relation between it and the radius—it being held, for example, that a is independent of r, also that a varies as $\sqrt{r}$.

The following values are given to afford a notion of the values of a in a few cases.

Rollers of elm on an oak track (Coulomb).......	0.032 in.
Iron or steel wheels on iron or steel rails........	0.007–0.020 "
" " " " " wood..................	0.06 –0.10 "

EXAMPLE.

If several rollers are used for rolling a heavy body along a horizontal roadway, show that their rolling resistance is given by $W(a'+a'')/2r$, W denoting the weight of the body, r the radii of the rollers, and a' and a'' their "coefficients of rolling resistance" for the surfaces at which the rolling occurs.

CHAPTER XIV.

WORK AND ENERGY.

§ I. WORK.

286. Work Defined.—Work is said to be done upon a body by a force when the application point is displaced so that the displacement has a component along the action line of the force.

If the force is constant in direction, the projection of the displacement on the action line of the force is called *effective displacement.* If the force changes its direction during a displacement of its application point, then the projection of the displacement which occurs during an element of time on the corresponding action line of the force is called the effective displacement for that elementary interval.

287. Expressions for Work Done by a Force.—I. The force is constant in magnitude and direction. The amount of work done is measured by the product of the force and the effective displacement. Thus, if AB (fig. 225) is a displacement of the application point of a force F, and if ϕ denotes the angle between AB and F, and w the work done by F, then

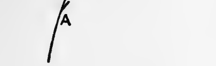

FIG. 225.

$$\mathrm{w} = \mathbf{F}(\overline{\mathbf{AB}} \cos \phi) = (\mathbf{F} \cos \phi)\overline{\mathbf{AB}}. \quad . \quad (1)$$

If $\phi = 0$, $w = F\,\overline{AB}$; if $\phi = 90°$, $w = 0$.

Observe that in the last form of (1), the expression for work is the product of the component of the force along the displacement and the displacement.

II. The force varies in magnitude or direction or both. Let AB (fig. 226) be a portion of the path of the application point of the force F, P being any intermediate position, ϕ the angle between F and the tangent to the path at P, and s the distance of P from some fixed origin in the path, it being measured positively in the direction of the motion. Then if dw denotes the work done by the force while its application point describes an elementary portion of the path ds including P,

$$dw = F\, ds \cos \phi;$$

and if w denotes the work done by F during the displacement AB in which s changes from s' to s'',

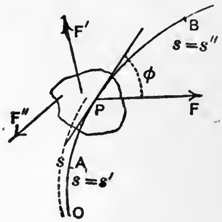

FIG. 226.

$$w = \int_{s'}^{s''} F\, ds \cos \phi = \int_{s'}^{s''} F_t ds. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $\phi = 0$, $F_t = F$; if $\phi = 90°$, $F_t = 0$ and $w = 0$.

288. Sign of a Work.—It is convenient to give sign to the work done by a force. The rule is as follows: A work is regarded as positive or negative according as the effective displacement agrees with or is opposed to the force in sense. It must be remembered that a displacement, and hence its projection also, is a vector quantity.

If the angle ϕ is always taken as in figs. 225 and 226, i.e., between the portions of the lines representing the force and the displacement toward which the arrows point, then the expressions for w in the preceding article give the correct sign of the work.

289. Unit of Work.—Equations (1) and (2), art. 287, imply as unit the work done by a unit force "acting through" unit distance. The value of the unit hence depends on the units used for force and distance. Thus we have, corresponding to the pound and foot, the foot-pound unit of work, to the kilogram and meter, the meter-kilogram, to the dyne and centimeter, the dyne-centimeter, etc. The last-named unit has also a special name, *erg*.*

290. Work Diagram.—If values of F_t and s be plotted on two rectangular axes (see fig. 227) for all positions of the application point of the force F, the curve joining the plotted points might be called a "tangential force-space (or F_t-s) curve." The por-

* For dimensions of a unit work, see Appendix C

tion of the figure between the curve, the s axis, and any two ordinates is called a work diagram.

Proposition.—The area of a work diagram represents the work done by the force during the displacement corresponding to the bounding ordinates.

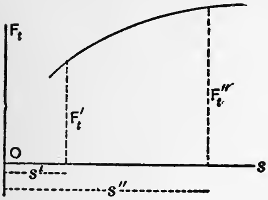

FIG. 227.

Proof: According to eq. (2), art. 287, $w = \int_{s'}^{s''} F_t \, ds$. Since F_t and s also denote coordinates of points on the F_t-s curve (fig. 227), the area of the work diagram is $\int_{s'}^{s''} F_t \, ds$, as is shown in works on calculus. Hence the area (according to some scale) equals w. Obviously the scale according to which the area of a work diagram is to be interpreted depends on the scales used in representing F_t and s. Thus if one-inch ordinates and abscissas represent 100 lbs. and 10 ft., respectively, one square inch of area represents 1000 ft.-lbs. of work.

Since the area of a work diagram equals the product of the average ordinate and the base, the work done by a force equals the average value of the tangential component and the length of the path described by the application point. If the F_t-s curve is straight, the average value of F_t is the mean of the initial and final values and the computation of the work is simple.

EXAMPLES.

1. Fig. 228 represents a body on a horizontal surface to which two forces (P and Q) are applied as shown. Compute the work

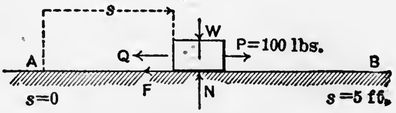

FIG. 228.

done by these forces, the weight, and the resistance of the surface while the body moves from A to B (5 ft.), Q being a variable force so that Q (in lbs.) $= 2s$ (in ft.) and the friction 30 lbs.

Solution: Since the displacements of the application points of W and N are normal to the action lines of the forces, W and N do no work. Since F and P are constant in magnitude and direction, we use eq. (1) art. 287 for computing the work done by them; for P, ϕ is zero and for F, ϕ is 180°, hence

the work done by $P = 100 \cdot 5 \cos 0 = 500$ ft.-lbs., and
" " " " $F = 30 \cdot 5 \cos 180° = -150$ ft.-lbs.

Since Q varies in magnitude we use eq. (2) art. 287, and ϕ being 180°, the expression for work done by Q is

$$\int_{s'}^{s''} Q \cos 180°\, ds = -\int_0^5 2s\, ds = -25 \text{ ft.-lbs.}$$

This value can be readily computed from the average value of Q which is, since Q varies uniformly with s, the mean of its initial and final values; these are 0 and 10 lbs. respectively. Hence the average value of Q is 5 lbs., and as it acts through 5 ft., the amount of work done by Q equals $5 \times 5 = 25$ ft.-lbs. The sign of the work is negative because the senses of the force and displacement are opposite.

2. Fig. 229 represents a body upon an inclined plane to which two forces (P and Q) are applied as shown. Compute the works done by them, the weight, and the reaction of the plane while the body moves from A to B (10 ft.), the frictional resistance of the plane and the weight being 10 and 100 lbs. respectively. *Ans.* Total work done $= -26.8$ ft.-lbs.

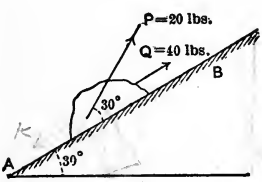

FIG. 229.

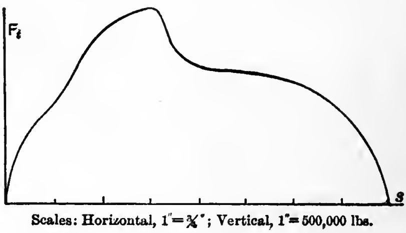

FIG. 230.

3. In punching a hole (2 in. diameter) in a certain iron plate (1½ in. thick), the pressure (F) between punch and plate varied as the ordinates to the curve of fig. 230 (the initial values of F are at the left). Estimate the area of the work diagram and

the amount of work done by the punch on the plate in the operation. Also compute the work from the average value of the force.

4. One end of an elastic cord whose natural length is 10 ft. is fastened to a body on a horizontal surface and the other end to a fixed point in the surface 20 ft. from the body. The tension in the cord is observed to be 30 lbs. When the body is released it moves toward the fixed point. Draw the work diagram for the tension in the cord and determine how much work is done by the tension in the first and second 5 feet of the displacement.

5. Suppose that a gas expands behind a piston in a cylinder according to the law $pv=C$, C being a constant, p the gas pressure per unit area, and v the volume of the gas. Show that the work done by the gas on the piston in an expansion from a volume v_1 to a volume v_2 equals $C \log_\epsilon(v_2/v_1)$.

6. Show that the work of a central force (one always directed toward a fixed point) in any displacement of its application point equals $-\int_{r_1}^{r_2} P\,dr$, in which P denotes the general value of the force, i.e., its value when its application point is any distance r from the fixed point and r_1 and r_2 denote the values of r at the beginning and end of the displacement respectively.

Solution: Let C, fig. 231, be the fixed point toward which P acts and OAB the path of the application point of P. The value of the work done by P is given by eq. (2) art. 287. Since (see the figure) $dr=-ds \cos \phi$, the value of the work as given by eq. (2) reduces to that given above.

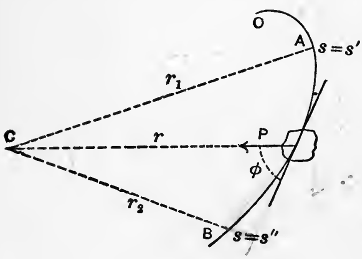

FIG. 231.

7. Suppose that the body described in ex. 4 is moved (after the cord is attached to it as described) so that the point of application of the cord moves in the circumference of a circle whose diameter is the cord in its first position. Compute the

work done by the tension while the application point describes the first quarter circle.

8. How much work is done by the cord up to the instant when it resumes its natural length?

291. Work Done by Gravity Upon a Body in Any Motion.—*Proposition.*—The work done by gravity upon a body in any motion equals the product of its weight and the vertical distance described by the centre of gravity, and the work is positive or negative according as the centre of gravity has descended or ascended.

Proof: Let w_1, w_2, etc., denote the weights of the particles of the body, y_1', y_2', etc., their distances above some datum plane (below which the body does not descend) at the beginning of the motion, and y_1'', y_2'', etc., their distances above that plane at the end of the motion (see fig. 232 where $a'a''$ is the path of the first particle, $b'b''$ that of the second, etc.). Also let W denote the weight of the body and $\bar{y}'$ and $\bar{y}''$ the initial and final heights of its centre of gravity above the plane. Then the sum of the works done by gravity on all the particles is

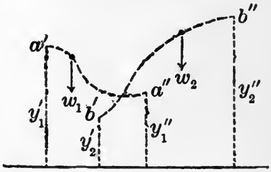

FIG. 232.

$$w_1(y_1'-y_1'')+w_2(y_2'-y_2'')+\ldots=(w_1y_1'+w_2y_2'+\ldots)-(w_1y_1''+w_2y_2''+\ldots)$$

According to art. 64,

$$w_1y_1''+w_2y_2''+\ldots=W\bar{y}'' \quad \text{and} \quad (w_1y_1'+w_2y_2'+\ldots)=W\bar{y}';$$

hence the sum of the works done on all the particles equals

$$W\bar{y}'-W\bar{y}''=W(\bar{y}'-\bar{y}''). \qquad \text{Q.E.D.}$$

292. Work Done by Concurrent Forces and by Their Resultant.—*Proposition.*—The work done by any number of concurrent forces in a displacement of their application point equals that done by their resultant in that displacement.*

Proof: Let F_1, F_2, etc., denote the forces, R their resultant, and ϕ_1, ϕ_2, etc., and ϕ respectively the angles which the forces

* It is assumed that the forces and their resultant have a common application point.

and their resultant make with the tangent to the path of the application point (taken as explained in art. 288). Then according to art. 36,

$$R \cos \phi = F_1 \cos \phi_1 + F_2 \cos \phi_2 + \ldots.$$

Therefore, $R \cos \phi \cdot ds = F_1 \cos \phi_1 \cdot ds + F_2 \cos \phi_2 \cdot ds + \ldots,$

and $\int R \cos \phi \cdot ds = \int F_1 \cos \phi_1 \cdot ds + \int F_2 \cos \phi_2 \cdot ds + \ldots;$

hence, etc.

293. Work Done by a Pair of Equal, Opposite, and Collinear Forces.—Suppose that A and B (fig. 233) are the application

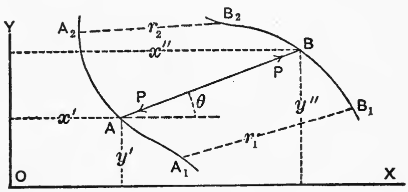

FIG. 233.

points of the forces at any instant during the motion and let P denote the value of the forces then. Also let x' and y' denote the coordinates of A and x'' and y'' those of B, and suppose that A moves from A_1 to A_2 and B from B_1 to B_2, i.e., assume for simplicity that the displacements are coplanar. The discussion can be easily extended to include non-coplanar displacements.

The work done by each force equals the sum of the works done by its x and y components; for the force P acting on A these are

$$-\int P \cos \theta \cdot dx' \quad \text{and} \quad -\int P \sin \theta \cdot dy',$$

and for the force P acting on B they are

$$\int P \cos \theta \cdot dx'' \quad \text{and} \quad \int P \sin \theta \cdot dy''.$$

The work done by both forces equals

$$\int P[\cos \theta \cdot (dx'' - dx') + \sin \theta \cdot (dy'' - dy')].$$

It is plain from the figure that

$$r^2 = (x'' - x')^2 + (y'' - y')^2,$$

hence $\quad r\,dr=(x''-x')(dx''-dx')+(y''-y')(dy''-dy')$,

or $\quad dr=\cos\theta\cdot(dx''-dx')+\sin\theta\cdot(dy''-dy')$.

Substituting according to this relation in the expression for total work we find that the latter becomes

$$\int_{r_1}^{r_2} P\,dr \text{ (when the force } P \text{ on } A \text{ acts from } B \text{ to } A\text{), but}$$

$$-\int_{r_1}^{r_2} P\,dr \text{ (when the force } P \text{ on } A \text{ acts from } A \text{ to } B\text{),}$$

as will be seen by changing the arrows on P in the figure and making the necessary changes in the discussion.

If the distance between the application points of the forces does not change during the displacement, $dr=0$, and the work done by the pair of forces equals zero. If P depends only on r, then the work done by P depends only on the initial and final values of r and not at all on the way in which r changes during the displacement.

EXAMPLE.

How much work is done by the steam in one cylinder of a locomotive during one stroke of the piston?

Solution: Consider a forward stroke of the piston and let P denote the pressure on the piston when its distance from the rear end of the cylinder is r. Then the work done by the pressures on the piston and rear end of the cylinder in one stroke equals

$$\int_{r_1}^{r_2} P\,dr=P_a(r_2-r_1)=P_a\,s,$$

r_1 and r_2 denoting the values of r at the beginning and end of the stroke, P_a the average value of the steam pressure, and s the length of stroke.

This value of the work might also be obtained by computing the work done by each pressure separately. Thus let R denote the radius of the driving-wheels, then the distance through which the locomotive moves in one stroke equals πR, and supposing that the locomotive is running forward, the work done by the steam on the rear end of the cylinder equals $-P_a\pi R$, and that done on the piston equals $P_a(\pi R+s)$; hence the work done by both pressures equals $P_a\,s$.

294. Work Done by a Body Against a Force.—It is convenient to use the expression "work done by a body against a force" applied to it; we mean by it the negative of the work done upon the body by the force. Thus if a body weighing 10 lbs. is made to rise 5 ft., gravity does -50 ft.-lbs. of work upon it and the body does $+50$ ft.-lbs. of work against gravity.

In accordance with the above, when the sense of a force and that of effective displacement of its application point are opposite, the work done by the body is positive; when they are the same, the work done by the body is negative.

§ II. ENERGY.

295. Energy Defined.—When the state or condition of a body is such that it can do work against forces applied to it, the body is said to possess energy. A stretched spring can do work against forces applied to it if they are such that it may contract; a body in motion can do work against an applied force which tends to stop it. The spring and the body therefore possess energy.

The amount of energy possessed by a body at any instant is the amount of work which it could do against applied forces while its state or condition changes from that of the instant to an assumed standard state or condition. The meaning of the standard condition is explained in subsequent articles.

The *unit of energy* must, in accordance with the above, be the same as the unit of work.

296. Kinetic Energy Defined. — Energy is classified into kinds depending on the state or condition of the body in virtue of which it has energy. Kinetic energy of a body is energy which it has by virtue of its velocity.

297. Kinetic Energy of a Particle.—The amount of kinetic energy possessed by a particle at any instant is the work which it could do while the velocity changes from its value at that instant to some other value taken as a standard. It is customary to take zero velocity as the standard one; this being understood, we may say that the amount of kinetic energy possessed by a particle is the work which it can do in "giving up its velocity."

Proposition.—The kinetic energy of a particle whose mass and velocity are m and v respectively equals $\frac{1}{2}mv^2$.

Proof: Let R denote the resultant of all the forces acting on the particle while it "gives up its velocity." Let A and B denote the beginning and end of the path, and s' its length. Then, according to arts. 292 and 287, the work (w) done by all the forces on the particle in the motion from A to B is given by

$$w = \int_0^{s'} R_t ds.$$

Hence the work done by the particle against the forces, or the kinetic energy (E) of the particle, is given by

$$E = -\int_0^{s'} R_t ds.$$

Now $\qquad R_t = ma_t = m\, dv/dt$ (see art. 236);

hence $\qquad E = -\int_v^0 mv\, dv = \frac{1}{2}mv^2.$ Q.E.D.

298. Kinetic Energy of any System of Particles.—The kinetic energy of a system of particles equals the sum of the kinetic energies of the separate particles. If m and v denote the mass and velocity respectively of any particle of a system, and E the kinetic energy of the system,

$$\mathbf{E} = \tfrac{1}{2}\Sigma \mathbf{mv}^2. \qquad (1)$$

I. *Translating Body.*—In this case all particles have at each instant equal velocities, hence

$$\tfrac{1}{2}\Sigma \mathbf{mv}^2 = \tfrac{1}{2}\mathbf{v}^2\Sigma \mathbf{m};$$

or, if M denote the mass of the body, the kinetic energy E is given by

$$\mathbf{E} = \tfrac{1}{2}\mathbf{Mv}^2. \qquad (2)$$

II. *Rotating Body.*—In this case the velocity of any particle of the body equals the product of its distance from the axis of rotation and the angular velocity of the body (art. 216). Let r denote the distance of any particle from the axis and ω the angular velocity of the body; then the value of the kinetic energy is

$$\tfrac{1}{2}\Sigma m(r\omega)^2 = \tfrac{1}{2}\omega^2\Sigma mr^2.$$

Now Σmr^2 is the moment of inertia of the rotating body with respect to the axis of rotation; if I be used to denote this quantity, the kinetic energy is given by

$$\mathbf{E} = \tfrac{1}{2}\mathbf{I}\omega^2. \qquad (3)$$

III. *Body having any Plane Motion.*—In this case the state of the motion at any instant may be regarded as rotational (see arts. 222 and 223). As shown in art. 223, the velocity of any particle of the body at any instant equals the product of its distance from the line which is the axis of rotation at that instant (instantaneous axis) and the angular velocity of the body. The reasoning in Case II applies here if the word "instantaneous" is inserted before the word "axis"; then $\frac{1}{2}I\omega^2$ is the expression for the kinetic energy of a body having any plane motion, I being the moment of inertia with respect to the instantaneous axis.

Since the instantaneous axis in general moves about in the moving body, the expression above is not always convenient to apply, and another, though not so simple in form, is simpler in its application. This expression may be deduced as follows: In addition to the notation employed in the preceding, let $\bar{I}$ denote the moment of inertia of the body with respect to a central axis perpendicular to the plane of the motion, $\bar{v}$ the velocity of the mass-centre at the instant considered, $\bar{r}$ the distance between the mass-centre and the instantaneous axis, and M the mass of the body. Then

$$I = \bar{I} + M\bar{r}^2 \text{ (art. 256)} \quad \text{and} \quad \bar{v} = \bar{r}\omega;$$

hence $$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\bar{I}\omega^2 + \tfrac{1}{2}M\bar{r}^2\omega^2,$$

or $$\mathbf{E} = \tfrac{1}{2}\bar{\mathbf{I}}\omega^2 + \tfrac{1}{2}\mathbf{M}\bar{\mathbf{v}}^2. \qquad (4)$$

Now $\frac{1}{2}\bar{I}\omega^2$ is the kinetic energy which the body would have if rotating about a fixed axis through its mass-centre with an angular velocity ω, and $\frac{1}{2}M\bar{v}^2$ is the kinetic energy which it would have if translating with a velocity $\bar{v}$. Hence the kinetic energy of a body having any plane motion is regarded as consisting of two parts, and they are called rotational and translational.

EXAMPLES.

1. Express the kinetic energy of a body weighing 1.5 tons and moving at a speed of 60 mi.-per-hr. in ft.-lbs.

2. Express the kinetic energy of a cylindrical disk weighing 3225 lbs. and rotating at an angular velocity of 300 rev.-per-min., the diameter of the disk being 4 ft.

3. Show that the kinetic energy of a rotating body equals $\frac{1}{2}Mv_k^2$, M denoting the mass of the body and v_k the velocity of a point of it whose distance from the axis of rotation equals the radius of gyration of the body with respect to that axis.

4. What is the kinetic energy of a homogeneous right circular cylinder which rolls so that the speed of its mass-centre is v, its mass being M? *Ans.* $\frac{3}{4}Mv^2$.

299. Potential Energy Defined.—A body may possess energy which is not due to velocity. Thus two mutually attracting bodies can do work against forces applied to either or both if allowed to move so that they approach each other; and as mentioned in art. 295, a compressed or stretched spring can do work against applied forces if permitted to resume its natural length. The "change of condition or state" in the first case is a change in configuration, i.e., a change in the positions of the bodies relative to each other, and in the second case, if we conceive of the spring as consisting of discrete particles, the change is also one in configuration.

Energy of a system of particles dependent on configuration of the system is called *energy of configuration* and, more commonly, *potential energy.*

300. The Amount of Potential Energy possessed by a system in any configuration is the work which it can do in passing from that configuration to any other taken as a standard, it being understood that no other change of condition takes place. The standard configuration may be chosen at pleasure, but it is convenient to so select it that in all other configurations considered the potential energy is positive.

Proposition.—The potential energy of a system in any configuration equals the amount of work done by the *internal forces* during the change to the standard configuration.

Proof: To determine the potential energy we are to compute the work done against the external forces while the system passes to the standard configuration, no other change of condition (as velocity) of the particles taking place so that in the passage to the standard configuration there is no change in the kinetic energy of the system. It is shown in art. 306 that the sum of the works done by the internal and the external forces during any change of configuration equals the increment in the kinetic energy of the system. Since in the case in hand there is no change in kinetic energy, the sum of the works done by the internal and external forces equals zero. Denoting the internal and external works by w_i and w_e respectively,

$$w_i + w_e = 0, \quad \text{or} \quad w_i = -w_e.$$

Now $-w_e$ is the work done by the system against the external forces during the passage to the standard condition, i.e., the potential energy of the system in its initial configuration; hence the last equation asserts the truth of the proposition.

301. Potential Energy of a System Not Always a Definite Quantity.—The amount of work done by the internal forces during a change of configuration (and hence the potential energy

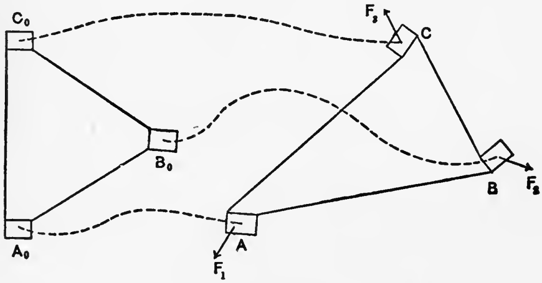

FIG. 234.

of the system) may or may not depend upon the way in which the change of configuration takes place. This is known to be true from direct experience, but it can also be proved. Thus, let A, B, and C (fig. 234) represent three bodies which may slide about on a table; imagine them connected by elastic cords as shown, and consider the three bodies, the table, and the earth as a system. The cords are introduced merely as a means of applying certain forces to the bodies, but the forces are to be thought of as exerted by one body directly upon another. Let A_0, B_0, and C_0 represent a selected standard configuration of the bodies and A, B, and C some other one; also, let F_1, F_2, and F_3 be such external forces acting upon them that the passage to the standard position is without change of velocity.

Now the only internal forces in the system which do work during the motion are the pulls exerted by the bodies upon each other and the frictions (if any) between them and the table. As shown in art. 293, the work done by the pulls exerted by any two of the bodies on each other does not depend on how the final positions are reached. The work done by the frictions, however, does depend on the manner of the motion; thus, if we suppose the friction on any body to be constant in value, the work done by that force equals the negative product of the force and the length of the path described by that body. Hence, in the system of bodies under consideration, the total work done by the internal forces depends on the way in which the change of configuration takes place and the potential energy is not a definite quantity. If, however, there is no friction, the work done by the internal forces is independent of the way in which the configuration takes place, and the potential energy is a definite quantity.

