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A Test for Conditional Heteroskedasticity In Time Series Models

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A Test for Conditional Heteroskedasticity
in Time Series Models

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ABSTRACT

While testing for conditional heteroskedasticity and nonlinearity, the power of the test in general depends on the functional forms of conditional heteroskedasticity and nonlinearity that are allowed under the alternative hypothesis. In this paper, we suggest a test for conditional heteroskedasticity/nonlinearity with the nonlinear autoregressive conditional heteroskedasticity (NARCH) model of Higgins and Bera (1989) as the nonlinear ARCH parameter is not identified under the null hypothesis. To resolve this problem, we apply the procedure recently proposed by Davies (1987). Power and size of the suggested test are investigated through simulation and an empirical application of testing for ARCH in exchange rates is also discussed.

Keywords. ARCH; bilinear; Davies' test; NARCH; Nonlinearity; Nonlinear time series models.

1. INTRODUCTION

Autoregressive conditional heteroskedasticity (ARCH), introduced by Engle (1982), is frequently used to model the changing volatility of economic time series. Such applications of the ARCH model can be found in Weiss (1984) and papers surveyed in Engle and Bollerslev (1986). When testing for conditional heteroskedasticity, the form of the test statistic, and hence the power of the test, in general depends on the functional form of the conditional variance specified under the alternative hypothesis. In this paper, we suggest a test in which the alternative is the nonlinear ARCH (NARCH) model proposed by Higgins and Bera (1989b). Consider the dynamic linear regression model

$$y_t = x_t'\beta + \epsilon_t \qquad (t = 1,\ldots,T) \tag{1.1}$$

where x_t is a vector of k predetermined variables which may include lagged values of the dependent variable y_t. Let Φ_t denote the information set at time t which includes current and all lagged values of ϵ_t. The error ϵ_t is generated by the ARCH process

$$\epsilon_t | \Phi_{t-1} \sim N(0, h_t)$$

where h_t is a function of the elements of Φ_{t-1}. Engle (1982) proposes several functional forms for h_t, but emphasizes the linear ARCH model

$$h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \ldots + \alpha_p\epsilon_{t-p}^2 \tag{1.2}$$

for its analytic convenience and ease of interpretation. Other functional forms, however, have been found to be useful [See Engle and Bollerslev (1986), Higgins and Bera (1989b), and Nelson (1989)]. For example, the linear ARCH model (1.2) requires $\alpha_0 > 0$ and $\alpha_i \geq 0$ $(i = 1,\ldots,p)$ to insure that the conditional variance is strictly positive. Geweke (1986) and Pantula (1986) suggest the logarithmic specification

$$\log(h_t) = \alpha_0 + \alpha_1 \log(\epsilon_{t-1}^2) + \ldots + \alpha_p \log(\epsilon_{t-p}^2) \tag{1.3}$$

for which the conditional variance is positive for all parameter values.

The Lagrange multiplier (LM) or score test principle provides an easily computed test for the presence of linear ARCH. Engle (1982) shows that the LM statistic for the null hypothesis H_0: $\alpha_1 = \ldots = \alpha_p = 0$ in (1.2) is equivalent to $T \cdot R^2$, where R^2 is the squared uncentered multiple correlation coefficient of the regression of $\tilde{\epsilon}_t^2$ on an intercept and $\tilde{\epsilon}_{t-i}^2$ $(i = 1,\ldots,p)$ and the $\tilde{\epsilon}_t$'s are the least squares residuals of (1.1). In general, however, the form of the LM statistic depends on the functional form of the ARCH process assumed under the alternative hypothesis. Conducting the above test for ARCH when the true alternative is, for example, the logarithmic model (1.3), may result in a significant loss of power. Furthermore, there is a general view that linear ARCH models do not provide a rich enough class of nonlinearities, and this necessitates a need for a more flexible parametric specification for the conditional heteroskedasticity [see Pagan and Wickens (1989, p. 983)]. In this paper we propose a test for ARCH in which the alternative is the nonlinear ARCH (NARCH) model suggested by Higgins and Bera (1989b). The NARCH model of order p, NARCH(p), specifies the conditional variance function

$$h_t = \left[\phi_0(\sigma^2)^\delta + \phi_1(\epsilon_{t-1}^2)^\delta + \dots + \phi_p(\epsilon_{t-p}^2)^\delta\right]^{1/\delta} \quad (1.4)$$

with the parameter space restricted to

$$\sigma^2 > 0; \quad \phi_i \geq 0, \ (i = 1,\dots,p); \quad \delta > 0; \quad \sum_{i=0}^{p} \phi_i = 1.$$

Rearranging (1.4)

$$\frac{h_t^\delta - 1}{\delta} = \phi_0 \frac{(\sigma^2)^\delta - 1}{\delta} + \phi_1 \frac{(\epsilon_{t-1}^2)^\delta - 1}{\delta} + \dots + \phi_p \frac{(\epsilon_{t-p}^2)^\delta - 1}{\delta}$$

the NARCH model is seen to be a Box-Cox power transformation of the terms of the linear ARCH model (1.2). The Box-Cox transformation is widely used in the selection of functional form of the mean of a regression model. In the present context, the NARCH model encompasses many of the functional forms used for ARCH. For example, when $\delta = 1$ the model is identical to the linear ARCH model (1.2). As δ approaches 0 from above, the model is equivalent to Geweke's logarithmic model (1.3). Higgins and Bera (1989b) discuss other ARCH specifications which NARCH encompasses. Furthermore, by adding Box-Cox transformations of lagged values of h_t to the right hand side of (1.4), the model can easily be generalized to include the GARCH model of Bollerslev (1986, 1988).

