A TEXT-BOOK

OF

MATHEMATICS AND MECHANICS

SPECIALLY ARRANGED

FOR THE USE OF STUDENTS QUALIFYING FOR

SCIENCE AND TECHNICAL EXAMINATIONS.

BY

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With Numerous Diagrams and Worked-out Examples.

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PREFACE.

The present work has been compiled with the object of assisting candidates for scientific and higher technical examinations. But little difficulty is experienced in the teaching of engineering subjects to students who have already a sound knowledge of mathematics. Unfortunately, however, the engineering student in Great Britain lags behind his Continental colleague in regard to the interest he takes in mathematics.

An engineer who desires to economise his time should, first of all, acquire a thorough knowledge of the elements of the calculus. To omit to do so is to find out, when too late, that he is improperly equipped for his work. In the present volume it has been assumed that the student has already acquired a thorough knowledge of algebra, including logarithmic series, and of plane trigonometry, including the solution of triangles. He is also assumed to have the usual acquaintance with geometry and elementary mechanics necessary to a proper understanding of the higher branches of the subjects.

The author has had experience in the teaching of mathematics in this country, extending since 1880, having acted as an assistant to the late Professor W. E. Ayrton, F.R.S., in the City and Guilds of London Institute. He has, therefore, been more than a mere spectator, and he has been struck by the low standard of mathematical knowledge which is deemed necessary for entrance to Technical Institutions, and by the indifference which is shown by many British engineers to the value of a sound knowledge of mathematics as a preliminary to higher technical studies. In Germany and other Continental countries, on the other hand, the greatest attention is paid to such preliminary education, and, in view of the increasing competition which is rapidly becoming international in character, the author feels strongly that the tacit opposition encountered in so many quarters to the spread of mathematical education is of the nature of a rearguard action, and that those who adopt this attitude will inevitably have, in the long run, to give way. In the meanwhile, much valuable time is lost, and the sooner the truth is realised the better will it be for the commercial and industrial development of the country.

In the section dealing with Analytical Geometry, the author has deviated from the course usually followed by British writers of treating the straight line and the circle separately, and of dealing subsequently with the parabola, ellipse, and hyperbola jointly as conic sections. He does not consider that this proceeding is logically sound. It is more natural to treat the five curves separately, and eventually to show that they all belong to one category—the conic section.

The section of Mechanics has been treated entirely from a dynamical point of view, and has been written with the intention of avoiding, as far as possible, the unscientific and erroneous expressions still employed by writers of the present day, which have done much to impede progress. It has been said that, in the teaching of any subject, "precision in the use of words and cogency in modes of thought" should be cultivated. For these reasons, the author has laid great stress on the principle of physical dimensions, and has applied those principles uniformly throughout the book.

The numerous worked-out examples, over 250 in number, should prove of great assistance to the student. Most of them have been chosen from examination questions of the Associate Membership of the Institution of Civil Engineers, and the qualifying examination of the Mechanical Science Tripos, Cambridge. They have been treated in a general way, and may be regarded as appendices.

The author's thanks are due to his friend, Mr F. F. Burrell, M.A., for his assistance in reading and correcting proof sheets, and for many valuable suggestions; and to the publishers for having spared neither trouble nor expense in preparing the book for the press.

In conclusion, the author would be grateful for any corrections or suggestions for improvement of any subsequent edition that might be called for.

CHARLES CAPITO.

LONDON, *December* 1912.

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Notation.		Reference.
$y=$ang. $(\sin=x)$ $y=$ang. $(\cos=x)$ $y=$ang. $(\tan=x)$ $y=$ang. $(\cot=x)$	Inverse trigonometrical functions.	Article 64, p. 65.
[M], [L], [T]	Physical dimensions.	Article 149, p. 172.
s	Length of path.	Article 156, p. 180.
u	Speed.	
$u_m,\ u_i,\ u_f$	Average, initial, final speed.	
v	Velocity.	Article 157, p. 181.
$v_m,\ v_i,\ v_f$	Average, initial, final velocity.	
v_r	Relative velocity.	Article 159, p. 183.
ω	Angular velocity.	P. 184.
$a_t,\ a_n,\ a$	Acceleration.	P. 188.
F_n	Normal force.	P. 190.
F_e	Effort.	
F_r	Resistance.	
E_p	Potential energy.	Article 167, p. 192.
E_k	Kinetic energy.	
E_e	Energy exerted.	
lbw.	Weight of a pound.	Chapter XXVI. p. 197.
tnw.	Weight of a ton.	
kgw.	Weight of a kilogramme.	
gmw.	Weight of a gramme.	
lb.	Mass of a pound.	
kg.	Mass of a kilogramme.	
gm.	Mass of a gramme.	
g	Acceleration of gravity at Greenwich.	
μ	Coefficient of friction.	P. 208.
I	Second mass-moment or moment of inertia.	Article 185, p. 240.
R_g	Radius of gyration.	
$(a.m.)_p$ $(a.m.)_y$ $(a.m.)_x$	First area-moments.	Article 192, p. 246.
$(A.M.)_p$ $(A.M.)_y$ $(A.M.)_x$	Second area-moments.	
r_m	First mean radius.	P. 248.
R_m	Second mean radius.	
$(s.m.)$	First arc-moment.	Article 215, p. 275.
M.C.	Mass-centre.	P. 284.

ANALYTICAL GEOMETRY.

CHAPTER I.

ERRATA.

Page 82 ; *read* : $\frac{d}{dx}\text{ang}(\sin = x) = \frac{1}{\sqrt{1-x^2}}$, and $\frac{d}{dx}\text{ang}(\cos = x) = -\frac{1}{\sqrt{1-x^2}}$

,, 107 ; *read* : $\frac{d(\text{ii.})}{dx} = \ . \ . \ . = \frac{x(2x^2+3x+4)}{2(1+x+x^2)^{\frac{3}{2}}}$

,, 110, (29) ; *read* : $f''(0) = -1{\cdot}4757$; . . . $-0{\cdot}738x^2$ . . . (*Ans.*).

,, 164, (55) numerator of X ; *read* : $9x$ instead of 9.

,, 235, (89) ; *read* : $E_1 =$ . . . $= 414{,}100$ ft.-lbws. ; $F_f = 246$ lbws., or 49 lbws. . . . (*Ans.*).

,, 236, (91) ; *read* : $a = 0{\cdot}18$ ft./s.^{2} (*Ans.*).

,, 314, (120) ; *read* : $\pi d =$ . . . (*Ans.*).

,, 320, (127) ; *read* : $R_g = 2$ ft. ; $M = 2935 \times \frac{400 \times 60}{4 \times 120^2} = 1223$ lbs. (*Ans.*).

,, 349, (139) ; *read* : P = 424 lbws. ; power . . . = 19·8 H.P. (*Ans.*).

,, 349, (140) ; *read* : kinetic energy = 2·5 in.-lbws. (*Ans.*).

,, 365 ; *read* : one metric atmosphere = 1 kgw. per cm.^{2}

,, 388, at top ; *read* : $P =$ . . . $= \frac{A(A+a)}{a} wx$ (*Ans.*).

,, 392, (158) ; *read* : . . . channel is 10 feet wide . . .

ANALYTICAL GEOMETRY.

CHAPTER I.

INTRODUCTION.

1. Rectangular co-ordinates.

It will be assumed that the student is perfectly familiar with the elementary process of plotting graphs on squared paper from simple equations containing two variables, such as x and y, or from experimental data of a series of corresponding values of two variable quantities, which have been determined by trial.

A graph provides a useful and practical illustration of the law of variation of the two variables, which would often not be com-

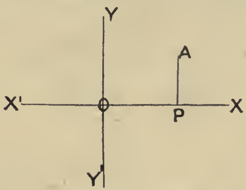


FIG. 1.

prehensible in any other way. The graph is also called the *locus*; it means the path traced by a point which moves in such a way as to always satisfy the conditions laid down in the equation.

For the purpose of drawing a graph two lines of reference, X′X and Y′Y (fig. 1), are conveniently chosen. The lines are termed the *axes of co-ordinates*, and in order to distinguish them X′X is called the *axis of abscissæ* or the *x-axis*; Y′Y is called the *axis of ordinates* or the *y-axis*. The point, O, at which the axes intersect is termed the *origin*. The angle X O Y is usually a right angle, and the system is then termed rectangular; but the system is called *oblique* if the angle X O Y differs from that of a right angle.

The position of a point A, referred to the axes, will be known when the length, b, of P A (the ordinate) drawn parallel to Y′ Y, and the length, a, of O P (the abscissa) are given; a and b are the co-ordinates of point A, and the position of the point is given by the symbol A(a, b).

The axes, however, divide the plane into four quadrants, and for the purpose of defining the quadrant in which the point is situated, the co-ordinates are not only given by their magnitudes, but also by their signs. It is conventional to take the directions X′ X and Y′ Y as positive, and the directions X X′ and Y Y′ as negative; hence the position of a point is perfectly determined by one of the following symbols, viz.:

1st quadrant (x, y); 2nd quadrant $(-x, y)$;
3rd quadrant $(-x, -y)$; 4th quadrant $(x, -y)$.

In the following, angle X O Y is understood to be a right angle, unless otherwise specified.

2. Polar co-ordinates.

The position of a point on a plane may often conveniently be referred to other systems of co-ordinates than the rectangular system.

A method which is often found useful consists in removing the y-axis. The position of a point, M (fig. 2), will be perfectly

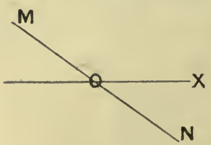


FIG. 2.

determined if the distance O M and the angle X O M are known.

The x-axis is then called the *fixed axis* or the *initial axis*, and the origin, O, is called the *pole*. The co-ordinates of point M are the length of O M—which is called the *radius vector*—and the angle X O M, which is called the *vectorial angle*; they are usually denoted by r and θ respectively. The position of a point referred to this system is denoted by (r, θ).

This method is known as the system of *polar co-ordinates.*

The vectorial angle is reckoned positive when measured in the anti-clockwise direction from O X.

The radius vector is positive when measured along the line bounding the vectorial angle, and it is negative when measured in the opposite direction.

Assuming that the distances O M and O N are equal, and the polar co-ordinates of point M are (r, θ), then those of point N will be $(-r, \theta)$.

If we choose angle X O N $= \theta + \pi$ as the vectorial angle, then the co-ordinates of the two points will be $M(-r, \theta+\pi)$ and $N(r, \theta+\pi)$.

The co-ordinates of a point may be changed from the one system into the other. Let the rectangular co-ordinates of point M be x and y, then it will be seen that

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta \qquad . \quad . \quad . \quad [1]$$

and conversely

$$r = \sqrt{x^2+y^2}, \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad . \quad . \quad . \quad [2]$$

For point N

$$x = -r\cos\theta, \quad y = -r\sin\theta,$$

and

$$r = -\sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}\cdot$$

3. Functions.

If a quantity (such as y) depends on another quantity (such as x), so that no change can be made in the value of the one (x) without making a corresponding change in the value of the other (y), then y is said to be a *function* of x. The usual symbol to express that y is a function of x is,

$$y = f(x) \qquad . \quad . \quad . \quad . \quad . \quad [3]$$

x in [3] is said to be the *independent variable* and y the *dependent variable*, because the value of y depends on the value we choose to give to x. In any equation involving x and y, y is a function of x if the latter is taken as the independent variable, or x is a function of y if y is taken as the independent variable.

The function is said to be *implicit* when expressed indirectly in terms of the independent variable, thus

$$x^2 - 4x + y^2 + 2y - 4 = 0 \qquad . \quad . \quad . \quad (1)$$

By solving (1) with respect to y, we have

$$y = -1 \pm \sqrt{5+4x-x^2} \qquad . \quad . \quad . \quad (2)$$

In (2) the function is expressed directly in terms of the independent variable, and it is said to be an *explicit* function.

The symbol for an implicit function is usually

$$f(x, y)=0 \quad . \quad . \quad . \quad . \quad . \quad [4]$$

whereas [**3**] is an explicit function.

The student need not be puzzled by the term "function." $f(x, y)=0$ stands practically for nothing else than a general symbol for an equation involving the two invariables x and y. He knows from experience, when plotting the graph of an equation, that he must choose one of the variables, say x, and assign a series of convenient values to it; he then works out the corresponding values of y. The value of y is thus dependent on the value he assigns to x.

4. Dimensions.

Geometrical quantities are measured by their dimensions in length.

A *curve* has no depth nor width and has therefore only one dimension; hence, it is measured by length, L, only. The symbol for the dimensions of a curve is [L].

A *surface* has two dimensions, width and length. It is always possible to construct a square whose area is equal to the area of the given surface; hence, a surface is measured, like the square, by L^2. The dimensions of a surface are $[L]^2$.

A mathematical *body* has three dimensions, length, width, and depth. There is always a cube whose volume is equal to that of the given body. A body is therefore measured, like the cube, by L^3. The dimensions of a mathematical body are $[L]^3$.

A *pure number* is the ratio of two quantities of the same dimensions; thus an angle expressed in radians is a pure number, because it is equal to the ratio of the length of a circular arc and the radius of the circle. A pure number has therefore no dimensions.

If the graph of an equation is to have any meaning in a geometrical sense, all the terms of the equation must be of the same dimensions, and the same system of units must be applied throughout the equation. It is obvious that one cannot add a volume to an area no more than one can add an apple to a penny; but it is possible to add 2 cubic feet to 4 cubic metres, only the two quantities must be referred to the same system of units.

If the equation $y=Ax^3+Bx^2+Cx+D$ is to have any geometrical meaning, then, as x and y are length, C must be a pure number, D a length, the dimensions of A must be $\frac{1}{L^2}$ or $[L]^{-2}$, and the dimensions of B must be $\frac{1}{L}$ or $[L]^{-1}$.

The student should always test his work by examining the dimensions of each term of his result.

The equation $y = \sin x$ cannot be plotted in a geometrical sense, unless, indeed, it is a special case of the general equation $y = a \sin \frac{x}{b}$, where a and b are lengths, x is then the length of a circular arc whose radius is b; the maximum value of y will be a, since the maximum value of sine is one.

In geometry x and y must be plotted to the same scale. If we were to plot the graph of $x^2 + y^2 = r^2$, taking the same scale for x and y, we should find that the curve is a circle. But if we were to take different scales for x and y, the graph would be an oval figure (ellipse).

5. Transformation of rectangular co-ordinates.

If we know the position of a point referred to a given set of

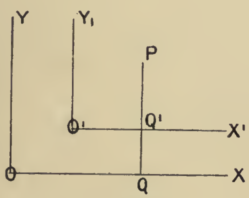


FIG. 3.

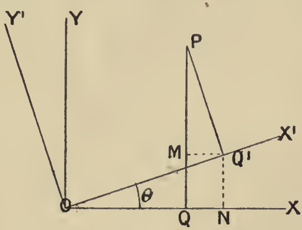


FIG. 4.

axes, we can deduce its position with reference to another set of axes.

(α) *The new axes are parallel to the original axes.*

Let the new origin (fig. 3) be point $O'(h, k)$, and let P be any point on the plane, then

$$OQ = h + O'Q', \quad \text{or} \quad x = h + x';$$

also $$QP = k + Q'P', \quad \text{or} \quad y = k + y'.$$

(β) *The direction of the new axes is changed without changing the origin.*

Let the new axes (fig. 4) be inclined at an angle, θ, to the original axes O X and O Y, then

$$OQ = ON - QN = \overline{OQ'}\cos\theta - \overline{PQ'}\sin\theta, \quad \text{or} \quad x = x'\cos\theta - y'\sin\theta;$$

also

$$QP = NQ' + MP = \overline{OQ'}\sin\theta + \overline{Q'P}\cos\theta, \quad \text{or} \quad y = x'\sin\theta + y'\cos\theta.$$

CHAPTER II.

THE STRAIGHT LINE.

6. The equation of a straight line in terms of the angle, ϕ, it makes with the x-axis and the intercept, c, which it cuts off from the y-axis, is

$$y = x \tan \phi + c \quad . \quad . \quad . \quad . \quad [5]$$

The position of a straight line referred to the axes of co-ordinates may be determined by the position of point M (fig. 5),

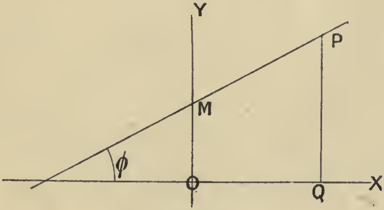


Fig. 5.

at which it intersects the y-axis, and the angle ϕ it forms with the x-axis. Taking any point, P, in the straight line, it will be seen that

$$QP - OM = \overline{OQ} \tan \phi, \text{ but } QP = y,\ OQ = x, \text{ and } OM = c,$$

hence,
$$y = x \tan \phi + c.$$

The symbol m is usually used instead of $\tan \phi$, hence, we may also write [5] as

$$y = mx + c \quad . \quad . \quad . \quad . \quad [6]$$

7. Any equation of the form $y = mx + c$ is the equation of a straight line (fig. 6).

The graph of the equation $y = mx + c$ will intersect the y-axis at a point M, and as $x=0$ makes $y=c$, $OM=c$. Take any three points $P_1(x_1, y_1)$; $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ on the graph, and draw a straight line $MN_1N_2N_3$ parallel to the x-axis. Further, draw the three lines MP_1, MP_2, and MP_3, then

$$\frac{N_1P_1}{MN_1}=\frac{y_1-c}{x_1},$$

which, according to the equation is equal to m,

hence, $$\frac{y_1-c}{x_1}=\frac{y_2-c}{x_2}=\frac{y_3-c}{x_3}=m,$$

i.e. the three triangles N M P are similar, hence, the curve $MP_1P_2P_3$ is a straight line.

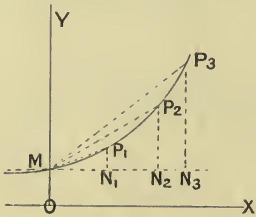


FIG. 6.

8. Any equation of first degree in x and y is the equation of a straight line.

Every equation of first degree in x and y may be reduced to

$$Ax+By+C=0 \quad . \quad . \quad . \quad . \quad [7]$$

[7] may also be written as

$$y=-\frac{A}{B}x-\frac{C}{B} \quad . \quad . \quad . \quad . \quad (1)$$

In (1) $-\frac{A}{B}$ must be a pure number and $-\frac{C}{B}$ must be a length, hence, (1) is an equation of the form $y=mx+c$, and [7] is therefore the equation of a straight line.

9. The equation of a straight line in terms of the angle, ψ, it makes with the y-axis and the intercept, γ, which it cuts off the x-axis is

$$x=y\tan\psi+\gamma \quad . \quad . \quad . \quad . \quad [8]$$

The proof is similar to that given in article 6.

If the equation of the straight line is given by $Ax + By + C = 0$, then the latter may be written as

$$x = -\frac{B}{A}y - \frac{C}{A} \qquad (1)$$

the intercept which the line cuts off from the x-axis is therefore $-\frac{C}{A}$.

10. The equation of a straight line in terms of the intercepts, a and b, which it cuts off from the axis is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad [9]$$

The equation $Ax + By + C = 0$ may also be written as

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \qquad (1)$$

According to articles **8** and **9** the two intercepts are $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$, hence [**9**] is the required equation.

11. To find the equation of a straight line parallel to one of the axes.

When a straight line is parallel to one of the axes, the angle which it makes with that axis will be zero; therefore, when the straight line is parallel to the y-axis, angle $\psi = 0$; and when the straight line is parallel to the x-axis, angle $\phi = 0$, hence,

$$x = \text{a constant} \qquad [10]$$

is the equation of a straight line parallel to the y-axis. The constant being the distance between [**10**] and the y-axis. And

$$y = \text{a constant} \qquad [11]$$

is the equation of a straight line parallel to the x-axis. The constant being the distance between [**11**] and the x-axis.

12. To find the equation of a straight line parallel to a given straight line.

Let the given straight line be given by

$$y = mx + c, \quad \text{or} \quad Ax + By + C = 0 \qquad (1)$$

and let the required equation be

$$y = m_1x + c_1, \quad \text{or} \quad A_1x + B_1y + C_1 = 0 \qquad (2)$$

Since (1) and (2) are to be parallel, they must form the same angle with the x-axis; hence, we must have $m = m_1$ and $\frac{A_1}{B_1} = \frac{A}{B}$ or, $\frac{A_1}{A} = \frac{B_1}{B} = k$ (a constant). The equations of two parallel straight lines are therefore

$$y = mx + c_1, \quad \text{and} \quad y = mx + c_2 \qquad . \quad . \quad [12]$$

or,
$$Ax + By + C_1 = 0, \quad \text{and} \quad kAx + kBy + C_2 = 0 \qquad . \quad [13]$$

13. To find the equation of a straight line, which is parallel to a given straight line, and which passes through a given point.

Let the given point be (a, b), and the given straight line be

$$y = mx + c, \quad \text{or} \quad Ax + By + C = 0 \qquad . \quad . \quad (1)$$

As the required straight line is to be parallel to (1), its equation may be written as

$$y = mx + c_1, \quad \text{or} \quad kAx + kBy + C_1 = 0 \quad . \quad . \quad (2)$$

As (2) is to pass through point (a, b), we must have

$$b = ma + c_1, \quad \text{or} \quad kAa + kBb + C_1 = 0 \quad . \quad . \quad (3)$$

By subtracting (3) from (2) we obtain

$$y - b = m(x - a), \quad \text{or} \quad A(x - a) + B(y - b) = 0 \quad . \quad [14]$$

14. To find the equation of a straight line through two given points.

Let the two given points be (a_1, b_1) and (a_2, b_2); and let the required equation be of the form

$$y = mx + c \qquad . \quad . \quad . \quad . \quad (1)$$

The constants of (1) are to be determined. Since (1) is to pass through point (a_1, b_1) we must have, according to [**14**],

$$y - b_1 = m(x - a_1) \quad . \quad . \quad . \quad . \quad (2)$$

and since (2) is to pass through point (a_2, b_2), we must have

$$b_2 - b_1 = m(a_2 - a_1) \quad . \quad . \quad . \quad . \quad (3)$$

By eliminating m between (2) and (3) we obtain

$$y = \frac{b_1 - b_2}{a_1 - a_2} x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \qquad . \quad . \quad . \quad [15]$$

which is the required equation.

15. To find the angle between two given straight lines.

The angles which the two straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$ form with the x-axis are ϕ_1 and ϕ_2 respectively. The angle between the lines will therefore be $\omega = \phi_2 - \phi_1$ (fig. 7), and

$$\tan \omega = \tan(\phi_2 - \phi_1) = \frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_1 \tan \phi_2} = \frac{m_2 - m_1}{1 + m_1 m_2}. \quad [16]$$

The lines will be parallel when ω is zero, which, according to [16], will take place when $m_2 = m_1$.

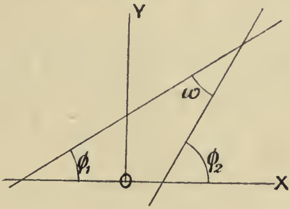


FIG. 7.

EXAMPLE.—Find the angle between the two lines $y = 2x + 3$ and $y = 0{\cdot}3x - 7$.

$$\tan \omega = \frac{2 - 0{\cdot}3}{1 + 2 \times 0{\cdot}3} = 1{\cdot}0625\text{; or } \omega = 46^\circ\ 44'.$$

16. To find the condition that two straight lines intersect at right angles.

In this case $\tan \omega$ in [16] must be equal to ∞, which requires that $1 + m_1 m_2 = 0$, or that

$$m_1 m_2 = -1 \quad . \quad . \quad . \quad . \quad [17]$$

In words: *Two straight lines intersect at right angles when the product of their m-constants is equal to minus one.*

Thus the two straight lines $y = 2x + 5$ and $y = -\frac{x}{2} - 3$ intersect at right angles as $2 \times -\frac{1}{2} = -1$.

17. To find the point (a, b) of intersection of two given straight lines.

Let the two lines be

$$y = m_1x + c_1, \quad \text{and} \quad y = m_2x + c_2 \quad . \quad . \quad (1)$$

The co-ordinates a and b of the point of intersection must satisfy (1), hence,

$$b = m_1 a + c_1, \quad \text{and} \quad b = m_2 a + c_2 \quad . \quad . \quad (2)$$

By solving (2) with regard to a and b we get the required co-ordinates, viz.:

$$a = \frac{c_1 - c_2}{m_2 - m_1}, \quad \text{and} \quad b = \frac{m_2 c_1 - m_1 c_2}{m_2 - m_1} \quad . \quad : \quad [18]$$

We have seen that the two straight lines will be parallel when $m_1 = m_2$ which makes both a and b in [18] equal to ∞, hence, two parallel straight lines intersect at an infinite distance.

18. To show that the equation of a straight line in terms of the length, p, of the perpendicular on it from the origin, and the angle, α, which this perpendicular makes with the x-axis, is

$$x \cos \alpha + y \sin \alpha = p \quad . \quad . \quad . \quad [19]$$

Let M L (fig. 8) be the given straight line, and point F the foot of the perpendicular O F $= p$. Angle X O F $= \alpha$. F is

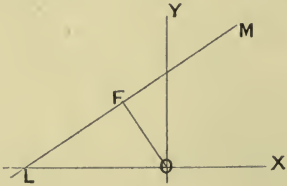


FIG. 8.

determined by its polar co-ordinates α and p, or by its rectangular co-ordinates ($p \cos \alpha$, $p \sin \alpha$).

Let the equation of M L be $y = mx + c$. The condition that it intersects O F at right angles is $m \tan \alpha = -1$, or $m = -\cot \alpha$, and as the line must pass through F, its equation is also

$$y - p \sin \alpha = -\cot \alpha (x - p \cos \alpha) \quad . \quad . \quad . \quad (1)$$

(1) can be reduced to

$$x \cos \alpha + y \sin \alpha = p,$$

which is the required equation.

If the equation of M L is given by

$$Ax + By + C = 0 \quad . \quad . \quad . \quad . \quad (2)$$

then the condition that it passes through point F is

$$A(x - p\cos\alpha) + B(y - p\sin\alpha) = 0,$$

hence,

$$C = -p(A\cos\alpha + B\sin\alpha) = -p\sqrt{A^2+B^2}\cos(\beta - \alpha),$$

where

$$\cos\beta = \frac{A}{\sqrt{A^2+B^2}}, \quad \sin\beta = \frac{B}{\sqrt{A^2+B^2}}, \quad \text{and} \quad \tan\beta = \frac{B}{A},$$

but the condition that M L intersects O F at right angles is, $\frac{A}{B} = \cot\alpha$, hence $\beta = \alpha$, and

$$p = -\frac{C}{\sqrt{A^2+B^2}} \quad . \quad . \quad . \quad . \quad [20]$$

The equation (2) can therefore be written

$$\frac{Ax + By}{\sqrt{A^2+B^2}} = p \quad . \quad . \quad . \quad . \quad [21]$$

EXAMPLE.—Let the straight line be $5x - 3y + 2 = 0$, then

$$p = -\frac{2}{\sqrt{25+9}} = -0{\cdot}343, \quad \cos\alpha = \frac{5}{\sqrt{34}} = +0{\cdot}8575,$$

$$\text{and} \quad \sin\alpha = -\frac{3}{\sqrt{34}} = -0{\cdot}5145,$$

hence $\alpha = 329^\circ\ 2'$. As p is negative, point F lies in the 2nd quadrant at a distance 0·343 from the origin.

19. To find the conditions that two parallel straight lines lie on the same side or on opposite sides of the origin.

(α) The equations of the two straight lines are

$$y = mx + c_1, \quad \text{and} \quad y = mx + c_2.$$

It is evident that when the signs of the intercepts c_1 and c_2 are the same, the two lines must lie on the same side of the origin, and when the signs of c_1 and c_2 are opposite, the two lines will lie on opposite sides of the origin.

(β) The equations of the two straight lines are

$$A_1x + B_1y + C_1 = 0, \quad \text{and} \quad A_2x + B_2y + C_2 = 0.$$

According to article **12** the lines will be parallel when $\frac{A_1}{B_1} = \frac{A_2}{B_2}$.

The intercepts cut off from the y-axis are $-\frac{C_1}{B_1}$ and $-\frac{C_2}{B_2}$ respectively.

This case is therefore reduced to that of (α).

(γ) The equations of the two straight lines are

$$x\cos\alpha_1 + y\sin\alpha_1 = p_1, \quad \text{and} \quad x\cos\alpha_2 + y\sin\alpha_2 = p_2.$$

The two lines will be parallel when $\cos\alpha_1 = \cos\alpha_2$ and $\sin\alpha_1 = \sin\alpha_2$, *i.e.* when the equations have the form of

$$x\cos\alpha + y\sin\alpha = p_1, \quad \text{and} \quad x\cos\alpha + y\sin\alpha = p_2 \quad . \quad (1)$$

One of the equations (1) might be given as

$$-x\cos\alpha - y\sin\alpha = p_2,$$

but the latter can be transformed into

$$x\cos\alpha + y\sin\alpha = -p_2.$$

We have stated in article **18** that p and α are the polar co-ordinates of the foot, F, of the perpendicular, hence the two parallel lines (1) lie on the same side of the origin when p_1 and p_2 have the same signs, and they lie on the opposite sides of the origin when p_1 and p_2 have opposite signs.

Example.—Let the two lines be

$$(1) \quad -3x + 4y + 7 = 0, \quad \text{and} \quad (2) \quad 6x - 8y + 5 = 0.$$

The lines are parallel, since $\frac{-3}{4} = \frac{6}{-8}$.

(2) may be written thus

$$-3x + 4y - 2\cdot5 = 0.$$

The intercepts cut off from the y-axis are respectively $-\frac{7}{4}$ and $\frac{2\cdot5}{4}$, hence, the two lines lie on opposite sides of the origin.

We might also have transformed the two equations into

$$(1)\ \frac{-3x+4y}{5} = -\frac{7}{5} = -1\cdot4 = p_1, \text{ and } (2)\ \frac{-3x+4y}{5} = \frac{2\cdot5}{5} = 0\cdot5 = p_2,$$

which also show that the two lines lie on opposite sides of the origin. We have further

$$\cos\alpha = -0\cdot6, \quad \text{and} \quad \sin\alpha = 0\cdot8, \quad \text{hence,} \quad \alpha = 126°\ 52'.$$

As p_1 is negative, its foot lies in the 4th quadrant and at a distance of 1·4 from the origin. p_2 being positive, its foot lies in the 2nd quadrant at a distance of 0·5 from the origin.

20. To find the distance, d, between two parallel straight lines.

(α) The equations of the two lines are

$$x\cos\alpha+y\sin\alpha=p_1, \quad \text{and} \quad x\cos\alpha+y\sin\alpha=p_2.$$

From what we have found in article **19** it is evident that

$$d=p_1-p_2 \quad . \quad . \quad . \quad . \quad [22]$$

where p_1 and p_2 must be taken with their signs.

(β) The equations of the two lines are

$$Ax+By+C_1=0, \quad \text{and} \quad Ax+By+C_2=0.$$

According to [**20**]

$$p_1=-\frac{C_1}{\sqrt{A^2+B^2}}, \quad \text{and} \quad p_2=-\frac{C_2}{\sqrt{A^2+B^2}},$$

hence,

$$d=\frac{-C_1+C_2}{\sqrt{A^2+B^2}} \quad . \quad . \quad . \quad . \quad [23]$$

Example.—Locate the following two straight lines and find the distance, d, between them.

(1) $7x-9y+23=0$, and (2) $-49x+63y+56=0$.

The two lines are parallel since $\frac{7}{-9}=\frac{-49}{63}$.

(2) may therefore be reduced to $7x-9y-8=0$,

hence,

$$d=\frac{8-(-23)}{\sqrt{49+81}}=2{\cdot}719.$$

With respect to the location of the two feet F_1 and F_2, it will be found that $\alpha=307°\ 52'$; F_1 is situated in the 2nd quadrant at a distance of 2·017 from the origin, and F_2 will be found in the 4th quadrant at a distance of 0·702 from the origin.

The position of the two lines may also be determined by finding the intercepts which they cut off from the axes. According to article **10** the equations of the two lines may be transformed into

(1) $$\frac{x}{-\frac{23}{7}}+\frac{y}{\frac{23}{9}}=1, \quad \text{or} \quad \frac{x}{-3{\cdot}29}+\frac{y}{2{\cdot}56}=1;$$

(2) $$\frac{x}{\frac{8}{7}}+\frac{y}{-\frac{8}{9}}=1, \quad \text{or} \quad \frac{x}{1{\cdot}14}+\frac{y}{-0{\cdot}89}=1,$$

from which we can see that the two given lines lie on opposite sides of the origin.

21. To find the length, d, of the perpendicular let fall from a given point, (h, k), upon a given straight line.

(α) Let the equation of the given straight line be

$$x \cos \alpha + y \sin \alpha = p \quad . \quad . \quad . \quad (1)$$

p should always be positive. The required length, d, will be equal to the distance between (1) and a straight line drawn through (h, k) and parallel to (1). The equation of the latter line is

$$(x - h) \cos \alpha + (y - k) \sin \alpha = 0,$$

which may also be written

$$x \cos \alpha + y \sin \alpha = h \cos \alpha + k \sin \alpha = p_1,$$

hence,

$$d = p_1 - p = h \cos \alpha + k \sin \alpha - p \quad . \quad . \quad [\mathbf{24}]$$

The length, d, of the required perpendicular is therefore obtained by substituting h and k for x and y in the given equation.

d will be positive when (1) lies between the origin and the given point, and d will be negative when the given point and the origin lie on the same side of (1).

(β) Let the equation of the straight line be

$$Ax + By + C = 0 \quad . \quad . \quad . \quad . \quad (2)$$

As the perpendicular, p, should be positive, (2) must be written so that C is a negative quantity.

The equation of a straight line through (h, k) and parallel with (2) is

$$A(x - h) + B(y - k) = 0, \quad \text{or} \quad Ax + By = Ah + Bk;$$

we have now that

$$p = \frac{-C}{\sqrt{A^2 + B^2}}, \text{ and } p_1 = \frac{Ah + Bk}{\sqrt{A^2 + B^2}},$$

hence,

$$d = p_1 - p = \frac{Ah + Bk + C}{\sqrt{A^2 + B^2}} \quad . \quad . \quad . \quad [\mathbf{25}]$$

The length, d, of the required perpendicular is therefore obtained by substituting h and k for x and y in the given equation, and dividing the result by the square root of the sum of the squares of the coefficients to x and y.

EXAMPLE.—Find the length of the perpendicular drawn from point $(2, -5)$ to $3x-5y+7=0$.

Write the equation of the straight line thus

$$-3x+5y-7=0.$$

The required length is

$$d=\frac{-3\times 2+5\times(-5)-7}{\sqrt{9+25}}=-6{\cdot}517.$$

As d is negative, the given point and the origin lie on the same side of the given straight line. This is also evident from the fact that $\cos\alpha$ being negative and $\sin\alpha$ positive, the foot of the perpendicular drawn from the origin to the given line lies in the 2nd quadrant, whereas the given point lies in the 4th quadrant.

22. To find the equations of the bisectors of the angles between the two straight lines.

$$A_1x+B_1y+C_1=0 \quad . \quad . \quad . \quad . \quad (1)$$

and

$$A_2x+B_2y+C_2=0 \quad . \quad . \quad . \quad . \quad (2)$$

The equations must be written so that both C_1 and C_2 are negative quantities. Let the lengths of the perpendiculars drawn from a point $M(h, k)$ to (1) and (2) be d_1 and d_2 respectively, then by [**25**]

$$d_1=\frac{A_1h+B_1k+C_1}{\sqrt{A_1^2+B_1^2}}, \text{ and } d_2=\frac{A_2h+B_2k+C_2}{\sqrt{A_2^2+B_2^2}}.$$

If point M is to be a point on one of the bisectors, we must have $d_1=d_2$, or

$$\frac{A_1h+B_1k+C_1}{\sqrt{A_1^2+B_1^2}}=\pm\frac{A_2h+B_2k+C_2}{\sqrt{A_2^2+B_2^2}},$$

$+$ or $-$ according to whether d_1 and d_2 have same or opposite signs. By substituting x for h and y for k, the required equations will be

$$\frac{A_1x+B_1y+C_1}{\sqrt{A_1^2+B_1^2}}\mp\frac{A_2x+B_2y+C_2}{\sqrt{A_2^2+B_2^2}}=0 \quad . \quad . \quad [\mathbf{26}]$$

The student will find that the two bisectors [**26**] intersect at right angles.

By article **21** d_1 and d_2 will have the same signs when point M and the origin lie on the same sides of the two straight lines, hence the upper sign in [**26**] gives the equation of the bisector of the angle in which the origin lies.

EXAMPLE.—Find the equations of the bisectors of the angles between the straight lines

$$-5x+7y-3=0, \quad \text{and} \quad 4x-9y+5=0.$$

The constant terms of the equations must have the same signs; we may therefore write them

$$-5x+7y-3=0, \quad \text{and} \quad -4x+9y-5=0.$$

The bisector of the angle in which the origin lies will be

$$\frac{-5x+7y-3}{\sqrt{25+49}}-\frac{-4x+9y-5}{\sqrt{16+81}}=0,$$

which can be reduced to

$$y=-1{\cdot}75x+1{\cdot}588 \quad . \quad . \quad . \quad (3)$$

The equation of the other bisector will be

$$\frac{-5x+7y-3}{\sqrt{25+49}}+\frac{-4x+9y-5}{\sqrt{25+49}}=0,$$

which can be reduced to

$$y=0{\cdot}571x+0{\cdot}496. \quad . \quad . \quad . \quad (4)$$

As $-1{\cdot}75\times 0{\cdot}571=-1$, the two bisectors (3) and (4) intersect at right angles.

CHAPTER III.

THE CIRCLE.

23. Definition.—A circle is the locus traced by a point which moves so that its distance from a given point, the centre, is constant.

The constant distance is called the *radius* of the circle.

24. To find the equation of a circle.

Let the centre be $C(h, k)$ and the radius be a. The distance of any point $P(x, y)$ from C is

$$\sqrt{(x-h)^2+(y-k)^2};$$

if point P is to lie on the circle we must have

$$\sqrt{(x-h)^2+(y-k)^2}=a$$

or

$$(x-h)^2+(y-k)^2=a^2 \qquad [27]$$

which is the equation of the circle.

25. To find the condition that the general equation of second degree may represent a circle.

The general equation of second degree is

$$Ax^2+By^2+2Cxy+2Dx+2Ey+F=0 \qquad (1)$$

[27] may also be written

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0 \qquad (2)$$

By comparing (1) and (2) it will be seen that if (1) is to represent a circle, we must have $A=B$ and $C=0$, hence, (1) must have the form

$$x^2+y^2-2Gx-2Hy+K=0 \qquad (3)$$

(3) may also be written

$$(x-G)^2+(y-H)^2=G^2+H^2-K \qquad (4)$$

If (3) is to represent a circle,

$$\sqrt{G^2 + H^2 - K}$$

must be the radius, or

$$a = \sqrt{G^2 + H^2 - K},$$

or we must have

$$K < G^2 + H^2.$$

EXAMPLE 1.—Does $x^2 + y^2 + 6x - 10y + 40 = 0$ represent a circle?

The equation may also be written

$$(x+3)^2 + (y-5)^2 = 9 + 25 - 40 = -6 \quad . \quad . \quad (1)$$

As the right-hand side of (1) is negative, the given equation does not represent a circle.

EXAMPLE 2.—Does $x^2 + y^2 + 8x - 6y + 9 = 0$ represent a circle?

The equation may also be written

$$(x+4)^2 + (y-3)^2 = 16 \quad . \quad . \quad . \quad (1)$$

which is a circle with radius equal to 4 and centre $(-4, 3)$. As the centre is situated at a distance 4 to the left of the y-axis and the radius is 4, the circle will touch the y-axis at a point which is situated at a distance $+3$ from the origin.

26. To find the equation of the tangent to a circle.

For this purpose we may choose a circle with centre at the origin. Let the equations of the tangent and the circle respectively be

$$y = mx + c, \quad \text{and} \quad x^2 + y^2 = a^2 \quad . \quad . \quad . \quad (1)$$

Let the point of contact be (x', y'); then its co-ordinates must satisfy both equations (1) or

$$y' = mx' + c, \quad \text{and} \quad x'^2 + y'^2 = a^2 \quad . \quad . \quad (2)$$

from (2) we obtain

$$x' = \frac{-mc \pm \sqrt{a^2(1+m^2) - c^2}}{1+m^2}, \text{ and } y' = \frac{c \pm m\sqrt{a^2(1+m^2) - c^2}}{1+m^2} \quad (3)$$

If $a^2(1+m^2) > c^2$, then there will be two points of intersection, and the straight line in (1) is a secant.

If $a^2(1+m^2) < c^2$, then x' and y' will be imaginary, or the straight line in (1) does not intersect the circle.

If, however, $a^2(1+m^2) = c^2$, then the two points in (3) become coincident, and the straight line is a tangent to the circle. In this case we obtain from (3)

$$m = -\frac{x'}{y'}, \text{ and } c^2 = a^2\left[1 + \left(\frac{x'}{y'}\right)^2\right] = \frac{a^4}{y'^2}, \text{ or } c = \frac{a^2}{y'}.$$

The equation of the tangent will therefore be

$$xx' + yy' = a^2 \quad . \quad . \quad . \quad . \quad [28]$$

The equation of the tangent is thus obtained from that of the circle by substituting xx' and yy' for x^2 and y^2 respectively.

CHAPTER IV.

THE PARABOLA.

27. Definition.—The parabola is the locus traced by a point moving so that it is always equidistant from a given straight line and a given point.

The given straight line is called the *directrix*, and the given point is called the *focus.*

28. To find the equation of the parabola (fig. 9).

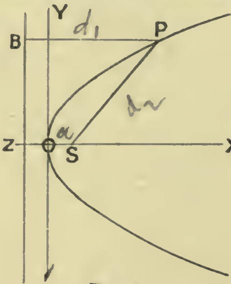


FIG. 9.

Let Z B be the directrix and point S the focus, then take the straight line S Z through the focus and perpendicular to the directrix as x-axis, and the middle point, O, of S Z as origin. The y-axis will thus be parallel to the directrix. Let $OS = ZO = a$, and the distances of a point $P(x, y)$ from the directrix and the focus be d_1 and d_2 respectively, then

$$d_1 = x + a, \quad \text{and} \quad d_2^2 = y^2 + (x - a)^2 \quad . \quad . \quad (1)$$

If the point P is to lie on the parabola we must have $d_1 = d_2$, or

$$(x + a)^2 = y^2 + (x - a)^2$$

or

$$y^2 = 4ax \quad . \quad . \quad . \quad . \quad . \quad . \quad [29]$$

which is the equation of the parabola.

The x-axis is called the *axis of the parabola*, and O is the *vertex.*

29. To find the points of intersection of a straight line and a parabola.

Let the straight line be $y = mx + c$ and the point of intersection be (x', y'). The co-ordinates of the point (x', y') must satisfy the equations of the straight line and the parabola,

hence, $$y' = mx' + c, \quad \text{and} \quad y'^2 = 4ax' \quad . \quad . \quad . \quad (1)$$

By solving the two simultaneous equations (1) we obtain

$$x' = \frac{2a - mc \pm 2\sqrt{a^2 - acm}}{m^2}, \quad \text{and} \quad y' = \frac{2a \pm 2\sqrt{a^2 - acm}}{m} \qquad (2)$$

From (2) follows that—

(α) If the quantity under the radical sign is negative, the straight line does not intersect the parabola.

(β) If $m = 0$, *i.e.* the straight line is parallel to the axis of the parabola, then there will be two points of intersection, viz. (∞, ∞) and $\left(\frac{0}{0}, \frac{0}{0}\right)$; the latter point can, however, easily be determined. When $m = 0$, the equation of the straight line is $y = c$, to which corresponds a point on the parabola whose abscissa is $\frac{c^2}{4a}$; hence, the second point of intersection is $\left(\frac{c^2}{4a}, c\right)$.

(γ) If the quantity under the radical sign is positive and m is not zero, then the straight line will intersect the parabola in two finite points.

(δ) If the quantity under the radical sign is zero, then the two points of intersection will coincide and the straight line becomes a tangent to the parabola.

30. To find the equation of the tangent to the parabola.

It was shown in the preceding article that the straight line will be a tangent when $cm = a$. The co-ordinates of the point of contact are

$$x' = \frac{a}{m^2}, \quad \text{and} \quad y' = \frac{2a}{m} \quad . \quad . \quad . \quad (1)$$

The equation of the tangent is

$$y = mx + \frac{a}{m}, \quad \text{or} \quad -\frac{x}{\frac{a}{m^2}} + \frac{y}{\frac{a}{m}} = 1,$$

or, according to (1),

$$\frac{x}{-x'} + \frac{y}{\frac{y'}{2}} = 1 \quad . \quad . \quad . \quad . \quad [30]$$

which is the equation of the tangent to the parabola in terms of the intercepts it cuts off the axes; [**30**] is the most practical form of the equation of the tangent, as it admits of an easy construction of the tangent to the parabola. Let P be a point on the parabola, then make O N = O Q. The straight line N P is then the tangent to the parabola at point P (fig. 10).

[**30**] may also be written

$$yy' = 2a(x + x'), \quad \text{or} \quad y = mx + \frac{a}{m} \quad . \quad . \quad (2)$$

31. To find the locus of the middle point C of a system of parallel chords of a parabola (fig. 11).

Let P Q be any one of the parallel chords, $P(x_1, y_1)$, and $Q(x_2, y_2)$ be the two points at which the chord intersects the

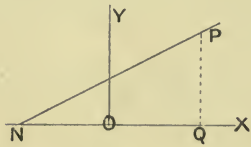


FIG. 10.

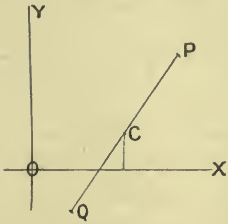


FIG. 11.

parabola, and let $C(h, k)$ be the middle point of the chord whose equation is $y = mx + c$. Taking the co-ordinates with their signs, it will be seen that

$$h = \frac{x_1 + x_2}{2}, \quad \text{and} \quad k = \frac{y_1 + y_2}{2}.$$

According to (2) in article **29** it will be found that

$$h = \frac{2a - mc}{m^2}, \quad \text{and} \quad k = \frac{2a}{m} \quad . \quad . \quad . \quad (1)$$

k is therefore constant for the same system of parallel chords, hence, *the locus is a straight line parallel to the x-axis*, and its equation is

$$y = \frac{2a}{m} \quad . \quad . \quad . \quad . \quad . \quad [\mathbf{31}]$$

A straight line passing through the middle points of a system of parallel chords is called a *diameter*.

The diameters of the parabola are parallel to the axis of the parabola.

The point at which the diameter meets the parabola is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$; the tangent at that point is therefore

$$\frac{x}{-\frac{a}{m^2}} + \frac{y}{\frac{a}{m}} = 1, \quad \text{or} \quad y = mx + \frac{a}{m},$$

hence, *the tangent to the point at which the diameter intersects the parabola is parallel to the chords.*

32. To show that the tangent at any point of the parabola bisects the angle between the diameter and the focal line through the point (fig. 12).

Let $P(\alpha, \beta)$ be a point on the parabola, and let PQ be the diameter, PM the tangent, and PS the focal line through P,

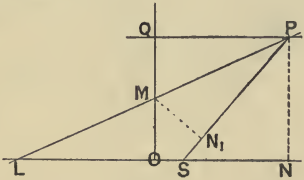


FIG. 12.

then it will be sufficient to show that the length of the perpendicular MN_1 is equal to $MQ = \frac{\beta}{2}$. The equation of SP is evidently

$$y = \frac{\beta}{\alpha - a}(x - a),$$

and as $\alpha = \frac{\beta^2}{4a}$, the equation may also be written

$$4a\beta x + (4a^2 - \beta^2)y - 4a^2\beta = 0 \quad . \quad . \quad . \quad (1)$$

As the co-ordinates of point M are $\left(0, \frac{\beta}{2}\right)$, the length of the perpendicular MN_1 is

$$\frac{(4a^2 - \beta^2)\frac{\beta}{2} - 4a^2\beta}{\sqrt{16a^2\beta^2 + (4a^2 - \beta^2)^2}} = \frac{\beta}{2}.$$

This principle is of great importance in applied physics. It shows that a ray emanating from the focus of a parabolic reflector will be reflected in a direction parallel to the axis; and *vice versâ*, a ray entering a parabolic reflector in a direction parallel to the axis will be reflected through the focus.

33. To trace the parabola.

The equation [**29**] of the curve gives

$$y = \pm 2\sqrt{ax} \quad . \quad . \quad . \quad . \quad (1)$$

which shows that the axis of the parabola is an axis of symmetry. (1) also shows that x cannot be negative; hence the curve is entirely situated on the right-hand side of the y-axis.

$x=0$ makes $y=0$, *i.e.* the curve passes through the origin.

When x increases in (1) y also increases, and when x becomes infinitely great, then y also becomes infinitely great.

The equation (2) in article **30** of the tangent shows that at the vertex the equation of the tangent becomes $2ax=0$, or $x=0$, which is the y-axis; hence the tangent to the curve at the vertex is perpendicular to the axis.

The tangent of the angle which the tangent makes with the axis is positive when y' is positive, and it is negative when y' is negative.

CHAPTER V.

THE ELLIPSE.

34. Definition.—**The ellipse is the locus traced by a point which moves so that the sum of its distances from two given points is constant.**

The two given points are called the *foci*, and the constant sum is denoted by $2a$.

35. To find the equation of the ellipse (fig. 13).

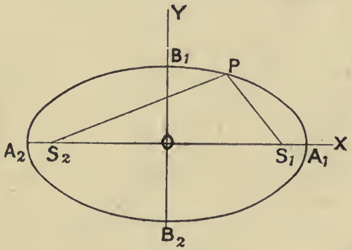


FIG. 13.

Let point S_1 and S_2 be the foci. Take line S_1S_2 as x-axis, and the middle point O of S_1S_2 as origin. Let point $P(x, y)$ be a point on the ellipse, then by definition

$$S_1P + S_2P = 2a \quad . \quad . \quad . \quad . \quad (1)$$

There will evidently be two points A_1 and A_2 of the curve on the x-axis; at these points we have $S_1A_1 + S_2A_1 = 2a = S_1A_2 + S_2A_2$, hence, $OA_1 = OA_2 = a$.

There will also be two points B_1 and B_2 on the y-axis, for which we have $S_1B_1 = S_2B_1 = S_1B_2 = S_2B_2 = a$.

The ratio

$$\frac{OS_1}{OA_1} = \frac{OS_2}{OA_2} = e < 1 \quad . \quad . \quad . \quad . \quad (2)$$

is called the *eccentricity* of the ellipse. By (2)

$$O S_1 = ea, \quad \text{and} \quad O S_2 = -ea \quad . \quad . \quad . \quad (3)$$

We have now $\overline{S_1P}^2 = y^2 + (x - ea)^2$, and $\overline{S_2P}^2 = y^2 + (x + ea)^2$, hence, $\overline{S_2P}^2 - \overline{S_1P}^2 = 4eax$, and as $S_1P + S_2P = 2a$, we have $S_2P - S_1P = 2ex$, hence, $S_2P = a + ex$ and $S_1P = a - ex$.

We have seen that $\overline{S_1P}^2 = y^2 + (x - ea)^2 = (a - ex)^2$,

hence,
$$y^2 = (1 - e^2)(a^2 - x^2) \quad . \quad . \quad . \quad [32]$$

which is the equation of the ellipse. Let $O B_1 = O B_2$ be denoted by b, then

$$b^2 = a^2 - e^2a^2 = a^2(1 - e^2), \quad \text{or} \quad (1 - e^2) = \frac{b^2}{a^2};$$

[**32**] may therefore be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad [33]$$

[**33**] is the most useful form of the equation of the ellipse. a is called the *semi-axis major* and b is called the *semi-axis minor* of the ellipse.

The above property of the ellipse permits of a simple construction for the curve.

Fasten a drawing-pin at each focus, and take a string of length $S_1S_2 + A_1A_2$. Pass the string round the two pins, then let the point of a pencil, always in contact with the string, move on the paper and keeping the string tight all the time. The pencil-point will then trace the ellipse.

36. To find the points of intersection of a straight line with the ellipse.

Let the straight line and the ellipse respectively be

$$y' = mx' + c, \quad \text{and} \quad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad . \quad . \quad (1)$$

The co-ordinates of the points of intersection are found by solving the two simultaneous equations (1). We get

$$x' = \frac{-mca^2 \pm ab\sqrt{m^2a^2 + b^2 - c^2}}{m^2a^2 + b^2}$$

and
$$y' = \frac{b^2c \pm mab\sqrt{m^2a^2 + b^2 - c^2}}{m^2a^2 + b^2} \quad . \quad . \quad . \quad (2)$$

(α) The straight line will not intersect the ellipse when $m^2a^2+b^2<c^2$.

(β) The straight line will intersect the ellipse when $m^2a^2+b^2>c^2$.

(γ) When $m^2a^2+b^2=c^2$ the two points of intersection will become coincident points, and the straight line is a tangent to the ellipse.

37. To find the equation of the tangent to the ellipse.

It has been shown in the preceding article that the straight line $y=mx+c$ will be a tangent to the ellipse when $m^2a^2+b^2-c^2=0$, hence, the equation of the tangent to the ellipse is

$$y=mx\pm\sqrt{m^2a^2+b^2} \quad . \quad . \quad . \quad [\mathbf{34}]$$

The two signs + and − in front of the radical sign show that there are always two tangents to the ellipse which are parallel to $y=mx+c$.

The co-ordinates x' and y' of the point of contact will, according to the preceding article, be

$$x'=-\frac{ma^2c}{m^2a^2+b^2}, \quad \text{and} \quad y'=\frac{b^2c}{m^2a^2+b^2}.$$

As $m^2a^2+b^2-c^2=0$, we have

$$m=-\frac{b^2}{a^2}\frac{x'}{y'} \quad . \quad . \quad . \quad . \quad (1)$$

and hence the equation of the tangent at point (x', y') is

$$y-y'=-\frac{b^2x'}{a^2y'}(x-x') \quad . \quad . \quad . \quad [\mathbf{35}]$$

By transformation of [**35**] the equation of the tangent may also be written

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \quad . \quad . \quad . \quad . \quad [\mathbf{36}]$$

The equation of the tangent is thus obtained from that of the ellipse by substituting xx' and yy' for x^2 and y^2 respectively.