302. *Conservative Systems.*—If the work done by the internal forces of a system during any change of configuration is independent of the way in which the change is made, the system is called conservative, and those internal force-pairs (action and reaction) whose work does not depend upon the way in which the change takes place are also called conservative. The potential energy of such a system in any definite configuration is a definite quantity.

It is characteristic of conservative forces that they are independent of the velocity of the particles on which they act. We consider only conservative forces which act along the line joining the particles between which they act and whose magnitudes depend on the distance between the particles. That such forces are conservative forces follows from art. 293.

303. *Non-Conservative Systems.*—If the total work done by the internal forces depends on the manner of the change of configuration, the system is called non-conservative and those internal force-pairs whose work depends on the manner of the change are also called non-conservative. The potential energy of a non-conservative system in any definite configuration is not a definite quantity.

It is a characteristic of non-conservative forces that their magnitudes or directions (or both) depend upon the velocity of the particles to which they are applied. Friction is the only one of this class herein considered.

304. Localization of Potential Energy.—Unlike the kinetic energy, the potential energy of a system cannot always be localized in detail, i.e., we cannot in all cases assign to parts of the system certain definite parts of its potential energy. As an example, consider two bodies A and B of the illustration in art. 301, and, for simplicity, neglect friction. As previously explained, the bodies can do a definite amount of work against external forces in passing to their standard positions, A_0B_0 (fig. 234), but the amount of work which each can do depends upon the way in which the passage is made. Thus, suppose that they move to their standard positions successively and that in one passage A moves first and in the other B moves first. Now the work which A can do in each case equals the work which the internal force acting on A does, and this work has different values in the two passages. For, let P denote the internal force acting on A (for simplicity assumed constant); then (see ex. 6, art. 290) the work done by P in the first case equals $\pm P(\overline{A_0B} - \overline{AB})$, and in the second case it equals $\pm P(\overline{A_0B_0} - \overline{AB_0})$; these values are in general unequal.

If, however, one of the bodies is always in its standard position, the potential energy of the system is rightly ascribed to the other. Thus, a magnet and a piece of soft iron attracting each other possess potential energy if separated, and if the magnet is regarded as fixed the energy is possessed by the iron. Similarly, the earth and an elevated body considered as a system possess potential energy, but it is practically necessary to regard the earth as fixed and hence to ascribe the energy to the elevated body.

305. Other Forms of Energy.—Kinetic and potential energies are often called *mechanical energy*. It is the opinion of some that all energy is mechanical, and some think that it is all kinetic. Whether either of these views be correct, it is practically necessary to recognize other forms. A mere enumeration of these with brief remarks is sufficient for the present purpose, since we shall deal mostly with energy known to be mechanical.

Thermal Energy.—A hot body may do work under favorable conditions; thus, if such a one is placed in a boiler containing water, the water will be heated and a part may be converted into steam which may drive a steam-engine, i.e., do work. By giving up its heat the hot body has done work, and hence by definition (art. 295) it possessed energy in its heated state. Not only is this fact well known, but also the fact that a given quantity of heat represents a definite amount of energy; the relation may be expressed thus:

$$\text{one British unit of heat}^* = 778+ \text{ ft.-lbs.}$$

Based on the molecular hypothesis, the common theory is that heat is due to the vibratory motion of molecules, i.e., that thermal energy is kinetic.

Chemical Energy.—Many substances combine chemically and their combination gives evidence that they possessed energy. Thus, coal and oxygen combine and produce heat which, as we have seen, is a form of energy. We rightly say, therefore, the coal and oxygen before combination possessed energy.

Based on the molecular hypothesis, the theory of chemical energy in cases where heat is generated in the chemical combination is that internal (molecular) forces of the substances do work during the combination, and hence (see art. 306) increase the kinetic energy of the molecules. According to this explanation the energy before combination is potential and after kinetic.

Electrical Energy.—If a storage battery charged with electricity is connected with a motor, work may be done by the latter. As the work is done, the electrical condition of the battery changes and we therefore ascribe the energy to the battery. The energy is called electrical because it is due to a change of electrical condition.

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* The amount of heat required to raise the temperature of one pound of water one degree Fahrenheit.

The nature of electrical energy is even less understood than that of thermal energy, and no commonly accepted explanation of it has yet been made.

§ III. PRINCIPLES OF WORK AND ENERGY.

306. Principle of Work and Kinetic Energy.—I. *For a Particle.*—The work done in any displacement of a particle by all the forces applied to it equals the increment in its kinetic energy during that displacement, or

$$\mathbf{w} = \Delta \mathbf{E}_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

w denoting the work done by the forces and ΔE_k the increment in the kinetic energy. The increment is positive or negative (and there is a gain or loss of kinetic energy) according as the final velocity is greater or less than the initial.

Proof: Let R denote the resultant of the forces acting on the particle, w the work done by them in the displacement, v_1 and v_2 the velocities of the particle at the beginning and end of the displacement respectively. Then, as shown in art. 292,

$$w = \int_{s_1}^{s_2} R_t ds,$$

where s is the distance of the moving particle from some fixed point in the path and s_1 and s_2 are the values of s at the beginning and end of the displacement. Since

$$R_t = ma_t = m\, dv/dt,$$

$$w = m\int_{s_1}^{s_2} ds\, dv/dt = m\int_{v_1}^{v_2} v\, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2.$$

Since $\frac{1}{2}mv_1^2$ and $\frac{1}{2}mv_2^2$ are the values of the kinetic energy of the particle at the beginning and end of the displacement respectively, the right-hand member above is the increment in the kinetic energy of the particle.

II. *For Any System of Particles.*—The work done by all the external and the internal forces in any displacement of a system of particles equals the increment in its kinetic energy during the displacement, or

$$\mathrm{w}_e + \mathrm{w}_i = \Delta \mathrm{E}_k, \quad \ldots \ldots \quad (2)$$

w_e and w_i denoting the "external" and the "internal" works respectively.

Proof: According to the principle of work and energy for a particle the total work done on a particle in any displacement equals the increment in its kinetic energy during that displacement. Hence the work done on all the particles of a system during any displacement equals the sum of the increments in their kinetic energies, i.e., the increment in the kinetic energy of the system.

III. *For a Rigid Body.*—The work done upon a rigid body by the external forces in any displacement equals the increment in its kinetic energy during that displacement, or

$$\mathrm{w}_e = \Delta \mathrm{E}_k. \quad \ldots \ldots \quad (3)$$

Proof: It is shown in art. 293 that the work done by two equal, opposite, and collinear forces is zero for any displacement of their application points if the distance between those points remains constant. As previously explained, the internal forces of any system of particles occur in pairs of equal, opposite, and collinear forces, and since in a rigid body the distances between the application points of the internal forces remain constant, the work done by the internal forces in any displacement of the body is zero; hence the work done by the external forces equals the increment in the kinetic energy (see II).

307. Principle of Work and Energy for Conservative Systems.—The work done upon a conservative system by external forces during any displacement equals the sum of the increments of its kinetic and potential energies, or

$$\mathrm{w}_e = \Delta \mathrm{E}_k + \Delta \mathrm{E}_p, \quad \ldots \ldots \quad (1)$$

ΔE_p denoting increment of potential energy.

Proof: Let C_1 and C_2 be the initial and final and C_0 the standard configuration of the system. Also let E_p' and E_p''

denote the potential energies in the configuration C_1 and C_2 respectively. The work done by the internal forces in the displacement from C_1 to C_0 would be E_p', and that from C_2 to C_0 would be E_p''; hence in the actual displacement (C_1 to C_2) the work done by the internal forces equals $E_p'-E_p''$. According to the preceding article,

$$w_e+(E_p'-E_p'')=\Delta E_k, \quad \text{or} \quad w_e=\Delta E_k+(E_p''-E_p').$$

Now $E_p''-E_p'$ is the increment in the potential energy of the system during the displacement; hence the principle is proved.

If the work done by the external forces is positive (that done by the system against the forces is negative), the system gains energy, and if their work is negative (that done by the system against them is positive), the system loses energy. The statement that a certain amount of positive work is done upon (or by) a system is equivalent to the statement that the system has gained (or lost) the same amount of energy.

308. Conservation of Energy.—In any change of condition of a material system which is isolated so that it neither receives nor gives out energy, its total energy (all forms included) remains constant in amount; or, as is sometimes stated, "energy is indestructible."

This principle is a generalization based on physical experience. It cannot be deduced in the general case from the laws of motion, at least not in the present state of knowledge regarding the constitution of matter and the nature of non-mechanical energy. For conservative systems the principle can be proved; thus, the system being isolated, there is no external work and equation (1), art. 307, becomes

$$\Delta \mathbf{E}_k+\Delta \mathbf{E}_p=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i.e., the sum of the increments of kinetic and potential energies in any change of condition equals zero; hence the sum of the kinetic and potential energies is constant.

If a system not isolated receives or loses energy, some other system must lose or receive an equal amount. For, let A be the first system and B the one from which A receives or to which it gives energy. If necessary, imagine B extended so that A and B together are an isolated system; then the total energy of A and B being constant, if A receives or loses energy, B must lose or receive an equal amount.

309. **Principle of Energy for Machines.**—The function of a machine is to transfer energy from a body or system of bodies (A) to another (B). The energy transferred may or may not have been transformed in the process. Thus, a dynamo receives mechanical and delivers electrical energy, while a water motor receives and delivers mechanical energy.

The energy received by the machine from A is called *input* and that delivered by it to B is called *output*. The amount of energy possessed by the machine at any instant is called its *stored energy* at that instant. It is a fact of experience that some of the energy miscarries, as it were, between A and B, and is delivered to other bodies than B; that energy is therefore called *lost energy* or simply the *loss*. This energy is lost principally as heat which is generated wherever there is a transformation or transference of energy.

Let E_i denote the input for any period,
E_o the output,
E_l " loss,
ΔE_s " increment in the stored energy;

then, since energy is indestructible,

$$E_i-(E_o+E_l)=\Delta E_s,$$

or

$$\mathbf{E}_i=\mathbf{E}_o+\mathbf{E}_l+\Delta\mathbf{E}_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the stored energy remains constant, or if at the beginning and end of the period the stored energies are equal (as at the beginning and end of a cycle through which the machine works), ΔE_s equals zero, and the equation of energy becomes

$$\mathbf{E}_i=\mathbf{E}_o+\mathbf{E}_l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

310. *Efficiency.*—The efficiency of a machine is the ratio of the output to input for a period at the beginning and end of

which the stored energy is the same. Thus, if e denotes efficiency,

$$e = E_o/E_i. \quad (1)$$

Since the input is always larger than the output, the efficiency of every machine is less than one.

311. *Power or Activity.*—The rate at which a machine or any "agent" does work is called its power or activity.

If the work is done uniformly, the power is constant; and if Δw denotes the work done in any period Δt and P the power,

$$P = \Delta w/\Delta t. \quad (1)$$

If the work is not done uniformly, the power is variable and the formula above gives the average value of the power; the actual value at any instant is the limit of $\Delta w/\Delta t$, or

$$P = dw/dt. \quad (2)$$

Units of Power.—Equations (1) and (2) imply as units of power a rate corresponding to unit work done in a unit time. Thus one foot-pound-per-second, one meter-kilogram-per-second, one erg-per-second, etc., are such units of power. These units are small for some purposes; the following are often more convenient:

the horse-power = 550 foot-pounds-per-second;
the force de cheval = 75 meter-kilograms-per-second;
the watt = 10^7 ergs-per-second.*

EXAMPLES.

1. Show that the rate at which steam does work in any engine is given by $p\,l\,a\,n$, the notation being:

p, average steam-pressure per unit area during a stroke;
l, length of stroke;
a, area of piston;
n, number of strokes per unit time.

* For dimensions of a unit of power see Appendix C.

2. Show that the rate at which steam does work in a locomotive whose velocity is v and drivers' diameter is d equals $4plav/\pi d$. for two cylinders

§ IV. Applications.

312. Computation of Velocity and Distance.—The equations expressing the principles of work and kinetic energy, (2) and (3), art. 306, contain "work terms" on one side and "kinetic energy terms" on the other. The factors in the work terms are force and distance, and those in the kinetic energy terms are mass and velocity. If all the factors except one are known, that one may be computed from the equation. In the following examples the unknown quantity is a velocity or a distance, and it can be most readily determined by the principle of work and kinetic energy.

EXAMPLES.

1. Suppose that in ex. 1, art. 290, the velocity of the body when at A is 2 ft.-per-sec. What is its velocity when it reaches B if the body weighs 160 lbs.?

Solution: The total work done by all the external forces in the motion from A to B was found to be 325 ft.-lbs. (see solution of ex. 1). The initial kinetic energy of the body (i.e., at A) is

$$\tfrac{1}{2}mv_1^2 = \tfrac{1}{2}\,4.97 \times 2^2 = 9.94 \text{ ft.-lbs.};$$

hence eq. (3), art. 306, becomes

$$325 = \tfrac{1}{2}\,4.97v_2^2 - 9.94, \quad \text{or} \quad v_2 = 11.61 \text{ ft./sec.}$$

2. Suppose that in ex. 2, art. 290, the velocity of the body when at A is 5 ft.-per-sec. What is its velocity when it reaches B?

3. Suppose that in ex. 4, art. 290, the body weighs 10 lbs. and that the horizontal surface is smooth. What is its velocity after having moved 10 ft.?

4. Solve the preceding example if the horizontal surface is rough, the coefficient of kinetic friction being one-fourth.

5. How far will the body of ex. 4 move supposing that after

the cord slackens the body moves under the influence of friction alone?

6. Suppose that in ex. 4, art. 266, the angular velocity of the rotating body is ω_1 at a certain instant. Determine its angular velocity when the suspended body has descended a distance h after that instant.

Solution: The external forces acting on the cord and rotating and suspended bodies considered as a system are the weights of the bodies and the "hinge reaction." Supposing the last to be frictionless, it does no work; neither does the weight of the rotating body, but the work done by that of the suspended body is Wh. The work done by forces internal to the two rigid bodies and that by the reactions between the cord and the two bodies is zero. The work of the forces internal to the cord is not zero, but it is small (see art. 315) and is negligible in this instance.

The initial kinetic energy of the system (neglecting the mass of the cord) is

$$\tfrac{1}{2}I\omega_1^2 + \frac{1}{2}\frac{W}{g}(r\omega_1)^2,$$

and if ω_2 denotes the final angular velocity, eq. (2), art. 306, becomes, since $\omega_i = 0$,

$$Wh = \tfrac{1}{2}I\omega_2^2 + \frac{1}{2}\frac{W}{g}(r\omega_2)^2 - \tfrac{1}{2}I\omega_1^2 - \frac{1}{2}\frac{W}{g}(r\omega_1)^2,$$

from which ω_2 can be computed.

7. Suppose that the disk of ex. 2, art. 298, is rotating on a shaft 4 in. in diameter and that the axle friction is 30 lbs. In how many turns would the frictional resistance stop it?

8. Let h denote the height of the centre of gravity of a pendulum when the velocity is zero above its lowest position. Show that the angular velocity of the pendulum when it reaches its lowest position is $(1/k)\sqrt{2gh}$, k denoting the radius of gyration of the pendulum with respect to the axis of suspension.

9. A body is suspended by two parallel cords of equal length and is allowed to swing in the plane of the cords under the influence of gravity. Show that the height (h) to which the centre of gravity rises above its lowest position and the velocity (v) in the lowest position are related as follows: $v^2 = 2gh$.

313. Train Resistance.—The resistance to motion experienced by a train moving on a track consists of rolling resistance, journal friction (at the car and locomotive axles), air resistance, and the frictional resistance at the working parts of the locomotive. For simplicity we may imagine these replaced by a single horizontal resistance equivalent to them all so far as the motion of the train is concerned. This single equivalent resistance is often called *train resistance*, but sometimes the term is intended not to include the locomotive and tender resistance.

It is customary to express train resistance as so many pounds for each ton of weight of train. Many experiments have been made to determine train resistance, and a number of formulas have been deduced to express its value. We give but three as illustrations and to furnish data for a few examples; the notation is as follows:

R, train resistance in lbs.-per-ton;
V, velocity in mi.-per-hour;
T, weight of train in tons (2000 lbs.).

Engineering News: $R = V/4 + 2$. . . . . (1)

This includes all resistances except the internal friction of the locomotive. For the higher velocities it was deduced from experiments on a fast passenger-train.

Crawford: $R = V^2/268 + 2.5$. . . . . (2)

This does not include resistance of locomotive and tender. It was deduced from experiments on passenger-trains.

Lundie: $R = 4 + V[0.2 + 14/(35 + T)]$. . . . . (3)

This was deduced from experiments on electric elevated street-railway trains and includes all resistances.

It will be noticed that only (3) contains T. The first two apply only to trains of approximately the same kind and weight as those experimented on.

EXAMPLES.

1. Plot the curves represented by the preceding equations, assuming T in the third to be one ton.

2. What is the relation between the "draw-bar pull" between tender and first car when the velocity of the train is constant?

Solution: The only forces doing work on the train are the draw-bar pull and the train resistance. Since the velocity of the train is constant, its kinetic energy remains constant and the total work done by the two forces must be equal to zero (see eq. (2), art. 306), i.e., the works done by them must be equal but of opposite sign. Since the forces act through equal distances in any motion of the train, the forces must be equal in amount.

Train resistance (for the cars) is determined by measuring the draw-bar pull in front of the first car when the velocity is constant.

3. Assume the train resistance for cars to be constant (as it is roughly below 10 mi.-per-hr., according to Crawford's formula) and that it equals 2.6 lbs.-per-ton. If the cars with load weigh 1000 tons, how much work does the engine do upon the cars to bring up the velocity of the train from 0 to 10 mi.-per-hr., if it is done in a distance of one mile on a level track?

4. Solve the preceding example supposing that the train runs up a "one-per-cent grade." *

5. Supposing that the speed in ex. 3 is increased uniformly, i.e., the acceleration is constant, what is the value of the draw-bar pull?

6. Express the rate at which the engine does work on the train in the preceding example in horse-powers when the train is starting and when its velocity is 10 mi.-per-hr.

7. Show that if P denotes the rate at which work is done by the steam in a locomotive, i.e., the rate at which energy is supplied to the engine, R the total train resistance, and v the speed, when v is constant, $P=Rv$.

* One ft. rise for every 100 ft. along the track.

Solution: The speed being constant, there is no change in the kinetic energy of the train; therefore the work done on the engine by the steam equals the work done by the train against the resistance during any period. If w denotes the first work during which the train moves any distance s, in a time t,

$$w=Rs, \quad \text{or} \quad w/t=Rs/t.$$

Now w/t is the rate at which work is done on the engine, and s/t is the velocity of the train; hence $P=Rv$.

8. If the Engineering News formula is correct at all speeds and gives practically the whole train resistance, show that the cylinder steam-pressure per unit area (average for one stroke) required to maintain any constant speed in a train on a level track is a linear function of the speed. (See ex. 2, art. 311.)

9. Show that when the speed of a train is changing, $P=Rv+mva$, m and a denoting the mass and acceleration of the train respectively and P, R, and v having meanings as in ex. 7. (The equation neglects the "rotational" component of the kinetic energy of the wheels, which is small compared to the kinetic energy of the train.)

314. Friction Brakes.—Fig. 235 represents a form of friction brake often used to measure the power of small motors. It consists essentially of a flat-faced pulley rigidly fastened to the shaft of the motor, a strap or rope partly encircling the pulley, one end of it being fastened to a spring-balance and the other sustaining a freely hanging body.

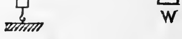


FIG. 235.

When the pulley rotates it drags the strap around with it a small distance and the spring tension (T) is greater or less than W according as the rotation is clockwise or counter-clockwise. Assuming it to be clockwise, the frictional resistance at the rim of the wheel equals $T-W$ and the work done by the motor against friction in one revolution of the wheel is $(T-W)2\pi r$. Hence, if the wheel makes n revolutions per unit time and all of the work done by the motor is thus expended against friction, the power of the motor equals $(T-W)2\pi rn$.

Fig. 236 represents a form of brake sometimes used on hoisting-drums for stopping the same. It consists essentially of a strap partially encircling the drum or a wheel fastened to the drum, one end being fastened to a fixed support, as A, and the other end to a lever, as CD. A force (P) applied as shown brings a frictional resistance to bear upon the drum at the strap and so controls the speed. We wish to find the relation between the pressure P and the weight of the descending load when its velocity is constant.

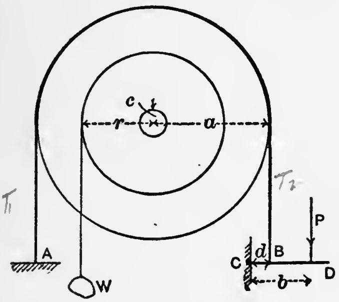


FIG. 236.

Let T_1 and T_2 denote the tensions in the strap at A and B respectively, F the frictional resistance at the brake-strap, f the coefficient of friction, F' the axle friction, and f' the coefficient of axle friction. Since the kinetic energy of the moving system is constant,

$$W2\pi r = F2\pi a + F'2\pi c. \quad . \quad . \quad . \quad (1)$$

As shown in art. 160,

$$T_2 = T_1 e^{f\pi}. \quad . \quad . \quad . \quad (2)$$

Considering the forces on the lever, it is seen that

$$T_2 d = Pb. \quad . \quad . \quad . \quad (3)$$

Also,

$$F' = f'(W + T_1 + T_2) \quad . \quad . \quad . \quad (4)$$

and

$$F = T_2 - T_1. \quad . \quad . \quad . \quad (5)$$

These five equations determine all the unknown quantities P, T_1, T_2, F, and F'.

315. Efficiency of Tackle.—Fig. 237 represents a fixed pulley. Let a denote its radius, r that of the axle, and d the diameter of the rope; also let P and Q denote the tensions on the two sides as shown.

Consider the pulley and as much of the rope as is shown together as a machine. In any motion, the external works are those done by P, Q, and the axle friction; the internal work is done by friction between the rope fibres where the rope winds on and off. In one revolution of the wheel the works done by P and Q respectively are $P2\pi a$ and $-Q2\pi a$, and that done by the axle friction is $-f'R2\pi r$, f' denoting the coefficient of axle friction and R the resultant axle pressure. Considering that numerical values of f' are uncertain, we may write $R=2Q$; then the work of axle friction is approximately $-f'Q4\pi r$. It has been found experimentally that the work "due to rigidity" of the rope is proportional to Q, i.e., to the tension on the following side; hence we may write for this work in one revolution $-cQ2\pi r$, c being an experimental coefficient.

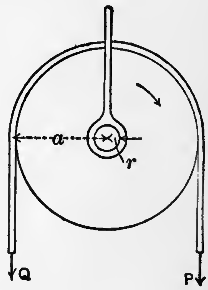


FIG. 237.

The equation of work and energy is (for one revolution)

$$P2\pi a-Q2\pi a-f'Q4\pi r-Q2\pi cr=0,$$

or

$$P=Q(1+cr/a+2f'r/a)=Qk, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k being an abbreviation for $(1+cr/a+2f'r/a)$, sometimes called "coefficient of resistance" for the pulley.

With equation (1) we can compute the relation between the load and the necessary pull to raise or lower it, and the efficiency of the tackle (see ex. 1). The value of the coefficient c has been found to vary directly as the square of the diameter of the rope (d) and inversely as the radius of the pulley. One formula (Eytelwein's) is

$$c=0.46d^2/a,$$

d and a being expressed in inches. If $a=4d$, $r=d/2$, and $f'=0.1$, the values of k are as follows:

$d=$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$
$k=$	1.08	1.11	1.14	1.20

EXAMPLES.

1. Determine the relation between F and W (fig. 83) and the efficiency when the load is being raised.

Solution: Call the tensions in the sections made by the horizontal line beginning at the left S_1, S_2, S_3, etc. Then $S_2=kS_1$, $S_3=k^2S_1$, $S_4=k^3S_1$, etc., and since

$$W=S_1+S_2+S_3+\text{ etc.},$$

$$W=S_1(1+k+k^2+k^3+k^4+k^5).$$

Also $$F=kS_6=k^6S_1;$$

hence $$F=Wk^6/(1+k+k^2+k^3+k^4+k^5).$$

During an ascent of the load equal to s, the point of application of F descends a distance $6s$. Hence for that motion the output and input are respectively

$$Ws \quad \text{and} \quad 6Fs,$$

and the efficiency equals

$$W/6F=(1+k+k^2+\ldots k^5)/6k^6.$$

2. Solve the preceding example supposing that the load is being lowered.

316. Efficiency of a Mine-hoist.—Fig. 238(a) represents a balanced vertical mine-hoist, consisting of an endless cable, two cages, a hoisting-drum above, and a pulley below. Ordinates from Ot to the various lines in fig. 238(b) represent various quantities involved in a power computation which is now to be made. Thus, ordinates to aa represent a frictional resistance (2 tons) applied to the surface of the drum equivalent to all the actual frictional resistances in the hoist; ordinates to bb represent the weight of the load (6 tons); and ordinates to cc represent the velocities of the cages. The assumed law of motion is (1) that the acceleration is greatest at the beginning of the motion and decreases to zero uniformly and so that the velocity acquired in 30 secs. is 60 ft.-per-sec., (2) then the velocity remains constant for 60 secs., (3) then a retardation follows which increases just as the acceleration decreased, i.e., uniformly and so that the cages are brought to rest in 30 secs.

