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TENSIONS IN TRACK CABLES AND LOGGING SKYLINES

The Catenary Loaded at One Point

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CORRIGENDA.

Page 7, equation (22) :—

$$e^{-\frac{x}{c}} = \frac{\sqrt{(s+s_0)^2+c^2} - (s+s_0)}{(\sqrt{s_0^2+c^2} - s_0)} \tag{22}$$

Page 8, equation (23) :—

$$y+\sqrt{s_0^2+c^2} = \frac{\sqrt{s_0^2+c^2}}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) + \frac{s_0}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right) \tag{23}$$

Page 10, equation (38) :—

$$y+y_0 = \frac{y_0}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) + \frac{s_0}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right) \tag{38}$$

Page 15, equation (55) :—

$$s-y=-(y_0-s_0)\left(e^{-\frac{x}{c}}-1\right) \tag{55}$$

TENSIONS IN TRACK CABLES AND LOGGING SKYLINES

The Catenary Loaded at One Point

INTRODUCTION

Two problems of the catenary, viz., the common catenary and the parabolic catenary, are generally treated in texts on Mechanics. A third problem, that of a *catenary loaded at one point,* is equally important, but either is not discussed at all or given a sketchy treatment that is far from satisfactory. This type of catenary is of considerable practical importance, as a conveying cableway or a cable tramway closely approximates in form to this curve. In the Western States such devices are much used in mining and logging operations for transporting loads from higher to lower levels by gravity. In this paper the equations of the *catenary loaded at one point* will be derived and the properties of such a catenary discussed. It will also be shown how the tensions may be computed and what the condition is for maximum tension.

It will be assumed that (1) the cable is perfectly flexible and (2) the load is fixed at one point, neither of which assumptions agree with practice. The cables in use are generally steel, sometimes two inches or more in diameter and very stiff in short lengths. However, when they are used in lengths of 2000 to 3000 feet and the vertical deflections are small compared to the length, the *relative* stiffness is negligible, and hence, the first assumption a legitimate one. As the load travels along, the conformation of the cable is constantly changing and likewise the tensions. But by assuming a fixed position of the load, it is possible to obtain the properties of the catenary for an *instantaneous position* of the load, from which the condition for maximum tension in the cable may be determined. To start with, it is assumed that the load is at *any* point, so that the treatment is general.

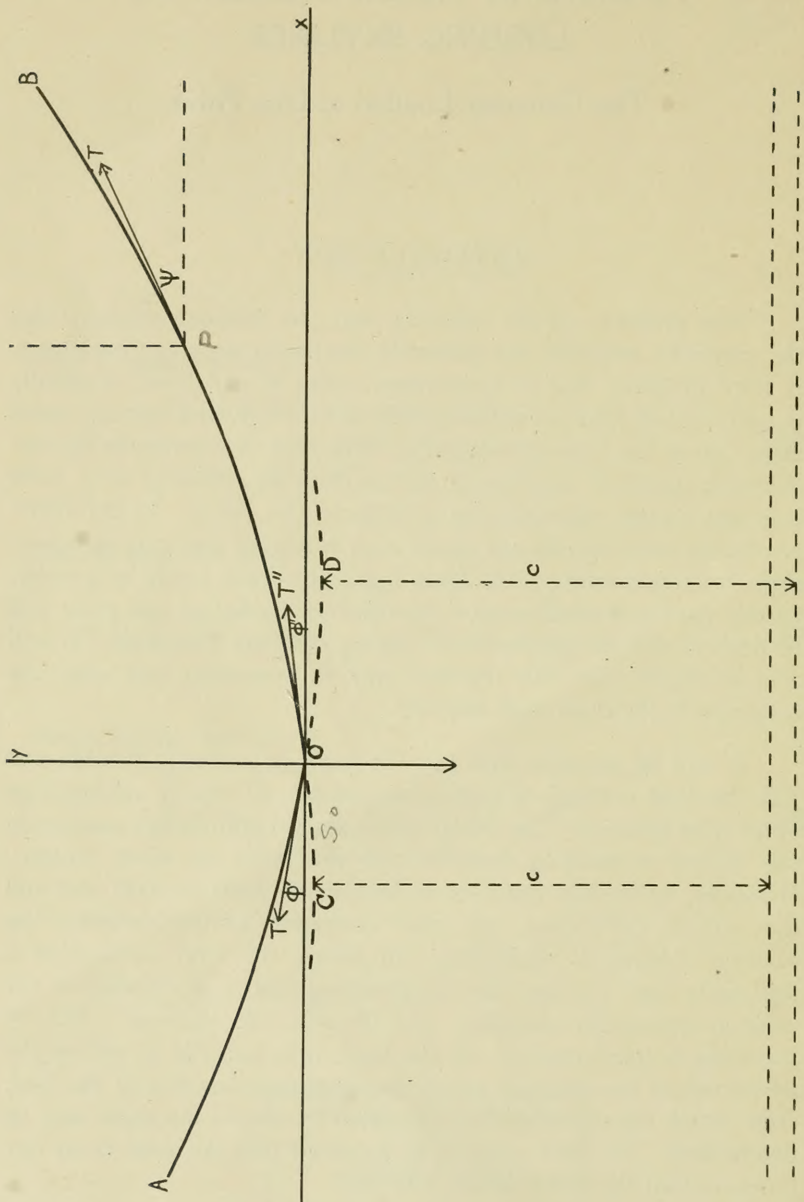

FIG. 1

I

Derivation of the Equation of the Catenary Loaded at One Point.