38. To find the condition that $x\cos\alpha+y\sin\alpha=p$ may touch the ellipse.

The straight line, $x\cos\alpha+y\sin\alpha=p$, meets the ellipse, $x^2b^2+y^2a^2=a^2b^2$, in two points, whose abscissæ are given by

$$x^2(a^2\cos^2\alpha+b^2\sin^2\alpha)-2pa^2(\cos\alpha)x+a^2(p^2-b^2\sin^2\alpha)=0 \quad (1)$$

The two points of intersection in (1) are coincident, and the straight line is a tangent, if

$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha \quad . \quad . \quad . \quad (2)$$

39. To find the locus of the middle points of parallel chords of the ellipse.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points at which a chord $y = mx + c$ intersects the ellipse, and let the middle point of PQ be $C(h, k)$ then, according to (2), article **36**,

$$h = \frac{x_1 + x_2}{2} = -\frac{ma^2c}{m^2a^2 + b^2}, \quad \text{and} \quad k = \frac{y_1 + y_2}{2} = \frac{b^2c}{m^2a^2 + b^2} \quad . \qquad (1)$$

By eliminating c between the two equations in (1), we obtain

$$k = -\frac{b^2}{ma^2}h,$$

and the locus is therefore

$$y = -\frac{b^2}{ma^2}x \quad . \quad . \quad . \quad . \quad [37]$$

which is a straight line through the origin, hence, *the diameters of an ellipse pass through the centre of the ellipse.*

We might write [**37**] as $y = m_1x$. The diameter of chords parallel to the latter would be

$$y = -\frac{b^2}{m_1a^2}x \quad . \quad . \quad . \quad . \quad (2)$$

but $m_1 = -\frac{b^2}{ma^2}$, which inserted in (2) gives

$$y = mx \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is parallel to chords PQ. Hence, each of the two diameters [**37**] and (3) bisects all chords parallel to the other. Such two diameters are called *conjugate diameters.*

40. The tangent at the extremity of any diameter is parallel to the chords which it bisects.

Let the equation of the diameter be

$$y = -\frac{b^2}{ma^2}x \quad . \quad . \quad . \quad . \quad (1)$$

and that of the tangent at the extremity (x', y') of the diameter be

$$y - y' = -\frac{b^2x'}{a^2y'}(x - x') \quad . \quad . \quad . \quad (2)$$

Point (x', y') is a point on (1),

hence, $$y' = -\frac{b^2}{ma^2}x', \quad \text{or} \quad \frac{x'}{y'} = -\frac{ma^2}{b^2},$$

which inserted in (2) gives

$$y - y' = m(x - x') \quad . \quad . \quad . \quad . \quad (3)$$

but (3) is parallel to the system of parallel chords $y=mx+c$ of which (1) is the diameter.

41. To find the locus of the point of intersection of tangents to an ellipse which meet at right angles.

The equation of one of the tangents is

$$y=mx+\sqrt{m^2a^2+b^2};$$

the second tangent is to be perpendicular on the latter, hence its equation must be

$$y=-\frac{x}{m}+\sqrt{\frac{a^2}{m^2}+b^2}, \quad \text{or} \quad y=\frac{-x+\sqrt{a^2+m^2b^2}}{m}.$$

Let the point of intersection of the two tangents be (h, k), then we must have

$$k=mh+\sqrt{m^2a^2+b^2}, \quad \text{and also} \quad k=\frac{-h+\sqrt{a^2+m^2b^2}}{m}, \quad \text{or}$$

$$k-mh=\sqrt{m^2a^2+b^2}, \quad \text{and} \quad mk+h=\sqrt{a^2+m^2b^2} \quad . \quad (1)$$

By squaring and adding the two in (1) we obtain $h^2+k^2=a^2+b^2$, or the locus is

$$x^2+y^2=a^2+b^2 \quad . \quad . \quad . \quad . \quad [38]$$

which is a circle with centre at the centre of the ellipse and radius $=\sqrt{a^2+b^2}$, which latter is the length of the line joining the ends of the major and minor axis. This circle is called the *director circle.*

42. To show that the tangent at any point of the ellipse bisects the angle between the two focal lines through the point.

Let P(x', y') (fig. 13) be a point on the ellipse. The equations of the focal lines through P are

$$y-y'=\frac{y'}{x'-ea}(x-x'), \quad \text{and} \quad y-y'=\frac{y'}{x'+ea}(x-x') \qquad (1)$$

The two equations in (1) may be reduced to

$$y'x-(x'-ea)y-eay'=0, \quad \text{and} \quad -y'x+(x'+ea)y-eay'=0 \quad (2)$$

The equation of the bisector of the angle in which the origin is not situated is, according to [26], article 22,

$$\frac{y'x-(x'-ea)y-eay'}{\sqrt{y'^2+(x'-ea)^2}}+\frac{-y'x+(x'+ea)y-eay'}{\sqrt{y'^2+(x'+ea)^2}}=0 \quad . \qquad (3)$$

The quantities under the radical signs are respectively

$$\overline{S_1P}^2 = (a - ex')^2, \quad \text{and} \quad \overline{S_2P}^2 = (a + ex')^2.$$

The equation (3) may be reduced to

$$y - y' = -\frac{b^2x'}{a^2y'}(x - x') \quad . \quad . \quad . \quad (4)$$

but (4) is the equation of the tangent at point P.

On account of this property of the ellipse a ray emanating from one of the foci of an elliptic reflector will be reflected through the other focus.

43. The auxiliary circle.

Definition.—The circle which is described on the major axis of an ellipse as diameter is called the *auxiliary circle* (fig. 14).

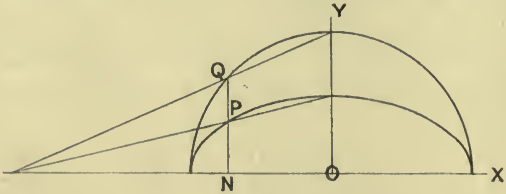


Fig. 14.

The equation of the auxiliary circle is

$$x^2 + y^2 = a^2, \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad . \quad . \quad (1)$$

The equation of the ellipse is

$$\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (2)$$

Let N Q be an ordinate on the circle, then N P is an ordinate on the ellipse. Q and P have the same abscissa O N $= x$; hence, from (1) and (2), we have

$$\frac{y^2}{a^2} = 1 - \frac{x^2}{a^2}, \quad \text{and} \quad \frac{y_1{}^2}{b^2} = 1 - \frac{x^2}{a^2} \quad . \quad . \quad (3)$$

Hence,
$$\frac{y^2}{a^2} = \frac{y_1{}^2}{b^2}, \quad \text{or} \quad \frac{y}{y_1} = \frac{a}{b} \quad . \quad . \quad . \quad [39]$$

We can therefore construct the ellipse by dividing the ordinates of the auxiliary circle in the ratio $\frac{a}{b}$.

44. To trace the ellipse.

The equation [**33**] of the ellipse shows that both axes are axes of symmetry. Hence, the two axes divide the curve into four equal quadrants; hence it becomes only necessary to investigate that part of the curve which is situated in the 1st quadrant. In the latter quadrant x is positive, and the ordinate is

$$y = +\frac{b}{a}\sqrt{a^2 - x^2} \quad . \quad . \quad . \quad . \quad (1)$$

(1) shows that the values of x lie between a and zero. For $x = a$ we have $y = 0$, and $x = 0$ gives $y = b$.

By article **39** all diameters pass through the point O at which the two axes intersect; and as conjugate diameters bisect each other, point O is a centre.

The equation [**35**] of the tangent shows that the tangent at the vertex (a, o) is perpendicular on the major axis; and the tangent at the end of the minor axis, point (o, b), is parallel to the major axis. Between these two points the angle which the tangent makes with the major axis is greater than 90°.

CHAPTER VI.

THE HYPERBOLA.

45. Definition.—The hyperbola is the locus traced by a point which moves so that the difference between its distances from two given points is constant.

The two given points are called the *foci*, and the constant difference is usually denoted by $2a$.

46. To find the equation of the hyperbola (fig. 15).

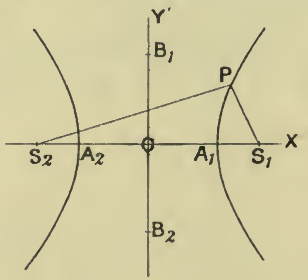


Fig. 15.

S_1 and S_2 are the foci. Line S_1S_2 is taken as x-axis and the middle point O of S_1S_2, called the *centre*, is taken as origin. Let point $P(x, y)$ be a point on the hyperbola, then by definition

$$S_2P - S_1P = 2a \quad . \quad . \quad . \quad . \quad (1)$$

There will evidently be two points A_1 and A_2 of the curve on the x-axis. For these points we have $S_2A_1 - S_1A_1 = S_1A_2 - S_2A_2 = 2a$; hence, $OA_1 = OA_2 = a$.

The ratio

$$\frac{OS_1}{OA_1} = \frac{OS_2}{OA_2} = e > 1 \quad . \quad . \quad . \quad (2)$$

is called the *eccentricity* of the hyperbola. By (2) we have

$$O\,S_1 = ea, \quad \text{and} \quad O\,S_2 = -ea \quad . \quad . \quad . \quad (3)$$

We have now

$$\overline{S_2P}^2 = y^2 + (x + ea)^2, \quad \text{and} \quad \overline{S_1P}^2 = y^2 + (x - ea)^2,$$

hence,

$$\overline{S_2P}^2 - \overline{S_1P}^2 = (S_2P - S_1P)(S_2P + S_1P) = 4eax, \quad S_2P - S_1P = 2a\,;$$

therefore

$$S_2P + S_1P = 2ex.$$

Hence,

$$S_2P = a + ex, \quad \text{and} \quad S_1P = ex - a.$$

We have now

$$\overline{S_1P}^2 = (ex - a)^2 = y^2 + (x - ea)^2.$$

Hence, the equation of the hyperbola is

$$y^2 = (e^2 - 1)(x^2 - a^2) \quad . \quad . \quad [\mathbf{40}]$$

Let $b^2 = a^2(e^2 - 1)$, then [**40**] becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad [\mathbf{41}]$$

For $x = 0$ in [**41**] we have $y = \pm b\sqrt{-1}$, which shows that the curve meets the y-axis in two imaginary points.

The points A_1 and A_2 are called the *vertices* of the hyperbola, A_1A_2 is the *first axis* or the *transverse axis*, and B_1B_2 is the *second axis* or the *conjugate axis*. $O\,B_1 = O\,B_2 = b$.

As the equations of the ellipse and the hyperbola only differ in the sign of b^2, many of the propositions for the hyperbola may be derived from those for the ellipse by substituting $-b^2$ for b^2.

47. To find the points of intersection of a straight line and the hyperbola.

As in article **36**, the co-ordinates of the points of intersection are found by solving the two simultaneous equations

$$y' = mx' + c, \quad \text{and} \quad \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \quad . \quad . \quad (1)$$

By eliminating y' in (1) we obtain

$$x'^2(b^2 - m^2a^2) - 2ma^2cx' - a^2(b^2 + c^2) = 0 \quad . \quad . \quad (2)$$

Hence,

$$x' = \frac{mca^2 \pm ab\sqrt{b^2 - m^2a^2 + c^2}}{b^2 - m^2a^2}, \text{ and } y' = \frac{b^2c \pm mab\sqrt{b^2 - m^2a^2 + c^2}}{b^2 - m^2a} \quad (3)$$

(α) The straight line will not intersect the hyperbola when $b^2 - m^2a^2 + c^2 < 0$.

(β) When $b^2 - m^2a^2 + c^2 > 0$, the straight line will intersect the hyperbola at two points.

(γ) When $b^2 - m^2a^2 + c^2 = 0$, the two points of intersection will become coincident points, and the straight line is a tangent.

(δ) The denominators in (3) become zero when $b^2 - m^2a^2 = 0$, or when $m = \pm\frac{b}{a}$. The equation of the straight line in (1) will then be

$$y = \frac{b}{a}x + c, \text{ or } y = -\frac{b}{a}x + c \quad . \quad . \quad . \quad (4)$$

Let us consider the general quadratic equation

$$Ax^2 + Bx + C = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Substitute $\frac{1}{y}$ for x in (5), then we get

$$A + By + Cy^2 = 0 \quad . \quad . \quad . \quad . \quad (6)$$

If $A = 0$, one root in (6) will be zero; if $A = 0$ and $B = 0$, both roots in (6) will be zero. But $y = 0$ makes $x = \infty$. Hence, if $A = 0$, there will be one infinite root in (5), and if $A = 0$ and $B = 0$, both roots in (5) will be infinite.

If now $b^2 = m^2a^2$, then there will be one infinite root in (2). The two straight lines (4) will therefore intersect the hyperbola, at two points of which one is at an infinite distance. The co-ordinates of the other point are

$$\text{For } m = +\frac{b}{a}, \quad x' = -\frac{a(b^2 + c^2)}{2bc}, \text{ and } y' = -\frac{b^2 - c^2}{2c} \quad . \quad . \quad (7)$$

$$\text{For } m = -\frac{b}{a}, \quad x' = \frac{a(b^2 + c^2)}{2bc}, \text{ and } y' = -\frac{b^2 - c^2}{2c} \quad . \quad . \quad . \quad (8)$$

48. To find the equation of the tangent to the hyperbola.

According to the preceding article, the straight line $y = mx + c$ will be a tangent to the hyperbola when $b^2 - m^2a^2 + c = 0$.

Hence,
$$y = mx \pm \sqrt{m^2a^2 - b^2} \quad . \quad . \quad . \quad [42]$$

is the equation of the tangent. [42] shows that there are two tangents parallel to $y = mx + c$.

By (3), article **47**, we have $m=\frac{b^2x'}{a^2y'}$, the equation of the tangent may therefore also be written

$$y-y'=\frac{b^2x'}{a^2y'}(x-x') \quad . \quad . \quad . \quad . \quad [43]$$

By transformation of [**43**] the equation of the tangent is also

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \quad . \quad . \quad . \quad . \quad [44]$$

[**42**] shows that, when $m^2>\frac{b^2}{a^2}$, there will be two tangents to the hyperbola which are parallel to $y=mx+c$. But there will be no tangents at all when $m^2<\frac{b^2}{a^2}$. When $m=\pm\frac{b}{a}$, then there will be two tangents to the hyperbola, both passing through the centre. Their equations are

$$y=+\frac{b}{a}x, \text{ and } y=-\frac{b}{a}x \quad . \quad . \quad . \quad (1)$$

The angle which the tangents in (1) make with the x-axis is smaller than that of any other tangent to the hyperbola. As $c=0$ in (1), it follows that the coefficients of x^2 and x in (2), article **47**, are both zero. Hence, both roots of the equation are infinite. The tangents (1) touch therefore the hyperbola at points which are situated at an infinite distance. Such a tangent is called an *asymptote*. *The hyperbola has therefore two asymptotes* (1).

49. To find the locus of the middle points of parallel chords of the hyperbola.

As in article **39**, the co-ordinates of the middle point are, according to (3), article **47**,

$$h=\frac{x_1+x_2}{2}=\frac{ma^2c}{b^2-m^2a^2}, \text{ and } k=\frac{y_1+y_2}{2}=\frac{b^2c}{b^2-m^2a^2} \quad . \quad (1)$$

By eliminating c in (1) we get

$$k=\frac{b^2}{ma^2}h,$$

and the locus

$$y=\frac{b^2}{ma^2}x \quad . \quad . \quad . \quad . \quad . \quad . \quad [45]$$

which is the equation of a straight line through the origin. Hence, the diameters of a hyperbola like those of the ellipse pass through the centre of the curve.

As in article **39**, the diameter of chords parallel to [**45**] would be $y = mx$. Hence, the two diameters are *conjugate diameters.*

50. The tangent at the extremity of any diameter is parallel to the chords which it bisects.

The equation of the diameter is

$$y = \frac{b^2}{ma^2}x \quad . \quad . \quad . \quad . \quad (1)$$

and that of the tangent through point (x', y') is

$$y - y' = \frac{b^2x'}{a^2y'}(x - x') \quad . \quad . \quad . \quad (2)$$

Point (x', y') lies on (1);

hence, $$y' = \frac{b^2}{ma^2}x', \quad \text{or} \quad \frac{x'}{y'} = \frac{ma^2}{b^2},$$

which, inserted in (2), gives

$$y - y' = m(x - x') \quad . \quad . \quad . \quad . \quad (3)$$

but (3) is parallel to the system of parallel chords of which (1) is the diameter.

51. To show that the tangent at any point on the hyperbola bisects the angle between the two focal lines through the point.

Let P (x', y') be a point on the hyperbola. The equations of the focal lines through P are

$$y'x - (x' - ea)y - eay' = 0, \text{ and } -y'x + (x' + ea)y - eay' = 0 \quad (1)$$

The equation of the bisector of the angle in which the origin is situated is, according to [**26**],

$$\frac{y'x - (x' - ea)y - eay'}{\sqrt{y'^2 + (x' - ea)^2}} - \frac{-y'x + (x' + ea)y - eay'}{\sqrt{y'^2 + (x' + ea)^2}} = 0 \quad . \quad (2)$$

The quantities under the radical signs are respectively

$$\overline{S_1P}^2 = (ex' - a)^2, \quad \text{and} \quad \overline{S_2P}^2 = (a + ex')^2.$$

(2) may be reduced to

$$y - y' = \frac{b^2x'}{a^2y'}(x - x') \quad . \quad . \quad . \quad (3)$$

but (3) is the equation of the tangent at (x', y').

On account of this property of the hyperbola, a ray emanating from one of the foci of a hyperbolic reflector will be reflected in a direction as if it emanated from the other focus.

52. The director circle of a hyperbola.

By the same process as applied in article **41**, the equation of the director circle will be found to be

$$x^2+y^2=a^2-b^2 \quad . \quad . \quad . \quad . \quad (1)$$

The radius of (1) is $\sqrt{a^2-b^2}$. The director circle will therefore only exist when

$$a^2>b^2, \quad \text{or} \quad 1>\frac{b^2}{a^2},$$

i.e. when the angle which the asymptote makes with the first axis is less than 45°.

53. To trace the hyperbola (fig. 16).

The equation [**41**] of the hyperbola shows that both axes are axes of symmetry; and therefore, as in the case of the ellipse,

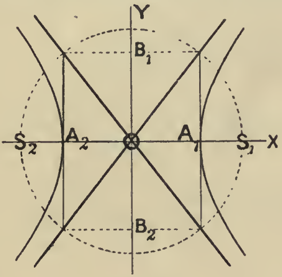


FIG. 16.

it will be sufficient to investigate that part of the curve which is situated in the 1st quadrant. In the latter quadrant x is always positive, and the ordinate is

$$y=+\frac{b}{a}\sqrt{x^2-a^2} \quad . \quad . \quad . \quad . \quad (1)$$

(1) shows that the values of x lie between $x=a$ and $x=\infty$. When $x=a$ we have $y=0$, and when $x=\infty$, $y=\infty$. As the curve is symmetrical about the origin, the latter point is a centre.

The equation [**43**] of the tangent shows that the tangent at the vertex (a, o) is perpendicular to the major axis. It has been shown in article **48** that the angle which the tangent makes with the x-axis is greater than $\tan^{-1}\dfrac{b}{a}$ and is smaller than 90°, except at the vertex. It is also useful to draw the asymptotes which the curve tends to meet, but only reaches at an infinite distance.

54. The equilateral or rectangular hyperbola.

In the particular case when $a=b$, the equation of the hyperbola becomes

$$x^2-y^2=a^2 \quad . \quad . \quad . \quad . \quad [46]$$

and the hyperbola is then called an *equilateral* or *rectangular* hyperbola. The equations of the asymptotes are

$$y=x, \quad \text{and} \quad y=-x \quad . \quad . \quad . \quad (1)$$

i.e. they are inclined at angles $\pm 45°$ to the x-axis, and are therefore at right angles, hence the name rectangular hyperbola.

55. The equation of the rectangular hyperbola referred to its asymptotes.

Angle θ in article **5** is in this case $-45°$; hence, $\cos\theta=+\sqrt{\frac{1}{2}}$, $\sin\theta=-\sqrt{\frac{1}{2}}$, $x=\sqrt{\frac{1}{2}}(x'+y')$, and $y=\sqrt{\frac{1}{2}}(y'-x')$. Inserting the latter values of x and y in [46] we get

$$x'y'=\frac{a^2}{2}$$

or, as it is usually written,

$$xy=c^2 \quad . \quad . \quad . \quad . \quad . \quad [47]$$

which is the required equation.

The equation [47] is of great importance in practical Physics and Engineering.

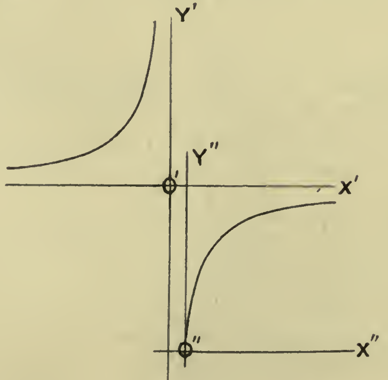


FIG. 17.

56. Another equation of the rectangular hyperbola.

In fig. 17 O′X′ and O′Y′ are the asymptotes. The equation of the curve referred to the asymptotes, and, in the position shown in fig. 17, is obtained by turning the original axes through an angle of $+45°$. Hence, as in the preceding article, the required equation is

$$x'y' = -\frac{a^2}{2} \quad . \quad . \quad . \quad . \quad (1)$$

We shall now proceed to find the equation of the curve referred to a pair of axes O″X″ and O″Y″, which are parallel to the asymptotes, and whose origin is $O''(h, -d)$.

By article **5** we have, $x' = x'' + h$ and $y' = y'' - d$; inserting these values in (1), we get $x''y'' - dx'' + hy'' - hd = -\frac{a^2}{2}$, but point O″ is a point on the curve; hence, $-hd = -\frac{a^2}{2}$. The required equation will therefore be

$$y'' = \frac{dx''}{h + x''}$$

or, as it is usually written,

$$y = \frac{ax}{b + x} \quad . \quad . \quad . \quad . \quad [48]$$

CHAPTER VII.

CONIC SECTIONS.

57. Definition.—The locus of a point, which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line, is called a conic section.

The fixed point is called the *focus.* The constant ratio is called the *eccentricity*, and is denoted by e. The fixed straight line is called the *directrix*, and the straight line through the focus and perpendicular to the directrix is called the *axis.*

It is evident that when the *eccentricity* is *unity* the conic section is a *parabola.*

58. The eccentricity is smaller than unity.

Let S_2 be the focus, K_2L_2 the directrix, and Z_2S_2 the axis (fig.18).

FIG. 18.

There will be two points A_1 and A_2 on the axis, such that

$$S_2A_2 = e\overline{A_2Z_2} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$S_2A_1 = e\overline{Z_2A_1} \quad . \quad . \quad . \quad . \quad (2)$$

Take the middle point O of A_1A_2 as origin, and let the length of A_1A_2 be $2a$; adding (1) and (2) we get

$$2a = 2e\overline{Z_2O}, \quad \text{or} \quad Z_2O = \frac{a}{e}.$$

Subtract (1) from (2) and we get

$$S_2O = ea.$$

Let $P(x, y)$ be a point on the curve, then

$$\overline{S_2P}^2 = e^2\overline{MP}^2, \quad \text{or} \quad y^2 + (x+ea)^2 = e^2\left(x + \frac{a}{e}\right)^2.$$

Taking $a^2(1-e^2) = b^2$, the equation of the locus will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of the ellipse. We may therefore also define *the ellipse* as *the locus of a point which moves so that its distance from a fixed point bears a constant ratio, which is less than unity, to its distance from a fixed line.*

On account of the symmetry of the ellipse with respect to the y-axis there will evidently be another directrix, L_1K_1, to the right of the y-axis and at the same distance from the latter as K_2L_2.

59. The eccentricity is zero.

In the case of the ellipse we have

$$b^2 = a^2(1-e^2).$$

If $e=0$, b will be equal to a and the equation of the ellipse will be

$$x^2 + y^2 = a^2,$$

which is a circle with centre at the origin and with radius equal to a. *The circle is* therefore *a conic whose eccentricity is zero.* In the case of the ellipse we saw that $Z_2O = \frac{a}{e}$ and $S_2O = ea$. As $e=0$ for the circle, it will be seen that the directrix of the circle is at an infinite distance, and that the foci of the circle coincide with the centre.

60. The eccentricity is greater than unity (fig. 19).

There will be two points A_1 and A_2 on the axis, such that

$$S_2A_1 = e\overline{Z_2A_1} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$A_2S_1 = e\overline{A_2Z_1} \quad S_2A_2 = e\overline{A_2Z_2} \quad . \quad . \quad . \quad (2)$$

Let the length A_2A_1 be $2a$, and let the origin be the middle point of A_2A_1. Subtracting (1) from (2) we get

$$Z_2O = \frac{a}{e}. \quad . \quad . \quad . \quad . \quad (3)$$

Adding (1) and (2) we get

$$S_2O = ea \quad . \quad . \quad . \quad . \quad (4)$$

Let point $P(x, y)$ be a point on the curve, then

$$\overline{S_2P}^2 = e^2\overline{PM}^2 = e^2\left(x + \frac{a}{e}\right)^2, \quad \text{or} \quad y^2 + (x + ea)^2 = (ex + a)^2.$$

Taking $a^2(e^2 - 1) = b^2$, the equation of the locus will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is the equation of the hyperbola. Hence, we may also define *the hyperbola* as *the locus of a point which moves so that its*

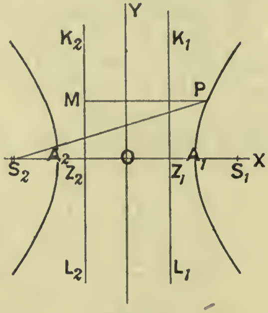


FIG. 19.

distance from a fixed point bears a constant ratio, which is greater than unity, to its distance from a fixed line.

As the y-axis is an axis of symmetry, there will be another directrix, L_1K_1, to the right of the y-axis and at the same distance from the latter as L_2K_2.

61. The focus lies on the directrix (fig. 20).

Let S be the focus as well as the origin. We have

$$SP = e\overline{MP}, \quad \text{or} \quad x^2 + y^2 = e^2x^2,$$

and the locus will be

$$x^2(1 - e^2) + y^2 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

(a), $e < 1$; (1) is reduced to the origin. Let us consider the equation of the ellipse, which may be written

$$x^2(1 - e^2) + y^2 = b^2 \quad . \quad . \quad . \quad . \quad (2)$$

Taking $b=0$, (2) will be reduced to (1), *i.e.* the ellipse reduced to its centre.

(β), $e=1$; the locus (1) will be $y^2=0$, or two straight lines $y=0$, *i.e.* two straight lines coinciding with the axis. This result may also be obtained from the equation of the parabola by making $a=0$; the equation will then be $y^2=0$, or the parabola reduced to its axis. The focus and the directrix of the parabola will respectively coincide with the vertex and the y-axis.

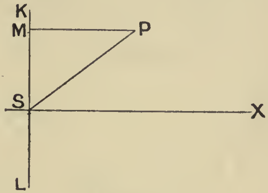


FIG. 20.

(γ), $e>1$; the locus (1) will be $y=\pm x\sqrt{e^2-1}$, or two straight lines through the origin. This result may be obtained from the equation of the hyperbola, which may be written

$$x^2-\frac{y^2}{\left(\frac{b}{a}\right)^2}=a^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

as $b^2=a^2(e^2-1)$ we have when $a=0$ also $b=0$, and hence, (3) becomes

$$y=\pm\frac{b}{a}x=\pm x\sqrt{e^2-1}. \quad . \quad . \quad . \quad . \quad (4)$$

or the hyperbola reduced to its asymptotes.

62. The expression conic section is derived from the cone.

Let us consider a cone of revolution (fig. 21), and a plane, P, passing through the apex.

(a) Let P cut the cone through the apex only. Any plane, Q, parallel to P will intersect the cone in an ellipse or a circle. As Q approaches P the curve becomes smaller and smaller, and when the planes coincide the curve will be reduced to its centre.

(β) Let P touch the cone along a generator. Any plane, Q, parallel to P will intersect the cone in a parabola, which will be reduced to its axis when the two planes coincide.

(γ) Let P cut the cone through two generators. Any plane, Q, parallel to P will intersect the cone in a hyperbola, whose asymptotes are parallel to the two generators through which P cuts the cone. The hyperbola will be reduced to its asymptotes when Q coincides with P.

Under the head of conic sections are therefore included—(i.) the ellipse, including the circle and a point; (ii.) the parabola,

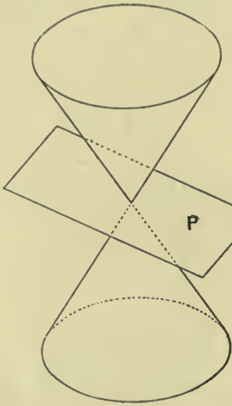


FIG. 21.

including two coincident straight lines; (iii.) the hyperbola, including two intersecting straight lines.

When the distance of the apex of the cone becomes infinite, then the cone becomes a cylinder. Any plane, Q, parallel to the axis of the cylinder cuts the cylinder through two parallel straight lines. For this reason two parallel straight lines are also included under the head of conic sections.

CHAPTER VIII.

EXAMPLES.

(1) Trace the graph of

$$x^2 + 3y^2 - 3xy - 12 = 0 \quad . \quad . \quad . \quad (1)$$

Solution.—(1) may be solved w.r.t. x,

$$x = \frac{3y \pm \sqrt{48 - 3y^2}}{2} \quad . \quad . \quad . \quad (2)$$

(1) may also be solved w.r.t. y,

$$y = \frac{x \pm \sqrt{16 - \frac{x^2}{3}}}{2} \quad . \quad . \quad . \quad (3)$$

It follows from (2) that x will be imaginary for values of y larger than $+4$ and smaller than -4; hence, the curve must lie between the two straight lines A B and C D (fig. 22) drawn parallel to the x-axis at distances $+4$ and -4 respectively.

(3) shows that y will be imaginary for values of x larger than $+6{\cdot}93$ and smaller than $-6{\cdot}93$; the curve is therefore situated between the two straight lines B C and A D drawn parallel to the y-axis at distances $+6{\cdot}93$ and $-6{\cdot}93$ respectively.

The entire graph lies therefore within the rectangle A B C D.

The work may be tabulated thus:—

x.	y.	x.	y.
imaginary	$y > +4$	$-6{\cdot}93$	$-3{\cdot}46$
$+6$	$+4$	$x < -6{\cdot}93$	imaginary
$x > +6{\cdot}93$	imaginary	$\pm 3{\cdot}46$	0
$+6{\cdot}93$	$+3{\cdot}46$	-6	-4
0	± 2	imaginary	$y < -4$

The straight lines AB, BC, CD, and DA are tangents to the curve, which is an ellipse (fig. 22).

(2) **Trace the graph of**

$$x^2+y^2+2xy-x-4=0 \quad . \quad . \quad . \quad (1)$$

Solution.—Solving (1) w.r.t. x we have

$$x=\frac{1-2y\pm\sqrt{17-4y}}{2} \quad . \quad . \quad . \quad (2)$$

By solving (1) w.r.t. y we obtain

$$y=-x\pm\sqrt{x+4} \quad . \quad . \quad . \quad . \quad (3)$$

(2) shows that x will be imaginary for values of y greater than $+4{\cdot}25$. The curve must therefore lie below the straight line AB

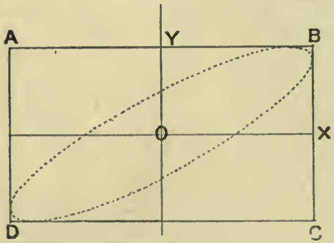


FIG. 22.

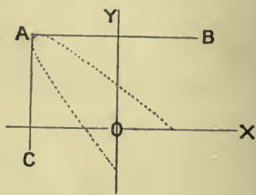


FIG. 23.

drawn parallel to the x-axis and at a distance $+4{\cdot}25$ from the latter (fig. 23).

Values of x smaller than -4 make y imaginary, see (3). The graph must therefore lie to the right of the straight line AC drawn parallel to the y-axis and at a distance -4 from the latter.

We may tabulate the work thus:—

x.	y.	x.	y.
imaginary	$y>+4{\cdot}25$	$x<-4$	imaginary
$-3{\cdot}75$	$+4{\cdot}25$	$+2{\cdot}56$	0
-4 }	$+4$	$-1{\cdot}56$	0
-3 }		0	-2
0	$+2$	$+\infty$	$-\infty$

The straight lines AB and AC are tangents to the curve, which is a parabola (fig. 23).

(3) Trace the graph of

$$y = a \sin \frac{x}{b} \quad . \quad . \quad . \quad . \quad (1)$$

Solution.—The maximum value of y is $+a$, since the maximum value of sine is $+1$; the minimum value of y is $-a$, as the minimum value of sine is -1. x is the length of a circular arc, and the radius of the circle is b. The angle $\frac{x}{b}$ is therefore expressed in radians.

All the values which can be assigned to sine lie between sin 0 and sin 2π. We need therefore only consider that part of the curve which is situated between $x=0$ and $x=2\pi b$; beyond these two values of x the curve will repeat itself.

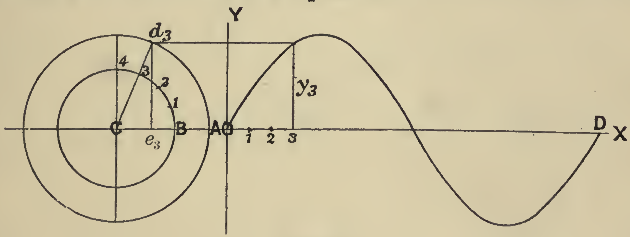


Fig. 24.

If we take an inch as unit of length, then we may proceed by dividing the length $2\pi b$ inches into a convenient number of equal parts, say 8. The work may be tabulated as follows :—

x inches.	$\frac{x}{b}$ radians.	Angle in degrees.	$\sin \frac{x}{b}$ pure number.	y inches.
0·00 πb	0·00 π	0·00	0·000	0·000 a
0·25 ,,	0·25 ,,	45	0·707	0·707 ,,
0·50 ,,	0·50 ,,	90	1·000	1·000 ,,
0·75 ,,	0·75 ,,	135	0·707	0·707 ,,
1·00 ,,	1·00 ,,	180	0·000	0·000 ,,
1·25 ,,	1·25 ,,	225	−0·707	−0·707 ,,
1·50 ,,	1·50 ,,	270	−1·000	−1·000 ,,
1·75 ,,	1·75 ,,	315	−0·707	−0·707 ,,
2·00 ,,	2·00 ,,	360	0·000	0·000 ,,

The curve may also be traced by means of a graphical method. At any convenient distance O C from O (fig. 24) on the x-axis, draw two concentric circles with centre at C and radii C A $=a$ and C B $=b$ respectively. Set off a distance O D $=2\pi b$ and divide it into any convenient number of equal parts at points 1, 2, 3 . . . Divide the circumference of the circle C B into the same number of equal parts at points 1, 2, 3, . . . then the lengths of the arcs B1, B2, B3 . . . are respectively equal to the lengths of O1, O2, O3 . . . measured along the x-axis. The ordinate corresponding to say $x_3 = $ O3 is then

$$y_3 = a \sin \frac{\text{B3}}{b} = e_3 d_3.$$

By continuing the process, a sufficient number of points of the graph may be plotted. Beyond point D to the right, and point O

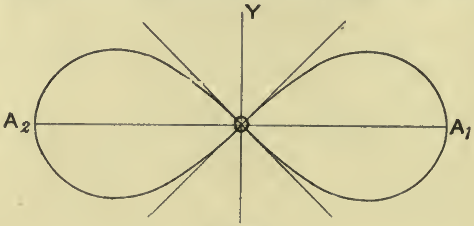


FIG. 25.

to the left, the curve will repeat itself. O D is called the *wave length* of the curve.

(4) Find the polar equation of $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ and plot the graph.

Solution.—By substituting r and θ for x and y, according to [1] the polar equation will be found to be

$$r^4 = 2a^2 r^2 \cos 2\theta \quad . \quad . \quad . \quad . \quad (1)$$

It will be seen that for each value of θ in (1) there are two equal values of r which are zero; the curve passes therefore through the pole. The two other values of r are

$$r = \pm a \sqrt{2 \cos 2\theta} \quad . \quad . \quad . \quad . \quad (2)$$

r will be imaginary when $\cos 2\theta$ is negative; hence the values of θ must lie between $+45°$ and $-45°$. As the two values of r in (2) are numerically equal, the curve must consist of two branches, one on each side of the pole. We need therefore only work out the values of $r = +a\sqrt{2 \cos 2\theta}$. But as $\cos(-2\theta)$

$=\cos(+2\theta)$, it follows that the fixed axis is an axis of symmetry; hence, we need only work out the values of $r= + a\sqrt{2\cos 2\theta}$ for values of θ between 0° and 45°.

It is evident that the polar equation is simpler than the equation in rectangular co-ordinates.

The curve, which is shown in fig. 25, is called the *lemniscate*.

(5) Prove that the three heights of any triangle meet at one point (fig. 26).

Solution.—Let us choose h_a as y-axis and side "a" as x-axis.

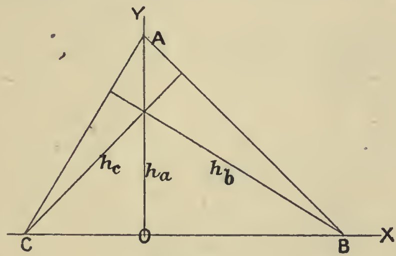


FIG. 26.

The co-ordinates of the vertices are: A(O, h_a), B(a_1, O), and C($-a_2$, O). Let the equations of the heights h_b and h_c be

$$y=m_1x+c_1, \quad \text{and} \quad y=m_2x+c_2 \quad . \quad . \quad (1)$$

respectively; it will suffice to show that $c_1=c_2$.

$$\tan C=\frac{h_a}{a_2}, \quad \text{and} \quad \tan(XBA)=-\frac{h_a}{a_1};$$

hence,
$$m_1=-\frac{a_2}{h_a}, \quad \text{and} \quad m_2=\frac{a_1}{h_a}.$$

But h_b passes through point B and h_c passes through point C, hence, the two equations (1) will give

$$O=-\frac{a_2}{h_a}a_1+c_1, \quad \text{and} \quad O=\frac{a_1}{h_a}(-a_2)+c_2,$$

hence,
$$c_1=c_2=\frac{a_1a_2}{h_a}.$$

(6) (Q. Nov. 1906).—**Prove that the angle between the lines whose equations referred to rectangular axes are** $ax+by+c=0$

and $a'x+b'y+c'=0$ respectively, is $\tan^{-1}\frac{ab'-a'b}{aa'+bb'}$. Find the angle between the lines $x+y=1$ and $2(y-x)=1$; also the area of the quadrilateral which they form with the co-ordinate axes.

Solution.—In article **15** it has been proved that the angle ω between the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is

$$\omega=\tan^{-1}\frac{m_2-m_1}{1+m_1m_2}. \quad . \quad . \quad . \quad (1)$$

In the two given lines $m_1=-\frac{a}{b}$ and $m_2=-\frac{a'}{b'}$; hence, (1) becomes

$$\omega=\tan^{-1}\frac{-\frac{a'}{b'}+\frac{a}{b}}{1+\frac{a}{b}\frac{a'}{b'}}=\tan^{-1}\frac{ab'-a'b}{aa'+bb'}.$$

In the given numerical example we get

$$\tan\omega=\frac{1\times2+2\times1}{-1\times2+1\times2}=\infty,$$

hence, $\omega=90°$. The equations of the two straight lines may be written thus, $y=-x+1$ and $y=x+\frac{1}{2}$, which shows that the

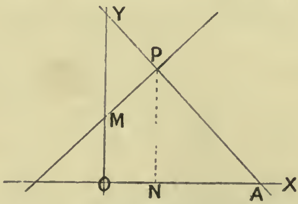


FIG. 27.

former makes an angle of 135° and the latter an angle of 45° with the x-axis.

The co-ordinates of P are $ON=\frac{1}{4}$, $NP=\frac{3}{4}$; also $OM=\frac{1}{2}$ and $NA=NP=\frac{3}{4}$; hence, area

$$OMPAO=\overline{ON}\frac{OM+NP}{2}+\frac{\overline{NA}\,NP}{2}=\frac{7}{16} \text{ (fig. 27).}$$

(7) (Q. Nov. 1906).—**Find the equations to the two circles, each of which passes through the origin and touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at an end of the major axis. Show that the condition that the circles meet the ellipse at other points is $\frac{a}{b}>\sqrt{2}$, and find the co-ordinates of these points.**

Solution.—There will be a circle on each side of the origin; their equations will be—

To the right of the origin:

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}, \quad \text{or} \quad x^2+y^2-ax=0\,. \qquad . \quad (1)$$

To the left of the origin:

$$\left(x+\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}, \quad \text{or} \quad x^2+y^2+ax=0\,. \qquad . \quad (2)$$

We need only find the intersection of (1) with the ellipse. By eliminating y between the two equations we get

$$x^2(b^2-a^2)+a^3x-a^2b^2=0,$$

the roots of which gives the abscissæ to the required points, viz.,

$$x'=a, \quad \text{and} \quad x'=\frac{ab^2}{a^2-b^2}.$$

The corresponding ordinates will be obtained by inserting the values of x' in (1). We get

$$y'=0, \quad \text{and} \quad y'=\pm\frac{ab\sqrt{a^2-2b^2}}{a^2-b^2} \qquad . \quad . \quad (3)$$

Therefore the three points in which (1) and the ellipse meet are: (a, o), at which the two curves touch, and

$$\left(\frac{ab^2}{a^2-b^2}, \frac{ab\sqrt{a^2-2b^2}}{a^2-b^2}\right), \text{ and } \left(\frac{ab^2}{a^2-b^2}, -\frac{ab\sqrt{a^2-2b^2}}{a^2-b^2}\right) \quad (4)$$

The three points in which (2) and the ellipse meet are: $(-a, o)$, at which the curves touch, and

$$\left(-\frac{ab^2}{a^2-b^2}, \frac{ab\sqrt{a^2-2b^2}}{a^2-b^2}\right), \text{ and } \left(-\frac{ab^2}{a^2-b^2}, -\frac{ab\sqrt{a^2-2b^2}}{a^2-b^2}\right) \quad (5)$$

The two last points in (4) and (5) will vanish when $a^2 - 2b^2 = 0$, or $\frac{a}{b} = \sqrt{2}$; hence, $\frac{a}{b}$ must be greater than $\sqrt{2}$ if the two circles shall meet the ellipse in more than one point each.

(8) (Q. Nov. 1907).—**State the formula for the length of the perpendicular drawn to the straight line $ax + by = 0$ from the point (h, k), distinguishing between the two cases in which the point is on one side or on the other side of the line, the axes being rectangular.**

Assuming the equation $xy = c^2$ for a rectangular hyperbola, prove that its equation referred to rectangular axes inclined at an angle of 45° to the original axes is $x^2 - y^2 = 2c^2$. Which line is here taken for axis of x?

P is any point on a rectangular hyperbola, C the centre, A the vertex of the branch on which P lies, and N the foot of the perpendicular drawn from P to the axis C A. Prove that $\overline{AP}^2$ is equal to $2\,\overline{CN}\,\overline{AN}$.

Solution—First part.—See article **21**.—If (h, k) is situated on the positive side of the given line, *i.e.* on the side where the perpendicular drawn from the origin to the line through (h, k) and parallel to the given line is positive, then the distance is

$$d = \frac{ah + bk}{\sqrt{a^2 + b^2}}.$$

If the point (h, k) is situated on the other side of the given line, then the perpendicular is negative and the distance is

$$d = -\frac{ah + bk}{\sqrt{a^2 + b^2}}.$$

The given line passes through the origin.

Second part.—The new x-axis makes 45° and the new y-axis makes 135° with the present x-axis. According to article **5** we have

$$x = (x' - y')\sqrt{\tfrac{1}{2}}, \quad \text{and} \quad y = (x' + y')\sqrt{\tfrac{1}{2}},$$

these values inserted in the given equation give

$$(x' - y')(x' + y') = 2c^2, \quad \text{or} \quad x'^2 - y'^2 = 2c^2.$$

Third part.—Draw a rectangular hyperbola with its axes along the co-ordinate axes and with its centre at the origin.

Its equation is then $x^2 - y^2 = a^2$. C N is the abscissa of P, and we have

$$\overline{AP}^2 = \overline{NP}^2 + \overline{AN}^2,$$

but according to the equation of the curve

$$\overline{CN}^2 - \overline{NP}^2 = a^2;$$

we have also

$$AN = CN - a;$$

hence,

$$\overline{AP}^2 = \overline{CN}^2 - a^2 + \overline{AN}^2 = (CN - a)(CN + a) + \overline{AN}^2$$
$$= \overline{AN}(CN + a + CN - a) = 2\,\overline{CN}\,\overline{AN}.$$

(9) Find the centre, axes, vertices, foci, asymptotes, and directrices of a hyperbola when the curve is drawn.

Solution. — Refer to figs. 15, 16, and 19.—(i.) Draw two parallel chords, one in each branch of the curve. Bisect both chords and draw the straight line through their middle points; the latter line is a diameter, whose middle point is the *centre* of the hyperbola.

(ii.) With the centre of the curve as centre, and with a convenient radius, draw a circle which meets the curve in four points. Draw the four chords (not the diameters) joining the four points, thus making two sets of two parallel chords, which meet at right angles. The diameters of the latter chords contain the axes of the curve. The *vertices*, the length of the first axis and therefore also that of the *semi-axis a*, are thus determined.

(iii.) We may determine the semi-axis b in the following manner: — Draw two convenient parallel chords and their diameter, then draw the tangent to the hyperbola at one of the points at which the diameter meets the curve. This tangent is parallel to the chords. The intercept c which the tangent cuts off the y-axis is by [**42**] equal to $\sqrt{m^2a^2 - b^2}$, hence, $b = \sqrt{m^2a^2 - c^2}$, ma is easily constructed. Hence, b and c are sides in a right-angled triangle whose hypotenuse is ma; b is thus determined.

(iv.) By article **46** we have that $b^2 = a^2(e^2 - 1)$ or $ae = \sqrt{a^2 + b^2}$, i.e. ae is the hypotenuse of a right-angled triangle whose sides are a and b; but $ae = OS_1 = OS_2$; hence, the *foci* are determined.

(v.) a and b being known, we can readily determine the position of the asymptotes. Draw the tangents at the vertices, and from the latter points set off a length b on each tangent to both sides of the axis. The two straight lines joining the diametrically opposite ends of the tangents are the *asymptotes.*

(vi.) By articles **46** and **60** we have that

$$e = \frac{O\,S_1}{O\,A_1} = \frac{S_1A_1}{Z_1A_1},$$

hence

$$\frac{O\,S_1}{O\,A_1} = \frac{O\,S_1 - S_1A_1}{O\,A_1 - Z_1A_1} = \frac{O\,A_1}{O\,Z_1},$$

or $O\,Z_1$ is third proportional to $O\,S_1$ and $O\,A_1$.

The position of the *directrices* is thus determined.

(10) Find the centre, axes, vertices, foci, and directrices of an ellipse when the curve is drawn.

Solution.—Refer to figs. 13 and 18.—(i.) Draw two parallel chords as far apart as convenient. Bisect the chords and draw a straight line through their middle points. The latter line is a diameter, whose middle point is the *centre* of the ellipse.

(ii.) With the centre of the ellipse as centre, and with a convenient radius, draw a circle which meets the curve in four points. Draw the four chords (not the diameters) joining the four points, thus making two sets of two parallel chords, which meet at right angles. The diameters of the latter chords are the *axes* of the ellipse, and the longest is the axis major. The two points A_1 and A_2 at which the axis major meets the curve are the *vertices* of the ellipse.

(iii.) With the centre at B_1 and a radius equal to the semi-axis major $O\,A_1 = a$, draw a circular arc, which meets the axis major at points S_1 and S_2; these two points are the *foci* of the ellipse.

(iv.) By articles **35** and **58** we have that the eccentricity of the ellipse is

$$e = \frac{O\,S_1}{O\,A_1} = \frac{S_1A_1}{Z_1A_1}, \quad \text{or} \quad \frac{O\,S_1}{O\,A_1} = \frac{O\,S_1 + S_1A_1}{O\,A_1 + Z_1A_1} = \frac{O\,A_1}{O\,Z_1},$$

i.e. the distance of the *directrix* from the centre of the curve is third proportional to $O\,S_1$ and $O\,A_1$, which can easily be constructed.

(11) Find the axis, focus, directrix, and vertex of a parabola when the curve is drawn.

Solution.—Refer to figs. 9 and 12.—(i.) Draw two parallel chords to the curve as far apart as possible; bisect the chords; the straight line through the middle points of the chords is a diameter and is parallel to the axis of the parabola.

(ii.) Draw a chord at right angles to the diameter, bisect the chord and draw a straight line through its middle point and parallel to the diameter. The straight line is the *axis* of the parabola, and the point at which it meets the curve is the *vertex* 0. The y-axis can now be drawn.

(iii.) Take any point P on the curve, draw the ordinate N P, then O N is the abscissa to P. Set off a length O L = O N, then L P is the tangent at P.

(iv.) Draw the diameter P Q through P, then make angle M P S equal to angle Q P M. The point S on the axis is the *focus*.

(v.) The *directrix* is parallel to the y-axis and is situated to the left of the vertex. Its distance from O is equal to O S.

The equation of the parabola referred to the usual axes is

$$y^2 = 4\overline{OS}\,x.$$

(12) (Q. Oct. 1909).—**Determine the co-ordinates of the centre of the circle which passes through the points** (1, 2), (2, 3), (4, 2), **and find the equation of the circle.**

Solution.—The equation of the circle is

$$(x-a)^2+(y-b)^2=r^2, \text{ or } x^2+y^2-2ax-2by+a^2+b^2-r^2=0 \quad (1)$$

By successively inserting the co-ordinates of the given points in the last of (1) we get

$$5-2a-4b+a^2+b^2-r^2=0 \quad . \quad . \quad . \quad (2)$$

$$13-4a-6b+a^2+b^2-r^2=0 \quad . \quad . \quad . \quad (3)$$

$$20-8a-4b+a^2+b^2-r^2=0 \quad . \quad . \quad . \quad (4)$$

Subtracting (2) from (4) we get $a=2{\cdot}5$.

Subtracting (2) from (3) we get $b=1{\cdot}5$.

Inserting the values of a and b in either of the equations (2), (3), or (4), we get $r^2=2{\cdot}5$; hence, the required equation is

$$(x-2{\cdot}5)^2+(y-1{\cdot}5)^2=2{\cdot}5, \quad \text{or} \quad x^2+y^2-5x-3y+6=0.$$

(13) Construct a rectangular hyperbola through a given point when the asymptotes are known.

Solution.—Let O X and O Y (fig. 28), be the asymptotes, and P(h, k) be the given point. Draw line M P L parallel to the x-axis.

Suppose we require to determine a point, $Q(x, y)$, on the hyperbola, which lies a given distance, M R, from the y-axis. Draw lines R N and P T both parallel to the y-axis, then draw O R and through point S draw a line parallel to the x-axis, point Q is then a point on the hyperbola.

Proof.—$\overline{NR} : \overline{ON} :: \overline{TS} : \overline{OT}$, or $k/x = y/h$.

Hence, $$xy = kh. \qquad (1)$$

(1) is the equation of the rectangular hyperbola through point P referred to its asymptotes (see article **55**).

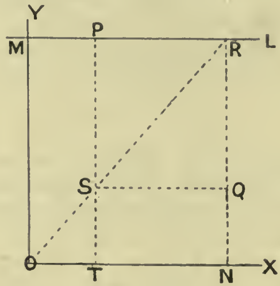


FIG. 28.

By this process any number of points on the hyperbola may be determined.

This construction of the rectangular hyperbola is frequently used in engineering.

(14) (Q. June 1909).—**An ellipse which has its centre at the origin and its axes along the co-ordinate axes, passes through the points $x = 2$, $y = 8$, and $x = 10$, $y = 3$. Find the equation to the ellipse.**

And find the length of the chord which passes through the origin and is parallel to the line joining the two given points.

Solution.—The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

(1) must be satisfied by the values of x and y of the two given points. Hence,

$$\frac{4}{a^2} + \frac{64}{b^2} = 1, \quad \text{and} \quad \frac{100}{a^2} + \frac{9}{b^2} = 1 \qquad (2)$$

By solving the two simultaneous equations (2) with respect to a and b, we get $a = 10{\cdot}76$ and $b = 8{\cdot}142$; hence, the equation of the ellipse is

$$\frac{x^2}{115{\cdot}7} + \frac{y^2}{66{\cdot}29} = 1 \quad . \quad . \quad . \quad . \quad (3)$$

According to [15] the constant m of the straight line through the two given points is $-\frac{5}{8}$. Hence, the equation of the chord through the origin and parallel to the straight line is

$$y = -\frac{5}{8}x \quad . \quad . \quad . \quad . \quad (4)$$

(4) intersects (3) at the two points $(-x', y')$ and $(x', -y')$; the length of the chord will therefore be $2\sqrt{x'^2 + y'^2}$. The values of x' and y' will be found by solving (3) and (4) w.r.t. x and y. We get

$$x' = 8{\cdot}29, \quad y' = 5{\cdot}18;$$

the *length of the chord* $= 19{\cdot}55$.

(15) (Q. Oct. 1909).—**Find the equation of the line which passes through the point** (1, 4) **and is perpendicular to the line** $x - 2y = 0$. **Show that the lines intersect in the point** $\left(\frac{12}{5}, \frac{6}{5}\right)$, **and that the point** (0, 2) **is equidistant from them.**

Solution.—The given line may be written $y = \frac{x}{2}$, which shows that it passes through the origin and that its m is $\frac{1}{2}$; hence, a line which is perpendicular to the given line must have its m equal to -2.

The equation of such a line is

$$y = -2x + c \quad . \quad . \quad . \quad . \quad (1)$$

The required line must also pass through point (1, 4); hence, its equation is

$$y - 4 = -2(x - 1), \quad \text{or} \quad 2x + y - 6 = 0 \quad . \quad . \quad (2)$$

The point of intersection (x', y') of the two lines is obtained by solving the two simultaneous equations

$$x' - 2y' = 0, \quad \text{and} \quad 2x' + y' - 6 = 0,$$

which give

$$x' = \frac{12}{5}, \quad \text{and} \quad y' = \frac{6}{5}.$$

According to article **21** the distances of point (0, 2) from the two lines are: from the given line

$$d_1 = \frac{0-4}{\sqrt{5}} = -\frac{4}{\sqrt{5}},$$

and from (2)

$$d_2 = \frac{0+2-6}{\sqrt{5}} = -\frac{4}{\sqrt{5}}.$$

Point (0, 2) is therefore equidistant from the two lines.

(**16**) (Q. June 1908).—**Give the equations to the tangents to the curves $x^2+y^2=4$ and $5x^2+y^2=5$ at the points where they intersect; and show that the angle at which they intersect is 37° 46′ to the nearest minute.**

Solution.—Write the general equations as

$$x^2+y^2=a^2, \quad \text{and} \quad b^2x^2+y^2=b^2,$$

which may also be written

$$x^2+y^2=a^2, \quad \text{and} \quad \frac{x^2}{1^2}+\frac{y^2}{b^2}=1 \qquad . \qquad . \qquad (1)$$

The first in (1) is a circle with centre at the origin and radius a.

The second in (1) is an ellipse with semi-axes unity and b.