The null hypothesis to be tested is $H_o: \phi_1 = \ldots = \phi_p = 0$. When these conditions are imposed, the conditional variance function (1.4) reduces to a constant and the model becomes the standard normal regression model. It is immediately noticed, however, that when the null hypothesis is imposed, δ drops out of the conditional variance function h_t. In other words, the nuisance parameter δ is identified only under the alternative hypothesis. It can be shown that under H_0, the information matrix is singular; thus invalidating the standard formulation of the LM test. Watson and Engle (1985) encounter the same problem in testing the constancy of a regression coefficient against the alternative that the parameter follows a first order autoregressive process. They consider the varying coefficient model

$$y_t = x_t'\gamma + z_t\beta_t + \epsilon_t$$

where x_t is a vector of exogenous variables, γ is vector of fixed parameters, z_t is an exogenous scalar and ϵ_t is a random disturbance. The stochastic parameter β_t is generated by

$$(\beta_t - \bar{\beta}) = \phi(\beta_{t-1} - \bar{\beta}) + u_t \qquad |\phi| < 1.$$

where ϕ and $\bar{\beta}$ are fixed parameters and u_t is a random disturbance with variance q. Since the unconditional variance of β_t is $q/(1-\phi^2)$, constancy of the parameter β can be examined by testing $q = 0$. When $q = 0$, however, ϕ is not identified. In order to proceed, they follow the suggestion of Davies (1977) and base a test on Roy's union-intersection principle. Like Watson and Engle (1985), we follow Davies (1977), and in addition make use of approximations provided by Davies (1987) to simplify the computation of p-values for the test. In Section 2, we briefly review the work of Davies (1977, 1987) and discuss its application to our testing problem. In Section 3 we derive the LM test required to implement Davies' procedure and in Section 4 we report Monte Carlo results on the finite sample null distribution and power of the proposed test. To illustrate the usefulness of the test, in Section 5 we present an application of testing for nonlinear ARCH in foreign exchange rates. Section 6 contains a few concluding remarks.

2. DAVIES' TEST

Davies (1977) considers a situation in which the density of the sample depends on two parameters α and θ. It is desired to test the hypothesis $\alpha = 0$ against the alternative $\alpha > 0$. It is assumed that, for a given value of θ, an appropriate Gaussian test statistic Z is available to test $\alpha = 0$. When the null hypothesis is true, however, it is assumed that the model is free of θ. In such a situation, in which the nuisance parameter θ is not identified under the null hypothesis, the asymptotic distribution theory of Z is invalid. Since the distribution of Z is correct for any arbitrarily assigned value of θ, Davies appeals to the "union-intersection principle" of Roy (1953) and suggests basing the test on a critical region of the form

$$\{\sup_{\theta} Z(\theta) > c\}.$$

Although the distribution of this test is unknown, Davies provides approximations for computing the p-values. The approximations require performing numerical integration of the continuous time autocorrelation function of $Z(\theta)$. Davies (1987) extends his results to statistics which are asymptotically χ^2 and provides a much simpler approximation for the p-values. It is this approximation which we make use of below.

To apply the results of Davies (1987), we first fix δ at an arbitrary value δ^* and derive a test for $\phi_1 = \ldots = \phi_p = 0$. In Section 3, we show that given $\delta = \delta^*$, the LM test statistic is

$$S(\delta^*) = T \cdot R^2$$

where R^2 is the squared multiple correlation coefficient of the regression of $\tilde{\epsilon}_t^2$ on an intercept and

$$\frac{(\tilde{\epsilon}_{t-i}^2)^{\delta^*} - 1}{\delta^*} \qquad (i = 1, \ldots, p),$$

where $\tilde{\epsilon}_t$ is the least squares residual. The actual test statistic is defined as

$$S = \sup_{\delta^*} S(\delta^*). \tag{2.1}$$

However, unlike $S(\delta^*)$, S will not have an asymptotic χ_p^2 distribution under the null hypothesis. Of course, it will be very difficult to find the exact critical values or p-values for S. Davies (1987) suggests an upper bound of the p-values which is described below.

For each value of $\delta \in \Delta \subset R^{+}$, we can express $S(\delta)$ as

$$S(\delta) = \sum_{i=1}^{p} Z_i^2(\delta) = Z'(\delta)Z(\delta)$$

where $Z(\delta) = (Z_1(\delta), Z_2(\delta), \ldots, Z_p(\delta))'$ is a $p \times 1$ vector. Under certain regularity conditions and the null hypothesis H_0: $\phi_1 = \ldots = \phi_p = 0$, $Z_i(\delta)$'s are asymptotically i.i.d. N(0,1). Define $Y(\delta) = \partial Z(\delta)/\partial\delta$ and denote $Var[Y(\delta)] = B(\delta)$ and $Cov[Z(\delta),Y(\delta)] = A(\delta)$. Let $\lambda_1(\delta)$, $\lambda_2(\delta)$, ..., $\lambda_p(\delta)$ be the eigenvalues of $B(\delta) - A'(\delta)A(\delta)$ and let $\eta(\delta) \sim N(0,\Lambda)$, where $\Lambda = diag(\lambda_1(\delta), \lambda_2(\delta), \ldots, \lambda_p(\delta))$.

Under the above setup, Davies (1987, p. 35) shows that

$$Pr[\{Sup\ S(\delta);\ \delta \in \Delta\} > u] \leq Pr(\chi_p^2 > u) + \int_\Delta \psi(\delta)d\delta$$

where

$$\psi(\delta) = E\left[\eta'(\delta)\eta(\delta)\right]^{1/2} \cdot \frac{e^{-u/2}u^{(p-1)/2}}{\pi^{1/2}2^{p/2}\Gamma((p+1)/2)}. \qquad (2.2)$$

As proved in Davies (1987, Theorem A.1)

$$\int_\Delta \psi(\delta)d\delta$$

is the expected number of upcrossing of the level u by $S(\delta)$ for $\delta \in \Delta$. This can also be viewed as the correction factor to the standard χ^2 p-value due to the scanning across a range of values of $\delta \in \Delta$. Theorem A.2 of Davies (1987) further shows that

$$E[\eta'(\delta)\eta(\delta)]^{1/2} = E[|\partial S^{1/2}(\delta)/\partial\delta|]\cdot\frac{\pi^{1/2}\Gamma((p+1)/2)}{\Gamma(p/2)}. \quad (2.3)$$

Combining (2.2) and (2.3), the upper bound of the significance level is given by

$$\Pr(\chi_p^2 > u) + \frac{e^{-u/2}u^{(p-1)/2}}{2^{p/2}\Gamma(p/2)}\int_\Delta E[|\partial S^{1/2}(\delta)/\partial\delta|]d\delta. \quad (2.4)$$