When the speed of the cages is constant the pull (F) of the hoisting-engine (assumed for simplicity as applied to the surface of the drum) just equals the sum of the load and the

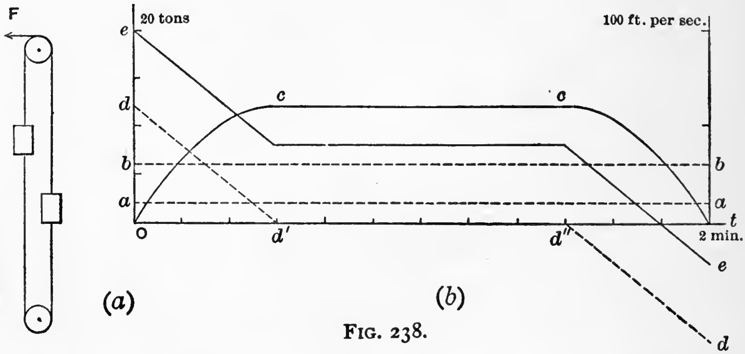


FIG. 238.

friction, but in the first 30 secs. the pull is increased by the "inertia force," and in the last 30 secs. it is decreased by the "inertia force." This inertia force depends upon the mass of the cable, load, and cages, the moment of inertia of the drum, and the acceleration or retardation. We suppose that these are such that the ordinates to dd' and to $d''d$ represent the values of the force as applied to the surface of the drum and that its maximum value is 12 tons. The total force then to be exerted by the engine on the drum at any instant is represented by the sum of the ordinates to the three lines aa, bb, and dd for that instant. Ordinates to the line ee represent all such sums.

The power of the engine P at the instant when the velocity is v and the work done by it w in a distance s or a time t from starting are given by

$$P=Fv, \quad w=\int_0^s Fds, \quad w=\int_0^t Pdt.$$

EXAMPLES.

1. Draw a curve showing how the power changes with the time.

2. Determine the entire work done by the engine in a single hoist, and also the efficiency of the hoist.

CHAPTER XV.

IMPULSE AND MOMENTUM.

§ I. IMPULSE.

317. Impulse of a Force whose Direction is Constant.—If the magnitude of the force is constant, the impulse of the force for any interval is the product of the force and the length of the interval, i.e., if F and $(t''-t')$ denote the force and the interval respectively, the impulse equals $\mathbf{F(t''-t')}$.

If the magnitude of the force varies, the impulse for any element of time equals the product of the value of the force at any instant of the element and the length of the element, i.e., Fdt; and the impulse for any finite interval $t''-t'$ equals $\int_{t'}^{t''}\mathbf{Fdt}$.

The unit of impulse depends on the units used to express force and time. If C.G.S. units be used, the unit of impulse is called a dyne-second; if the pound and second are used as units of force and time respectively, the unit of impulse is called a pound-second.*

318. Impulse of a Force whose Direction Varies.—An impulse should be regarded as a vector quantity, its direction being the same as that of the force if that is constant.

The meaning of impulse of a force whose direction varies may be explained as follows: Imagine a force to change its direction and magnitude five times in a certain interval, and that the values of the force and the corresponding portions of the interval are F', F'', etc., and $(\Delta t)'$, $(\Delta t)''$, etc., respectively. Then the impulses of the force for these portions are $F'(\Delta t)'$, $F''(\Delta t)''$, etc. Now if Aa, ab, etc. (fig. 239), represent these impulses, the impulse of the force for the entire interval is the vector sum of Aa, ab, etc., or AB.

* For dimensions of a unit impulse see Appendix C.

If the force changes by small amounts and many times in the interval, its variation may resemble that of a continuously varying force. By impulse of a continuously varying force is meant the limit toward which the impulse of a suddenly varying force tends as the manner of variation of the latter approaches that of the former.

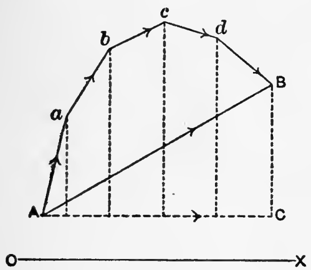


FIG. 239.

Employing the language of the infinitesimal calculus again, we state that the impulse of a force for an element of time equals $F \cdot dt$ if F denotes the value of the force at any instant of the element, and its direction is that of the force. The impulse of the force for a finite interval is $\int F dt$, the integration being not ordinary but vectorial.*

319. Component of an Impulse.—Since an impulse is a vector quantity it can be resolved; thus let AB (fig. 239) represent an impulse and OX an x axis; then the x component of the impulse is represented by AC.

Proposition.—The component of the impulse of a force along any line equals the impulse of the component of the force along that line.

Proof: Let F denote the value of the force at any instant and α its angle with the line (the x axis, say). Then for any element of time (dt) including the instant, the x component of the impulse of the force is $(Fdt) \cos \alpha$. Evidently the x component of the impulse for a finite interval $t''-t'$ equals the sum of the x components of the elementary impulses, i.e., $\int_{t'}^{t''} Fdt \cdot \cos \alpha$. But

$$\int_{t'}^{t''} Fdt \cdot \cos \alpha = \int_{t'}^{t''} F_x dt,$$

F_x denoting the x component of F. Since the second integral

* While the student may not be able to compute the impulse of a force in this way, it is desirable that he should understand the principle of the method as above given.

is the impulse of the x component of the force for the interval, the proposition is proved.

320. Moment of the Impulse of a Force.—We regard the impulse of a force as having not only magnitude and direction, but also "position." If the action line of the force is fixed, then that line is also the position line of the impulse. If the action line changes, then the position line of the impulse for an element of time coincides with the action line of the force at any instant of the interval.

I. The Force is Constant and its Action Line is Fixed.—Let ab (fig. 240) represent such a force (F) and OX an axis of moments. The impulse of the force for a period $t''-t'$ is $F(t''-t')$, its position line being ab.

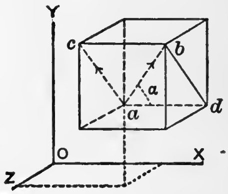


FIG. 240.

The moment of the force about OX is defined in art. 28 as the product of the component of the force which is perpendicular to OX (the other being parallel to OX) and the distance between the perpendicular component and the axis. That is, if M denotes the moment and p the distance,

$$M=(F \sin \alpha)p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In an analogous way we define the moment of the impulse of the force (or the angular impulse of the force, as it is also called) to be the product of that component of the impulse which is perpendicular to the axis and the distance between that component and the axis. If ab is taken to represent the impulse, ac represents the perpendicular component, and the angular impulse equals

$$\mathbf{F}(\mathbf{t}''-\mathbf{t}') \sin \alpha \cdot \mathbf{p}=\mathbf{M}(\mathbf{t}''-\mathbf{t}'), \quad . \quad . \quad . \quad . \quad (2)$$

i.e., for any period the moment of the impulse about any axis of a constant force whose action line is fixed equals the product of the moment of the force about that line and the length of the period.

II. The Force Varies in Any Way.—The moment of the impulse about any line for an element of time is the product of the value of the moment of the force about that line at any instant of the element and the element, i.e., Mdt. The moment of the impulse for a finite interval $t''-t'$ is the sum of the angular impulses for all elements in the interval, i.e., $\int_{t'}^{t''} Mdt$.

The Unit of an Angular Impulse is the angular impulse of a force whose impulse is such that its component perpendicular to a moment axis equals a unit impulse and has an arm of unit length. There are no names in use for these units. To describe the unit of any numerical value of an angular impulse, or moment of an impulse, we name the units of impulse and length used in the computation.

The rule of signs for moments of impulses is like that for moments of forces (see art. 28), i.e., we give the same sign to the moment of an impulse of a force with respect to an axis as we give to the moment of the force with respect to that axis.

§ II. MOMENTUM.

321. Momentum of a Particle.—The momentum of a particle is the product of the mass (m) and the velocity (v) of a particle.

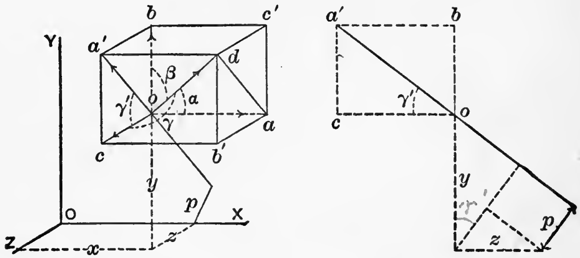
Unit of Momentum.—The definition implies as unit the momentum of a particle of unit mass moving with unit velocity. The magnitude of the unit hence depends upon the units of mass and velocity employed. No single words have been generally accepted as names for any units of momentum. It is shown in Appendix C that the dimensions of a unit momentum are the same as those of a unit impulse; hence it is not inappropriate to call these units by the same name. Thus in the C.G.S. system the unit of momentum is called a dyne-second, in the English engineers' system it is called a pound (force)-second, etc.

322. Components of a Momentum.—Momentum should be regarded as a vector quantity, its direction being the same as that of the velocity of the particle. Like any other vector quantity, a momentum may be resolved; thus if od (fig. 241a) represents the velocity of a particle, to some scale it also represents the momentum (mv), and oa and oa' represent two components of the momentum, their values being $mv \cos \alpha$ and $mv \sin \alpha$

respectively. Also *oa*, *ob*, and *oc* are the *x*, *y*, and *z* components of the momentum, and their values are

$$\mathbf{mv}\cos\alpha=\mathbf{mv}_x,\quad \mathbf{mv}\cos\beta=\mathbf{mv}_y,\quad \mathbf{mv}\cos\gamma=\mathbf{mv}_z.$$

323. Moment of Momentum.—Momentum should be regarded as having not only magnitude and direction, but also "position."



(*a*) (*b*)

Fig. 241.

The tangent line along which the velocity at any instant is directed is the position line of the momentum.

Analogous to the moment of a force with respect to an axis (art. 28) we define the moment of momentum of a particle (or angular momentum, as it is also called) to be the product of that component of the momentum which is perpendicular to the axis (the other being parallel to it) and the distance between the perpendicular component and the axis. Thus, the momentum represented by *od* (fig. 241*a*) has a moment about the line *OX* equal to $oa' \times p$, p denoting the distance from the axis to oa'.

The unit of angular momentum is the angular momentum of a particle whose momentum is such that its perpendicular component equals a unit momentum and has a unit "arm." There are no names in use for these units. To describe the unit of any numerical value of a moment of momentum we name the units of momentum and length used in the computation.

The rule of signs for moments of momentum is similar to that for moments of forces (art. 28). We imagine the "perpendicular component" to be a force and then the sign of the moment of momentum is like that of the moment of that force. Thus, in the preceding illustration, the angular momentum about the x axis is positive.

Proposition.—If the momentum of a particle be resolved into three rectangular components, the moment of momentum with respect to a line parallel to one of the components equals the sum of the moments of the other two components with respect to that line. (Compare Prop. I, art. 28.)

Proof: Let od (fig. 241) represent the momentum, OX, OY, and OZ directions of resolution, and OX the moment axis. As explained in the foregoing, the moment of momentum with respect to that axis is

$$oa' \times p = (mv \sin \alpha)p.$$

From fig. 241(b), which represents the lines in the plane boc in their true relations, it is plain that

$$p = y \cos \gamma' - z \sin \gamma';$$

hence the moment of momentum equals

$$mv \sin \alpha \ (y \cos \gamma' - z \sin \gamma'),$$

or

$$(mv_z)y - (mv_y)z.$$

324. Momentum of a System of Particles.—By momentum of a system of particles is meant the resultant * of the momenta of the particles. We shall not need a general expression for this resultant, but will deduce the value of its component along any line and the value of the resultant in special cases (art. 326).

Just as in a system of forces, the component of the resultant of any number of momenta along a line equals the algebraic sum of the components of the momenta along that line. Thus the x component of the resultant equals, if m', m'', etc., denote the masses of the particles and v', v'', etc., their velocities,

$$m'v_x' + m''v_x'' + \ldots = \Sigma mv_x = M\bar{v}_x \text{ (see eq. (2), art. 239)},$$

* Computed according to the methods for compounding forces as given in Chap. II.

M being the mass of the system and $\bar{v}$ the velocity of its mass-centre. That is, the component of the momentum of a system of particles along any line is the same as if the entire mass were concentrated at the mass-centre.

325. Moment of the Momentum of a System of Particles.—By moment of momentum of any system of particles about any line is meant the algebraic sum of the moments of the momenta of the particles about that line.

As explained in art. 323, the moment of the momentum of a particle whose mass, velocity, and coordinates are m', v', (x', y', z') about the x axis is

$$(m'v_z'y' - m'v_y'z);$$

hence the moment of momentum of the system about the x axis is

$$\Sigma(\mathbf{mv_z y - mv_y z}).$$

326. Momentum of a Rigid Body in Special Cases.—I. *A Translating Body.*—The velocities of all particles being the same in magnitude and direction, the momentum equals, if v denotes the common velocity, $(dm)_1$, $(dm)_2$, etc., the masses of the particles, and m the mass of the body,

$$\mathbf{(dm)_1v + (dm)_2v + \ldots = v\Sigma dm = mv},$$

and the direction of the momentum is the same as that of the velocity.

The position line of the momentum contains the mass-centre, as can be shown by a method similar to that employed in art. 244. Hence the moment of the momentum about any line is the same as if the entire mass were concentrated at the mass-centre.

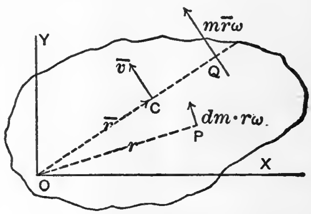


FIG. 242.

II. *A Rotating Body.*—We assume as in art. 262 that the rotating body is homogeneous and has a plane of symmetry perpendicular to the axis of rotation. Let fig. 242 represent that section of symmetry, C the mass-centre, and O the centre of rotation. As in art. 242, imagine the body divided into elementary rods parallel to the axis of rotation. Evidently the position line of the momentum of each rod is in the plane of symmetry; hence that of the momentum of all the rods (or the body) must be in that plane. Since the components of the momentum of the body parallel to the x and y axis equal $m\bar{v}_x$ and $m\bar{v}_y$ respectively (see art. 324), the resultant momentum equals

$$[(m\bar{v}_x)^2+(m\bar{v}_y)^2]^{\frac{1}{2}}=m\bar{v}=m\bar{r}\omega$$

(ω denoting the angular velocity of the body), and its direction is the same as that of the velocity of the mass-centre. The position line of the momentum passes through a point Q in the line OC, whose distance (q) from the axis of rotation is given by

$$q=k^2/\bar{r},$$

k being the radius of gyration of the body with respect to the axis of rotation. This fact can readily be proved from the result of the remainder of this article.

The moment of the momentum about the axis of rotation might be computed from art. 325, but the following method is as simple and preferable: Let P (fig. 242) represent any particle, dm its mass, and r its distance from the axis of rotation; then the momentum of that particle is $dmr\omega$, its position line as shown, and the moment of its momentum equals $(dmr\omega)r$. Hence the moment of the momentum of the body equals

$$\int dmr^2\omega=\omega\int dm\cdot r^2=I\omega,$$

I denoting the moment of inertia of the body with respect to the axis of rotation. This resultant is independent of the assumption made in the first paragraph.

§ III. PRINCIPLES OF IMPULSE AND MOMENTUM.

327. Principles for a Particle.—Let a, v, and (x, y, z) denote the acceleration, velocity, and coordinates at any instant of a particle whose mass is m, and let R denote the resultant of all the forces applied to it. Then, according to art. 236,

$$R_x=ma_x=mdv_x/dt,$$

or

$$R_xdt=mdv_x.$$

Let v' and v'' denote values of the velocity at times t' and t'' respectively, then

$$\int_{t'}^{t''} \mathbf{R}_x \mathbf{dt} = \mathbf{mv}_x'' - \mathbf{mv}_x'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The integral is the x component of the impulse of the resultant for the interval $(t''-t')$, and the right-hand member is the increment in the momentum of the particle along the x axis during that interval. Now the component along any line of the impulse of the resultant of any number of forces applied to a particle equals the algebraic sum of the components of the impulses of the forces along the same line;* hence

> The algebraic sum of the components along any line of the impulses of the forces applied to a particle equals the increment in the component of the momentum of the particle along that line.

From art. 236 we have also

$$R_y = m dv_y/dt \quad \text{and} \quad R_z = m dv_z/dt;$$

hence $\quad R_y z = mz dv_y/dt, \quad R_z y = my dv_z/dt,$

and $\quad (R_z y - R_y z) = (my dv_z - mz dv_y)/dt.$

Now the left-hand member is the moment of the resultant about the x axis, and we will replace it by M_x. The right-hand member equals $d(mv_z y - mv_y z)/dt$, i.e., the time-rate of increase of the moment of momentum about the x axis (see art. 323), and for convenience we will denote this moment by U. Then

$$M_x = dU/dt, \quad \text{or} \quad M_x dt = dU,$$

and

$$\int_{t'}^{t''} \mathbf{M}_x \mathbf{dt} = \mathbf{U}'' - \mathbf{U}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The left-hand member equals the moment of the impulse of the resultant about the x axis (equals also the sum of the moments of the impulses of the forces acting on the particle), and the right-hand member is the increment in the moment of the momentum of the particle about the x axis. Hence

* The proof can be supplied by the reader (see art. 319)

The algebraic sum of the moments of the impulses of the forces applied to a particle about any line equals the increment in the moment of the momentum of the particle about that line.

328. Principles for a System of Particles.—The internal forces in a system of particles occur in pairs, the forces of each being at each instant equal, opposite, and collinear. Therefore the impulses of each pair of forces for any interval are equal and opposite; also the moments of the impulses of each pair about any line are equal and opposite. It follows that the internal forces contribute nothing to any change in the momentum or moment of momentum of a system of particles, and it can be readily shown from the results reached regarding a single particle that

A. For any period the algebraic sum of the components of the impulses of the external forces acting on a system along any line equals the increment in the component of the momentum of the system along that line.

B. For any period the sum of the moments of the impulses of the external forces acting on any system about any line equals the increment in the moment of the momentum about that line.

Special Case: No External Forces Applied to the System.—It follows from the foregoing that

(*a*) The component of the momentum of the system along any line remains constant; this principle is called that of "conservation of linear momentum."

(*b*) The moment of the momentum of the system with respect to any line remains constant; this principle is called that of "conservation of angular momentum."

§ IV. APPLICATIONS.

329. Computation of Velocity and Time.—The equations expressing the principles of impulse and momentum (*A* and *B*, art. 328) contain impulse (or moment of impulse) terms on one side and momentum (or moment of momentum) terms on the other. The factors in the impulse or moment of impulse terms are time and force or moment of force, and those in the momentum or moment of momentum terms are velocity and mass or moment of mass. If all the factors except one in one of the equations are known, that one may be computed. In the following examples the unknown quantity is a velocity or a time, and the examples can be most readily solved by the principles of impulse and momentum.

EXAMPLES.

1. A body weighing 80 lbs. is moved along a horizontal surface by a horizontal pull of 100 lbs. If the frictional resistance is 20 lbs., how much velocity does the body acquire in 10 seconds?

Solution: There are three external forces acting upon the body, the pull, the weight, and the reaction of the surface. The components of the impulses of these along the path are respectively

$$100 \times 10, \quad 0, \quad \text{and} \quad -20 \times 10 \text{ lb.-secs.}$$

Calling the velocity of the body at the beginning of the 10-sec. period v_1, and that at the end of it v_2, the equation of impulse and momentum for the period is

$$1000 - 200 = 2.45 v_2 - 2.45 v_1,$$

2.45 being the mass of the body in geepounds. Hence the increment of the velocity is

$$v_2 - v_1 = 800/2.45 = 326.5 \text{ ft.-per-sec.}$$

2. Solve ex. 1, supposing that $v_1 = 0$, and that the pull instead of being constant equals $20 + 10t$ (t being time in secs. after the instant of starting).

3. Supposing that v_2 in ex. 1 is 300 ft.-per-sec., determine how long the body would slide after the tenth second, the motion taking place under the influence of friction alone.

4. Suppose that the disk of ex. 2, art. 298, is rotating on a shaft 4 in. in diameter, and that the axle friction is 30 lbs. In how many seconds would the frictional resistance stop it? (Use B, art. 328.)

5. Two pulleys are mounted on the same shaft, one being

"fast" and the other "loose." Suppose that the shaft (and the fast pulley) to be turning at a certain instant with an angular velocity ω, and that the loose pulley is quickly made fast so that it also turns with the shaft. If the shaft turns in smooth bearings, determine the subsequent angular velocity of the pulleys and shaft.

Solution: Under the supposition there are no external forces acting on the rotating system having moments about the axis of the shaft. Then the moment of momentum of the three bodies about the axis of the shaft remains constant.

Let I denote the moment of inertia of the shaft and the fast pulley, I' that of the loose pulley, and ω_2 the final velocity. Before and after the loose pulley is made fast, the moments of momentum of the three bodies about the axis are respectively

$$I\omega_1 + 0 \quad \text{and} \quad I\omega_2 + I'\omega_2;$$

and since these are equal as above explained,

$$\omega_2 = I\omega_1/(I+I').$$

6. Two spheres whose masses and radii equal m and r respectively rotate on a light frame about a vertical axis as shown in fig. 243 with an angular velocity ω_1. Suppose that, in some way

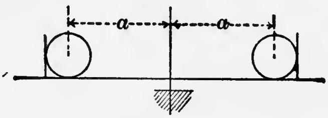


FIG. 243.

without interfering directly with the motion, the distances a are increased to b. Determine the angular velocity after the change.

330. Pressures Due to Jets.—Pressure due to a jet of liquid can often be most readily determined by the principles of impulse and momentum. How this is done is explained in the solution of some of the following

EXAMPLES.

1. Fig. 244(a) represents a jet impinging on a flat surface so that the direction of the jet is changed 90°. If W is the weight

(of water) impinging per unit time and v the impinging velocity, show that the pressure of the water on the plate is Wv/g.

Solution: Consider the motion of the amount of water represented in the figure for a small period dt, during which it moves

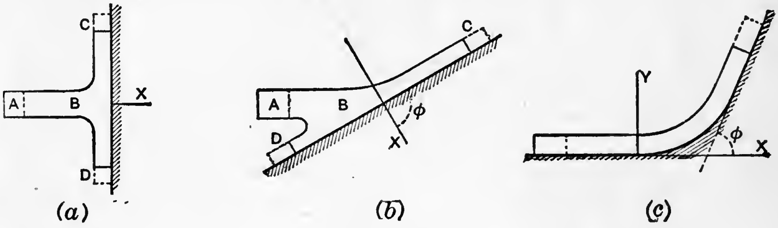


FIG. 244.

into the position indicated by the dotted lines. Let M_x' and M_x'' denote the initial and final momentums in the x direction of the water whose motion is being considered, then

$$M_x' = \text{the "}x\text{ momentum" of } B + (Wdt/g)v;$$

also
$$M_x'' = \text{the "}x\text{ momentum" of } B,$$

because C and D have no x momentum. Hence the change in the x momentum of the body of water is

$$M_x' - M_x'' = (Wdt/g)v.$$

The impulse of the force producing this change (the pressure of the surface on the water, which call F) is Fdt; hence

$$Fdt = (Wdt/g)v,$$

or
$$F = Wv/g.$$

2. Fig. 244(b) represents a jet impinging on an inclined surface. If W is the weight of the water impinging per unit time, and v the impinging velocity, show that the normal pressure on the jet equals $(Wv/g)\cos\phi$.

Solution: Consider the motion of the body of water represented for a small period during which it moves into the position indicated by the dotted lines. With notation as in the preceding solution,

$$M_x' = x \text{ momentum of } B + (Wdt/g)v\cos\phi$$

and
$$M_x'' = x \text{ momentum of } B.$$

Hence the change in the x momentum of the body of water is

$$M_x' - M_x'' = (Wdt/g)v \cos \phi.$$

Calling the normal pressure of the surface on the water F_n, the impulse of F_n is $F_n dt$, and

$$F_n dt = (Wdt/g)v \cos \phi, \quad \text{or} \quad F_n = (Wv/g) \cos \phi.$$

3. Fig. 244(*c*) represents a jet impinging on a guide or vane which suppose smooth, so that the speed of the water is not changed. Determine the x and y components of the pressure of the jet. *Ans.* The x component is $Wv(1 - \cos \phi)/g$.

4. Fig. 245 represents in plan and elevation a simple reaction-wheel. Water is poured in at the top and escapes through two orifices O and O in a horizontal direction. Such a wheel is caused to revolve by the "re-action" of the jets. It is required to determine the moment of these reactions.

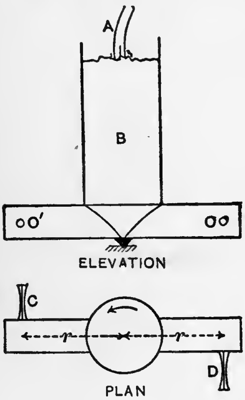


FIG. 245.