The co-ordinates of the points of intersection will be obtained by solving the two simultaneous equations in (1). We get

$$x' = \pm\sqrt{\frac{b^2-a^2}{b^2-1}}, \quad \text{and} \quad y' = \pm b\sqrt{\frac{a^2-1}{b^2-1}} \qquad . \qquad (2)$$

There will therefore be four points of intersection, viz. in

1st quadrant (x', y'); 2nd quadrant $(-x', y')$;
3rd quadrant $(-x', -y')$; 4th quadrant $(x', -y')$.

The equations of the tangents are—

To the circle:

$$xx'+yy'=a^2;\ -xx'+yy'=a^2;\ -xx'-yy'=a^2;\ xx'-yy'=a^2 \quad (3)$$

To the ellipse:

$$\frac{xx'}{1^2}+\frac{yy'}{b^2}=1;\ -\frac{xx'}{1^2}+\frac{yy'}{b^2}=1;\ -\frac{xx'}{1^2}-\frac{yy'}{b^2}=1;\ \frac{xx'}{1^2}-\frac{yy'}{b^2}=1 \qquad (4)$$

As the tangents in the 1st quadrant are parallel to those in the 3rd quadrant, and the tangents in the 2nd quadrant are parallel to those in the 4th quadrant, it follows that the four sets of tangents intersect at the same angle ω. We need therefore only consider the tangents in the 1st quadrant, which may be written—

To the circle: $$y = -\frac{x}{b}\sqrt{\frac{b^2 - a^2}{a^2 - 1}} + \frac{a^2}{b}\sqrt{\frac{b^2 - 1}{a^2 - 1}},$$

and to the ellipse:

$$y = -xb\sqrt{\frac{b^2 - a^2}{a^2 - 1}} + b\sqrt{\frac{b^2 - 1}{a^2 - 1}} \quad . \quad . \quad (5)$$

According to [**16**]

$$\tan \omega = \frac{\sqrt{b^2 - a^2}\sqrt{a^2 - 1}}{b} \quad . \quad . \quad . \quad (6)$$

Inserting the given numerical values $a = 2$ and $b = \sqrt{5}$, we get the tangents to the circle:

$$0\cdot5x + 1\cdot94y = 4; \quad -0\cdot5x + 1\cdot94y = 4;$$
$$-0\cdot5x - 1\cdot94y = 4; \quad 0\cdot5x - 1\cdot94y = 4.$$

The tangents to the ellipse:

$$0\cdot5x + 0\cdot39y = 1; \quad -0\cdot5x + 0\cdot39y = 1;$$
$$-0\cdot5x - 0\cdot39y = 1; \quad 0\cdot5x - 0\cdot39y = 1.$$

And $\tan \omega = \sqrt{0\cdot6}$; hence, $\omega = 37° \, 46'$ (about).

(**17**) (Q. May 1907).—**Find the equations to the two straight lines which are parallel to the line $4x + 3y + 1 = 0$, and at a distance 2 from it; also the areas of the triangles which they make respectively with the co-ordinate axes.**

Solution.—The length of the perpendicular drawn to the given line from the origin is, according to [**21**], $p = -\frac{1}{5}$; the lengths of the perpendiculars to the required lines will respectively be

$$p_1 = -\frac{1}{5} - 2 = -\frac{11}{5}, \quad \text{and} \quad p_2 = -\frac{1}{5} + 2 = \frac{9}{5};$$

the equations of the two lines will therefore be

$$4x + 3y + 11 = 0, \quad \text{and} \quad 4x + 3y - 9 = 0 \quad . \quad . \quad (1)$$

The two equations (1) may, according to (1), article **10**, also be written

$$\frac{x}{-\frac{11}{4}} + \frac{y}{-\frac{11}{3}} = 1, \quad \text{and} \quad \frac{x}{\frac{9}{4}} + \frac{y}{\frac{9}{4}} = 1 \quad . \quad (2)$$

The areas of the two triangles will respectively be

$$a_1 = \frac{1}{2} \times \frac{11}{4} \times \frac{11}{3} = 5{\cdot}04, \quad \text{and} \quad a_2 = \frac{1}{2} \times \frac{9}{4} \times \frac{9}{3} = 3{\cdot}375;$$

a_1 lies below the x-axis and to the left of the y-axis, or, in other words, in the 3rd quadrant; a_2 lies in the 1st quadrant.

(18) (Q. Nov. 1908).—**A rod of length 5″ is moved so that one end moves along each of two lines at right angles. Show that the path traced out by any point on the rod, other than the middle point, or either end, is an ellipse, and find the semi-axes of the ellipse traced out by a point 2″ from the end (fig. 29).**

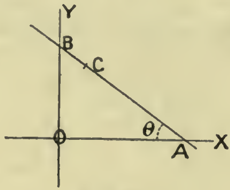


FIG. 29.

Solution.—Let A B = L and A C = l, and let C(x, y) be the point whose locus we require to trace out. We have

$$\frac{x^2}{(L-l)^2} = \cos^2\theta, \quad \text{and} \quad \frac{y^2}{l^2} = \sin^2\theta \quad . \quad . \quad (1)$$

By adding the two equations in (1), we obtain the required equation

$$\frac{x^2}{(L-l)^2} + \frac{y^2}{l^2} = 1,$$

which is an ellipse whose semi-axes are L − l and l. If C be the middle point of A B, then $l = \frac{L}{2}$, and the equation of the ellipse becomes

$$x^2 + y^2 = \frac{L^2}{4},$$

which is a circle with centre at the origin and radius $\frac{L}{2}$.

In the given example $l = 2''$ and L = 5″, the ellipse will therefore be

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

and its semi-axes are 3 and 2.

(19) (Q. Nov. 1907).—**Find the points of intersection of the straight line $4x-2y=3$ with the parabola $y^2=4x$. Also write down the equations to the tangents to the parabola at these points of intersection.**

Find also the angles which the tangents make with the given line.

Solution.—It has been shown in article **29** how to find the points of intersection of a straight line and a parabola. We now proceed to solve the two simultaneous equations $4x-2y=3$ and $y^2=4x$. The two points of intersection will be

$$(2{\cdot}25,\ 3), \quad \text{and} \quad (0{\cdot}25, -1) \quad . \quad . \quad . \quad (1)$$

According to (2), in article **30**, the equations of the two tangents will respectively be

$$3y=2(x+2{\cdot}25), \quad \text{and} \quad -y=2(x+0{\cdot}25),$$

which may also be written

$$y=\frac{2}{3}x+1{\cdot}5, \quad \text{and} \quad y=-2x-0{\cdot}5 \quad . \quad . \quad (2)$$

The angles ω_1 and ω_2 between the given line and the two lines in (2) will, according to [**16**], be respectively

$$\tan\omega_1=\frac{2-\frac{2}{3}}{1+2\times\frac{2}{3}}=\frac{4}{7}=0{\cdot}57143,$$

and
$$\tan\omega_2=\frac{-2-2}{1-4}=\frac{4}{3}=1{\cdot}33333,$$

hence, $\omega_1=29^\circ\ 45'$ (nearly), and $\omega_2=53^\circ\ 8'$ (nearly).

(20) (Q. May 1907).—**Find the equations to the four circles of radius unity, each of which touches both co-ordinate axes; indicating on a sketch which circle is represented by each equation.**

Solution.—Let radius of the circle be a, then according to whether the circle lies—

(i.) In the 1st quadrant, the centre is (a, a), and the equation of the circle is

$$(x-a)^2+(y-a)^2=a^2, \quad \text{or} \quad x^2+y^2-2a(x+y)+a^2=0.$$

(ii.) In the 2nd quadrant, the centre is $(-a, a)$, and the circle

$$(x+a)^2+(y-a)^2=a^2, \quad \text{or} \quad x^2+y^2+2a(x-y)+a^2=0.$$

(iii.) In the 3rd quadrant, the centre is $(-a, -a)$, and the circle

$$(x+a)^2+(y+a)^2=a^2, \quad \text{or} \quad x^2+y^2+2a(x+y)+a^2=0.$$

(iv.) In the 4th quadrant, the centre is $(a, -a)$ and the circle

$$(x-a)^2+(y+a)^2=a^2, \quad \text{or} \quad x^2+y^2-2a(x-y)+a^2=0.$$

In the present case $a=1$; hence, the four circles are

$$(x-1)^2+(y-1)^2=1; \ (x+1)^2+(y-1)^2=1;$$
$$(x+1)^2+(y+1)^2=1; \ (x-1)^2+(y+1)^2=1.$$

(21) Plot the graph of $y^2-6x^2+xy-9x-8y+15=0$.

Solution.—Solving the equation w.r.t. x, we get

$$x=\frac{y-9\pm\sqrt{25y^2-210y+441}}{12};$$

but
$$25y^2-210y+441=25\left(y-\frac{21}{5}\right)^2.$$

Hence,
$$x=\frac{y-9\pm 5\left(y-\frac{21}{5}\right)}{12},$$

or
$$y=2x+5 \quad \text{and} \quad y=-3x+3 \quad . \quad . \quad (1)$$

The graph is two straight lines which intersect at point $(-0{\cdot}4, 4{\cdot}2)$.

(22) Plot the graph of $9x^2+y^2-6xy-12x+4y-45=0$.

Solution.—Solving the equation w.r.t. y, we get

$$y=3x-2\pm 7,$$

or
$$y=3x+5, \quad \text{and} \quad y=3x-9,$$

which are the equations of two parallel lines.

DIFFERENTIAL CALCULUS.

CHAPTER IX.

FUNCTIONS—DIFFERENTIAL COEFFICIENTS.

63. Known functions.

The species of functions which we have met with hitherto are classed as follows :—

(i.) *Algebraical functions,* which consist of powers of variables and constants connected by the signs $+$, $-$, $\times$, $\div$. If the function is of the first degree with respect to the variables, it is said to be *linear*, because the graph of the function is a straight line. If any of the powers of the variables are fractional, the function is said to be *irrational*, otherwise the function is *rational.*

(ii.) *Transcendental functions* which are—(1) *the exponential function* $y=a^x$, where a is a constant; (2) *the logarithmic function* $y=\log_a x$, where a is the base of the system; (3) *the trigonometrical functions* $y=\sin x$, $y=\cos x$, etc.; (4) *the inverse trigonometrical functions* $y=\sin^{-1} x$, $y=\cos^{-1} x$, etc.

64. Inverse functions.

$y=f_1(x)$ is said to be the inverse function of $y=f_2(x)$ if by solving the latter function w.r.t. x we obtain the result $x=f_1(y)$. Thus $y=\log_a x$ is the inverse function of $y=a^x$, because from the latter function we have $x=\log_a y$.

The graph of the inverse function is the same as that of the original function, only its position w.r.t. the axes of co-ordinates is different.

In the mathematical nomenclature the Continental symbols used for denoting the inverse trigonometrical functions differ from those used in Great Britain. Thus, the Continental symbols are— $y=\text{arc}\,(\sin=x)$; $y=\text{arc}\,(\cos=x)$, etc.; and read, "$y=$ the arc whose sine is equal to x," etc. By "the arc" is meant the length of the arc corresponding to a radius equal to unity. Those expressions are evidently meaningless. If y were an arc its dimensions would be [L], but y is an angle expressed in radians, hence a pure number whose dimensions are zero.

On the other hand, the British symbols are misleading, thus—$y = \sin^{-1} x$ might be understood to be $y = 1/\sin x = \operatorname{cosec} x$. I have suggested the adoption of the symbols: $y = \text{ang.}(\sin = x)$; $y = \text{ang.}(\cos = x)$, etc., which read, "$y =$ the angle in radians whose sine is equal to x," etc. The latter symbols will be used in the following.

65. Continuity and discontinuity.

Let us consider the ellipse (fig. 13, p. 27) as being traced by a point starting at the vertex A_2 and moving towards B_1. While the abscissa increases from $-a$ to zero, the ordinate will gradually go

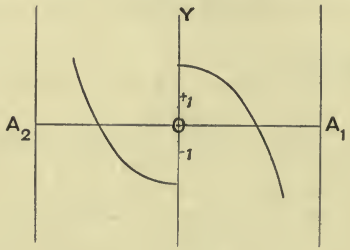


FIG. 30.

through all the possible values between zero and b, and by continuing the motion towards the vertex A_1, the abscissa will steadily increase from zero to $+a$, and the ordinate will steadily decrease, going through all the possible values between b and zero.

The m-constant of the tangent, $y = mx + c$, to the curve will, during the motion from A_2 to B_1, pass gradually through all the possible positive values between infinity and zero; and again, while the point is tracing the curve from B_1 to A_1, the value of m will gradually go through all the possible negative values which exist between zero and infinity.

Let us next trace the lemniscate (fig. 25, p. 50), starting at point A_2 and keeping above the x-axis. As the point traces the curve the abscissa will gradually increase from $-A_2O$ to zero, and at the same time the ordinate will gradually increase from zero to a maximum, and then again decrease towards zero. The m-constant of the tangent will go through all the possible positive values between infinity and zero, and then go through all the possible negative values between zero and -1. The moving point has now arrived at the origin, and it is a question whether it will cross the x-axis or not. If it crosses the x-axis and traces the portion of the curve which lies below the latter axis, nothing remarkable happens; but if the point remains above the x-axis, the value of m will suddenly change from -1 to $+1$ without going through the intermediate values. If we plot the graph (fig. 30) with the same abscissæ as those of the lemniscate, but with ordinates representing the corresponding values of m, then the point tracing the latter graph must, when $x=0$, leave the curve and suddenly jump from a distance -1 to a distance $+1$ from the origin. The curve is *not continuous* at the origin, but there is *discontinuity* in the value of m, for $x=0$.

Hence, we say that a *function is continuous* between any two values x_1 and x_2 of the independent variable, if the point tracing the graph of the function can remain on the curve while the abscissa gradually goes through all the intermediate values between x_1 and x_2.

66. Limiting values.

Suppose we have two quantities of which the first one is an approximate value of the second one. If we, however, by a certain operation can bring the value of the first quantity to become a closer and closer approximation of the value of the second quantity, so that their values ultimately become equal, then the second quantity is said to be the limiting value of the first quantity.

Example 1.—If the length of the side of an equilateral polygon inscribed in a closed curve be decreased indefinitely, and at the same time the number of sides be increased indefinitely, then the area and the shape of the polygon will gradually approach those of the curve, and ultimately the polygon and the curve will coincide. The *curve* is thus the *limit of the polygon.*

Example 2.—We say that $\frac{1}{3}=0{\cdot}333$ . . . and mean that, by increasing the number of threes to the right of the decimal point, we can make the decimal fraction to approach the value of $\frac{1}{3}$ to any degree we please. $\frac{1}{3}$ is therefore the *limiting value* of $0{\cdot}333$. . . .

Example 3.—The function

$$\frac{x(3a^2+2x^2)}{3(a^2+x^2)^{\frac{3}{2}}}$$

will ultimately become zero as the value of x decreases towards zero. When x is increased indefinitely the function becomes $\frac{\infty}{\infty}$, but we may write it

$$\frac{\frac{3a^2}{x^2}+2}{3\left(\frac{a^2}{x^2}+1\right)^{\frac{3}{2}}},$$

which for $x=\infty$ becomes $\frac{2}{3}$; hence,

$$\lim_{x=0}.\frac{x(3a^2+2x^2)}{3(a^2+x^2)^{\frac{3}{2}}}=0, \quad \text{and} \quad \lim_{x=\infty}.\frac{x(3a^2+2x^2)}{3(a^2+x^2)^{\frac{3}{2}}}=\frac{2}{3},$$

or the two limiting values of the function are 0 and $\frac{2}{3}$.

In the following we shall use the symbols *lim.* for $\lim_{x=0}$., and *Lim.* for $\lim_{x=\infty}$.

67. Undetermined forms.

It sometimes happens that for a certain assigned value of the independent variable the function will be reduced to one of the following forms, viz. :—

$$\frac{0}{0};\ \frac{\infty}{\infty};\ 0\times\infty\ ;\ \infty-\infty\ ;\ 0^0;\ \infty^0;\ 1^{\infty}. \qquad (1)$$

In either case the function cannot be determined by direct substitution of the assigned value of the independent variable. The forms in (1) are called undetermined forms, because they are apparently equal to any finite quantity a; thus, we may say $\frac{0}{0}$ is equal to a because $0\times a=0$. As a rule, they have a value which can be determined. The most common of the forms in (1) is $\frac{0}{0}$.

EXAMPLE 1.

$$y=\frac{x^2-a^2}{x-a};$$

the value of y becomes $\frac{0}{0}$ by direct substitution of a for x. The value, however, is $2a$, because

$$\frac{x^2-a^2}{x-a}=\frac{(x-a)(x+a)}{x-a}.$$

EXAMPLE 2.

$$y = \frac{\sin\theta}{\theta};$$

y becomes $\frac{0}{0}$ for $\theta = 0$, but a figure will show that

$$\sin\theta < \theta < \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{or} \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta};$$

substituting 0 for θ we have

$$1 < \frac{0}{\sin 0} < 1,$$

which is impossible unless

$$\text{lim.} \frac{\sin\theta}{\theta} = 1.$$

EXAMPLE 3.—Suppose we have a circular arc whose radius is r and whose angle at the centre is θ. Let the length of the arc be a, and the length of the chord be $ch.$, then the ratio $\frac{ch.}{a}$ will be $\frac{0}{0}$ when $\theta = 0$. But a figure will show that

$$\frac{ch.}{a} = \frac{2r\sin\frac{\theta}{2}}{r\theta} = \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}; \text{ hence, lim. } \frac{ch.}{a} = 1.$$

This subject will be treated more fully in article **94**.

68. The tangent to a curve.

We have, in some preceding chapters, determined the tangent to the various kinds of conic sections. The process consisted in finding the conditions which make the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, in which the straight line $y = mx + c$ intersects the curve, coincident points on the curve.

The value of m is always $\frac{y_1 - y_2}{x_1 - x_2}$, and when the two points become coincident, then we reach the limiting value of m, which corresponds to the tangent. By the process hitherto adopted in this work, we have found the limiting value of m in an indirect way. We shall now proceed to explain a principle by which the limiting value of m may be found in a direct manner.

Let $y = f(x)$ be the equation of the curve shown in fig. 31, and let the problem be to determine the direction of the tangent to any point, $P(x, y)$, on the curve. N is another point on the curve at a very short distance from P; the co-ordinates of N being $x+\delta x$ and $y+\delta y$, where δx and δy are small corresponding increments of the two variables. The chord P N makes an angle ψ with the x-axis, and it is evident that $\tan\psi=\frac{\delta y}{\delta x}$. Let now N travel on the curve towards P, and let it ultimately coincide with P. During the travelling of N, δx will gradually go through all the intermediate values between Q R and zero, and δy will at the same time go through all the intermediate values between S N and zero, but always so that $y+\delta y=f(x+\delta x)$; hence, the ratio $\frac{\delta y}{\delta x}$ depends on the law of generation of the curve. The

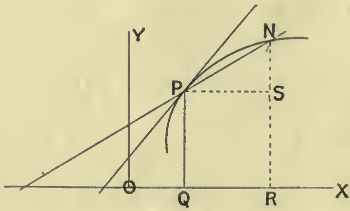


FIG. 31.

limiting value of $\frac{\delta y}{\delta x}$ is evidently $\tan\phi$, where ϕ is the angle which the tangent at P makes with the x-axis; we may therefore write $\tan\phi=\lim.\frac{\delta y}{\delta x}$: the latter is called the *differential coefficient* of $y=f(x)$, and is denoted by $\frac{dy}{dx}$, or $f'(x)$. $\frac{\delta y}{\delta x}$ is a ratio of two finite quantities δy and δx, but $\lim.\frac{\delta y}{\delta x}=\frac{dy}{dx}$ is of the form $\frac{0}{0}$, but it can always be determined. It must therefore be understood that $\frac{dy}{dx}$ is not a fraction whose denominator is dx and whose numerator is dy, but that it is a notation denoting that a certain operation is to be undertaken with $y=f(x)$. It might have been better never to have introduced the notation $\frac{dy}{dx}$ but to have adopted the notation $f'(x)$ only.

The differential coefficient is also called the *derived function* of $y=f(x)$, and the latter is called the *primitive function* of $f'(x)$.

DEFINITION.—*The differential coefficient of a function is the* *limiting value of the ratio of the corresponding and simultaneously vanishing increments of the dependent and the independent variables.*

For the purpose of saving space it has become conventional, instead of writing $\frac{dy}{dx}=f'(x)$, to write $dy/dx=f'(x)$, and even $dy=f'(x)dx$.

69. The differential coefficient as a measurer of the growth of the function.

The differential coefficient being lim. $\frac{\delta y}{\delta x}$ measures the gradient or slope of the curve w.r.t. the x-axis. If we consider δx to be always positive, then $\frac{dy}{dx}$ will be positive or negative, according as the function (y) is growing or diminishing; hence, the gradient at a point of the curve is upwards (in the direction of the positive y) when $\frac{dy}{dx}$ is positive, and downwards when $\frac{dy}{dx}$ is negative. The maximum steepness corresponds to a gradient, $\frac{dy}{dx}$, equal to infinity; and the minimum steepness corresponds to a gradient, $\frac{dy}{dx}$, which is zero.

Let δt be the time it takes to increase the abscissa by δx, then

$$\lim. \frac{\delta x}{\delta t} \quad \text{and} \quad \lim. \frac{\delta y}{\delta t}$$

measure the rates at which the abscissa and the ordinate are respectively growing. But we may write

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \div \frac{\delta x}{\delta t},$$

and therefore

$$\lim. \frac{\delta y}{\delta x} = \frac{dy}{dx} = \lim. \frac{\delta y}{\delta t} \div \lim. \frac{\delta x}{\delta t};$$

hence, the differential coefficient measures the ratio of the rates of growth of the function and the independent variable.

CHAPTER X.

DIFFERENTIATION.

70. Definition.—The operation of determining the differential coefficient of a given function, $y=f(x)$, is called differentiation.

General process.—The co-ordinates of point N (fig. 31) are $x+\delta x$ and $y+\delta y$, which satisfy the equation, $y=f(x)$, of the curve; hence, $y+\delta y=f(x+\delta x)$, but as $y=f(x)$, we have

$$\delta y=f(x+\delta x)-f(x),$$

and
$$\frac{dy}{dx}=\lim.\frac{f(x+\delta x)-f(x)}{\delta x} \quad . \quad . \quad . \quad [49]$$

EXAMPLE 1.—*To find* $\frac{dy}{dx}$ *to the parabola* $y^2=4ax$.

$$(y+\delta y)^2=4a(x+\delta x), \quad \text{or} \quad y^2+\delta y^2+2y\,.\,\delta y=4ax+4a\,.\,\delta x \quad (1)$$

Subtract $y^2=4ax$ from (1) and we get

$$\delta y^2+2y\,.\,\delta y=4a\,.\,\delta x, \quad \text{or} \quad \frac{\delta y}{\delta x}(\delta y+2y)=4a\,;$$

hence,
$$\frac{dy}{dx}=\lim.\frac{\delta y}{\delta x}=\frac{4a}{2y}=\frac{y}{2x},$$

which we also found in article **30.**

EXAMPLE 2.—*To find* $\frac{dy}{dx}$ *of the circle* $x^2+y^2=a^2$.

$$(x+\delta x)^2+(y+\delta y)^2=a^2,$$

or
$$x^2+\delta x^2+2x\,.\,\delta x+y^2+\delta y^2+2y\,.\,\delta y=a^2 \quad . \quad . \quad (1)$$

Subtract $x^2+y^2=a^2$ from (1) and we get

$$\delta x^2+2x\,.\,\delta x+\delta y^2+2y\,.\,\delta y=0, \quad \text{or} \quad \delta x+2x+\frac{\delta y}{\delta x}(\delta y+2y)=0\,;$$

hence, $$\frac{dy}{dx} = \lim. \frac{\delta y}{\delta x} = -\frac{x}{y},$$

which we also found in article **26.**

71. Differential coefficient of a constant is zero.

Suppose $y = f(x) = a$. As the value of y corresponding to $x + \delta x$ is always equal to a, it follows that $\delta y = a - a = 0$;

hence, $$\frac{da}{dx} = 0 \quad . \quad . \quad . \quad . \quad [50]$$

72. The differential coefficient of $\log_e x$.

It has been shown in algebra that

$$\log_e(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad . \quad . \quad . \quad (1)$$

$$\frac{d}{dx}(\log_e x) = \lim. \frac{\log_e(x + \delta x) - \log_e x}{\delta x}$$

$$= \lim. \frac{\log_e \frac{x + \delta x}{x}}{\delta x} = \lim. \frac{\log_e\left(1 + \frac{\delta x}{x}\right)}{\delta x};$$

hence, according to (1),

$$\frac{d}{dx}(\log_e x) = \lim. \frac{\frac{\delta x}{x} - \frac{\delta x^2}{2x^2}}{\delta x} = \frac{1}{x} \quad . \quad . \quad . \quad [51]$$

73. The differential coefficient of a product of a constant and a function, is equal to the constant into the differential coefficient of the function.

Let $y = a \,.\, f(x)$, where a is a constant.

$$\frac{dy}{dx} = \lim. \frac{a \,.\, f(x + \delta x) - a \,.\, f(x)}{\delta x} = \lim. a \frac{f(x + \delta x) - f(x)}{\delta x} = a \,.\, f'(x),$$

hence, $$\frac{d}{dx}[a \,.\, f(x)] = a\frac{df(x)}{dx} = a \,.\, f'(x) \quad . \quad . \quad . \quad [52]$$

74. The differential coefficient of $\log_a x$.

$$x = e^{\log_e x} = a^{\log_a x} \quad . \quad . \quad . \quad . \quad (1)$$

Taking the natural log of (1) we get

$$\log_e x \log_e e = \log_a x \log_e a, \quad \text{or} \quad \log_e x = \log_e a \log_a x;$$

hence, $$\frac{d}{dx}(\log_e x) = \log_e a \frac{d}{dx}(\log_a x)$$

or $$\frac{d}{dx}(\log_a x) = \frac{1}{\log_e a}\,\frac{1}{x} = \frac{M}{x} \quad . \quad . \quad . \quad [53]$$

where M is the modulus of system a.

75. The differential coefficient of a function of a function.

Suppose that we have given $y = \log z$, where $z = \sin x$. By eliminating z between the two equations we get

$$y = \log(\sin x) \quad . \quad . \quad . \quad . \quad (1)$$

If it were required to plot the graph of (1) we should have to begin by plotting the sine curve, $z = \sin x$; and then to plot the logarithmic curve, $y = \log z$. From these two curves we could obtain the required curve. y in (1) is thus a function of a function of x, because z is a function of x, and y is a function of z.

Let δy, δz, and δx be the three corresponding and simultaneously vanishing increments of the three variables in $y = f_1(z)$, where $z = f_2(x)$, and let the angles which the tangents to the curves $y = f_1(z)$, $z = f_2(x)$, and $y = f_1[f_2(x)]$ make with their respective axis of abscissæ be ϕ_1, ϕ_2, and ϕ, then

$$\tan\phi_1 = \frac{dy}{dz} = \text{lim.}\,\frac{\delta y}{\delta z}, \quad \text{and} \quad \tan\phi_2 = \frac{dz}{dx} = \text{lim.}\,\frac{\delta z}{\delta x};$$

but $$\frac{\delta y}{\delta z}\,\frac{\delta z}{\delta x} = \frac{\delta y}{\delta x},$$

and this holds for any value of δx, however small, and therefore also at the lim.; hence, $\tan\phi = \tan\phi_1 \tan\phi_2$, or

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx} \quad . \quad . \quad . \quad . \quad [54]$$

This principle may be extended to any number of functions; hence,

$$\frac{dy}{dx} = \frac{dy}{d(\text{1st function})}\,\frac{d(\text{1st function})}{d(\text{2nd function})} \cdots \frac{d(\text{last function}}{dx} \quad . \quad [55]$$

76. The differential coefficient of x^n.

By taking the $\log_e$ of $y = x^n$ we get

$$\log_e y = n \log_e x \quad . \quad . \quad . \quad . \quad (1)$$

By differentiating (1) w.r.t. x we obtain

$$\frac{d}{dx}(\log_e y) = \frac{d}{dy}(\log_e y)\frac{dy}{dx} = n\frac{d}{dx}(\log_e x)$$

or
$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x}, \quad \text{or} \quad \frac{dy}{dx} = \frac{ny}{x} = nx^{n-1},$$

hence,
$$\frac{d}{dx}(x^n) = nx^{n-1} \quad . \quad . \quad . \quad . \quad [56]$$

77. The differential coefficient of a^x.

By taking the $\log_e$ of $y = a^x$ we get

$$\log_e y = x \log_e a \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating (1) w.r.t. x we obtain

$$\frac{d}{dx}(\log_e y) = \log_e a \frac{dx}{dx}, \quad \text{or} \quad \frac{d}{dy}(\log_e y)\frac{dy}{dx} = \log_e a\,;$$

hence,
$$\frac{1}{y}\frac{dy}{dx} = \log_e a, \quad \text{or} \quad \frac{dy}{dx} = y \log_e a\,;$$

hence,
$$\frac{d}{dx}(a^x) = (\log_e a)\, a^x \quad . \quad . \quad . \quad [57]$$

In the special case when $a = e$ we have

$$\frac{d}{dx}(e^x) = e^x \quad . \quad . \quad . \quad . \quad [58]$$

78. The differential coefficient of a sum of functions of x is equal to the sum of the differential coefficients of the functions with respect to x.

Suppose we have $y = y_1 + y_2 + y_3 \ . \ . \ .$, where $y_1 = f_1(x)$; $y_2 = f_2(x) \ . \ . \ .$, then

$$y + \delta y = y_1 + \delta y_1 + y_2 + \delta y_2 \ . \ . \ . = f_1(x + \delta x) + f_2(x + \delta x) \ . \ . \ . ,$$

and
$$\delta y = f_1(x + \delta x) - f_1(x) + f_2(x + \delta x) - f_2(x) \ . \ . \ . \ ;$$

hence,

$$\frac{d}{dx}(y_1 + y_2 + y_3 \ . \ . \ .) = \text{lim.}\,\frac{\delta y}{\delta x}$$

$$= \text{lim.}\left[\frac{f_1(x + \delta x) - f_1(x)}{\delta x} + \frac{f_2(x + \delta x) - f_2(x)}{\delta x} \ . \ . \ .\right],$$

or
$$\frac{d}{dx}(y_1 + y_2 + y_3 \ . \ . \ .) = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \frac{dy_3}{dx} \quad . \quad . \quad [59]$$

EXAMPLE.—*Find the differential coefficient of* $y = log_e x^n + a^x - 5\ log_e a^x$.

We may also write $y = n \log_e x + a^x - 5x \log_e a$; hence,

$$\frac{dy}{dx} = n\frac{d \log_e x}{dx} + \frac{da^x}{dx} - 5 \log_e a \frac{dx}{dx}$$

$$= \frac{n}{x} + a^x \log_e a - 5 \log_e a = \frac{n}{x} + (a^x - 5) \log_e a.$$

79. The differential coefficient of a product of two functions of x is equal to the first function into the differential coefficient of the second function w.r.t. x plus the second function into the differential coefficient of the first function w.r.t. x.

Suppose we have $y = y_1 y_2$, where $y_1 = f_1(x)$, $y_2 = f_2(x)$, then

$$y + \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2) = y_1 y_2 + y_1 \,.\, \delta y_2 + y_2 \,.\, \delta y_1 + \delta y_1 \,.\, \delta y_2,$$

or

$$\delta y = y_1 \,.\, \delta y_2 + y_2 \,.\, \delta y_1 + \delta y_1 \,.\, \delta y_2 ;$$

hence,

$$\frac{d(y_1 y_2)}{dx} = y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx} \quad . \quad . \quad . \quad [60]$$

[60] may also be written

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} \quad . \quad . \quad . \quad (1)$$

80. The differential coefficient of a product of functions of x divided by the product is equal to the sum of the differential coefficients of the functions w.r.t. x, each divided by its function.

Let $y = y_1 y_2 y_3$, then according to (1), article **79**, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2 y_3}\frac{d(y_2 y_3)}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \frac{1}{y_3}\frac{dy_3}{dx},$$

which may be extended to any number of functions ; thus, if

$$y = y_1 y_2 y_3 \ . \ . \ . \ y_n,$$

then

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \frac{1}{y_3}\frac{dy_3}{dx} + \ . \ . \ . \ + \frac{1}{y^n}\frac{dy_n}{dx} \quad . \quad [61]$$

81. The differential coefficient of a quotient of two functions of x is equal to the denominator into the differential coefficient of the numerator w.r.t. x minus the numerator into the differential coefficient of the denominator w.r.t. x, the whole divided by the square of the denominator.

Let $y = \frac{y_1}{y_2}$, hence, $y_1 = y\ y_2$, and therefore

$$\frac{1}{y_1}\frac{dy_1}{dx} = \frac{1}{y}\frac{dy}{dx} + \frac{1}{y_2}\frac{dy_2}{dx},$$

or

$$\frac{dy}{dx} = \frac{y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}}{{y_2}^2} \quad . \quad . \quad . \quad . \quad [62]$$

which may also be written

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} - \frac{1}{y_2}\frac{dy_2}{dx} \quad . \quad . \quad . \quad (1)$$

(1) may be extended to any number of functions of x; thus, by differentiating

$$y = \frac{y_1 y_2 y_3 \ . \ . \ . \ y_n}{z_1 z_2 z_3 \ . \ . \ . \ z_m},$$

we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1 y_2 y_3 \ . \ . \ . \ y_n}\frac{d(y_1 y_2 y_3 \ . \ . \ . \ y_n)}{dx} - \frac{1}{z_1 z_2 \ . \ . \ . \ z_m}\frac{d(z_1 z_2 \ . \ . \ . \ z_m)}{dx},$$

or

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} \ . \ . \ . \ + \frac{1}{y_n}\frac{dy_n}{dx} - \frac{1}{z_1}\frac{dz_1}{dx} - \frac{1}{z_2}\frac{dz_2}{dx} \ . \ . \ . - \frac{1}{z_m}\frac{dz_m}{dx} \quad . \quad (2)$$

82. The differential coefficient of sin x.

We have given $y = \sin x$;

hence, $\frac{d}{dx}(\sin x) = \lim. \frac{\sin(x + \delta x) - \sin x}{\delta x}$

$$= \lim. \frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x} = \cos x \lim. \frac{\sin \delta x}{\delta x};$$

but, according to article **67**,

$$\lim. \frac{\sin \delta x}{\delta x} = 1;$$

hence,

$$\frac{d}{dx}(\sin x) = \cos x \quad . \quad . \quad . \quad . \quad [63]$$

83. The differential coefficient of cos x.

$$\frac{d}{dx}(\cos x) = \text{lim.}\frac{\cos(x+\delta x) - \cos x}{\delta x}$$

$$= \text{lim.}\frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x} = -\sin x \,\text{lim.}\frac{\sin \delta x}{\delta x},$$

or

$$\frac{d}{dx}(\cos x) = -\sin x\,. \qquad . \qquad . \qquad . \qquad [64]$$

84. The differential coefficient of tan x.

$\tan x = \dfrac{\sin x}{\cos x}$, therefore, by article **81**, we have

$$\frac{d}{dx}(\tan x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} \qquad . \qquad [65]$$

85. The differential coefficient of cot x.

$\cot x = \dfrac{\cos x}{\sin x}$; hence, by article **81**, we have

$$\frac{d}{dx}(\cot x) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} \qquad . \qquad [66]$$

86. The differential coefficient of an inverse function.

If $x = f_2(y)$ be the inverse function of $y = f_1(x)$, then, according to fig. 31 and article **68**, if $\dfrac{dy}{dx} = \tan\phi$, $\dfrac{dx}{dy}$ must be equal to $\cot\phi$; hence,

$$\frac{dy}{dx}\frac{dx}{dy} = 1 \qquad . \qquad . \qquad . \qquad . \qquad [67]$$

87. The differential coefficients of the inverse trigonometrical functions.

(i.) $y = \text{ang}(\sin = x)$; hence, $x = \sin y$, and $\dfrac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$ $= \sqrt{1 - x^2}$, or

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}};$$

hence,

$$\frac{d}{dx}\text{ang}(\sin = x) = \frac{1}{\sqrt{1-x^2}} \qquad . \qquad . \qquad . \qquad [68]$$

(ii.) $y = \text{ang}(\cos = x)$, or $x = \cos y$, $\frac{dx}{dy} = -\sin y$, $\frac{dy}{dx} = -\frac{1}{\sin y}$;

hence, $$\frac{d}{dx}\text{ang}(\cos = x) = -\frac{1}{\sqrt{1-x^2}} \quad . \quad . \quad . \quad [69]$$

(iii.) $y = \text{ang}(\tan = x)$, or $x = \tan y$; hence,

$$\frac{dx}{dy} = \frac{1}{\cos^2 y}, \text{ and } \frac{dy}{dx} = \cos^2 y = \frac{1}{1+\tan^2 y};$$

hence, $$\frac{d}{dx}\text{ang}(\tan = x) = \frac{1}{1+x^2} \quad . \quad . \quad . \quad [70]$$

(iv.) $y = \text{ang}(\cot = x)$, or $x = \cot y$; hence,

$$\frac{dx}{dy} = -\frac{1}{\sin^2 y}, \text{ and } \frac{dy}{dx} = -\sin^2 y = -\frac{1}{1+\cot^2 y};$$

hence, $$\frac{d}{dx}\text{ang}(\cot = x) = -\frac{1}{1+x^2} \quad . \quad . \quad . \quad [71]$$

88. The differential coefficient of $f(x+z)$.

Let x and z in $y = f(x+z)$ be two independent variables, such that the one is constant while the other varies. We may then differentiate $f(x+z)$ w.r.t. x while z is constant or *vice versâ*. Hence,

$$\frac{d}{dx}f(x+z) = \frac{df(x+z)}{d(x+z)}\frac{d(x+z)}{dx} = f'(x+z), \quad \text{as} \quad \frac{d}{dx}(x+z) = 1;$$

also

$$\frac{d}{dz}f(x+z) = \frac{df(x+z)}{d(x+z)}\frac{d(x+z)}{dz} = f'(x+z), \quad \text{as} \quad \frac{d}{dz}(x+z) = 1.$$

Hence, $$\frac{d}{dx}f(x+z) = \frac{d}{dz}f(x+z) = f'(x+z) \quad . \quad . \quad [72]$$

89. Successive differentiation.

As $\frac{dy}{dx} = f'(x)$ is a function of x, it follows that it may be differentiated w.r.t. x; we have

$$\frac{d\frac{dy}{dx}}{dx} = \frac{df'(x)}{dx} = \frac{d^2y}{dx^2} = f''(x) \quad . \quad . \quad . \quad (1)$$

(1) is called the *second differential coefficient* of y w.r.t. x, or the *second derived function* of $f(x)$. In the same manner we obtain

$$\frac{d^3y}{dx^3}=f'''(x) \;.\;.\;.\; \frac{d^ny}{dx^n}=f^n(x) \quad . \quad . \quad . \qquad (2)$$

The last in (2) is called the n^{th} *differential coefficient* of y w.r.t. x, or the n^{th} *derived function* of $f(x)$.

EXAMPLE 1.—*Find the n^{th} differential coefficient of e^x.*

As $\dfrac{de^x}{dx}=e^x$, it follows that $\dfrac{d^ne^x}{dx^n}=e^x$.

EXAMPLE 2.—*Find the n^{th} differential coefficient of x^n.*

$$\frac{dx^n}{dx}=nx^{n-1}\,;\;\; \frac{d^2x^n}{dx^2}=n\frac{dx^{n-1}}{dx}=n(n-1)x^{n-2}\,;$$

hence,
$$\frac{d^nx^n}{dx^n}=n(n-1)(n-2)\;.\;.\;.\;x^{n-n}=\underline{|n}.$$

EXAMPLE 3.—*Find the second differential coefficient of sin x.*

$$\frac{d}{dx}\sin x=\cos x,\; \frac{d^2}{dx^2}\sin x=\frac{d}{dx}\cos x=-\sin x.$$

EXAMPLE 4.—*Find the n^{th} differential coefficient of a^x.*

$$\frac{da^x}{dx}=a^x\log_e a\,;\;\; \frac{d}{dx}(a^x\log_e a)=a^x(\log_e a)^2=\frac{d^2a^x}{dx^2}\,;$$

hence,
$$\frac{d^na^x}{dx^n}=a^x(\log_e a)^n.$$

90. Partial differentiation.

The terms in $f(x, y)=0$ may be divided into four kinds, viz.:

(i.) The terms which contain neither x nor y but only constants; let the sum of these terms be denoted by K.

(ii.) The terms which only contain constants and the variable x; let X denote the sum of these terms.

(iii.) The terms which only contain constants and the variable y; the sum of which we will denote by Y.

(iv.) The terms which contain constants as well as products of x and y; the sum of these terms we will denote by XY.

We may now write

$$f(x, y)=\mathrm{K}+\mathrm{X}+\mathrm{Y}+XY=0 \quad . \quad . \quad . \qquad (1)$$

By articles **75, 78,** and **79** we get

$$\frac{d}{dx}f(x, y) = \frac{dK}{dx} + \frac{dX}{dx} + \frac{dY}{dy}\frac{dy}{dx} + Y\frac{dX}{dx} + X\frac{dY}{dy}\frac{dy}{dx} = 0,$$

or $$\frac{d}{dx}f(x, y) = \left(\frac{dX}{dx} + Y\frac{dX}{dx}\right) + \left(\frac{dY}{dy} + X\frac{dY}{dy}\right)\frac{dy}{dx} = 0 \quad . \quad (2)$$

The first portion of (2) may be obtained by differentiating $f(x, y) = 0$ w.r.t. x, considering y as a constant. The second portion of (2) may be obtained by differentiating $f(x, y) = 0$ w.r.t. y, considering x as a constant and multiplying the result by $\frac{dy}{dx}$. The two bracketed portions of (2) are called *partial differential coefficients*, and in order to distinguish them from the ordinary differential coefficients we will use the letter Δ instead of d. Hence, (2) should be written thus

$$\frac{d}{dx}f(x, y) = \frac{\Delta}{\Delta x}f(x, y) + \frac{\Delta}{\Delta y}f(x, y)\frac{dy}{dx} = 0 . \quad . \quad (3)$$

For the sake of abbreviation we may write $u = f(x, y) = 0$. Hence, (3) becomes

$$\frac{du}{dx} = \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta y}\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{\Delta u}{\Delta x} \div \frac{\Delta u}{\Delta y} \quad . \quad [73]$$

EXAMPLE. $u = f(x, y) = y^2 - x^2 + xy - 2x + 4 = 0$;

$$\frac{\Delta u}{\Delta x} = y - 2x - 2 ; \quad \frac{\Delta u}{\Delta y} = 2y + x ;$$

hence, $$\frac{dy}{dx} = \frac{2x + 2 - y}{2y + x} .$$

91. Table of results to be remembered.

$$\frac{dy}{dx} = \lim . \frac{f(x + \delta x) - f(x)}{\delta x} ;$$

$$\frac{d}{dx}(y_1 + y_2 + \ . \ . \ . \ + y_n) = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \ . \ . \ . \ + \frac{dy_n}{dx} ;$$

$$\frac{d}{dx}(y_1 y_2) = y_1\frac{dy_2}{dx} + y_2\frac{dy_1}{dx} ;$$

$$\frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}}{y_2{}^2} ;$$

$$\frac{1}{y_1 y_2 \ldots y_n} \frac{d}{dx}(y_1 y_2 \ldots y_n) = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \ldots + \frac{1}{y_n}\frac{dy_n}{dx};$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \ldots + \frac{1}{y_n}\frac{dy_n}{dx} - \frac{1}{z_1}\frac{dz_1}{dx} - \ldots - \frac{1}{z_m}\frac{dz_m}{dx},$$

where $y = \dfrac{y_1 y_2 \ldots y_n}{z_1 z_2 \ldots z_m}$.

$$\frac{d}{dx}a = 0;$$

$$\frac{d}{dx}x^n = nx^{n-1};$$

$$\frac{d}{dx}\log_e x = \frac{1}{x};$$

$$\frac{d}{dx}\log_a x = \frac{M}{x};$$

$$\frac{d}{dx}a^x = a^x \log_e a;$$

$$\frac{d}{dx}e^x = e^x;$$

$$\frac{d}{dx}\sin x = \cos x;$$

$$\frac{d}{dx}\cos x = -\sin x;$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x};$$

$$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x};$$

$$\frac{dy}{dx}\frac{dx}{dy} = 1;$$

$$\frac{d}{dx}\text{ang}(\sin = x) = \frac{1}{\sqrt{1+x^2}};$$

$$\frac{d}{dx}\text{ang}(\cos = x) = -\frac{1}{\sqrt{1+x^2}};$$

$$\frac{d}{dx}\text{ang}(\tan = x) = \frac{1}{1+x^2};$$

$$\frac{d}{dx}\text{ang}(\cot = x) = -\frac{1}{1+x^2};$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx};$$

$$\frac{d}{dx}f(x+z) = \frac{d}{dz}f(x+z) = f'(x+z);$$

$$\frac{du}{dx} = \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta y}\frac{dy}{dx} = 0.$$

CHAPTER XI.

EXPANSION.

92. Taylor's theorem.

Suppose we have given $y=f(x)$, and let the problem be to find the expansion of $f(x+\delta x)$ in ascending integral positive powers of δx.

Let

$$f(x+\delta x)=a_0+a_1 . \delta x+a_2 . \delta x^2+a_3 . \delta x^3+ \ . \ . \ . +a_n . \delta x^n+ \ . \ . \ . \quad (1)$$

As (1) must be satisfied by any value of δx, we may take $\delta x=0$; hence, $a_0=f(x)$, which inserted in (1) gives

$$\frac{f(x+\delta x)-f(x)}{\delta x}=a_1+a_2 . \delta x+a_3 . \delta x^2+ \ . \ . \ . +a_n . \delta x^{n-1} \ . \ . \ . ,$$

and therefore

$$\lim . \frac{f(x+\delta x)-f(x)}{\delta x}=f'(x)=a_1 ;$$

hence, (1) becomes

$$f(x+\delta x)=f(x)+f'(x)\frac{\delta x}{1}+a_2 . \delta x^2 \quad . \quad . \quad (2)$$

Differentiating (2) w.r.t. δx, according to article **88**, we get

$$f'(x+\delta x)=0+f'(x)+2a_2 . \delta x+3a_3 . \delta x^2+ \ . \ . \ . +na_n . \delta x^{n-1} \quad (3)$$

By dividing (3) by δx we get

$$\frac{f'(x+\delta x)-f'(x)}{\delta x}=2a_2+3a_3 . \delta x+ \ . \ . \ . +na_n . \delta x^{n-2} \quad (4)$$

hence,

$$\lim . \frac{f'(x+\delta x)-f'(x)}{\delta x}=f''(x)=2a_2, \quad \text{or} \quad a_2=\frac{f''(x)}{1\times 2};$$

in the same manner we get

$$a_3 = \frac{f'''(x)}{\lfloor 3}, \quad \text{and} \quad a_n = \frac{f^n(x)}{\lfloor n};$$

hence,

$$f(x+\delta x) = f(x) + f'(x)\frac{\delta x}{1} + f''(x)\frac{\delta x^2}{\lfloor 2} + \ . \ . \ . + f^n(x)\frac{\delta x^n}{\lfloor n} \ . \ . \ . \ . \quad [74]$$

which is Taylor's theorem.

[74] will be true as long as none of the derived functions is infinite.

EXAMPLE.—*Expand $(a+x)^n$ in ascending powers of x; n may be any positive or negative integral number or fraction.*

We may write

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n(1+h)^n \quad \text{where} \quad h = \frac{x}{a}.$$

By applying Taylor's theorem to the expansion of $(1+h)^n$, we must take

$$f(x) = 1\,;\ f'(x) = n\,;\ f''(x) = n(n-1)\ .\ .\ .$$
$$f^r(x) = n(n-1)(n-2)\ .\ .\ .\ (n-r+1);$$

hence,

$$(1+h)^n = 1 + \frac{n}{1}h + \frac{n(n-1)}{\lfloor 2}h^2 + \frac{n(n-1)(n-2)}{\lfloor 3}h^3$$
$$+ \ . \ . \ . + \frac{n(n-1)\ .\ .\ .\ (n-r+1)}{\lfloor r}h^r + \ . \ . \ . \ . \quad [75]$$

[75] is convergent when $h<1$ or $x<a$, and the number of terms is finite when n is a positive integral number, in which case the last term will be (by making $r=n$)

$$\frac{n(n-1)\ .\ .\ .\ (n-n+1)}{\lfloor n}h^n = h^n.$$

By multiplying [75] by a^n and substituting $\frac{x}{a}$ for h we get

$$(a+x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{\lfloor 2}a^{n-2}x^2$$
$$+ \ . \ . \ . + \frac{n(n-1)\ .\ .\ .\ (n-r+1)}{\lfloor r}a^{n-r}x^r + \ . \ . \ . \ . \quad [76]$$

[76] is the *binomial theorem.* In engineering [75] is the most useful form.

93. Maclaurin's theorem.

Suppose we substitute δx for x and x for δx in [**74**], then we have

$$f(\delta x+x)=f(\delta x)+f'(\delta x)\frac{x}{1}+f''(\delta x)\frac{x^2}{\underline{|2}}+\ldots+f^n(\delta x)\frac{x^n}{\underline{|n}}+\ldots \quad (1)$$

Make δx in (1) equal to 0, and we get

$$f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{\underline{|2}}+\ldots+f^n(0)\frac{x^n}{\underline{|n}}+\ldots \quad [77]$$

which is *Maclaurin's theorem.*

EXAMPLE 1.—*Expand* a^x.

$f(x)=a^x$, hence, $f(0)=1$, $f'(0)=\log_e a$,

$f''(0)=(\log_e a)^2 \ldots f^n(0)=(\log_e a)^n$, etc.;

hence,

$$a^x=1+\frac{\log_e a}{1}x+\frac{(\log_e a)^2}{\underline{|2}}x^2+\ldots+\frac{(\log_e a)^n}{\underline{|n}}x^n+\ldots \quad (2)$$

Let $a=e$ in (2), and we get

$$e^x=1+\frac{x}{1}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^n}{\underline{|n}}+\ldots \quad (3)$$

Let $x=1$ in (3), and we obtain

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}+\ldots \quad (4)$$

(4) is convergent, and we get $e=2{\cdot}718281828\ldots$

EXAMPLE 2.—*Expand* $\sin x$.

$f(x)=\sin x$, $f(0)=0$, $f'(0)=1$; $f''(0)=0$,

$f'''(0)=-1$, $f^{iv}(0)=0$, etc.;

hence, $$\sin x=\frac{x}{1}-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}\ldots \quad (5)$$

94. Undetermined forms.

The undetermined forms in article **67** can all be reduced to $\frac{0}{0}$, thus

$$\frac{\infty}{\infty}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}},\text{ and } 0\times\infty=\frac{0}{\frac{1}{\infty}},\text{ etc.,}$$

both of which have the form $\frac{0}{0}$.

If now $f(x)=\dfrac{\phi(x)}{\psi(x)}$, and for $x=a$ both $\phi(a)$ and $\psi(a)$ are 0, then $f(a)=\dfrac{0}{0}$. For the purpose of determining the actual value of $f(a)$ we will expand $f(a+h)$ and then determine lim. $f(a+h)$.

(i.) If *Taylor's theorem can be used, i.e.* when the differential coefficients of $\phi(a)$ and $\psi(a)$ are finite. By substituting h for δx in [**74**] we get

$$f(a+h)=\frac{\phi(a)+\phi'(a)\frac{h}{1}+\phi''(a)\frac{h^2}{\underline{|2}}+\ldots}{\psi(a)+\psi'(a)\frac{h}{1}+\psi''(a)\frac{h^2}{\underline{|2}}\ldots},$$

but $\phi(a)=0$ and $\psi(a)=0$;

hence,
$$f(a+h)=\frac{\phi'(a)+\phi''(a)\frac{h}{\underline{|2}}+\ldots}{\psi'(a)+\psi''(a)\frac{h}{\underline{|2}}+\ldots};$$

hence,
$$\text{lim. } f(a+h)=\frac{\phi'(a)}{\psi'(a)} \quad . \quad . \quad . \quad [78]$$

EXAMPLE 1.—*Find the value of* $\dfrac{\sin x}{x}$ *for* $x=0$.

$\phi(0)=\sin 0=0$, $\psi(0)=0$; but $\phi'(x)=\cos x$ and $\psi'(x)=1$; hence, $\phi'(0)=\cos 0=1$ and $\psi'(0)=1$ and $\dfrac{\sin 0}{0}=\dfrac{1}{1}=1$; which we have found before in a different manner.

EXAMPLE 2.—*Find the value of* $\dfrac{e^x-e^{-x}}{\sin x}$ *for* $x=0$.

We have $\phi(0)=0$ and $\psi(0)=0$. $\phi'(x)=e^x+e^{-x}$, $\psi'(x)=\cos x$; hence, $\phi'(0)=2$ and $\psi'(0)=1$, therefore the value of $\dfrac{e^x-e^{-x}}{\sin x}$ for $x=0$ is 2.

If $x=a$ makes $\phi'(a)=0$ and $\psi'(a)=0$, then

$$f(a+h)=\frac{\phi''(a)\frac{h^2}{\underline{|2}}+\phi'''(a)\frac{h^3}{\underline{|3}}+\ldots}{\psi''(a)\frac{h^2}{\underline{|2}}+\psi'''(a)\frac{h^3}{\underline{|3}}+\ldots}$$

$$=\frac{\phi''(a)+\ldots}{\psi''(a)+\ldots}, \text{ and } \text{lim. } f(a+h)=\frac{\phi''(a)}{\psi''(a)}.$$

EXAMPLE 3.—*Find the value of* $\dfrac{1-\cos x}{x^2}$ *for* $x=0$.

We have $\phi(0)=0$ and $\psi(0)=0$. $\phi'(x)=\sin x$ and $\psi'(x)=2x$; hence, $\phi'(0)=0$ and $\psi'(0)=0$; but $\phi''(x)=\cos x$ and $\psi''(x)=2$; hence, $\phi''(0)=1$ and $\psi''(0)=2$; hence, the value of $\dfrac{1-\cos x}{x^2}$ for $x=0$ is equal to $\dfrac{1}{2}$.

EXAMPLE 4.—*Find the value of* $\dfrac{e^x-e^{-x}-2x}{x-\sin x}$ *for* $x=0$.

We have $\phi(0)=0$ and $\psi(0)=0$. $\phi'(x)=e^x+e^{-x}-2$ and $\psi'(x)=1-\cos x$; hence, $\phi'(0)=0$ and $\psi'(0)=0$; $\phi''(x)=e^x-e^{-x}$ and $\psi''(x)=\sin x$; hence, $\phi''(0)=0$ and $\psi''(0)=0$; but $\phi'''(x)=e^x+e^{-x}$ and $\psi'''(x)=\cos x$; hence, $\phi'''(0)=2$ and $\psi'''(0)=1$; hence, the value of $\dfrac{e^x-e^{-x}-2x}{x-\sin x}$ for $x=0$ is 2.