Let OAB, figure 1, represent a heavy, flexible cord or cable, fastened at A and B and supporting a known load W at O. Assume B to be at a higher horizontal level than A. At O there will be three forces in equilibrium: (1) the weight, (2) the tension in the part of the cable OA which will be called T′ and (3) the tension in OB which will be called T″. Let ϕ' and ϕ'' be the angles that T′ and T″ respectively make with the horizontal. By Lame's theorem we have

$$\frac{W}{\sin[180^\circ-(\phi'+\phi'')]}=\frac{T'}{\sin(90^\circ+\phi'')}=\frac{T''}{\sin(90^\circ+\phi')}$$

or

$$\frac{W}{\sin(\phi'+\phi'')}=\frac{T'}{\cos\phi''}=\frac{T''}{\cos\phi'} \tag{1}$$

Solving for W

$$W=T'\sin\phi'+T'\cos\phi'\tan\phi'' \tag{2}$$

and

$$W=T''\tan\phi'\cos\phi''+T''\sin\phi'' \tag{3}$$

Resolving the forces vertically and horizontally, respectively, we have

$$W=T'\sin\phi'+T''\sin\phi'' \tag{4}$$

and

$$T'\cos\phi'=T''\cos\phi''=T_h \tag{5}$$

where T_h is the horizontal component of the tension of the cable and is the same at all points for a given load and position of that load. Combining equations (2), (3), (4) and (5) we get

$$2W=T'\sin\phi'+T''\sin\phi''+T_h\tan\phi'+T_h\tan\phi'' \tag{6}$$

$$W=T_h\tan\phi'+T_h\tan\phi'' \tag{7}$$

It is evident that in form the cable is divided into two parts the point of division being at O, where the load is hung. From a mathematical consideration, this is a point of discontinuity which means that in passing from one portion of the cable to the other the change in the properties of the catenary with respect to the length is very abrupt at this point. It will be necessary, then, to consider the two parts separately and derive the equations for each part.

We will consider first the arc OB. Let P be any point of this arc, and let T be the tension at this point making an angle ψ with the horizontal. The arc OP is in equilibrium under three forces: (1) the tension at O, which with respect to P is opposite and equal to T″ of figure 1; (2) the weight of the cable between O and P acting at the center of gravity of OP; (3) the tension T at P. If s be the length of the cable from O to P measured along this arc and w the weight per unit length, then the second force is ws. Resolving these three forces vertically, we have

$$T\sin\psi = ws + T''\sin\phi'' \tag{8}$$

and resolving horizontally

$$T\cos\psi = T_h = T''\cos\phi'' \tag{9}$$

Dividing equation (8) by (9) we have

$$\tan\psi = \frac{dy}{dx} = \frac{ws}{T_h} + \tan\phi'' \tag{10}$$

This is the differential equation of the curve OB which is the conformation the cable assumes under the action of the three forces. The solution of this gives the desired analytical equation.

It is evident that in equation (10) w, T_h, and $\tan\phi''$ are all constants for a given cable, a given load and fixed position of that load. We will write for these three constants

$$c = \frac{T_h}{w} \tag{11}$$

and

$$s_0 = c\tan\phi'' \tag{12}$$

Equation (11) conforms with the usual method of treatment of the common catenary, but it will be noticed that no assumption is made regarding the form of the curve OB. That is, c is considered nothing more than a constant at this stage of procedure. Introducing these two constants equation (10) becomes

$$\frac{dy}{dx} = \frac{s+s_0}{c} \tag{13}$$

Since s is a variable dependent upon x and y it is necessary to eliminate one of the three before the equation can be solved. This may be done by the relationship

$$ds^2 = dy^2 + dx^2 \tag{14}$$

First eliminataing x we have

$$\frac{dy}{ds} = \frac{s+s_o}{\sqrt{(s+s_o)^2+c^2}} \tag{15}$$

the solution of which is

$$y = \sqrt{(s+s_o)^2+c^2} + A$$

where A is a constant of integration. If we choose the origin at 0 so that $y=0$ when $s=0$, then

$$A = -\sqrt{s^2_o+c^2}$$

and (16) becomes

$$y = \sqrt{(s+s_o)^2+c^2} - \sqrt{s^2+c^2} \tag{17}$$

On eliminating y between (13) and (14) we have

$$\frac{ds}{dx} = \sqrt{\frac{(s+s_o)^2}{c^2} + 1} \tag{18}$$

the solution of which is

$$\log\,[(s+s)_o + \sqrt{(s+s_o)^2+c^2}] = \frac{x}{c} + B \tag{19}$$

where B is a constant of integration. Using the same conditions as for evaluating the constant A,

$$B = -\log(s_o + \sqrt{+s^2_o+c^2})$$

Substituting this value of B (19) becomes

$$\frac{x}{c} = \log\frac{(s+s_o)+\sqrt{(s+s_o)^2+c^2}}{s_o+\sqrt{s^2_o+c^2}} \tag{20}$$

This may be written in the exponential form,

$$e^{\frac{x}{c}} = \frac{(s+s_o)+\sqrt{(s+s_o)^2+c^2}}{s_o+\sqrt{s^2_o+c^2}} \tag{21}$$

By inverting equation (21) we get

$$e^{\frac{-x}{c}} = \frac{\sqrt{(s+s_o)^2+c^2}-(s+s_o)}{(\sqrt{s^2+c^2}-s_o)} \tag{22}$$

We may obtain an expression for y in terms of x and the constant c and s_o by adding (21) and (22) and combining with (17).

$$y+\sqrt{s_o^2+c^2} = \frac{\sqrt{s_o^2+c^2}}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) = \frac{s_o}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right) \qquad (23)$$

By substracting (21) from (22) we get a symmetrical expression for s.

$$s+s_o = \frac{\sqrt{s_o^2+c^2}}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right) + \frac{s_o}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) \qquad (24)$$

Equation (23) is the equation of the curve OB in rectangular co-ordinates, and (24) gives the length of the arc measured from O in terms of the abscissa and the constants s_o and c. These equations represent a *general case* of the catenary of which the *common* catenary is a special case. This may be seen from the following consideration. Suppose that $\tan\phi''=0$, which could be attained by removal of the load W or by shifting to such a point that the arc OB became horizontal at O. Then by (12) $s_o=0$ and equations (23) and (24) reduce to

$$y + c = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \qquad (25)$$

and

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \qquad (26)$$

which are the well known equations of the common catenary.