Solution: Let W denote the weight of the water escaping and entering per unit time, and v the velocity of the escaping water relative to the orifices. Then if ω denotes the angular velocity of the wheel, the absolute velocity of the jets is $r\omega - v$. Consider the motion, for a short period, of the water in the wheel and the amount about to flow in for that period (Wdt); that water is represented in the elevation fig. 245. At the end of the period the entering water is all in the wheel and an amount Wdt has escaped, as shown in the plan. Let M' and M'' denote the moment of momentum of the body of water being considered about the axis at the beginning and end of the period dt Then supposing that the water enters vertically or so that the jet is bisected by the axis of rotation,

M' = the moment of momentum of B

and M'' = " " " " " $B + (Wdt/g)(r\omega - v)r$;

the change of moment of momentum hence equals

$$(Wdt/g)(r\omega - v)r.$$

Let M_a denote the moment (about the axis) of the pressure of the wheel on the water; then the moment of the impulse of the pressure is $M_a dt$ and

$$M_a dt = (W dt/g)(r\omega - v)r,$$

or

$$M_a = (W/g)(r\omega - v)r,$$

and the moment of the water-pressure (turning the wheel) is

$$(W/g)(v - r\omega)r.$$

5. Show that when water issues from an orifice in a vessel at rest, the water exerts on the vessel a force equal to $(W/g)v$ in a direction opposite to that of the velocity of the jet. (W denotes the weight of water escaping per unit time, and v its velocity.)

331. Sudden Impulses.—The impulse of a force which acts for a very short time is called a sudden impulse.

If a body is subjected to a blow and to an ordinary or steady force at the same time, the impulse of the latter during the blow is often negligible compared with the sudden impulse. Thus in the case of a ball thrown against a wall there are two forces acting on the former during the impact, namely, the weight of the ball and the reaction of the wall. Supposing the ball to be thrown horizontally, the reaction of the wall is horizontal, and its impulse equals the change in the horizontal component of the momentum of the ball. The impulse of the weight equals the change in the vertical component of the momentum. Now this latter change, as we see from observation, is practically zero compared to the first change; hence the impulse of the weight is also practically zero compared to the impulse of the blow.

The principles of impulse and momentum are especially adapted to questions involving sudden changes of motion, and the remainder of this chapter relates to changes of this kind.

332. Force of a Blow and Recoil of a Gun.—When one body strikes another we name the act "a blow," and by force of the blow we mean the pressure between the bodies during the blow. This pressure is variable, changing from zero to a maximum and back to zero. If t denotes the duration of the blow and F the variable value of the force, the impulse is $\int_0^t Fdt$. By average force of the blow * is meant a constant force which, acting for a time equal to the duration of the blow, has an impulse equal to that of the actual force. Thus if F_a denotes the average force of the blow,

$$F_a t = \int_0^t Fdt.$$

When a gun is fired, the powder-gases exert a backward force on the gun as well as a forward one on the shot. If the mass of the powder were negligible, the two forces at each instant and their impulses would be equal. Thus let m_1 and m_2 denote masses of the shot and gun respectively, and v_1 and v_2 their velocities just after the shot leaves the barrel. Then, since the impulses producing the velocities are equal, the momenta of shot and gun must be equal, i.e.,

$$m_1 v_1 = m_2 v_2.$$

If a gun is suspended in a horizontal position by means of two parallel cords of equal length, the velocity of recoil (v_2) and the height (h) to which the gun rises during the recoil are related thus: $v_2{}^2 = 2gh$ (see ex. 9, art. 312); hence

$$m_1 v_1 = m_2 \sqrt{2gh}, \quad \text{or} \quad v_1 = (m_2/m_1)\sqrt{2gh}.$$

EXAMPLES.

1. A body whose weight is W is dropped twice from a height h, once striking on a pile of hay and once upon the ground. If the times of the impacts are t' and t'', compute the average force of the blow in each case.

2. If a 2-oz. lead bullet strikes a plate with a velocity of 1000 ft.-per-sec., and is "flattened out" in 1/100 sec., what is the average force of the blow?

333. Collision or Impact.—Consider two bodies in a collision such that the only sudden impulses involved are those of the pressures which they exert upon each other. Then the impulse of other forces (as gravity) which may act on the bodies during

* This average, it should be noted, is a time-average, and the average force of the chapter on Work and Energy (XIV) is a space-average.

collision are negligible and the principles of conservation apply. In such a collision there is no change in the momentum or moment of momentum of both bodies together.

Definitions: If the mass-centres of two bodies before collision move along the same straight line, the impact is called *direct*. If the forms of bodies are such that the pressures which they exert upon each other are directed along the line joining their mass-centres, the impact is called *central*.

334. *Direct Central Impact.*—Let the common path of the mass-centres be taken as an x axis, then the momentum of each body has no y or z component. Since the impulses due to the collision are directed along the x axis, the increment in the momentum of each body due to the collision is directed along the x axis and the momenta of the bodies after the collision have no y or z components, i.e., the mass-centres of the bodies move along the x axis after the impact.

The moment of the impulse of the force exerted upon either body about any axis through its mass-centre is zero; hence the moment of momentum of each body about any axis through its mass-centre is unchanged in the collision. In particular, if the motion of either body before collision is translatory it will be so after collision.

Let A and B be two translating bodies in collision and let

m_1 and m_2 denote the masses of A and B;
v_1 and v_2 " " velocities of A and B before impact;
v_1' and v_2' " " " " A and B after impact.

The velocities should be regarded as having sign, those in one direction being positive and those in the other negative. As previously explained, the total momentum of the two bodies is not changed by the impact; hence

$$m_1v_1+m_2v_2=m_1v_1'+m_2v_2'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It has been determined experimentally that when two spheres collide directly and centrally the velocity of either relative to the other is reversed by the impact and diminished, and that the diminution depends only on the materials. That is,

$$(v_1 - v_2) = -e(v_1'-v_2'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

e being a common fraction and called "coefficient of restitution."

We assume that relative velocities of any two bodies before and after a direct and central impact are related as in the case of spheres.

Equations (1) and (2) determine the final velocities of two colliding bodies in terms of their masses, initial velocities, and their coefficient of restitution. Thus we find from the equations that

$$v_1' = v_1 - (1+e)(v_1 - v_2)m_2/(m_1+m_2), \quad . \quad . \quad . \quad (3)$$

$$v_2' = v_2 - (1+e)(v_2 - v_1)m_1/(m_1+m_2). \quad . \quad . \quad . \quad (4)$$

The value of the impulse equals the change in the momentum of either body. This change is

$$m_1v_1' - m_1v_1 = -(1+e)(v_1 - v_2)m_1m_2/(m_1+m_2). \quad . \quad (5)$$

During a collision each body is first compressed, and after full compression has been reached it begins to recover its natural form unless the body is perfectly inelastic. The time occupied by the compression is called the period of compression, and it is assumed to be alike for both bodies. The remainder of the time of an impact is called the period of restitution. When the compression ends and restitution begins the velocities of the mass-centres are the same; let v denote this common velocity. Then, from the principle of conservation,

$$(m_1v_1 + m_2v_2) = (m_1+m_2)v = m_1v_1' + m_2v_2';$$

hence
$$v = \frac{m_1v_1 + m_2v_2}{m_1+m_2} = \frac{m_1v_1' + m_2v_2'}{m_1+m_2}. \quad . \quad . \quad . \quad . \quad (6)$$

For the case of a body (m_1) impinging on a fixed one (m_2), we substitute for m_2 and v_2 respectively ∞ and 0; then

$$v_1' = -ev_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

335. *Loss of Energy in an Impact.*—There is always a loss of kinetic energy in an impact (unless $e=1$), its value being found as follows: The kinetic energy before and after impact equals

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \quad \text{and} \quad \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2.$$

Subtracting the latter from the former and substituting for v_1' and v_2' their values from (3) and (4), we have as the loss

$$\tfrac{1}{2}(1-e^2)\frac{m_1m_2}{m_1+m_2}(v_1 - v_2)^2.$$

EXAMPLES.*

✓ 1. Show that if two bodies of equal mass and perfectly elastic ($e=1$) collide, they exchange velocities.

2. If a ball falls from a height h upon a horizontal plane, show that the height of rebound equals e^2h.

3. Two bodies of unequal mass with momenta numerically equal meet. Show that their momenta after impact are still numerically equal.

4. A body whose mass is 10 lbs., moving with a velocity of 87 ft.-per-sec., overtakes a body whose mass is 40 lbs., moving with a velocity of 6 ft.-per-second. The coefficient of restitution being 3/4, compute the final velocity of each and the impulse of the forces due to impact. *Ans.* $v_2'=6.7$ ft./sec.

5. Suppose that in ex. 4 the bodies meet and collide; then solve.

6. Suppose that in ex. 4 the bodies are inelastic, and solve.

7. Deduce values of the impulses for the periods of compression and restitution, and show that the former is independent of the coefficient of restitution.

336. Ballistic Pendulum and Centre of Percussion.—Fig. 246 represents a ballistic pendulum (for determining the velocity of a projectile); let M denote its mass, k its radius of gyration with respect to its axis of rotation, m the mass of the shot, v its striking velocity, and ω the angular velocity produced in the pendulum by the impact. The velocity of the shot just after it is imbedded and has come to rest relative to the pendulum equals $r\omega$. Since the time of the impact (during which the angular velocity is generated) is very short, the angular displacement of the pendulum during that time is practically zero, and the direction of the velocity of the shot just after imbedding is practically horizontal. Hence the impulse exerted on the shot by the pendulum and that exerted on the pendulum by the shot equal

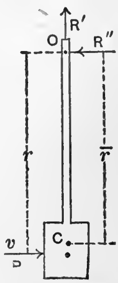


FIG. 246.

$$m(v-r\omega).$$

* Direct central impact is implied in all the examples.

The moment of the impulse of the shot on the pendulum with respect to the axis is $m(v-r\omega)r$, and the change in the moment of momentum of the pendulum (with respect to the axis) during the time of the impulse equals $Mk^2\omega$ (see art. 326). According to B, art. 328,

$$m(v-r\omega)r=Mk^2\omega, \quad \text{or} \quad v=(Mk^2+mr^2)\omega/mr.$$

If h is the height to which the centre of gravity of the pendulum rises, $\omega=(1/k)\sqrt{2gh}$ (see ex. 8, art. 312); hence

$$v=\frac{Mk^2+mr^2}{mrk}\sqrt{2gh},$$

and from this equation v may be computed, the quantities on the right side being readily measured in any actual c se.

Centre of Percussion.—Let R' and R'' denote the average values of the vertical and horizontal components f the hinge reaction on the ballistic pendulum during an impact, P the average value of the force of the blow, and t the duration of the blow which is assumed to be so short that the displacement of the pendulum for the time t is practically zero.

The momentum of the pendulum after the blow is $M\bar{r}\omega$ and its direction is horizontal, and the moment of its momentum about the axis is $Mk^2\omega$ (see art. 326); hence, according to the principles of impulse and momentum (see fig. 246),

$$R't-Wt=0,$$
$$Pt-R''t=M\bar{r}\omega,$$
$$(Pt)r=Mk^2\omega.$$

From these, $R'=W$ and $R''=M\omega(k^2/r-\bar{r})/t$.

Observe that R' is independent of the blow and that R'' equals zero *if* the blow is applied at a distance equal to $k^2/\bar{r}$ below the axis. The point in OC produced, whose distance from O equals $k^2/\bar{r}$, is called *centre of percussion;* it coincides with the centre of oscillation of the pendulum (see art. 267), and the methods given in art. 267 may be employed to locate the centre of percussion of a ballistic pendulum.

APPENDIX A.

VECTORS.

A 1. Scalar and Vector Quantities.—A quantity which has magnitude or magnitude and sign only is called a scalar quantity. An amount of money, a volume, etc., are examples of scalar quantities.

A quantity which has magnitude and direction is called a vector quantity. A step, a force, and a velocity are examples of vector quantities.

The methods of ordinary algebra are sufficient for purposes of analysis in mechanics when only scalar quantities are concerned, but insufficient in general for dealing with vector quantities. For example, the algebraic sum of two forces is in general meaningless or at any rate without mechanical significance, but their vectorial sum (to be explained) has a very important significance. There is a branch of mathematics sometimes called Vector Algebra, the methods of which are especially adapted for dealing with vector quantities. We proceed to a brief explanation of Addition and Subtraction by those methods.

A 2. Vector Defined.—A straight line of definite length and direction is called a vector. The word direction here refers not only to the inclination (or "clinure") of the line, but also to its

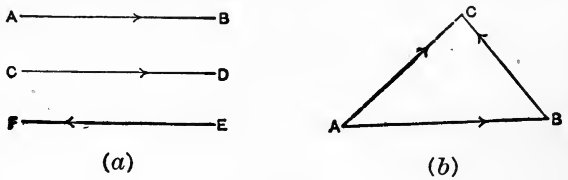


FIG. A 1.

"sense" ("right- or left-ness" or "up- or down-ness" along the line). The sense is usually indicated by an arrow-head placed on the line. The lines of fig. A 1 are vectors.

Two vectors in order to be equal must be equal in length and of the same direction. The first and second vectors of fig. A 1 (a) (counting downwards) are equal because they agree in length and direction, but the third is not equal to either of the others because its direction is different from theirs.

A 3. Addition of Vectors.—*Definition.*—The sum of the vectors AB and BC is the vector AC (fig. A 1 b). Notice that this definition does not conflict (as at first sight it may appear) with the proposition of geometry which states that the length of any side of a triangle is less than the sum of the lengths of the other two.

To add more than two vectors we proceed as in the ordinary algebra, i.e., we add any two, then to that sum another vector, and so on until all have been combined. As in the addition of scalars, the sum of several vectors does not depend on the order in which they are added; thus the sum of the four vectors a, b, c, and d (fig. A 2) is found to be AB by adding them in a certain order and CD

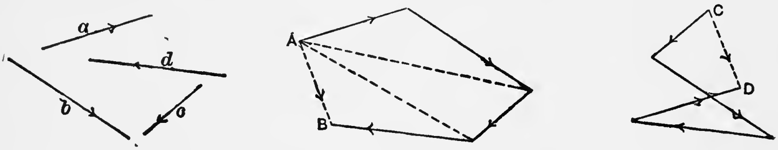


FIG. A 2.

by adding in another order, and AB and CD are found to agree in length and direction. If they are added in any other order, the sum will be found to be equal to AB or CD.

EXAMPLE.

Reverse the arrows on vectors c and d (fig. A 2) and add the four vectors in at least two ways.

A 4. Negative of a Vector.—In ordinary algebra, the negative of a quantity is one which added to the quantity gives a sum equal to zero. So, too, the negative of a vector is one which added to the vector gives a sum equal to zero. Now the sum of two vectors which are equal in length, parallel, and opposite in sense equals zero; hence either is the negative of the other.

A 5. Subtraction of Vectors.—In ordinary algebra, a quantity is subtracted from another by adding its negative. So, too, to subtract a vector from another we add the negative of the former to the latter.

EXAMPLES.

1. Subtract vector a (fig. A 2) from vector b.
2. Subtract vector b from vector a.

APPENDIX B.

RATES.*

THE term rate is in common use, but usually in an inexact sense In the following a precise meaning is given to it which, it will be observed, does not conflict with the popular notions in so far as they are exact.

B 1. Kinds of Variable Quantities.—Let x and y be two quantities which are related to each other, a change in x producing a change in y. If all equal changes in x (large or small) produce equal changes in y, y is said to vary uniformly with respect to x and it is called a *uniform variable.* If equal changes in x produce unequal changes in y, it is said to vary non-uniformly with respect to x, and it is called a *non-uniform variable.*

B 2. Rate of a Uniform Scalar.—If y is a uniform variable, then the locus representing the relation between x and y is a straight line, for in such a locus equal changes in x produce equal changes in y.

In this simple case, the popular meaning of "the rate of y" is definite, it being the change in y per unit change in x. If Δx, $x_2 - x_1$ (see fig. B 1), denotes any change in x, and Δy, $y_2 - y_1$, the corresponding change in y, then the change in y per unit change in x is $\Delta y/\Delta x$, or $(y_2 - y_1)/(x_2 - x_1)$. Hence if r denotes rate,

$$r = \Delta y/\Delta x. \quad . \quad . \quad . \quad . \quad (1)$$

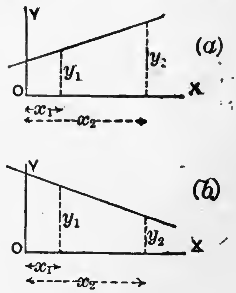


FIG. B 1.

Evidently r is the same for all values of Δx, i.e., the rate of a uniform variable is constant.

B 3. Rate of a Non-Uniform Scalar.—If y denotes a non-uniform variable, then the locus representing the relation between x and y obviously must be curved.

According to popular notions, the rate of a non-uniform vari-

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* This appendix is intended for students who do not associate the idea of rate with dy/dx, but do understand that dy/dx is a "limit."

able is not constant and has a definite value at each value of the variable, but these definite values of the rate are not clearly discerned in the popular mind.

Average Rate Defined.—Let Δy denote the change in y due to a change Δx in x (see fig. B 2 *a*). Evidently the rate of a uniform variable, also depending on x, might be such that the change in that variable due to a change Δx in x equals Δy. The *average rate of the non-uniform variable* for the range $x_2 - x_1$ is defined as the rate of such a uniform variable as just mentioned. Now the rate of the uniform variable is $\Delta y/\Delta x$; hence if r_a denotes average rate of the non-uniform variable,

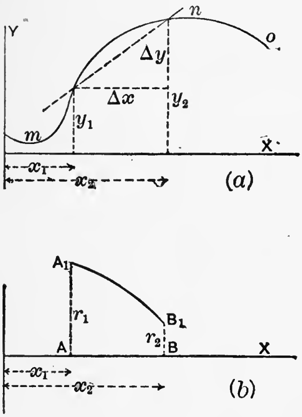

FIG. B 2.

$$\mathbf{r}_a = \Delta \mathbf{y}/\Delta \mathbf{x}. \quad . \quad . \quad . \quad (2)$$

True or Instantaneous Rate Defined.—The value of the average rate of a variable is generally different for different values of $x_2 - x_1$, but (see the figure) the average rate, $\Delta y/\Delta x$, approaches a definite value as Δx approaches zero (x_2 approaches x_1, for instance). The rate of the variable y at the value $y = y_1$ is defined as the limit of the average rate of y for a range $x_2 - x_1$ as x_2 approaches x_1. Now the limit of $\Delta y/\Delta x$ is dy/dx, i.e., the rate of y at the value $y = y_1$ equals the value of dy/dx corresponding to $y = y_1$ (and $x = x_1$). Hence if r denotes the rate of y at any value of y,

$$\mathbf{r} = \mathbf{dy}/\mathbf{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Observe that the rate of y at the value $y = y_1$ is represented by the slope of the tangent to the locus representing the relation between x and y at the point $x_1 y_1$.

Consistency of the Definitions.—The average value of a number of quantities is computed by adding them and dividing the sum by their number—the quotient being the average sought. It can be shown that the above definitions of average and instantaneous rates agree with this method of computing averages. Thus let r_1 and r_2 denote the values of the rate of y when $x = x_1$ and x_2 respectively (see fig. B 2 *b*), and let ordinates to the curve represent values of the rate at intermediate values of x. Then since the average ordinate represents the average rate, the latter is given by

$$r_a = (\text{area } ABB_1A_1)/(x_2 - x_1).$$

It is shown in works on calculus that the area ABB_1A_1 equals $\int_{x_1}^{x_2} r\,dx$, hence

$$r_a = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} r\,dx = \frac{1}{x_2 - x_1}\int_{y_1}^{y_2} dy = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

Thus we find that the ordinary method of computing averages leads to a result identical with that given by eq. (2).

The definition of "instantaneous rate" further agrees with popular notions, as may be explained thus: Let y and z denote uniform and non-uniform variables respectively, both depending on x, and suppose that their relations to x are represented by the straight and curved loci of fig. B 3. From the figure it is plain that for any change Δx between A and B (at A_z the tangent to the curve is parallel to the straight line) the change, or increment, in z is smaller than that in y, and that they become more nearly equal the nearer B is to A. Also for any change Δx between A and C the change in z is greater than that in y, and they become more nearly equal the nearer C is to A. Popularly stated: up to A, the rate of z is less than the rate of y, beyond A the rate of z is greater than that of y, and at A they are equal. Now our formulas for rate should agree with this popular expression, and it is readily shown that they do. Thus the rates of z and y are dz/dx and dy/dx; and it will be seen from the figure that for values of x less than x' the former is the lesser, for values of x greater than x' the former is the greater, and for $x = x'$ they are equal.

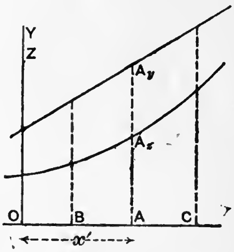

FIG. B 3.

B 4. Sign of a Rate.—Both $\Delta y/\Delta x$ and dy/dx may be negative as well as positive; hence in order that equations (1), (2), and (3) may be true as to sign, it is necessary to regard a rate as having sign and the sign must be the same as that of the expression for the rate ($\Delta y/\Delta x$ or dy/dx). Now $\Delta y/\Delta x$ (for a uniform variable) and dy/dx are positive when y increases as x increases, and they are negative when y decreases as x increases.*

> Hence the rate of y with respect to x at any particular value of y is positive or negative according as y increases or decreases at that value as x increases.

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* Algebraic increase and decrease are meant.

Thus in fig. B 1 (*a*) the rate of y is positive, in fig. B 1 (*b*) it is negative, and in fig. B 2 (*a*) the rate has different signs at different values of x or y, it being positive from m to n, negative from n to o.

B 5. Unit of Rates.—The expressions for rates in eqs. (1), (2), and (3) imply a certain unit, namely, the rate of a uniform variable y which changes a unit in amount for a unit change in x. Thus if y denotes volume and x distance, the unit rate might be one cubic foot per foot, one gallon per inch, etc.; if y denotes distance and x time, the unit rate might be one foot per second, one mile per hour, etc., etc.

B 6. Rate of a Uniform Vector.—Let y denote a vector which is related to x and let Oa, Ob, Oc, etc. (fig. B 4), represent y at values of x equal to x_1, x_2, x_3, etc., the differences between successive values of x being equal ($x_2 - x_1 = x_3 - x_2$, etc.). Now if $ab = bc = cd$ etc., y is a uniform variable, for the changes (or increments) in y (vectors) for equal changes in x are equal.

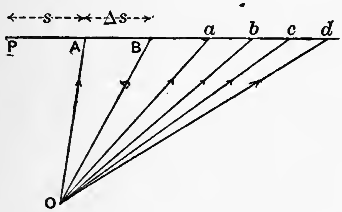


FIG. B 4.

By rate of a uniform vector y is meant its change per unit change in x; hence the rate equals any change in y (a vector) divided by the corresponding change in x, as (vector ac)/$(x_3 - x_1)$. If OA and OB represent y for any values of x as x and $x + \Delta x$ respectively, then the rate of y is

$$(\text{vector } AB)/\Delta x.$$

Let P be a fixed point in ad in the direction BA from A, and let s denote the variable distance PA. The increment of s due to a change Δx in x equals AB as shown, and since y is a uniform variable, s evidently is also; hence the rate of s is $\Delta s/\Delta x$. Finally, since Δs = length AB, the rate of y is a vector

> whose *direction* is that of the increment of y and
> whose *magnitude* equals ds/dx.

B 7. Rate of a Non-Uniform Vector.—Let Oa, Ob, Oc, etc. (fig. B 5), represent a vector y at values of x equal to x_1. x_2, x_3, etc., the successive differences in x being equal. The changes in y corresponding to changes $x_2 - x_1$, $x_3 - x_2$, etc., are the vectors ab, bc, etc. Then if, in fig. B 5 (*a*), ab, bc, etc., are unequal and if, in Fig. B 5 (*b*), ab, bc, etc., are equal or unequal, y is a non-uniform variable; for the changes, or increments, in y due to equal changes in x are not equal.*

Average Rate Defined.—The average rate of a vector y for any range Δx in x is defined as the rate of a uniform vector whose change, for the same range in x, equals that of y. Thus if OA and OB repre-

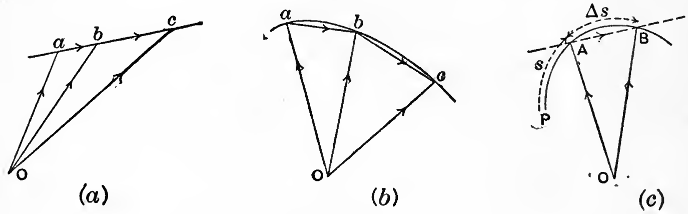


FIG. B 5.

sent y for values x and $x+\Delta x$, the change in y for the range Δx is the vector AB. Now the rate of a uniform vector whose change is also AB for the range Δx is (vector $AB)/\Delta x$, and this, by the definition, is the average rate of y during the change Δx, i.e.,

the average rate is a vector whose direction is AB and whose magnitude equals (length $AB)/\Delta x$.

Actual or Instantaneous Rate Defined.—The value of the average rate of y is different in magnitude and in direction for different values of Δx. The average rate approaches a definite direction and a definite magnitude as Δx approaches zero. The rate of the variable y at the value $y=OA$ is defined as the limit of the average rate of y as Δx approaches zero (B approaches A). The limiting direction of the average rate is the limiting direction of AB, which is the direction of the tangent at A. The limiting value of the average rate is the limit of (chord $AB)/\Delta x$, which is the same as the limit of (arc $AB)/\Delta x$. Now let P be any fixed point on the curve AB in the direction BA from A, and s the distance from P to A; then the change in s, Δs, due to a change Δx in x is the arc AB, and

$$\lim \frac{\text{arc } AB}{\Delta x} = \text{limit } \frac{\Delta s}{\Delta x} = \frac{ds}{dx}.$$

Finally, the rate of y at the value $y=OA$ is a vector

whose *direction* is that of the tangent at A, and whose *magnitude* $= ds/dx$.