(ii.) *Taylor's theorem cannot be used, i.e.* one of the differential coefficients is ∞. In this case $\phi(a+h)$ and $\psi(a+h)$ must be expanded in some other manner.

EXAMPLE.—*Find the value of* $\dfrac{\sqrt{a+x}-\sqrt{a-x}-\sqrt{2a}}{\sqrt{ax-x^2}}$ *for* $x=a$.

$$\phi(a)=0 \text{ and } \psi(a)=0;\quad \phi'(a)=\infty \text{ and } \psi'(a)=\infty;$$

hence, Taylor's theorem cannot be applied. But by applying the binomial theorem we have

$$\frac{\phi(a+h)}{\psi(a+h)}=\frac{(a+a+h)^{\frac{1}{2}}-(a-a-h)^{\frac{1}{2}}-(2a)^{\frac{1}{2}}}{[a(a+h)-(a+h)^2]^{\frac{1}{2}}}=\frac{(2a+h)^{\frac{1}{2}}-(-h)^{\frac{1}{2}}-(2a)^{\frac{1}{2}}}{[-h(a+h)]^{\frac{1}{2}}}$$

$$=\frac{(2a)^{\frac{1}{2}}+\frac{h}{2}(2a)^{-\frac{1}{2}}\ldots-(-1)^{\frac{1}{2}}h^{\frac{1}{2}}\ldots-(2a)^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}h^{\frac{1}{2}}(a^{\frac{1}{2}}+\frac{h}{2}a^{-\frac{1}{2}}\ldots)}=\frac{-(-1)^{\frac{1}{2}}h^{\frac{1}{2}}+\frac{1}{2}(2a)^{-\frac{1}{2}}h\ldots;}{(-1)^{\frac{1}{2}}a^{\frac{1}{2}}h^{\frac{1}{2}}+\frac{1}{2}(-1)^{\frac{1}{2}}a^{-\frac{1}{2}}h^{\frac{3}{2}}\ldots};$$

hence, $$\lim.\frac{\phi(a+h)}{\psi(a+h)}=-\frac{1}{\sqrt{a}},$$

which is the required value.

CHAPTER XII.

PLANE CURVES.

95. Equation of the tangent.

Let the equation of the curve be $y=f(x)$, and let Y and X be the current co-ordinates of the tangent; then, as the tangent passes through point $P(x, y)$ on the curve, the equation of the tangent to the curve at point $P(x, y)$ is

$$Y-y=\frac{dy}{dx}(X-x) \quad . \quad . \quad . \quad . \quad [79]$$

Hence, to draw the tangent through any point on the curve, we must draw a straight line through the point, which makes an angle ϕ with the x-axis so that $\tan \phi$ is equal to the value of $\frac{dy}{dx}$ for the given point. $\frac{dy}{dx}$ in [**79**] is therefore the same as the constant m in the equation of a straight line.

If the equation of the curve be given as

$$u=f(x, y)=0 \quad . \quad . \quad . \quad . \quad (1)$$

then by article **90**

$$\frac{du}{dx}=\frac{\Delta u}{\Delta x}+\frac{\Delta u}{\Delta y}\frac{dy}{dx}=0,$$

and the equation of the tangent is

$$(X-x)\frac{\Delta u}{\Delta x}+(Y-y)\frac{\Delta u}{\Delta y}=0 \quad . \quad . \quad . \quad [80]$$

Example 1.—*Find the equation of the tangent to parabola $y^2=4ax$.*

$$2y\frac{dy}{dx}=4a, \quad \text{or} \quad \frac{dy}{dx}=\frac{2a}{y}=\frac{y}{2x};$$

hence, the equation of the tangent

$$Y - y = \frac{y}{2x}(X - x),$$

which we have found before.

EXAMPLE 2.—*Find the equation of the tangent to* $x^2 + 4y^2 - 3xy + 8 = 0$.

$$\frac{\Delta u}{\Delta x} = 2x - 3y\ ;\ \frac{\Delta u}{\Delta y} = 8y - 3x\ ;$$

hence, the equation of the tangent to the curve at any point (x, y) is

$$(X - x)(2x - 3y) + (Y - y)(8y - 3x) = 0.$$

96. Definition.—The *normal* at any point on a curve is a straight line drawn through the point and perpendicular on the tangent to the curve at the point.

97. Equation of the normal.

Since the equation of the tangent is

$$Y - y = \frac{dy}{dx}(X - x),$$

and since the normal is perpendicular on the tangent, the equation of the normal must be

$$Y - y = -\frac{dx}{dy}(X - x), \quad \text{or} \quad (X - x) + (Y - y)\frac{dy}{dx} = 0 \quad . \quad [81]$$

If the equation of the curve be $u = f(x, y) = 0$, the equation of the normal is

$$\frac{X - x}{\frac{\Delta u}{\Delta x}} = \frac{Y - y}{\frac{\Delta u}{\Delta y}} \quad . \quad . \quad . \quad . \quad [82]$$

EXAMPLE.—*Find the equations of the tangent and the normal to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (1)$$

By differentiating (1) w.r.t. x and y respectively, as in article **90**, we get

$$\frac{\Delta u}{\Delta x} = \frac{2x}{a^2}, \quad \text{and} \quad \frac{\Delta u}{\Delta y} = \frac{2y}{b^2}\ ;$$

hence, the equation of the tangent is

$$(X-x)\frac{2x}{a^2}+(Y-y)\frac{2y}{b^2}=0, \quad \text{or} \quad \frac{Xx}{a^2}+\frac{Yy}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the same as we found in article **37**. The equation of the normal is

$$\frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}}, \quad \text{or} \quad \frac{(X-x)a^2}{x}=\frac{(Y-y)b^2}{y}.$$

98. Subtangent and subnormal.

Let the equation of the curve be $y=f(x)$; and let the straight lines PM (fig. 32) and PQ be respectively the tangent and the normal to the curve at point P. The length N_1M is called the *subtangent*, and the length N_1Q is called the *subnormal*. These

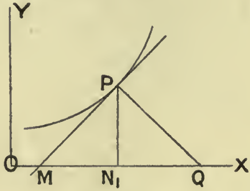


FIG. 32.

lines are reckoned positive and negative according to their direction from N_1. In the above figure N_1M is negative and N_1Q is positive. Hence, length of subtangent

$$S_t=-y\frac{dx}{dy} \quad . \quad . \quad . \quad . \quad [83]$$

length of subnormal,

$$S_n=y\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad [84]$$

length of tangent PM denoted by

$$T=y\sqrt{1+\left(\frac{dx}{dy}\right)^2} \quad . \quad . \quad . \quad [85]$$

length of normal PQ,

$$N=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \quad . \quad . \quad . \quad [86]$$

EXAMPLE 1.—*Find the length of the subtangent to* $y=ae^{\frac{x}{a}}$.

$$\frac{dy}{dx}=a\frac{1}{a}e^{\frac{x}{a}}=e^{\frac{x}{a}}; \text{ hence, } S_t=-ye^{-\frac{x}{a}}=-ae^{\frac{x}{a}}e^{-\frac{x}{a}}=-a,$$

or the subtangent is constant, *i.e.* it has the same value for all points on the curve.

EXAMPLE 2.—*Find the length of the subnormal to* $y^2 = 4ax$.

$$\frac{dy}{dx} = \frac{2a}{y}; \text{ hence, } S_n = y\frac{dy}{dx} = 2a,$$

hence the subnormal to the parabola is constant.

99. Definition.—An *asymptote* is the limiting position of the tangent to a curve when the point of contact moves away along the curve to an infinite distance, while the tangent itself does not ultimately lie wholly at infinity.

100. The equation of the asymptote.

By definition the equation of an asymptote is

$$Y = X\left(\text{Lim.}\frac{dy}{dx}\right) + \text{Lim.}\, c \quad . \quad . \quad . \quad [87]$$

EXAMPLE.—*To find the equation of the asymptote to* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The equation of the tangent to point (x, y) on the curve is

$$Y = \frac{b^2 x}{a^2 y} X - \frac{b^2}{y}, \quad \text{but} \quad x = \pm\frac{a}{b}\sqrt{b^2 + y^2};$$

hence, the equation of the tangent may also be written thus,

$$Y = \pm\frac{b}{a}\sqrt{1 + \frac{b^2}{y^2}}\, X - \frac{b^2}{y};$$

but $$\text{Lim.} \pm\frac{b}{a}\sqrt{1 + \frac{b^2}{y^2}} = \pm\frac{b}{a}, \text{ and Lim.}\frac{b^2}{y} = 0.$$

Hence, there are two asymptotes to the hyperbola, both through the origin; their equations are

$$Y = \frac{b}{a}X, \quad \text{and} \quad Y = -\frac{b}{a}X.$$

We arrived at the same result in article **48**, but in a more complicated manner.

101. Definition.—In the immediate neighbourhood of any given point on a curve the latter is said to be convex as seen from the tangent to the curve through the given point, and the curve is concave as seen from a point situated on the side of the curve opposite to that on which the tangent lies.

102. Geometrical investigation of concavity.

By article **69** the ordinates of a curve increases with increasing abscissæ when the gradient $\left(\frac{dy}{dx}\right)$ of the curve is positive, and diminishes when the gradient is negative.

Let us, instead of plotting the curve

$$y = f(x) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

plot the curve

$$\frac{dy}{dx} = f'(x) \quad . \quad . \quad . \quad . \quad (2)$$

i.e. if we trace the curve whose ordinates are $\frac{dy}{dx}$ instead of y. The gradient to curve (2) is $\frac{d^2y}{dx^2}$, and by article **69** the ordinate $\left(\frac{dy}{dx}\right)$ will increase with increasing values of x when the gradient $\left(\frac{d^2y}{dx^2}\right)$ is positive, and will diminish when the gradient is negative.

Returning to curve (1), and assuming that x is always increasing, we have just found that $\frac{d^2y}{dx^2}$ is positive when $\frac{dy}{dx}$ is increasing, and $\frac{d^2y}{dx^2}$ is negative when $\frac{dy}{dx}$ is decreasing.

Example 1.—Consider the parabola (fig. 9, p. 22), and trace the curve, beginning at the vertex. As the point moves along the portion of the curve whose ordinates are positive, $\frac{dy}{dx}$ will steadily diminish from $+\infty$; hence, $\frac{d^2y}{dx^2}$ is always negative. Take now any point on the curve and draw the ordinate. If we wish to see the concave side of the curve, we must move from the point on the curve in the negative direction along the ordinate.

Trace now the other portion of the parabola. As the point moves from the vertex the values of $\frac{dy}{dx}$ increase (absolutely) from $-\infty$ to any negative value; hence, $\frac{d^2y}{dx^2}$ is always positive. By moving from the curve along the ordinate in the positive direction we shall see the concave side of the curve.

Example 2.—Let us trace the ellipse (fig. 13, p. 27), starting from the vertex A_2. As the point moves towards B_1 and thence towards A_1, $\frac{dy}{dx}$ is steadily diminishing from $+\infty$ to $-\infty$; hence, $\frac{d^2y}{dx^2}$ is always negative, and in order to see the concave side of the curve the point must move from the curve in the negative direction of the ordinates.

Next let the point start again from A_2 and move towards B_2 and thence to A_1. During this motion $\frac{dy}{dx}$ is steadily increasing from $-\infty$ to $+\infty$; hence, $\frac{d^2y}{dx^2}$ is positive throughout the motion, and the point will be on the concave side of the curve, if it moves from the curve in the positive direction of the ordinates.

From these two examples it will be seen that, *when* $\frac{d^2y}{dx^2}$ *is positive the curve* (1) *is concave seen from a point moving from the curve in the positive direction of the ordinate*, and *when* $\frac{d^2y}{dx^2}$ *is negative the curve is concave seen from a point moving from the curve in the negative direction of the ordinate.*

A continuous function cannot change its sign without passing through 0 or ∞; for instance, $\frac{dy}{dx}$ to the ellipse (fig. 13, p. 27) changes sign at the vertices where its value is ∞, and it also changes sign at points B_1 and B_2 where its value is 0. For this reason we may expect that the concavity of a curve will change its direction at points where $\frac{d^2y}{dx^2}$ is either 0 or ∞. A point of this kind is called a *point of inflexion.* The centre of the lemniscate (fig. 25, p. 50) is a point of inflexion.

It does not, however, follow that because d^2y/dx^2 is 0 or ∞, that the point is a point of inflexion. A closer examination of this subject is made in the next article.

103. Analytical investigation of concavity.

Let the equation of the curve in fig. 33 be $y=f(x)$. The equation of the tangent through point $M(a, b)$ is

$$Y-b=f'(a)(X-a), \text{ or as } b=f(a),\ Y-f(a)=f'(a)(X-a) \quad (1)$$

For $X=OP_1=a+\delta x$ we have $Y=P_1Q_1=f(a)+f'(a).\delta x$; for δx negative we have $X=OP_2$ and $Y=P_2Q_2$.

We have also

$$P_1R_1 = f(a+\delta x) = f(a) + f'(a)\,.\,\delta x + f''(a)\frac{\delta x^2}{\lfloor 2} + \cdot \cdot \cdot$$

A. The difference between the two ordinates P Q and P R is

$$q = -\delta x^2\left[f''(a)\frac{1}{\lfloor 2} + f'''(a)\frac{\delta x}{\lfloor 3} + f^{\text{iv}}(a)\frac{\delta x^2}{\lfloor 4} + \ldots\right] \quad . \quad (2)$$

δx may be taken so small that the sign of (2) will depend on the sign of $f''(a)$ only.

(i.) If $f''(a)$ *is positive*, both q_1 and q_2 will be negative (fig. 34).
(ii.) If $f''(a)$ *is negative*, both q_1 and q_2 will be positive (fig. 33).

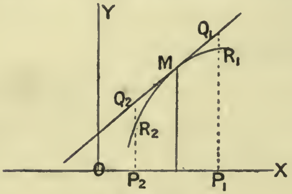


FIG. 33.

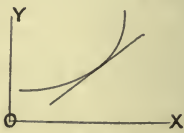


FIG. 34.

B. If $f''(a)$ *is equal to* 0, then

$$q = -\delta x^3\left[f'''(a)\frac{1}{\lfloor 3} + f^{\text{iv}}(a)\frac{\delta x}{\lfloor 4} + f^{\text{v}}(a)\frac{\delta x^2}{\lfloor 5} + \ldots\right] \quad . \quad (3)$$

δx may be taken sufficiently small so that the sign of (3) will depend on the signs of $f'''(a)$ and δx.

(i.) If $f'''(a)$ *is positive* then q_1 will be negative and q_2 will be positive. We have then a point of inflexion (fig. 35).

(ii.) If $f'''(a)$ is *negative* then q_1 will be positive and q_2 will be negative. We have then a point of inflexion as shown in fig. 36.

(iii.) If $f'''(a)$ is 0, then

$$q = -\delta x^4\left[f^{\text{iv}}(a)\frac{1}{\lfloor 4} + f^{\text{v}}(a)\frac{\delta x}{\lfloor 5} + \cdots\right] \quad . \quad . \quad (4)$$

If δx be taken sufficiently small the sign of (4) will depend on the sign of $f^{\text{iv}}(a)$ only; the curve in the immediate neighbourhood of point M will be similar to that in fig. 34 when $f^{\text{iv}}(a)$ is positive, and it will be like that in fig. 33 when $f^{\text{iv}}(a)$ is negative.

(iv.) If $f^{\text{iv}}(a)$ is 0, then

$$q = -\delta x^5\left[f^{\text{v}}(a)\frac{1}{\lfloor 5} + f^{\text{vi}}(a)\frac{\delta x}{\lfloor 6} \ldots\right] \quad . \quad . \quad (5)$$

The curve near point M will be like that in fig. 35 when $f^{\text{v}}(a)$ is positive, and like that in fig. 36 if $f^{\text{v}}(a)$ is negative.

Rule.—*When the first of the differential coefficients in* (2), *which is not zero, is of even index, the curve in the neighbourhood of point* M *lies entirely on one side of the tangent; and if the said differential coefficient be positive, the curve is concave seen from a point moving from the curve in the positive direction of the ordinate; but if it be negative the curve will be concave seen from a point moving from the curve in the negative direction of the ordinate. If the said differential coefficient is of odd index, then the point is a point of inflexion.*

If any of the differential coefficients in (2) are infinite, then (2) cannot be used. In such case the curve may be examined by

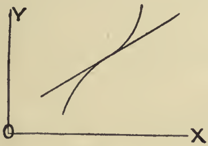


FIG. 35.

FIG. 36.

taking y as abscissa and x as ordinate. The equation of the curve will then be $x=\phi(y)$, and instead of (2) we shall have

$$q=-\delta y^2\left[\phi''(b)\frac{1}{\underline{\lvert 2}}+\phi'''(b)\frac{\delta y}{\underline{\lvert 3}}+\phi^{\text{iv}}(b)\frac{\delta y^2}{\underline{\lvert 4}}\ \cdots\right] \quad . \quad (6)$$

104. Maxima and minima.

When the tangent to a point on a curve is parallel to the x-axis, and the point is not a point of inflexion, then it is evident that the ordinate to the point has reached a maximum, or a minimum value. A figure will show that the ordinate is a maximum if the curve in the neighbourhood of the point is concave seen from a point moving from the curve in the negative direction of the ordinate, and that it is a minimum if the curve is concave in the opposite direction. Hence,

$$\left.\begin{array}{l}\text{the ordinate is a maximum when}\\ \qquad\qquad \dfrac{dy}{dx}=0 \quad \text{and} \quad \dfrac{d^2y}{dx^2} \text{ is negative,}\\ \text{and the ordinate is a minimum when}\\ \qquad\qquad \dfrac{dy}{dx}=0 \quad \text{and} \quad \dfrac{d^2y}{dx^2} \text{ is positive.}\end{array}\right\} \quad . \quad . \quad (1)$$

This subject can, however, be investigated more thoroughly by means of analysis.

A function is a maximum for $x=a$ if the values of the function in the neighbourhood of (a, b) are smaller than b; and a function is a minimum if the values of the function in the neighbourhood of (a, b) are greater than b. Hence,

Maxima of $y=f(x)$ will occur when

$$f(a-\delta x)<f(a)>f(a+\delta x).$$

Minima of $y=f(x)$ will occur when

$$f(a-\delta x)>f(a)<f(a+\delta x).$$

Maxima of $x=\phi(y)$ will occur when

$$\phi(b-\delta y)<\phi(b)>\phi(b+\delta y).$$

Minima of $x=\phi(y)$ will occur when

$$\phi(b-\delta y)>\phi(b)<\phi(b+\delta y).$$

By Taylor's theorem we have

$$f(a+\delta x)=f(a)+f'(a)\,.\,\delta x+f''(a)\frac{\delta x^2}{\lfloor 2}+\ .\ .\ .\ ,$$

or
$$f(a+\delta x)-f(a)=\delta x\left[f'(a)+f''(a)\frac{\delta x}{\lfloor 2}+f'''(a)\frac{\delta x^2}{\lfloor 3}+\ .\ .\ .\right] \qquad (2)$$

Whether δx is positive or negative, *i.e.* whether the neighbouring point is to the right or to the left of point (a, b), the expression (2) must be negative when $b=f(a)$ is a maximum, and it must be positive when $b=f(a)$ is a minimum. This, however, is impossible unless $f'(a)$, $\frac{dy}{dx}$ to (a, b), is equal to 0. Hence, we must have

$$f(a+\delta x)-f(a)=\delta x^2\left[f''(a)\frac{1}{\lfloor 2}+f'''(a)\frac{\delta x}{\lfloor 3}\ .\ .\ .\right] \quad . \qquad (3)$$

which must be negative in the case of a maximum and positive in the case of a minimum. Hence, it follows that *if $x=a$ makes $f'(a)=0$ and at the same time makes $f''(a)$ negative, then $b=f(a)$ is a maximum; whereas if $x=a$ makes $f''(a)$ positive, then $b=f(a)$ is a minimum.* This is the same rule as given in (1).

In the case that $x=a$ also makes $f''(a)$ equal to 0, then there can be no maximum nor minimum unless $f'''(a)$ is also equal to 0, because

$$f(a+\delta x)-f(a)=\delta x^3\left[f'''(a)\frac{1}{\underline{|3}}+f^{\text{iv}}(a)\frac{\delta x}{\underline{|4}}\ .\ .\ .\right]$$

changes sign with δx, we must therefore have

$$f(a+\delta x)-f(a)=\delta x^4\left[f^{\text{iv}}(a)\frac{1}{\underline{|4}}+f^{\text{v}}(a)\frac{\delta x}{\underline{|5}}\ .\ .\ .\right] \quad . \quad (4)$$

According as $f^{\text{iv}}(a)$ in (4) is negative or positive, $b=f(a)$ is a maximum or a minimum. Hence, the above rule may now be extended to: *When $x=a$ makes $f'(a)=0$ and at the same time the first differential coefficient, which is not 0, is of even index, then $b=f(a)$ is a maximum or a minimum according as the latter differential coefficient is negative or positive. If the latter differential coefficient is of odd index, then $b=f(a)$ is neither a maximum nor a minimum, but the point (a, b) is a point of inflexion.*

Exactly in the same manner we may show that $y=b$ makes $a=\phi(b)$ a maximum or a minimum when $\frac{dx}{dy}=\phi'(b)=0$ $\left(i.e.\ \frac{dy}{dx}=f'(a)=\infty\right)$, and at the same time $\frac{d^2x}{dy^2}=\phi''(b)$ is negative or positive, etc.

Example.—*Find the maximum and minimum values of the ordinate and the abscissa to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

(i.) $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$, and $\frac{d^2y}{dx^2}=-\frac{a^2b^2y-xb^2a^2\frac{dy}{dx}}{a^4y^2}=-\frac{b^4}{a^2y^3}$.

$\frac{dy}{dx}=0$ for $x=0$ and $y=\pm b$; hence,

$y=+b$, point $(0, b)$, is a maximum as $\frac{d^2y}{dx^2}$ is negative.

$y=-b$, point $(0, -b)$, is a minimum as $\frac{d^2y}{dx^2}$ is positive.

(ii.) $\frac{dx}{dy}=-\frac{a^2y}{b^2x}$, and $\frac{d^2x}{dy^2}=-\frac{b^2xa^2-a^2yb^2\frac{dx}{dy}}{b^4x^2}=-\frac{a^4}{b^2x^3}$.

$\frac{dx}{dy}=0$ for $y=0$ and therefore $x=\pm a$; hence,

$x = +a$, point $(a, 0)$, is a maximum as $\frac{d^2x}{dy^2}$ is negative.

$x = -a$, point $(-a, 0)$, is a minimum as $\frac{d^2x}{dy^2}$ is positive.

105. Contact.

Let the equations of the two curves I and II (fig. 37), which intersect at M (a, b), respectively be

$$y = f(x), \quad \text{and} \quad Y = \phi(X) \quad . \quad . \quad . \quad (1)$$

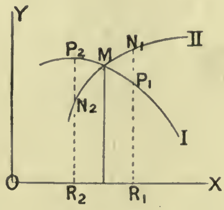


FIG. 37.

The ordinate

$$R_1P_1 = f(a + \delta x) = f(a) + f'(a)\frac{\delta x}{1} + f''(a)\frac{\delta x^2}{\underline{|2}} + \ldots \quad (2)$$

The ordinate

$$R_1N_1 = \phi(a + \delta x) = \phi(a) + \phi'(a)\frac{\delta x}{1} + \phi''(a)\frac{\delta x^2}{\underline{|2}} + \ldots \quad (3)$$

as the two curves intersect at M (a, b), we have $b = f(a) = \phi(a)$;

$$\text{hence,}\ P_1N_1 = q_1 = [f'(a) - \phi'(a)]\frac{\delta x}{1} + [f''(a) - \phi''(a)]\frac{\delta x^2}{\underline{|2}} + \ldots \quad (4)$$

If also $f'(a) = \phi'(a)$ (*i.e.* the two curves have common tangent at M), $f''(a) = \phi''(a)$, $f'''(a) = \phi'''(a)$ . . . $f^n(a) = \phi^n(a)$, then

$$q_1 = [f^{n+1}(a) - \phi^{n+1}(a)]\frac{\delta x^{n+1}}{\underline{|n+1}} + [f^{n+2}(a) - \phi^{n+2}(a)]\frac{\delta x^{n+2}}{\underline{|n+2}} + \ldots \quad (5)$$

The expression for $P_2N_2 = q_2$ is the same as (5), only δx is negative. When δx is sufficiently small the sign of q_1 will be the same as the sign of the first term in (5). The values of q_1 and q_2 will be the smaller the greater n is, and therefore the two curves will approach each other the more the greater n is. When n is even, q_1 and q_2 have opposite signs and the two curves intersect at M (although they have common tangent at M). Whereas when n is odd, q_1 and q_2 have the same sign, and also the same sign as $(f^{n+1}(a) - \phi^{n+1}(a))$, the two curves do not intersect. The two curves are said to have *contact of the* n^{th} *order*. When $n=0$, the two curves *intersect* and have *no contact*. When $n=1$, the curves *do not intersect*, but as $f'(a)=\phi'(a)$, the two curves have common tangent at M, and the curves have *contact of first order*. When $n=2$, the curves *intersect*, and as $f''(a)=\phi''(a)$, the curves have *contact of second order*.

106. Osculation.

Let $y=f(x)$ be the equation of a given curve, and $Y=\phi(X)$ be the equation of a curve of a certain kind with $n+1$ constants; the latter are to be determined so that the two curves may have contact of the n^{th} order at point (a, b) on the curve $y=f(x)$; $Y=\phi(X)$ is then said to be the curve of its kind which has the closest contact with $y=f(x)$ at point (a, b). $Y=\phi(X)$ is also said to have osculation with $y=f(x)$ at point (a, b).

The straight line, $Y=mX+c$, contains only two constants (m and c), and its closest contact with curve $y=f(x)$ can therefore only be of the first order. $X=a$ requires that $Y=f(a)=b$ and $m=f'(a)$; *the osculating straight line* is therefore $Y-b = f'(a)(X-a)$, i.e. *the tangent.*

At some special points on the curve, $y=f(x)$, the osculating curve may have contact with $y=f(x)$ of higher order than the n^{th}. For after we have determined the $(n+1)$ constants of $Y=\phi(X)$ so as to get contact of the n^{th} order, it may happen, quite apart from our control, that, for the special value of the abscissa, $f^{n+1}(x)=\phi^{n+1}(x)$, $f^{n+2}(x)=\phi^{n+2}(x)$, etc., in which case we should have contact of the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, . . . order.

EXAMPLE. $y=x-x^5$.

$$\frac{dy}{dx}=1-5x^4\,;\ \frac{d^2y}{dx^2}=-20x^3\,;\ \frac{d^3y}{dx^3}=-60x^2\,;\ \frac{d^4y}{dx^4}=-120x.$$

Hence, at the origin (0, 0) the tangent $(Y=X)$ to the given curve will have contact of the fourth order with the curve.

107. The osculating circle.

The equation of the circle whose centre is (α, β) and whose radius is ρ, is $(X-\alpha)^2+(Y-\beta)^2=\rho^2$;

hence, $$(Y-\beta)\frac{dY}{dx}+(X-\alpha)=0,$$

and $$(Y-\beta)\frac{d^2Y}{dx^2}+\left(\frac{dY}{dx}\right)^2+1=0.$$

As the equation of the circle contains three constants (α, β, and ρ), the osculating circle can, as a rule, only have contact of the second order with a curve, or the osculating circle intersects the curve at the point of contact. In order to determine the three constants we must substitute x, y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ of the curve for the corresponding terms of the circle. The following equations will therefore determine α, β, and ρ, viz.:

$$(x-\alpha)^2+(y-\beta)^2=\rho^2;\ (y-\beta)\frac{dy}{dx}+(x-\alpha)=0,$$

and
$$(y-\beta)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0;$$

hence,
$$x-\alpha=\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]\div\frac{d^2y}{dx^2};\ y-\beta=-\left[1+\left(\frac{dy}{dx}\right)^2\right]\div\frac{d^2y}{dx^2};$$

and the radius of the osculating circle

$$\rho=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad [88]$$

EXAMPLE 1.—*Find the radius of the osculating circle of the parabola* $y^2=4ax$.

$$\frac{dy}{dx}=\frac{2a}{y};\ \frac{d^2y}{dx^2}=-\frac{4a^2}{y^3};$$

hence,
$$\rho=\frac{(y^2+4a^2)^{\frac{3}{2}}}{4a^2}.$$

At the *vertex* $\rho=2a$. The centre of the osculating circle at the vertex lies on the axis at a distance $2a$ from the vertex.

EXAMPLE 2.—*Find the radius of the osculating circle of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y};\ \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}$$

hence,
$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

For the points on the minor axis we have $x=0$, $y=\pm b$; hence, $\rho=\frac{a^2}{b}$; for the vertices we have $x=\pm a$, $y=0$; hence, $\rho=\frac{b^2}{a}$.

It is evident that at the vertices of a conic and at the ends of the minor axis of an ellipse the osculating circle cannot intersect the curve as the axis is an axis of symmetry. Hence, the contact of the osculating circle at the latter points must be of a higher order than the second order, and must also be of an odd index. This might be investigated by determining the higher differential coefficients of the circle as well as those of the curve.

108. Curvature.

The curvature of a circle is said to be measured by the reciprocal of its radius. Hence, we might conclude that the curvature of a curve at any point on the curve might be measured by the reciprocal of the radius of the osculating circle. This conclusion, however, is not justifiable without a

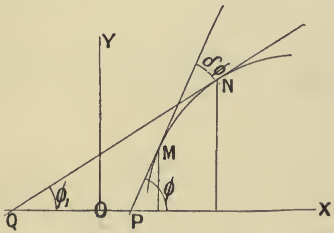


FIG. 38.

further investigation, as the osculating circle has only contact of second order with the curve and it intersects the curve.

The direction of the curve at any of its points is determined by the direction of the tangent at the point. The greater the rate at which the direction of the tangent changes the greater is the average bending of the curve.

For a certain length $MN = \delta s$ of the curve (fig. 38) the average curvature of the arc will increase with the angle of contingency $\delta\phi = \phi - \phi_1$, P M and Q N being the tangents respectively to points M and N on the curve. Hence, the average curvature is $\epsilon_m = \frac{\delta\phi}{\delta s}$; hence, the curvature of the curve at any point on the curve is

$$\epsilon = \lim. \frac{\delta\phi}{\delta s} = \frac{d\phi}{ds} \quad . \quad . \quad . \quad . \quad [89]$$

δx, δy, $\delta\phi$, and δs are corresponding and simultaneously vanishing increments. Since lim. [length of the arc ÷ length of the chord] = 1 (see article **67**) we have lim. $[\delta s^2 \div (\delta x^2 + \delta y^2)] = 1$;

hence,
$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad . \quad . \quad . \quad . \qquad [90]$$

ϕ is the angle whose $\tan = \frac{dy}{dx}$; hence, $\phi = \text{ang}\left(\tan = \frac{dy}{dx}\right)$, and therefore

$$\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2};$$

since $\epsilon = \frac{d\phi}{dx} \div \frac{ds}{dx}$ we have

$$\epsilon = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \div \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2} \div \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{1}{\rho} \qquad [91]$$

Hence, *the curvature* at any point of a curve *is equal to the reciprocal of the radius of the osculating circle*; for this reason the latter is also called the *circle of curvature* and ρ is called the *radius of curvature.*

CHAPTER XIII.

POLAR CO-ORDINATES.

109. Differential coefficients.

Let the equation of the curve A M (fig. 39) be referred to polar co-ordinates, point O being the pole and points N and M two neighbouring points on the curve whose co-ordinates are respectively (r, θ) and $(r+\delta r, \theta+\delta\theta)$. Further, let the length of

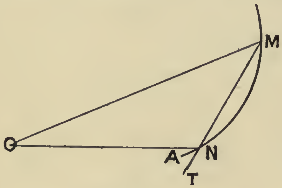


FIG. 39.

the chord N M be δc and that of the arc N M be δs; δr, $\delta\theta$, δc, and δs being corresponding and simultaneously vanishing increments. From triangle M O N we get

$$\delta c^2 = r^2 + (r+\delta r)^2 - 2r(r+\delta r)\cos\delta\theta = 2r(r+\delta r)(1-\cos\delta\theta) + \delta r^2 \quad (1)$$

but $1-\cos\delta\theta = 2\sin^2\frac{\delta\theta}{2}$; hence, (1) becomes

$$\delta c^2 = 4r(r+\delta r)\sin^2\frac{\delta\theta}{2} + \delta r^2;$$

hence,
$$\left(\frac{\delta c}{\delta s}\right)^2\left(\frac{\delta s}{\delta r}\right)^2 = 4r(r+\delta r)\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 \times \frac{1}{4} \times \left(\frac{\delta\theta}{\delta r}\right)^2 + 1 \quad . \quad (2)$$

but lim. $\frac{\delta c}{\delta s}$ and lim. $\left(\sin\frac{\delta\theta}{2} \div \frac{\delta\theta}{2}\right)$ are both equal to unity; hence, lim. (2) is

$$\left.\begin{aligned}&\left(\frac{ds}{dr}\right)^2 = r^2\left(\frac{d\theta}{dr}\right)^2 + 1,\\ &\text{and similarly}\\ &\left(\frac{dr}{ds}\right)^2 = 1 - r^2\left(\frac{d\theta}{ds}\right)^2, \quad \text{and} \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2\end{aligned}\right\} \quad . \quad [92]$$

110. The angle between the tangent and the radius vector.

The secant T N M (fig. 39) will ultimately become the tangent to the curve at N; and the angle O N T, ψ, will ultimately become the angle, ϕ, which the tangent makes with the radius vector. By triangle M O N (fig. 39) we have

$$(r+\delta r)^2 = r^2 + \delta c^2 + 2r\ \delta c\ \ \cos\psi, \text{ or } \cos\psi = \frac{1}{2r}\left(-\delta c + \delta r\ \frac{\delta r}{\delta c} + 2r\frac{\delta r}{\delta c}\right);$$

hence, $\cos\phi = \text{lim.}\cos\psi = \frac{dr}{ds}$; $\sin\phi = \sqrt{1-\cos^2\phi}$; hence, by [92]

$\sin\phi = r\frac{d\theta}{ds}$. $\quad \text{Tan}\,\phi = \frac{\sqrt{1-\cos^2\phi}}{\cos\phi}$, but $\cos^2\phi = 1 \div \left[r^2\left(\frac{d\theta}{dr}\right)^2 + 1\right]$;

hence, $\tan\phi = r\frac{d\theta}{dr}$.

CHAPTER XIV.

EXAMPLES.

(23) Differentiate the following functions w.r.t. x.

(1) $e^y = \log_e x$; (2) $y = \dfrac{e^x(\cos x + \sin x)}{\cos x - \sin x}$; (3) $y = \log_e \tan x$;

(4) $y = \sqrt{a^2 - x^2}$; (5) $y = \log_e \sqrt{x + \sqrt{1 + x^2}}$; (6) $y = \sec x$;

(7) $y = \operatorname{cosec} x$; (8) $\sin y = \tan x$.

Solution.

(1) $$\frac{de^y}{dy}\frac{dy}{dx} = \frac{d(\log_e x)}{dx},\ e^y\frac{dy}{dx} = \frac{1}{x};$$

hence, $$\frac{dy}{dx} = \frac{1}{x \log_e x}\ (Ans.).$$

(2) $$\frac{\cos x + \sin x}{\cos x - \sin x} = \frac{1 + \tan x}{1 - \tan x} = \frac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} = \tan(45^\circ + x);$$

hence,

$$y = e^x \tan(45^\circ + x);\ \frac{dy}{dx} = e^x \frac{1}{\cos^2(45^\circ + x)} + \tan(45^\circ + x)\, e^x$$

$$= e^x\left[\frac{1}{\cos^2(45^\circ + x)} + \frac{\sin(45^\circ + x)}{\cos(45^\circ + x)}\right] = e^x \frac{2 + \cos 2x}{2\cos^2(45^\circ + x)},$$

but $$\cos^2(45^\circ + x) = \frac{1 - \sin 2x}{2};$$

hence, $$\frac{dy}{dx} = e^x \frac{2 + \cos 2x}{1 - \sin 2x}\ (Ans.).$$

(3) $$\frac{dy}{dx}=\frac{d(\log_e \tan x)}{d\tan x}\,\frac{d\tan x}{dx}=\frac{1}{\tan x}\,\frac{1}{\cos^2 x};$$

hence, $$\frac{dy}{dx}=\frac{2}{\sin 2x}\ (Ans.).$$

(4) $$\frac{dy}{dx}=\frac{d\sqrt{a^2-x^2}}{d(a^2-x^2)}\,\frac{d(a^2-x^2)}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\times -2x;$$

hence, $$\frac{dy}{dx}=-\frac{x}{\sqrt{a^2-x^2}}\ (Ans.).$$

(5) $$\frac{dy}{dx}=\frac{d\log_e\sqrt{x+\sqrt{1+x^2}}}{d\sqrt{x+\sqrt{1+x^2}}}\,\frac{d\sqrt{x+\sqrt{1+x^2}}}{d(x+\sqrt{1+x^2})}\,\frac{d(x+\sqrt{1+x^2})}{dx},$$

or

$$\frac{dy}{dx}=\frac{1}{\sqrt{x+\sqrt{1+x^2}}}\,\frac{1}{2\sqrt{x+\sqrt{1+x^2}}}\left[1+\frac{d\sqrt{1+x^2}}{d(1+x^2)}\,\frac{d(1+x^2)}{dx}\right];$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1+x^2}}\ (Ans.).$$

(6) $$y=\sec x=\frac{1}{\cos x}=(\cos x)^{-1};$$

hence, $$\frac{dy}{dx}=-1\times(\cos x)^{-2}\,\frac{d\cos x}{dx};$$

$$\frac{dy}{dx}=\frac{\sin x}{\cos^2 x}\ (Ans.).$$

(7) $y=\operatorname{cosec} x=\dfrac{1}{\sin x}$; hence, $\dfrac{dy}{dx}=-1\times(\sin x)^{-2}\,\dfrac{d\sin x}{dx}$;

$$\frac{dy}{dx}=-\frac{\cos x}{\sin^2 x}\ (Ans.).$$

(8) $$\frac{d\sin y}{dy}\,\frac{dy}{dx}=\frac{d\tan x}{dx},\quad\text{or}\quad \cos y\,\frac{dy}{dx}=\frac{1}{\cos^2 x},$$

but $\cos y=\sqrt{1-\tan^2 x}$, and $\cos^2 x=\dfrac{1}{1+\tan^2 x}$;

$$\frac{dy}{dx}=\frac{1+\tan^2 x}{\sqrt{1-\tan^2 x}}\ (Ans.).$$

(24) (Q. Oct. 1909).—**Differentiate (i.)** $e^{\sin^2 x}$; **(ii.)** $\dfrac{x^2}{\sqrt{1+x+x^2}}$.

Find the fourth differential coefficient of $e^x \sin x$.

Solution.—Differentiating (i.) w.r.t. x we get

$$\frac{de^{\sin^2 x}}{dx} = \frac{de^{\sin^2 x}}{d(\sin^2 x)}\,\frac{d(\sin^2 x)}{d(\sin x)}\,\frac{d(\sin x)}{dx} = e^{\sin^2 x} 2 \sin x \cos x = e^{\sin^2 x} \sin 2x.$$

Differentiating (ii.) w.r.t. x we get

$$\frac{d(\text{ii.})}{dx} = \frac{\sqrt{1+x+x^2} \times 2x - x^2 \dfrac{1+2x}{2\sqrt{1+x+x^2}}}{(1+x+x^2)} = \frac{x(2x^2+3x+4)}{(1+x+x^2)^{\frac{3}{2}}}.$$

Let $y = e^x \sin x$ then, as $\dfrac{de^x}{dx} = e^x$, we get

$$\frac{dy}{dx} = e^x\left(\sin x + \frac{d \sin x}{dx}\right) = e^x(\sin x + \cos x);$$

$$\frac{d^2y}{dx^2} = e^x\left[\sin x + \cos x + \frac{d(\sin x + \cos x)}{dx}\right] = 2e^x \cos x;$$

$$\frac{d^3y}{dx^3} = 2e^x\left(\cos x + \frac{d \cos x}{dx}\right) = 2e^x(\cos x - \sin x);\ \text{and, finally,}$$

$$\frac{d^4y}{dx^4} = 2e^x\left[\cos x - \sin x + \frac{d(\cos x - \sin x)}{dx}\right] = -4e^x \sin x = -4y.$$

(25) (Q. Nov. 1907). — **Prove that in any triangle** $a^2 = b^2 + c^2 - 2bc \,.\, \cos \text{A}$. **Prove that if the angle A is increased by one minute, the sides** b **and** c **being unchanged, the increase in the side** a **is approximately** $\dfrac{\pi}{10800}p$, **where** p **is the length of the perpendicular drawn from the angle A to the side** a.

Solution.—It is shown in trigonometry that

$$a^2 = b^2 + c^2 - 2bc \cos \text{A} \quad . \quad . \quad . \quad (1)$$

A figure will show that

$$pa = bc \sin \text{A}, \quad \text{or} \quad p = \frac{bc \sin \text{A}}{a} \quad . \quad . \quad (2)$$

a and A in (1) are two variables; we may therefore differentiate a w.r.t. A. We get

$$2a\frac{da}{d\text{A}} = 2bc \sin \text{A}, \quad \text{or approximately} \quad \frac{\delta a}{\delta \text{A}} = \frac{bc}{a} \sin \text{A}. \quad (3)$$

By (2) and (3) we have

$$\delta a = p \ \delta A \quad . \quad . \quad . \quad . \quad (4)$$

$\delta A = 1' = \frac{\pi}{10800}$ radians ; hence, (4) becomes

$$\delta a = \frac{\pi}{10800} p \ (Ans.).$$

(26) (Q. Nov. 1906).—**Prove that as x increases, $\frac{a \sin x + b \cos x}{c \sin x + e \cos x}$ either increases for all values of x or else decreases for all values of x; and find the condition to be satisfied by the constants a, b, c, and e to provide that it shall always increase.**

Solution.—When x increases, δx is positive ; hence, by article **69**, the function increases when $\frac{dy}{dx}$ is positive, and diminishes when $\frac{dy}{dx}$ is negative. We must therefore find the differential coefficient w.r.t. x of the given function. By article **81** we have

$$\frac{d}{dx}\left(\frac{a \sin x + b \cos x}{c \sin x + e \cos x}\right)$$

$$= \frac{(c \sin x + e \cos x)(a \cos x - b \sin x) - (a \sin x + b \cos x)(c \cos x - e \sin x)}{(c \sin x + e \cos x)^2}$$

$$= \frac{ae - bc}{(c \sin x + e \cos x)^2}.$$

The denominator is positive for all values of x; if therefore $ae > bc$, the function will increase for all values of x, and when $ae < bc$ the function will diminish for all values of x.

(27) (Q. June 1909).—**Write down the form of Taylor's series which expresses $f(a+x)$ in a series of ascending powers of x, and derive from it the first four terms in the expansion of** $\sin (a+x)$.

Apply this to determine $\sin 31°$ **correctly to four places, being given** $\sin 30° = 0{\cdot}5$ **and** $\cos 30° = 0{\cdot}866$.

Solution.—Substituting a for x and x for δx in [**74**] we get

$$f(a+x) = f(a) + f'(a)\frac{x}{1} + f''(a)\frac{x^2}{\underline{|2}} + f'''(a)\frac{x^3}{\underline{|3}} \ . \ . \ . \ ;$$

hence, $$\sin (a+x) = \sin a + \cos a \ \frac{x}{1} - \sin a \ \frac{x^2}{\underline{|2}} - \cos a \ \frac{x^3}{\underline{|3}};$$

hence, $$\sin 31° = \sin 30° + \cos 30° \frac{x}{1} - \sin 30° \frac{x^2}{\underline{|2}} - \cos 30° \frac{x^3}{\underline{|3}} \ . \quad (1)$$

x in (1) must be expressed in radians, hence, $x=\frac{\pi}{180}$, and inserting this value of x in (1) we get

$$\sin 31^\circ = \sin 30^\circ\left(1-\frac{\pi^2}{2\times 180^2}\right)+\cos 30^\circ\left(1-\frac{\pi^2}{6\times 180^2}\right)\frac{\pi}{180}$$

$$=0{\cdot}5\times 0{\cdot}99985+0{\cdot}866\times\frac{\pi}{180}\times 0{\cdot}99994$$

$$=0{\cdot}4999+0{\cdot}0151=0{\cdot}5150.$$

$$\sin 31^\circ = 0{\cdot}515 \ (\textit{Ans.}).$$

(28) (Q. Oct. 1909).—**Use Maclaurin's series to show that the expansion of $\log_e(1+x)$ in terms of x is $x-\frac{x^2}{2}+\frac{x^3}{3}\ldots$**

Calculate to four figures of decimals the value of $\log_{10} 1{\cdot}08$, having given that $\log_e 10 = 2{\cdot}3026$.

Solution. — Applying Maclaurin's theorem, article **93**, to $\log_e(1+x)$ we have

$$f(x)=\log_e(1+x)\,;\ f'(x)=\frac{1}{1+x}\,;\ f''(x)=-\frac{1}{(1+x)^2}\,;$$

$$f'''(x)=\frac{2}{(1+x)^3}\,;\ f^{\text{iv}}(x)=-\frac{6}{(1+x)^4}\,;$$

hence,

$$f(0)=1\,;\ f'(0)=0\,;\ f''(0)=-1\,;\ f'''(0)=2\,;\ f^{\text{iv}}(0)=-6\,;$$

hence, $$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ \ldots \qquad (1)$$

In the present case $1+x=1{\cdot}08$, hence, $x=0{\cdot}08$, and as we require $\log_{10} 1{\cdot}08$ with four decimals, we ought to calculate $\log_e 1{\cdot}08$ to six decimals. By substituting $0{\cdot}08$ for x in (1) we get

$$\log_e 1{\cdot}08=0{\cdot}08-\frac{0{\cdot}0064}{2}+\frac{0{\cdot}000512}{3}-\frac{0{\cdot}00004096}{4}=0{\cdot}076960\,;$$

as $\log_e x=\log_e 10\log_{10} x$, we have $\log_{10} x=\frac{\log_e x}{\log_e 10}$;

hence, $$\log_{10} 1{\cdot}08=\frac{0{\cdot}07696}{2{\cdot}3026}=0{\cdot}0334 \ (\textit{Ans.}).$$

(**29**) (Q. Oct. 1909).—**Find the first three terms in the expansion of $(4+x)^{\frac{1}{2}}(1+2x)^{\frac{1}{3}}$ in ascending powers of x.**

Solution.—Let $(4+x)^{\frac{1}{2}}(1+2x)^{\frac{1}{3}}=f(x)$

then
$$f'(x)=\frac{d}{dx}\left[(4+x)^{\frac{1}{2}}(1+2x)^{\frac{1}{3}}\right],$$

and
$$f''(x)=\frac{d^2}{dx^2}\left[(4+x)^{\frac{1}{2}}(1+2x)^{\frac{1}{3}}\right].$$

By differentiation we get
$$f'(x)=(4+x)^{\frac{1}{2}}\frac{2}{3(1+2x)^{\frac{2}{3}}}+(1+2x)^{\frac{1}{3}}\frac{1}{2(4+x)^{\frac{1}{2}}}.$$

By substituting 0 for x we get
$$f(0)=2\,;\ f'(0)=1{\cdot}583\,;\ f''(0)=-1{\cdot}6212.$$

By Maclaurin's theorem, article **93**, we get
$$(4+x)^{\frac{1}{2}}(1+2x)^{\frac{1}{3}}=2+1{\cdot}583x-0{\cdot}811x^2\ \ldots\ (\textit{Ans.}).$$

(**30**) (Q. Nov. 1907).—**Find the condition to be satisfied by the coefficients to secure that x^3+ax^2+bx+c shall increase as x increases for all values of x.**

Solution.—Let us write
$$y=x^3+ax^2+bx+c\,. \qquad (1)$$

y will always increase as x increases if $\frac{dy}{dx}$ is always positive.
$$\frac{dy}{dx}=3x^2+2ax+b=3\left(x+\frac{a}{3}\right)^2+b-\frac{a^2}{3} \qquad (2)$$

As c does not appear in (2), it may be any positive or negative integral or fractional quantity.

$\left(x+\frac{a}{3}\right)^2$ is always positive, its lowest value is 0, viz. when $x=-\frac{a}{3}$; hence, $b-\frac{a^2}{3}$ must be positive.
$$b-\frac{a^2}{3}>0\ (\textit{Ans.}).$$

(**31**) (Q. Nov. 1907).—**A variable quantity θ is specified as a function of x by the equation $\theta=a+x\sin\theta$, where a is a constant. Prove that the value of $\frac{d\theta}{dx}$ when $x=0$ is $\sin a$, and that the value of $\frac{d^2\theta}{dx^2}$ when $x=0$ is $\sin 2a$.**

Show that $\alpha + x\sin\alpha + \frac{1}{2}x^2\sin 2\alpha$ **is an approximate value of** θ **when** x **is small.**

Solution.—We have given

$$\theta = \alpha + x\sin\theta \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating (1) w.r.t. θ we get

$$\frac{d\theta}{dx} = 0 + x\cos\theta\frac{d\theta}{dx} + \sin\theta, \quad \text{or} \quad \frac{d\theta}{dx} = \frac{\sin\theta}{1 - x\cos\theta} \quad . \quad (2)$$

For $x=0$ in (1) we get $\theta = \alpha$; hence, (2) will be, for $x=0$, $\frac{d\theta}{dx} = \sin\alpha$.

Differentiating (2) w.r.t. x we get

$$\frac{d^2\theta}{dx^2} = \frac{(1 - x\cos\theta)\cos\theta\frac{d\theta}{dx} - \sin\theta\left(x\sin\theta\frac{d\theta}{dx} - \cos\theta\right)}{(1 - x\cos\theta)^2} \quad . \quad (3)$$

For $x=0$ we have found that $\theta=\alpha$ and $\frac{d\theta}{dx} = \sin\alpha$; inserting these values in (3) we get

$$\frac{d^2\theta}{dx^2} = \sin 2\alpha.$$

By Maclaurin's theorem we get

$$\theta = f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{\underline{|2}} + f'''(0)\frac{x^3}{\underline{|3}} \dots \quad . \quad (4)$$

$$x=0, f(0) = \alpha\,; \; f'(0) = \sin\alpha\,; \; f''(0) = \sin 2\alpha\,;$$

neglecting the remaining terms of (4) we get

$$\theta = \alpha + x\sin\alpha + \frac{x^2}{2}\sin 2\alpha.$$

(32) (Q. Nov. 1906).—**Prove that in any triangle** $c^2 = a^2 + b^2 - 2ab\cos C$.

A straight line of given length moves with its extremities on two fixed lines. Prove that the area of the triangle formed by the fixed lines and the moving line is a maximum when that triangle is isosceles.

Solution.—It is proved in trigonometry that

$$c^2 = a^2 + b^2 - 2ab\cos C \quad . \quad . \quad . \quad (1)$$

Let $CB = a$ and $CA = b$ be the two fixed lines, and $BA = c$ be the line of given length. Hence a and b are two variable quantities. The area of triangle BCA is

$$T = \tfrac{1}{2}ab \sin C \quad . \quad . \quad . \quad . \quad (2)$$

We will now find the value of a which makes T maximum, b being a function of a.

Differentiating (2) w.r.t. a we get

$$\frac{dT}{da} = \frac{b}{2} \sin C + \frac{a}{2} \frac{db}{da} \sin C \quad . \quad . \quad . \quad (3)$$

Differentiating (1) w.r.t. a we get

$$0 = 2a + 2b\frac{db}{da} - 2b \cos C - 2a\frac{db}{da} \cos C \quad . \quad . \quad (4)$$

From (4) we get

$$\frac{db}{da} = \frac{a - b \cos C}{a \cos C - b};$$

$\frac{dT}{da}$ must be equal to 0; hence, $\frac{db}{da} = -\frac{b}{a}$. By equating the two values of $\frac{db}{da}$, we have

$$\frac{a - b \cos C}{a \cos C - b} = -\frac{b}{a}, \text{ or } a = b.$$

As a is equal to b when $\frac{dT}{da} = 0$, we have $\frac{db}{da} = -1$ in (3);

hence,
$$\frac{d^2T}{da^2} = -\frac{\sin C}{2} - \frac{\sin C}{2},$$

which is negative. The area of the triangle is therefore maximum when $a = b$, *i.e.* when the triangle is isosceles.

(33) (Q. May 1907).—**Sketch the curve $y^2x = (x+1)^2$. Find the value of x for which the tangent is parallel to the axis of x; and show that, for values of x greater than this, the angle which the tangent makes with the axis of x does not exceed the value $\tan^{-1}\frac{1}{3\sqrt{3}}$, which occurs when $x = 3$.**

Solution.—The knowledge which we have gained by studying the differential calculus is of very great value to us when tracing the graph of an equation. We would naturally begin by searching for points which at once would give us a general idea of the curve. Such points are called *singular points*, and include maxima, minima, points of inflexion, etc.

As the x-axis is an axis of symmetry of the given curve, we need only determine that part of the curve which is situated above the x-axis and for which the ordinates are positive. Hence,

$$y=\frac{(x+1)\sqrt{x}}{x};\ \frac{dy}{dx}=\frac{(x-1)\sqrt{x}}{2x^2};\ \frac{d^2y}{dx^2}=\frac{(3-x)\sqrt{x}}{4x^3}.$$

x.	y.	$\frac{dy}{dx}$.	$\frac{d^2y}{dx^2}$.	Remarks.
neg.	imag.	...	...	The curve lies entirely to the right of the y-axis.
0	∞	∞	...	The y-axis is an asymptote.
$x<1$	...	neg.	pos.	
1	2	0	pos.	Minimum.
$3>x>1$	...	...	pos.	
3	$\frac{4}{\sqrt{3}}$	$\frac{1}{3\sqrt{3}}$	0	Point of inflexion.
$x>3$	...	pos.	neg.	
∞	∞	0	...	The tangent tends ultimately to become parallel to the x-axis.

$\frac{dy}{dx}$ is a maximum when $x=3$ as $\frac{d^2y}{dx^2}=0$. The curve can now easily be sketched.

(34) (Q. June 1908).—**Find the maximum and the minimum values, and the least value, of the function of x which is equal to $2x(x-1)$ when x is less than one. and is equal to $(x-1)(x-2)(x-3)$ when x is greater than one. Is the slope of the graph continuous?**

Solution.

(i.) $x \not> 1$; $y_1 = 2x(x-1)$; $\frac{dy_1}{dx} = 2(2x-1)$; $\frac{d^2y_1}{dx^2} = 4$.