In order to interpret the constants s_o and $\sqrt{s_o^2+c^2}$ of equations (23) and (24) let us assume that the curve OB is extended to the left of O to a point C where the tangent is horizontal. This is represented in figure 1 by the broken line curve CO. Let the weight per unit length of the imaginary cable be the same for the arc OB, and let the horizontal tension be T_h. Call the co-ordinates of O, referred to C, p and q. The arc CO will be in equilibrium under three forces, T_h, the weight of the cable wr where r is the length of the arc CO, and the tension T″ at O. Resolving these forces horizontally and vertically we have the following relations:

$$T_h = T''\cos\phi'' \qquad (27)$$

and

$$T''\sin\phi'' = wr \qquad (28)$$

Dividing equation (28) by (27) we obtain

$$\tan\phi'' = \frac{dq}{dp} = \frac{r}{c} \qquad (29)$$

This is the differential equation of the arc CO, and is of the same form as that of the common catenary, so that the solution may be immeditely written.

$$q = \sqrt{r^2+c^2} - c \tag{30}$$

$$p = c \log\left\{\frac{r+\sqrt{r^2+c^2}}{c}\right\} \tag{31}$$

Since the curve COB is continuous the equations for *any* point referred to C as the origin will be of the same form as (30) and (31). Let x′, y′ be the co-ordinates of any point referred to C. Then

$$y' = \sqrt{(s+r)^2 + c^2} - c \tag{32}$$

$$x' = c \log\left\{\frac{(s+r) + \sqrt{(s+r)^2 + c^2}}{c}\right\} \tag{33}$$

where (s+r) is the length of the arc measured from C.

Now let us transform the last two equations by the following relationship,

$$y' = x + p$$
$$y' = y + q$$

and substituting values of p and q from (30) and (31).

$$y = \sqrt{(s+r)^2 + c^2} - c + \sqrt{r^2 + c^2} + c \tag{34}$$

$$\frac{x}{c} = \log\left\{\frac{(s+r) + \sqrt{(s+r)^2 + c^2}}{r - \sqrt{r^2 + c^2}}\right\} \tag{35}$$

Equations (34) and (35) must be identical with (17) and (20) respectively, since in both cases x, y are the co-ordinates of any point of the arc OB referred to O as an origin. Hence

$$r = s_0$$

That is, the constant s_0 of equations (17) and (18) is the length of the fictitious arc CO. This means that *the curve OB is a portion of a common catenary, the lowest point of which is a distance s_0 from O measured along the curve.* While OB is the only part of this catenary that represents the conformation of a part of the cable, nevertheless, the part of the catenary CO has a physical interpretation which will be given later.

From a consideration of equations (17), (34), and (30) it is evident that

$$\sqrt{s_o^2 + c^2} = \sqrt{r^2 + c^2} = q + c \qquad (37)$$

In the common catenary c is the distance from the lowest point to a horizontal line which may be called the directrix. Hence $\sqrt{s_o^2+c^2}$ is the distance of O from this same directrix. We will call this y_o. Making this substitution in equations (23) and (24) we get

$$y + y_o = \frac{y_o}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + \frac{s_o}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \qquad (38)$$

$$s + s_o = \frac{y_o}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) + \frac{s_o}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \qquad (39)$$

Substituting the hyperbolic functions for exponential, these equations become

$$y + y_o = y_o \cosh\frac{x}{c} + s_o \sinh\frac{x}{c} \qquad (40)$$

$$s + s_o = y_o \sinh\frac{x}{c} + s_o \cosh\frac{x}{c} \qquad (41)$$

These equations involve three constants, c, s_o, and y_o, the last of which is dependent upon the other two.

$$y_o = \sqrt{s_o^2 + c^2} \qquad (42)$$

All three are quantities of the fictitious arc CO; c being the parameter, s_o the length of the arc, and y_o the ordinate of the terminal point O measured from the directrix. By assigning different values to c, s_o, and y_o (40) and (41) become the equations of a family of curves which represent the conformation the cable will take under various loads and horizontal tensions.

In the same manner we may obtain equations of the arc OA which is the conformation assumed by the part of the cable to the left of the load W. The differential equation is,

$$\frac{dy}{dx} = \frac{ws}{T_h} + \tan\phi' \qquad (43)$$

which differs from (10) in that x and s are taken to left of O and ϕ' replaces ϕ''. Hence the analytical equations of the arc OA will be the same as (40) and (41) except the constants s_o and y_o will have different values. The constant c is the same for the two curves, since it depends upon the horizontal tension and the weight of the cable per unit length, which are the same for all parts of the cable. Hereafter when s_{o1} and y_{o1} are used in equations (40) and (41) they apply to the arc OA to the left of the origin and hence the values assigned to s and x are negative only; and when s_{o2} and y_{o2} are used the above equations apply to the arc OB to the right of the origin and the values assigned to s and x are positive only. Since the constant c is the same for both arcs it is evident that OA and OB are arcs of the *same* common catenary having the parameter c. If in figure 1 the curve DOA were shifted to the left and slightly upward until the point D coincided with C, AODCOB would form a continuous curve which is the common catenary from which OA and OB are taken to form the discontinuous curve AOB. The length of the arc DO is the constant s_{o1} of the curve OA and the length of COO is the constant s_{o2} of the curve OB.

II.

Determination of the Tensions.

Arbitrarily we have set

$$T_h = wc \tag{11}$$

and found that c is a perameter of both curves OB and OA.

The vertical components of the tensions T′ and T″ at O may be obtained by combining equations (5), (11) and (12).

$$T' \sin\phi' = ws_{o1} \tag{44}$$
$$T'' \sin\phi'' = ws_{o2} \tag{45}$$

Hence *the vertical component of the tension in OA at O is equal to the weight of the fictitious arc OD; and likewise, the tension in OB at O is equal to the weight of OC.* If these expressions for the vertical components of the tensions are substituted in equation (4), an expression is obtained giving the relation between the load W, the weight per unit length of the cable, and the fictitious arcs.

$$W = w(s_{o1} + s_{o2}) \tag{46}$$

This shows that the constants s_{o1} and s_{o2} are dependent upon the nature of the cable and the load.

$$T' = w\sqrt{c^2 + s_{o1}^2} = wy_{o1} \tag{47}$$
$$T'' = w\sqrt{c^2 + s_{o2}^2} = wy_{o2} \tag{48}$$

The vertical component of the tension at *any* point in the curve OB is given by equation (8). Combining this with (45) we get

$$T_2 \sin\psi_2 = w(s + s_{o2}) \tag{49}$$

And in like manner we have for any point in the curve OA

$$T' \sin\psi_1 = w(s + s_{o1}) \tag{50}$$

Now $(s+s_{o2})$ is the length of the arc COP measured from the lowest point of the fictitious arc OC, and hence *the vertical component of the tension in the cable at any point is w times the length of the arc of the common catenary of which OB or OA is a part.*

By combining (11) with (50) and (49) respectively we obtain the total tension at any point in OA or OB respectively.

$$T_1 = w(y + y_{o1}) \tag{51}$$
$$T_2 = w(y + y_{o2}) \tag{52}$$

Hence *the total tension at any point is w times the verticle distance of that point above the directrix of the common catenary.*

Equations (11), (44), (45), (47), (48), (49), (50), (51) and (52) give completely the tension in the cable. In every case the tension depends upon the weight per unit length, w, and one of the three constants, c, s_o, or y_o. If these tensions are known, the constants can then be determined and by the aid of equations (40) and (41) the position of the cable determined. And conversely, if the constants can be determined from the data available and equations (40) and (41), the tension for any point in the cable can be found.