Davies proposes to estimate

$$\int_\Delta E[|\partial S^{1/2}(\delta)/\partial\delta|]d\delta$$

from the total variation

$$\begin{aligned} V &= \int_\Delta |\partial S^{1/2}(\delta)/\partial\delta|d\delta \\ &= |S^{1/2}(\delta_1) - S^{1/2}(L)| + |S^{1/2}(\delta_2) - S^{1/2}(\delta_1)| + \ldots \\ &\quad + |S^{1/2}(U) - S^{1/2}(\delta_M)| \end{aligned} \quad (2.5)$$

where L and U are the lower and upper bounds for δ and $\delta_1, \delta_2, \ldots, \delta_M$ are the turning points of $S^{1/2}(\delta)$. Therefore, from (2.4), the significance level of our test based on S will be approximately

$$\Pr(\chi_p^2 > S) + V\cdot\frac{e^{-S/2}S^{(p-1)/2}}{2^{p/2}\Gamma(p/2)}. \quad (2.6)$$

Although (2.6) is only an approximation, we expect it to perform better than basing the test on just the first term in (2.6). In the second term, one part is essentially the χ_p^2 density function and the other part, V, reflects the variation in $S^{1/2}(\delta)$ over values of δ corresponding to different alternative hypotheses. Davies (1987) presents numerical results which show that this type of approximation performs very well.

Here we should note that the set Δ need not coincide with the theoretical range for δ; it could be any subset of that range. The only constraint is that the same set should be used for maximizing $S(\delta)$ in (2.1) and in calculating V in (2.5). Also, Davies (1977, p. 253) mentions that for the the procedure to be useful, $S(\delta)$ cannot have spurious peaks. To see that spurious peaks of $S(\delta)$ is not very likely in our case, let us write $S(\delta)$ as

$$T \cdot R^2 = T \cdot \frac{\xi' W(W'W)^{-1} W' \xi}{\xi' \xi}$$

where ξ is the vector of $\tilde{\epsilon}_t^2$ and W includes an unit vector and Box-Cox transformations of $\tilde{\epsilon}_{t-1}^2, \tilde{\epsilon}_{t-2}^2, \ldots, \tilde{\epsilon}_{t-p}^2$. Therefore, this is a standard Box-Cox regression with transformation only in the non-constant independent variables. Later, in the simulation study, we present graphs of realizations of $S(\delta)$ under both the null and alternative hypotheses which indicate that $S(\delta)$ is in fact very smooth.

To see the behavior of the test under H_0 and the alternative hypothesis H_A: at least one $\phi_i \neq 0$, let us first note that

$$\operatorname*{plim}_{T \to \infty} \frac{\xi' \xi}{T} < \infty, \qquad \operatorname*{plim}_{T \to \infty} \frac{W'W}{T} = \text{a finite non-null matrix}$$

and under H_0 that $\operatorname*{plim}_{T\to\infty} W'\xi/T = 0$ for any value of δ. While under the alternative hypothesis H_A, $\operatorname*{plim}_{T\to\infty} W'\xi \neq 0$ for any δ. Therefore, the test is consistent. However, we cannot claim that our test has any optimality property. For a weak optimality property of this kind of test, see Davies (1977, p. 252).

3. LM TEST FOR FIXED δ

We now derive the LM test with δ fixed at a pre-assigned value δ^*. The conditional variance function (1.4) becomes

$$h_t = \left[\phi_0(\sigma^2)^{\delta^*} + \phi_1(\epsilon_{t-1}^2)^{\delta^*} + \ldots + \phi_p(\epsilon_{t-p}^2)^{\delta^*}\right]^{1/\delta^*}.$$

Let $\phi' = (\phi_1,\ldots,\phi_p)$, $\nu' = (\sigma^2,\phi')$ and $\theta' = (\beta',\nu')$. The log-likelihood for the NARCH regression model can be written, omitting a constant,

$$\mathcal{L}(\theta) = -\frac{1}{2}\sum \log(h_t) - \sum \frac{\epsilon_t^2}{2h_t} \tag{3.1}$$

where the summations are over t. Higgins and Bera (1989b) show that the information matrix is block diagonal between the regression parameters β and the variance parameters ν. Furthermore, since H_0 does not impose restrictions on β, the LM test reduces to [see e.g. Bruesch and Pagan (1980)]

$$LM = d(\tilde{\theta})_\nu' I(\tilde{\theta})_{\nu\nu}^{-1} d(\tilde{\theta})_\nu \tag{3.2}$$

where $d(\theta)_{\nu}$ and $I(\theta)_{\nu\nu}$ are the score function and information matrix with respect to the variance parameters and "~" denotes quantities evaluated at the restricted maximum likelihood estimators (MLE's). Differentiating (3.1) with respect to the variance parameters, the score function is

$$\frac{\partial \mathcal{L}}{\partial \nu} = \sum \frac{1}{2h_t} \cdot \frac{\partial h_t}{\partial \nu} \left(\frac{\epsilon_t}{h_t} - 1 \right)$$

and the hessian of the log-likelihood is

$$\frac{\partial^2 \mathcal{L}}{\partial \nu \partial \nu'} = - \sum \frac{1}{h_t^2} \cdot \frac{\epsilon_t^2}{h_t} \cdot \frac{\partial h_t}{\partial \nu} \cdot \frac{\partial h_t}{\partial \nu'} + \sum \left(\frac{\epsilon_t^2}{h_t} - 1 \right) \cdot \frac{\partial}{\partial \nu'} \left(\frac{1}{2h_t} \cdot \frac{\partial h_t}{\partial \nu} \right).$$