* Equal vectors are equal in length and the same in direction.

It is implied that the fixed point from which s is measured is so taken that Δs is positive. The arrow on the tangent, giving the sense of the rate, points in the positive s direction.

B 8. Descriptive Terms.—A variable y may depend on several quantities. If so, it has as many rates, and we say for brevity the "x rate of y" to distinguish the rate of y with respect to x from its other rates. The rate dy/dx is often described by naming the kind of a quantity represented by x; thus if x denotes distance, dy/dx is called a "space rate" of y; if x denotes time, dy/dx is called a "time rate" of y. The time rate of y is sometimes indicated thus, $\dot{y}$.

If y and x denote certain quantities, the rate is given a single name; thus if y and x denote distance and time respectively, dy/dx is called velocity; if y denotes mass and x volume, dy/dx is called density.

APPENDIX C.

DIMENSIONS OF UNITS.

C 1. Magnitude of a Quantity.—The magnitude of a quantity is expressed by stating how many times larger it is than a standard quantity of the same kind and naming the standard. Thus, we say that a certain distance is 10 miles, meaning that the distance is 10 times as great as the standard distance, the mile.

The number expressing the relation between the magnitude of the quantity and the standard (the number 10 in the illustration) is called the *numeric* (or numerical value) of the quantity, and the standard is called the *unit*.

C 2. Fundamental and Derived Units.—A unit for measuring any kind of quantity may be selected arbitrarily, but it must of course be a quantity of the same kind as the quantity to be measured (e.g., a unit for measuring lengths must be a length). Thus, as unit of velocity we might select the velocity of light, as unit of area the area of one face of a silver dollar, etc. Many units in use are arbitrarily chosen, i.e., without reference to another unit (e.g., the bushel and the degree), but it is convenient practically to define them with reference to each other. All mechanical and nearly all physical quantities can be defined in terms of three arbitrarily selected units, i.e., ones not dependent on any other units. These are called *fundamental units*, and the others, defined with reference to them, *derived units*. It is customary in works on theoretical mechanics and physics to choose as fundamental the units of

length, mass, and time,

but it is sometimes more convenient to take as fundamental the units of

length, force, and time.

In the following article we give a discussion of derived units with reference to each of these sets of fundamentals, and on pages 360 and 361 there appear summaries in which the absolute units are referred to the first set of fundamentals and the gravitational units to the second set. But either set might serve as fundamentals for all absolute and gravitational units.

C 3. "Dimensions" of Units.—A statement of the way in which a derived unit depends on the fundamental ones is called a statement of its dimensions.

Obviously an area depends only on the unit of length, definitely as the square of the unit of length. Thus

$$(\text{one sq. yd.})/(\text{one sq. ft.}) = (\text{one yd. or three ft.})^2/(\text{one ft.})^2 = 9.$$

This relation is expressed in the form of a "dimensional" equation, thus

$$(\text{unit area}) = (\text{unit length})^2,$$

and briefly a unit area is said to be "two dimensions in length." Similarly a unit volume is said to be three dimensions in length.

Velocity.—According to the definition of velocity (art. 167), a unit velocity is directly proportional to the unit length and inversely to the unit time; hence if **V**, **L**, and **T** denote units of velocity, length, and time respectively, the dimensional equation is

$$\mathbf{V} = \mathbf{L}/\mathbf{T} = \mathbf{LT}^{-1},$$

and a unit velocity is one dimension in length and minus one in time.

Acceleration.—According to the definition of acceleration (art. 173), a unit acceleration is proportional directly to the unit velocity and inversely to the unit of time; hence if **A** denotes unit acceleration, the dimensional equation is

$$\mathbf{A} = \mathbf{V}/\mathbf{T} = \mathbf{L}/\mathbf{T}^2 = \mathbf{LT}^{-2},$$

and a unit acceleration is one dimension in length and minus two in time.

Angular Velocity.—According to the definition of angular velocity (art. 210), a unit angular velocity is proportional directly to the unit of angle and inversely to the unit of time; hence if ω and θ denote units of angular velocity and angle respectively, the dimensional equation is

$$\omega = \theta/\mathbf{T}, \quad \text{or} \quad \omega = \mathbf{T}^{-1},$$

since units of angle, (degree, radian, etc.) are independent of the fundamental units. A unit angular velocity is therefore minus one dimension in time.

Angular Acceleration.—According to the definition of angular velocity (art. 213), a unit angular acceleration is proportional directly to the unit angular velocity and inversely to the unit time; hence if α denotes unit angular acceleration, the dimensional equaion is

$$\alpha = \omega/\mathbf{T} = \mathbf{T}^{-2},$$

and a unit angular acceleration is minus two dimensions in time.

Force.—In accordance with the equation of motion of a particle (art. 236), $R = ma$, or

"force = mass × acceleration",

i.e., the unitforce is directly proportional to the units of mass and acceleration; hence if **F** and **M** denote units of force and mass respectively, the dimensional equation is

$$\mathbf{F} = \mathbf{MA} = \mathbf{LMT}^{-2},$$

and a unit force is one dimension in length, one in mass, and minus two in time.

Mass.—If we regard length, force, and time as fundamental units, then the last equation written as follows is the dimensional equation for a unit mass:

$$\mathbf{M} = \mathbf{FT}^2/\mathbf{L} = \mathbf{L}^{-1}\mathbf{FT}^2,$$

and a unit mass is minus one dimension in length, one in force, and two in time.

Work.—According to the definition of work (art. 286), the unit of work is directly proportional to the units of force and length; hence if **W** denotes unit work, the dimensional equation is

$$\mathbf{W} = \mathbf{LF} = \mathbf{L}^2\mathbf{MT}^{-2},$$

and a unit work is one dimension in length, one in force, or two in length, one in mass, and minus two in time.

Power.—According to the definition of power (art. 311), a unit of power is proportional directly to the unit of work and inversely to the unit of time; hence if **P** denotes unit of power, the dimensional equation is

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = \mathbf{LFT}^{-1} = \mathbf{L}^2\mathbf{MT}^{-3},$$

and a unit power is one dimension in length and force and minus one in time, or two in length, one in mass, and minus three in time.*

C 4. Applications of the Theory of Dimensions.—A knowledge of the theory of dimensions is probably of most value to the beginner as a help to a clear understanding of the different mechanical quantities and the relations between them. The theory is useful practically in other ways, two of which we mention.

(1) *As a test of the accuracy of equations between mechanical quantities.*—Such an equation if *rationally* and *correctly* deduced must be homogeneous, i.e., the terms in it must be the same in kind. To

* Determination of the dimensions of the other units mentioned in the following tables is left to the student.

ABSOLUTE SYSTEMS ("SCIENTIFIC").

Names of Quantities.	Dimensional Formulas.	Names of Units.	
		C.G.S.	F.P.S.
Length	L	centimeter (cm)	foot (ft)
Mass	M	gram (gr)	pound (lb)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	cm/sec ("kine")	ft/sec
Acceleration	LT^{-2}	cm/sec² ("spoud")	ft/sec²
Angular Velocity	T^{-1}	rad/sec	rad/sec
Angular Acceleration	T^{-2}	rad/sec²	rad/sec²
Force	LMT^{-2}	dyne	poundal (pdl)
Weight	LMT^{-2}	dyne	pdl
Moment of Mass	LM	gr-cm	lb-ft
Moment of Inertia (Body)	L^2M	gr-cm	lb-ft
Moment of Force	L^2MT^{-2}	cm-dyne	ft-pdl
Work	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Energy	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Power	L^2MT^{-3}	erg/sec	ft-pdl/sec
Impulse	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Momentum	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Density	$L^{-3}M$	gr/cm³	lb/ft³
Specific Weight	$L^{-2}MT^{-1}$	dyne/cm³	pdl/ft³
Moment of Area	L^3	cm³	ft³
Moment of Inertia (Area)	L^4	cm⁴	ft⁴
Stress	LMT^{-2}	dyne	pdl
Stress Intensity	$L^{-1}MT^{-2}$	dyne/cm²	pdl/ft²

ascertain whether terms are the same in kind we write the dimensional form of the equation, reduce the terms to their simplest forms and compare; if they are alike, the terms are the same in kind. To illustrate, consider eq. (1), art. 250,

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+q^2y=q^2A \sin \omega t, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which y and A denote lengths, t and $1/p$ time, and q and ω angular velocity. Then d^2y/dt^2 is an acceleration and dy/dt a velocity, and the dimensional equation is

$$\mathbf{LT^{-2}+(T^{-1})(LT^{-1})+(T^{-1})^2(L)=(T^{-1})^2L.}$$

An abstract number, as the sine of an angle, is independent of all units and hence does not affect a dimensional equation. Reducing the terms in the last equation we get

$$\mathbf{LT^{-2}+LT^{-2}+LT^{-2}=LT^{-2}};$$

i.e., the terms are alike and the original equation is homogeneous.

Showing that the equation is homogeneous does not prove that

GRAVITATION SYSTEMS ("ENGINEERS'").

Names of Quantities.	Dimensional Formulas.	Names of Units.	
		F.P. (force) S.	M.K. (force) S.
Length	L	foot (ft)	meter (m)
Force	F	pound (lb)	kilogram (kg)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	ft/sec	m/sec
Acceleration	LT^{-2}	ft/sec²	m/sec²
Angular Velocity	T^{-1}	rad/sec	rad/sec
Angular Acceleration	T^{-2}	rad/sec²	rad/sec²
Mass	$L^{-1}FT^2$	"geepound" (glb)	"geekilogram" (gkg)
Weight	F	lb	kg
Moment of Mass	FT^2	glb-ft	gkg-m
Moment of Inertia	LFT^2	glb-ft²	gkg-m²
Moment of a Force	LF	ft-lb	m-kg
Work	LF	ft-lb	m-kg
Energy	LF	ft-lb	m-kg
Power	LFT^{-2}	ft-lb/sec	m-kg/sec
Impulse	FT	lb-sec	kg-sec
Momentum	FT	lb-sec	kg-sec
Density	L^2FT^2	glb/ft³	gkg/m³
Specific Weight	$L^{-3}F$	lb/ft³	kg/m³
Moment of Area	L^3	ft³	m³
Moment of Inertia	L^4	ft⁴	m⁴
Stress	F	lb	kg
Stress Intensity	$L^{-2}F$	lb/ft²	kg/m²

it is correct, but that it may be correct; showing that an equation is non-homogeneous shows it to be incorrect. Since abstract numbers do not appear in the dimensional form of an equation, the test for homogeneity does not discover errors in numerical coefficients and terms, nor of course errors in signs.

As another illustration consider eq. (2), art. 250,

$$y = \frac{q^2 \sin\varepsilon}{\omega p} A \sin(\omega t + \varepsilon). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It was deduced from eq. (1) by two integrations. If we find that (2) is non-homogeneous, we may be sure that a mistake was made in the deduction from (1) to (2). The dimensional form of (2) is

$$\mathbf{L} = \frac{(\mathbf{T})^2}{(\mathbf{T}^{-1})(\mathbf{T}^{-1})}\mathbf{L} = \mathbf{L},$$

i.e., the equation is homogeneous and it may be correct.

Not only must rational mechanical equations be homogeneous, but every factor or expression in it which is the sum of several terms must also be homogeneous. For example, in eq. (2) the expression

$(\omega t + \epsilon)$ must be homogeneous, and since ϵ denotes an angle, it is readily seen to be so.

(2) *To express a magnitude in different units.*—Obviously the numerical value of a given quantity changes inversely as the magnitude of the unit used; thus a certain distance may be expressed as

10 mi., 17,600 yds., and 52,800 ft.,

and plainly the numerics are respectively as 1, 1760, and 5280, while the corresponding units are as 5280, 1760, and 1.

Let q_1 be the known numerical value of a quantity when expressed in the unit Q_1, and q_2 the numeric (to be found) of the same quantity expressed in the unit Q_2; then

$$q_1/q_2 = Q_2/Q_1, \quad \text{or} \quad q_1 = q_2 Q_2/Q_1.$$

The ratio Q_1/Q_2 can be easily computed by substituting for Q_1 and Q_2 their *equivalents* in terms of fundamental units; thus if a, b, and c are the dimensions of Q_1 (and Q_2),

$$Q_1 = k_1(L_1^a M_1^b T_1^c) \quad \text{and} \quad Q_2 = k_2(L_2^a M_2^b T_2^c),$$

where L_1, M_1, and T_1 are the *particular* fundamentals for Q_1, L_2, M_2, and T_2 those for Q_2, and k_1 and k_2 numerical coefficients (very often unity). Finally,

$$q_1 = q_2 \frac{k_1}{k_2} \left(\frac{L_1}{L_2}\right)^a \left(\frac{M_1}{M_2}\right)^b \left(\frac{T_1}{T_2}\right)^c.$$

As an example, let us determine how many watts in 10 horsepower. Since Q_1 (horse-power) $= 550$ ft.-lb.-sec.$^{-1}$ and Q_2 (watt) $= 10^7$ ergs per sec. $= 10^7$ cm.-dyne-sec.$^{-1}$,

$$q_1 = 10 \frac{550}{10^7} \frac{\text{ft.}}{\text{cm.}} \frac{\text{lb.}}{\text{dyne}} \frac{\text{sec}^{-1}}{\text{sec}^{-1}} = 10 \frac{550}{10^7}(30.48)(4.45 \times 10^5)(1) = 7640.$$

(3) *To ascertain the unit of the result of a numerical calculation*—Substitute for the quantities the names of the units in which they are expressed, and then repeat the calculation, treating the names as though they were algebraic quantities. The reduced answer is the name of the unit of the numerical answer. Thus in the formula for the elongation of a rod due to a pull at each end, Pl/AE, wherein P denotes pull, l length of the rod, A area of cross-section, and E Young's modulus for the material, suppose that $P = 10{,}000$ lbs., $l = 50$ in., $A = 0.5$ in.^{2}, $E = 30{,}000{,}000$ lbs./in.^{2}; the calculations for elongation and name of unit are

$$\frac{10{,}000 \times 50}{0.5 \times 30{,}000{,}000} = 0.33, \quad \text{and} \quad \frac{\text{lbs.} \times \text{in.}}{\text{in.}^2 \times \text{lbs./in.}^2} = \frac{\text{lbs.} \times \text{in.} \times \text{in.}^2}{\text{in.}^2 \times \text{lbs.}} = \text{in.}$$

APPENDIX D.

SECOND MOMENTS OF AREAS (MOMENT OF INERTIA, ETC.).*

In the subject of strength of materials especially, the student of engineering meets with quantities expressed by integrals of the kind and form $\int dA \cdot x^2$ and $\int dA \cdot xy$, A denoting area and x and y distance. Such quantities have been called "moments of area of the second order," or briefly "second moments of area," the terms being in line with "first moments of area," which term is applied to quantities expressed by integrals like $\int dA \cdot x$ (see art. 83). We distinguish between second moments of area employing special names for the kinds.

§ I. Moment of Inertia.

1. Moment of Inertia Defined.—The moment of inertia of a plane area with respect to any axis is the sum of the products obtained by multiplying each elementary part of the area by the square of its distance from the axis.

The axis of reference will often be called "inertia-axis" to distinguish it from other axes, coordinate, geometrical, etc. We consider only moments of inertia with respect to axes in or normal to the plane area; the latter are called *polar moments of inertia*, and the corresponding axes *polar axes*.

Expression for Moment of Inertia.—Let dA_1, dA_2, dA_3, etc., denote elementary parts of an area and ρ_1, ρ_2, ρ_3, etc., respectively their distances from some axis; then according to the definition, the moment of inertia of the area with respect to that axis is

$$(dA_1)\rho_1^2 + (dA_2)\rho_2^2 + \ldots$$

Or, if I † denotes the moment of inertia, dA any elementary por-

* Writers on Strength of Materials usually refer to works on Mechanics for a treatment of these second moments, and for that reason this appendix is herein included.

† A subscript affixed to the symbol refers to the inertia-axis; thus I_x stands for moment of inertia with respect to the x axis.

tion of the area all points of which are equally distant from the axis, and ρ that distance,

$$I = \int dA \cdot \rho^2.$$

[The term moment of inertia of an area is unfortunate because beginners are prone to seek reasons for its appropriateness which do not exist. They should recognize at the outset that an area has no inertia and hence, in the ordinary sense of the words, no moment of inertia. The reason why this second moment of an area was so called lies in the fact that the moment is closely analogous to another quantity (a second moment of mass, see art. 254) which had previously been called moment of inertia of a body.]

D 2. Units of Moment of Inertia.—Each term in the preceding series is the product of four lengths; hence a moment of inertia of an area is four "dimensions" in length. The numerical value of a moment of inertia of an area is usually computed with the inch as unit length, and the corresponding unit moment of inertia is called a "biquadratic inch," abbreviated thus: $in.^4$

D 3. Radius of Gyration.—Since any moment of inertia of an area is four "dimensions" in length, it can be expressed as the product of an area and a length squared. It is sometimes convenient to so express it.

Definition.—The radius of gyration of an area with respect to an axis is such a length whose square multiplied by the area equals the moment of inertia with respect to that axis. That is, if k and I denote the radius of gyration and moment of inertia of an area A with respect to the same axis,

$$k^2A = I, \quad \text{or} \quad k = \sqrt{I/A}.$$

The square of the radius of gyration of an area with respect to an axis is the mean of the squares of the distances of all the *equal* elementary parts of the area from that axis. For let ρ_1, ρ_2, etc., be the distances from the elements (dA) to the axis, and let n denote their number (infinite); then the mean of the squares is

$$(\rho_1^2 + \rho_2^2 + \rho_3^2 + \ldots)/n = (\rho_1^2 dA + \rho_2^2 dA + \ldots)/ndA = I/A.$$

But I/A is the square of the radius of gyration, hence, etc.

[Like moment of inertia, the term radius of gyration when applied to areas is strictly inappropriate. Its use in this connection is justified by analogy, the quantity k being closely analogous to another quantity which had been previously called radius of gyration (art. 255).]

EXAMPLES.

1. Show that the moment of inertia and radius of gyration of a *rectangle with respect to a central axis parallel to the base* are respectively

$$\tfrac{1}{12}ba^3 \quad \text{and} \quad a\sqrt{1/12},$$

b and a denoting the base and altitude respectively.

Solution: We will take a horizontal strip as elementary area (see fig. D 1); then $dA = bdy$, and

$$I_x = \int_{-a/2}^{+a/2} bdy \cdot y^2 = \tfrac{1}{12}ba^3.$$

Also, $A = ba$, therefore $k^2 = \tfrac{1}{12}ba^3/ba$, or $k = a\sqrt{1/12}$.

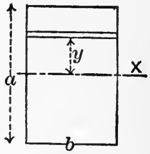


FIG. D 1.

2. Show that the moment of inertia and radius of gyration of a *triangle with respect to a central axis parallel to the base* are respectively

$$\tfrac{1}{36}ba^3 \quad \text{and} \quad a\sqrt{1/18},$$

b and a denoting the base and the altitude respectively.

Solution: We will take a horizontal strip as elementary area (fig. D 2), calling the length of any strip u, then $dA = udy$; also,

$$\frac{u}{b} = \frac{\frac{2}{3}a - y}{a}; \text{ therefore } dA = (2/3 - y/a)bdy.$$

Hence

$$I_x = \int_{-a/3}^{2a/3} (2/3 - y/a)bdy \cdot y^2 = \tfrac{1}{36}ba^3.$$

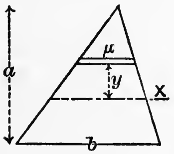


FIG. D 2.

Also, since $A = \frac{1}{2}ba$, $k^2 = \frac{1}{36}ba^3/\frac{1}{2}ba$ or $k = a\sqrt{1/18}$.

3. Show that the moment of inertia and radius of gyration of a *circle with respect to a diameter* are respectively

$$\tfrac{1}{4}\pi r^4 \quad \text{and} \quad \tfrac{1}{2}r,$$

r denoting radius of the circle.

Solution: We will take a horizontal strip as elementary area (see fig. D 3 a); then if u denotes the length of any strip,

$$dA = udy = 2\sqrt{r^2 - y^2}dy.$$

Hence

$$I_x = 2\int_{-r}^{+r} \sqrt{r^2 - y^2}dy \cdot y^2 = \tfrac{1}{4}\pi r^4;$$

and since $A = \pi r^2$,

$$k_x^2 = \tfrac{1}{4}\pi r^4/\pi r^2, \quad \text{or} \quad k_x = \tfrac{1}{2}r.$$

4. Show that the moment of inertia and radius of gyration of a *circle with respect to a central polar axis* are respectively

$$\tfrac{1}{2}\pi r^4 \quad \text{and} \quad r\sqrt{1/2}.$$

Solution: We choose as elementary area one as represented in fig. D 3 (*b*); then $dA = \rho d\theta d\rho$, and

$$I_z = \int_0^{2\pi} \int_0^r (\rho d\theta \cdot d\rho)\rho^2 = \tfrac{1}{2}\pi r^4, \text{ etc.}$$

In the preceding examples, the inertia-axes are central, but moments of inertia of such areas can be readily computed by integra-

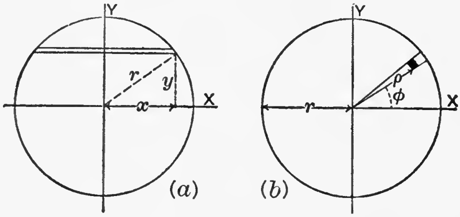


FIG. D 3.

tion for any axes which are simply situated with reference to a line of the figure. Such cases are rectangle and triangle with inertia-axis parallel to a side, circle with inertia-axis parallel to a diameter, etc. In the next article it is shown how to determine moments of inertia without integration in these and similar cases, but, to test his understanding of the integration method, the student should determine the moment of inertia of

(*a*) a rectangle with respect to a side;
(*b*) " triangle " " " " " ;
(*c*) " circle " " " " tangent.

D 4. Relations between Moments of Inertia and between Radii of Gyration with Respect to (a) Two Parallel Axes, (b) Three Rectangular Axes.—*Proposition I.*—The moment of inertia (I) of an area with respect to any axis equals its moment of inertia ($\bar{I}$) with respect to a parallel centroidal axis plus the product of its area (A) and the square of the distance (d) between the axes, or, symbolically,

$$\mathbf{I} = \bar{\mathbf{I}} + \mathbf{Ad}^2. \quad (1)$$

Proof: (*a*) The inertia-axis is in the plane area. Let the area be that represented in fig. D 4 (*a*), U being the inertia axis and C the centroid. Then

$$I = \int dA \cdot v^2 = \int dA (y + d)^2 = \int dA\ y^2 + 2d\int dA \cdot y + d^2 \int dA.$$

But $\int dA \cdot y^2 = \bar{I}$, $\int dA \cdot y = A\bar{y} = 0$, and $\int dA = A$; hence, etc.

(*b*) The inertia-axis is normal to the area. Let the area be that represented in fig. D 4 (*b*), *O* the point where the inertia-axis pierces the plane of the area, and *C* the centroid. Then

$$I=\int dA(y^2+\overline{x+d}^2)=\int dA(y^2+x^2)+2d\int dA\cdot x+d^2\int dA.$$

Now

$$\int dA(y^2+x^2)=\bar{I},\quad \int dA\cdot x=A\bar{x}=0,\quad \text{and}\quad \int dA=A;\quad \text{hence, etc.}$$

Corollary: Dividing both sides of eq. (1) by A, we have

$$I/A=\bar{I}/A+d^2, \quad \text{or} \quad \mathbf{k}^2=\bar{\mathbf{k}}^2+\mathbf{d}^2, \qquad (2)$$

k denoting the radius of gyration of the area with respect to any axis, and $\bar{k}$ that with respect to a parallel centroidal axis.

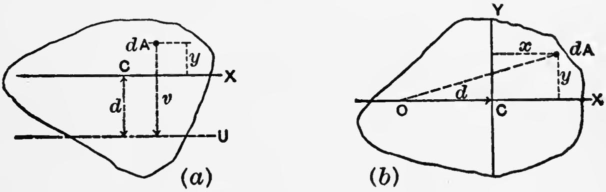


FIG. D 4.

Equations (1) and (2) show that the moment of inertia and radius of gyration of an area with respect to a centroidal axis are respectively less than those for any other parallel axis. Also, that with respect to any axis the radius of gyration (k) is always greater than the ordinate (d) from that axis to the centroid, but if $\bar{k}$ is small compared to d,

$$k=d \quad \text{and} \quad I=Ad^2, \textit{ approximately.}$$

Proposition II.—A polar moment of inertia of an area (I_z) equals the sum of its moments of inertia (I_x and I_y) with respect to any rectangular axes in the area which intersect the polar axis, or

$$\mathbf{I}_z=\mathbf{I}_x+\mathbf{I}_y. \qquad (3)$$

Proof: In accordance with the notation above, the inertia-axes in the plane of the area must be called x and y coordinate axes and the polar one the z axis. Then the distance of any point of the area from the z axis is $\sqrt{x^2+y^2}$, and therefore

$$I_z=\int dA(x^2+y^2)=\int dA\cdot x^2+\int dA\cdot y^2.$$

But $\int dA\cdot x^2=I_y$ and $\int dA\cdot y^2=I_x$; hence, etc.