Equations (51) and (52) show that for a given load and given position of that load the tension in either part of the cable is greatest at the point where $(y+y_o)$ is a maximum, that is, at the point of fastening at the end of the cable. So that in future considerations of the effect of the load and of the position of the load the tensions at the ends *only* need be taken account of. At which of the two ends the tension is the greater depends upon the position of the load. If in equations (51) and (52) y_1 is the verticle height of the lower end above the position of the load and y_2 that of the upper end, y_2 will be greater than y_1 for any position of the load

Page 12, insert before equation (47),

The tensions T′ and T″ can be expressed in terms of w, c, s_o, and y_o. Combining (11) with (44) and (45) respectively we have,

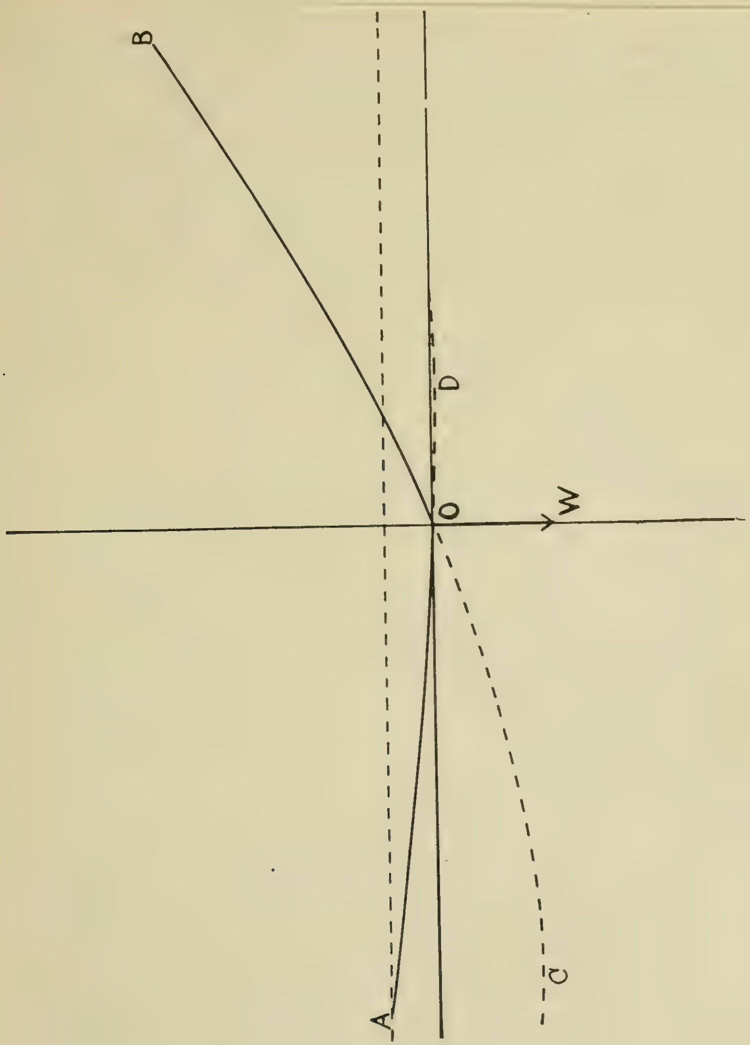

FIG. 2

along the cable, but y_{02} will be considerably less than y_{01} when the load is near the lower end and hence T_2 will be less that T_1. But for the load near the middle of the span and any place between the middle and the higher end T_2 will be greater than T_1. (See figure 2).

In practice, information is wanted regarding the tensions in various parts of the cable for a certain load and different positions of that load, or more specifically, the maximum load the cable will carry. To answer the latter question equation (46) must be used which involves the constants s and s_o. It is desirable, then, to express s in terms of other quantities on which it depends. This may be done by eliminating y_o between equations (40) and (41). Writing these in the following form

$$y - s_o \sinh\frac{x}{c} = y_o\left(\cosh\frac{x}{c} - 1\right) \tag{40}$$

$$s + s_o\left(1 - \cosh\frac{x}{c}\right) = y_o \sinh\frac{x}{c} \tag{41}$$

and dividing the first by the second we get,

$$s_o = \frac{y}{2}\,\frac{\sinh\frac{x}{c}}{\cosh\frac{x}{c} - 1} - \frac{s}{2}$$

which may be simplified to

$$s_o = \frac{y}{2}\coth\phi - \frac{s}{2} \tag{53}$$

where

$$\phi = \frac{x}{2c} \tag{54}$$

If x_1, y_1 are the coordinates of one end of the cable (the lower) referred to the position of the load as origin, s_1 is the length of the portion of the cable extending from the load to the end, i.e., OA of figure 1. But the value of s_1 depends upon the parameter c. An infinite number of catenaries may be drawn thru the points A and O, and the *particular one* which represents the conformation of the cable is determined by the value of c taken. In an *unloaded* cable the value of c depends alone upon the horizontal tension which may be made anything desirable. In the *loaded* cable c may be computed from the value of the horizontal tension, but this latter depends in part upon the load which is placed on the cable. A relationship between x, y, s and c, which is independent of the other constants s_o and y_o, may be obtained which is of considerable use. Subtracting equation (38) from (39) we get

$$s - y = -(y_o - s_o)(e^{\frac{x}{c}} - 1) \quad (55)$$

and by adding

$$s + y = (y_o + s_o)(e^{\frac{x}{c}} - 1) \quad (56)$$

On multiplying (55) by (56) and substituting

$$c^2 = y^2_o - s^2_o$$

we have

$$s^2 - y^2 = c^2(e^{\frac{x}{c}} - 2 + e^{-\frac{x}{c}})$$

which may also be expressed in terms of the hyperbolic function,

$$\sqrt{s^2 - y^2} = 2c \sinh\frac{x}{2c} \quad (57)$$

This may be thrown into a more useful form by substituting from equation (54).