The information matrix with respect to the variance parameters is then given by

$$I_{\nu\nu} = -E\left(\frac{\partial^2 \mathcal{L}}{\partial \nu \partial \nu'} \right)$$

$$= E\left[E\left(\frac{\partial^2 \mathcal{L}}{\partial \nu \partial \nu} \middle| \Phi_{t-1} \right) \right]$$

$$= - \sum \frac{1}{2h_t^2} \cdot \frac{E(\epsilon_t^2)}{h_t} \cdot \frac{\partial h_t}{\partial \nu} \cdot \frac{\partial h_t}{\partial \nu'} + \sum \left(\frac{E(\epsilon_t^2)}{h_t} - 1 \right) \cdot \frac{\partial}{\partial \nu'} \left(\frac{1}{2h_t} \cdot \frac{\partial h_t}{\partial \nu} \right)$$

$$= \frac{1}{2} \sum E\left(\frac{1}{h_t^2} \cdot \frac{\partial h_t}{\partial \nu} \cdot \frac{\partial h_t}{\partial \nu'} \right)$$

which can be consistently estimated by

$$I_{\nu\nu} = \frac{1}{2} \sum \left(\frac{1}{h_t} \cdot \frac{\partial h_t}{\partial \nu}\right)\left(\frac{1}{h_t} \cdot \frac{\partial h_t}{\partial \nu'}\right).$$

Therefore, the LM statistic (3.2) is

$$LM = \frac{1}{2}\left[\sum \frac{\partial \tilde{h}_t}{\partial \nu}\left(\frac{\tilde{\epsilon}_t^2}{\tilde{\sigma}^2} - 1\right)\right]' \left[\sum \frac{\partial \tilde{h}_t}{\partial \nu} \cdot \frac{\partial \tilde{h}_t}{\partial \nu'}\right]^{-1} \left[\sum \frac{\partial \tilde{h}_t}{\partial \nu}\left(\frac{\tilde{\epsilon}_t}{\tilde{\sigma}^2} - 1\right)\right] \tag{3.3}$$

where $\tilde{\epsilon}_t$ is the least squares residual and $\tilde{\sigma}^2$ is the usual MLE of the variance of the error in the standard normal linear regression model. Now define f to be a column vector whose elements are $(\tilde{\epsilon}_t^2/\tilde{\sigma}^2) - 1$ $(t = 1,\ldots,p)$ and let $z_t = \partial\tilde{h}_t/\partial\nu$ and $Z' = (z_1,\ldots,z_T)$. The LM statistic (3.3) can then be expressed in matrix form as

$$LM = \frac{1}{2} \cdot f'Z(Z'Z)^{-1}Z'f$$

which is 1/2 the regression sums of squares from the regression of f on Z. Furthermore, since under the null hypothesis $\underset{T\to\infty}{\text{plim}}\ f'f/T = 2$ and the arithmetic mean of the elements of f is 0, an asymptotically equivalent form of the test statistic is

$$LM = S(\delta^*) = T \cdot \frac{f'Z(Z'Z)^{-1}Z'f}{f'f} = T \cdot R^2$$

where R^2 is the squared multiple correlation coefficient from the regression of f on Z. The elements of z_t are easily shown to be

$$\frac{\partial \tilde{h}_t}{\partial \sigma^2} = 1$$

and

$$\frac{\partial \tilde{h}_t}{\partial \phi_i} = \tilde{\sigma}^2 \left(\frac{(\epsilon_{t-i}^2/\tilde{\sigma}^2)^{\delta^*} - 1}{\delta^*} \right) \qquad (i = 1,\ldots,p).$$

Since a linear transformation of the variables in a regression does not affect the R^2, the test can be computed by regressing $\tilde{\epsilon}_t^2$ on an intercept and

$$\frac{(\tilde{\epsilon}_{t-i}^2)^{\delta^*} - 1}{\delta^*} \qquad (i = 1,\ldots,p).$$

The independent variables of the auxiliary regression are seen to be Box-Cox transformations of $\tilde{\epsilon}_t^2$ where δ^* is the Box-Cox parameter. When $\delta^* = 1$, the statistic is equivalent to Engle's ARCH test. As $\delta^* \longrightarrow 0$, the test would be based on the regression of $\tilde{\epsilon}_t^2$ on an intercept and $\log(\tilde{\epsilon}_{t-i}^2)$ $(i = 1,\ldots,p)$. This limiting case corresponds to the test for ARCH when the alternative is the logarithmic model (1.2). If a regression package is available for estimating a Box-Cox model with transformation only of the independent variables, S could be obtained very easily.

4. SIMULATION EXPERIMENTS

In this section we conduct a simulation study to determine the accuracy of the approximation (2.2). We also consider the power of Davies' test and Engle's ARCH test when the alternative model is in the class of NARCH models.

Lastly, we compare the power of these two tests under a bilinear alternative, to the LM test for that specific alternative.

To determine the accuracy of the approximate level of significance (2.6), for various sample sizes ranging from 25 to 200, 500 random normal samples were generated and Davies' test for NARCH (D-N) was computed. Therefore, the maximum standard error of the estimates of type 1 error probabilities and power in the following tables would be $\sqrt{.5\times.5/500} \approx .022$. The D-N test is based on the alternative hypothesis that the series is generated with a conditional mean zero and NARCH(1) heteroskedasticity, that is

$$y_t|\Phi_{t-1} \sim N(0,h_t) \tag{4.1}$$

where

$$h_t = \left[\phi_0(\sigma^2)^\delta + \phi_1(y_{t-1}^2)^\delta\right]^{1/\delta}. \tag{4.2}$$

Computing the approximate level of significance requires finding the supremum of $S(\delta)$ and the total variation of $S^{1/2}(\delta)$ over the permissible range of δ. Since the parameter space only imposes $\delta > 0$, an upper bound for δ must be chosen. We only present results for $0 < \delta < 2$. Below we discuss the consequences of varying the upper bound. The supremum of $S(\delta)$ and the turning points of $S^{1/2}(\delta)$ were found using a grid search with step length .01. Davies' test provides only an approximate p-value for the statistic. To examine the quality of this approximation, we choose a nominal level of significance, then compute the p-value for each sample. When a computed p-value is less than the nominal significance level, a rejection is recorded. Estimates of the type 1 error probabilities are obtained by counting the number of times the null hypothesis is rejected and dividing by 500. We then compare the those estimates to the chosen nominal significance level.