As $\frac{d^2y_1}{dx^2}$ is always positive there can be no maximum value of y_1. $\frac{dy_1}{dx} = 0$, when $x = \frac{1}{2}$, and as $\frac{d^2y_1}{dx^2}$ is positive the minimum value of y_1 is equal to $-\frac{1}{2}$, which is also the least value of y_1 as $\frac{d^2y_1}{dx^2}$ is always positive.

When $x = 1$, $\frac{dy_1}{dx} = 2$ and $y_1 = 0$.

(ii.) $x \not< 1$; $y_2 = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$.

$\frac{dy_2}{dx} = 3x^2 - 12x + 11 = 3(x - 1{\cdot}423)(x - 2{\cdot}577)$; $\frac{d^2y_2}{dx^2} = 6(x-2)$.

x.	y_2.	$\frac{dy_2}{dx}$.	$\frac{d^2y_2}{dx^2}$.	Remarks.
1	0	2	neg.	
1·423	0·385	0	neg.	Maximum.
2	0	neg.	0	Point of inflexion as $\frac{d^3y_2}{dx^3}$ is not 0.
2·577	−0·385	0	pos.	Minimum, and also the least value of y_2 as $\frac{dy_2}{dx}$ and $\frac{d^2y_2}{dx^2}$ both remain positive when $x > 2{\cdot}577$.

(iii.) $x = 1$, $y_1 = y_2 = 0$, and $\frac{dy_1}{dx} = \frac{dy_2}{dx} = 2$; the slope of the graph is therefore continuous. But $\frac{d^2y_1}{dx^2}$ is positive, whereas $\frac{d^2y_2}{dx^2}$ is negative; hence the point is a point of inflexion.

(35) (Q. June 1908).—**The x, y co-ordinates of a point are given by the expressions $x = 1 - \cos\theta$, $y = \cos\theta + \frac{1}{4}\cos 2\theta$.**

Prove that $\frac{dy}{dx}$ is equal to $-2\cos^2\frac{\theta}{2}$; and sketch the path described by the point as θ increases from 0 to 180°. What is the path of the point as θ increases from 180° to 360°?

Solution. $\frac{dy}{dx}=\frac{dy}{d\theta}\frac{d\theta}{dx}=-\frac{d\theta}{dx}\left(\sin\theta+\frac{\sin 2\theta}{2}\right)$; but $\frac{dx}{d\theta}=\sin\theta$.

Hence, $$\frac{dy}{dx}=-(1+\cos\theta)=-2\cos^2\frac{\theta}{2}.$$

As $\cos\theta=1-x$, the equation of the curve is

$$2y=x^2-4x+\frac{5}{2}=(x-3{\cdot}225)(x-0{\cdot}775)\,. \qquad (1)$$

We have also $\frac{dy}{dx}=x-2$ and $\frac{d^2y}{dx^2}=1$. The graph of (1) will be found to be a parabola with vertex at $(2, -\frac{3}{4})$ and its axis parallel to the y-axis. But we only require the portion of the curve which lies between $x=0$, or $\theta=0$, and $x=2$, or $\theta=180°$. For this portion of the curve we have

x.	y.	$\frac{dy}{dx}$.	$\frac{d^2y}{dx^2}$.	Remarks.
0	1·25	−2	pos.	
$x<0{\cdot}775$	pos.	neg.	,,	
0·775	0	−1·23	,,	
$2>x>0{\cdot}775$	neg.	neg.	,,	
2	$-\frac{3}{4}$	0	,,	Minimum.

While θ increases gradually from 0 to 180°, the point will move on the curve from (0, 1·25) to $(2, -\frac{3}{4})$. While θ increases from 180° to 360°, the values of x will be repeated backwards; hence, the point will move back on the curve from $(2, -\frac{3}{4})$ to (0, 1·25). As θ continues to increase, the tracing-point will oscillate between (0, 1·25) and $(2, -\frac{3}{4})$. A complete oscillation will take place while θ increases an amount 2π.

(36) (Q. June 1908).—**The equation to a curve being given in terms of polar co-ordinates, find an expression for the angle ϕ between the radius vector of any point on the curve and the tangent to the curve at the point.**

Find the co-ordinates of the points on the curve $r^2 = a^2 \sin 2\theta$ at which the tangent is at right angles to the initial line from which θ is measured.

Solution.—The first part of the question is answered by article **110**.

The graph of $r^2 = a^2 \sin 2\theta$ is the lemniscate (see (**4**) p. 50); in the latter example the polar equation of the curve is $r^2 = 2a^2 \cos 2\theta$; but by turning the x-axis (fig. 25), which is also the initial axis, through an angle of minus 45°, the polar equation of the curve will be found to be $r^2 = 2a^2 \sin 2\theta$; and by taking a^2 for $2a^2$ the equation of the curve is $r^2 = a^2 \sin 2\theta$, which is the equation given in the present question.

In article **110** it is shown that $\tan \phi = r\frac{d\theta}{dr}$. Let now α be the angle which the tangent makes with the initial line in the positive direction, then it will be seen that $\alpha = \theta + \phi$; hence, $\tan \alpha = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$. In the present case $\alpha = \frac{\pi}{2}$, or $\tan \alpha = \infty$, which will occur when $1 - \tan \theta \tan \phi = 0$, or $\tan \phi = \cot \theta = r\frac{d\theta}{dr}$. Differentiating the equation of the curve w.r.t. θ we get

$$2r\frac{dr}{d\theta} = 2a^2 \cos 2\theta,$$

or
$$r\frac{d\theta}{dr} = \frac{r^2}{a^2 \cos 2\theta} = \frac{a^2 \sin 2\theta}{a^2 \cos 2\theta} = \tan 2\theta = \cot \theta,$$

or
$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{\tan \theta};$$

hence,
$$\tan \theta = \pm \sqrt{\frac{1}{3}}, \text{ or } \theta = \begin{cases} 30° \\ 210° \end{cases}.$$

The numerical value of r is $0·9306a$. Hence, the two points are

$$(0·9306a,\ 30°), \text{ and } (0·9306a,\ 210°)\ (\textit{Ans.});$$

the notation may also be written thus

$$(0·9306a,\ 30°), \text{ and } (-0·9306a,\ 30°).$$

(37) (Q. Nov. 1906).—**Find the value of $\frac{dy}{dx}$ at any point x, y of the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, and write down the equation of the tangent to the curve at the point. If the tangent cuts the axes Ox, Oy, in P and Q respectively, show that O P + O Q is constant, and equal to a.**

Take a equal to 4 inches, and draw a number of straight lines cutting the axes and having the sum of their intercepts equal to this. Show that the curve may then be sketched in to touch these lines.

Solution.—Differentiating the given equation w.r.t. x we get

$$\tfrac{1}{2}x^{-\frac{1}{2}}+\tfrac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0, \quad \text{or} \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}};$$

hence, the equation of the tangent is

$$\mathrm{Y}-y=-\frac{\sqrt{y}}{\sqrt{x}}(\mathrm{X}-x),$$

$$\text{or} \quad \frac{\mathrm{X}}{\sqrt{x}(\sqrt{x}+\sqrt{y})}+\frac{\mathrm{Y}}{\sqrt{y}(\sqrt{x}+\sqrt{y})}=1, \quad \text{or} \quad \frac{\mathrm{X}}{\sqrt{ax}}+\frac{\mathrm{Y}}{\sqrt{ay}}=1\,;$$

hence, by article **10**, the intercepts which the tangent cuts off the axes are O P = $\sqrt{ax}$ and O Q = $\sqrt{ay}$.

Hence, $\quad \mathrm{OP}+\mathrm{OQ}=\sqrt{ax}+\sqrt{ay}=\sqrt{a}(\sqrt{x}+\sqrt{y})=a.$

The second part of the question follows from the first part.

(38) (Q. Nov. 1908, second part).—**A circular cylinder of a given capacity, closed at each end, is to be made of thin sheet metal of a uniform thickness. Find the ratio of the diameter to the length of the cylinder such that the weight of the vessel shall be a minimum.**

Solution.—Let V be the constant volume, l the length, and d the diameter of the cylinder. As the walls of the cylinder are very thin, we may take the weight as being proportional to the total surface S. S is therefore to be a minimum. We have

$$\mathrm{V}=\frac{\pi d^2}{4}l. \qquad (1)$$

$$\mathrm{S}=2\frac{\pi d^2}{4}+\pi dl \qquad (2)$$

Let $\frac{d}{l}=x$, then $d=lx$, which, inserted in (1) and (2), gives

$$V=\frac{\pi}{4}l^3x^2, \text{ and } S=\frac{\pi}{2}l^2x^2+\pi l^2x \quad . \quad . \quad (3)$$

Hence, $$\frac{dV}{dx}=0=\frac{\pi}{4}\left(l^3 2x+x^2 3l^2\frac{dl}{dx}\right), \text{ or } \frac{dl}{dx}=-\frac{2l}{3x} \quad . \quad (4)$$

$$\frac{dS}{dx}=\pi\left(\tfrac{1}{2}l^2 2x+\frac{x^2}{2}2l\frac{dl}{dx}+l^2+x2l\frac{dl}{dx}\right)=\frac{\pi}{3}l^2(x-1) \quad . \quad (5)$$

$$\frac{dS}{dx}=0 \text{ for } x=1, \text{ or } l=d;$$

$$\frac{d^2S}{dx^2}=\frac{\pi}{3}l\left(l+2(x-1)\frac{dl}{dx}\right),$$

which is positive for $x=1$.

$$l=d \ (Ans.).$$

(**39**) (Q. Nov. 1907).—**Find the maximum and minimum values of $\frac{x(x+1)}{x-1}$, distinguishing between them.**

Solution.—Let $y=\frac{x(x+1)}{x-1}$, then

$$\frac{dy}{dx}=\frac{(x-1)(x+x+1)-x(x+1)}{(x-1)^2}=\frac{x^2-2x-1}{(x-1)^2}=\frac{(x-1-\sqrt{2})(x-1+\sqrt{2})}{(x-1)^2}.$$

$$\frac{d^2y}{dx^2}=\frac{4}{(x-1)^3};\ \frac{dy}{dx}=0 \text{ when } x=1+\sqrt{2} \text{ and } x=1-\sqrt{2}.$$

The function is a maximum when $x=1-\sqrt{2}$ as $\frac{d^2y}{dx^2}$ is negative.

The function is a minimum when $x=1+\sqrt{2}$ as $\frac{d^2y}{dx^2}$ is positive. (*Ans.*).

(**40**) (Q. May 1907).—**A certain function of x is equal to ax^2 for values of x less than 1, and to $-ax^2+bx-1$ for values of x greater than 1. Find the values of the constants a and b in order that there may be no discontinuity or abrupt change of slope in the graph of the function at the point $x=1$.**

With these values of a and b find the values of x for which the function is zero.

Solution.—The values of the ordinates of the two portions of the given function, for $x=1$, are respectively $y_1=a$ and $y_2=-a+b-1$.

The slopes for $x=1$ are $\frac{dy_1}{dx}=2a$ and $\frac{dy_2}{dx}=-2a+b$.

If there is to be no discontinuity we must have

$$y_1=y_2, \quad \text{or} \quad 2a=b-1 \quad . \quad . \quad . \quad . \quad (1)$$

and if there is to be no abrupt change of slope we must have

$$\frac{dy_1}{dx}=\frac{dy_2}{dx}, \quad \text{or} \quad 4a=b \quad . \quad . \quad . \quad (2)$$

The simultaneous equations in (1) and (2) give

$$a=\tfrac{1}{2}, \quad \text{and} \quad b=2 \quad . \quad . \quad . \quad . \quad (3)$$

Inserting the values (3) in the function we get

$$\tfrac{1}{2}x^2, \quad \text{and} \quad -\tfrac{1}{2}x^2+2x-1.$$

The function will therefore be zero for $x=0$, $x=2+\sqrt{2}=3{\cdot}414$ and $x=2-\sqrt{2}=0{\cdot}586$ (*Ans.*).

(41) (Q. June 1908).—**Apply Taylor's theorem to express $\sin\left(\frac{\pi}{4}+x\right)$ in the form of a series in ascending powers of x; and show exactly how many terms must be included, in order to give a result which is correct to five places of decimals for all positive values of x not exceeding the circular measure of 2°.**

Solution —Substituting $\frac{\pi}{4}$ for x, and x for δx in [**74**], we get

$$\sin\left(\frac{\pi}{4}+x\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\frac{x}{1}-\sin\frac{\pi}{4}\frac{x^2}{\lfloor 2}-\cos\frac{\pi}{4}\frac{x^3}{\lfloor 3}+\sin\frac{\pi}{4}\frac{x^4}{\lfloor 4}\cdot\cdot\cdot$$

$$=\sqrt{0{\cdot}5}\left(1+\frac{x}{1}-\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}\cdot\cdot\cdot\right). \quad (1)$$

x must not exceed the circular measure of 2°, but 2° is equal to $\frac{\pi}{90}=0{\cdot}0349666$ radians. As the result is required to be correct to five decimals, we must work with seven decimals.

Substituting 0·0349666 for x in (1) we get

$$\sin\left(\frac{\pi}{4}+0\cdot0349666\right)=0\cdot7071066(1+0\cdot0349666$$
$$-0\cdot0006112-0\cdot0000071+0\cdot0000000)\ldots$$
$$=0\cdot73135 \quad . \quad . \quad . \quad . \quad (2)$$

(2) shows that we only require four terms when x is not to exceed 0·0349666 radians.

$$\sin\left(\frac{\pi}{4}+x\right)=0\cdot7071066\left(1+\frac{x}{1}-\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}\right)\ (Ans.).$$

(42) (Q. June 1909).—**A conical tent is to have a given volume. Find what is the ratio of the height of the tent to the radius of its base when the least possible amount of canvas is used in the conical surface.**

If in this case the canvas is spread out on the ground, what fraction of a complete circle is it?

Solution.—The cone is a cone of revolution. Let r be the radius of the base, h the height of the cone, and l the length of the generator.

$$\text{Volume of cone}=\frac{\pi}{3}r^2h=\text{K (a constant)} \quad . \quad (1)$$

Area of curved surface

$$=\text{A}=\tfrac{1}{2}\times 2\pi rl=\pi r\sqrt{h^2+r^2} \quad . \quad . \quad . \quad . \quad (2)$$

Let $\dfrac{h}{r}=x$, or $h=rx$. By inserting the latter value for h in (1) and (2) we get

$$\text{K}=\frac{\pi}{3}r^3x, \quad \text{and} \quad \text{A}=\pi r^2\sqrt{1+x^2} \quad . \quad . \quad (3)$$

r is a function of x in (3). By differentiating the first in (3) w.r.t. x, we get

$$0=\frac{\pi}{3}r^3+\pi r^2x\frac{dr}{dx}, \quad \text{or} \quad \frac{dr}{dx}=-\frac{r}{3x}.$$

By differentiating A in (3) w.r.t. x, we get

$$\frac{d\text{A}}{dx}=\frac{\pi r^2(x^2-2)}{3x\sqrt{1+x^2}}; \ \frac{d\text{A}}{dx} \text{ is 0 for } x=\sqrt{2}.$$

Inserting $x=\sqrt{2}$ in $\dfrac{d^2\text{A}}{dx^2}$, it will be found that the latter quantity will be positive.

A : area of complete circle : : πrl : πl^2; hence, for $x=\sqrt{2}$, $\frac{\pi rl}{\pi l^2}=0{\cdot}577$.

$$\frac{h}{r}=\sqrt{2};\ \text{A}=0{\cdot}577 \text{ of a complete circle } (\textit{Ans.}).$$

(43) (Q. June 1910).—**Find the differential coefficients of (i.)** $\log_e \tan x$; **(ii.)** $\frac{e^{x^2}+1}{x^2+1}$.

If $y=ax/(x+b)$, **prove that** $\frac{2}{y}\frac{dy}{dx}-\frac{d^2y}{dx^2}\Big/\frac{dy}{dx}=\frac{2}{x}$.

Solution.

(i.) $$\frac{d\log_e \tan x}{dx}=\frac{d\log_e \tan x}{d\tan x}\,\frac{d\tan x}{dx}=\frac{1}{\tan x}\,\frac{1}{\cos^2 x}=\frac{2}{\sin 2x}.$$

(ii.) $$\frac{de^{x^2}}{dx}=\frac{de^{x^2}}{dx^2}\,\frac{dx^2}{dx}=2xe^{x^2}.$$

Hence,

$$d\frac{e^{x^2}+1}{x^2+1}\Big/dx=\frac{(x^2+1)2xe^{x^2}-(e^{x^2}+1)2x}{(x^2+1)^2}=\frac{2x(x^2e^{x^2}-1)}{(x^2+1)^2}.$$

(iii.) $$\frac{dy}{dx}=d\frac{ax}{x+b}\Big/dx=\frac{ab}{(x+b)^2};\ \frac{d^2y}{dx^2}=d\frac{ab}{(x+b)^2}\Big/dx=-\frac{2ab}{(x+b)^3}.$$

Hence, $$\frac{2(x+b)}{ax}\,\frac{ab}{(x+b)^2}+\frac{2ab}{(x+b)^3}\,\frac{(x+b)^2}{ab}=\frac{2}{x}\ (\textit{Ans.}).$$

INTEGRAL CALCULUS.

CHAPTER XV.

INTEGRALS—AREAS.

111. Integration.

As subtraction and division are respectively the inverse operations of addition and multiplication, so is *integration the inverse operation of differentiation.*

If we have given the slope, $\frac{dy}{dx}=f'(x)$, of a curve as a function of x, the equation of the curve is found by integrating $f'(x)$. Just as the result of division is tested by multiplying the divisor by the quotient, so we test the result obtained by integration by differentiating the latter result, and if we thus obtain the function which has been integrated, then the integration has been correctly performed. *The student is advised always to make this test.*

That a function is to be integrated w.r.t. x is denoted by $\int f(x)\,dx$. Hence, we have

$$\frac{dy}{dx}=f'(x), \quad \text{and} \quad y=\int f'(x)\,dx=f(x) \qquad [93]$$

The second expression in **[93]** reads "*y equal to the integral of $f'(x)$ w.r.t. x.*"

Example 1.—Integrate $y=\int x^m dx$.

Since $\frac{dy}{dx}=x^m$, we have $y=\frac{x^{m+1}}{m+1}$; hence, $\int x^m dx=\frac{x^{m+1}}{m+1}$.

Example 2.—Integrate $y=\int e^x dx$.

Clearly $y=e^x$ because $\frac{dy}{dx}=e^x$; hence, $\int e^x dx=e^x$.

112. Table of results to be remembered.
(Refer to table in article **91**.)

$$\int a\,dx = a\int dx = ax\,;\qquad \int \frac{\sin x}{\cos^2 x}\,dx = \sec x\,;$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}\,;\qquad \int \frac{\cos x}{\sin^2 x}\,dx = -\operatorname{cosec} x\,;$$

$$\int a^x\,dx = \frac{a^x}{\log_e a}\,;\qquad \int \frac{dx}{\sqrt{1-x^2}} = \text{ang}(\sin = x) = -\text{ang}(\cos = x)\,;$$

$$\int e^x\,dx = e^x\,;$$

$$\int \frac{dx}{x} = \log_e x\,;\qquad \int \frac{dx}{1+x^2} = \text{ang}(\tan = x) = -\text{ang}(\cot = x)\,;$$

$$\int \cos x\,dx = \sin x\,;$$

$$\int \sec^2 x\,dx = \tan x\,;\qquad \int [f_1(x) + f_2(x)]dx$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x\,;\qquad = \int f_1(x)dx + \int f_2(x)dx.$$

113. Determination of an area.

Let the problem be to find the area L B C M L = A (fig. 40),

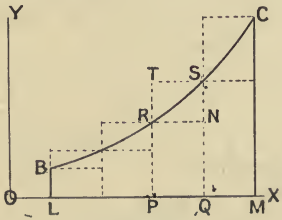


Fig. 40.

which is enclosed by the axis of x, the ordinates to points B and C, and the curve B C, whose equation is $y = f(x)$.

Let O P be the abscissa to point R on the curve, and $PQ = \delta x$, then $PR = f(OP)$ and $QS = f(OP + \delta x)$, or generally $y = f(x)$ and $y + \delta y = f(x + \delta x)$. Divide now L M into a number of parts each of which is equal to δx, and draw the ordinates y and $y + \delta y$ to each division corresponding to P R and Q S of the division P R S Q, then clearly

$$\sum(\text{rectangles P T S Q P}) > \text{A} > \sum(\text{rectangles P R N Q P}) \qquad (1)$$

By diminishing δx indefinitely, and at the same time increasing the number of divisions indefinitely, the two areas to the right and left of A in (1) will ultimately be equal, and will therefore also both be equal to A. Area L B C M L is therefore the limiting value of $\sum$(rectangles P R N Q P).

Let us consider division P R S Q P, and let the area L B R P L be A_x, corresponding to $OP = x$, then area P R S Q P is δA, *i.e.* δx, δy, and δA are corresponding and simultaneously vanishing increments of x, y, and A_x. We have now $(y+\delta y)\delta x > \delta A > y\ \delta x$, or $y + \delta y > \frac{\delta A}{\delta x} > y$, and ultimately

$$y > \lim. \frac{\delta A}{\delta x} = \frac{dA}{dx} > y;\ \text{hence,}\ \frac{dA}{dx} = y \qquad . \qquad . \quad [94]$$

and

$$A = \int y\,dx = \int f(x)dx = F(x) \ . \qquad . \qquad . \quad [95]$$

We thus see that the area $A = F(x)$ is determined by integrating the equation of the curve by which the area is bounded.

Integration may therefore also be defined as the operation of determining the area under the graph of the function which is to be integrated.

EXAMPLE.—To find the area of parabola $y^2 = 4ax$.

$y = \pm 2\sqrt{ax}$; hence, the area of the portion of the parabola, which lies between $x = 0$ and $x = x$, is

$$A = 2\int y\,dx = 2\int 2\sqrt{ax}\,dx = 4\sqrt{a}\int\sqrt{x}\,dx = \frac{4}{3}\sqrt{a}\,x^{\frac{3}{2}} = \frac{4}{3}xy,$$

or the area is equal to $\frac{2}{3}$ of the rectangle $x2y$.

114. Arbitrary constant.

We have
$$\frac{df(x)}{dx} = f'(x),$$

but we have also
$$\frac{d[f(x)+C]}{dx} = f'(x),$$

where C is a constant. Hence, we may write

$$\int f'(x)dx = f(x) + C,$$

where C may or may not be equal to 0. For this reason it is necessary to add a constant C to any function which is produced by integration. This constant is called an *arbitrary constant*, and $\int f'(x)dx$ is called an *indefinite*, or *uncorrected* integral.

115. Definite and corrected integrals.

The curve (fig. 40) may be continued beyond points B and C, but in article **113** we only required the area under the portion B C of the curve. Let O M $= a$ and O L $= b$, then by [**95**] $A = F(a) - F(b)$, which we write

$$A = \int_b^a f(x)dx = F(a) - F(b) \quad . \quad . \quad . \quad [96]$$

and read, "*the integral of $f(x)$ between the limits of a and b.*" a is called the upper limit and b is called the lower limit, and the integral in [**96**] is called a *definite integral.* $a - b$ is termed the *interval.*

If we require an expression for the area from L B to a distance x from the origin, we would write

$$A = \int_b^x f(x)dx = F(x) - F(b) \quad . \quad . \quad . \quad [97]$$

The upper limit is undetermined, and the integral is called a *corrected integral.*

Examples.

$$\int_{-\infty}^0 e^x\,dx = e^0 - e^{-\infty} = 1\,; \qquad \int_{-\infty}^x e^x\,dx = e^x - e^{-\infty} = e^x\,;$$

$$\int_0^x e^x\,dx = e^x - e^0 = e^x - 1\,; \qquad \int_0^x \cos x\,dx = \sin x - \sin 0 = \sin x\,;$$

$$\int_0^{\frac{\pi}{2}} \cos x\,dx = \sin\frac{\pi}{2} - \sin 0 = 1.$$

The limits of a definite integral may be interchanged by changing the sign of the integral.

Example 1.

$$\int_b^a f(x)dx = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f(x)dx.$$

Example 2.

$$\int_a^c f(x)dx - \int_b^c f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx = \int_a^b f(x)dx.$$

Example 3.

$$\int_a^c f(x)dx - \int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^a f(x)dx = \int_b^c f(x)dx.$$

116. Approximate determination of an area.

If the integration of [**96**] can be performed, then the area A will be determined with absolute accuracy. If, however, [**96**] cannot be integrated, or the equation of the curve, $y=f(x)$, is unknown, then we must be satisfied with an approximate value of the area A.

(i.) *The trapezoid rule.*—Let B C (fig. 41) be the curve whose equation is unknown, or which is too complicated to be integrated. For the purpose of determining the area L B C M L = A, the base line L M may be divided into a number of equal portions, each of length h; the latter must be chosen so small that the arc of the curve of any division is so short that it may be considered

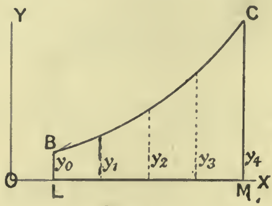


FIG. 41.

approximately a straight line. Area A is then approximately equal to the sum of the areas of the divisions, but each division is a trapezoid; hence,

$$A=\frac{y_0+y_1}{2}h+\frac{y_1+y_2}{2}h+\frac{y_2+y_3}{2}h+\frac{y_3+y_4}{2}h$$

$$=h\left(\frac{y_0+y_4}{2}+y_1+y_2+y_3\right),$$

or generally

$$A=h\left(\frac{y_0+y_n}{2}+y_1+y_2+\ .\ .\ .\ +y_{n-1}\right)=h\left(\frac{y_0+y_n}{2}+\sum_{r=1}^{r=n-1}y_r\right) \quad [98]$$

where $h=\frac{LM}{n}$, and y_0 and y_n are respectively the first and the last ordinate. This method is known as the *trapezoid rule.*

(ii.) *Simpson's rule.*—Divide the base line L M (fig. 41) into an even number, $2n$, of equal portions, each of length h; hence, $h=\frac{LM}{2n}$. Let point L be the origin, and let us first determine the area A_2 of the two divisions between y_0 and y_2. Instead of considering the arcs of the curve between y_0 and y_2 straight lines, let us substitute for the given curve another curve whose equation is $y = a + bx + cx^2$; hence,

$$A_2 = \int_0^{2h} (a + bx + cx^2)dx = \left[x\left(a + \frac{bx}{2} + \frac{cx^2}{3}\right)\right]_{x=0}^{x=2h}$$

$$= 2h\left(a + bh + \frac{4}{3}ch^2\right) \quad . \quad (1)$$

but $x = 0$, $y_0 = a$; $x = h$, $y_1 = a + bh + ch^2$;

$$x = 2h, \; y_2 = a + 2bh + 4ch^2 \quad . \quad (2)$$

By eliminating a, b, and c between (1) and (2) we get

$$A_2 = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

In the same manner we get the area between y_2 and y_4,

$$A_4 = \frac{h}{3}(y_2 + 4y_3 + y_4).$$

The last area, between $y_{2(n-1)}$ and y_{2n} is

$$A_{2n} = \frac{h}{3}(y_{2(n-1)} + 4y_{2n-1} + y_{2n}).$$

Hence,

$$A = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ . \ . \ . + 2y_{2(n-1)} + 4y_{2n-1} + y_{2n}),$$

or

$$A = \frac{h}{3}\left(y_0 + y_{2n} + 4\sum_{r=1}^{r=n} y_{2r-1} + 2\sum_{r=1}^{r=n-1} y_{2r}\right) \quad . \quad . \quad [99]$$

which is *Simpson's rule.* Results obtained by **[99]** are more accurate than those obtained by **[98]**.

Example.—Find the area under the curve from the data given in the following table:—

$y_0 = 4{\cdot}47$	$y_3 = 5{\cdot}66$	$y_6 = 6{\cdot}63$	$y_9 = 7{\cdot}48$	$h = 1$
$y_1 = 4{\cdot}90$	$y_4 = 6{\cdot}00$	$y_7 = 6{\cdot}93$	$y_{10} = 7{\cdot}75$	
$y_2 = 5{\cdot}29$	$y_5 = 6{\cdot}32$	$y_8 = 7{\cdot}21$		

(α) By **[98]** we get $A = 62{\cdot}53$ units.
(β) By Simpson's rule **[99]** we get $A = 62{\cdot}55$ units.

117. Determination of an area referred to polar co-ordinates. Let the problem be to determine the area A (fig. 42), bounded by O L, O Q, and the curve L N P Q, whose equation is $r=f(\theta)$.

Let $ON=r$, angle $XON=\theta$, and angle $NOP=\delta\theta$, then $OP=r+\delta r$. δA will thus be equal to area N O P N, and it is obvious that the value of δA must lie between those of the areas of the two circle-sectors $MOPM=\frac{1}{2}(r+\delta r)^2\delta\theta$ and $NORN=\frac{1}{2}r^2\delta\theta$; we have therefore $\frac{1}{2}(r+\delta r)^2\delta\theta>\delta A>\frac{1}{2}r^2\delta\theta$, and in the limit

$$\frac{dA}{d\theta}=\tfrac{1}{2}r^2, \text{ and therefore } A=\tfrac{1}{2}\int_{r_1}^{r_2} r^2 d\theta=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}[f(\theta)]^2 d\theta \quad [\mathbf{100}]$$

where $r_1=OL$, $r_2=OQ$, $\theta_1=$ angle XOL, and $\theta_2=$ angle XOQ.

If [**100**] cannot be integrated, or $r=f(\theta)$ is not known, then we may resort to the approximate methods shown in the preceding article. For this purpose we divide the angular interval

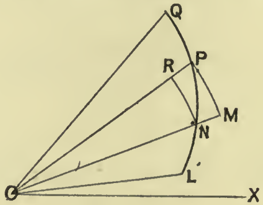


FIG. 42.

$\theta_2-\theta_1$ into a number of equal portions each equal to ϵ radians, and measure the lengths of the radii vectores r_0, r_1, r_2 . . . corresponding to angles 0, ϵ, 2ϵ . . . reckoned from O L. The values of r_0, r_1 . . . must be squared.

(i.) When applying [**98**] we have $\epsilon=\dfrac{\theta_2-\theta_1}{n}$, and the area will approximately be

$$A=\frac{\epsilon}{2}\left(\frac{r_0^{\,2}+r_n^{\,2}}{2}+\sum_{q=1}^{q=n-1} r_q^{\,2}\right) \quad . \quad . \quad . \quad [\mathbf{101}]$$

(ii.) When we apply Simpson's rule, we have $\epsilon=\dfrac{\theta_2-\theta_1}{2n}$, and the area is approximately

$$A=\frac{\epsilon}{6}\left(r_0^{\,2}+r_{2n}^2+4\sum_{q=1}^{q=n} r_{2q-1}^2+2\sum_{q=1}^{q=n-1} r_{2q}^2\right) \quad . \quad . \quad [\mathbf{102}]$$

EXAMPLE.—Find the area bounded by $r=a(1-\cos\theta)$ and the radii vectores corresponding to $\theta=0$ and $\theta=\gamma$.

By [**100**] we have

$$A=\tfrac{1}{2}a^2\int_0^\gamma(1-\cos\theta)^2\,d\theta=\frac{a^2}{2}\int_0^\gamma(1+\cos^2\theta-2\cos\theta)\,d\theta$$

$$=\frac{a^2}{2}\int_0^\gamma\left(1+\frac{1+\cos 2\theta}{2}-2\cos\theta\right)d\theta=\frac{a^2}{4}\left(3\theta+\frac{\sin 2\theta}{2}-4\sin\theta\right)_0^\gamma$$

$$=\frac{a^2}{4}(3\gamma+\sin\gamma\cos\gamma-4\sin\gamma).$$

Let $\gamma=\frac{\pi}{2}$ and $a=8$, then $r=8(1-\cos\theta)$ and A = 11·4 units.

If we use Simpson's rule and take $\epsilon=\frac{\pi}{20}=9°$, our calculated values may be tabulated thus :—

θ degrees.	r.		r^2.	θ degrees	r.		r^2.
0	r_0	0	0	54	r_6	3·30	10·89
9	r_1	0·0984	0·00968	63	r_7	4·37	19·097
18	r_2	0·391	0·1529	72	r_8	5·53	30·581
27	r_3	0·872	0·7604	81	r_9	6·75	45·563
36	r_4	1·53	2·3409	90	r_{10}	8·00	64·00
45	r_5	2·34	5·480				

$A=\frac{\pi}{120}(0+64+283{\cdot}64+87{\cdot}93)=11{\cdot}4$, which agrees with the value we found by integration.

118. The average value of a function.

By the mean ordinate of curve BC (fig. 41) is understood the height of the rectangle whose base is $LM=a-b$, and whose area is equal to the area LBCML. The mean ordinate y_m is therefore the average value of $y=f(x)$, the equation of the curve, in the interval $a-b$. Hence,

$$y_m=\frac{\text{area LBCML}}{\text{LM}}=\frac{\int_b^a f(x)\,dx}{a-b} \quad . \quad . \quad [103]$$

In practical physics we often speak of the average value of a cause, which has produced a certain effect during a certain given interval of time T. Let the variable cause be C, and the variable time be t, then $C=f(t)$.

The total effect produced by C during the time-interval T is, $\int_0^T f(t)\,dt$, and the average value of C during the same interval is therefore

$$C_m=\frac{1}{T}\int_0^T f(t)\,dt \quad . \quad . \quad . \quad . \quad . \quad [104]$$

C_m is that constant cause which would produce the same effect during the time-interval T as the variable cause has produced during the same interval.

CHAPTER XVI.

VOLUMES.

119. The volume of any solid.

Let M N (fig. 43) be any solid whatever standing on the plane P, and let O X be perpendicular to P. Planes at distances x and $x + \delta x$ and parallel to P cut the solid through sections whose

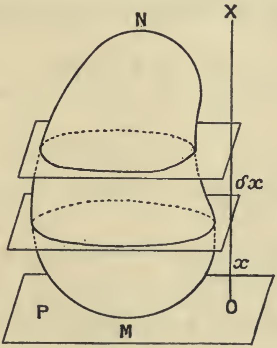


FIG. 43.

areas are A_x and $(A_x + \delta A_x)$ respectively. Let δV be the volume of the portion of the solid which lies between the two planes. The value of δV must lie between the cylindrical volumes $(A_x + \delta A_x)\delta x$ and $A_x\,\delta x$; therefore $\frac{\delta V}{\delta x}$ lies between $A_x + \delta A_x$ and A_x; hence, in the limit

$$\frac{dV}{dx} = A_x, \quad \text{and} \quad V = \int_0^H A_x\,dx \qquad . \qquad . \quad [105]$$

where H is the height of the solid measured along O X. The volume V is therefore determined as if it were the area under the graph of $A_x = f(x)$, A_x being the ordinate to the graph. Hence, if the variation of A_x with x, *i.e.* $A_x = f(x)$, is known and integration is possible, V can be determined with absolute accuracy; but if $f(x)$ is unknown, or cannot be integrated, we must apply the approximate methods. The rule **[98]** becomes

$$V = h\left(\frac{A_0 + A_n}{2} + \sum_{r=1}^{r=n-1} A_r\right) \quad . \quad . \quad . \quad [106]$$

where $h = \dfrac{H}{n}$.

Simpson's rule **[99]** becomes

$$V = \frac{h}{3}\left(A_0 + A_{2n} + 4\sum_{r=1}^{r=n} A_{2r-1} + 2\sum_{r=1}^{r=n-1} A_{2r}\right) \quad . \quad . \quad [107]$$

where $h = \dfrac{H}{2n}$. The areas A_0, A_1, etc., may be determined by planimeter.

Example.—To find the volume of the solid from the data given below.

$H = 10$, $h = 1$ and

A_x.		A_x.		A_x.	
A_0	25·16	A_4	50·27	A_8	75·43
A_1	31·37	A_5	56·48	A_9	81·71
A_2	37·61	A_6	62·77	A_{10}	87·90
A_3	43·94	A_7	69·10		

(i.) By **[107]** we have

$$V = \tfrac{1}{3}(25{\cdot}16 + 87{\cdot}90 + 1130{\cdot}40 + 452{\cdot}16) = 565{\cdot}21 \text{ units.}$$

(ii.) By **[106]** we have

$$V = (56{\cdot}53 + 508{\cdot}68) = 565{\cdot}21 \text{ units.}$$

120. The volume of a cone or a pyramid.

In fig 44, O C B is a cone or a pyramid. O is the apex and C B the base of the solid. The base may be any closed curve or polygon. Turn the solid about O until the plane of the base C B is perpendicular to O X. Let the height of the solid be H and the area of the base be A. A plane parallel to the base and

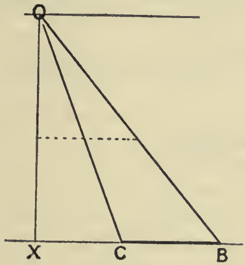


FIG. 44.

at a distance x from O will cut the solid at a section whose area is A_x. But we have

$$\frac{A_x}{x^2}=\frac{A}{H^2}; \quad \text{hence,} \quad A_x=\frac{A}{H^2}x^2=f(x),$$

and the volume of the solid is

$$V=\int_0^H A_x\,dx=\frac{A}{H^2}\int_0^H x^2\,dx=\frac{1}{3}\,A\,H \qquad . \quad . \quad [108]$$

The volume of any cone or pyramid is therefore equal to one-third of the volume of a cylinder whose height and base are the same as those of the cone or the pyramid.

121. The volume of any truncated cone or pyramid.

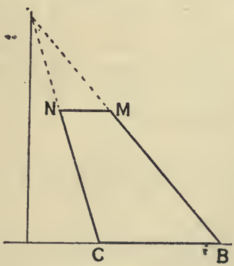


FIG. 45.

Let B C N M B (fig. 45) be any truncated cone or pyramid, and let A_1 and A_2 be the areas of the two plane ends. The volume of the solid is

$$V = \frac{1}{3}(A_2H_2 - A_1H_1), \text{ but } \frac{A_1}{A_2} = \frac{H_1^2}{H_2^2} \text{ and } H_1 = H_2 - H;$$

hence,
$$V = \frac{H}{3}(A_1 + \sqrt{A_1A_2} + A_2) \quad . \quad . \quad . \quad [109]$$

A truncated cone or pyramid is also called a *frustum*.

122. Volume of revolution.

Let it be required to find the volume formed by the revolution of a given curve about an axis in its own plane. Take the axis of revolution as x-axis, for instance, O X in fig. 40, p 123. Planes at right angles to the axis of revolution will cut the solid in circles, and if the equation of the revolving curve be $y = f(x)$, then the area A_x will be equal to πy^2; hence, the volume of the solid will be

$$V = \pi\int_0^H y^2\,dx \quad . \quad . \quad . \quad . \quad [110]$$

If [110] can be integrated, then the value of V can be determined accurately, otherwise the values of $\pi y^2 = \pi[f(x)]^2$ must be calculated for $x = 0$, $x = h$, $x = 2h$, etc., and one of the approximate methods must be applied. Formulæ [98] and [99], pp. 126, 127, will respectively be

$$V = \pi h\left(\frac{y_0^2 + y_n^2}{2} + \sum_{r=1}^{r=n-1} y_r^2\right) \quad . \quad . \quad . \quad [111]$$

and
$$V = \frac{\pi h}{3}\left(y_0^2 + y_{2n}^2 + 4\sum_{r=1}^{r=n} y_{2r-1}^2 + 2\sum_{r=1}^{r=n-1} y_{2r}^2\right). \quad . \quad [112]$$

123. The sphere.

The sphere is produced by the revolution of a circle about its diameter; take the latter as x-axis and the centre of the circle as origin, and let the equation of the circle be $x^2 + y^2 = R^2$.

(i.) The volume of the *sphere* is

$$V = \pi\int_{-R}^{+R} y^2\,dx = 2\pi\int_0^{+R} y^2\,dx = 2\pi\int_0^{+R}(R^2 - x^2)dx = 2\pi R^2\int_0^{+R} dx - 2\pi\int_0^{+R} x^2\,dx$$

$$= \left(2\pi R^2 x\right)_{x=0}^{x=R} - \left(\frac{2\pi x^3}{3}\right)_{x=0}^{x=R} = 2\pi R^2(R - 0) - \frac{2\pi}{3}(R^3 - 0) = \frac{4}{3}\pi R^3.$$

(ii.) The portion of the sphere which is cut off by a plane is called a spherical segment. Let us choose the diameter which is perpendicular to the plane as x-axis, and let the height of the segment be H, then the volume of the *spherical segment* is

$$V=\pi\int_{R-H}^{R}y^2\,dx=\pi\int_{R-H}^{R}(R^2-x^2)dx=\frac{\pi}{3}H^2(3R-H).$$

(iii.) The portion of a sphere which is cut off by two parallel planes is called a spherical zone (fig. 46). Let H be the distance

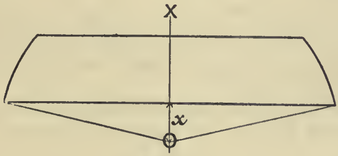


FIG. 46.

between the two parallel planes, and let the radii of the two circles in which the planes intersect the sphere be R_1 and R_2. The volume of the *spherical zone* is

$$V=\pi\int_{x}^{x+H}y^2\,dx=\pi\int_{x}^{x+H}(R^2-x^2)dx=\pi\int_{x}^{x+H}R^2\,dx-\pi\int_{x}^{x+H}x^2\,dx=\Big(\pi R^2x\Big)_{x=x}^{x=x+}$$

$$-\left(\frac{\pi x^3}{3}\right)_{x=x}^{x=x+H}=\left[\pi R^2(x+H)-\pi R^2x\right]-\left[\frac{\pi}{3}(x+H)^3-\frac{\pi}{3}x^3\right].$$

But $R_1{}^2=R^2-x^2$, and $R_2{}^2=R^2-(x+H)^2$;

hence, $$V=\frac{\pi}{2}(R_1{}^2+R_2{}^2)H+\frac{\pi}{6}H^3.$$

124. The paraboloid of revolution.

Let the parabola, $y^2=4ax$, revolve about its axis; the solid thus formed is a paraboloid. The volume of the paraboloid is

$$V=\pi\int_0^H y^2\,dx=\pi\int_0^H 4ax\,dx=4\pi a\left(\frac{H^2}{2}-0\right)=2\pi aH^2,$$

where H is the height of the paraboloid. Let b be the radius of the base of the solid, then $b^2=4aH$; hence, $V=\frac{\pi}{2}b^2H$, *or equal to half the volume of the cylinder whose base and height are the same as those of the paraboloid.*

125. The ellipsoid and the spheroid.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

(i.) The solid formed by the revolution of an ellipse about the major axis is called an *ellipsoid* of revolution. The volume of the latter solid is

$$V = \pi\int_{-a}^{+a} y^2\,dx = 2\pi\int_0^a y^2\,dx = 2\pi\frac{b^2}{a^2}\int_0^a (a^2 - x^2)dx$$

$$= 2\pi b^2\int_0^a dx - 2\pi\frac{b^2}{a^2}\int_0^a x^2\,dx = 2\pi ab^2 - \frac{2}{3}\pi ab^2 = \frac{4}{3}\pi ab^2;$$

when $a = b$ the ellipse becomes a circle and the solid becomes a sphere whose volume is $\frac{4}{3}\pi a^3$, which we found in article **123.**

(ii.) The solid formed by the revolution of an ellipse about its minor axis is called a *spheroid*, whose volume is

$$V = \pi\int_{-b}^{+b} x^2\,dy = 2\pi\int_0^b x^2\,dy = 2\pi\frac{a^2}{b^2}\int_0^b (b^2 - y^2)dy$$

$$= 2\pi a^2\int_0^b dy - 2\pi\frac{a^2}{b^2}\int_0^b y^2\,dy = 2\pi a^2 b - \frac{2}{3}\pi a^2 b = \frac{4}{3}\pi a^2 b.$$

126. The hyperboloid of revolution.

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(i.) *The hyperboloid of two sheets.*—The volume of the solid formed by the revolution of the curve about the first axis is

$$V = \pi\int_a^{+x} y^2\,dx + \pi\int_{-x}^{-a} y^2\,dx = \pi\int_{-x}^{+x} y^2\,dx = 2\pi\int_a^x y^2\,dx = 2\pi\frac{b^2}{a^2}\int_a^x (x^2 - a^2)dx$$

$$= 2\pi\frac{b^2}{a^2}\left(\frac{x^3}{3} - a^2x\right)_{x=a}^{x=x} = \frac{2}{3}\pi\frac{b^2}{a^2}\left(x^3 - 3a^2x + 2a^3\right)$$

$$= \frac{2}{3}\pi\frac{b^2}{a^2}(x - a)^2(x + 2a),$$

where x is always positive.

(ii.) *The hyperboloid of one sheet, or the skew hyperboloid.*—The volume of the solid formed by the revolution of the curve about the second axis is

$$V = \pi\int_{-y}^{+y} x^2\,dy = 2\pi\int_0^y x^2\,dy = 2\pi\frac{a^2}{b^2}\int_0^y (y^2 + b^2)dy = 2\pi\frac{a^2}{b^2}\left(\frac{y^3}{3} + b^2y\right)$$

$$= \frac{2}{3}\pi\frac{a^2}{b^2}(y^2 + 3b^2)y,$$

where y is always positive.

127. The equation of the revolving curve is unknown.

Let us find the volume of the solid which is formed by the revolution of the curve of which data are given in the following table :—

y.		y^2.	y.		y^2.	y.		y^2.
y_0	2·83	8·01	y_4	4·00	16·00	y_8	4·90	24·01
y_1	3·16	9·99	y_5	4·24	17·98	y_9	5·10	26·01
y_2	3·46	11·97	y_6	4·47	19·98	y_{10}	5·29	27·98
y_3	3·74	13·99	y_7	4·69	22·00			

The distance h is equal to 1; hence, we have by [**112**]

$$V = \frac{\pi}{3}(35{\cdot}99 + 359{\cdot}88 + 143{\cdot}92) = 565{\cdot}27 \text{ units.}$$

The curve is really the portion of the parabola, $y^2 = 2x$, which lies between $x = 14$ and $x = 4$; hence,

$$V = \pi\int_4^{14} y^2\,dx = 2\pi\int_4^{14} x\,dx = \pi(14^2 - 4^2) = 565{\cdot}44 \text{ units.}$$

128. Guldin's theorem.

Let B C D F B (fig. 47) be any closed plane curve which does

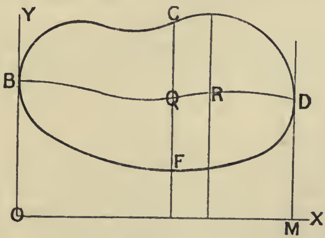


FIG. 47.

not cut the axis of x; let the axis of y be chosen so that it is tangential to the curve, and draw also tangent M D parallel to the latter axis. Further, let curve B Q R D be the locus of the middle points of chords parallel to the y-axis. The ordinates to points C and F are respectively y_2 and y_1; hence, the ordinate to point Q is $\eta=\frac{y_2+y_1}{2}$. If A be the area enclosed by B C D F B, and L be the distance of its centroid from the axis of x, then the first area-moment w.r.t. the axis of x is

$$(a.\,m.)=\mathrm{A\,L} \quad . \quad . \quad . \quad . \quad (1)$$

We have also

$$\mathrm{A}=\int_0^b y_2\,dx-\int_0^b y_1\,dx=\int_0^b(y_2-y_1)dx, \quad \text{where} \quad b=\mathrm{O\,M} \quad (2)$$

The centroid of δA must lie somewhere between Q and R on the curve B Q R D when Q R is indefinitely small; hence, we must have

$$\delta\mathrm{A}\,(\eta+\delta\eta)\gtrless\delta(a.\,m.)\gtrless\delta\mathrm{A}\,\eta;$$

hence, in the limit

$$\frac{d(a.\,m.)}{d\mathrm{A}}=\eta=\frac{y_1+y_2}{2}=\frac{d(a.\,m.)}{dx}\;\frac{dx}{d\mathrm{A}};$$

hence, we have

$$\mathrm{A\,L}=(a.\,m.)=\int^b\frac{d(a.\,m.)}{d\mathrm{A}}\;\frac{d\mathrm{A}}{dx}dx=\tfrac{1}{2}\int_0^b(y_1+y_2)\,(y_2-y_1)dx$$

$$=\tfrac{1}{2}\int_0^b(y_2{}^2-y_1{}^2)dx, \quad \text{or} \quad 2\pi\mathrm{L\,A}=\pi\int_0^b(y_2{}^2-y_1{}^2)dx,$$

but the latter integral is the volume of the solid which is formed by the revolution of curve B C D F B about the x-axis. Hence, *when any closed curve revolves about an axis in its own plane, and which does not cut the curve, the volume of the ring so formed is equal to that of a cylinder whose base is the curve and whose height is equal to the length of the path described by the centroid of the area enclosed by the curve*

$$\mathrm{V}=2\pi\mathrm{L\,A} \quad . \quad . \quad . \quad . \quad [\mathbf{113}]$$

EXAMPLE.—A circle is revolving about an axis in its plane. The distance, L, of the centre from the axis of rotation is greater than radius, r, of the circle. Find the volume of the ring formed by the revolution of the circle. By [**113**] we have

$$\mathrm{V}=2\pi\mathrm{L}\pi r^2=2\pi^2\mathrm{L}r^2 \quad . \quad . \quad . \quad (3)$$

This article may be left out until the student has studied Chap. XXXV., p. 275.

CHAPTER XVII.

SURFACES—RECTIFICATION.

129. Surface of revolution.

Let S be the area of the curved surface formed by the revolution of a curve M B C N (fig. 48) about O X; and let δs be the length of the curve-element B C. δS will thus be the area of the surface-element of the solid between the sections through B and C.

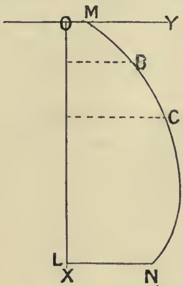


FIG. 48.

It is obvious that the value of δS must lie between those of the areas of the cylindrical surfaces $2\pi y\,\delta s$ and $2\pi(y+\delta y)\delta s$. Hence, we have in the limit

$$\frac{dS}{ds}=2\pi y, \quad \text{and therefore} \quad S=2\pi\int_{x=0}^{x=H} y\,ds \qquad . \quad [114]$$

where $H = O\,L$,

but
$$\frac{dS}{ds}=\frac{dS}{dx}\,\frac{dx}{ds}=\frac{dS}{dy}\,\frac{dy}{ds},$$

therefore
$$\frac{dS}{dx}=2\pi y\frac{ds}{dx} \quad \text{and} \quad \frac{dS}{dy}=2\pi y\frac{ds}{dy};$$

hence,
$$S=2\pi\int_0^H y\frac{ds}{dx}dx=2\pi\int_r^R y\frac{ds}{dy}dy\,. \qquad . \qquad . \quad [115]$$

where $r = \mathrm{OM}$ and $\mathrm{R} = \mathrm{LN}$. If now either of the three functions $y = \phi(s)$, or $y_1 = y\frac{ds}{dx}$, or $y_2 = y\frac{ds}{dy}$ is known, and the integration in [**114**] or [**115**] can be performed, then the value of S can be found accurately. It is evident that S is equal to 2π times the area under either of the graphs whose equations are $y = \phi(s)$, or $y_1 = y\frac{ds}{dx}$, or $y_2 = y\frac{ds}{dy}$. Hence, if we find it necessary to resort to approximate methods, and we choose the graph of $y = \phi(s)$, then we must rectify the curve MBCN as accurately as possible; which may be done by dividing the curve into a number of small parts, each so small that it may be considered approximately a straight line, and then add all the parts together by setting them off along a straight line.

(i.) When applying [**98**], p. 126, the rectified curve MBCN must be divided into n equal parts each of length h, and the values of y corresponding to $s = 0$, $s = h$, $s = 2h$, etc., must be determined from the curve MBCN. We have then approximately

$$\mathrm{S} = 2\pi h\left(\frac{y_0 + y_n}{2} + \sum_{r=1}^{r=n-1} y_r\right) \quad . \quad . \quad . \quad [\mathbf{116}]$$

(ii.) If we desire to apply Simpson's rule, we must divide the rectified curve into $2n$ equal parts each of length h, and determine the values of y corresponding to $s = 0$, $s = h$, etc.

The curved surface of the solid will approximately be

$$\mathrm{S} = \frac{2\pi h}{3}\left(y_0 + y_{2n} + 4\sum_{r=1}^{r=n} y_{2r-1} + 2\sum_{r=1}^{r=n-1} y_{2r}\right). \quad . \quad [\mathbf{117}]$$

130. The cone of revolution.

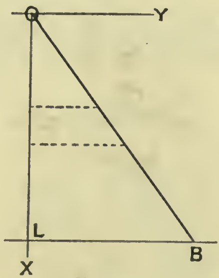


FIG. 49.

Let l be the length of the generator OB (fig. 49), then $\frac{ds}{dx} = \frac{l}{\mathrm{H}}$

and $\frac{y}{x} = \frac{r}{H}$, where $OL = H$ and $LB = r$. Hence, by [**115**], we have

$$S = 2\pi\frac{l}{H}\,\frac{r}{H}\int_0^H x\,dx = \pi rl.$$

i.e. S is equal to half the curved surface of a cylinder with the same base and the same generator.

131. Truncated cone of revolution.

In fig. 50 OX is the axis of the cone, $NM = r_1$, $PB = r_2$,

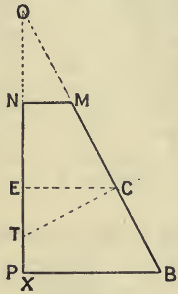


FIG. 50.

$MB = l$, and $NP = H$. It will be found that $OB = \frac{r_2 l}{r_2 - r_1}$; hence, by the preceding article,

$$S = \pi r_2 \frac{r_2 l}{r_2 - r_1} - \pi r_1 \frac{r_1 l}{r_2 - r_1} = \pi(r_1 + r_2)l \quad . \qquad . \quad (1)$$

Let EC be parallel to, and equidistant from, NM and PB, and draw line CT perpendicular to MB; then, if $CT = R$, we have $l : H :: R : \frac{r_1 + r_2}{2}$, or $\pi(r_1 + r_2)l = 2\pi RH$; hence, (1) becomes

$$S = \pi(r_1 + r_2)l = 2\pi RH.$$

132. The sphere.

By articles **129** and **108** we have

$$\frac{dS}{dx} = 2\pi y\frac{ds}{dx} = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

let the equation of the great circle be $x^2 + y^2 = R^2$;

then $$\frac{dy}{dx} = -\frac{y}{x}, \quad \text{or} \quad \frac{dS}{dx} = 2\pi R;$$

hence, $$S = 2\pi R\int_{-R}^{+R} dx = 4\pi R^2,$$

or *the area of the surface of a sphere is equal to four times the area of the great circle.*

133. The spherical segment and the spherical zone.

By the preceding article we have that $\frac{dS}{dx} = 2\pi R$; hence, the area of the curved surface of

(i.) the spherical segment, whose height is H, is

$$S = 2\pi R \int_{R-H}^{R} dx = 2\pi R\,H.$$

(ii.) the spherical zone (fig. 46, p. 135) is

$$S = 2\pi R \int_{x}^{x+H} dx = 2\pi R\,H.$$

Hence, when the heights of a spherical segment and of a spherical zone, cut off from the same sphere, are equal, then the areas of their curved surfaces are also equal.