Corollary.—Dividing both sides of eq. (3) by A, we get

$$I_z/A = I_x/A + I_y/A, \quad \text{or} \quad k_z^2 = k_x^2 + k_y^2, \quad . \quad . \quad . \quad (4)$$

k_z denoting the radius of gyration with respect to the polar axis, k_x and k_y those with respect to any rectangular axes in the plane of the area cutting the polar one.

Equations (3) and (4) show that the sums $I_x + I_y$ and $k_x^2 + k_y^2$ are the same for all directions of the axis x (and y). Hence if we imagine the x and y axes (90° apart) to turn about the polar axis, when I_x and k_x reach a maximum or minimum value, I_y and k_y are a minimum or maximum.

EXAMPLES.

1. Show without integration that the moment of inertia and radius of gyration of a *rectangle with respect to its base* are respectively

$$\tfrac{1}{3}ba^3 \quad \text{and} \quad a\sqrt{1/3},$$

b and a denoting the base and altitude (see ex. 1, art. D 3).

2. Show that the moment of inertia and radius of gyration of a *triangle with respect to its base* are respectively

$$\tfrac{1}{12}ba^3 \quad \text{and} \quad a\sqrt{1/6},$$

b and a denoting base and altitude respectively (see ex. 2, art. D 3).

3. Show that the moment of inertia and radius of gyration of a *square with respect to a central polar axis* are respectively

$$\tfrac{1}{6}b^4 \quad \text{and} \quad b\sqrt{1/6},$$

b denoting the length of its sides.

4. From the result of ex. 3, art. D 3, deduce the expression for the moment of inertia of a circle with respect to a central polar axis.

5. Show that the moments of inertia of a square with respect to a diagonal and a central axis parallel to any side are equal.

6. Compute the moments of inertia and radii of gyration of a rectangle 1×12 in. with respect to axes 7 inches from the centre and parallel to the sides. Compare the radii of gyration with the distance from the axes to the centre. *Ans.* Greater $I = 732$ in.^{4}

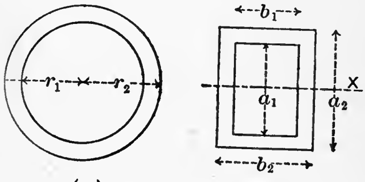
D 5. Composite Areas.—We refer now to areas which can be divided into simple component parts; e.g., a trapezoid divisible into two triangles, a circular annulus, consisting of a circle minus a smaller one, etc. The moment of inertia of such an area with respect to any axis can be computed by adding algebraically the moments of inertia of its parts with respect to the same axis, the moments of inertia of the "negative component parts" being given the minus sign.

EXAMPLES.

1. Show that the moment of inertia and radius of gyration of a *circular annulus with respect to a diameter* (fig. D 5 *a*) are respectively

$\frac{1}{4}\pi(r_2^4 - r_1^4)$ and $\frac{1}{2}\sqrt{r_2^2 + r_1^2}$.

Solution: The moment of inertia of the larger circle is $\frac{1}{4}\pi r_2^4$ (see ex. 3, art. D 3) and that of the smaller is $\frac{1}{4}\pi r_1^4$. Hence the moment of inertia of the annulus is $\frac{1}{4}\pi(r_2^4 - r_1^4)$, etc.



(*a*) (*b*)

FIG. D 5.

2. Show that the radius of gyration of a *circular annulus with respect to a central polar axis* is $\sqrt{(r_2^2 + r_1^2)/2}$.

3. Show that the moment of inertia and radius of gyration of a *hollow rectangle with respect to a central axis parallel to the base* (fig. D 5 *b*) are respectively

$$\tfrac{1}{12}(b_2a_2^3 - b_1a_1^3) \quad \text{and} \quad [(b_2a_2^3 - b_1a_1^3)/12(b_2a_2 - b_1a_1)]^{\frac{1}{2}}.$$

4. Compute the moments of inertia of the "angle" section of fig. D 6 with respect to the *x* and *y* axes respectively.

Solution: Consider that the section consists of the rectangle *ABCD* minus the rectangle *A'B'C'D*. The moments of inertia of these with respect to the line *CD* are (see ex. 1, art. D 4)

$$\tfrac{1}{3}\, 3.5 \times 7^3 = 400.16 \quad \text{and} \quad \tfrac{1}{3}\, 2.5 \times 6^3 = 180 \text{ in.}^4$$

Hence the moment of inertia of the section with respect to *CD* is $400.16 - 180 = 220.16$. The area of the section being 9.5 in.^{2}, the moment of inertia sought is (see eq. (1), art. D 4),

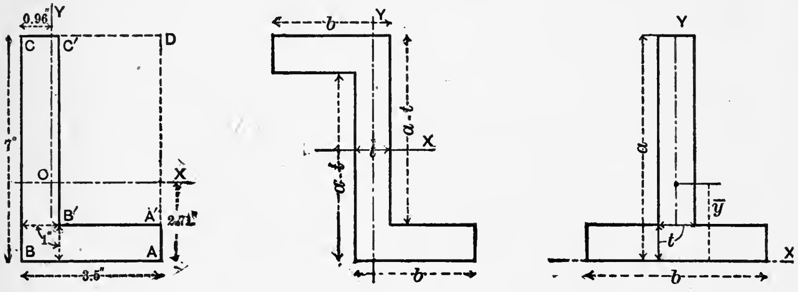
$$220.16 - 9.5(7 - 2.71)^2 = 45.32 \text{ in.}^4$$

5. Show that for the Z section of fig. D 6

$$I_x = \tfrac{1}{12}[ba^3 - (b - t)(a - 2t)^3].$$

6. Deduce an expression for the moment of inertia of the T-section of fig. D 6 with respect to a central axis parallel to the base.

7. Compute the moment of inertia of the area represented in fig. D 7 (a section of a "built-up" steel beam) consisting of a "web plate," two "side plates," and four Z bars) with respect



(a) (b) (c)

FIG. D 6.

to a horizontal central axis, the moment of inertia of one Z section with respect to its horizontal central axis being given as 50.22 in.^{4} (Such numbers and similar data are obtainable from "handbooks" published by steel-manufacturing companies.)

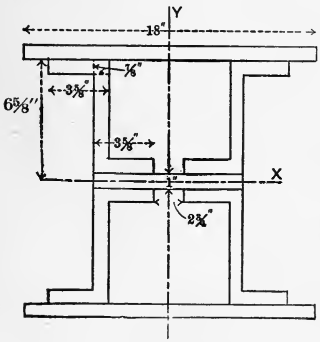


FIG. D 7.

Solution: We compute the moment of inertia of the parts separately. Side-plate section: moment of inertia with respect to its horizontal central axis $= \frac{1}{12}\ 18 \times 1^3 = 1.5$ in.^{4}; area $= 18$ in.^{2}; distance from its centroid to the inertia-axis specified $= 7\frac{1}{8}$ in. Hence for two side plates

$$I_x = 2(1.5 + 18 \times 7\tfrac{1}{8}^2) = 1830.52 \text{ in.}^4$$

Z section: moment of inertia with respect to its horizontal central axis $= 50.22$ in.^{4}; area $= 10.17$ in.^{2}; distance between its centroid and the inertia-axis $= 3\frac{9}{16}$ in.; hence for the four Z bars

$$I_x = 4(50.22 + 10.17 \times 3\tfrac{9}{16}^2) = 717.16 \text{ in.}^4$$

Web-plate section: moment of inertia with respect to the x axis is

$$\tfrac{1}{12}\ 10 \times 1^3 = 0.83 \text{ in.}^4$$

For the entire section:

$$I_x = 1830.52 + 717.16 + 0.83 = 2548.51 \text{ in.}^4$$

8. The moment of inertia of one Z section (fig. D 7) with respect to its vertical central axis being 19.18 in.[4], compute the moment of inertia of the composite area with respect to the y axis.

9. The radii of gyration of the "angle" of fig. D 6 with respect to its horizontal and vertical central axes are 2.19 and 0.89 in. respectively. Compute the radii of gyration of a pair of such sections with respect to their horizontal and vertical central axes, they being placed so that AB is an axis of symmetry for the pair.

§ II. Product of Inertia.

D 6. Product of Inertia Defined.—The product of inertia of a plane area with respect to a pair of coordinate axes in the plane is the sum of all the products obtained by multiplying each elementary area by its coordinates.

Expression for Product of Inertia.—Let dA_1, dA_2, dA_3, etc., denote elementary parts of an area and (x_1, y_1), (x_2, y_2), etc., their coordinates respectively; then, according to the definition, the product of inertia of the area with respect to the coordinate axes is

$$(dA_1)x_1y_1 + (dA_2)x_2y_2 + \text{etc.},$$

or, if J * denotes the product of inertia, dA any element of the area (dA must be of the second order, as $dx\,dy$), and x and y its coordinates, then

$$\mathbf{J} = \int \mathbf{dA \cdot xy},$$

the limits of integration being assigned so that the products $dAxy$ for all elements are included in the integration.

D 7. Units of Product of Inertia.—It is plain from the definition and expression of the preceding article that a unit product of inertia is four "dimensions" in length. Like moments of inertia we will express products of inertia in biquadratic inches.

EXAMPLES.

1. Deduce an expression for the product of inertia of the rectangle (fig. D 8) with respect to the coordinate axes.

Solution: $J = \int dA \cdot xy = \int_0^b \int_0^a (dx\,dy)xy = \frac{1}{4}b^2a^2 = \frac{1}{4}A^2$.

* When it is necessary to specify the axes with respect to which the product of inertia is taken, they are indicated by subscripts thus: J_{xy}.

2. Deduce expressions for the products of inertia of the rectangle with respect to coordinate axes through the other corners and centre.

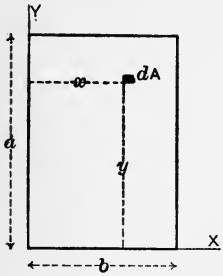


FIG. D 8.

D 8. Zero Products of Inertia. — Unlike a moment of inertia, a product of inertia may be negative or zero. The pair of axes passing through a given point with respect to which the product of inertia of an area equals zero is often of practical importance; in art. D 14 a method is given for finding such axes. The following proposition enables one to find those axes more readily in certain cases.

Proposition.—If an area has an axis of symmetry, its product of inertia with respect to that axis and any other perpendicular to it is zero.

Proof: Let the elementary parts of the area be grouped into pairs, the elements of each pair being symmetrically situated with reference to the axis of symmetry. Obviously the product of inertia of each pair is zero, and hence the product of inertia of all the pairs (the entire area) is also zero.

EXAMPLES.

1. With respect to which axes are the products of inertia of the following zero: (1) rectangle; (2) isosceles triangle; (3) circle: (4) "channel," "tee," and "angle of equal legs" (fig. 49)?

D 9. Relation between Products of Inertia for Parallel Pairs of Axes. *Proposition.*—The product of inertia (J) of an area with respect to any pair of coordinate axes equals its product of inertia ($\bar{J}$) with respect to a parallel pair of central axes plus the area (A) times the product of the coordinates of the centroid ($\bar{x}$, $\bar{y}$) referred to the first set of axes, or

$$J = \bar{J} + A\bar{x}\bar{y}. \quad . \quad . \quad . \quad . \quad (1)$$

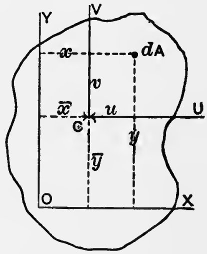


FIG. D 9.

Proof: Let OX and OY (fig. D 9) be the axes and C the centroid. Then

$$J = \int dA \cdot xy, \quad \text{and} \quad \bar{J} = \int dA \cdot uv.$$

Since $x = u + \bar{x}$ and $y = v + \bar{y}$,

$$J = \int dA(u + \bar{x})(v + \bar{y}) = \int dA \cdot uv + \bar{y}\int udA + \bar{x}\int vdA + \bar{x}\bar{y}\int dA.$$

But $\int udA = \bar{u}A = 0$, $\int vdA = \bar{v}A = 0$, and $\int dA = A$; hence, etc.

D 10. Composite Areas.—The products of inertia of such an area with respect to any pair of coordinate axes equals the algebraic sum of the products of inertia of its component parts with respect to the *same* axes.

EXAMPLES.

1. Compute the product of inertia of the angle section of fig. D 6 with respect to axes OX and OY.

Solution: Divide the figure into two rectangles as shown in fig. D 10; the areas of the parts are 3.5 and 6 in.^{2}, and their centroids are at C_1 and C_2 respectively, C being the centroid of the entire figure.

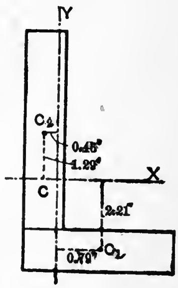


FIG. D 10.

For the first rectangle the product of inertia with respect to horizontal and vertical axes through C_1 equals zero (see prop., art. D 8), and hence, according to the preceding article,

$$J_{xy} = 0 + 3.5(0.79)(-2.21) = -6.11 \text{ in.}^4$$

For the second rectangle the product of inertia with respect to horizontal and vertical axes through C_2 equals zero, and hence

$$J_{xy} = 0 + 6(-0.46)(1.29) = -3.56 \text{ in.}^4$$

For the entire figure, therefore,

$$J_{xy} = (-6.11) + (-3.56) = -9.67 \text{ in.}^4$$

2. Show that the product of inertia of the Z section of fig. D 6 with respect to the x and y axes is -11.54 in.^{4}, a, b, and t being 6, $3\frac{1}{2}$, and $\frac{3}{8}$ in. respectively.

§ III. Relation between Moments of Inertia with Respect to Axes Inclined to Each Other.

D 11. General Equations.—Let OX and OY (fig. D 11) be any two rectangular axes and OU and OV another pair, XOU being any angle α. From the figure it is plain that

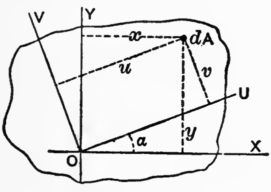


FIG. D 11.

$$u = y \sin \alpha + x \cos \alpha,$$

and

$$v = y \cos \alpha - x \sin \alpha.$$

If these values for u and v be substituted in

$$I_u = \int dA \cdot v^2 \quad \text{and} \quad J_{uv} = \int dA \cdot uv,$$

it will be found that

$$I_u = I_x \cos^2\alpha + I_y \sin^2\alpha - J_{xy} \sin 2\alpha. \quad . \quad . \quad . \quad . \quad (1)$$

and $$J_{uv} = \tfrac{1}{2}(I_x - I_y) \sin 2\alpha + J_{xy} \cos 2\alpha. \quad . \quad . \quad . \quad . \quad (2)$$

With these equations it is possible to find the moment of inertia with respect to any axis through any point and the product of inertia with respect to any pair of axes through that point, if the moments and the product of inertia of the area with respect to two rectangular axes through the point are known.

EXAMPLES.

1. Take u and v axes through O in fig. D 6 (a) inclined at 45° and 135° to the x axis, and compute I_u and J_{uv}, it being given that I_x, I_y, and J_{xy} equal 45.37, 7.53, and −9.67 in.⁴ respectively.

Solution: Substituting in eqs. (1) and (2), we find that

$$I_u = 45.37 \cos^2 45° + 7.53 \sin^2 45° + 9.67 \sin 90° = 36.12 \text{ in.}^4$$

and $$J_{uv} = \tfrac{1}{2}(45.37 - 7.53) \sin 90° - 9.67 \cos 90° = 18.92 \text{ in.}^4$$

2. Consider a rectangle whose base and altitude equal 4 and 6 in. respectively, and compute its moment of inertia with respect to a line perpendicular to either diagonal at its end; also its product of inertia with respect to that line and the diagonal.

D 12. Geometrical Constructions.—Equations (1) and (2) of the preceding article can be solved graphically. There are two graphical methods; in one a certain ellipse, the "inertia-ellipse," is employed, and in the other a certain circle, which will herein be called "inertia-circle." The former possesses more elegance and is probably more powerful, but the latter is as much simpler to apply as is the drawing of a circle compared to that of an ellipse. Only a brief discussion of the inertia-circle method can be given herein.*

D 13. *Inertia-Circle.*—Let it be required to ascertain the moments and product of inertia of the area of fig. D 12 with reference to any coordinate axes u and v through the point O. Let Ox and Oy be two axes with respect to which the moments and product of inertia of the area are known. Then lay off

$$OX = I_x, \quad OY = I_y, \quad \text{and} \quad YA = J_{xy},$$

OX and OY along the positive x axis and YA in the positive or negative y direction according as J_{xy} is positive or negative. The circle

* The invention of the inertia-circle is due to Prof. Culmann. For a full treatment see his "Graphische Statik" or that of Müller-Breslau, or a paper by Prof. L. J. Johnson in the Jour. Assoc. Eng. Soc., No. 5, Vol. XXVIII.

whose centre is midway between X and Y (the point C) and passing through A is the inertia-circle for the area corresponding to the axes x and y.

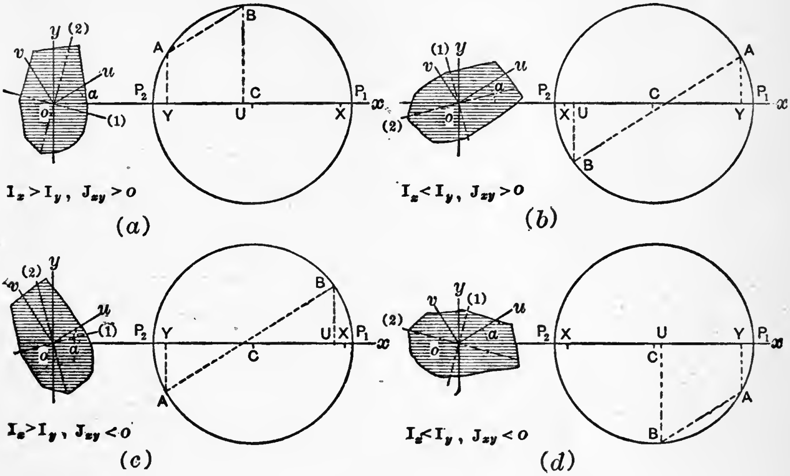


FIG. D 12.

To determine I_u and J_{uv}.—From A draw a secant parallel to the u axis, thus determining a point B, and from B drop a perpendicular to the x axis, thus determining a point U. Then

$$OU = I_u \quad \text{and} \quad UB = J_{uv}$$

to the scale used in laying off OX, OY, and YA.* J_{uv} is positive or

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* Proof: In part, fig. D 13 is a reproduction of a portion of fig. D 12 (*a*); XA' is made equal to YA, hence $A'Bb$ is perpendicular to AB and to the u axis. The construction of the other dotted lines is obvious, they being either parallel or perpendicular to the u or x axis.

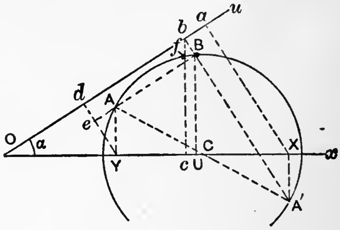

FIG. D 13.

Equation (1), art. D 11 can be written thus:

$$I_u = (I_x \cos\alpha - J_{xy} \sin\alpha) \cos\alpha$$
$$+ (I_y \sin\alpha - J_{xy} \cos\alpha) \sin\alpha,$$

and this expression for I_u can be readily evaluated from the figure. It is plain that since $OX = I_x$ and $XA' = J_{xy}$,

$$I_x \cos\alpha = Oa, \quad J_{xy} \sin\alpha = ab, \quad I_x \cos\alpha - J_{xy} \sin\alpha = Ob,$$

and $$(I_x \cos\alpha - J_{xy} \sin\alpha) \cos\alpha = Oc;$$

negative according as UB is along the positive or negative y direction. If it should happen that J_{xy} equals zero, the construction would simplify; for in that case A coincides with Y, and hence XY is a diameter of the inertia-circle.

EXAMPLES.

1. Solve ex. 1, art. D 11 by means of the inertia-circle.

Solution: Lay off on the x axis (fig. D 14) OX, OY, and YA to

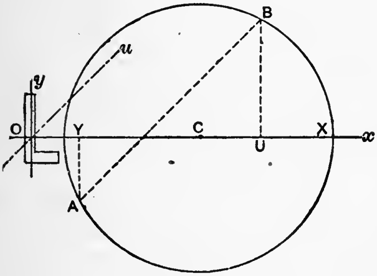

FIG. D 14.

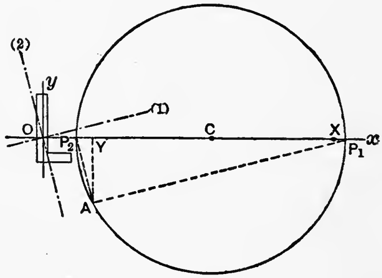

FIG. D 15.

represent I_x, I_y, and J_{xy} (scale 1 in. = 30 in.[4]). Then draw a circle with centre at C and radius equal to CA; this is the inertia-circle corresponding to the axis Ox and Oy. Next draw a line through A parallel to the u axis, mark its intersection with the circle B, and drop the perpendicular BU; then OU (1.2 in.) and UB (0.62 in.) represent the moment of inertia and product of inertia sought.

2. Solve ex. 2, art. D 11 by means of the inertia-circle.

---

also that, since $Oy = I_y$ and $YA = J_{xy}$,

$$I_y \sin \alpha = Yd, \quad J_{xy} \cos \alpha = Ye, \quad I_y \sin \alpha - J_{xy} \cos \alpha = de = Bb,$$

and
$$(I_y \sin \alpha - J_{xy} \cos \alpha) \sin \alpha = Bf = cU.$$

Hence
$$I_u = Oc + cU = OU. \qquad \text{Q.E.D.}$$

Equation (2), art. D 11 can be written thus:

$$J_{uv} = (I_x \cos \alpha - J_{xy} \sin \alpha) \sin \alpha - (I_y \sin \alpha - J_{xy} \cos \alpha) \cos \alpha,$$

and this expression for J_{uv} can be readily evaluated from the figure. As already explained,

$$I_x \cos \alpha - J_{xy} \sin \alpha = Ob,$$

hence
$$(I_x \cos \alpha - J_{xy} \sin \alpha) \sin \alpha = bc;$$

and since, as explained, $I_y \sin \alpha - J_{xy} \cos \alpha = Bb$,

$$(I_x \sin \alpha - J_{xy} \cos \alpha) \cos \alpha = bf.$$

Hence
$$J_{xy} = bc - bf = fc = BU. \qquad \text{Q.E.D.}$$

D 14. Principal Axes and Principal Moments of Inertia.—As the angle α (fig. D 12) increases from 0 to 360°, the point U moves along the x axis, its extreme positions being P_1 and P_2, and for these two positions UB equals zero. Hence

(*a*) the maximum and minimum values of I_u are given by OP_1 and OP_2;

(*b*) the corresponding inertia-axes are parallel to AP_1 and AP_2 (therefore rectangular);

(*c*) the product of inertia with respect to those axes equals zero.

If I_x equals I_y and $J_{xy}=0$, the inertia-circle vanishes, and hence I_u is constant and J_{uv} equals zero for all values of α.

Definitions.—The two axes through a point with respect to which the moments of inertia of an area are greater and less than for any other axes through that point are called the principal axes of the figure at that point, and the corresponding moments of inertia and radii of gyration are called the principal moments of inertia and radii of gyration of the area at that point. We will denote these maximum and minimum moments of inertia and radii of gyration by I_1, I_2, k_1, and k_2 respectively, and will mark the corresponding inertia-axes (1) and (2) (see fig. D 12).

According to the above the product of inertia of an area with respect to the principal axes at a point equals zero. This proposition leads at once to an algebraic method for finding principal axes and moments of inertia. Thus let α' denote the value of α which makes J_{uv} zero, then (see eq. (2), art. D 11)

$$\tfrac{1}{2}(I_x-I_y)\sin 2\alpha'+J_{xy}\cos 2\alpha'=0,$$

or

$$\tan 2\alpha'=2J_{xy}/(I_y-I_x). \qquad (1)$$

By means of this equation we may locate the principal axes at a point (i.e., determine α'), and then determine the principal moments by substituting the two values of α' given by that equation in eq. (1), art. D 11.

EXAMPLES.

1. Determine the central principal axes of the angle section of fig. D 6 and the corresponding moments of inertia, I_x, I_y, and J_{xy}, being 45.37, 7.53, and -9.67 in.[4] respectively.

Solutions: (1) Graphical. Fig. D 15 is the inertia-circle for the area corresponding to the axis Ox and Oy (constructed as explained in the solution of ex. 1, art. D 13). $O(1)$ and $O(2)$, parallel to AP_1 and AP_2, are the principal axes at O, and OP_1 and OP_2 represent (by the scale used) the greater and lesser principal moments respectively.