In Table 1, we present the estimates of the type 1 error probabilities of D-N for different sample sizes and different nominal levels of significance. For comparison, we also report the corresponding values for Engle's LM test for ARCH (LM-A) against an ARCH(1) alternative. A rejection for LM-A is recorded when the computed value of the statistic exceeds the χ_1^2 critical value determined by the nominal level of significance. The results for D-N indicate that the approximation (2.6) works well. All estimated probabilities are quite close to the specified nominal significance levels. The quality of the approximation for D-N is certainly no worse than the approximation provided by the asymptotic distribution theory for LM-A.

We also give in Table 1 estimates of probabilities of type 1 error obtained by comparing the supremum of $S(\delta)$ to the χ_1^2 critical value for the specified nominal significance level. This is the test which results from omitting the second term in (2.6). The simulations indicate, as expected, that the null hypothesis is rejected too frequently when the simple χ_1^2 critical value is used. Figure 1 is a plot of the the χ_1^2 density function and a nonparametric estimate of the density function of D-N based on the 500 samples of size 100 from Table 1. A kernel estimator was used with a window width of 1 and a biweight kernel function [see Silverman (1986)]. As seen in Figure 1, the density of D-N is skewed further right than the χ_1^2 density.

Again, this indicates that a critical value based on the χ_1^2 density will lead to too frequent rejection of the null hypothesis.

The supremum search of $S(\delta)$ was conducted over the interval $0 < \delta < 2$. All quantities in Table 1 were also computed using 5 and 10 as an upper bound for δ. The results indicated that the choice of the upper bound does not affect the quality of the approximation of the significance level of D-N. The performance of the test using the χ_1^2 critical value became worse. Since the supremum of $S(\delta)$ cannot decrease when the upper bound of δ increases, the likelihood of rejecting the null can get larger as the upper bound for δ increases.

To determine the power of D-N, samples were generated from the model (4.1) and (4.2). Experiments were conducted with points in the parameter space at $\phi_1 \in \{.3,.5,.8\} \times \delta \in \{.01,.1,.3,.5,.8,1.0,1.5\}$. To reduce the computational burden, the step length for both the supremum search of $S(\delta)$ and the computation of the total variation of $S^{1/2}(\delta)$ was increased to .025. Again, all experiments were based on 500 replications. We also compute the empirical power of LM-A for comparison. Results for samples of size 50, 100 and 150 are presented in Table 2.

The results indicate that D-N will significantly increase the ability to detect conditional heteroskedasticity when the data are generated under NARCH. For a given sample size T and given value of ϕ_1, as δ declines to 0, that is as the alternative moves away from the linear ARCH model, the power of D-N systematically increases relative to LM-A. In some instances, there is a gain in power of more than 15%. Equally important, when $\delta = 1$, that is when the true model is precisely Engle's linear ARCH model, there is little if any loss in power from using the D-N test relative to LM-A.

Regarding the computation of the D-N test statistic $S = \sup S(\delta)$, as we mentioned earlier, it would be undesirable if $S(\delta)$ possesses spurious peaks. In Figures 2 and 3, we present plots from two random replications under the null and alternative hypotheses, respectively. In Figure 2, although $S(\delta)$ has a clear maximum, the graph is somewhat flat. This may be due to the fact that δ is not identified under the null hypothesis. In Figure 3, $S(\delta)$ has a maximum very close to the true value .5. The value of δ for which $S(\delta)$ is a maximum, provides an empirical estimate of δ. For maximum likelihood estimation after the testing this could provide a starting value.

It would also be desirable if the D-N test has good power against other types of nonlinear models. Recently there has been interest in the ability of different tests to detect a variety of nonlinear models [see Keenan (1985), and Luukkonen, Saikkonen and Teräsvirta (1988)]. In a preliminary attempt to investigate this possibility, we generated samples from the single term bilinear model

$$y_t = \beta y_{t-2}\epsilon_{t-1} + \epsilon_t \tag{4.3}$$

where

$$\epsilon_t \sim N(0,1).$$

As discussed in Higgins and Bera (1989a), this process has second moments which are very similar to the ARCH model. In Table 3, we present the estimated power, based on 500 replications, of the D-M test for $\beta \in \{.1,.3,.5,.8\}$ and samples of size 50, 100 and 150. For comparison, we report the estimated power of LM-A and the LM test (LM-B) for the specific alternative given by (4.3). The results in Table 5 are self explanatory. When β is only of moderate size, $\beta \leq .5$, the power of D-N and LM-A are very similar. At $\beta = .8$, D-N does appear to be slightly more powerful than LM-A. Unfortunately, when compared to LM-B, neither test does well. This, however, is not unexpected since LM-B is based upon the specific alternative model given in (4.3). These preliminary results do indicate that D-N can detect types of nonlinearity other than those encompassed in the NARCH specification.

5. AN APPLICATION

We motivate our test by suggesting that the LM test for the linear ARCH model (LM-A) may not readily detect different kinds of nonlinearity and conditional heteroskedasticity. We suggest a LM test (D-N) based on a broader alternative, the NARCH model, which may be able detect a wider range of nonlinearity. To illustrate this possibility, in this section we examine the spot exchange rates between the U.S. dollar and the French franc (Ff), German mark (Gm), Italian lire (Il), Japanese yen (Jy), Swiss franc (Sf) and British pound (Bp). The data are monthly from January 1973 to December 1986. The series actually analyzed are the first differences of the logarithms center about their means. These particular series were chosen because their conditional means can be represented by a simple autoregressive (AR) model.

The sample autocorrelation and partial autocorrelation functions of the series indicate that an AR(1) process is a adequate model for the conditional mean of each of the series. Since the presence of conditional heteroskedasticity is anticipated, the significance of the autocorrelations were tested using standard errors and a portmanteau test robust to the presence of linear ARCH [see Milhøj (1985) and Diebold (1986)]. The AR(1) models were estimated by least squares and the least squares residuals were used to compute Engle's LM test for linear ARCH and our test for NARCH for orders 1 through 12. The p-values for each test statistic are reported in Table 4. Examining the p-values reveal that the two tests can give very different impressions about the presence of ARCH and nonlinearity. LM-A does not indicate any ARCH, at conventional levels of significance, for Ff, Il, Jy and Sf; however, D-N finds ARCH significant at the 10% level for at least one order for each of these series. For Gm and Bp, the two tests are in close agreement through all orders of the test.