134. Guldin's theorem.

Let the first moment of the perimeter (fig. 51) w.r.t. O X be

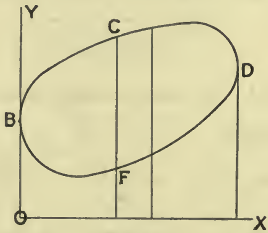


FIG. 51.

$(s.\,m.) = sl$, where l is the distance of the centroid of the perimeter from O X and s is the length of the perimeter B C D F B. Let the length of the portion B C D of the perimeter be s_2, and its current ordinate be y_2; and, similarly, let the length of B F D be s_1, and its current ordinate be y_1, then $s = s_1 + s_2$. It is obvious that $\delta(s.\,m.)_2$ lies between $(y_2 + \delta y_2)\delta s_2$ and $y_2\,\delta s_2$; hence, in the limit

$$\frac{d(s.\,m.)_2}{ds_2} = y_2, \quad \text{and similarly} \quad \frac{d(s.\,m.)_1}{ds_1} = y_1;$$

hence, $$(s.\,m.)_2 = \int_{x=0}^{x=b} y_2\,ds_2, \quad \text{and} \quad (s.\,m.)_1 = \int_{x=0}^{x=b} y_1\,ds_1.$$

But we have that

$$(s.\,m.)_1 + (s.\,m.)_2 = (s.m.) = sl = \int_{x=0}^{x=b} y_1\,ds_1 + \int_{x=0}^{x=b} y_2\,ds_2 \quad . \qquad (1)$$

The area of the surface of the ring which is formed by the revolution of B C D F B about O X is, according to [**114**],

$$S = 2\pi\int_{x=0}^{x=b} y_1\,ds_1 + 2\pi\int_{x=0}^{x=b} y_2\,ds_2\,;$$

hence, by (1) we have

$$S = 2\pi ls \qquad . \quad . \quad . \quad . \quad [118]$$

Hence, *when any closed curve revolves about an axis in its own plane, and which does not cut the curve, the area of the surface thus formed is equal to that of the cylinder whose base is the curve and whose height is equal to the length of the path described by the centroid of the perimeter of the curve.*

This article may be left out until the student has studied Chap. XXXV., p. 275.

135. Rectification.

The operation of finding the length of a curved line is called rectification.

When the curve is drawn on the paper its length can always be found approximately by the method described in article **129**. In some cases, however, a great amount of work may be saved by the application of calculus. By article **108** we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,;\ \text{hence, } s = \int_b^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx\,. \quad [119]$$

s in [**119**] is the length of the portion of the curve between the values a and b of x.

CHAPTER XVIII.

INTEGRATION RULES

136. Nature of the problem.

The results given in the table, article **112**, follow obviously from the examples tabulated in article **91**, and require, therefore, no further comment. It has been shown in Chap. XV. that, if the integral is definite and the constants are numerical, the integration may be performed with great accuracy by means of approximate rules, but the saving of labour, however, is very marked when true integration can be performed.

Generally the functions to be integrated are much more complicated than the simple functions given in article **112**. The process consists, then, in trying to simplify the given function, and reduce it to one or more of the forms which we know. In the present and in the next chapters we will endeavour to establish some general rules by which the integration of a given function may be performed.

137. Integration by parts.

Let $u=f(x)$ and $v=\phi(x)$ be two functions of x; then, by article **79**, we have

$$\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx},$$

and, conversely,

$$uv=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx;$$

hence,

$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx \quad . \quad . \quad . \quad [120]$$

which is known as the principle of *integration by parts*. The process consists in resolving the function to be integrated into two factors, of which one (dv/dx) must be integrable.

EXAMPLE 1.

$$\int \log_e x\,dx = x\log_e x - \int x\frac{1}{x}dx = x\log_e \frac{x}{e} + C$$

where $u = \log_e x$ and $dv/dx = 1$; hence, $du/dx = 1/x$ and $v = \int dx = x$.

(i.) *The integral may be reduced to a known form, or to a simpler one of the same form.*

In Example 1 the integral is reduced to a known form.

EXAMPLE 2. To find $\int a^x x^m dx$.

Take $$u = x^m,\ v = \int a^x dx = \frac{a^x}{\log_e a};$$

then $$du/dx = mx^{m-1},\ \text{and}\ dv/dx = a^x,$$

we get $$\int a^x x^m dx = \frac{a^x x^m}{\log_e a} - \frac{m}{\log_e a}\int a^x x^{m-1}\,dx \qquad . \qquad . \qquad (1)$$

The last integral in (1) is of the same form as the given integral, but is reduced to a simpler one. If m is positive the continued integration will lead to $\int a^x\,dx = a^x/\log_e a$. Thus, by continued integration, we get

$$\int e^x x^2\,dx = e^x x^2 - 2\int e^x\,x\,dx = e^x(x^2 - 2x + 2) + C.$$

EXAMPLE 3. To find $\int \frac{dz}{(1+z^2)^r}$.

By taking $u = \frac{1}{(1+z^2)^r}$, $v = z$, and therefore $\frac{du}{dz} = -\frac{2rz}{(1+z^2)^{r+1}}$,

we get $$\int \frac{dz}{(1+z^2)^r} = \frac{z}{(1+z^2)^r} + 2r\int \frac{z^2}{(1+z^2)^{r+1}}dz$$

$$= \frac{z}{(1+z^2)^r} + 2r\int \frac{dz}{(1+z^2)^r} - 2r\int \frac{dz}{(1+z^2)^{r+1}};$$

hence, $$\int \frac{dz}{(1+z^2)^{r+1}} = \frac{z}{2r(1+z^2)^r} + \frac{2r-1}{2r}\int \frac{dz}{(1+z^2)^r};$$

by substituting r for $(r+1)$ and $(r-1)$ for r, we obtain the following *reduction formula*:—

$$\int \frac{dz}{(1+z^2)^r} = \frac{z}{2(r-1)(1+z^2)^{r-1}} + \frac{2r-3}{2(r-1)}\int \frac{dz}{(1+z^2)^{r-1}} \quad . \qquad (2)$$

By applying this method of reduction $(r-1)$ times we get, finally,

$$\int\frac{dz}{1+z^2}=\text{ang}\,(\tan=z)+C.$$

EXAMPLE 4.—By exactly the same method as those used in the preceding example, we obtain the *reduction formula.*

$$\int\frac{dz}{(z^2-1)^r}=-\frac{z}{2(r-1)(z^2-1)^{r-1}}-\frac{2r-3}{2(r-1)}\int\frac{dz}{(z^2-1)^{r-1}} \qquad (3)$$

By applying (3) $(r-1)$ times we get

$$\int\frac{dz}{z^2-1}=-\frac{1}{2}\int\frac{dz}{z+1}+\frac{1}{2}\int\frac{dz}{z-1}=\frac{1}{2}\log_e\frac{z-1}{z+1}+C.$$

(ii.) *The given integral may be expressed by itself through one or more integrations.*

EXAMPLE 5.

$$\int e^x\cos x\,dx=e^x\sin x-\int e^x\sin x\,dx \quad . \quad . \quad (4)$$

by taking $u=e^x$ and $v=\sin x$.

Similarly we have

$$\int e^x\sin x\,dx=-e^x\cos x+\int e^x\cos x\,dx \quad . \quad . \quad (5)$$

by taking $u=e^x$ and $v=-\cos x$.

Hence, we get by (4) and (5)

$$\int e^x\sin x\,dx=\frac{e^x(\sin x-\cos x)}{2}+C, \text{ and } \int e^x\cos x\,dx$$

$$=\frac{e^x(\sin x+\cos x)}{2}+C \quad . \quad . \quad . \quad (6)$$

138. Methods of substitution.

Let $y=f(z)$ where z is a function of x, then, by article **75**, we have

$$\frac{df(z)}{dx}=\frac{df(z)}{dz}\cdot\frac{dz}{dx}=f'(z)\frac{dz}{dx};$$

conversely $$\int f'(z)\frac{dz}{dx}dx=f(z),$$

but we have also $f(z)=\int f'(z)dz$; hence,

$$\int f'(z)\frac{dz}{dx}dx=\int f'(z)dz=f(z) \quad . \quad . \quad . \quad [121]$$

When applying a substitution, the limits of a definite integral must be changed to those corresponding to the new variable. If $x=a$ corresponds to $z=b$, then

$$\int_a^x f'(z)\frac{dz}{dx}dx = \int_b^z f'(z)dz = f(z) - f(b) \quad . \quad . \quad [122]$$

EXAMPLE 1.

$$\int\frac{dx}{\sqrt{x+a}} = \int\frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{x+a} + C$$

where $z=x+a$ and $dz/dx=1$.

EXAMPLE 2.

$$\int\frac{dx}{(x+a)^2} = \int\frac{dz}{z^2} = -\frac{1}{z} = -\frac{1}{x+a} + C$$

where $z=x+a$ and $dz/dx=1$.

EXAMPLE 3.

$$\int\frac{2ax+b}{ax^2+bx+c}dx = \int\frac{dz}{z} = \log_e z = \log_e\left[C(ax^2+bx+c)\right]$$

where $z=ax^2+bx+c$ and $dz/dx=2ax+b$.

EXAMPLE 4.

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int\frac{dz}{\sqrt{1-z^2}} = \text{ang}\left(\sin = \frac{x}{a}\right) + C.$$

EXAMPLE 5.

$$\int\frac{x\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}\int\frac{-2x\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}\int\frac{d(a^2-x^2)}{\sqrt{a^2-x^2}}$$

$$= -\frac{1}{2}\int\frac{dz}{\sqrt{z}} = -\sqrt{a^2-x^2} + C.$$

EXAMPLE 6.

$$\int\sin(ax)\,dx = \frac{1}{a}\int\sin(ax)\,d(ax)$$

$$= \frac{1}{a}\int\sin z\,dz = -\frac{\cos z}{a} = -\frac{\cos(ax)}{a} + C.$$

139. Integration of trigonometrical functions.

EXAMPLE 1.

$$\int\tan x\,dx = \int\frac{\sin x}{\cos x}dx = -\int\frac{d(\cos x)}{\cos x} = -\log_e\cos x + C = \log_e\frac{C}{\cos x}.$$

EXAMPLE 2.

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \log_e(C \sin x).$$

EXAMPLE 3.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\cos \frac{x}{2} \, d\left(\frac{x}{2}\right)}{\sin \frac{x}{2} \cos^2 \frac{x}{2}}$$

$$= \int \frac{d\left(\tan \frac{x}{2}\right)}{\tan \frac{x}{2}} = \log_e\left(C \tan \frac{x}{2}\right).$$

EXAMPLE 4.

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2} + x\right)} = \log_e\left[C \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right],$$

from Example 3.

EXAMPLE 5.

$$\int \frac{dx}{\tan x} = \int \cot x \, dx, \quad \text{and} \quad \int \frac{dx}{\cot x} = \int \tan x \, dx.$$

See Examples 1 and 2.

EXAMPLE 6.

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x}$$

$$= \tan x - \cot x = -2 \cot 2x + C.$$

EXAMPLE 7.

$$\int \sin^2(nx) \, dx = \frac{1}{n} \int \sin^2 z \, dz = \frac{1}{n} \int \frac{1 - \cos 2z}{2} dz$$

$$= \frac{1}{2n} \int dz - \frac{1}{2n} \int \cos(2z) \, dz = \frac{z}{2n} - \frac{\sin 2z}{4n} = \frac{nx - \sin(nx)\cos(nx)}{2n} + C.$$

EXAMPLE 8.

$$\int \cos^2(nx) \, dx = \frac{1}{n} \int \cos^2 z \, dz = \frac{1}{2n} \int (1 + \cos 2z) dz$$

$$= \frac{nx + \sin(nx)\cos(nx)}{2n} + C.$$

EXAMPLE 9.

$$\int \frac{dx}{(a \sin x + b \cos x)^2}.$$

Let $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$, and $\tan\theta = \frac{b}{a}$; hence, $\theta = \text{ang}\left(\tan = \frac{b}{a}\right)$.

By substituting $\sqrt{a^2+b^2}\cos\theta$ and $\sqrt{a^2+b^2}\sin\theta$ for a and b, we get

$$\int \frac{dx}{(a\sin x + b\cos x)^2} = \frac{1}{a^2+b^2}\int \frac{dx}{\sin^2(x+\theta)}$$

$$= \frac{1}{a^2+b^2}\int \frac{d(x+\theta)}{\sin^2(x+\theta)} = -\frac{1}{a^2+b^2}\cot(x+\theta) + \text{C}.$$

EXAMPLE 10.

$$\int \frac{\sin x}{a+b\cos x}dx = -\int \frac{d(\cos x)}{a+b\cos x}$$

$$= -\frac{1}{b}\int \frac{d(a+b\cos x)}{a+b\cos x} = -\frac{1}{b}\log_e[c(a+b\cos x)].$$

The integration of algebraical functions may often be facilitated by substituting trigonometrical functions for x.

EXAMPLE 11.

$$\int \sqrt{a^2-x^2}\,dx.$$

Take $x = a\sin\theta$, then $dx/d\theta = a\cos\theta$; hence, $\sqrt{a^2-x^2}\,dx = a^2\cos^2\theta\,d\theta$, and

$$\int \sqrt{a^2-x^2}\,dx = a^2\int \cos^2\theta\,d\theta = \frac{(\theta+\sin\theta\cos\theta)a^2}{2}$$

$$= \frac{a^2}{2}\text{ang}\left(\sin = \frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + \text{C}.$$

CHAPTER XIX.

ALGEBRAICAL FRACTIONAL FORMS.

140. Rational forms.

A fraction, $\frac{\phi(x)}{f(x)}$, whose numerator and denominator both are algebraic rational functions, can be integrated by resolving it into partial fractions :—

EXAMPLE 1. $\frac{1}{(x-1)^3x}$ can be resolved into the following partial fractions,

$$\frac{1}{(x-1)^3x} = \frac{k_1}{(x-1)} + \frac{k_2}{(x-1)^2} + \frac{k_3}{(x-1)^3} + \frac{k_4}{x}. \qquad (1)$$

To find the constants k_1, k_2 . . ., clear (1) of fractions; we get

$$1 = k_1x(x-1)^2 + k_2x(x-1) + k_3x + k_4(x-1)^3 \qquad (2)$$

(2) is an identity and must be satisfied by any value of x; thus, $x=1$ in (2) gives $k_3 = 1$; then, dividing (2) by $(x-1)$, we get

$$0 = k_1x(x-1) + k_2x + 1 + k_4(x-1)^2 \qquad (3)$$

$x=1$ in (3) gives $k_2 = -1$, and dividing (3) by $(x-1)$, we get

$$0 = k_1x - 1 + k_4(x-1) \qquad (4)$$

$x=1$ in (4) gives $k_1 = 1$ and $k_4 = -1$.

Hence,
$$\frac{1}{(x-1)^3x} = \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} - \frac{1}{x}. \qquad (5)$$

All the partial fractions in (5) are integrable. Hence,

$$\int\frac{dx}{(x-1)^3x} = \int\frac{dx}{x-1} - \int\frac{dx}{(x-1)^2} + \int\frac{dx}{(x-1)^3} - \int\frac{dx}{x} = \log_e(x-1)$$
$$+\frac{1}{x-1} - \frac{1}{2(x-1)^2} - \log_e x = \frac{2x-3}{2(x-1)^2} + \log_e\frac{c(x-1)}{x}$$

where c is the arbitrary constant.

EXAMPLE 2.—Resolve $\dfrac{x^3-2x+3}{(x^2-x+1)(x^2+2x+4)}$ into partial fractions. A further resolution into factors of the denominator of the given fraction would lead to imaginary results. Hence,

$$\frac{x^3-2x+3}{(x^2-x+1)(x^2+2x+4)}=\frac{k_1x+l_1}{x^2-x+1}+\frac{k_2x+l_2}{x^2+2x+4} \quad . \qquad (6)$$

Clearing (6) of fractions, we get

$$x^3-2x+3=(k_1+k_2)x^3+(2k_1+l_1-k_2+l_2)x^2 + (4k_1+2l_1+k_2-l_2)x+(4l_1+l_2) . \quad . \qquad (7)$$

As (7) is an identity, we require that

$$\left.\begin{array}{c} k_1+k_2=1 ;\ 2k_1+l_1-k_2+l_2=0 ; \\ 4k_1+2l_1+k_2-l_2=-2 ;\ 4l_1+l_2=3 \end{array}\right\} \quad . \quad . \qquad (8)$$

By solving the four simultaneous equations in (8) we get

$$k_1=-\frac{4}{9};\ l_1=\frac{2}{9};\ k_2=\frac{13}{9} \text{ and } l_2=\frac{19}{9}.$$

Hence, $$\frac{x^3-2x+3}{(x^2-x+1)(x^2+2x+4)}=\frac{-4x+2}{9(x^2-x+1)}+\frac{13x+19}{9(x^2+2x+4)} \qquad (9)$$

The integration of the given fraction requires the integration of the two partial fractions in (9).

$$\int\frac{-4x+2}{9(x^2-x+1)}dx=-\frac{2}{9}\int\frac{d(x^2-x+1)}{x^2-x+1}=-\frac{2}{9}\log_e(x^2-x+1. \qquad (10)$$

$$\int\frac{13x+19}{9(x^2+2x+4)}dx=\int\frac{13x+13+6}{9(x^2+2x+4)}dx=\frac{13}{18}\int\frac{d(x^2+2x+4)}{x^2+2x+4} + \frac{2}{3\sqrt{3}}\int\frac{d\left(\frac{x+1}{\sqrt{3}}\right)}{1+\left(\frac{x+1}{\sqrt{3}}\right)^2}=\frac{13}{18}\log_e(x^2+2x+4) + \frac{2}{3\sqrt{3}}\operatorname{ang}\left(\tan=\frac{x+1}{\sqrt{3}}\right) \quad . \quad . \quad . \qquad (11)$$

(9), (10), and (11) give

$$\int\frac{x^3-2x+3}{(x^2-x+1)(x^2+2x+4)}dx=-\frac{2}{9}\log_e(x^2-x+1) + \frac{13}{18}\log_e(x^2+2x+4)+\frac{2\sqrt{3}}{9}\operatorname{ang}\left(\tan=\frac{x+1}{\sqrt{3}}\right)+C.$$

RULE.—If the given rational algebraic fraction, $\frac{\phi(x)}{f(x)}$, is to be integrated, the denominator, $f(x)$, must be resolved into factors, none of which must be imaginary. The given fraction is then thrown into partial fractions. The dimensions of $\phi(x)$ must be less than those of $f(x)$; if this is not the case, then divide $\phi(x)$ by $f(x)$ until the dimensions of the remainder are less than those of $f(x)$.

As shown in Examples 1 and 2, the partial fractions will take either of the following forms, viz.:—

$$\frac{k}{x-a};\ \frac{k_r}{(x-a)^r};\ \frac{k(x-a)+l}{(x-a)^2+\beta^2};\ \frac{k_r(x-a)+l_r}{[(x-a)^2+\beta^2]^r} \quad . \quad (12)$$

It is obvious that the first two in (12) are integrable. The third one

$$\int\frac{k(x-a)+l}{(x-a)^2+\beta^2}dx = k\int\frac{x-a}{(x-a)^2+\beta^2}dx + l\int\frac{dx}{(x-a)^2+\beta^2}$$

$$= \frac{k}{2}\int\frac{d[(x-a)^2+\beta^2]}{(x-a)^2+\beta^2} + \frac{l}{\beta}\int\frac{d\left(\frac{x-a}{\beta}\right)}{1+\left(\frac{x-a}{\beta}\right)^2}$$

$$= \frac{k}{2}\log_e[(x-a)^2+\beta^2] + \frac{l}{\beta}\operatorname{ang}\left(\tan = \frac{x-a}{\beta}\right).$$

The fourth one in (12),

$$\int\frac{k_r(x-a)+l_r}{[(x-a)^2+\beta^2]^r}dx = k_r\int\frac{(x-a)}{[(x-a)^2+\beta^2]^r}dx + l_r\int\frac{dx}{[(x-a)^2+\beta^2]^r} \quad (13)$$

The first in (13),

$$k_r\int\frac{(x-a)}{[(x-a)^2+\beta^2]^r}dx = \frac{k_r}{2}\int\frac{d[(x-a)^2+\beta^2]}{[(x-a)^2+\beta^2]^r}$$

$$= -\frac{k_r}{2(r-1)}\,\frac{1}{[(x-a)^2+\beta^2]^{r-1}}.$$

The second in (13),

$$l_r\int\frac{dx}{[(x-a)^2+\beta^2]^r} = \frac{l_r}{\beta^{2r-1}}\int\frac{dz}{(1+z^2)^r},$$

where $z = \frac{x-a}{\beta}$. The solution of (13) depends therefore on $\int\frac{dz}{(1+z^2)^r}$, which can be integrated by applying the reduction method (2), article **137**. Hence, the four forms in (12) are all integrables.

141. Irrational forms.

It has been shown in the preceding article that rational algebraic fractional forms can always be integrated. If, therefore, the function X, which is to be integrated, contains irrational fractional forms, the integration may be performed if we can convert the irrational forms into rational forms. This may be done by substitution.

EXAMPLE 1. $X = \dfrac{\sqrt{2x-1}}{\sqrt[3]{2x-1}-1}$.

By taking $z^6 = 2x - 1$ the irrational forms in X will disappear.

Hence, $$z = \sqrt[6]{2x-1} \quad \text{and} \quad \frac{dx}{dz} = 3z^5.$$

We have

$$\int \frac{\sqrt{2x-1}}{\sqrt[3]{2x-1}-1}dx = \int \frac{z^3}{z^2-1}3z^5dz = 3\int \frac{z^8}{z^2-1}dz = 3\int \frac{z^8-1+1}{z^2-1}dz$$

$$= 3\int \frac{z^8-1}{z^2-1}dz + 3\int \frac{dz}{z^2-1} = 3\int (z^6+z^4+z^2+1)dz + \frac{3}{2}\int \frac{dz}{z-1} - \frac{3}{2}\int \frac{dz}{z+1}$$

$$= 3\left[\frac{z^7}{7}+\frac{z^5}{5}+\frac{z^3}{3}+z+\frac{1}{2}\log_e\left(C\frac{z-1}{z+1}\right)\right] \quad . \quad . \quad (1)$$

$\sqrt[6]{2x-1}$ must be substituted for z in (1).

EXAMPLE 2. $X = \dfrac{1}{\sqrt{1+x^2}}$.

Suppose we put $\sqrt{1+x^2} = z$, then $1 + x^2 = z^2$; this substitution would obviously lead to no result, as we have simply changed x to z.

But take $\sqrt{1+x^2} = z - x$, then

$$z = x + \sqrt{1+x^2}\,;\; x = \frac{z^2-1}{2z}\,;\; \frac{dx}{dz} = \frac{z^2+1}{2z^2}\,;\; \sqrt{1+x^2} = \frac{z^2+1}{2z}.$$

Hence, $$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{2z}{z^2+1}\,\frac{z^2+1}{2z^2}dz = \int \frac{dz}{z} = \log_e(Cz)$$

$$= \log_e[C(x+\sqrt{1+x^2})] \quad . \quad . \quad . \quad (2)$$

EXAMPLE 3. $X = \dfrac{1}{\sqrt{x^2-8x+15}}$.

This case may be treated in the same manner as the preceding one by taking $\sqrt{x^2-8x+15} = z - x$.

It will, however, be observed that $x^2 - 8x + 15 = (x-3)(x-5)$, and we may therefore use the substitution $z = \sqrt{\frac{x-3}{x-5}}$.

Hence, $x = \frac{5z^2 - 3}{z^2 - 1}$, $\frac{dx}{dz} = -\frac{4z}{(z^2-1)^2}$; $\sqrt{x^2 - 8x + 15} = \frac{2z}{z^2 - 1}$.

Hence,

$$\int \frac{dx}{\sqrt{x^2 - 8x + 15}} = 2\int \frac{dz}{1 - z^2} = \int \frac{dz}{1 - z} + \int \frac{dz}{1 + z} = \log_e\left(C\,\frac{1+z}{1-z}\right)$$

$$= \log_e[C(x - 4 + \sqrt{x^2 - 8x + 15})] \quad . \quad . \quad (3)$$

CHAPTER XX.

EXAMPLES.

(44) (Q. May 1907).—**Evaluate** $\int_0^1 \frac{x^2}{4-x^2}dx$ **and** $\int_0^a \sin^2 x\, dx$. **Find the average value of the latter integral for all values of** a **from** 0 **to** π.

Solution.

(i.) $$\frac{x^2}{4-x^2} = \frac{4+x^2-4}{4-x^2} = \frac{4}{4-x^2} - 1 = \frac{1}{2-x} + \frac{1}{2+x} - 1;$$

hence, $$\int \frac{x^2}{4-x^2}dx = -\int \frac{d(2-x)}{2-x} + \int \frac{d(2+x)}{2+x} - \int dx$$

$$= -\log_e(2-x) + \log_e(2+x) - x + C = \log_e \frac{2+x}{2-x} - x + C;$$

hence, $$\int_0^1 \frac{x^2}{4-x^2}dx = \log_e 3 - \log_e 1 - 1 = \log_e 3 - 1.$$

$$0{\cdot}098612 \ (Ans.).$$

(ii.) By article 139

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C;$$

hence, $$\int_0^a \sin^2 x\, dx = \frac{a}{2} - \frac{\sin 2a}{4}.$$

If we draw the curve $y = \frac{x}{2} - \frac{\sin 2x}{4}$, the area under the curve is

$$A = \int_0^\pi \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)dx = \frac{\pi^2}{4};$$

the required average value is the mean ordinate of the said curve, viz.: $\frac{\pi^2}{4} \div \pi = \frac{\pi}{4}$.

$$\frac{a}{2} - \frac{\sin 2a}{4}, \quad \text{and} \quad \frac{\pi}{4} \ (Ans.).$$

(45) **Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Solution.—It has been shown in article **43** that if the equation of the auxiliary circle be $x^2 + y_1^2 = a^2$, and the ordinate of the ellipse is y, then $\frac{y}{y_1} = \frac{b}{a}$. Let A be the area of the ellipse, and $A_1 = \pi a^2$ be the area of the circle, then

$$\frac{A}{A_1} = \frac{4\int_0^a y\,dx}{4\int_0^a y_1 dx} = \frac{\frac{b}{a}\int_0^a y_1 dx}{\int_0^a y_1 dx}, \quad \text{or} \quad A = \frac{b}{a}A_1 = \frac{b}{a}\pi a^2.$$

Hence, $$A = \pi ab \qquad [123]$$

(46) (Q. Oct. 1909).—**Evaluate the integrals: (i.) $\int x^2 \log_e x\,dx$; (ii.) $\int \frac{dx}{9-4x^2}$; (iii.) $\int_0^\pi \sin^2 2\theta\,d\theta$.**

A line rotates uniformly through a complete revolution about a fixed axis at one end, and its length increases uniformly from 1 inch to 2 inches during the revolution. Find the area swept out by the line.

Solution.

(i.) Take $v = \int x^2 dx = \frac{x^3}{3}$; $u = \log_e x$, and $\frac{du}{dx} = \frac{1}{x}$. Then,

$$\int x^2 \log_e x\,dx = \frac{x^3}{3}\log_e x - \frac{1}{3}\int x^2\,dx + C = \frac{x^3}{9}\left(\log_e x^3 - 1\right) + C \ (Ans.).$$

(ii.) $$\frac{1}{9-4x^2} = \frac{1}{3+2x}\,\frac{1}{3-2x} = \frac{1}{6}\left(\frac{1}{3+2x} + \frac{1}{3-2x}\right); \text{ hence,}$$

$$\int \frac{dx}{9-4x^2} = \frac{1}{6}\left[\frac{1}{2}\int\frac{d(3+2x)}{3+2x} - \frac{1}{2}\int\frac{d(3-2x)}{3-2x}\right] + C$$

$$= \log_e\left(C\,\frac{3+2x}{3-2x}\right)^{\frac{1}{12}} \ (Ans.).$$

(iii.)
$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2};$$

hence, $\int_0 \sin^2 2\theta d\theta = \frac{1}{8}\int_0^\pi (1-\cos 4\theta)d(4\theta) = \frac{\pi}{2}$ (*Ans.*).

(iv.) Let $r=a$ when $\theta=0$, and $r=a+b$ when $\theta=2\pi$, then $r=a+\frac{b}{2\pi}\theta$ when $\theta=\theta$; and the equation of the curve described by the end of the rotating line is $r=a+\frac{b}{2\pi}\theta$.

The area swept out by r during one revolution is

$$A=\frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi}\left(a^2+\frac{b^2}{4\pi^2}\theta^2+\frac{ab}{\pi}\theta\right)d\theta = \pi\left(a^2+\frac{b^2}{3}+ab\right).$$

Hence, when $a=b=1$ inch, the area will be

$$A = 2{\cdot}333\ \ldots\ \pi \text{ inches}^2 \ (\textit{Ans.}).$$

(47) (Q. Nov. 1908).—**Evaluate the integrals:** (i.) $\int e^{ax}x^2 dx$ and (ii.) $\int_0^\infty \frac{r^4 dx}{(r^2+x^2)^{\frac{5}{2}}}$.

Solution.

(i.) Take $v=\int e^{ax}dx = \frac{1}{a}\int e^{ax}d(ax) = \frac{e^{ax}}{a}$; $u=x^2$ and $\frac{du}{dx}=2x$.

Hence,
$$\int e^{ax}x^2 dx = \frac{x^2}{a}e^{ax} - \frac{2}{a}\int e^{ax}x\, dx \quad . \quad . \quad . \quad (1)$$

Take
$$v=\int e^{ax}dx = \frac{1}{a}e^{ax};\ u=x \text{ and } \frac{du}{dx}=1;$$

hence,
$$\int e^{ax}x\, dx = \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax}.$$

Hence,
$$\int e^{ax}x^2 dx = e^{ax}\frac{a^2x^2-2ax+2}{a^3} + C \ (\textit{Ans.}).$$

(ii.) $\frac{r^4 dx}{(r^2+x^2)^{\frac{5}{2}}} = \frac{d\frac{x}{r}}{\left[1+\left(\frac{x}{r}\right)^2\right]^{\frac{5}{2}}}$; take $\frac{x}{r}=\tan\theta$, then $\frac{d\tan\theta}{d\theta}=\frac{1}{\cos^2\theta}$

and

$\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)^5 = \cos^5\theta$. When $x=\infty$, $\sin\theta=1$; $x=0$, $\sin\theta=0$.

Hence, $\int_0^\infty \frac{r^4\,dx}{(r^2+x^2)^{\frac{5}{2}}} = \int_0^{\frac{\pi}{2}} \cos^3\theta\,d\theta = \int_0^1 (1-\sin^2\theta)d\sin\theta = \frac{2}{3}$ (*Ans.*).

(48) (Q. June 1909, 2nd part).—**Integrate the expressions** $\int ae^{ax}dx$ **and** $\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$.

Solution.

(i.) $$\int ae^{ax}\,dx = \int e^{ax}d(ax) + \mathrm{C} = e^{ax} + \mathrm{C} \ (Ans.).$$

(ii.) $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \mathrm{C}.$$

Hence, $$\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1 \ (Ans.).$$

(49) (Q. Nov. 1906).—**Two rods A B, B C, of equal length** a, **are hinged together at B; the rods being initially in one straight line, the rod A B rotates in one plane about the point A which is fixed, and the rod B C rotates in the same plane with angular velocity twice that of A B and in the same direction. Sketch the curve traced out by the point C, and calculate the whole area passed over by the rods, counting each part of it only once.**

Solution.

Let angle C_1AB (fig. 52) be θ and the length of AC_1 be r.

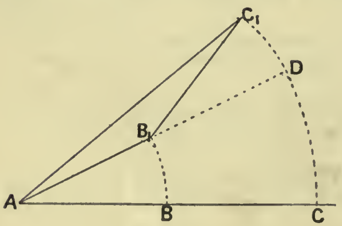


Fig. 52.

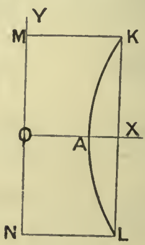


Fig. 53.

Angle C_1B_1D = angle $B_1AB = \alpha$; hence, $\theta = \alpha + \frac{\alpha}{2} = \frac{3\alpha}{2}$, and $r = 2a\cos\frac{\alpha}{2}$; hence, the equation of the curve traced by point C is

$$r = 2a\cos\frac{\theta}{3} \quad . \quad . \quad . \quad . \quad (1)$$

When A B has made half a revolution about A, B C will have made half a revolution about B and $\theta = \frac{3\pi}{2}$. The area passed over by r will be

$$Area = \tfrac{1}{2}\int_0^{\frac{3\pi}{2}} r^2 d\theta = 2a^2\int_0^{\theta=\frac{3\pi}{2}} \cos^2\frac{\theta}{3} d\theta = 6a^2\int_0^{\frac{\pi}{2}} \cos^2 z\, dz = \frac{3\pi}{2} a^2.$$

(50) (Q. Nov. 1908).—**Fig. 53 shows a portion of the rectangular hyperbola $x^2 - y^2 = a^2$ cut off by a chord K L, parallel to O Y and at a distance $2a$ from O Y. Show that the volume of the solid cap formed by the revolution of K A L about O X is equal to the volume of a sphere of radius a.**

Show also that if the figure be rotated about the axis O Y, the solid generated by the revolution of the figure M K A L N has half the volume of the cylinder generated by the revolution of the rectangle M K L N.

Solution.—Let the volume of the solid cap formed by the revolution about O X be V_1, then

$$\delta V_1 = \pi y^2\, \delta x = (\pi x^2 - \pi a^2)\delta x;$$

hence, $$V_1 = \int_a^{2a}(\pi x^2 - \pi a^2)dx = \pi\int_a^{2a} x^2\, dx - \pi a^2\int_a^{2a} dx = \frac{4}{3}\pi a^3,$$

which is the volume of a sphere of radius a.

Let V_2 be the volume of the solid generated by the revolution about O Y, then

$$\delta V_2 = \pi x^2\, \delta y = \pi(y^2 + a^2)\delta y;$$

hence,

$$V_2 = 2\pi\int_0^{a\sqrt{3}}(y^2 + a^2)dy = 2\pi\int_0^{a\sqrt{3}} y^2 dy + 2\pi a^2\int_0^{a\sqrt{3}} dy = 4\pi a^3\sqrt{3}.$$

Volume of cylinder generated by the revolution of M K L N about O Y is

$$V_3 = 4\pi a^2 \times 2a\sqrt{3} = 8\pi a^3\sqrt{3}. \quad \text{Hence,} \quad V_3 = 2V_2.$$

(51) (Q. June 1908).—**Sketch the curve $a^4y^2 = (a^2 - x^2)^3$, and find the volume of the solid which is formed by the revolution of this curve about the axis of x.**

Solution.—The curve is symmetrical w.r.t. the two co-ordinate axes; hence, we need only work out the data of the portion which lies in the 1st quadrant. We have

$$y = \frac{(a^2 - x^2)^{\frac{3}{2}}}{a^2}; \; \frac{dy}{dx} = -\frac{3x\sqrt{a^2 - x^2}}{a^2}; \; \frac{d^2y}{dx^2} = -\frac{3(a^2 - 2x^2)}{a^2\sqrt{a^2 - x^2}}.$$

Hence, the following singular points :—

x.	y.	$\frac{dy}{dx}$.	$\frac{d^2y}{dx^2}$.	Remarks.
0	a	0	neg.	Maximum of y.
$x<\frac{a}{\sqrt{2}}$	pos.	neg.	neg.	
$\frac{a}{\sqrt{2}}$	$\frac{a}{\sqrt{8}}$	neg.	0	Point of inflexion.
$x>\frac{a}{\sqrt{2}}$	pos.	neg.	pos.	
a	0	0	$+\infty$	Maximum of x.

The above portion of the curve is shown in fig. 54.

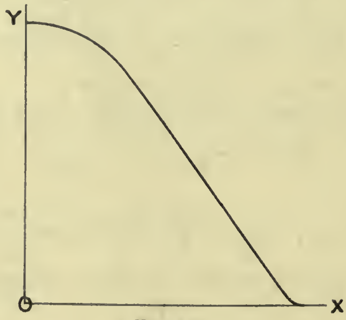


FIG. 54.

Let V be the volume of the solid formed by the revolution of the whole curve about the axis of x, then

$$\delta V = \pi y^2\,\delta x, \quad \text{and} \quad V = \frac{2\pi}{a^4}\int_0^a (a^2 - x^2)^3 dx$$

$$= \frac{2\pi}{a^4}\int_0^a (a^6 - x^6 + 3a^2x^4 - 3a^4x^2)dx.$$

$$V = \frac{32}{35}\pi a^3 \ (Ans.).$$

(52) (Q. June 1908).—**Sketch the curve $a^2y^2 = x^2(a^2 - x^2)$ and show that the area of either loop is equal to $\frac{2}{3}a^2$.**

Solution.—Both axes of co-ordinates are axes of symmetry, and we need therefore only work out the data for the portion of the

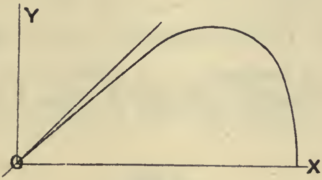


FIG. 55.

curve which is situated in the 1st quadrant, and which is shown in fig. 55.

$$y = \frac{x}{a}\sqrt{a^2 - x^2}\,; \quad \frac{dy}{dx} = \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}}\,; \quad \frac{d^2y}{dx^2} = \frac{x(2x^2 - 3a^2)}{a(a^2 - x^2)^{\frac{3}{2}}}\,.$$

Hence the following data:—

x.	y.	$\frac{dy}{dx}$.	$\frac{d^2y}{dx^2}$.	Remarks.
0	0	+1	0	Point of inflexion.
$x < \frac{a}{\sqrt{2}}$	pos.	pos.	neg.	
$\frac{a}{\sqrt{2}}$	$\frac{a}{2}$	0	,,	Maximum.
$a > x > \frac{a}{\sqrt{2}}$	pos.	neg.	,,	
a	0	∞	∞	$\frac{d^2y}{dx^2}$ changes sign.
$x > a$	imaginary			

The area A of one loop is

$$A = 2\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx = -a^2\int_{x=0}^{x=a}\left[1-\left(\frac{x}{a}\right)^2\right]^{\frac{1}{2}} d\left[1-\left(\frac{x}{a}\right)^2\right]$$
$$= -\frac{2}{3a}\left(\left[a^2 - x^2\right]^{\frac{3}{2}}\right)_{x=0}^{x=a} = \frac{2}{3}a^2.$$

(53) (Q. Nov. 1907).—**Evaluate** $\int xe^x\,dx$. **Sketch the curves** $y = e^x$ **and** $y = xe^x$ **between** $x = -1$ **and** $x = 1$, **and calculate the area enclosed between portions of these two curves and the axis of** y.

Solution.

(i.) Take $v = \int e^x\,dx = e^x$; $u = x$ and $\frac{du}{dx} = 1$. Hence,

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1) + C \quad (Ans.).$$

(ii.) Let $y_1 = e^x$; hence, $\frac{dy_1}{dx} = e^x$ and $\frac{dy_1{}^2}{dx^2} = e^x$.

Let also $y_2 = xe^x$; $\frac{dy_2}{dx} = (x+1)e^x$; $\frac{d^2y_2}{dx^2} = (x+2)e^x$.

Data for tracing the two curves are given in the following table :—

x.	y_1.	$\frac{dy_1}{dx}$.	$\frac{d^2y_1}{dx^2}$.	y_2.	$\frac{dy_2}{dx}$.	$\frac{d^2y_2}{dx^2}$.	Remarks.
-1	$\frac{1}{e}$	$\frac{1}{e}$		$-\frac{1}{e}$	0		Minimum of y_2.
neg.	pos.	pos.		neg.	pos.		
0	+1	+1	positive	0	+1	positive	
pos.	pos.	pos.		pos.	pos.		
+1	e	e		e	$2e$		

When the student has sketched the curves he will find that the required area is

$$A=\int_0^1(y_1-y_2)dx=\int_0^1(e^x-xe^x)dx=e-2=0{\cdot}71828 \text{ units } (\textit{Ans.}).$$

(54) (Q. Nov. 1906). — **Prove that the area between the curve $y=ae^{-bx}\sin x$ and the axis of x, from $x=0$ to $x=\pi$, is $\frac{a}{b^2+1}(e^{-b\pi}+1)$.**

If $a=1$ and $b=0{\cdot}1$, prove that the decrease of area, when b is increased by a comparatively small quantity λ, is approximately $2{\cdot}6\ \lambda$.

Solution.—Take

$$v=\int\sin x\,dx=-\cos x;\ u=e^{-bx},\ \text{and}\ \frac{du}{dx}=-be^{-bx};$$

hence, $$\int e^{-bx}\sin x\,dx=-e^{-bx}\cos x-b\int e^{-bx}\cos x\,dx.$$

Take $$v=\int\cos x\,dx=\sin x;\ u=e^{-bx},\ \text{and}\ \frac{du}{dx}=-be^{-bx};$$

hence, $$\int e^{-bx}\cos x\,dx=e^{-bx}\sin x+b\int e^{-bx}\sin x\,dx;$$

hence, $$\int e^{-bx}\sin x dx=-e^{-bx}\cos x-be^{-bx}\sin x-b^2\int e^{-bx}\sin x\,dx,$$

or $$\int e^{-bx}\sin x dx=-\frac{e^{-bx}(\cos x+b\sin x)}{1+b^2};$$

hence, the required area is

$$A=a\int_0^\pi e^{-bx}\sin x\,dx=\frac{a}{b^2+1}(e^{-b\pi}+1)\ (\textit{Ans.}).\qquad(1)$$

Differentiating (1) w.r.t. b we get

$$\frac{dA}{db}=-\frac{a[(\pi b^2+\pi+2b)e^{-b\pi}+2b]}{(b^2+1)^2}$$

as $\delta b=\lambda$, we have approximately

$$\delta A=-\frac{a[(\pi b^2+\pi+2b)e^{-b\pi}+2b]}{(b^2+1)^2}\lambda;$$

substituting 1 for a and $0{\cdot}1$ for b, we get

$$\delta A=2{\cdot}6\lambda\ (\textit{Ans.}).$$

(55) **Integrate** $X = \dfrac{x^7 + 3x^5 + 4x^3 + 5x^2 + 9}{x^2 + 2x + 5}$ w.r.t. x.

Solution. — The denominator in X cannot be resolved into factors. By dividing the numerator by the denominator we get

$$X = x^5 - 2x^4 + 2x^3 + 6x^2 - 18x + 11 + \frac{11(7x - 5)}{x^2 + 2x + 5} \quad . \quad (1)$$

$$\int X\,dx = \frac{x^6}{6} - \frac{2}{5}x^5 + \frac{1}{2}x^4 + 2x^3 - 9x^2 + 11x + 11\int\frac{(7x-5)dx}{x^2+2x+5} \quad (2)$$

$$\int\frac{(7x-5)}{x^2+2x+5}dx = \int\frac{7x+7-12}{(x+1)^2+4}dx = 7\int\frac{x+1}{(x+1)^2+4}dx - 12\int\frac{dx}{(x+1)^2+4}$$

$$= \frac{7}{2}\int\frac{d[(x+1)^2+4]}{(x+1)^2+4} - 6\int\frac{d\left(\frac{x+1}{2}\right)}{1+\left(\frac{x+1}{2}\right)^2}$$

$$= \frac{7}{2}\log_e[(x+1)^2+4] - 6\,\text{ang}\left(\tan = \frac{x+1}{2}\right).$$

Hence,
$$\int X\,dx = \frac{x^6}{6} - \frac{2x^5}{5} + \frac{x^4}{2} + 2x^3 - 9x^2 + 11x$$
$$+ \frac{77}{2}\log_e[(x+1)^2+4] - 66\,\text{ang}\left(\tan = \frac{x+1}{2}\right) + C.$$

(56) **To find the area of the rectangular hyperbola.**

Solution.—The equation of the rectangular hyperbola is

$$\text{(i.) } x^2 - y^2 = a^2, \quad \text{or} \quad \text{(ii.) } xy = c^2, \quad \text{or} \quad \text{(iii.) } y = \frac{ax}{1+bx},$$

accordingly to the position of the curve w.r.t. the axes of coordinates (see p. 40). Find, in all three cases, the area enclosed by the curve and the axis of x in the interval $x_2 - x_1$.

(i.) $$y^2 = x^2 - a^2, \quad \text{or} \quad y = \pm\sqrt{x^2 - a^2}.$$

Hence, the required area is $A = 2\int_{x_1}^{x_2}\sqrt{x^2 - a^2}\,dx.$

As $$\sqrt{x^2 - a^2} = \sqrt{(x+a)(x-a)},$$

we take $$z = \sqrt{\frac{x+a}{x-a}}; \; x = \frac{a(z^2+1)}{z^2-1}; \; \frac{dx}{dz} = -\frac{4az}{(z^2-1)^2}$$

and $$\sqrt{x^2 - a^2} = \frac{2az}{z^2-1}.$$

Hence, $$\int \sqrt{x^2-a^2}\,dx = -8a^2\left[\int\frac{dz}{(z^2-1)^2}+\int\frac{dz}{(z^2-1)^3}\right] \quad . \quad (1)$$

$$\int\frac{dz}{(z^2-1)^3} = -\frac{z}{4(z^2-1)^2}-\frac{3}{4}\int\frac{dz}{(z^2-1)^2} \quad . \quad . \quad (2)$$

and $$\int\frac{dz}{(z^2-1)^2} = -\frac{z}{2(z^2-1)}-\frac{1}{2}\int\frac{dz}{z^2-1} \quad . \quad . \quad (3)$$

but

$$\int\frac{dz}{z^2-1} = \int\frac{dz}{(z-1)(z+1)} = -\frac{1}{2}\int\frac{dz}{z+1}+\frac{1}{2}\int\frac{dz}{z-1} = \frac{1}{2}\log_e\frac{z-1}{z+1} \quad (4)$$

By (2), (3), and (4) we get

$$\int \sqrt{x^2-a^2}\,dx = a^2\left[\frac{z(z^2+1)}{(z^2-1)^2}-\frac{1}{2}\log_e\frac{z+1}{z-1}\right]$$

$$=\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\log_e\frac{x+\sqrt{x^2-a^2}}{a}\right) \quad . \quad . \quad (5)$$

Hence, $$A = \frac{1}{2}\left(x_2y_2-x_1y_1+a^2\log_e\frac{x_1+y_1}{x_2+y_2}\right) \quad . \quad . \quad [124]$$

(ii.) $y=\frac{c^2}{x}$; hence, the required area is

$$A = \int_{x_1}^{x_2} y\,dx = c^2\int_{x_1}^{x_2}\frac{dx}{x} = c^2\log_e\frac{x_2}{x_1} = x_2y_2\log_e\frac{x_2}{x_1} \quad . \quad [125]$$

(iii.) The required area is

$$A = \int_{x_1}^{x_2} y\,dx = a\int_{x_1}^{x_2}\frac{x}{1+bx}dx = \frac{a}{b}\int_{x_1}^{x_2}\frac{1+bx-1}{1+bx}dx$$

$$=\frac{a}{b}\int_{x_1}^{x_2}dx - \frac{a}{b^2}\int_{x_1}^{x_2}\frac{d(1+bx)}{1+bx} = \frac{a}{b}(x_2-x_1)-\frac{a}{b^2}\log_e\frac{1+bx_2}{1+bx_1}$$

$$=\frac{a}{b}(x_2-x_1)-\frac{a}{b^2}\log_e\frac{x_2y_1}{x_1y_2} \quad . \quad . \quad . \quad [126]$$

(57) To find the length of a parabolic arc.

Solution.—Let the equation of the parabola be $y^2=4ax$, then $\left(\frac{dy}{dx}\right)^2=\frac{a}{x}$ and $\frac{ds}{dx}=\sqrt{\frac{a+x}{x}}$. The length of the arc from the vertex to point (x, y) on the curve is

$$s = \int_0^x \sqrt{\frac{a+x}{x}}\,dx \quad . \quad . \quad . \quad . \quad (1)$$

The integration of (1) can be performed by substitution.

Take $z^2 = \frac{a+x}{x}$; hence, $x = \frac{a}{z^2-1}$ and $\frac{dx}{dz} = -\frac{2az}{(z^2-1)^2}$. We have

$$s = -2a\int\frac{z^2}{(z^2-1)^2}dz + C$$

$$= -2a\int\left(\frac{1}{4(z-1)} - \frac{1}{4(z+1)} + \frac{1}{4(z-1)^2} + \frac{1}{4(z+1)^2} + C\right)dz$$

$$= -2a\left(\frac{1}{4}\log_e\frac{z-1}{z+1} - \frac{z}{2(z^2-1)} + C\right).$$

Inserting x for z we get

$$s = \sqrt{(a+x)x} - a\log_e\frac{\sqrt{a+x} - \sqrt{x}}{\sqrt{a}} \quad . \quad . \quad [127]$$

$C = 0$, because $s = 0$ for $x = 0$.

When $\frac{x}{a}$ is small $\frac{dx}{dy}$ will also be small, and the length of the parabolic arc may be determined approximately in a simple manner. We have $\frac{dx}{dy} = \frac{y}{2a}$, and

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}} = 1 + \frac{y^2}{8a^2} - \frac{y^4}{128a^4} + \ . \ . \ . ;$$

hence, $$s = \int dy + \frac{1}{8a^2}\int y^2\,dy - \frac{1}{128a^4}\int y^4\,dy + \ . \ . \ . ,$$

or $$s = y + \frac{y^3}{24a^2} - \frac{y^5}{640a^4} + \ . \ . \ . \quad . \quad . \quad [128]$$

[128] may be applied to determine the lengths of the cables of a suspension bridge or of a suspended telegraph wire. There are two cases to be considered.

(i.) *The two points of suspension are in the same level.*

Let A and B (fig. 56) be the points of suspension, $l = \text{A B}$, the length of the span, and $h = \text{O D}$, the dip. To find a we have that $x = h$, $y = \frac{l}{2}$. Hence, $\frac{l^2}{4} = 4ah$, or $a = \frac{l^2}{16h}$. The length of the whole cable is approximately

$$2s = l + \frac{8}{3}\frac{h^2}{l} \quad . \quad . \quad . \quad . \quad (2)$$

(ii.) *The two points of suspension are not in the same level.*

In this case the position of the vertex is not known, but we have given the span $\text{E D} = l$ and the height $\text{A D} = h_a$ and $\text{B E} = h_b$.

Let O D $= m$ and O E $= n$ (fig. 57).

The equation of the parabola is $y^2 = 4ax$. When $x = h_a$, $y = m$; and $x = h_b$, $y = n$. Hence, $4a = \frac{m^2}{h_a} = \frac{n^2}{h_b}$ and $\frac{m^2}{n^2} = \frac{h_a}{h_b}$; but $m + n = l$.

Hence, $$m = \frac{l\sqrt{h_a}}{\sqrt{h_a} + \sqrt{h_b}}, \text{ and } n = \frac{l\sqrt{h_b}}{\sqrt{h_a} + \sqrt{h_b}}.$$

The lengths of the two portions of the cable are

$$s_1 = m + \frac{2}{3}\frac{h_a^2}{m}, \quad \text{and} \quad s_2 = n + \frac{2}{3}\frac{h_b^2}{n} \quad . \quad . \quad (3)$$

The form assumed by a suspended telegraph wire is a catenary, but when the ratio of the dip to the length of the span is very

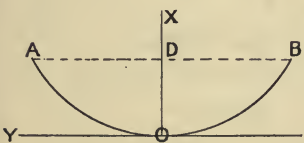


FIG. 56.

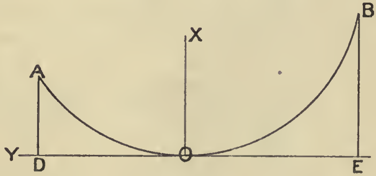


FIG. 57.

small then dx/dy is also very small throughout the length of the wire; the curve may then be considered a parabolic arc, and [128] may be applied.

The curve of equilibrium of the cables of a suspension bridge is a parabola. The length of the cable may therefore be determined accurately by [127]; but when the dip is small [128] may be applied.

(58) (Q. Nov. 1909).—**Sketch the shape of the curve $y = a\cos^2\frac{x}{b}$ and show that the area of each of the regions bounded by the curve and the x-axis is $0{\cdot}5\,\pi ab$.**

Solution.

$$\frac{dy}{dx} = a2\cos\frac{x}{b} \times -\sin\frac{x}{b}\frac{1}{b} = -\frac{a}{b}\sin\frac{2x}{b}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{b}\cos\frac{2x}{b}\,\frac{2}{b} = -\frac{2a}{b^2}\cos\frac{2x}{b}.$$

y is always positive, and the y-axis is an axis of symmetry.

Hence, we need only consider that portion of the curve which is situated to the right of the y-axis. $\frac{dy}{dx}$ is zero when $\frac{2x}{b}=p\pi$, where p is any positive integral number; $\frac{dy}{dx}$ cannot be ∞; $\frac{d^2y}{dx^2}$ is zero when $\frac{2x}{b}=\frac{2p+1}{2}\pi$, and the latter points are points of inflexion. The curve is a periodical curve.

The shape of the curve may be sketched from the values found below:—

x.	y.	$\frac{dy}{dx}$.	$\frac{d^2y}{dx^2}$.	Remarks.
0	a	0	neg.	Maximum.
$\frac{\pi}{4}b$	$\frac{a}{2}$	$-\frac{a}{b}$	0	Point of inflexion.
$\frac{\pi}{2}b$	0	0	pos.	Minimum.
$\frac{3\pi}{4}b$	$\frac{a}{2}$	$\frac{a}{b}$	0	Point of inflexion.
πb	a	0	neg.	Maximum.

The curve now repeats itself. πb is the wave-length.

The area enclosed by one wave and the x-axis is

$$A=a\int_0^{\pi b}\cos^2\frac{x}{b}\,dx=\frac{a}{2}\int_{x=0}^{x=\pi b}\left(1+\cos\frac{2x}{b}\right)dx$$

$$=\frac{a}{2}\int_0^{\pi b}dx+\frac{ab}{4}\int_{x=0}^{x=\pi b}\cos\frac{2x}{b}\,d\frac{2x}{b}=0\cdot5\pi ab\ (Ans.).$$

(59) (Q. June 1910).—**Evaluate (i.)** $\int\frac{x+1}{(x+2)^2}dx$, **(ii.)** $\int x\sin x\,dx$.

Prove that $\int_1^2\frac{1}{x(x+1)}dx=\cdot288$.

Solution.