(2) Algebraic. Substituting in equation (1), we find that

$$\tan 2\alpha' = 2(-9.67)/(7.53-45.32) = 0.5118,$$

i.e., $2\alpha' = 27^\circ\ 6'$ or $207^\circ\ 6'$; hence $\alpha' = 13^\circ\ 33'$ or $103^\circ\ 33'$.

Substituting these two values successively in eq. (1), art. D 11, we find as the two values of I_u

$$I_1 = 47.70 \text{ in.}^4 \quad \text{and} \quad I_2 = 5.20 \text{ in.}^4$$

2. In fig. D 6 (b) let a, b, and t equal 6, $3\frac{1}{2}$, and $\frac{3}{8}$ in. respectively; then

$$I_x = 25.32, \quad I_y = 9.11, \quad \text{and} \quad J_{xy} = -11.54 \text{ in.}^4$$

Determine the central principal axes and the corresponding moments of inertia.

3. In fig. D 6(a) let $AB = BC = 3$ in., $AA' = CC' = \frac{1}{4}$ in., then the centroid is 0.84 in. from AB and BC and $I_x = I_y = 1.24$ in.^{4} Determine the central principal axes and the corresponding radii of gyration. *Ans.* $k_2 = 0.59$ in.

D 15. Graphical Determination of Moment of Inertia of a Plane Figure.*—Let *abba* (fig. D 16) be the given figure whose moment of inertia with respect to OX, say, is required. (1) Draw O_1X_1 and O_2X_2 at any convenient distance m from OX, and through any convenient point P draw a y axis, marking its intersection with O_1X_1 and O_2X_2 M and N respectively. (2) Draw a line aa cutting the figure and parallel to the x axis, marking its intersections with the perimeter a, and that with the y axis α. (3) Lay off $M\alpha'$ equal to $P\alpha$, and determine the intersections of the lines $a\alpha'$ with O_1X_1, marking them a'. (4) Determine the intersections of the lines Pa' with aa, marking them a''. (5) Repeat the construction for other lines like aa, as bb, thus locating points b'. (6) Draw a smooth curve through all points a'', b'', c'', etc. (7) Measure the area of the loops † (shaded in the figure); if A'' denotes that area, then the desired moment of inertia equals $A''m^2$.

Proof: The moment of inertia is given by

$$I = \int_A dA \cdot y^2 = \int_{-c_2}^{+c_1} y^2 w dy,$$

w denoting any width of the figure, as aa or bb. Let w' and w''

---

* Logically this article should appear in section 1 of this appendix.

† If the inertia axis does not cut the plane figure, then there will be only one loop.

respectively denote the lengths $a'a'$ and $a''a''$ (or $b'b'$ and $b''b''$); then from the geometry of the figure

$$\pm y/w'' = m/w', \quad \text{and} \quad \pm y/w' = m/w,$$

the positive or negative sign being taken according as the w's

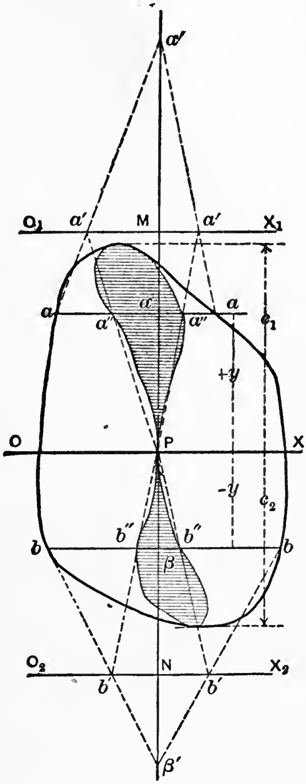

FIG. D 16.

refer to widths above or below the x axis. Multiplication of the equations gives

$$y^2/w'w'' = m^2/w'w, \quad \text{or} \quad y^2w = m^2w''.$$

Substituting this value of y^2w in the first equation gives

$$I = m^2\int_{-c_2}^{+c_1} w''dy = m^2A''.$$

EXAMPLE.

Draw in succession three circular arcs of 30° each, 3, 2, and 1 in. in radius respectively, and tangent to each other so that they form approximately an elliptical quadrantal arc. Compute the moment of inertia of the elliptic quadrant with respect to the major axis of the ellipse. [Suggestion: Take the point *P* at the centre of the ellipse.]

APPENDIX E.

VIRTUAL WORK.

E 1. Definitions.—Any imaginary displacement of a point is called a virtual displacement of that point. The work which a force would do in a real displacement of its application point like some particular virtual displacement is called the virtual work of that force for that virtual displacement.

The virtual work of a force may be computed by the methods explained in arts. 287 and 288 for computing real work.

E 2. Principle of Virtual Work for a Particle.—If a particle is in equilibrium, then the algebraic sum of the virtual works of all the forces acting upon it for any infinitesimal virtual displacement equals zero.*

Proof: It follows from art. 292 that the algebraic sum of the virtual works of the forces equals the virtual work of their resultant. In an infinitesimal displacement, the work of each force would generally be an infinitesimal of the first order if the displacement is regarded as of that order; and since the magnitude of the resultant would not become finite in such a displacement, the work of the resultant would be an infinitesimal of the second order. Hence, according to the theory of infinitesimals, the algebraic sum of the virtual works of the forces acting on the particle may be written equal to zero.

Equations of virtual work for forces acting on a particle in equilibrium are easily reduced to equations of equilibrium and therefore are not specially useful. They are here explained as a preliminary to other practically useful methods.

EXAMPLES.

1. A bead *A* (fig. E 1) is to be supported on a smooth circular wire (plane vertical) in the position shown by means of a horizontal force. Compute the value of the force.

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* This principle holds also for finite displacements if the forces are in equilibrium at all stages of the displacement, as is evident from the first statement in the proof above.

Solution: The forces acting on the bead are the pull P, the weight of the bead W, and the reaction of the wire R. Assuming a virtual displacement from A to B (perpendicular to R), the forces remaining unchanged in magnitude and direction, then their virtual works are respectively

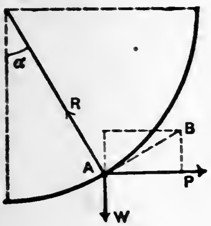

FIG. E 1.

$$P\overline{AB} \cos \alpha, \; -W\overline{AB} \sin \alpha, \text{ and } 0.$$

The equation of virtual work is therefore

$$P\overline{AB} \cos \alpha - W\overline{AB} \sin \alpha + 0 = 0;$$

hence $$P = W \tan \alpha.$$

The reason for choosing a virtual displacement perpendicular to R should be noted. So selected, the virtual work of R is zero, and therefore R will not appear in the equation of virtual work, leaving but one unknown quantity, P, in that equation. Any other direction for the virtual displacement would introduce R into the equation of virtual work, making two unknowns in the equation; to determine either unknown would necessitate the use of a second equation of virtual work, the virtual displacement being chosen in some direction not coincident with the first.

It should be noticed also that the foregoing equation of virtual work after division by AB is an ordinary equilibrium equation—the one obtained by resolving the three forces along AB and taking the algebraic sum of the components.

2. Show by the principle of virtual work that $R = W \sec \alpha$ in ex. 1.

3. Solve ex. 5, art. 129, by the principle of virtual work.

E 3. Principle of Virtual Work for a System of Particles.—If every particle of a system is in equilibrium, then the algebraic sum of the works of all the forces (external and internal) acting on the system for any infinitesimal virtual displacement equals zero. If the system is a rigid one, then the algebraic sum of the virtual works of the external forces equals zero.

Proof: According to art. E 2, the algebraic sum of the virtual works of all the forces acting upon any one particle for any infinitesimal virtual displacement equals zero; hence the sum of the virtual works of the forces acting on all the particles also equals zero.

If the system is a rigid one, then the algebraic sum of the virtual works of all the internal forces equals zero because the virtual work of each pair of internal forces constituting an action and reaction equals zero (see art. 293), and hence the virtual work for all pairs, or all the internal forces, equals zero. It follows therefore that the algebraic sum of the virtual works of all the external forces equals zero.

In applying this principle, the virtual displacements should be chosen so as to make the virtual work of the unknown forces not desired equal to zero, thus eliminating those unknown forces from the equation of virtual work.

EXAMPLES.

1. A heavy bar rests against a smooth wall and on a smooth floor as shown in fig. E 2, and is prevented from slipping by a horizontal cord tied to its lower end and to the foot of the wall. The inclination of the bar being α, its weight W, and its centre of gravity at the middle, compute the tension in the cord by the principle of virtual work.

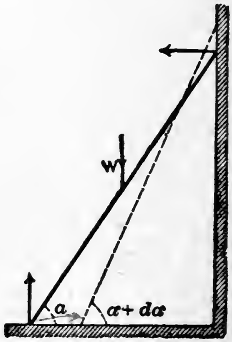

FIG. E 2.

Solution: The external forces acting on the bar consist of the pull of the string P, the weight of the bar W, and the reactions of the wall and floor, horizontal and vertical respectively. If a displacement of the bar to the position represented by the dotted line is assumed, then the virtual work of each reaction will equal zero. If the new inclination of the bar be called $\alpha+d\alpha$ and the length $2l$, then the virtual work of the pull will be

$$P[2l\cos\alpha-2l\cos(\alpha+d\alpha)]=-P2l\,d\cos\alpha,$$

and the virtual work of W will be

$$-W[l\sin(\alpha+d\alpha)-l\sin\alpha]=-Wl\,d\sin\alpha.$$

The equation of virtual work is therefore

$$-P2l\,d\cos\alpha-Wl\,d\sin\alpha=0;$$

hence

$$P=\tfrac{1}{2}W\cot\alpha.$$

2. Solve ex. 6, art. 137, by the principle of virtual work.

3. Determine W of ex. 2, art. 162, by the principle of virtual work.

Solution: Consider the forces acting on the collection of bodies (fig. 144*b*) consisting of the cross-head, connecting-rod, crank-axle and drum, hoisting-chain, and the suspended load. The forces external to this system consist of P, the weights of the parts, the reactions of the cross-head guides, and those of the axle bearings. The internal forces consist of the mutual pressures at the pins A and B, those between the links of the chain, and the forces "within" each rigid body. Now if all rubbing surfaces are regarded smooth, then the only forces whose virtual work is not zero for a slight displacement of the cross-head forward are P and the weights of the parts. If the displacement of the cross-head be called ds, then that of the suspended body is $2.1ds$, as can be shown by the trigonometry of the figure. Disregarding the weights of the connecting-rod and the chain, the equation of virtual work becomes

$$1000ds - W2.1ds \quad \text{or} \quad W = 1000/2.1 = 476 \text{ lbs.}$$

E 4. Application of the Principle of Virtual Work to Statically Indeterminate Problems.—By a statically indeterminate problem is meant one relating to a system of forces in equilibrium which cannot be solved by the principles of statics alone. Ex. 2, art. 138, is of this class; the determination of the reactions on a beam supported at three points is another. Trusses with superfluous or redundant members or supports are also statically indeterminate (see art. F 11, Appendix F). The statically indeterminate forces in the cases just mentioned depend upon the way in which the beam or truss or their supports deform, and the additional principles needed relate to the elasticity of materials. For a fuller treatment of this subject the student is referred to works on "Strength of Materials" and "Bridges." The solution to a simple example only is given here to satisfy the curiosity of the student who has wondered over these cases.

EXAMPLES.

1. A square board is suspended in a horizontal position by means of vertical wires fastened at its corners; then a heavy body is placed on the board at its centre. It is required to determine the tensions in the wires supposing that the body weighs 100 lbs., and that the wires fastened at A, B, C, and D (fig. E 3) are 0.01, 0.02, 0.03, and 0.4 sq. in. in cross-section respectively.

Solution: The forces acting on the board are the weight and the pulls of the four wires. If any one of these four were known, the other three could be obtained from the three conditions of

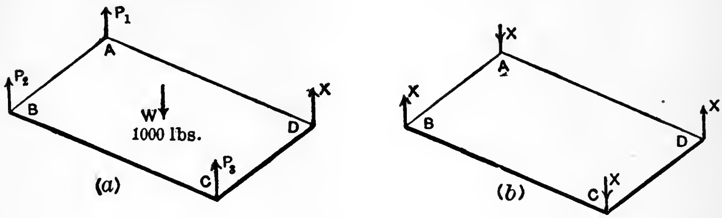

FIG. E 3.

equilibrium for such a system, parallel non-coplanar. Call the pull first determined the statically indeterminate one and denote it by X, and the others by P_1, P_2, and P_3 (fig. E 3*a*). Then, by the conditions of equilibrium (art. 125), it is easy to show that

$$P_2 = X, \quad \text{and} \quad P_1 = P_3 = 500 - X.$$

Now instead of applying the principle of virtual work to the five forces acting on the board, take the force X applied at D and three other vertical forces at A, B, and C, which together with X would constitute a system in equilibrium; that system is represented in fig. E 3*b*. Assume a virtual displacement like the displacement which the board *actually* underwent when the body was placed upon it. This displacement was permitted by the elongations of the wires, and it is now necessary to compute those elongations. The elongation of a wire, not strained beyond its "elastic limit" (assumed to be the case here), is given by the expression Pl/AE, in which P denotes pull, l length, A area of cross-section, and E Young's modulus for the material of the wire. Hence the elongations are as recorded below:

Wires.	Elongations.	Virtual Works.
A	$(500-X)l/0.01E$	$X(500-X)l/0.01E$
B	$Xl/0.02E$	$-X^2l/0.02E$
C	$(500-X)l/0.03E$	$X(500-X)l/0.03E$
D	$Xl/0.04E$	$-X^2l/0.04E$

Multiplying the forces by the corresponding virtual displacements of their application points and affixing the proper signs, we get the expressions recorded in the last column of the tabulation. The equation of virtual work after cancellation and reduction becomes

$$(500 - X)(1/0.01 + 1/0.03) = X(1/0.02 + 1/0.04);$$

hence $X = 400$ lbs., and, according to the foregoing,

$$P_2 = X = 400, \quad P_1 = P_2 = 100 \text{ lbs.}$$

APPENDIX F.

SUPPLEMENT TO STATICS.

Arts. 1 to 11 of this appendix relate to analysis of Roof Trusses, and arts. 12 to 16 deal with some important properties of the funicular polygon, useful in various lines of applied mechanics.

F 1. Truss Loads.—Roof- and bridge-truss loads may be classified into *permanent*, or *dead*, and *temporary*, or *live*. A permanent or dead load is one always on the truss, while a temporary or live load is one not always on the truss.

A roof-truss commonly sustains dead loads only, as its own weight and that of the roof covering; weight of snow and wind pressure are live loads.

A bridge truss sustains both dead and live loads; he first consists of the weights of the truss, floor, etc., and the second of the weights of passing crowds, cars, and wagons, snow, wind pressure, etc.

F 2. *Weight of Roofing.*—The weight of this can be closely estimated for any kind of covering. The following are weights of some roofing materials in pounds per square foot of roof surface.

Shingling: tin, 1; wood shingles, 2 to 3; corrugated iron, 1 to 3; slate, 8 to 10; tiles, 10 to 25.

Sheathing: Boards, 3 to 5.

Rafters: Wood, 1.5 to 3.

Purlins: Wood, 1 to 3; steel, 2 to 4.

Rafters and purlins are beams whose loads in a given case are approximately known, hence their necessary size and weight can be fairly accurately computed.

F 3. *Weight of Trusses.*—The actual weight of a truss can be determined only after it is designed. Its probable weight must be known for the analysis, and this is estimated from the weights of similar existing trusses or computed from a formula derived from the actual weights of existing trusses. The following is such a formula for the weights of steel trusses:*

$$W = al(1 + l/25),$$

* From "Modern Framed Structures," Johnson, Bryan, and Turneaure.

W denoting the weight of a truss in pounds, a the distance between adjacent trusses, and l the span, both in feet.

The probable weight of a wooden truss is about three-fourths of that given by the above formula.

F 4. *Weight of Snow.*—The probable weight of snow which may have to be borne by a roof- russ depends, of course, on location. In that part of the United States where it is necessary to allow for a snow load the assumed weight varies from 10 to 30 lbs. per sq. ft. of area covered by the roof.

F 5. *Wind Pressure.*—The intensity of the pressure of a wind blowing normally against a plane surface is approximately proportional to the square of the velocity of the wind; it may be computed from the formula

$$p = 0.005v^2,$$

p denoting intensity in lbs./ft.^{2}, and v velocity in mi./hr.

An intensity of 30 lbs./ft.^{2} corresponds by this formula to a velocity of 77.5 mi./hr.

The intensity of the pressure of a wind blowing obliquely against a surface depends on the inclination as well as the velocity. The following formula for intensity on plane surfaces is generally regarded as reliable:

$$p_i = p \sin i / (1 + \sin^2 i);$$

i denoting inclination of the surface to the wind,
p_i intensity on the inclined surface, and
p intensity on a surface normal to the wind.

For an intensity on a normal surface (p) of 30 lbs. per sq. ft., the intensities on oblique surfaces according to the foregoing formula are as follows:

$i =$ 10,	20,	30,	40,	50,	60–90 degrees
$p_i =$ 10,	18,	24,	27,	29,	30 lbs./ft.^{2}

The direction of the pressure of a wind blowing obliquely against an inclined plane surface is practically normal to the surface. In computing wind pressures on roofs, the wind is supposed to blow horizontally and at right angles to the axis of the building.

F 6. **Computation of "Apex Loads."**—The weight of the roof covering, including rafters and purlins, comes upon the trusses at the points where they support the purlins; likewise the pressure due to wind and snow. Sometimes all the purlins are supported at the joints of the trusses; in such cases the loads mentioned act upon the trusses at their joints. However, the roof, snow, and wind loads are always assumed to be applied to the truss at its upper joints. This assumption is equivalent to neglecting the bending effect due to the pressure of those purlins which are not supported at joints; this effect can be computed separately.

The weight of the truss itself is assumed to come upon the truss at its upper joints; this, of course, is not exactly correct. Most of the weight does come upon the upper joints, for the upper members are much heavier than the lower, and the assumption is sufficiently correct in most cases.

EXAMPLE.

It is required to compute the apex loads for the truss of fig. F 1 due to dead load, snow, and wind. Assume that the truss is a steel one, total roofing weighs 15 lbs./ft.^{2}, snow weight is 10 lbs./ft.^{2}, horizontal, and the normal wind pressure is 30 lbs./ft.^{2}

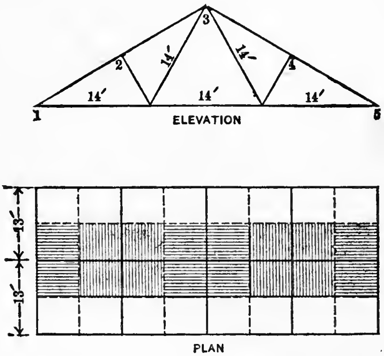

FIG. F 1.

Solution: The formula (art. F 3) gives for the probable weight of truss (see fig. F 1 for dimensions)

$$13\times42(1+42/25),\quad \text{or} \quad 1463 \text{ lbs.}$$

The area of roofing sustained by one truss is about $48\frac{1}{2}\times13$, or $630\frac{1}{2}$ ft.^{2}; hence the weight of it is $630\frac{1}{2}\times15$, or 9457 lbs. The permanent or dead load therefore equals

$$1463+9457,\quad \text{or} \quad 11{,}920 \text{ lbs.}$$

The area covered by the roofing supported by one truss is

42×13, or 546 ft.^{2}; hence the snow load borne by one truss equals 546×10, or 5460 lbs.

The angle which the wind, directed as assumed in art. F 5, makes with the roof is 30°; hence the intensity of wind pressure on the roof is 24 lbs./ft.^{2} (see art. F 5), and the total wind pressure borne by one truss is

$$(24\tfrac{1}{4} \times 13)24, \quad \text{or} \quad 7566 \text{ lbs.}$$

The dead load is proportioned among the five upper joints, but joints (1) and (5) sustain only one-half as much as the others. Hence for joints (1) and (5) the apex load is one-eighth, and for joints (2), (3), and (4) one-fourth of the total load.

The snow load is proportioned among the joints just like the dead, i.e. one-eighth at joints (1) and (5) and one-fourth at joints (2), (3), and (4).

The wind load coming upon one-half of the roof only, the left half, say, is proportioned among joints (1), (2), and (3); one-fourth at joints (1) and (3), and one-half at joint (2).

F 7. Determination of Reactions. — To analyze a truss, it is generally necessary to determine the reactions due to the dead and wind loads separately, and sometimes those due to the snow load. Each of these load systems together with the reactions due to it constitutes a system of forces in equilibrium, and the unknowns in the system (the reactions) can be determined by the principles of statics in all ordinary cases. The determination usually falls under arts. 139, 140, or 141. Two cases, however (arts. F 8 and F 9), need further explanation.

F 8. *Reactions on a Rigid Truss due to Wind Pressure.*—(*a*) If one end of the truss rests on rollers * and one end is fixed, the reactions are statically determinate. The one at the roller end is practically vertical, the direction of the other is unknown at the outset; the reactions may be completely determined by methods of art. 140.

(*b*) If both ends of the truss are fixed, the reactions are statically indeterminate, for the magnitude and direction of each are unknown and there are but three conditions of equilibrium available. Usually one of the following assumptions is made to determine the reactions:

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* Rollers are usually placed under one end of a long truss to allow it to expand and contract freely.

(1) They are parallel to the resultant wind pressure on the roof.
(2) Their horizontal components each equal one-half of the total horizontal wind pressure, or, otherwise stated, each support takes one-half of the horizontal thrust of the wind.

Once the first assumption is made, the problem of determining the reactions is reduced to that of art. 139. When the second assumption is made, the total horizontal wind pressure or thrust should be computed first—easily done from the force polygon for the wind loads; then imagine each reaction resolved into its horizontal and vertical components, and noting that the horizontal components equal one-half of the thrust, there remain but two unknown forces in the wind system, namely, the vertical components of the reactions. Their determination also falls under art. 139.

Obviously the first of these two assumptions is wrong if the resultant wind pressure is horizontal and does not pass through the supports, for then the two reactions (both horizontal) could

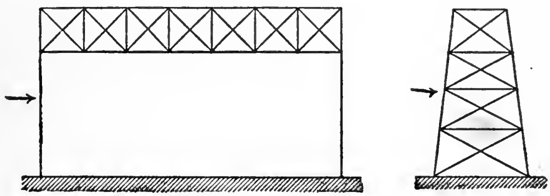

FIG. F 2.

not possibly balance the resultant. In such cases and in approximations thereto the second assumption is made; examples are a "bent" of a mill building and a framed tower (fig. F 2).

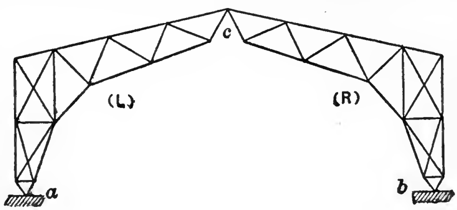

FIG. F 3.

F 9. *Reactions on a Three-hinged Arch.*—A three-hinged arch consists of two rigid trusses pinned together and each to a support; see fig. F 3, pins at a, b, and c. The reactions at the supports are unknown at the outset, in magnitude and direction, even for vertical loads. At first thought this appears to be like case (*b*) of the preceding article, but the third hinge really makes the reactions statically determinate.

There are many solutions of this problem—two are here given; each may be carried out graphically or algebraically, but they are respectively adapted to graphic and algebraic methods.

(1) Determine the reactions due to the loads on each half of the arch separately, and then combine these partial reactions. Let A and B denote the total reactions at a and b respectively, A_r and B_r the reactions at those points due to the loads on (R), and A_l and B_l those due to loads on (L).

Supposing first that only (R) is loaded, the part (L) is under the action of two forces only, A_r and the pressure at c; these two being in equilibrium must be collinear, and hence they act along ac. The external forces on the entire arch now (loads on (R) only) are all known except two, A_r and B_r, but the action line of A_r and the application point of B_r are known; these unknown forces may hence be determined by methods of art. 140.

Similarly it can be shown that B_l acts along bc, and hence B_l and A_l may be determined like B_r and A_r. Finally, composition of A_r and A_l and of B_r and B_l gives A and B

The pressures exerted by (R) and (L) upon each other can be easily obtained from the component reactions at a and b. The pressure which (R) exerts on (L) is identical with the resultant of A_r reversed and B^l, and the pressure which (L) exerts on (R) is identical with the resultant of B_l reversed and A_r; these pressures are of course equal, opposite, and collinear.

(2) Imagine the total reactions at a and b resolved into two components one of which acts along the line ab; these components may readily be computed and then compounded to get the actual reactions. Thus, call the components acting along ab, A_x and B_x respectively, and the others A_y and B_y. "Taking moments" of all the forces on the arch with origin at a and b gives A_y and B_y. Taking moments for all the forces on the part (R) with origin at c gives B_x, and for all the forces on (L) with origin at c gives A_x. If the loads are vertical, and A_y and B_y are taken vertical, then A_x and B_x are equal and opposite.

The pressures at c can now be readily determined by consideration of all the forces acting on either (R) or (L).

F 10. Maximum Stresses.—The complete analysis of a truss should include a determination of the stress in each member due to (*a*) the dead load, (*b*) the snow load, (*c*) wind pressure right, and (*d*) wind pressure left.