To further illustrate that LM-A may fail to detect nonlinearity when the conditional heteroskedasticity is not linear, both the linear ARCH and NARCH models were estimated for the Il series. The Il series was chosen because the discrepancy between the two test seems greatest. The smallest p-value of LM-A is .43, while the p-values for D-N are less than .10 at orders 3, 4, 8, 11 and 12. Table 5 shows maximum likelihood estimates for the linear ARCH(3) model (L-ARCH) and the NARCH(3) models. Other order models were also estimated, but these gave the best fit. In spite of LM-A being insignificant at all orders, it is evident from Table 5 that some form of nonlinearity is present in Il. The parameter α_2 is significant in L-ARCH and the parameters α_2 and α_3 are significant in NARCH. The salient result in Table 5 is that the nonlinearity parameter δ is estimated as 4.00 in the NARCH specification. The high degree of nonlinearity in the conditional variance function may explain why LM-A does not detect conditional heteroskedasticity in this series.

6. CONCLUSIONS

Our Monte Carlo results present evidence that the approximation given by Davies (1987) is sufficiently accurate in small samples so as to be able to confidently use D-N. The power studies indicate that D-N can be significantly more powerful than the LM test for linear ARCH when the alternative is NARCH. As illustrated by our simulation study, this is particularly true when the nonlinearity parameter δ is quite small. From our experience in estimating NARCH models with exchange rate data, small values of δ are frequently encountered. Also in the empirical example presented here, LM-A could not detect heteroskedasticity when the estimated value of δ was quite high. However, the D-N test was able to pick up this nonlinear conditional heteroskedasticity. Hence, D-N should be a useful test when some type of conditional heteroskedasticity is suspected.

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TABLE 1

ESTIMATES OF THE TYPE 1 ERROR PROBABILITIES

SAMPLE SIZE	10%			5%			1%		
	D-N	LM-A	χ_1^2	D-N	LM-A	χ_1^2	D-N	LM-A	χ_1^2
25	.082	.056	.152	.038	.022	.092	.018	.008	.030
50	.066	.050	.178	.032	.016	.086	.006	.004	.024
75	.076	.082	.158	.032	.024	.090	.014	.008	.022
100	.072	.088	.150	.030	.034	.080	.010	.004	.016
150	.094	.078	.166	.056	.036	.114	.010	.012	.036
200	.102	.104	.196	.058	.044	.126	.020	.018	.038

TABLE 2

ESTIMATED POWER OF D-N AND LM-A AGAINST NARCH(1)

		ϕ_1					
		.3		.5		.8	
SAMPLE SIZE	δ	D-N	LM-A	D-N	LM-A	D-N	LM-A
50	.01	.466	.374	.876	.690	.998	.946
	.10	.404	.318	.806	.642	.994	.902
	.30	.334	.314	.640	.536	.936	.826
	.50	.298	.302	.542	.482	.890	.774
	.80	.276	.272	.524	.490	.790	.748
	1.00	.278	.292	.444	.448	.712	.660
	1.50	.350	.364	.494	.498	.694	.674
100	.01	.872	.668	1.000	.938	1.000	.994
	.10	.646	.538	.946	.796	1.000	.982
	.30	.488	.458	.858	.758	.998	.958
	.50	.402	.446	.792	.720	.962	.896
	.80	.376	.396	.720	.704	.932	.888
	1.00	.404	.424	.678	.680	.910	.868
	1.50	.448	.490	.662	.670	.840	.812
150	.01	.970	.838	1.000	.992	1.000	1.000
	.10	.746	.612	.990	.904	1.000	.996
	.30	.624	.566	.954	.858	1.000	.978
	.50	.544	.526	.888	.818	.998	.974
	.80	.482	.498	.814	.790	.980	.974
	1.00	.540	.546	.808	.808	.954	.936
	1.50	.606	.638	.818	.816	.938	.914

TABLE 3

ESTIMATED POWER OF D-N, LM-A AND LM-B AGAINST BILINEARITY

SAMPLE SIZE		β .1	.3	.5	.8
50	D-N	.100	.133	.167	.267
	LM-A	.033	.167	.133	.200
	LM-B	.167	.367	.567	.733
100	D-N	.038	.126	.304	.456
	LM-A	.034	.124	.332	.404
	LM-B	.154	.692	.938	.928
150	D-N	.050	.172	.404	.602
	LM-A	.056	.184	.436	.478
	LM-B	.224	.886	.994	.964

TABLE 4

P-VALUES OF LM-A AND D-N FOR EXCHANGE RATE DATA

ORDER OF ARCH	Ff		Gm		Il		Jy		Sf		Bp	
	LM-A	D-N	LM-A	D-N	LM-A	D-N	LM-A	D-N	LM-A	D-N	LM-A	D-N
1	.47	.10	.01	.00	.85	.24	.37	.46	.41	.32	.01	.02
2	.50	.16	.03	.01	.52	.33	.42	.14	.62	.48	.02	.05
3	.46	.14	.02	.01	.43	.06	.61	.15	.23	.05	.05	.12
4	.63	.30	.03	.01	.50	.08	.64	.06	.30	.08	.09	.21
5	.40	.13	.07	.07	.59	.13	.70	.11	.24	.03	.16	.33
6	.59	.21	.17	.13	.67	.11	.69	.19	.18	.04	.12	.26
7	.38	.11	.06	.05	.53	.11	.78	.17	.21	.05	.18	.36
8	.40	.12	.05	.05	.63	.09	.86	.23	.21	.05	.12	.26
9	.52	.18	.32	.35	.70	.11	.87	.32	.33	.11	.11	.22
10	.59	.24	.35	.34	.69	.18	.91	.44	.43	.17	.13	.14
11	.63	.66	.58	.45	.66	.09	.80	.22	.55	.16	.16	.20
12	.79	.67	.85	.61	.70	.09	.83	.17	.67	.22	.12	.29

TABLE 5

ESTIMATED MODELS FOR DOLLAR/LIRE EXCHANGE RATE*

L-ARCH:	ϕ	α_0	α_1	α_2	α_3	
	.396 (.095)	3.722 (.686)	.020 (.054)	.091 (.035)	.048 (.073)	
NARCH:	ϕ	σ^2	α_1	α_2	α_3	δ
	.397 (.0006)	3.723 (.0006)	.0001 (.0004)	.006 (.0006)	.0008 (.00006)	4.00 (.0005)

*Standard errors of the estimates are shown in parenthesis.