(i.) $$\frac{x+1}{(x+2)^2}=\frac{x+2-1}{(x+2)^2}=\frac{1}{x+2}-\frac{1}{(x+2)^2}.$$

Hence, $\int \frac{x+1}{(x+2)^2} dx = \int \frac{1}{x+2}\, d(x+2)$

$$- \int \frac{1}{(x+2)^2} d(x+2) = \log_e c(x+2) + \frac{1}{x+2}.$$

(ii.) Taking $u = x$ and $v = \int \sin x\, dx = -\cos x$ we get

$$\int x \sin x\, dx = -x \cos x + \int \cos x\, dx = -x\cos x + \sin x + c.$$

(iii.) $$\frac{1}{x(x+1)} = \frac{1+x-x}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Hence, $\int \frac{1}{x(x+1)} dx = \int \frac{1}{x}\, dx - \int \frac{1}{x+1}\, d(x+1) = \log_e \frac{cx}{x+1}$,

and

$$\int_1^2 \frac{1}{x(x+1)} dx = \log_e \frac{4}{3} = 1{\cdot}3863 - 1{\cdot}0986 = {\cdot}288 \ (Ans.).$$

MECHANICS.

CHAPTER XXI.

FUNDAMENTAL UNITS.

142. Matter—Mass.

By matter is understood everything that makes an impression upon us through our senses. Thus all we smell, feel, taste, and see is matter, and it is only through vibration of matter that sound can be produced and transmitted to our ear; in a perfect vacuum there would be no sound.

A bounded portion of matter is called a *material body*, and a bounded portion of space is called a *mathematical body*. Hence, any physical body possesses its mathematical double or mathematical body. The amount of space which a body occupies is called the *volume* of the body. The amount of matter contained in a physical body is called the *mass* of the body.

In practice the mass of a body is measured by comparing it with the mass of a standard body. Thus in Great Britain the standard mass is called a *pound*.

In France the standard mass is a *kilogramme*, and it was originally the mass of a cubic decimetre of pure water at the standard temperature, but now it is the mass of a piece of platinum, which is as nearly as possible equal to the original kilogramme.

In the C.G.S. system the standard mass is a *gramme*, which is one-thousandth part of a kilogramme.

The determination of the mass of a body is done by means of a pair of scales, whereby is found the number of times that the mass of the body is greater than the standard mass. This is the only way in which the mass of a body can be found direct. For very accurate measurements it may be necessary to take into account the mass of the air which is displaced by the body.

The measurement of mass by means of a pair of scales and a standard mass can be made at any place in the world.

143. Length—Time.

The British standard length is a *yard*, being the distance at the standard temperature between two marks on a bar kept in the Standard Office of the Board of Trade.

The French standard length is a *metre*, and is the distance between two marks on a certain bar at the standard temperature.

The standard length used in science is a *centimetre*, which is one-hundreth part of a metre.

Time.—The standard time is a *second.*

144. Density—Specific density—Specific volume.

By the *density* of a substance is understood the *mass contained in unit volume* of the substance. In Great Britain density is expressed in *pounds per cubic foot*, and in France in *kilogrammes per cubic metre.*

In science density is expressed in *grammes per cubic cm.*

The density of a substance varies with the temperature and the pressure to which it is subjected.

By *relative density*, or *specific density* (sometimes called specific gravity), of a substance at a given temperature is understood the ratio of the density of the substance at the given temperature to that of pure water at the standard temperature.

By *specific volume*, or *bulkiness* of a substance, is understood the number of units of volume occupied by a portion of the substance, of which the mass is equal to the standard mass. It is therefore measured by the *number of cubic feet to the pound*, or the *number of cubic metres to the kilogramme*, or the *number of cubic cms. to the gramme.*

145. Particle.

A particle or a molecule is the smallest conceivable portion of a material body, and may be defined as *the limit of division of a mass by mechanical means.* The geometrical dimensions of a particle are so small that for all practical purposes a particle may be considered as a point with a mass.

146. Homogeneous and heterogeneous bodies.

When any two equal portions of a body, however small, are in every respect similar, the body is said to be homogeneous, otherwise the body is said to be heterogeneous. In this book we shall consider the bodies as being homogeneous unless otherwise specified.

147. Rigid bodies.

A physical body will slightly yield when subjected to a pressure or a change of temperature, *i.e.* the distances between its particles will change to some extent. For theoretical reasons it is convenient, even necessary, to neglect this internal change of the body and to conceive a body which cannot yield to any cause applied to it; such a theoretical body is called a *rigid body*. In this part of the book all bodies will be considered rigid unless otherwise specified.

148. Gravitation.

Matter possesses a characteristic property called gravitation, *i.e.* two masses m_1 and m_2 at a distance, d, apart attract each other and exert a mutual pull which is equal to km_1m_2/d^2, where k is a constant called the *gravitation constant.*

The pull set up between a body in the immediate neighbourhood of the surface of the earth is called the *weight* of the body. The pull might just as truly be called the weight of the earth w.r.t. the given body.

Different masses placed at the same spot will have weights which are directly proportional to their masses; but the weight of a given mass varies with its position relative to the earth. For this reason, by weighing a body on a pair of scales, we do not determine the weight but the mass of the body.

Example 1.—If we buy a ton of rice at the equator, measuring it on a pair of scales, and then bring the rice to London, and again measure it on a pair of scales, the last measure will be the same as that at the equator. But if we measure the rice in London on a spring-balance, calibrated at Greenwich, the rice will measure more than a ton, because the spring-balance measures the pull between the rice and the earth.

Example 2.—If the safety-valve of a boiler is loaded by weights, the valve will open at a smaller effective pressure at the equator than at the pole; but if the valve is loaded by a spring, the effective pressure, which is required to open the valve, will be the same at any place of the world.

149. Physical dimensions.

Just as a geometrical quantity depends on its dimensions in length, so does a physical quantity depend on its dimensions in mass, length, and time. Thus the density of a body varies directly as its mass and inversely as its volume; hence, the dimensions of density is *plus* one in mass and *minus* three in length, or *density* $=[M]\,[L]^{-3}$.

The specific volume of a substance is proportional to the volume of a body made of the substance, and inversely proportional to the mass of that body; hence, the dimensions of specific volume is *plus* three in length and *minus* one in mass, or *specific volume* $=[M]^{-1}\,[L]^3$.

The specific density, however, is a pure number, as it is expressed by the ratio of two quantities of the same dimensions; hence, the dimensions of *specific density* are *zero*.

For these reasons the unit of mass, length, and time are called *fundamental units*, whereas the units of other physical quantities are called *derived units.*

The correctness of an equation between any number of physical quantities may be tested by examining the dimensions in mass, length, and time of each term. It is obvious that the dimensions in mass, length, and time must be the same in each term.

The student should always test his results by means of dimensions.

CHAPTER XXII.

VECTORS.

150. Vector quantity.

When a person tells us that he has walked 10 miles, the only information he has given us is that the arithemetical sum of all the distances he has walked is equal to 10 miles.

If he tells us further, that he started from a certain given point and always walked in the same direction, say in a line stretching south-north, and always in the same sense, say from north towards south and never from south towards north, then we know exactly what he has done.

The latter example has been chosen to illustrate that a quantity may not always be determined by its *magnitude* only, but may also involve the idea of *direction* and *sense* of action. A quantity involving the latter three ideas is called a *vector quantity.* Quantities which only involve the idea of magnitude are called *scalar quantities.*

A B (fig. 58) represents a vector quantity whose magnitude is

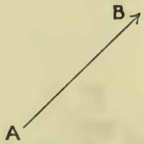


FIG. 58.

given by the length A B drawn to scale, the direction of which is identical with that of A B and whose sense of action is from A towards B, the latter being denoted by an arrow-head pointing towards B. If + A B is a given vector quantity, then a vector quantity − A B is denoted by reversing the arrow-head, which would also denote the vector quantity + B A.

151. Displacement.

When a particle has changed its position, due to some cause, it is said to have been *displaced.* A displacement is determined, (i.) by the original position of the particle; (ii.) by the direction in which the displacement has taken place; (iii.) by the magnitude of the displacement, *i.e.* by the length of the straight line joining the initial and final positions of the particle; and, finally (iv.), by the sense of the displacement in the given direction.

Displacement is therefore *a vector quantity*; hence, we may illustrate the properties of vectors by the displacement of a particle.

Suppose a particle (fig. 59) originally at A is displaced to B and thence to C. The displacement *A C* which would have placed the particle at the same point C is called *the equivalent of A B and B C.*

The displacement A B (fig. 60) is equivalent to the two displacements A D and D B, the latter being perpendicular to A C

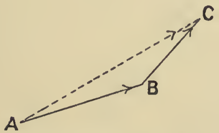


Fig. 59.

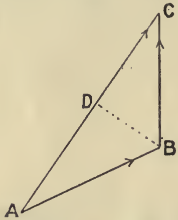


Fig. 60.

and similarly B C is equivalent to B D and D C; but A C = A D + D B + B D + D C, because B D = − D B.

As, however, A D and D B together, and B D and D C together, are equivalents to A B and B C respectively, it is said, that

$$\text{the vector sum } [AB + BC] = \text{vector } AC \quad . \quad . \quad (1)$$

$$\text{and the vector difference } [AC - AB] = \text{vector } BC \quad . \quad (2)$$

Hence, the sum of two given vectors is found by choosing any point A and setting out the two vectors as A B and B C in fig. 59; A C is then the required vector sum. The order in which the vectors are to be set out is very important. They must be set out w.r.t. their senses, so that they form a continuous sense-circuit which is only broken by the equivalent, which belongs to a sense-circuit going in the opposite direction.

The difference of two given vectors is found by setting them out from point A, such as A C and A B (fig. 59); B C is then the required difference. The sense of B C is found by remembering that it must be in sense-circuit with the vector which is to be subtracted.

The above principle may be extended to any number of displacements. Let the successive displacements of the particle be A B, B C, C D, and D E (fig. 61). Let A X be a straight line

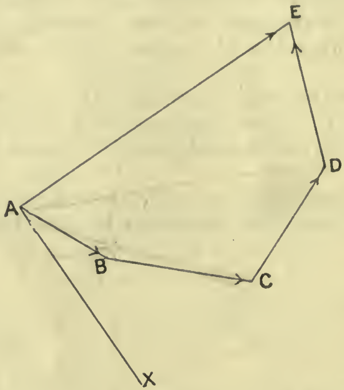


FIG. 61.

perpendicular to A E, and let the vectors form angles α_1, α_2, α_3, and α_4 respectively with A X in the positive direction, then

$$AE = AB \sin\alpha_1 + BC \sin\alpha_2 + CD \sin\alpha_3 + DE \sin\alpha_4 ;$$

we have also

$$AB \cos\alpha_1 + BC \cos\alpha_2 + CD \cos\alpha_3 + DE \cos\alpha_4 = 0 ;$$

hence,

$$AE = AB \sin\alpha_1 + AB \cos\alpha_1 + BC \sin\alpha_2 + BC \cos\alpha_2 + CD \sin\alpha_3 + CD \cos\alpha_3 + DE \sin\alpha_4 + DE \cos\alpha_4,$$

but A B is equivalent to $AB \sin\alpha_1 + AB \cos\alpha_1$, etc. Hence, the vector sum

$$(AB + BC + CD + DE) = \text{vector } AE = -\text{vector } EA . \quad (1)$$

152. Simultaneous displacements.

Two or more agents may displace a particle simultaneously in two or more directions. The latter may be illustrated by a person walking on the deck of a moving ship. Let the original position of the particle be at A (fig. 62), and let A B and A C be the two simultaneous displacements. The final position of the particle will obviously be reached if it be displaced to B and thence to D, B D being parallel and equal to A C; or if the particle be displaced to C and thence to D, C D being equal and parallel to A B. The ultimate position of the particle will be point D, which is the opposite corner to A in the parallelogram A B D C A.

If the two simultaneous displacements take place in a regular manner, *i.e.* that equal displacements, however small, always take place during equal intervals of time, then there will correspond to any displacement AB_1, along A B, a displacement AC_1, along A C; AB_1 and AC_1 being sides in the parallelogram $AB_1D_1C_1A$. Hence, point D_1 lies on the straight line A D, and the latter is thus the locus traced by the particle during the two simultaneous

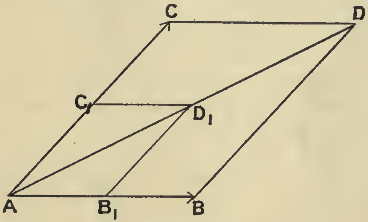


FIG. 62.

displacements A B and A C. A D is called the *resultant* of A B and A C, and is in every respect the substitute for the two simultaneous displacements A B and A C.

153. Composition.

A vector quantity has been defined as a quantity which possesses direction, sense of action, and magnitude; it must therefore be able to produce or tend to produce a displacement. Hence, two like vectors acting simultaneously at a point must produce or tend to produce displacements along their lines of action which are proportional to the magnitudes of the respective vectors.

Hence, we may extend the rule which we have just found for the two simultaneous displacements A B and A C in fig. 62 to any vector quantity, or we may say:

Any two like vector quantities Δ_1 and Δ_2 acting simultaneously at the same point, may be substituted in every respect by a third vector quantity Δ_r of the same kind, which is represented in magnitude and direction by the diagonal of the parallelogram of *which the sides are formed by* Δ_1 *and* Δ_2. Δ_r *is called the resultant of* Δ_1 *and* Δ_2.

We may now determine the resultant of any number of simultaneously acting vectors whose lines of action lie in the same plane, but which do not necessarily pass through one point. Let Δ_1, Δ_2, Δ_3, and Δ_4 be four such vectors, whose resultant it is required to find. Choose any point A in the plane (fig. 63), draw a vector A B equal to Δ_1, and similarly draw vectors B C, C D, and D E equal to Δ_2, Δ_3, and Δ_4 respectively, then A C is equal to the resultant of Δ_1 and Δ_2; likewise A D is equal to the resultant of Δ_1, Δ_2, and Δ_3, and, finally, A E, which is the vector sum of the given vectors, is equal to the resultant of all four vectors. The polygon A B C D E A is called the *vector-polygon.* The resultant, however, is still to be localised; this is done by producing the

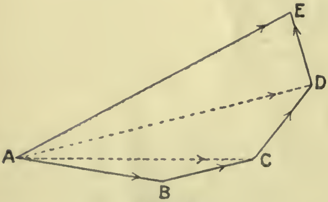


FIG. 63.

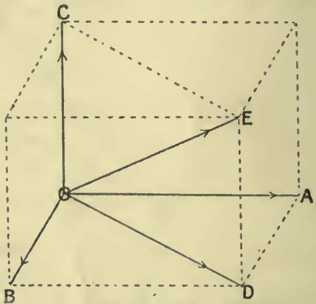


FIG. 64.

lines of action of Δ_1 and Δ_2 until they meet, and through the point of intersection draw a line parallel to A C, then produce the latter line until it meets the line of action of Δ_3, and through their point of intersection draw a line parallel to A D. Finally, produce the latter line and the line of action of Δ_4, the resultant, which is parallel to and equal to A E, will pass through their point of intersection.

We may now also determine the resultant of any number of simultaneously acting vectors passing through one point, but whose lines of action do not lie in one plane. In fig. 64 O A, O B, and O C are three such vectors. The resultant of O A and O B is O D, and the resultant of O D and O C is O E, which is also the resultant of the three given vectors. To find the resultant O E we might simply have drawn the lines O A, A D, and D E. The latter principle may obviously be extended to any number of vectors passing through one point.

The operation of determining the resultant vector of a number of given simultaneously acting vectors is called *composition*; we compound the given vectors to obtain the resultant. The given vectors are called the *components* of the resultant.

154. Resolution.

We may obviously also perform the operation of determining the components in given directions of a given vector. The latter operation is called *resolution*, since we resolve the given vector in the given directions to obtain the components in those directions.

When the directions of the components of a given vector are at right angles to each other, the components are called the *resolved parts*, or the *resolutes* of the given vector in those directions.

Let us take a system of rectangular co-ordinates. A vector lying in the co-ordinate plane can be resolved into components parallel to the axes. The latter components are the resolutes in the directions of the axes. Suppose we have a number of vectors $\Delta_1, \Delta_2 \ldots \Delta_n$ forming angles $\alpha_1, \alpha_2 \ldots \alpha_n$ respectively with the axis of x. The components in the direction of the x-axis are $\Delta_1 \cos \alpha_1, \Delta_2 \cos \alpha_2 \ldots \Delta_n \cos \alpha_n$; and those in the direction of the y-axis are $\Delta_1 \sin \alpha_1$, $\Delta_2 \sin \alpha_2 \ldots \Delta_n \sin \alpha_n$. Hence, the vector sum in the direction of the x-axis is $\sum_{r=1}^{r=n} \Delta_r \cos \alpha_r = X$, and that in the direction of the y-axis $\sum_{r=1}^{r=n} \Delta_r \sin \alpha_r = Y$.

The resultant of X and Y is $R = \sqrt{X^2 + Y^2}$. R is the vector sum of all the given vectors $\Delta_1, \Delta_2 \ldots \Delta_n$, and is also equal to their resultant. If θ be the angle which R forms with the x-axis, then $\tan \theta = Y/X$.

CHAPTER XXIII.

SPEED—VELOCITY.

155. Motion.

A body is said to be in *absolute motion* when it changes its position in space; and it is said to be in *relative motion* when it changes its position relatively to other bodies. We do not know any instance of absolute motion; all motion which we can observe is relative.

When a body does not change its position in space it is said to be at *absolute rest*; the latter condition, however, does not exist so far as we know, and if it did exist and we could see the body, we would say that the body was moving at a great rate, because we are always changing our position in space. A stone lying on the road is said to be at rest, while it is really moving at a great rate, the stone is at *relative rest.* There is no rest, everything is in motion, and all the laws of mechanics can be deduced from the laws of motion.

A particle which is in continuous motion traces out a curve, which is called the *path* of the particle.

156. Speed.

The speed of a particle is the rate at which it describes its path.

When we say that a train is travelling at a rate of 32 miles an hour, amidst bodies at relative rest, we mean that if the speed remained the same as at the particular moment, we would travel a distance of 32 miles in one hour. The speed may vary considerably during the journey, but when a distance of, say, 63·5 miles has been travelled in 2 hours 10 minutes, we say that the average speed of the train has been 29·3 miles an hour. The average speed is therefore the distance travelled divided by the time occupied. The *dimensions of speed are* $[L][T]^{-1}$.

Speed does not involve the ideas of direction or of sense, but simply of magnitude, and *is* therefore *not a vector quantity.*

Let δs and δt respectively be the length of an arc of the path, and the interval of time occupied by describing δs, then the average speed $u_m = \delta s/\delta t$. If the speed be not constant during time, δt, there must be a minimum speed, u, and a maximum speed, $u+\delta u$; hence, u_m must lie between u and $u+\delta u$, and therefore in the limit

$$u=\frac{ds}{dt} \quad . \quad . \quad . \quad . \quad [129]$$

The unit speed is the unit of length travelled in the unit of time, thus: 1 foot per second; 1 metre per second; 1 cm. per second, etc. The latter units, however, are for various reasons not always convenient; hence, in practice other units are used, such as: 1 mile per hour; 1 kilometre per hour; 1 knot = 1 nautical mile per hour, etc.

Referring to [129] we have

$$s=\int_{t_1}^{t_2} u\,dt \quad . \quad . \quad . \quad . \quad [130]$$

When the relation between speed and time is given the distance travelled is found by [130]; s is thus the area under the speed-time curve. If [130] cannot be integrated, then draw the speed-time curve and calculate the area by [98] or [99]. The mean ordinate of the curve is the average speed.

When the relation between distance and time of a moving particle is given the speed at any time may be found by [129], or by plotting the distance-time curve and carefully drawing the tangent to the curve at the point corresponding to the given time. If the tangent forms an angle α with the time-axis the speed is proportional to tan α.

157. Velocity.

Velocity is defined as *the rate of change of displacement.*

Velocity is therefore a vector quantity possessing both magnitude, direction, and sense. In fig. 63 the path described by the particle is the broken line A B C D E; if t be the time occupied by the motion it is evident that, the average velocity of the particle must be $v_m = \text{A E}/t$, because v_m is the average rate of change of displacement. But A E is the vector sum of the various displacements from A to E; hence, *the average velocity v_m is equal to the* vector sum *of the displacements divided by the time occupied by the motion.* The average velocity is therefore not the same as *the average speed*, the latter quantity *is equal to the* arithmetical sum *of the displacements divided by the time occupied by the motion*, or $u_m = \dfrac{\text{AB}+\text{BC}+\text{CD}+\text{DE}}{t}$.

If the displacement changes direction continuously, then the path of the particle becomes a curved line such as A B C in fig. 65. Let t_1 and t_2 respectively be the intervals of time occupied in describing the arcs $AB = s_1$ and $AC = s_2$. The chords A B and A C are respectively the vector sums of the displacements represented by arcs s_1 and s_2; hence, vector B C is equal to the vector difference (A C − A B), which is the change of displacement of the particle while describing arc B C. Hence, the average velocity of the particle, while moving along arc B C, is equal to the length of the chord B C divided by time $(t_2 - t_1)$, whereas the average speed is $(s_2 - s_1)/(t_2 - t_1)$. But lim. chord $BC/(t_2 - t_1) =$ lim. $(s_2 - s_1)/(t_2 - t_1)$, and if C T be the tangent to the path at point C then lim. (angle T C B) is zero. Hence, *the magnitude of the velocity v, of a particle is equal to the speed ds/dt, of the particle,*

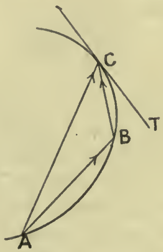


FIG. 65.

the direction of the velocity is along the tangent to the path, and its sense is that of the motion. The *dimensions of velocity* are the same as those of speed, viz. $[L][T]^{-1}$.

$$\text{Magnitude of } v = \frac{ds}{dt} \quad . \quad . \quad . \quad [131]$$

A particle possesses the unit of velocity when it undergoes a displacement of the unit of length in the unit of time. But the particle moves with the unit of speed when it describes the unit of length in the unit of time.

158. Resultant velocity.

Being a vector quantity all the laws regarding vectors also apply to velocity. A particle may thus be subjected to two simultaneous velocities v_1 and v_2, whose resultant may be found by constructing the triangle of velocities as in fig. 59, taking $AB = v_1$ and $BC = v_2$; A C is the resultant velocity and the particle will move as if it only possessed the latter velocity.

If a particle be possessed of more than two simultaneous velocities, all in one plane, then the resultant of all the velocities may be found by drawing the polygon of velocities as in fig. 63, taking $AB = v_1$, $BC = v_2$, etc.; AE is the resultant velocity and the motion of the particle will be the same in every respect as if it were only subjected to the velocity AE.

If the directions of the velocities to which the particle is subjected do not lie in one plane, then the resultant velocity will be found by constructing the parallelopiped of velocities, as shown in fig. 64, taking $OA = v_1$, $OB = v_2$, etc. The particle will move in space in every respect as if it only possessed the velocity OE.

Conversely, a velocity may be resolved into components in given directions.

Let the resolutes of a velocity v in the directions of the axes of a rectangular system of co-ordinates be v_x in the direction of the axis of x, and v_y in the direction of the axis of y; and let δx, δy, and δt be corresponding increments, then $v_x = \text{lim.}\ \delta x/\delta t = dx/dt$ and $v_y = \text{lim.}\ \delta y/\delta t = dy/dt$. Also

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2} \quad . \quad . \quad [132]$$

159. Relative velocity.

Let A_1 and A_2 (fig. 66) be two particles moving in the same

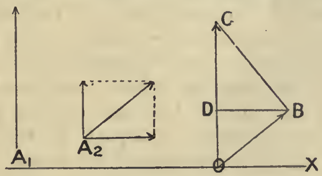


FIG. 66.

plane with velocities v_1 and v_2 respectively. Let O be the origin of a system of rectangular co-ordinates, whose y-axis is parallel to v_1. Set off OC equal to v_1, OB equal to v_2, and DB parallel to the x-axis. Finally, let angle BOC be equal to β.

Seen from A_1 the motion of A_2 appears to be as if A_2 moved with a velocity $(v_2 \cos\beta - v_1) = CD$ in the direction of the y-axis, and with a velocity of $v_2 \sin\beta = DB$ in the direction of the x-axis. Hence, the relative motion of A_2 w.r.t. A_1 has a velocity $v_r = CB$.

Seen from A_2 the motion of A_1 seems to possess a velocity $(v_1 - v_2 \cos\beta) = DC$ in the direction of the y-axis, and with a velocity of $-v_2 \sin\beta = BD$ in the direction of the x-axis. Hence, $v_r = BC$.

From the above it follows that

$$v_r = \text{the vector difference } (v_1 - v_2) \quad . \quad . \quad [133]$$

W.r.t. sense, v_r is in circuit with the velocity of the particle from which the motion is observed.

160. Angular velocity.

Let a straight line O A turn in a plane about one of its points O, and let t be the time it takes the line to turn through an angle θ radians, then θ/t is called the average angular velocity of the line during time t. Hence, the angular velocity of the line is

$$\omega = \lim. \frac{\theta}{t} = \frac{d\theta}{dt}, \text{ and its dimensions are } [\mathrm{T}]^{-1} \quad . \quad [134]$$

The *unit of angular velocity* is evidently 1 *radian per second.*

In accordance with the convention adopted in trigonometry the angular velocity is positive when the line turns in an anti-clockwise sense.

EXAMPLE.—*A particle moves with constant speed on a circle with radius r. Find the angular velocity of the radius.*

As the speed is constant equal sectors will be described by the radius in equal times; hence, the angular velocity of the radius is constant. During the time T of one revolution the radius will turn through an angle of 2π radians. Hence, the angular velocity of the radius is $2\pi/\mathrm{T}$ radians per second.

As a circular arc is equal to the central angle which it subtends into the radius, it follows that the speed of the particle is equal to the radius into the angular velocity independent of the sense of the latter. It also follows that in a given position of radius the velocity of the particle is equal to the radius into the angular velocity, taking into account the sense of the latter.

CHAPTER XXIV.

ACCELERATION—FORCE.

161. Acceleration.

The velocity of a moving particle is always changing except in one case, viz. when the particle describes a straight line at constant speed.

The rate of change of velocity is called acceleration.

Let A B (fig. 67) be an arc of the path described by the

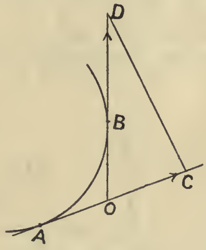


FIG. 67.

particle which is moving in a plane, and let $v_1 = OC$ and $v_2 = OD$ be the velocities of the particle at points A and B respectively. C D is then the change of velocity, and if it takes the particle time t to describe arc A B, then CD/t is the average acceleration. If the arc A B be diminished indefinitely, then we have δv and δt instead of C D and t. Hence, in the limit

$$\text{Acceleration} = \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad (1)$$

Acceleration is a vector quantity, and the resultant of two or more accelerations can be found in the same manner as the resultant of two or more velocities. Conversely, an acceleration may be resolved into components in given directions.

Let the resolutes of an acceleration, a, in the directions of a pair of rectangular axes be a_x in the direction of the axis of x, and a_y in the direction of the axis of y. According to [132] and (1), $a_x = dv_x/dt = d^2x/dt^2$ and $a_y = dv_y/dt = d^2y/dt^2$. Also

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(d^2x/dt^2)^2 + (d^2y/dt^2)^2} \qquad . \quad [135]$$

A particle moves with the unit of acceleration when its velocity changes by the unit of velocity in the unit of time. The practical units of acceleration are: 1 foot per sec. per sec.; 1 metre per sec. per sec.; 1 cm. per sec. per sec.

The dimensions of acceleration are $[L][T]^{-2}$.

The *rate of change of angular velocity* is called *angular acceleration*, which is obviously

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \text{ and its dimensions are } [T]^{-2}. \qquad . \quad [136]$$

The *unit of angular acceleration* is 1 *radian per second per second.*

162. Hodograph.

The relation between velocity and time and between acceleration and time cannot be graphically represented by a curve

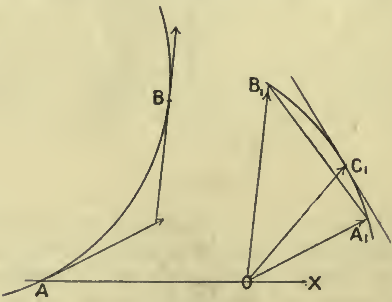


FIG. 68.

referred to rectangular co-ordinates, such as the relation between speed and time. Velocity and acceleration are not determined by their magnitudes only, but their directions are also required to be known; hence, the term velocity-time curve is erroneous, it really means speed-time curve.

Let A B (fig. 68) be an arc of the path described by a moving particle. Take any point O in the plane of the curve and draw O A_1 and O B_1 equal to the velocities v_a and v_b at the points A and B respectively. Similarly, draw radii vectores from O equal to the velocities of the particle at various points between A and B. The curve $A_1C_1B_1$ which connects the ends of the velocity vectors drawn from O is called a *hodograph*. Chord A_1B_1 is the vector-difference of v_a and v_b; hence, if t be the time it takes the particle to describe arc A B, then chord A_1B_1/t is the average acceleration of the particle during the time-interval t. But chord A_1B_1 is also the vector-difference of the displacements of the point describing the hodograph, while the latter point passes over arc A_1B_1. Hence, A_1B_1/t is also the average velocity of the point on the hodograph while it describes arc A_1B_1 or, *the total acceleration of the particle when passing the point on A B, at which the velocity is* OC_1, *is equal to the velocity of point* C_1 *on the hodograph.*

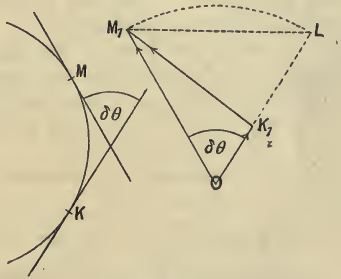


FIG. 69.

When the tangent to the hodograph is not perpendicular to the velocity-vector, then the acceleration of the particle can be resolved into two resolutes, one at right angles to and another in the direction of the velocity of the particle. The former is called the *normal acceleration*, and the latter is called the *tangential acceleration* of the particle. When the tangent at a point of the hodograph is at right angles to the velocity-vector, then the acceleration is totally normal and the particle moves at that moment with constant speed.

K M in fig. 69 is a small arc, δs, of the path described by the particle, and K_1M_1 is the corresponding arc of the hodograph. Hence, OM_1 is the velocity, v, at point M; $K_1M_1 = \delta v$, and $K_1L = \delta u$. The total average acceleration is $= \frac{K_1M_1}{\delta t}$; the average normal acceleration $= \frac{M_1L}{\delta t} = \frac{v\,\delta\theta}{\delta t} = v\frac{\delta\theta}{\delta s}\frac{\delta s}{\delta t}$, and the average tangential acceleration $=\frac{\delta u}{\delta t}$. But lim. $\frac{\delta\theta}{\delta s}=\frac{1}{\rho}$ (see article **108**), and lim. $\frac{\delta s}{\delta t}$ = the magnitude of v.

Hence, total acceleration, $a_t=\frac{dv}{dt}$ . . . [137]

normal acceleration, $a_n=\frac{v^2}{\rho}$ . . . [138]

tangential acceleration, $a=\frac{du}{dt}=\frac{d^2s}{dt^2}$ . . [139]

The direction of the total acceleration [137] is that of the tangent to the hodograph, and the direction of the tangential acceleration [139] is that of the tangent to the path.

The acceleration is negative when the speed diminishes. A negative acceleration is called a *retardation.*

If the speed-time curve be known the magnitude of the tangential acceleration can be found by drawing the tangent to the curve at the given point. Let α be the angle which the tangent forms with the axis of time, then the numerical value of $a=\tan\alpha$.

Example.—*A particle moves with constant speed on a circle with radius r. Find the acceleration.*

As the speed u is constant $\frac{du}{dt}=0$, and the hodograph is also a circle. Hence, the acceleration is totally normal.

Hence, $a_t=a_n=\frac{v^2}{r}$; $a=\frac{du}{dt}=0$ . . . [140]

163. Motion with uniform (constant) **acceleration.**

When the acceleration is constant and in the direction of the motion, then $a_n=0$, $a_t=a$, and the path is a straight line. Hence, $\frac{du}{dt}$ is constant, and the speed-time curve is therefore also a straight line, the slope of which is equal to the numerical value of the acceleration. In fig. 70 O A is the initial speed of the particle, A B the speed-time curve, and tan C A B = the magnitude of a. The velocity, v, at the end of time-interval O D $=t$ is therefore

$$v=v_i+at \quad . \quad . \quad . \quad . \quad [141]$$

According to article **156**, the area O A B D O represents the distance, s, traversed in the time-interval, t. But O A B D O = O A C D O + A B C A, or

$$s=v_it+\tfrac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad [142]$$

The same results are obtained by integrating [137] and (1) in article **161** thus,

$$\int_{v_i}^{v} dv = a\int_0^t dt, \quad \text{and} \quad \int_0^s ds = \int_0^t v\,dt = \int_0^t (v_i + at)dt.$$

If the mean velocity be v_m, then $v_m t = s$.

Hence,
$$v_m = \frac{v + v_i}{2} \quad . \quad . \quad . \quad . \quad [143]$$

The magnitude of v_m is graphically represented by H K in fig. 70. By eliminating t between [141] and [142], we get

$$v = \sqrt{v_i^2 + 2as} \quad . \quad . \quad . \quad . \quad [144]$$

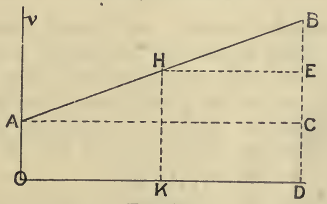


FIG. 70.

By [141] and [142], we get

$$t = \frac{v - v_i}{a} = \frac{\sqrt{v_i^2 + 2as} - v_i}{a} \quad . \quad . \quad . \quad [145]$$

164. The law of inertia, or Newton's first law of motion, may be stated thus:

Every mass remains in its state of rest (relative or absolute), *or continues to move with constant velocity in a straight line as long as no external cause occurs to change that state.*

This characteristic property of matter which shows itself by the apparent reluctance of a mass to change its velocity is called *inertia*, which means inertness or sluggishness. Inertia manifests itself to us whenever we happen to be in a train. The sudden starting and stopping or moving on a curve appear to us as something which our body is not willing to partake in. It might seem, at first, to us that inertia is a property which one would wish that matter did not possess, but we shall see that it is a very useful quality.

165. Force.

The external cause which produces, or tends to produce, a change of velocity is called force. On account of the inertia of the mass the change of velocity will be gradual, and the force will therefore manifest itself by producing acceleration. It is obvious that a force must be proportional to the acceleration it produces on a given mass, and also that it must be proportional to the mass on which it produces a given acceleration. Hence, by adopting proper units, we have

$$\text{Force, F} = \text{Mass into acceleration} \quad . \quad . \quad [\mathbf{146}]$$

In other words, *unit of force acting on unit of mass produces unit of acceleration.* Thus, in the C.G.S. system, the unit of force is that force which acting on a mass of one gramme produces an acceleration of one centimetre per second per second; this force is called a *dyne.*

A force is a vector quantity, as it must have direction and sense of action in order to be able to change velocity. Hence, a force can be resolved into components in given directions, and the resultant of a number of given forces can be found by constructing the polygon of forces, the triangle of forces, or the parallelopiped of forces, as the case may be. It is evident that *the dimensions of force are* [M] [L] $[T]^{-2}$.

A force which acts in the same direction and sense as the velocity is called an *effort* (F_e), and a force which acts in the same direction, but in the opposite sense of the velocity, is known as a *resistance* (F_r).

A force producing a curvilinear motion must have two components (article **162**), one in the direction of the velocity, *the tangential force*, and a second component in the direction of the normal; the latter force is called *the deviating force*, because it causes the mass to deviate from the straight line motion, or *the normal force* (F_n), because it acts along the normal to the curve. By [**139**] and [**138**] we get

$$\text{magnitude of tangential force} = m\frac{du}{dt}, \quad \text{and} \quad F_n = m\frac{v^2}{\rho} \quad [\mathbf{147}]$$

When the velocity of the body changes from v_1 to v_2 during time-interval t, the average force which has caused the change of velocity is F_m = mass of body into vector difference $(v_2 - v_1)$ divided by t.

166. Momentum.

Newton's second law of motion states: *The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

The momentum or quantity of motion of a body is measured by the product of the mass and the velocity of the body. Momentum is a vector quantity, its direction being the same as that of the velocity. The dimensions of *momentum* are $[M][L][T]^{-1}$, which are the same as those of force into time. Hence, *force* may be measured by *rate of change of momentum.*

An *instantaneous force*, such as that produced by an explosion, may be considered as having been constant during the short time-interval t in which it has existed, and is therefore measured by the momentum it has produced, divided by t.

CHAPTER XXV.

ENERGY—POWER.

167. Energy.

Newton's third law of motion states: *To every action there is an equal and opposite reaction.*

The manner in which a force acts on a body is by push, although it may sometimes be transmitted to the body by pull. The steam in the cylinder of an engine acts by pushing the piston. A horse before a cart seems to pull, but it really pushes the harness.

The body always resists the push by an equal force in the same direction but in the opposite sense of the push. Thus, the total steam pressure on the piston is always met with an equal resistance, consisting of the back pressure, the useful and wasteful resistances, and the inertia resistance of the moving masses of the engine. There is, however, a great difference between the push and some of the resistances, such as the inertia resistance. The push is an *active force*, which is kept up by a source possessing capacity for producing activity, whereas the inertia resistance is only a *reaction* or *passive force*, which can produce no motion. As long as there is steam at the proper pressure in the boiler, so long will the push on the piston be able to overcome the reaction.

The capacity for producing activity is called *energy*. The primary source of the energy in the case of the steam-engine is in the fuel, which is used for firing the boiler. As long as there is fuel, so long can also the push be maintained.

The energy stored in the fuel is latent until it is converted into heat by being burnt in the furnace of the boiler. Latent energy is called *potential energy* (E_p). It is manifested by the motion of the piston against all the resistances. The push on the piston is an effort, and the amount of steam required is proportional to the distance through which the piston has travelled.

Hence, *the energy* manifested by the motion of the piston *is equal to the effort into the distance through which it travels.*

Suppose that a constant effort F_e is acting on a particle of mass m. At a certain time t_1 the speed of the particle is u_1, and at a later time t_2 the speed is u_2. As F_e is constant the acceleration is also constant. Hence, see [139],

$$a = \frac{u_2 - u_1}{t_2 - t_1}, \quad \text{and therefore} \quad F_e = m\frac{u_2 - u_1}{t_2 - t_1} \quad . \quad (1)$$

Let the distance traversed by the particle during time-interval $(t_2 - t_1)$ be s, then, as the mean speed is $(u_2 + u_1)/2$, the amount of energy manifested by the effort is

$$F_e\, s = m\frac{u_2 - u_1}{t_2 - t_1}\,\frac{u_2 + u_1}{2}(t_2 - t_1) = m\frac{u_2^2 - u_1^2}{2}. \quad . \quad (2)$$

Energy may thus be measured by $mu^2/2$. The quantity $F_e\, s$ *is called the energy exerted* (E_e) *by the effort* F_e *while travelling through the distance s.* The energy exerted is therefore the amount of the potential energy contained in the source which has been manifested.

If F_e be variable the elementary energy exerted by the effort while traversing distance δx, is

$$\delta E_e = F_e\, \delta x = m\frac{du}{dt}u\,\delta t\,; \text{ hence, } dE_e/dt = mu\frac{du}{dt} = \frac{m}{2}\frac{du^2}{dt}.$$

Hence,
$$E_e = \frac{m}{2}\int_{t_1}^{t_2}\frac{du^2}{dt}dt = m\frac{u_2^2 - u_1^2}{2} \quad . \quad . \quad [148]$$

The quantity $mu^2/2$ is called *kinetic energy* or *energy of motion.*

The *mean effort* is that constant effort which exerts the same amount of energy as the variable effort while travelling through the same distance. The mean effort must not be confounded with the average force in article **165**. The two forces are not necessarily equal.

The potential energy of the source may, according to [**148**], be converted into kinetic energy due to the inertia of matter. We may therefore say that, *inertia is that property of matter which enables a mass to become an accumulator of kinetic energy.*

Kinetic energy can also be converted into potential energy. This statement may be exemplified by the following simple case:

A cylindrical drum can turn without friction about its axis, which is horizontal and in fixed bearings. A weight is attached to one end of a string which is coiled round the drum. The weight is let go when it is at its highest level. Air resistance and resistance of the string to being bent and straightened are neglected.

The weight of the falling body is an effort, because it acts in the direction and sense of the velocity which the mass of the body is gaining while falling. As the body is descending the energy exerted by its weight will be converted into kinetic energy, which is manifested by the rotation of the drum, and the motion of the following mass.

The body will be at its lowest level when the string is completely uncoiled; its vertical velocity is then zero. Let the magnitude of the weight be W and the vertical distance through which it has fallen be h, then the energy exerted by the weight is Wh, which is therefore also the total available potential energy E_p stored in the body when it is at its top-level. E_p has during the fall of the body been gradually converted into kinetic energy E_k, which is now accumulated in the rotating mass of the drum.

In virtue of E_k the drum will continue its rotation, lifting the body, whose weight is now a resistance as it acts in the opposite sense of the velocity of the body. The weight is therefore not exerting energy but is consuming energy; the latter is manifested by the rotation of the drum becoming slower and slower. At last the total kinetic energy, which was accumulated in the mass of the drum, is exhausted and the drum stops. The body is then again at its top-level, containing all its original potential energy.

The body will now fall again turning the drum, and the process of converting potential energy into kinetic energy and *vice versâ* will be repeated. The sum of the kinetic energy accumulated in the moving masses and the available potential energy stored in the weight is at any time equal to $E_p = Wh$.

The unit of energy must be expressed by unit of force exerted through unit of distance. In the C.G.S. system the unit of energy, equal to the energy exerted by one dyne through one centimetre, is called an *erg*.

The dimensions of energy are $= [M][L]^2[T]^{-2}$.

168. Work (Wh).

By work is understood *the production of motion against a resistance.* Let us consider a body, such as the piston of a steam-cylinder. On the one side of the piston we have the pressure of the steam, which is an effort (F_e); on the other side there are the back-pressure of the exhaust steam, the useful and wasteful resistances to be overcome; let their resultant be F_r. Both F_e and F_r act along the same line and hence move through the same distance. The force—the effective force—which is left to produce motion is $F_e - F_r$; hence, *the energy exerted by the effective force = the energy exerted by the effort minus the work done on the resistance.*

When F_e is different from F_r the effective force will produce acceleration and the speed of the moving piston will be changed from u_1 to u_2. Let m be the mass of all the moving parts of the engine reduced to the piston, then the change of kinetic energy is $m(u_2^2 - u_1^2)/2$, and the energy-equation (see [**148**]) is

$$\text{Energy exerted} = \text{Work done} + \text{Change of kinetic energy} \qquad [149]$$

[**149**] expresses the *principle of conservation of energy* and also the *impossibility of perpetual motion.*

It is obvious that work done is measured by the same units as energy, *work done is energy consumed.*

169. Work represented by an area.

As work done and energy exerted are products of two quantities, force and length, they can be represented by the area of a plane figure, which is also the product of two quantities.

The area under the effort-distance curve represents the energy exerted, and the area under the resistance-distance curve represents the work done. The difference of the two areas determines the change of kinetic energy.

170. Energy in terms of pressure and volume.

A force is said to exert itself as a pressure when it is applied over an area. The ratio of the force to the area is called the *intensity* of the pressure. Steam acts on a piston by pressure, and the force of the steam is equal to the intensity of the steam pressure into the area of the piston.

The *dimensions of intensity of pressure* are $\frac{F}{S} = [M][L]^{-1}[T]^{-2}$.

Suppose the area of a cylinder-piston is A and the intensity of the pressure is P. The total force pushing the piston is thus P A, and when the piston has moved through a distance l, the energy exerted is equal to $PAl = PV$, where V is the volume swept by the piston. Hence, *the energy exerted by the force pushing the piston is equal to the intensity of the push multiplied by the volume swept by the piston.*

171. Power.

Power is defined as *the rate of doing work.* The terms energy and power must not be confounded. Power has relation to time. Thus, one ton of coal contains a certain amount of latent energy, which can be realised by burning the coal in a steam-boiler. The rate at which the coal is consumed may, however, be varied to almost any extent. The fuel may thus be consumed in one hour in a large boiler feeding a large engine; or it may be consumed say in fifteen hours in a small boiler supplying a correspondingly smaller engine. The total energy is the same in both cases, but the power of the larger engine is said to be fifteen times greater than that of the smaller one, assuming that the efficiencies of the two engines and boilers are the same.

The unit of power is therefore expressed in unit of work in unit of time. Hence, in the C.G.S. system the unit of power is 1 *erg per sec.*

The dimensions of power are $\frac{Wh}{T} = [M]\,[L]^2\,[T]^{-3}$.

Energy may be expressed as power into time.

CHAPTER XXVI.

PRACTICAL UNITS.

172. Absolute systems.

It has been shown, in the preceding chapters of this part of the book, that the dimensions of all mechanical quantities are dependent, and dependent only, on mass, length, and time. Hence, when the question arises to invent a system of units it is obviously logical to settle on the units of mass, length, and time first of all. A system which is built up on this principle is called an *absolute system*.

The *C.G.S.* (centimetre, gramme, second) system is an absolute system, and is the only international system of units. Some of its units have already been defined, but may, with advantage, be repeated here.

Unit of mass is called a *gramme* (gm.), and is 0·001 of the mass (at 0° C.) of a standard piece of platinum kept at Paris to represent the mass of a kilogramme (kg.). It was, however, originally defined as the mass of a cubic centimetre of pure water at its highest density.

Unit of length is called a *centimetre* (cm.), and is 0·01 of the distance (at 0° C.) between two marks on a standard bar kept at Paris. Originally, however, a centimetre was defined as 10^{-9} of an earth-quadrant measured on an arc of 90° of the meridian.

Unit of time is called a *second* (sec.), and is 1/86164 of the time it takes the earth to make one revolution about its axis.

Unit of velocity is 1 *cm. per sec.* (1 cm./s.).

Unit of acceleration is 1 *cm. per sec. per sec.* (1 cm./s.^{2}).

Unit of force is called a *dyne*, and is that force which, acting on the mass of one gramme, produces an acceleration of 1 cm. per sec. per sec.

Unit of energy is called an *erg* and is, the amount of energy which an effort of 1 dyne exerts when travelling through the distance of 1 cm.

Unit of power is 1 *erg per sec.*

Derived units :

Unit of energy = 10^7 *ergs* is called a *joule.*

Unit of power = 10^7 *ergs per sec.* = 1 *joule per sec.* is called a *watt.*

10 metres = 1000 cms.	is called	1 deca-metre.
100 ,,	,,	1 hecto-metre.
1000 ,,	,,	1 kilo-metre.
0·1 of a metre	,,	1 deci-metre.
0·01 ,,	,,	1 centi-metre.
0·001 ,,	,,	1 milli-metre.
1000 dynes	,,	1 kilo-dyne.
10^6 ,,	,,	1 mega-dyne.

173. Engineering systems.

Engineers do not often use the C.G.S. units, the reason being the difficulty of measuring forces readily in terms of a dyne. Apparatus such as a spring-balance or a safety-valve can be calibrated and tested in the most simple manner, by the use of ordinary weights. Hence, the fundamental units in the engineering systems are: unit of force, unit of length, and unit of time. A system of units built up on this principle is also called a *gravitation system.*

A. British engineering system.

Unit of force is the weight of 1 pound, or 1 *pound-weight* (lbw.), and is the force which balances the gravitation pull on 1 pound-mass (lb.) at the sea-level at Greenwich.

Unit of length = 1 *foot*, which is the distance (at 0° C.) between two marks on a standard bar kept at the Board of Trade.

Unit of time = 1 *sec.*

Unit of velocity = 1 *foot per sec.* (1 ft./s.).

Unit of acceleration = 1 *foot per sec. per sec.* (1 ft./s.^{2}).

Unit of mass = the mass of 32·187 *lbs.* (See note below.)

Unit of energy = 1 *foot-pound* (ft.-lbw.), *i.e.* the energy exerted by an effort of 1 pound-weight when travelling through a distance of 1 foot.

Unit of power = 1 *ft.-lbw. per sec.*

Derived units :

A *horse-power* (H.P.) is a unit of power, and is, = 550 *ft.-lbws. per sec.* = 33,000 *ft.-lbws. per min.*

1 *mile per hour* = 22/15 *feet per sec.*; hence, 1 *foot per sec.* = 15/22 *mile per hour.*

A *ton-weight* (tnw.) is often used as *unit of force.*

B. French engineering system.

Unit of force is the *weight of* 1 *kilogramme*, or 1 *kilogramme-*

weight (kgw.); it is the force which balances the gravitation-pull on the mass of 1 kilogramme (kg.) at the sea-level at Paris.

Unit of length = 1 *metre.*

Unit of time = 1 *sec.*

Unit of velocity = 1 *metre per sec.* (1 m./s.).

Unit of acceleration = 1 *metre per sec. per sec.* (1 m./s.^{2}).

Unit of mass = the mass of 9·8087 *kilogrammes.* (See note below.)

Unity of energy is called 1 *kilogramme-metre* (kgwm.), and is the energy exerted by an effort of 1 kgw. when travelling through the distance of 1 metre.

Unit of power = 1 *kgwm. per sec.*; also a *French horse-power* (F.H.P.) = 75 *kgwms. per sec.*

NOTE.—If unit of force is equal to the weight of the standard mass at a standard place, where the acceleration due to gravity is g units, then unit of mass m in terms of the standard mass, is

$$m = \frac{g \text{ units of acceleration}}{\text{unit of acceleration}} \text{ standard mass.}$$

The acceleration due to gravity is always denoted by the letter "g." At Greenwich, $g = 32{\cdot}187$ feet/sec^2. Hence, the British engineering unit of mass = the mass of 32·187 lbs.

At Paris, $g = 9{\cdot}8087$ metres/sec^2. Hence, the French engineering unit of mass = the mass of 9·8087 kgs.

174. Relation between the units of the different systems.

Mass.—1 kg. = 1000 gms. = 2·20463 lbs.; 1 lb. = 0·45359 kg. = 453·59 gms.

Length.—1 ft. = 0·3048 metre = 30·48 cms.; 1 metre = 3·2809 ft. = 100 cms.; 1 cm. = 0·3937 inch; 1 inch = 2·54 cms. = 25·4 millimetres.

Area.—1 sq. ft. (ft.^{2}) = 0·092904 sq. metre (m.^{2}); 1 sq. metre = 10·764 sq. ft.; 1 sq. cm. (cm.^{2}) = 0·155 sq. inch (inch^2); 1 sq. inch. = 6·452 sq. cms.

Volume.—1 cubic metre (m.^{3}) = 35·314 cubic ft. (ft.^{3}); 1 cubic ft. = 0·028317 cubic metre.

Field-length.—1 kilometre = 0·62138 mile; 1 mile = 5280 ft. = 1·6093 kilometres.

$$\textit{Force.}\text{—1 kgw.} = \frac{2{\cdot}20463 \times 9{\cdot}8087 \times 3{\cdot}2809}{1 \times 32{\cdot}187} = 2{\cdot}20460 \text{ lbw.}$$

$$= \frac{1000 \times 980{\cdot}87}{1 \times 1} = 980{,}870 \text{ dynes.}$$

$$1 \text{ lbw.} = 0{\cdot}4536 \text{ kgw.} = 444{,}995 \text{ dynes.}$$

Work and energy.—1 ft.-lbw. = 0·1382 kgwm. = 13,560,000 ergs = 1·356 joules.

1 kgwm. = 7·233 ft.-lbws. = 98,087,000 ergs = 9·8087 joules.

Power.—1 British H.P. = 550 ft.-lbws. per sec. = 76 kgwms. per sec. = 1·014 F.H.P. = 746 watts.

1 F.H.P. = 75 kgwms. per sec. = 0·986 British H.P. = 736 watts = 542·5 ft.-lbws. per sec.

CHAPTER XXVII.

MOTION UNDER GRAVITY.

175. Acceleration due to gravity.

The weight of a body with mass m is $W = mg$. If the earth did not revolve about its axis, g would only depend on the distance from the centre of the earth, and would be inversely proportional to the square of this distance. Let R be the radius of the earth and d the distance of the mass from the surface of the earth, then $g \propto 1/(R+d)^2$. In practice, however, d is always small compared with R and may therefore be neglected; the magnitude of g may thus be taken as being independent of d.

g, however, varies with the latitude of the place of observation, a fact which is due to the rotation of the earth about its axis. The body is forced to describe a circle whose radius at the equator is equal to that of the earth, and at the pole is zero. The circular motion requires a normal force; hence, part of the gravitation-force is applied as a normal force. g is therefore smallest at the equator and greatest at the pole, the ratio being about 978 : 983.

176. Vertical motion under gravity.

Neglecting any resistance due to the atmosphere, a particle, which is let fall, will describe a straight vertical line and will move with a constant acceleration g. The weight of the particle is an effort; hence, the total acceleration is tangential and $g = dv/dt$. As g is constant the laws of motion of the particle are found by substituting g for a in article **163**. Hence,

$$v = v_i + gt;\ s = v_i t + 0{\cdot}5gt^2;\ v_m = (v + v_i)/2;$$
$$v = \sqrt{v_i^2 + 2gs};\ t = (v - v_i)/g. \qquad [150]$$

Let us next consider that the particle is projected vertically upwards starting with a velocity v_i. The force of gravity (weight) will act on the particle as a resistance and the velocity will gradually diminish. The laws of motion of the particle are obtained by substituting $-g$ for g in all the formulæ in **[150]**. At the highest point the velocity v is zero; hence, the greatest height, h, attained is found by $0 = v_i^2 - 2gh$, or

$$h = \frac{v_i^2}{2g} \quad . \quad . \quad . \quad . \quad [151]$$

[151] may also be found graphically. In fig. 71 tan D C A is numerically equal to $-g$, O A $= v_i$. v is zero at the end of time-

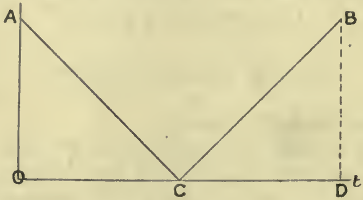


FIG. 71.

interval T = O C; hence, at that moment the greatest height h has been reached, h = area C O A C = $0·5\ \overline{OA}\ \overline{OC} = v_i^2/2g$. The particle starts now on its downward path, and when C D = O C the particle will be at the starting-point, having attained a velocity v_i and has fallen through the height h = area C D B C.