$$\sqrt{s^2 - y^2} = x\frac{\sinh \phi}{\phi} \quad (58)$$

By this last equation the length of the cable between the load and terminal point may be quickly found for any value of c. To faciiltate computation, it is desirable to have a curve in which $\frac{\sinh \phi}{\phi}$ is plotted as ordinates against ϕ as abscissas. Such a curve is given in figure 3. In the next section it will be shown how these computations may be made.

Another equation giving s in terms of x, y, and c may be obtained by eliminating s_o between equations (40) and (41).

$$s = y + 2y_o \tanh\frac{x}{2c} \quad (59)$$

III

Practical Solution of the Problem.

In sections I and II equations have been derived giving the relations among all the factors involved in stresses due to loading. But when an actual computation is undertaken, it is generally found that not sufficient data is available. Perhaps this may be best illustrated by taking a numerical problem, which was presented to the writer by a logging company. It is typical of those the engineers encounters.

It is required to know the maximum load a cable will carry when used as a logging "sky-line" under the following conditions:

1. Breaking strength—112 tons.
2. Weight of cable—5.38 lbs./linear ft.
3. Horizontal span—2500 ft.
4. Vertical height of upper end above lower (tail spar tree above head spar tree)—625 ft.
5. Factor of safety—3.5.

The maximum tension allowable is

$$\frac{112 \times 2000}{3.5} = 64{,}000 \text{ lbs.}$$

Calling this T_2 and substituting in equation (52) we find

$$y_2 + y_{02} = \frac{64{,}000}{5.38} = 11{,}896 \text{ ft.}$$

This may be used in equation (40) for finding s_{02}, but before any computation is possible values of x, y, s and c are necessary. Since the value of T_2 used above is the maximum that the cable is to stand, the value of x to be used should be the horizontal distance of the load from the upper end *where the tension is the greatest.* In other words, before we can proceed further it is required to know the condition of maximum tension in the cable for a given load. So far no *general* method of treating this phase of the problem has been found, but it has been determined for a special case, which will be taken up in detail.

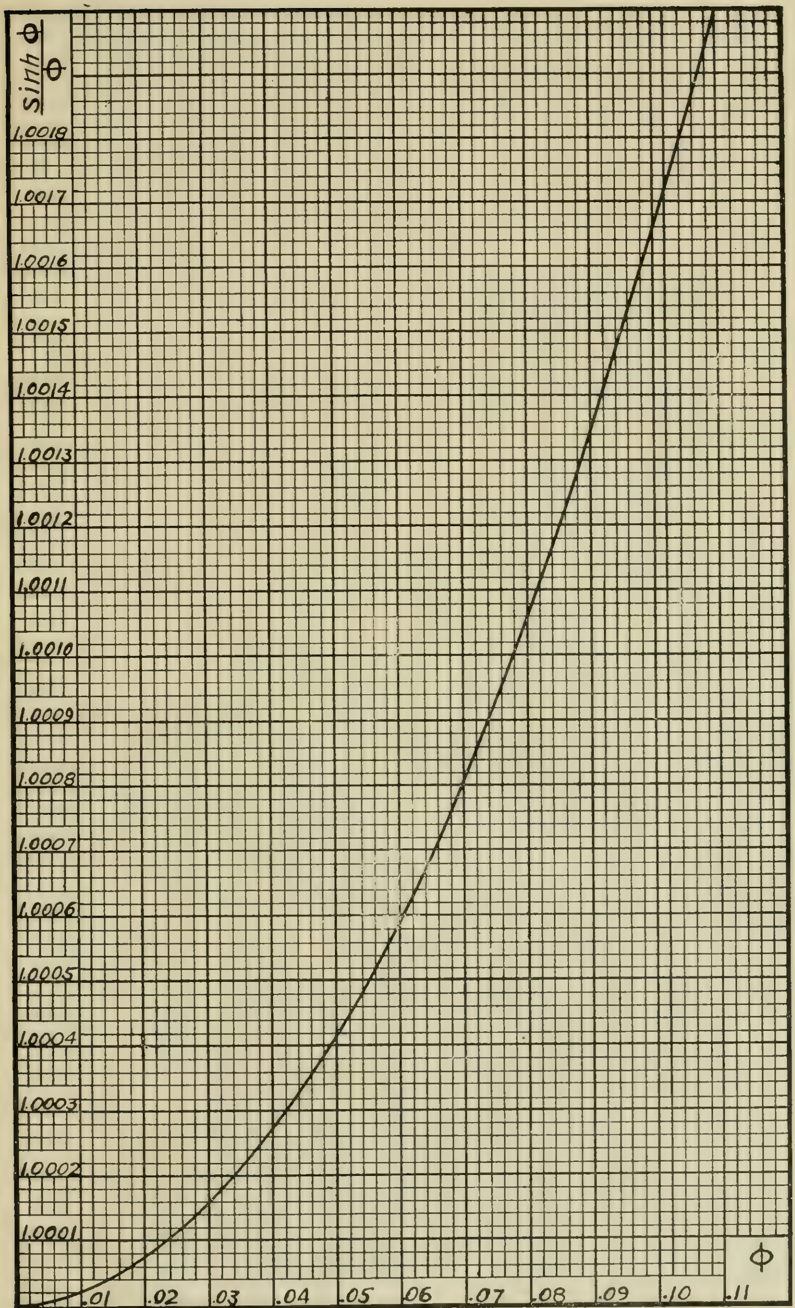
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FIG. 3

The first step necessary in this calculation is the assumption of some value of the parameter c. A common usage* is to take the horizontal tension equal to one-fourth of the breaking strength of the cable, and the "deflection" of the cable when the maximum gross load is at the center to be one-twentieth of the span. By "deflection" here is meant the vertical distance of the loaded point of the cable below the lower fixed end of the cable.