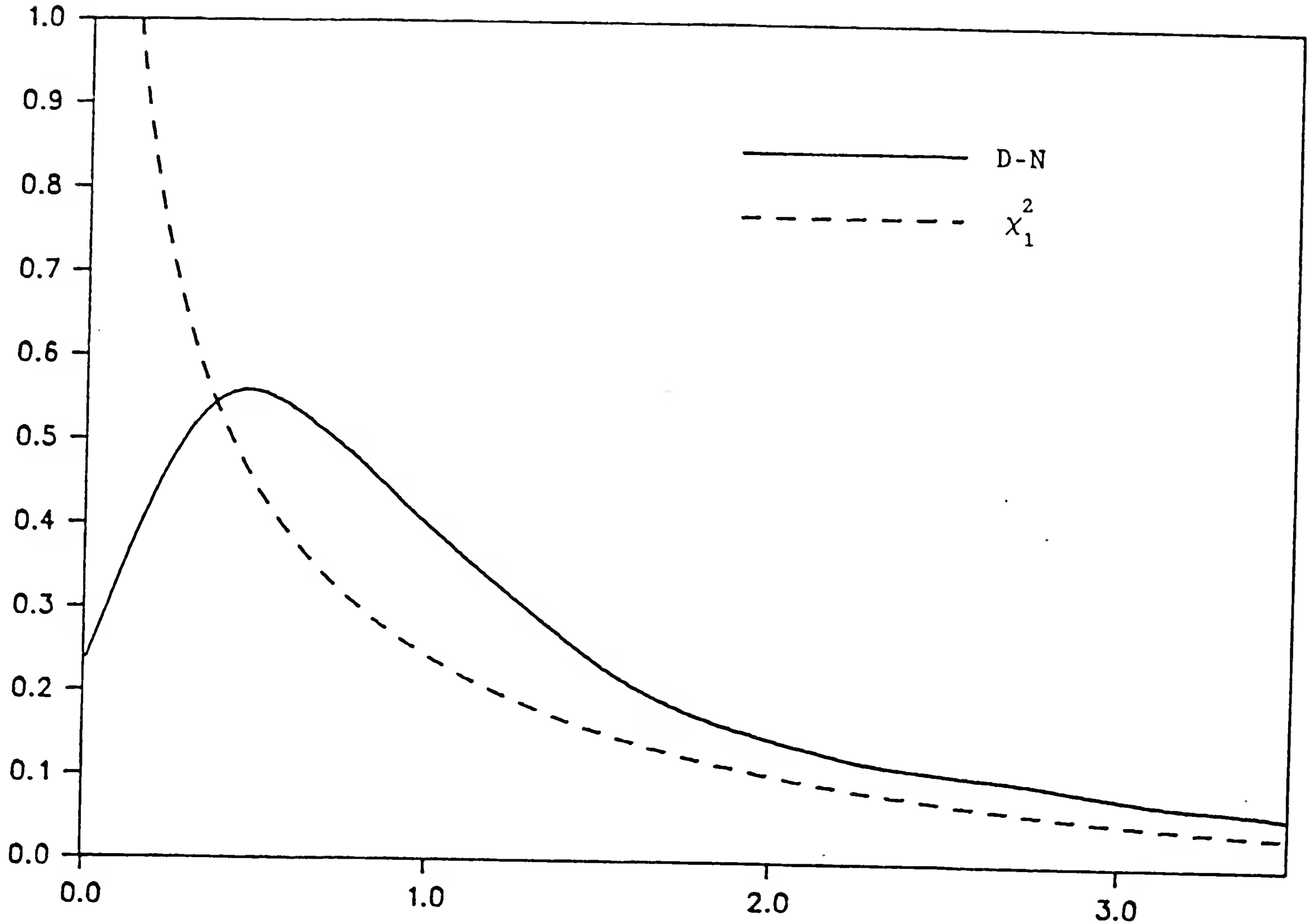

FIGURE 1. Nonparametric density estimate for D-N.