When all the snow apex loads are the same fractional part of the dead apex loads, the "snow stress" in any member equals that same fractional part of the permanent stress in that member; no stress diagram for snow loads is therefore needed, but the snow stresses are readily computed from the permanent stresses, conveniently by slide-rule.

If the truss is symmetrical and fastened at both ends, then the stress in any member when the wind blows from the left is just like that in the symmetrical member when the wind blows from the right; only one stress diagram is therefore necessary, "wind left stresses" being obtainable from a "wind right stress diagram." But when one end of the truss rests on rollers and one end is fixed, then separate analyses for wind right and left are necessary.

A record of the stresses should be made as fast as they are determined, the kind of stress being indicated as well as the amount. The following is a convenient form:

Member.	Stress.				
	D. L.	S. L.	W. R.	W. L.	Maximum.
ab	+14,000	+7,000	+10,600	+1,400	+31,600
bc	−3,200	−1,600	+4,000	−1,000	−5,800, +800
....	....	......	....	......	....

The abbreviations are: D. L., dead load; S. L., snow load; W. R., wind right; W. L., wind left; +, tension; −, compression.

By maximum stress for any member is meant the greatest stress, tension or compression, to which it may be subjected, due to any possible combination of loads which may come upon the truss. The possible combinations of the usual loads are:

1. Dead.
2. Dead and snow.
3. Dead and wind right.
4. Dead and wind left.
5. Dead, snow, and wind right.
6. Dead, snow, and wind left.

For member *ab* the combination of D.L., S.L., and W.R. produces the greatest stress, 31,600 lbs., tension. No combination produces a compression in that member. For member *bc*, the combination of D.L., S.L., and W.L. produces the greatest compression, 5800 lbs., and the combination of D.L. and W.R. produces the greatest tension, 800 lbs.

Some assume that great snow and wind loads will not come upon the truss at the same time, and they exclude combinations 5 and 6 in a computation for maximum stresses.

F 11. A Classification of Frames.—A frame may be either (*a*) complete, (*b*) incomplete, or (*c*) redundant. In the following definitions, it is assumed that the frame is pin-connected; a riveted frame is classified as though it were pin-connected.

(*a*) A complete frame is one which would not maintain its shape under all conditions of loading with any member removed, i.e., it is indeformable and without superfluous or redundant members. See fig. F 4(*a*) for an example. Trusses of this class

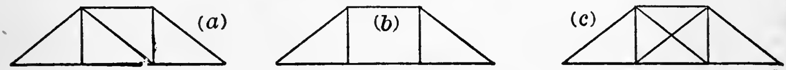

FIG. F 4.

are statically determinate.

(*b*) An incomplete frame is one which might maintain its shape under certain conditions of loads, but to do so for all conditions would require one or more additional members, i.e., it is generally deformable. See fig. F 4(*b*) for an example. This truss would maintain its shape for symmetrical loading, but not for any other distribution. Incomplete frames loaded so that they will maintain their shape are s'atically determinate.

(*c*) A redundant frame is one which maintains its shape under all conditions of loads and would do so with fewer members, i.e., it is indeformable, but has superfluous or redundant members.* See fig. F 4(*c*) for an example: removal of either diagonal of the rectangle would still enable the truss to sustain any load. Trusses of this kind are a'ways statically indeterminate.

Criteria for Classification of a Frame.—Let m denote the number of members in the frame and j the number of joints; then for a frame which is

complete, $m = 2j - 3$;
inc mplete, $m < 2j - 3$;
redundant, $m > 2j - 3$.

* These members are, or ought to be, structurally useful, but are so named because the frame is indeformable without them.

These criteria assume (see proof below) that every member of the truss can sustain bo h tension or compression. If the truss has members which can sustain only tension or compression and if any one of those members has a counter,* then both the main member and its counter should not be included in m.

Proo : Evidently the simplest complete frame is a triangle; it has three joints and three members. It can be extended so as to remain complete only by the addition of two members for each new joint; thus, if m' and j' denote the number of new members and joints respectively,

$$m' = 2j'.$$

Also, $$m' + 3 = 2j' + 3 = 2(j' + 3) - 3,$$

or $$m = 2j - 3.$$

This being the relation for a complete frame, those already given for incomplete and redundant frames follow.

F 12. Moments of Forces Determined Graphically.—Let ab (fig. F 5) be the action line of a force, AB the magnitude and direction, and P a moment origin. From any convenient pole O

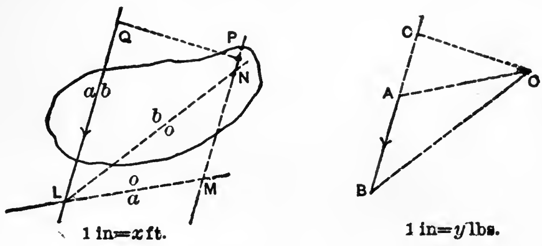

FIG. F 5.

draw the rays OA and OB, and then from any point on ab the corresponding strings oa and ob. Next draw a line parallel to ab

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* A counter is a member whose main function is to relieve another member from a stress of a kind for which the latter was not designed. Thus, if both diagonal members in the quadrilateral of fig. F4(c) consist of rods, neither can sustain a compression, and any load which tends to put a compression on one will put tension on the other; hence either member may be regarded as a counter to the other, but the one which would be stressed under the permanent load would be called the main member, and the other the counter.

through P, and note its intersections M and N with the strings oa and ob. Then measure the intercept MN by the scale of the space diagram and the perpendicular OC from the pole to AB by the scale of the vector diagram; so measured, their product equals the moment of the force. For, by definition, the moment of the force equals

$$(\text{the force})\times(\text{the arm})=(AB\cdot y)(PQ\cdot x),$$

and from the similar triangles OAB and LMN,

$$AB/OC=MN/PQ, \quad \text{or} \quad (MN)(OC)=(AB)(PQ);$$

hence

$$(MN\cdot x)(OC\cdot y)=(AB\cdot y)(PQ\cdot x).$$

OC is called the *pole distance* of the force AB, and if we call MN the *intercept* of the force with respect to P, then we may state that the moment of a force with respect to a point equals the product of its intercept with respect to that point and its pole distance, it being understood that the intercept and pole distance are properly scaled as explained.

F 13. *Sum of the Moments of any Number of Forces.*—This is most conveniently computed graphically by determining its equal, the moment of their resultant. Thus, to find the sum of the moments of the four forces represented in fig. F 6(a) with respect

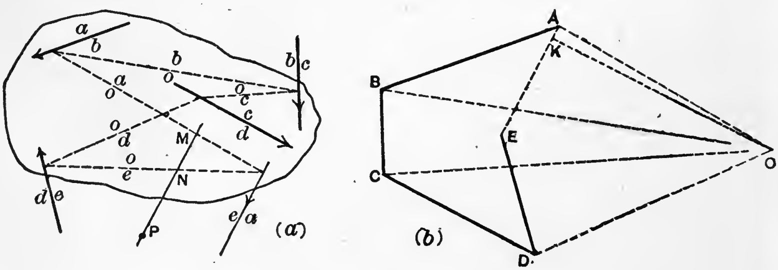

FIG. F 6.

to P, first determine the resultant (AE and ae) and then the "intercept" (MN) and the "pole distance" (OK) of the resultant. The product of the intercept and pole distance is the moment sought. Notice that the intercept is that cut off on the line through the moment origin parallel to the resultant by the strings "corresponding" to the resultant. It is unnecessary to actually draw the action line of the resultant, but its approximate position and the sense of the resultant must be known in order to fix the sign of the moment, here negative.

When once a force and a funicular polygon for any system of forces have been constructed, then the algebraic sum of the moments of any number of them *which are represented consecutively in the force polygon* can be computed as just explained.

F 14. *Parallel Forces in Equilibrium.*—The algebraic sum of the moments of all the forces on either side of any moment origin can be easily read from any funicular polygon of the system provided that these same forces occur consecutively in the corresponding force polygon.*

Thus let the loads and reactions on the beam represented in fig. F 7 be the system considered; *ABCDEA* is a force polygon

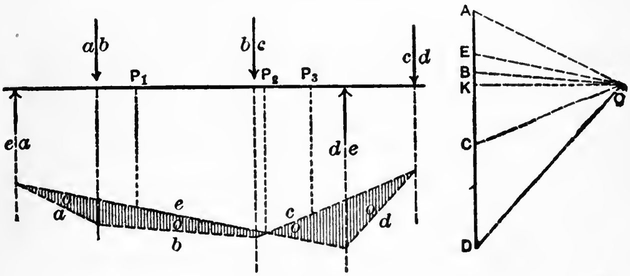

FIG. F 7.

satisfying the requirement specified, and *oa*, *ob*, *oc*, *od*, and *oe* a funicular polygon. The resultant of the forces to the left of P_1 is *EB*, and according to the preceding article the algebraic sum of the moments of those two forces with respect to *P* equals the intercept cut off by the strings *oe* and *ob* on the vertical line through *P* multiplied by the pole distance *OK*.

Since this pole distance is the same for all resultants of groups of the forces, the algebraic sums of the moments of the forces to the left or right of different moment origins are proportional to the intercepts, or ordinates, in the funicular polygon immediately below the origins; the figure formed by the funicular polygon

* The property of the funicular polygon here being described is utilized mostly in connection with beam problems—loads and reactions vertical. If the loads and reactions are represented in the force polygon in the order in which their application points occur around the beam, then the forces appear consecutively as above described.

is therefore called a "moment diagram." If the scale of the space diagram is 1 in. $= x$ ft. and the pole distance scales z lbs., then the scale of the moment diagram is 1 in. $= xz$ ft.-lbs. Selecting the scale and the pole so as to make x and z "round numbers," then the scale of the moment diagram will be simple, and by it moments may be read directly from the diagram.

EXAMPLES.

1. Compute graphically the moment of each force of ex. 1, art. 45, and also the moment of their resultant. Compare the last with the algebraic sum of the moments of the forces.

2. A beam 20 ft. long rests on end supports and sustains loads of 8000, 6000, and 10,000 lbs. 2, 8, and 18 ft. respectively from the left end. Construct a moment diagram for all the forces acting on the beam; state its scale and give the algebraic sum of the moments of all the forces on the left half with respect to the middle.

3. Solve ex. 2 supposing that the supports are 3 ft. from the ends.

F 15. To "Pass" a Funicular Polygon through Three Points.—By this is meant the construction of a funicular polygon for a given system of forces so that three specified strings shall pass through three given points, a construction often made, especially in the design of a masonry arch.

It will first be shown how to pass a funicular polygon through two points. Let *ab*, *bc*, *cd*, and *de* (fig. F 8) be four forces; re-

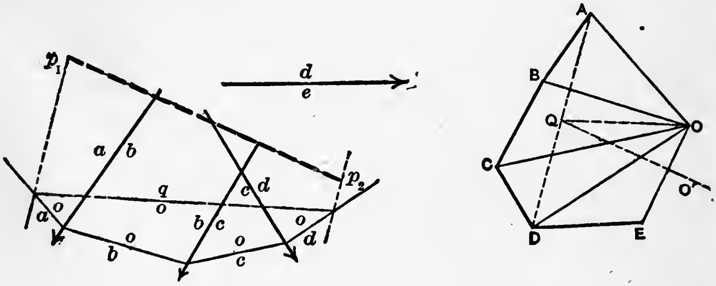

FIG. F 8.

quired to construct a funicular polygon so that the strings *oa* and *od* shall pass through p_1 and p_2 respectively. Imagine the forces included within the rays *OA* and *OD* in the force polygon to act upon a beam supported at p_1 and p_2, the supports being such that the reactions are statically determinate. For simplicity imagine that the supports are such that the reactions are parallel to each other and therefore to the resultant of the loads applied to the beam; then determine the magnitudes of those reactions. Thus, having drawn the force polygon, join A and D to get the direction of the reactions; then draw their action lines, parallel to AD. Next construct a funicular polygon and note the intersections of the first and last strings, *oa* and *od*, with the action lines of the reactions through p_1 and p_2 respectively. Join these intersections—the line is the closing string (see art. 139)—and draw a ray parallel to that line, noting its intersection Q with AD; then QA and DQ denote the reactions at p_1 and p_2 respectively. Next choose a new pole O' *anywhere* on a line through Q parallel to p_1p_2, and construct a corresponding funicular polygon beginning at p_1 or p_2; this new polygon will pass through both points.

Proof: The three loads AB, BC, and CD and the reactions are in equilibrium, and every funicular polygon for those forces must close. Hence if p_1p_2 is taken as the string $o'q$ of a new funicular polygon (and hence a new pole O' on the line through Q parallel to p_1p_2) and then the strings $o'a$, $o'b$, etc., be drawn, $o'd$ must pass through p_2 to close the polygon.

To pass a funicular polygon through three points: By the method explained in the foregoing, determine a ray $Q'O'$, the funicular polygon corresponding to which will pass through any pair of the three points as prescribed; then determine another ray $Q''O''$, the funicular polygon corresponding to which will pass through another pair of the three points as prescribed. Then with the intersection of $Q'O'$ and $Q''O''$ as a pole, construct a funicular polygon drawing one of the specified strings through its specified point; it will be found that the polygon passes as required through the other two points.

As an illustration, take five parallel forces, *ab*, *bc*, *cd*, *de*, and *df* (fig. F 9), the requirement being to make *oa*, *od*, and *of* pass through p_1, p_2, and p_3 respectively.*

* This is the form in which the practical problem appears, i.e., the forces are parallel, two of the points embrace all the forces and the third is somewhere between; the first and last strings are to pass through the first two points, and that string is to pass through the third point which extends between the action lines of the two forces adjacent to that point.

The first, or preliminary funicular polygon, is the upper one, corresponding to the pole O_1; its strings o_1a, o_1d, and o_1f with

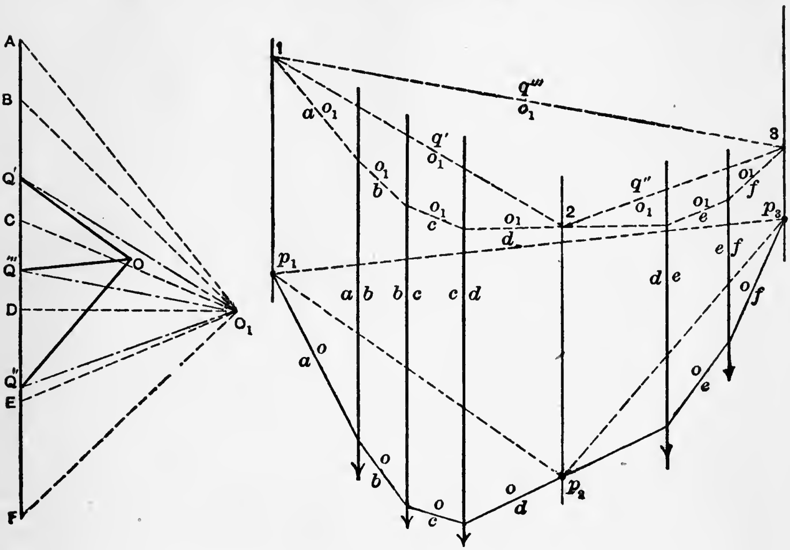

FIG. F 9.

the verticals through p_1, p_2, and p_3 respectively determine the closing strings o_1q' and o_1q''. Lines parallel to these through O_1 determine Q' and Q'', and lines through these points parallel to p_1p_2 and p_2p_3 respectively determine the pole O, the funicular polygon corresponding to which if started at p_1, p_2, or p_3, as prescribed, will pass through the other two points.

Before beginning the construction of the final funicular polygon, it may be well to check the location of the pole O by means of the third closing line $\overline{13}$; this line serves to locate Q''' (FQ''' and $Q'''A$ being reactions on the imaginary beam supported at p_1 and p_3 and sustaining all the forces as loads); a line through Q''' parallel to p_1p_3 will pass through O if no error has been made.

EXAMPLE.

Assume completely six parallel forces and select three points in the space diagram not in the same straight line, and then construct a funicular polygon so that three of its strings will pass through the points.

F 16. Relation between Two Funicular Polygons for a Given Force System Drawn from Different Poles.—The intersections of corresponding strings of such polygons lie on a straight line parallel to the line joining the two poles.

Proof: Let *ab*, *bc*, and *cd* (fig. F 10) be three forces, *oa*, *ob*, *oc*,

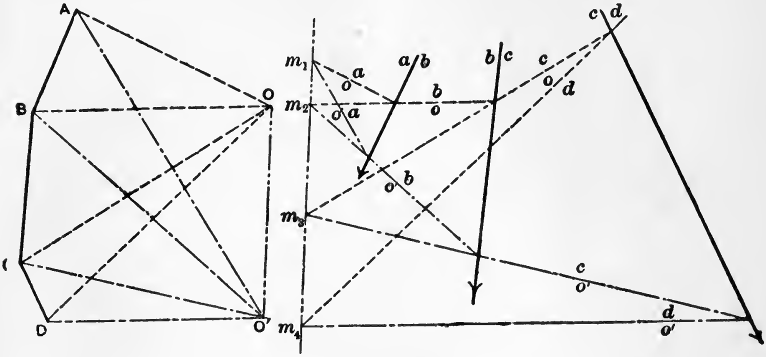

FIG. F 10.

and *od* being their funicular polygon for the pole *O*, and *o'a*, *o'b*, *o'c*, and *o'd* their funicular polygon for the pole *O'*; the intersections of corresponding strings of these polygons are m_1, m_2, m_3, and m_4.

The forces *BO* and *OC* are equivalent to *BC*; so also are *BO'* and *O'C*.* Therefore *BO*, *OC*, *O'B*, and *CO'* are in equilibrium, and the resultant of any pair of these four balances the resultant of the other pair. Now the resultant of *O'B* and *BO* acts through m_2, and the resultant of *OC* and *CO'* acts through m_3; since these resultants balance, they must coincide, i.e., each acts through m_2 and m_3. The direction of the resultant of the first pair is *O'O* (and that of the second is *OO'*), and since the action line of a force is parallel to its direction, m_2m_3 is parallel to *OO'*.

In a similar manner it might be shown that m_1m_2 and m_3m_4 are parallel to *OO'*; thus the proposition is proved.

By means of the property discussed in the foregoing, a string may readily be drawn to an inaccessible intersection, e.g., one which falls beyond the limits of a drawing. Thus, let *ab*, *bc*, and *cd* (fig. F 11) be three forces, *oa* and *ob* two strings of a funicular

* The sequence of these letters indicates the sense of the force designated.

polygon which it is desired to complete. The string *oc* ordinarily would be drawn parallel to *OC* through the intersection of *ob* and *bc*,

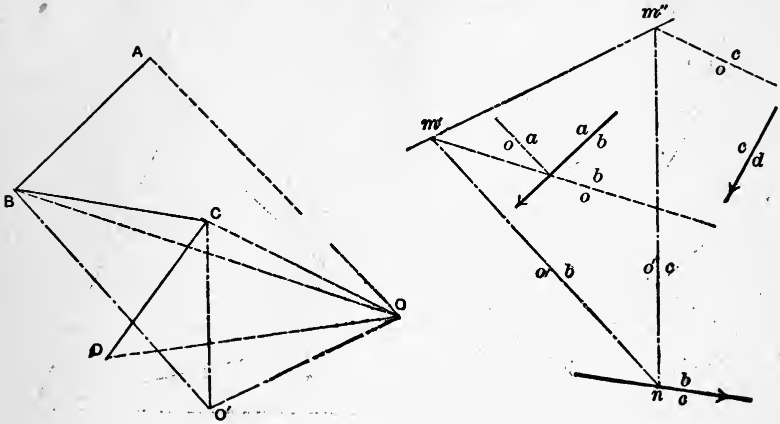

FIG. F 11.

here beyond the limits of the drawing.* Choose a new pole O'; from any point n in bc, draw the strings $o'b$ and $o'c$, and note the intersection m' of $o'b$ and ob; draw through m' a parallel to OO', and note its intersection m'' with $o'c$; finally draw through m'' a line parallel to OC. This is the desired string oc.

Notice that it will always be possible to choose O' and n so that the points m' and m'' will fall upon the drawing.

F 17. To Close a Gauche † Polygon with Three Sides whose Directions are Given.—This is a geometrical construction used in the graphical solution of the following problem: A system of concurrent non-coplanar forces is in equilibrium and all the forces are known except three, whose action lines only are known; required to determine their magnitude and senses.

Let there be four forces in the system, F being the wholly known force and F_1, F_2, and F_3 the partially known ones, and let their action lines be as represented in fig. F 12 (*a*).‡ The polygon

* In such a case a draughtsman usually tacks a sheet of paper to his board which will take the intersection desired; often, however, such an intersection is very acute—therefore indefinite—and this special construction is advantageous.

† A gauche polygon is one whose sides are non-coplanar.

‡ See art. 49 for notation and scheme of representing concurrent non-coplanar forces.

for the system closes, so let it be called $ABCDA$, AB, BC, CD, and DA representing the magnitudes and directions of F, F_1, F_2,

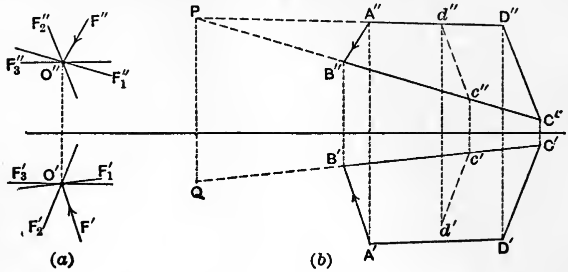

FIG. F 12.

and F_3 respectively, and let $A'B'C'D'A'$ and $A''B''C''D''A''$ be its horizontal and vertical projections—their construction will be explained presently.

The point d' (of any polygon such as $c'c''d''d'$, similar to $C'C''D''D'$), the point Q (construction for which is apparent), and the point D' are in the same straight line, for regard $A''D''$, $B''C''$, and $B'C'$ (indefinite in length) as the action lines of three forces such that $D'D''C''C'D'$ and $d'd''c''c'd'$ are funicular polygons for them. The action line of the resultant of those forces passes through D' and d', the intersections respectively of the extreme strings of those polygons; it also passes through Q because that point is the limiting position of the extreme strings of $d'd''c''c'd'$ as c'' or d'' is taken nearer and nearer to P. Hence the three points are on a straight line, which fact leads to the following construction for the projections of the force polygon:

First draw the projections $A'B'$ and $A''B''$ (of F) and the indefinite projections $B'C'$ and $B''C''$ (of F_1) and $A'D'$ and $A''D''$ of (F_3); then guess at the position of C, as at c', c'', and supposing the guess to be correct, draw the projections $c''d''$ and $c'd'$ of F_2; next prolong $A''d''$ and $B''c''$ to their intersection P, through which draw a vertical and note its intersection Q with $B'C'$; finally draw $Q'd'$, and note its intersection with $A'D'$ (indefinite in length)—this is the required point D' from which the projection $D'C'$ may be drawn and then $D''C''$.

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Numbers refer to pages.

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EXPLANATION OF PRINCIPAL SYMBOLS.

(The numbers refer to pages, where additional explanation may be found.)

A . . . area.
a . . . arm of a force; linear acceleration.
a_x . . . arm with respect to x axis; x component of an acceleration.
$\bar{a}$ . . . acceleration of mass-centre.
a_n . . . normal component of acceleration.
a_t . . . tangential component of acceleration.
α . . . angular acceleration (205); angle of repose (150).
C . . . couple.
C_x . . . component of C perpendicular to an x axis (46).
δ . . . density.
E . . . energy.
E_k . . . kinetic energy.
E_p . . . potential energy.
e . . . efficiency (317).
ϵ . . . lead, lag, epoch, or epoch angle (177, 197).
F . . . force; friction.
F_x . . . x component of a force (15).
F' . . . limiting friction (150).
ϕ . . . angle of friction (150).
I . . . moment of inertia (250, 263).
I_x . . . I with respect to x axis.
$\bar{I}$ . . . I with respect to a centroidal axis (253, 366).
J . . . product of inertia (257, 371).
J_{xy} . . . J with respect to x and y axes.
$\bar{J}$ . . . J with respect to central axes (372).
k . . . radius of gyration (250, 364); gravitation constant (74).
k_x . . . k with respect to x axis.
$\bar{k}$ . . . k with respect to a centroidal axis.
l . . . length.
M . . . moment of a force; sometimes mass.
M_0 . . . M with respect to O.
m . . . mass.
N . . . normal pressure (149).
n . . . frequency (177).
P . . . force; power (318).
p . . . intensity of stress (86).
Q . . . force.
R . . . resultant; reaction.
R_n . . . normal component of R (42, 275).
R_t . . . tangential component of R (42, 275).
$\bar{r}$ . . . radius to mass-centre (275).
ρ . . . radius of curvature; surface density (83).
S . . . force.
s . . . length of arc.
T . . . tension; period (177).
t . . . time.
θ . . . angular displacement (204); angle.
U . . . moment of momentum (337).
V . . . volume.
v . . . linear velocity.
v_x . . . x component of v.
$\bar{v}$ . . . velocity of mass-centre.
W . . . weight.
w . . . work; sometimes weight.
ω . . . angular velocity (204); solid angle (79).
$\bar{x}$ . . . coordinate of centroid.
$\bar{y}$ . . . " " "
$\bar{z}$ . . . " " "
x_c . . . coordinate of centre of stress (88).
y_c . . . coordinate of centre of stress (88).