Following thse two specifications we may find values of c, x and y. From equation (11)

$$c = \frac{T_h}{w} = \frac{112 \times 2000}{4 \times 5.38} = 10{,}409$$

One-twentieth of the span is 125 feet, so that

$y_2 = 750$ ft.
$x_2 = 1250$ ft.

By equation (54) we find

$$\phi = \frac{1250}{2 \times 10409} = .06004$$

From figure 3 $\dfrac{\sinh \phi}{\phi} = 1.0012$.

Substituting these values of x_2, y_2, and $\dfrac{\sinh \phi}{\phi}$ in equation (58) we find

$s_2 = 1458$ ft.

To find s_1 the same procedure is followed.

$x_1 = 1250$ ft.
$y_1 = 125$ ft.
$\dfrac{\sinh \phi}{\phi} = 1.0012$

The value of $\dfrac{\sinh \phi}{\phi}$ is the same for the two parts of the cable *only* for the middle of the span where $x_1 = x_2$. This greatly simplifies calculations. It is found that

$s_1 = 1257$ ft.

* Mark's Handbook of Mechanical Engineering, page 1160.

The length of cable required for this span of 2500 feet is then
$s_1+s_2=2715$ ft.

We may now compute s_{o1} and s_{o2}, using equation (53).

$s_{o1}=414$ ft.
$s_{o2}=5524$ ft.

The load which satisfies the above conditions if found by equation (46).

$W=5.38(414+5524)=31{,}946$ lbs.

To find the tensions at the two ends the constants y_{o1} and y_{o2} must be found, using equation (42).

$y_{o1}=10{,}419$.
$y_{o2}=11{,}784$.

Then by equations (51) and (52) the tensions are computed.

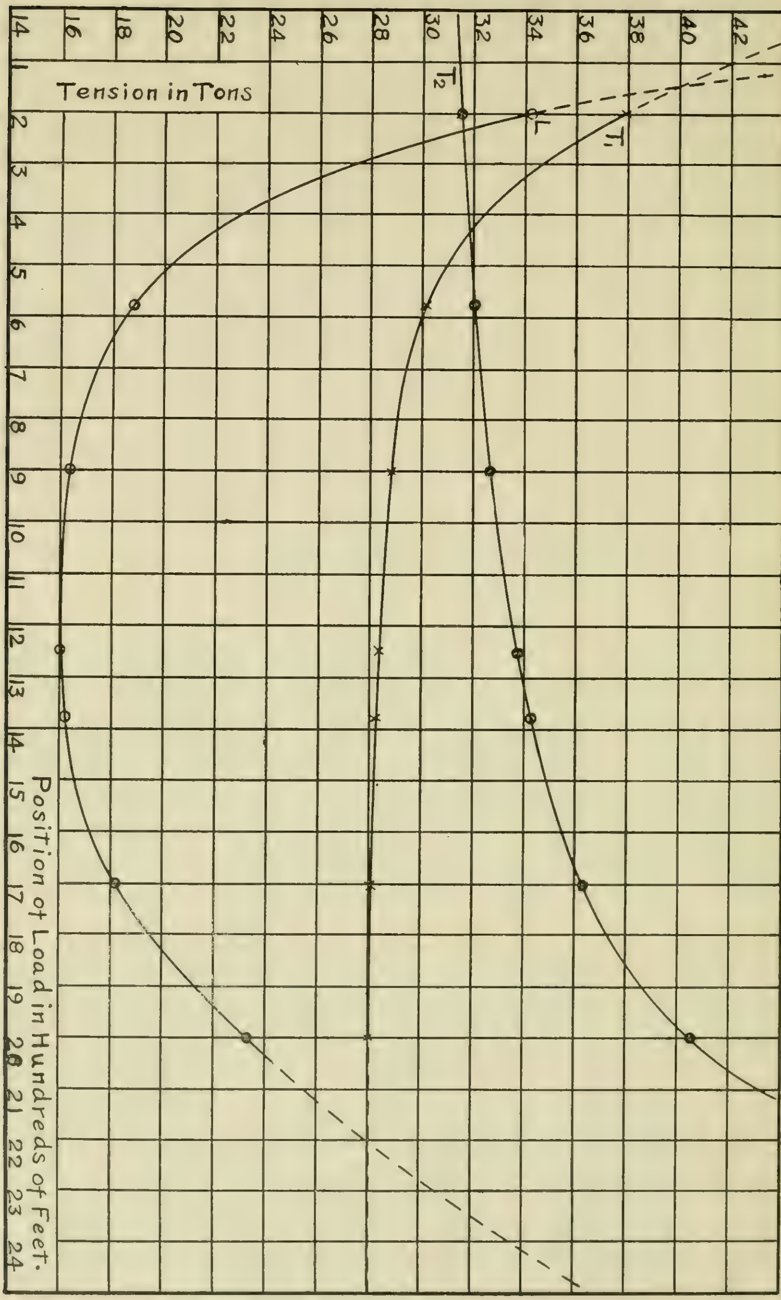
$T_1=5.38(10{,}419+125)=56{,}726$ lbs.
$T_2=5.38(11{,}784+750)=68{,}203$ lbs.

Perhaps attention should be called to the fact that in this calculation no use has been made of the factor of safety specified above. Arbitrarily a horizontal tension was chosen equal to one-fourth the breaking strength, and this gave a value for the constant c upon which is based the calculation of the load and tensions as just given above. The factor of safety comes out 3.29 instead of 3.5. It is apparent that, in using the horizontal tension recommended in Mark's Handbook, too small a factor of safety is provided for. Reference will be made to this discrepancy later.

In order that we may know where the load produces the greatest tension, the computation must be made for several different positions of the load in the same manner as above. In carrying out this computation one of three proceedures must be followed:

(1). We may assume the exact position of the load measured horizontally and vertically from one end, and the horizontal tension of the cable as was used above.

(2). We may take the length of the cable fixed (say, the length found above for the load at the middle, 2715 feet), and assume *the horizontal tension the same,* one-fourth of the breaking strength.