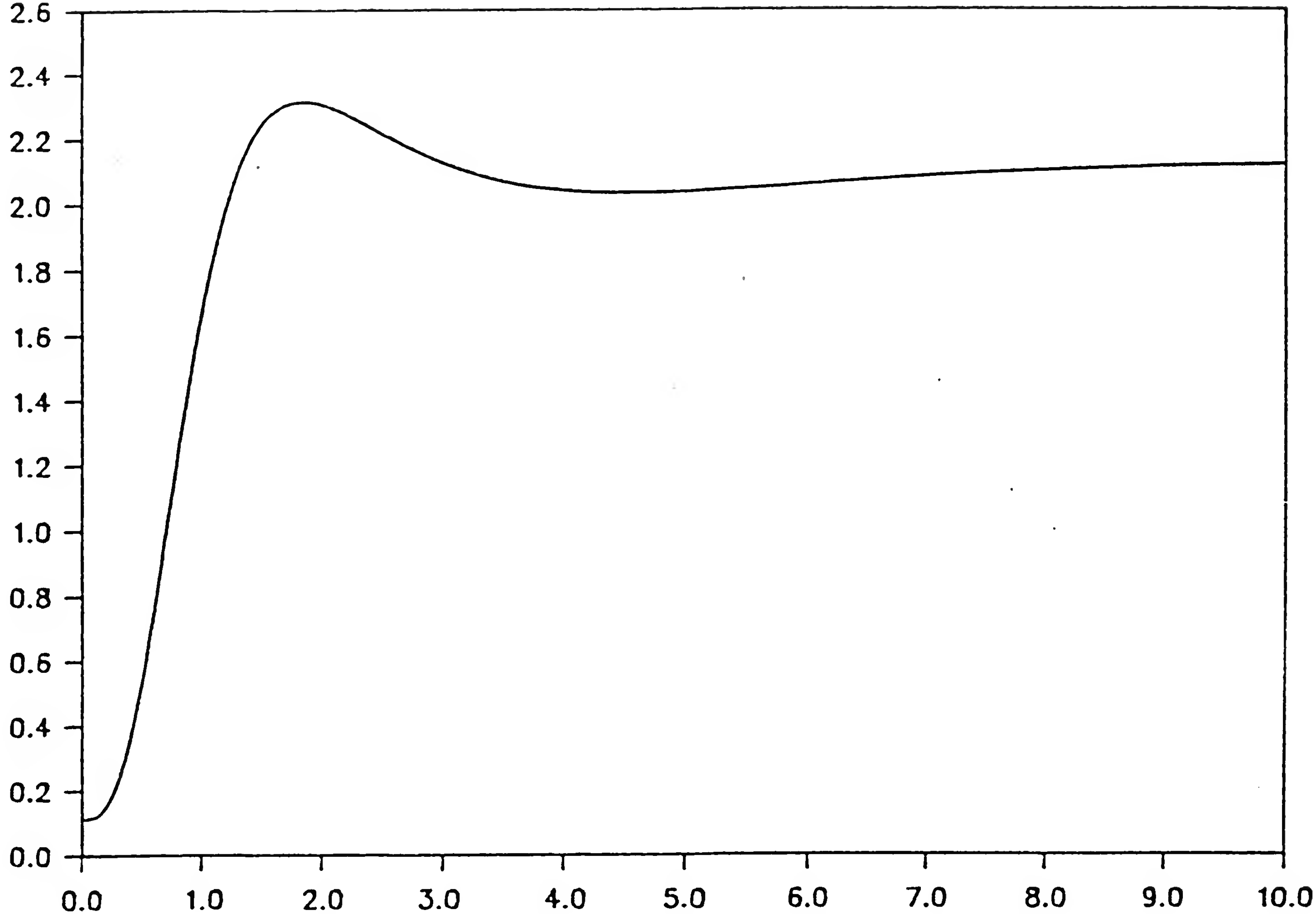

FIGURE 2. Plot of $S(\delta)$ under null hypothesis.

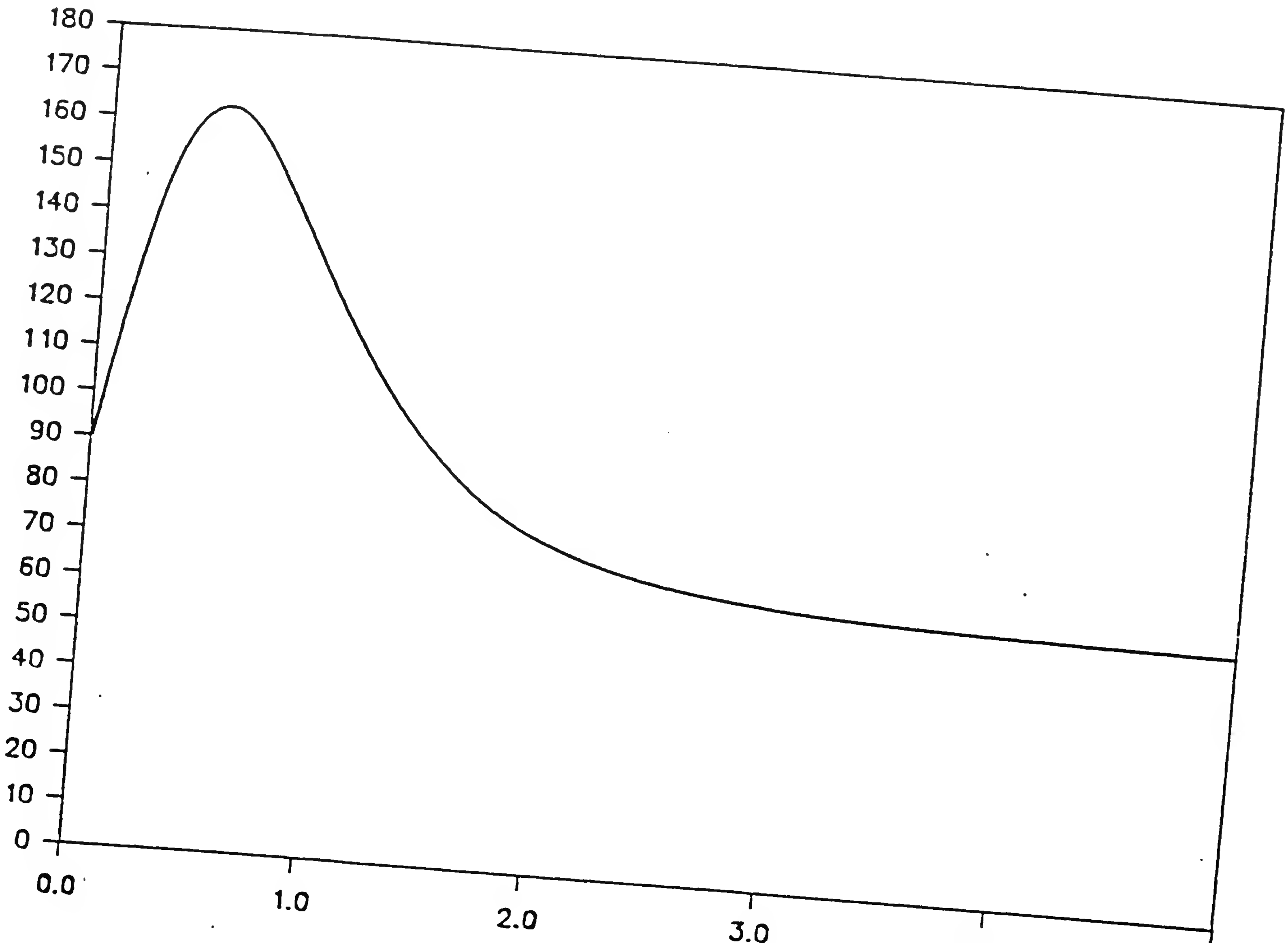

FIGURE 3. Plot of $S(\delta)$ under alternative hypothesis ($\phi = .8$, $\delta = .5$).

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