Tension in Tons
14 16 18 20 22 24 26 28 30 32 34 36 38 40 42
T_2
L
T_1
Position of Load in Hundreds of Feet.
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

FIG. 4

(3). We may take the length of the cable fixed, and assume *the load the same* as we found above for the middle point of the span.

Of these three proceedures the first is the easiest, but of no value, because the length of the cable found in this manner will be different for each position of the load. In practice the length of the cable is generally constant while the load is being transported. (Although in some logging operations the cable is let down, given a big sag, in picking up the load). The third method will give the most useful result, but is the most difficult to carry out. In fact, it was found impossible to make this calculation until, by the second proceedure, the tensions and positions of the load had been found for a number of points.

The exact proceedure by the second method is as follows:

(1). Assume a value of c, say 10,409.

(2). Assume certain values of x_1 and y_1, which seem to be reasonable according to one's best judgment.

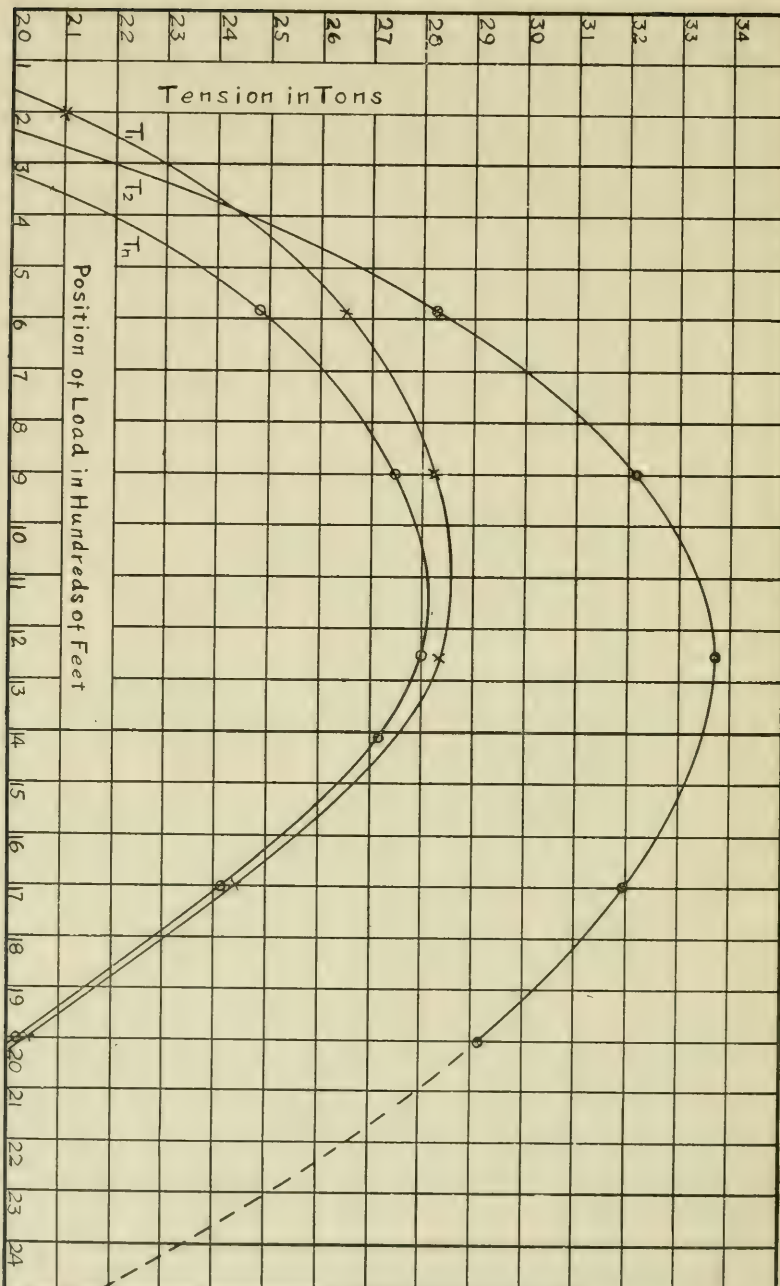
(3). By equation (58) compute s_1.

(4). By the same equation compute s_2, using the same value of c, and values of x_2 and y_2, which may be found by the following relations,

x_1+x_2=horizontal span.
y_2-y_1=verticle height of higher end above lower.

(5). The sum of s_1 and s_2 so found should be the length of cable 2715 feet. If it does not, new values of x_1 and y_1 must be assumed and the same proceedure gone thru with again.

It is thus seen that this method is one of "trial and error" which involves a vast amount of work. The computations were carried out for seven positions or points in the span and the results plotted in the curves of figure 4. Abscissas for all the curves are values of x_1, i.e., the horizontal distance of the load from the lower end support. In curve L the ordinates show the load necessary at any point in order that the horizontal tension may be constant, 56,000 lbs., giving a constant value of c of 10,409. T_1 shows the total tension in the cable at the lower end fastening, and T_2 the total tension at the upper end for any position of the load.


Tension in Tons
Position of Load in Hundreds of Feet
T_1
T_2
T_h
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
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22
23
24

FIG. 5

Curve L shows the rather surprising result that the required load is a minimum at the center of the span. And since the load is larger near the ends, the tensions in the cable are larger also for the load near the ends. The curves of this figure are of little practical value except as a guide in selecting values of the coordinates to be used in the computations made for the curves of figure 5.

In computing the data for the curves of this latter figure, we assume:

(1). That the length of the cable is constant and equal to 2715 feet.

(2). That the load is constant and equal to 31,950 lbs., which is the load (approximately) at the middle of the span as first found.

The proceedure followed is one of trial and error as described in method 2, but with this additional requisite: $s_{01}+s_{02}$ must equal a certain constant value, viz., 5938, which is obtained by dividing the load by the weight per unit length of the cable, 5.38 lbs./ft. This makes the computations for these curves twice as laborious as for figure 4.

As in the former figure, the abscissas of figure 5 are values of x_1. Curve T_h shows how the horizontal tension varies with the position of the load. Curves T_1 and T_2 give the tensions in the cable at the lower and upper ends respectively. The form of all these curves might have been predicted from those of figure 4. The maximum tension at the higher end occurs when the load is at the center of the span. The maximum point in the other two curves is slightly displaced toward the lower end.

Although these curves give the relation between the position of the load and the tensions for a *special* case, they may be safely taken as a guide in predicting in *general* the conditions of maximum tensions in the cable. It is evident that if the vertical distance between the end supports of the cable are less than the value taken above (one-fourth of the span) the tension curves will be more nearly symmetrical, and when the two supports are at the same level they will be perfectly symmetrical about an ordinate thru a point corresponding to the center of the span. Hence for all cases in which the difference in level between the ends is one-fourth the length of the span or less, the maximum point on the tension curves will occur at a position corresponding to the center of the span.

For greater differences in height between the ends, the maximum point may be displaced a little toward the lower end. But as the curves are rather flat on top, no great error will be involved in assuming that *the maximum tension in each branch of the cable occurs when the load is at the center of the span.* This conclusion is of the greatest importance, for as it has been shown the calculation of the tensions is the simplest when the load is at the center of the span, that is when $x_1=x_2$; and when this computation is made we may be certain that the *maximum tension* is known.

Futhermore, the conclusion stated above enables us to examine the most favorable conditions for the use of a track cable. As was previously stated, the use of a horizontal tension of one-fourth the breaking strength is inconsistent with a large factor of safety. Mark's Handbook of Mechanical Engineering recommends a factor of safety of 4. This is absolutely impossible with a horizontal tension of one-fourth the breaking strength. The effect of the sag and the horizontal tension on the load and factor of safety is shown in table 1.

Table 1.

Sag	*Horizontal Tension.*	*Load*	T_1	T_2	*Factor of Safety*
1/20th	1/4th	31,946	56,726	68,200	3.29
1/10th	1/4th	41,800	63,150	70,800	3.17
1/20th	1/5.25th	30,602	43,300	56,000	4.0
1/10th	1/5.25th	33,700	47,900	56,000	4.0

With a sag of one-twentieth the horizontal span and a horizontal tension of one-fourth the breaking strength, the factor of safety is 3.29, which is lower than is advisable to use. By using a greater sag, one-tenth of the span, and the same horizontal tension the load will be increased 30%. But this decreases the factor of safety to 3.17. By keeping the sag one-twentieth of the span and making the factor of safety 4 the load is reduced by 4.2 per cent and the horizontal tension to a little less than one-fifth of the breaking strength. If a greater sag is used, say one-tenth of the span, the load is greater by 5.25 percent, and the horizontal tension is the same.

Hence if it is desired to use a factor of safety of 4, the sag should be about one-tenth of the span and the horizontal tension one-fifth of the breaking strength. If it is necessary to keep the sag small because of the topography of the ground, the horizontal tension should be made as small as possible and still keep the load at a sufficient height above the ground.

It will be observed that in the above discussion no consideration is given to the bending stress which the cable undergoes in passing around pulleys. Of course this is an important feature of cable transportation and should be given proper weight in the design of a track cable, but it is not a part of this particular problem which treats only of stresses due to loading.

IV

Directions for Computing the Load.

In conclusion the proceedure for finding the safe load that a track cable will carry will be outlined. It is assumed that the following data is given:

(a) Breaking strength of the cable.

(b) Linear density.

(c) Horizontal span.

(d) Vertical distance between end fastenings.

(1) Take the horizontal tenions to be one-fifth of the breaking strength and compute the parameter, c, by

$$c = \frac{T_h}{w} \tag{11}$$

(2) Take the sag at the center of the span to be one-tenth, (or as near that as feasible) of the span. This gives the ordinate y_1 of the lower end. The ordinate y_2 of the upper end is y_1 plus the vertical height between the two ends. The absicassas, x_1 and x_2, are equal and each equal to one-half of the span.

(3) Compute the length of each part of cable between the load and the lower end, and between the load and the upper end respectively, i.e., s_1 and s_2, by

$$\sqrt{s^2 - y^2} = x \frac{\sinh \phi}{\phi} \tag{58}$$

(4) The constant s_{o1} and s_{o2} may now be computed by

$$s_o = \frac{y}{2} \coth\phi - \frac{s}{2} \tag{53}$$

(5) The load is given by

$$W = w(s_{o1} + s_{o2}) \tag{46}$$

(6) The constants y_{o1} and y_{o2} may be found by

$$y_o = \sqrt{c^2 + s^2_o} \tag{42}$$

(7) The tension at each end, T_1 and T_2, respectively, may be computed by

$$T = w(y + y_o) \qquad (51)\ (52)$$

using y_1 and y_{o1} for T_1, and y_2 and y_{o2} for T_2. Since y_2 is taken greater than y_1, T_2 is the greater tension and the *maximum for any position of the load.*

Altho the computations required by the outline above are somewhat longer than those called for by the approximate formulas given in handbooks, they are by no means laborious, *and have the advantage of giving exactly the load which may be carried and the maximum tension involved.*

SUMMARY.

(a). When a cable is loaded at one point, the conformation which the cable assumes is that of two arcs of the same common catenary, with the point of intersection at point of loading.

(b). The equations of these two arcs have been derived. They involve three constants or parameters: (1) c, which is the parameter of the common catenary, of which the two arcs are part, and which depends implicitly upon the horizontal tension and linear density of the cable; (2) s_0, which is equal to the length of an arc of the common catenary extending from the lowest point of the common catenary to the point of intersection (the loaded point) of the two arcs which represent the actual conformation of the cable, and which may be called the "fictitious arc" and depends upon the load and the linear density of the cable; (3) y_0, which is dependent upon the other two parameters, and is the vertical distance of the loaded point above the directrix of the common catenary.

(c). Expressions for the horizontal tension, vertical tension and total tension at *any* point in the cable have been derived.

(d). It has been shown that in either one of the arcs the point of maximum tension is the highest point, i.e., the point of fastening at the end, and that in most cases the higher end of the cable has the greater tension.

(e). It has been shown for a special case that the maximum tension occurs in the cable when the load is at the center of the span, and that the point of maximum tension is the higher end of the cable. Furthermore, it has been shown that very probably the maximum tension *always* occurs, in a catenary loaded at one point, when the load is at the center of the span.

(f). A method of computing *exactly* the load and tensions in a given track cable has been outlined in section IV. To the logging engineer this will probably prove the most interesting and useful part of the paper.