177. Inclined plane.

A plane surface which is not horizontal is called an inclined plane. The *line of greatest steepness* of an inclined plane is the

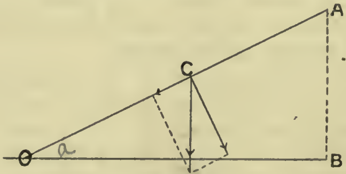


FIG. 72.

path which a smooth body would describe when allowed to move freely down the smooth plane; this path is a straight line making a right angle with the line of intersection of the inclined plane and the horizontal plane. The angle which the line of greatest steepness makes with the horizontal plane is called the *angle of inclination* (α), and the steepness of the plane is measured by tan α, called the *gradient*. The level from which the inclined plane rises is called the *datum-level*. In fig. 72 O A is the line of greatest steepness, O B is the datum-line which is horizontal, and angle B O A is the angle of inclination.

The acceleration, due to gravity, down the inclined plane is found by resolving g into two resolutes, one in the direction of A O and the other in the direction perpendicular on A O. The former is $g \sin \alpha$, *the tangential acceleration*, and the latter is $g \cos \alpha$, *the normal acceleration.* Hence, if the mass of the particle C be m, and its weight be $W = mg$, the force by which the particle is pulled downwards is $P = W \sin \alpha$ and the pressure of W on the plane is $Q = W \cos \alpha$.

The motion of a particle on a smooth inclined plane is determined by substituting $g \sin \alpha$ or $-g \sin \alpha$ for g in the formulæ of the preceding article according as the motion is down or up the plane.

$W \sin \alpha$ is the effort urging the particle down the plane, and $-W \sin \alpha$ is the resistance opposing the motion up the plane. The energy exerted by $W \sin \alpha$ is $W \sin \alpha \, \overline{AO} = W \, \overline{AB} = Wh$. Hence, the energy exerted by gravity while the particle moves down the plane, and the work done against gravity while moving the particle up the plane, are of the same magnitude as when the particle moves through the vertical height of the plane.

178. The speed due to gravity is independent of the path.

Let A B (fig. 73) be the path which the particle must describe,

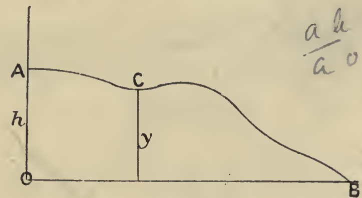


FIG. 73.

and let it arrive at C with a speed u, and leave the element δs with a speed $(u + \delta u)$. If α be the angle of slope at point C, then the energy equation is

$$W \sin \alpha \, \delta s = \frac{W[(u + \delta u)^2 - u^2]}{2g} \quad . \quad . \quad . \quad (1)$$

but $\delta s \sin \alpha = -\delta y$. In the limit (1) becomes $u(du/dy) = -g$.

Hence, $$\int_{u_i}^{u} u \, du = -g \int_{h}^{y} dy, \quad \text{or} \quad u^2 = u_i^2 + 2g(h - y) \quad . \quad [152]$$

u is thus independent of the path, and is equal to the speed which the particle would have attained in falling through the height

$h - y$. Multiplying [**152**] by the weight, W, of the particle, we get

$$\frac{W}{g}\frac{u^2 - u_i^2}{2} = W(h - y) \qquad . \quad . \quad . \quad (2)$$

The left-hand side of (2) is the change of kinetic energy of the particle in describing path A C, and the right-hand side of (2) is the energy exerted by W in falling through height $(h - y)$, or the work done in raising W through $h - y$ as the case may be. Hence, *energy exerted by gravity and work done against gravity are independent of the path described by the particle.*

The laws of motion due to gravity which have been developed in the present and preceding articles, can only be applied to a particle or to a body placed in vacuum. A body moving in the atmosphere is subjected to air-resistance, which increases with the square of the speed of the body and even with the cube of the speed, when the latter is very high. A bullet and a feather, when dropped simultaneously from the same level, will not reach the ground at the same time, the feather having a much larger surface exposed to the air than the bullet for the same weight. The formulæ, which we have just found, can therefore only be applied in practical cases when the body is moving at a moderately slow rate.

179. Projectiles in vacuum.

We will now consider the motion of a particle projected in a direction which makes an angle α with the horizontal plane. In

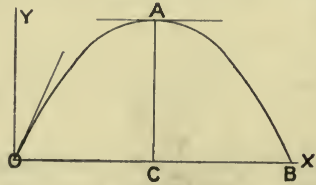


FIG. 74.

fig. 74 point O is the point of projection, the path O A B, which is described by the particle, is called the *trajectory*, the horizontal displacement O B of the projectile is known as the *range*, and the time occupied in describing the path is called the *time of flight* (T).

The determination of the trajectory is most difficult unless we may neglect all disturbances due to the air. Hence, we shall consider that the projectile moves in a vacuum. The only force which acts upon the particle is its weight; hence, the motion will take place in the vertical plane which contains the direction of the initial velocity (v_i). The displacement of the particle at any time may be resolved into a horizontal component x, and a vertical component y.

(i.) *The horizontal motion.*

$$\text{Velocity, } v_x = v_i \cos \alpha \quad . \quad . \quad . \quad (1)$$

$$\text{Displacement, } x = v_i \cos \alpha \, t \quad . \quad . \quad . \quad (2)$$

As v_x is constant the motion will be uniform, and the range will be equal to $v_i \cos \alpha \, T$.

(ii.) *The vertical motion.*

$$\text{Velocity, } v_y = v_i \sin \alpha - gt \quad . \quad . \quad . \quad (3)$$

$$\text{Displacement, } y = v_i \sin \alpha \, t - \tfrac{1}{2} g t^2 \quad . \quad . \quad (4)$$

The vertical motion is therefore the same as when the particle is projected vertically upwards with an initial velocity $v_i \sin \alpha$ (see article **176**); it will therefore attain a maximum height $CA = h = (v_i \sin \alpha)^2 / 2g$. By eliminating t between (2) and (4) we obtain the equation of the trajectory,

$$y = x \tan \alpha - \frac{g}{2 v_i^2 \cos^2 \alpha} x^2 \quad . \quad . \quad . \quad (5)$$

y is zero at points O and B; therefore, $y = 0$ in (4) gives $t = 0$ and $t = T$. Hence, the time of flight,

$$T = \frac{2 v_i \sin \alpha}{g} \quad . \quad . \quad . \quad . \quad (6)$$

$$\text{and the range, } OB = \frac{v_i^2 \sin 2\alpha}{g} \quad . \quad . \quad (7)$$

For a given velocity v_i the range is maximum for $\alpha = 45°$. As $\sin 2(45 + \beta) = \sin 2(45 - \beta)$ it follows that, for a given range, there are two directions of projection which are equally inclined to the direction of maximum range.

The tangent to the trajectory at point A is horizontal, because the vertical component of the velocity is zero. The ordinate to the latter point is h, and its abscissa is determined by substituting the value of h for y in (5).

Hence, the highest point is

$$A\left(\frac{v_i^2 \sin 2\alpha}{2g}, \frac{v_i^2 \sin^2 \alpha}{2g}\right) \quad . \quad . \quad . \quad (8)$$

Let it be required to refer the equation of the trajectory to point A as origin and to a pair of axes parallel to the given ones. By article **5** we have

$$x = \frac{v_i^2 \sin 2\alpha}{2g} + x_1, \quad \text{and} \quad y = \frac{v_i^2 \sin^2 \alpha}{2g} - y_1 \qquad . \quad (9)$$

Inserting the values of (9) in (5) we obtain the equation of the trajectory referred to the new system of co-ordinates

$$x_1^2 = \frac{2v_i^2 \cos^2 \alpha}{g} y_1 \quad . \quad . \quad . \quad . \quad (10)$$

which is a parabola whose vertex is point A, and whose axis is the vertical line A C.

The trajectory is thus symmetrical w.r.t. the straight line C A.

CHAPTER XXVIII.

FRICTION.

180. Friction.

Experience shows that a body may rest on an inclined plane although, apparently, it is perfectly free to move. There must therefore be a force which prevents the sliding of the body, and which the force $W \sin \alpha$ cannot overcome. If the inclined plane can be made to incline at any angle, such as a board, it will be found that by increasing the angle α sufficiently the body will begin to slide. If we now gradually diminish the angle of inclination, and at the same time tap the board gently, a value ϕ of α will be reached, when the body just ceases to slide. The angle ϕ is called the *angle of repose*, and the force resisting the sliding is called the *force of friction*; it is obvious that the force of friction is equal to $W \sin \phi$.

The force of friction is therefore a force which resists the sliding of two bodies on each other at their surfaces of contact (bearing-surfaces).

The surface of a body is never perfectly smooth, although it may appear to be so, since, by examining it with a magnifying glass, it will be found to be more or less undulated. To overcome friction, therefore, consists in lifting the sliding body over the undulations.

The degree of smoothness which can be given to a surface depends on the material; thus, the surface of a metal can be made smoother than the surface of leather. By using a lubricant, which will adhere to the surfaces and which can be squeezed into and fill up the undulations, the smoothness of the bearing-surfaces will be greatly increased.

As long as the force P, by which the two sliding bodies (not perfectly rigid) are pressed together, is so small that it does not cause any indentations in the surfaces of the bodies, but leaves them in their original condition, then the force of friction has been found to be proportional to P and independent of the area of the bearing-surfaces and the speed of the motion. Hence, in this case we have

$$\text{the force of friction} = \mu P \quad . \quad . \quad . \quad [153]$$

where μ is a factor called the *coefficient of friction*, whose value depends on the state of the bearing-surfaces as to smoothness and lubrication.

If P be so great that it causes the bodies to grind into each other, then μ will increase with P. It is, therefore, of great importance in constructing machinery to make the bearing-surfaces large enough, so as to prevent P from producing excessive friction and injuring the bearing-surfaces.

We have seen that the force, which is to overcome the force of friction on an inclined plane, is $W \sin\phi$, and as P in [153] is $= W \cos\phi$, we have

$$W \sin\phi = \mu W \cos\phi, \quad \text{or} \quad \tan\phi = \mu \quad . \quad . \quad [154]$$

hence, the coefficient of friction is equal to the tangent of the angle of repose.

Friction must not be confounded with adhesion; the latter is a force which resists the separation of two bodies at their surfaces of contact without the bodies being pressed together. Adhesion increases with the area of the surfaces of contact; hence, lubrication increases adhesion, but diminishes friction. At small pressures between the surfaces of contact adhesion may be greater than friction, but with great pressures the former is very small in comparison with the latter.

While two surfaces of contact—of bodies which are not perfectly rigid—are at relative rest the undulations will have time to fit better into each other, and the lubricant, if there is any, will gradually be squeezed out between the surfaces. For these reasons the friction between two surfaces at the moment of starting the sliding motion is sensibly greater than that between the same two surfaces while sliding on each other. The excess, however, of the *friction of rest or static friction* over the *friction of motion or kinetic friction* is soon destroyed by vibration.

181. Motion on a rough surface.

Let P be a particle which can slide on the rough surface S S′ (fig. 75). Let P A be the normal and P B the tangent-plane to the surface at the given point. Draw the straight line P C, making an angle ϕ (the angle of repose) with the normal, and let P C revolve about the normal, thus describing a cone of revolution called the *cone of friction*.

Let R, which makes an angle α with the normal, be the resultant reaction between the particle and the surface SS'. Resolve R along the normal and the tangent, the two components being $N = R\cos\alpha$ and $T = R\sin\alpha$ respectively. If there were no friction there would be acceleration, when T is greater than zero; but the surface is rough; hence, there will be acceleration when the force of friction is less than T, *i.e.* when $R\sin\alpha > R\cos\alpha\tan\phi$ or $\alpha > \phi$. Hence, the motion of the particle can only be continued when α is at least equal to ϕ, or when R falls on or outside the surface of the cone of friction. If R falls inside the cone no motion of the particle can take place. If,

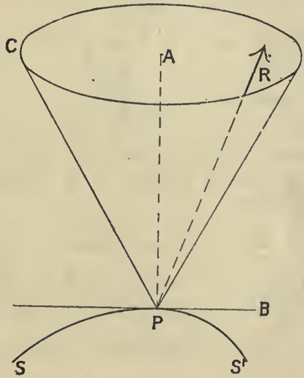


FIG. 75.

therefore, the particle be at rest, it will remain at rest until $\alpha > \phi$, or R falls outside the cone of friction.

182. Motion on a rough inclined plane.

The pressure on the plane (fig. 72) due to gravity is $W\cos\alpha$; hence, the force of friction is $W\cos\alpha\tan\phi$, which resists the motion of the particle, whether the motion is down or up the plane.

(i.) When the motion is down the plane the resultant force is $W\sin\alpha - W\cos\alpha\tan\phi = W\cos\alpha(\tan\alpha - \tan\phi)$. Hence, the acceleration, which is constant, is

$$g\cos\alpha(\tan\alpha - \tan\phi) \quad . \quad . \quad . \quad (1)$$

By substituting (1) for g in article **176** we obtain all the formulæ for the motion. If α is greater than ϕ in (1) there will be acceleration, and the velocity of the particle will increase as it moves down the plane. For $\alpha = \phi$ there is no acceleration, and the motion is uniform. When $\alpha < \phi$ the motion will be retarded, and the particle may come to rest before reaching point O. The velocity v of the particle is by article **176**

$$v = \sqrt{v_i^2 + 2g\cos\alpha(\tan\alpha - \tan\phi)s} \quad . \quad . \quad (2)$$

where s is the displacement from A along A O, v is zero when $v_i^2 + 2g\cos\alpha(\tan\alpha - \tan\phi)s = 0$, which can only take place when $\alpha < \phi$. Hence, in the latter case the particle will come to rest when

$$s = \frac{v_i^2}{2g\cos\alpha(\tan\phi - \tan\alpha)},$$

or when the particle has descended height $h - y = s\sin\alpha$.

Hence, $$y = h - \frac{v_i^2}{2g}\,\frac{\tan\alpha}{\tan\phi - \tan\alpha} \quad . \quad . \quad . \quad (3)$$

(ii.) When the particle is projected along the surface of the plane from point O the resultant resistance is $W\sin\alpha + W\cos\alpha\tan\phi$. Hence, the acceleration is

$$-g\cos\alpha(\tan\alpha + \tan\phi) \quad . \quad . \quad . \quad (4)$$

By substituting (4) for g in article **176** we obtain the formulæ for the motion. The velocity, v, of the particle is

$$v = \sqrt{v_i^2 - 2g\cos\alpha(\tan\alpha + \tan\phi)s} \quad . \quad . \quad (5)$$

where s is the displacement along O A from O. The particle will come to rest when $v_i^2 = 2g\cos\alpha(\tan\alpha + \tan\phi)s$, or

$$s = \frac{v_i^2}{2g}\,\frac{1}{\cos\alpha(\tan\alpha + \tan\phi)}, \quad \text{and} \quad y = \frac{v_i^2}{2g}\,\frac{\tan\alpha}{\tan\alpha + \tan\phi} \quad (6)$$

(iii.) The work done in raising a particle a distance l up a rough inclined plane is obviously

$$(W\sin\alpha + W\cos\alpha\tan\phi)l = Wl\sin\alpha + Wl\cos\alpha\tan\phi \quad (7)$$

but $l\sin\alpha$ is the height h of the plane, and $l\cos\alpha$ is the base b.

Hence, $$\text{the work done} = Wh + Wb\tan\phi \quad . \quad . \quad [155]$$

or: *The work done in raising a body up a rough inclined plane is equal to the work done in raising the body through the vertical height of the plane* plus *the work done in moving the body along the base, the latter being of the same roughness as the plane.*

CHAPTER XXIX.

EXAMPLES.

(60) (Q. Nov. 1907).—**State the law of composition of vectors. The wind blows from a direction 30° south of east at 5 miles an hour, and a flag on the mast of a ship, which is sailing due north, points in a direction 30° west of south; calculate the speed of the ship.**

Solution.—The first part of the question follows from Chap. XXII. The triangle of vectors can easily be constructed. The wind-vector and the ship-vector meet at an angle of 60°, and the flag-vector and the wind-vector intersect at an angle of 90°. Hence the triangle of vectors is a right-angled triangle, of which the one side, the wind-vector, is given. The velocity v of the ship is, therefore, found by $v \cos 60^\circ = 5$, or

$$v = 10 \text{ miles an hour } (\textit{Ans.}).$$

(61) (Q. Nov. 1908).—**If two vectors A and B are given, show how to find the vector sum A + B, and the vector difference A − B. Calling the sum S, and the difference D, show further from your construction that the vector sum, S + D, is equal to vector 2A, and that the vector difference, S − D, is equal to the vector 2B.**

If A be represented by a straight line 4 inches long, making an angle of 30° with a fixed line O X, and B by a straight line 3 inches long, making an angle of 90° with O X, (OX, A and B all lying in one plane), find the values of S and D by a construction and verify by calculation.

Solution.—Draw O*a* and O*b* (fig. 76) equal to A and B respectively; *ba* is then the vector difference D. Complete parallelogram O*a*C*b*O, then O C is the vector sum S. Draw O*d* equal and parallel to *ba*, then *d*C is the vector S − D which is equal to 2B because the two triangles *a*O*d* and C*ba* are equal in every respect. Complete parallelogram O*de*CO, then O*e* is the vector S + D, which is equal to 2A since Oe and dC are the diagonals of the parallelogram OdeCO.

Let angle $a\text{O}b = \alpha$, then we have

$$S = \sqrt{A^2 + B^2 + 2AB\cos\alpha}, \quad \text{and} \quad D = \sqrt{A^2 + B^2 - 2AB\cos\alpha}.$$

Inserting the given numerical values of A, B and α, we get

$$S = 6{\cdot}08 \text{ ins.}, \quad \text{and} \quad D = 3{\cdot}61 \text{ ins. } (Ans.).$$

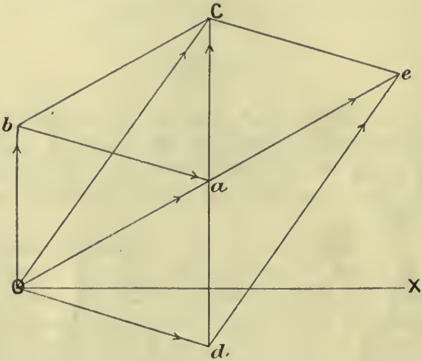


FIG. 76.

(62) (Q. Oct. 1909).—**A battleship is steaming due north at 12 knots and is observed by a submarine which is 5 sea miles north-west of her. The latter closes to attack at 9 knots; what course should she steer to close as quickly as possible, and how long time will elapse before the vessels are within 2000 yards of one another? 1 knot = 1 sea mile per hour, and the sea mile may be taken as 2000 yards.**

Solution.—Let A and B (fig. 77) be the positions of the submarine and the battleship respectively, B A being 5 sea miles. Let us find the motion of B relative to A. If B C had been zero the two ships would have collided; hence in the present case the relative velocity v_r must be in the direction B A.

Graphical Method.—Set off B C = 12 in the northerly direction, and from C draw a line C D F parallel to A B. With centre at B draw a circular arc with radius = 9, then B F is the course which the submarine must steer in order to close as quickly as possible. If it were to take the course B D it would take more time, as $v_r = $ F C is greater than $v_r = $ D C.

Calculation.—From the triangle of velocities we have

$$9^2 = 12^2 + v_r^2 - 2 \times 12 \times v_r \cos 45°, \quad \text{or} \quad v_r = \begin{Bmatrix} 11{\cdot}49 \\ 5{\cdot}49 \end{Bmatrix} \text{ knots};$$

we must take $v_r = 11{\cdot}49$ knots. The two vessels will be within 2000 yards = 1 sea mile when the battleship has sailed 4 sea miles at the rate of 11·49 knots relative to the submarine.

Hence, time $t = \frac{4}{11{\cdot}49} = 20$ mins. 53 secs.

Let angle C B F $= \alpha$, $\frac{\sin \alpha}{\sin 45°} = \frac{11{\cdot}49}{9}$;

hence $\alpha = 64° \; 31'$ east of north, which is the course to be taken by the submarine.

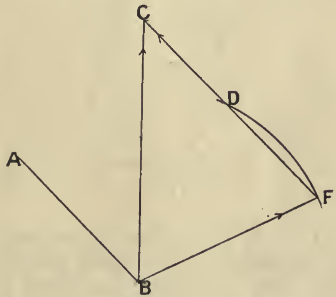


FIG. 77.

(63) (I.C.E., Oct. 1906).—A man, standing on a train which is moving with a speed of 36 miles per hour, shoots at an object running away from the railway at right angles at a speed of 12 miles per hour. If the bullet, which is supposed to move in a horizontal straight line, has a velocity of 880 feet per second, and if the line connecting man and object makes an angle of 45° with the train when he fires, find at what angle to the train he must aim in order to hit the object.

Solution.—A speed of 880 ft./sec. = a speed of 600 miles/hr. For the sake of generality let the velocities of the train, the object, and the bullet be respectively V_1, V_2, and V_3; let also V_r be the velocity of the object relative to the man and t be the time which must elapse from the moment the man fires till the bullet hits the object.

In fig. 78 O is the point from which the man fires and A is the point at which the object is seen. Set off $CA = V_1 t$ and $CB = V_2 t$, then $AB = V_r t$, and point B is thus the point where the object is hit. $OB = V_3 t$ is therefore the direction in which the man should fire. Let angle $CAB = \alpha$ and $BOA = \beta$, then

$$\frac{\sin\beta}{\sin(45^\circ+\alpha)} = \frac{AB}{OB} = \frac{V_r}{V_3}; \text{ hence, } \sin\beta = \frac{V_r}{V_3}(\cos\alpha + \sin\alpha) \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2}\frac{V_r}{V_3}\left(\frac{V_1}{V_r} + \frac{V_2}{V_r}\right) = \frac{\sqrt{2}}{2}\frac{V_1+V_2}{V_3} = 0{\cdot}05656.$$

Hence, $\beta = 3^\circ\ 15'$, and the angle to the train the man must aim is therefore $48^\circ\ 15'$ (*Ans.*).

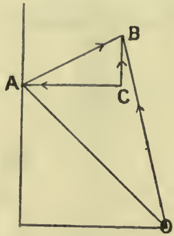


FIG. 78.

(64) (I.C.E., Feb. 1909).—**Two ships start simultaneously from two ports situated on the same meridian and 50 miles apart. The ship from the more northerly port steers south-west at 12 knots and the other steers due west at 15 knots. How near will the two ships approach each other, and at what time after departure will they be closest? 1 knot = 6080 feet per hour.**

Solution.—Let A B (fig. 79) be the meridian and A and B be the two ports, the latter being the more southerly one. Let v_r be the velocity of the ship starting from A as seen from the other ship. v_r is the vector difference of the velocities of the two ships.

Graphical Method.—Set off from a point D a velocity of 15 (D A) in a westerly direction, and a velocity of 12 (D F) in a south-westerly direction, then v_r is represented by A F. Produce A F and draw B C perpendicular to A C, then B C is the shortest distance between the two ships, and A C is the distance the ship from A must travel to reach the point where the two ships are closest together. If fig. 79 be drawn to scale, A B being 50 miles, then B C and A C will be determined in miles. v_r will be found in knots and must be reduced to miles per hour. The required time t in hours is equal to A C in miles divided by v_r in miles per hour.

Calculation.—Let angle D A F be α, then $AC = 50 \sin\alpha$ miles, and $BC = 50 \cos\alpha$ miles. $v_r^2 = 12^2 + 15^2 - 2 \times 12 \times 15 \cos 45°$; hence, $v_r = 10{\cdot}67$ knots $= 12{\cdot}29$ miles per hour. From the triangle of velocities we have

$$\frac{\sin\alpha}{\sin 45°} = \frac{12}{10{\cdot}67};$$

hence, $\alpha = 52°\ 41'$, and $AC = 39{\cdot}8$ miles, $BC = 30{\cdot}3$ miles, and $t = 3$ hrs. 14 mins. 7 secs.

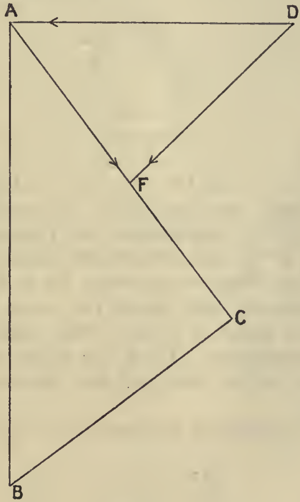


FIG. 79.

(65) (I.C.E., Oct. 1898).—A body of 30 lbs. moves towards south at 30 feet per second, in 2 minutes it moves towards the south-west at 40 feet per second; what is the added velocity? Find the average acceleration. What constant force would produce this change?

Solution.—The body is moving on a curve (fig. 80); at point A it is moving towards south with a velocity v_1, and at point B towards south-west with a velocity v_2. It takes the body 2 mins. = 120 secs. to move from A to B. Draw $OC = v_1$ and $OD = v_2$, then $CD = v_3$ is the added velocity. The average acceleration is $a_m = v_3/120$. v_3 may be found graphically or by calculation.

$$v_3^2 = v_1^2 + v_2^2 - 2v_1v_2 \cos 45°.$$

Hence, $v_3 = 28{\cdot}34$ ft./s. $a_m = 0.236$ ft./s.^{2},

and the constant force $= 0{\cdot}236 \times 30/g = 0{\cdot}22$ lbws.

The direction of the force is the same as that of v_3. Let angle

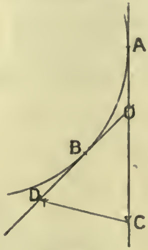


FIG. 80.

O C D be θ, then $\sin\theta : \sin 45° :: v_2 : v_3$. Hence, $\theta = 86° \ 24'$, or the direction of the force is west by $3° \ 36'$ north.

(66) (Q. June 1909).—**An electric car runs 440 yards between one stop and the next, and covers the distance in 84 seconds. At starting it is uniformly accelerated up to a speed of 12 miles per hour; it maintains this speed for some time, and is then retarded uniformly until it stops. The time of acceleration is twice that of retardation. Find the value of each of them in foot-second-second units, and find the distance the car runs at its highest speed.**

Solution.—The speed-time diagram of the car is shown in fig. 81.

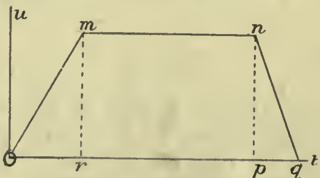


FIG. 81.

The time between stops $= \mathrm{T} = \mathrm{O}q$ (known); time of acceleration $= \mathrm{O}r = t_1$; time of highest speed $= rp = t_2$; time of retardation $= pq = t_3$; highest speed $= rm = u$ (known); distance between stops $= s =$ area $\mathrm{O}mnq\mathrm{O}$ (known); distance car runs at highest speed $= s_2 =$ area $rmnpr$.

$\mathrm{T} = t_1 + t_2 + t_3$; $t_3 = 2t_1$; $s = ut_1/2 + ut_2 + ut_3/2 = u(2\mathrm{T} - 3t_3)/2$. Hence, $t_3 = 2(u\mathrm{T} - s)/3u$; magnitude of acceleration $= u/t_1$; magnitude of retardation $= u/t_3$; $s_2 = ut_2 = (\mathrm{T} - 3t_3)u$.

The given values are: $s = 1320$ ft.; $u = 17{\cdot}6$ ft./s.; $T = 84$ secs. Hence,

$t_1 = 12$ secs.; $t_2 = 66$ secs.; $t_3 = 6$ secs.; magnitude of acceleration $= 1{\cdot}47$ ft./s.^{2}; magnitude of retardation $= 2{\cdot}93$ ft./s.^{2}; $s_2 = 1162$ ft. (*Ans.*).

(67) (Q. June 1908).—**A train passes a station, A, at 30 miles an hour, maintains this speed for $4\frac{1}{2}$ miles, and is then uniformly retarded, stopping at B, which is 5 miles from A. A second train starts from A at the instant the first train passes, and, being uniformly accelerated for part of the journey and uniformly retarded for the rest, reaches B at the same instant as the first train. What is the greatest speed on the journey?**

If the second train, after a certain uniform acceleration, runs at a constant speed for 1 mile and is then uniformly retarded, so that it reaches B with the first train; what is the value of the constant speed?

Solution—Question 1.—The speed-time diagrams of the two trains are shown in fig. 82. It takes the trains time $Oq = t_2$ to

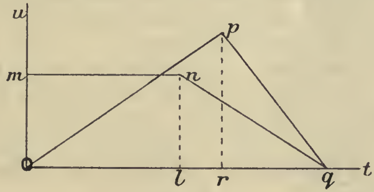


FIG. 82.

travel from A to B. $Omnq$ is the diagram of the first train. $Om = u_i$, $Ol = t_1$, area $OmnlO = s_1$ miles, and area $OmnqO = s$ miles. Opq is the diagram of the second train, area $OpqO = s$ miles and $rp = u_{max.}$ is the speed to be determined.

$$t_1 u_i = s_1\ ;\ (t_2 - t_1)u_i = 2(s - s_1). \quad \text{Hence, } t_2 = (2s - s_1)/u_i \qquad (1)$$

$$t_2 u_{max.} = 2s. \quad \text{Hence, } u_{max.} = 2su_i/(2s - s_1) \quad . \quad . \qquad (2)$$

The numerical values are: $u_i = 30$ miles/hr.; $s = 5$ miles; $s_1 = 4{\cdot}5$ miles.

Hence, $u_{max.} = 54{\cdot}55$ miles/hr. (*Ans.*).

Question 2.—$Omnq$ (fig. 81) is the speed-time diagram of the train, $mr = u_c$ is the constant speed to be determined, area $OmnqO = 5$ miles, $Oq = t_2$ and $rq = t_3$.

$$t_3 u_c = 1\ ;\ (t_2 - t_3)u_c = 2 \times 4 = 8\ ;$$

hence, $$u_c = \frac{9 \times 54{\cdot}55}{10} = 49{\cdot}1 \text{ miles/hr. } (Ans.).$$

(68) (Q. June 1909).—**The graph of the speed of a train during a 10 minutes' run is given below. Tabulate the values of the average acceleration or retardation during each minute and plot the acceleration-time curve, pointing out the significance of the abrupt change at the highest point of the speed curve. Also estimate the total distance run in the 10 minutes.**

Time in minutes.	0	1	2	3	4	5	6	7	8	9	10
Miles per hour.	0	10·00	18·05	25·00	28·95	31·25	19·15	11·40	5·00	1·60	0

Solution.—The student should plot the graph from this table. The scales are 1 min. = 1 cm., and 5 miles/hr. = 1 cm.; he may, of course, take 1 in. instead of 1 cm.

The average tangential acceleration during an interval of time t is $a_m = (u_f - u_i)/t$, where u_f and u_i are respectively the final and the initial speeds of the interval. If we take an hour as unit of time, the acceleration will be expressed in miles per hour per hour. The magnitude of the average acceleration is tabulated in the following table :—

Interval of one minute.	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Average acceleration in miles per hour per hour.	600	483	417	237	138	minus 726	minus 465	minus 384	minus 204	minus 96

Negative acceleration means retardation. During the first 5 minutes run the speed of the train has been accelerated and during the last 5 minutes the speed has been retarded. At the end of the former period the driver began to cut steam off or perhaps cut it off altogether; at any rate, the effective force pulling the train became suddenly negative, *i.e.* it became a resistance. If the effective force had remained constant the speed-time curve during retardation would have been a straight line, but the speed fell quicker. It is probable that the steam was cut off entirely and the brakes applied at the end of the first period. The distance run in the 10 minutes is represented by the area under the curve; for this purpose we may apply [98].

Hence, the distance run

$$= \tfrac{1}{60} \times (0 + 10{\cdot}00 + 18{\cdot}05 + 25{\cdot}00 + 28{\cdot}95 + 31{\cdot}25 + 19{\cdot}15 + 11{\cdot}40 + 5{\cdot}00 + 1{\cdot}60 + 0) = 2{\cdot}51 \text{ miles.}$$

(69).—A train starts from rest and moves in a straight line; the velocity, in miles per hour, at the end of successive minutes is 5·55, 9·82, 13·41, 16·58, 19·88, 23·78, 28·28, 33·08, 37·88; the acceleration varies without abrupt change during the first 7 minutes, and remains constant during the two next. Sketch velocity-time and acceleration-time diagrams for the first 9 minutes of the journey on the following scales :—

2 mins. = 1 in.; 10 miles an hour = 1 in.; 5 miles an hour per min. = 1 in. **Also find the distance travelled in 7 mins. from the start.**

Solution.—Draw the speed-time curve, then draw tangents to the latter curve at successive minutes. If α be the angle which the tangent forms with the time-axis, then $a = \tan \alpha$ if the scales for speed and time were the same, but in the present case $a = 5 \tan \alpha$. The values of a are tabulated in the following table :—

Minutes . .	0	1	2	3	4	5	6	7	8	9
Miles an hour per minute.	6·70	4·74	3·87	3·35	3·00	3·60	4·20	4·80	4·80	4·80

There is a point of inflexion on the speed-time curve at the end of the fourth minute, at which moment the value of the acceleration is a minimum. The acceleration-time curve (giving the magnitudes of the acceleration only) is a continuous curve until at the end of the seventh minute, when the curve suddenly changes into a horizontal straight line. This change is due to the driver suddenly changing the supply of steam to the cylinders.

We have now to find the distance s through which the train travels in the first 7 minutes. Bearing in mind that 1 mile an hour $= \tfrac{1}{60}$ mile a minute, and applying the trapezoidal rule [98], we have

$$s = \tfrac{1}{60}\left(\frac{0 + 28{\cdot}28}{2} + 5{\cdot}55 + 9{\cdot}82 + 13{\cdot}41 + 16{\cdot}58 + 19{\cdot}88 + 23{\cdot}78\right) = 1{\cdot}72 \text{ miles.}$$

(70).—A truck weighing 1000 lbws. is started from rest on a level track. The speed of the truck in terms of the time is recorded in the table below. If the frictional resistance to the motion of the truck be constant and equal to 10 lbws., find the force which is being applied in pulling the truck at the end of 5 seconds.

Estimate also how much work is done in the first 4 seconds in overcoming the frictional resistance.

Seconds . . .	0	1	2	3	4	5	6	7	8
Feet per second . .	0	1·66	2·93	4·00	4·94	5·80	6·58	7·30	8·00

The student should draw the speed-time curve, using the same scale for speed and time.

Solution.—The force applied in pulling the truck is F_e = the resistance *plus* the mass of the truck into the acceleration in the direction of the motion, or

$$F_e = 10 + \frac{1000}{g}a \quad . \quad . \quad . \quad . \quad (1)$$

$a = \tan \alpha$, where α is the angle which the tangent to the curve for time = 5 secs. makes with the time-axis. $\tan \alpha$ will be found to be 0·82. Hence, $a = 0·82$ ft./s.² Hence,

$$F_e = 10 + \frac{1000 \times 0·82}{g} = 35·5 \text{ lbws. } (\textit{Ans.} \text{ to Question 1}).$$

Let s be the distance through which the truck has moved in the first 4 seconds, then the work done in overcoming the frictional resistance has been $10s$ ft.-lbws. But s is the area enclosed by the curve, the time-axis, and the ordinate 4·94. Applying Simpson's rule [99], we have

$$s = [0 + 4·94 + 4(1·66 + 4·00) + 2 \times 2·93]\tfrac{1}{3} = 11·15 \text{ ft.}$$

Hence, work done = 111·5 ft.-lbws. (*Ans.* to Question 2).

(71) (I.C.E., Oct. 1903).—In an electric railway the average distance between stations is 0·5 mile, the running time from start to stop 1·5 minutes, and the constant speed between the end of acceleration and beginning of retardation 25 miles an hour. If the acceleration and retardation be taken as uniform and numerically equal, find their values; and if the weight of

the train be 150 tons and the frictional resistance 11 lbs. per ton, find the tractive force necessary to start on the level.

Solution.—In fig. 81 $Oq = 1{\cdot}5$ mins. $= 90$ secs.; area $OmnqO =$ 0·5 mile $= 2640$ ft.; $rm = pn = u = 25$ miles/hr. $= 36{\cdot}7$ ft./s.; angle $rOm =$ angle $pqn = \alpha$; $\tan\alpha =$ numerical values of acceleration and retardation; $Or = pq = t_1$; $rp = t_2$. We have

$$at_1 = u = 36{\cdot}7\,;\ 2640 = u(t_1 + t_2),\ \text{or}\ \ t_1 + t_2 = 72\,;\ 2t_1 + t_2 = 90 \quad (1)$$

Eliminating t_1 and t_2 between the three equations in (1), we get $a = 2{\cdot}04$ ft./s.^{2}.

The tractive force F to start the train on the level is

$$F = \frac{11 \times 150}{2240} + \frac{150 \times 2{\cdot}04}{g} = 10{\cdot}2 \text{ tnws. } (\textit{Ans.}).$$

(72) (Q. Nov. 1906).—**A body weighing 1 ton, starting with the velocity of 10 miles an hour, moves in a straight line, the power applied (tending to increase its velocity) being constant, namely one horse-power. Find the time that will elapse before the acceleration will be reduced to one-half of its initial value. Find also the ratio of the initial acceleration to that of gravity.**

Solution.—Let W be the weight of the body, then

$$\text{the power applied} = \frac{W}{g}av = \frac{W}{g}\frac{dv}{dt}v = K \quad . \quad . \quad (1)$$

where K is a constant. Hence,

$$\frac{a_i}{g} = \frac{K}{Wv_i} \quad . \quad . \quad . \quad . \quad (2)$$

as the final acceleration is $= a_i/2$ we have, $v_f = 2v_i$. Integrating (1) we get

$$\frac{W}{gK}\int_{v_i}^{v_f} v\,dv = \int_0^t dt. \quad \text{Hence, } t = \frac{(v_f^2 - v_i^2)W}{2gK} \quad . \quad (3)$$

where t is the required time.

(3) may be obtained direct from the equation: Power × time = change of kinetic energy, or

$$Kt = \frac{W}{g}\frac{(v_f^2 - v_i^2)}{2}.$$

The given numerical values are: $W = 2240$ lbws.; $K = 550$ ft.-lbws./s.; $v_i = 10$ miles/hr. $= 14{\cdot}67$ ft./s.

Hence, $\qquad a_i/g = 0{\cdot}0167\,;\ t = 40{\cdot}8\ (\textit{Ans.}).$

(73) (Q. June 1908).—**Two men exerting together a force of 90 lbs. weight put a railway waggon into motion. The waggon weighs 6 tons and the resistance to motion is 10 lbs. per ton. How far does the waggon advance in 1 minute; and at what rate, in horse-power, are the men working at the end of the minute?**

If the men can at most do work at the rate of 0·8 horse-power, at what constant speed can they keep the waggon moving?

Solution—Question 1.—We will assume that the waggon is to be moved on the level. Let the weight of the waggon be W tnw. and the force of friction be fW lbws. If the men exert a force of P lbws., then the effective force is $P-fW$ assuming that P is horizontal. We have

$$P-fW=a\frac{2240W}{g}\text{; hence, } a=\frac{g(P-fW)}{2240W}\text{ ft./s.}^2 \quad . \quad (1)$$

At the end of the t secs.,

$$s=\tfrac{1}{2}at^2=\frac{g(P-fW)}{4480W}t^2\text{ ft.} \quad . \quad . \quad . \quad (2)$$

and

$$u=at=\frac{g(P-fW)t}{2240W}\text{ ft./s.} \quad . \quad . \quad . \quad (3)$$

and the men work at the rate of

$$\frac{Pu}{550}=\frac{Pg(P-fW)t}{550\times 2240W}\text{H.P.} \quad . \quad . \quad . \quad (4)$$

The numerical values are: $P=90$ lbws.; $fW=60$ lbws.; $t=60$ secs.; $W=6$ tnws.

$s=128·6$ ft.; rate of work at the end of 1 min. $=0·7$ H.P. (*Ans.*).

Question 2.—Let the constant speed be $=u$ ft./s. As u is to be constant, a must be zero, *i.e.* $P=fW$.

Hence,
$$\frac{fWu}{550}=0·8,\quad\text{and}\quad u=\frac{440}{fW}.$$

$$u=7\tfrac{1}{3}\text{ft./s. } (\textit{Ans.}).$$

(74) (Q. Nov. 1907).—**A car weighing 1 ton starts from rest on a level road. The tractive force acting on it is initially 80 lbws., and this falls, the decrease being proportional to the distance travelled, until its value is 30 lbws., at the end of 200 yards, after which it remains constant. There is a constant**

frictional resistance of 30 lbws. Find the speed of the car at the end of 200 yards, and plot a curve, on a distance base, showing the gradual rise of the speed from the start.

Solution.—The ordinates to the straight line D F (fig. 83) represent the tractive force, and A C = B F represent the constant frictional resistance Q = 30 lbws. which acts against the motion and in the opposite direction of the tractive force. The car starts at point A and moves in direction A B. The initial tractive force is A D = P = 80 lbws., and A B = l = 200 yds. = 600 ft. The tractive force at distance x from B is qm. We may consider that the force $F_x = nm$ (the effective force) is the force which moves the car, and it is due to this force that the car gains speed. If u_x be

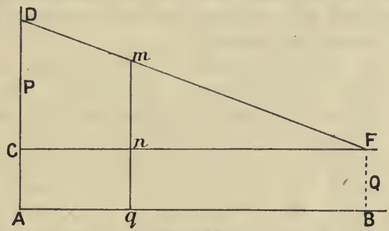


FIG. 83.

the speed when the car has moved through the distance $l - x$, and W = 2240 lbws. be the weight of the car, then the energy equation is

$$\text{Energy represented by area } \mathrm{CD}mn\mathrm{C} = \frac{W}{g}\frac{u_x^2}{2} \quad . \quad (1)$$

The diagram gives $F_x = x(P - Q)/l$. Hence, area $\mathrm{CD}mn\mathrm{C} = (F_x + P - Q)(l - x)/2 = (l^2 - x^2)(P - Q)/2l$, which, inserted in (1), gives

$$u_x^2 = \frac{l^2 - x^2}{l} g \frac{P - Q}{W} \quad . \quad . \quad . \quad (2)$$

We obtain the speed u_l at B by making $x = 0$ in (2).

Hence,
$$u_l^2 = gl\frac{P - Q}{W} \quad . \quad . \quad . \quad . \quad (3)$$

By (2) and (3) we get the equation of the required speed-distance curve, viz. :

$$\frac{x^2}{l^2} + \frac{u_x^2}{u_l^2} = 1 \quad . \quad . \quad . \quad . \quad (4)$$

(4) is the equation of an ellipse with centre at F and with l and u_l as horizontal and vertical axes respectively.

Inserting the given numerical values we get

$$u_l = 20\cdot8 \text{ ft./sec.}; \text{ equation of the ellipse } \frac{x^2}{36\times10^4}+\frac{u_x^2}{431}=1 \ (\textit{Ans.}).$$

The student should draw the ellipse.

(75) (Q. Oct. 1909).—**Given the initial velocity and the mass of the body, show how to find its velocity at any instant when the curve connecting force and distance travelled is given.**

A gun of 6-inch calibre fires a projectile weighing 100 lbs., the travel of the shell in the gun being 18 feet. If the curve of gas pressure in the bore be as in the figure, and if friction be neglected, calculate the velocity in feet per second with which the projectile leaves the muzzle.

Solution.—The following table is given here instead of the curve. The student should draw the curve. Horizontal scale 2 ft. = 1 cm., and vertical scale a pressure of 3 tnws. per inch² = 1 cm.

Shot travel in ft.	0	1	2	3	4	6	8	10	12	14	16	18
Pressure in tnws. per inch².	1·17	7·05	11·58	14·40	14·25	12·24	10·47	9·00	7·50	6·24	5·28	4·80

Question 1.—The area under the force-distance curve is $\int_{s=0}^{s=s} F_e ds$ which is the energy exerted by the effort while travelling through distance s. Let E_e be the latter energy, then

$$E_e = m(v^2 - v_i^2)/2, \quad \text{or} \quad v = \sqrt{v_i^2 + 2E_e/m}\,. \qquad (1)$$

Applying the trapezoidal rule [98] to calculate the area under the curve, we get

$$1\left(\frac{1\cdot17+14\cdot25}{2}+7\cdot05+11\cdot58+14\cdot40\right)$$
$$+2\left(\frac{14\cdot25+4\cdot80}{2}+12\cdot24+10\cdot47+9\cdot00+7\cdot50+6\cdot24+5\cdot28\right)$$
$$=161\cdot24 \text{ ft.-tnw. per inch}^2 \qquad (2)$$

As the calibre is 6 ins. the area exposed to the pressure of the gas is 28·27 inch². Hence, $E_e = 161\cdot24 \times 28\cdot27 \times 2240 = 10{,}210{,}500$ ft.-lbws. As $v_i = 0$ and $m = 100/g$, we have

$$v = 2564 \text{ ft./s. } (\textit{Ans.}).$$

(76) (Q. Nov. 1907).—**A particle, mass 10 lbs., moves uniformly in a horizontal circle of 3 feet, suspended by a cord of length 5 feet attached to a fixed point; calculate the velocity of the particle and the tension of the cord.**

Solution.—O B (fig. 84) is the cord of length l; A B is the radius r of the circle. Let angle A O B be θ and W be the weight of the particle. The circular motion of the particle requires a normal force F_n (here horizontal) which is furnished by gravity.

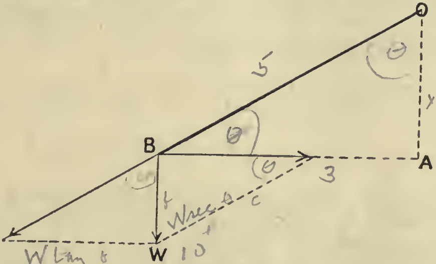


FIG. 84.

Resolve W into W $\tan\theta$ and W $\sec.\theta$; the former is F_n and the latter is the tension T in the cord. Hence, the magnitude of

$$F_n = W\tan\theta = \frac{u^2}{r}\frac{W}{g};\ T = W\sec.\theta;\ r = l\sin\theta \quad . \quad (1)$$

where u is the speed of the particle. Eliminating θ in (1) we get

$$u^2 = \frac{gr^2}{\sqrt{l^2 - r^2}};\ T = \frac{Wl}{\sqrt{l^2 - r^2}} \quad . \quad . \quad . \quad (2)$$

Inserting the given values in (2) we get

$$u = 8{\cdot}5 \text{ ft./sec.};\ T = 12{\cdot}5 \text{ lbws. } (Ans.).$$

(77) (I.C.E., Feb. 1908).—**A body of weight 2 tons rotates on rails in a vertical circle of radius 30 feet, without friction. Find the least speed at the highest point at which it will retain contact with the rails there, and find then the speed at the lowest point.**

Solution.—Let W be the weight of the body, r the radius of the circle, u_1 the least speed at the highest point, u_2 that at the lowest point, and $\tan\alpha$ the slope of the circle at any point.

The acceleration, g, of gravity may be resolved into $g\sin\alpha$ and $g\cos\alpha$. Hence, $a_n \geq g\cos\alpha$, which latter is maximum when cos $\alpha = 1$, or $\alpha = 0$, *i.e.* at the highest point of the vertical circle. Hence, for the latter point we have

$$a_n = \frac{u_1^2}{r} = g, \quad \text{or} \quad u_1 = \sqrt{rg} \quad . \quad . \quad . \quad (1)$$

The acceleration of the weight along the circle is $g \sin \alpha$, the same as on an inclined plane. Hence, the energy exerted by the weight while moving from the highest point to the lowest point of the circle is $W2r$. At the moment the weight reaches the lowest point the energy-equation is

$$W2r = \frac{W}{g}\frac{u_2^2 - u_1^2}{2}, \quad \text{or} \quad u_2 = \sqrt{5rg} \quad . \quad . \quad (2)$$

$$u_1 = 31{\cdot}1 \text{ ft./s.}; \; u_2 = 69{\cdot}5 \text{ ft./s. } (Ans.).$$

(78) (I.C.E., Oct. 1908).—**A large hollow sphere, of 20 feet in internal radius, is pivoted on a vertical axis and rotated at 20 revolutions per minute. To what portion on the inside surface would a ball roll if displaced from the lowest point of the surface? If the coefficient of friction between a wooden block and the inside surface is 0·2, at what minimum speed must the sphere revolve if the block is to remain at rest in the horizontal diametral plane?**

Solution—Question 1.—Let B (fig. 85) be the position of the

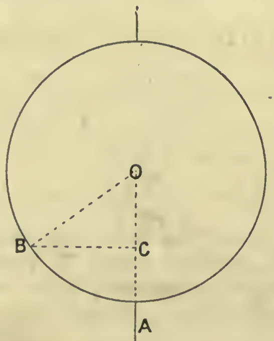


FIG. 85.

ball, A the lowest point of the sphere, $AC = y$, $CB = x$, angle $AOB = \alpha$, and ω the angular velocity of the sphere.

Neglecting friction, the normal acceleration at B, $a_n = x\omega^2$, must be equal to $g \tan \alpha$, g being resolved in the directions O B and B C.

Hence, $$a_n = x\omega^2 = g \tan \alpha = g\frac{x}{r-y}, \text{ or } y = \frac{r\omega^2 - g}{\omega^2} \quad . \quad (1)$$

$$r = 20\,;\; g = 32{\cdot}2\,;\; \omega = 2\pi/3 \text{ rads. per sec.}$$

Hence, $$y = 12{\cdot}66 \text{ ft. } (\textit{Ans.}).$$

Question 2.—The force of friction between the block and the inside surface of the sphere must be equal to the weight, W, of the block. Hence, $\frac{W}{g}r\omega^2\mu = W$, or $\omega = \sqrt{\frac{g}{r\mu}}$ = minimum speed of rotation.

$$g = 32{\cdot}2,\; r = 20,\; \mu = 0{\cdot}2\,;$$

hence, $$\omega = 2{\cdot}84 \text{ rads./s.} = 27{\cdot}12 \text{ revs./min. } (\textit{Ans.}).$$

(79) (I.C.E., Feb. 1906).—**A train is travelling at a uniform speed on the level. The weight of the brake-van at the rear of the train is 10 tons and the weight of the remaining part of the train is 90 tons. If the brakes are applied to the brake-van, what will be the force on the brake-van couplings, the coefficient of friction of the wheels on the rails being 0·1?**

Solution.—Let W_1 = the weight of the brake-van; W_2 = the weight of the remaining part of the train; $\mu = 0{\cdot}1$.

The retarding force of the brake-van when the brakes are applied is $= \mu W_1 = ma$, where $m = (W_1 + W_2)/g$ is the mass of the train and a is the retardation. Hence, $a = \frac{\mu W_1}{W_1 + W_2}g$, but the retarding force μW_1 must be communicated to the remaining part of the train through the couplings. Hence, the force on the couplings will be aW_2/g, we have therefore

$$a\frac{W_2}{g} = \frac{\mu W_1 W_2}{W_1 + W_2} \quad . \quad . \quad . \quad . \quad (1)$$

Inserting the given values in (1),

$$\text{force on couplings} = 0{\cdot}9 \text{ tnw. } (\textit{Ans.}).$$

(80) (I.C.E., Oct. 1901).—**A man weighing 140 lbs. stands on the floor of a lift. Find the pressure he exerts on the floor (*a*) when the lift ascends and descends with uniform velocity; (*b*) when it ascends with a velocity which decreases by the**

acceleration of 0·1 g; (c) when it descends with a velocity which increases at the rate of 8 feet per second per second. Under what conditions can the pressure be (i.) zero, (ii.) greater than the weight of the man?

Solution.—Let W be the weight of a body standing on the floor of the lift. The motion of the body will be unchanged if we remove the floor and add a force P equal to the pressure on the floor (fig. 86). If the acceleration of the body be a, we have

$$W - P = \frac{W}{g}a \quad . \quad . \quad . \quad . \quad (1)$$

When the velocity of the body is uniform, *i.e.* when the lift

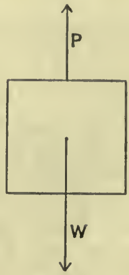


FIG. 86.

ascends and descends with uniform velocity, then $a = 0$. Hence, $P = W$.

When the sense of a is the same as that of gravity, *i.e.* when the lift descends with increasing velocity, or ascends with decreasing velocity, then a is positive, and $P = W - \frac{W}{g}a$ is less than the weight of the body.

When the sense of a is opposite to that of gravity, *i.e.* when the lift ascends with increasing velocity, or descends with decreasing velocity, then a is negative, and $P = W + \frac{W}{g}a$ is greater than the weight of the body.

In question (a), $P = W$; question (b), $P = W + \frac{W}{g} \times 0{\cdot}1g = 1{\cdot}1 \times 140 = 154$ lbws.; question (c), $P = W - \frac{W}{g} \times 8 = 0{\cdot}75W$ (about); question (i.) when $a = g$; question (ii.) when a is negative.

(81) (I.C.E., Oct. 1907). — **In a colliery winding-plant the weight of the cage and its load is 2·7 tons, and the rope is balanced. The depth of the shaft is 500 yards. The cage ascends with a uniform acceleration of 4·5 feet per second per second for 9 seconds; it then ascends at uniform speed, and at the top the retardation is also 4·5 feet per second per second for 9 seconds. Find the time taken to make a journey, and the tensions in the rope during acceleration and retardation.**

Solution.—Let W be the weight of the cage, T_1 and T_2 the tensions in the rope respectively during acceleration and retardation. Fig. 87 is the speed-time diagram, $AF = ED = t_1$ is the

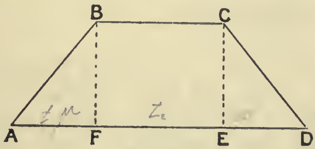


FIG. 87.

time of acceleration and retardation, $FE = t_2$ the time of constant speed u, and angle FAB = angle $EDC = \alpha$. We have

$$u = t_1 \tan\alpha,\ t_1^2 \tan\alpha + t_2 t_1 \tan\alpha = d \text{ (the depth of the shaft).}$$

Hence,
$$t_2 = \frac{d - t_1^2 \tan\alpha}{t_1 \tan\alpha},$$

and the time taken to make a journey, $t = 2t_1 + t_2$.

During acceleration the cage is urged on by a force $T_1 - W$ and with an acceleration $\tan\alpha$. During retardation the cage is urged on by a force $W - T_2$ and with a retardation $\tan\alpha$.

Hence,
$$T_1 - W = \frac{W}{g}\tan\alpha, \text{ and } W - T_2 = \frac{W}{g}\tan\alpha.$$

Hence,
$$T_1 = W + \frac{W}{g}\tan\alpha;\ T_2 = W - \frac{W}{g}\tan\alpha.$$

We have $t_1 = 9$; $W = 2{\cdot}7$; $g = 32{\cdot}2$, and $\tan\alpha = 4{\cdot}5$. Hence, $t = 46$ secs.; $T_1 = 3{\cdot}08$ tnws.; $T_2 = 2{\cdot}32$ tnws. (*Ans.*).

(82) (I.C.E., Oct. 1906).—**The final blow on a 9-inch diameter pile 20 feet long, from a weight of 1800 lbs., dropping 25 feet, drives it in 1 inch. Calculate the force of the blow, and the frictional resistance on the side of the pile, per square foot, if all the energy of the blow is thus spent.**

Solution.—The energy of the blow itself is equal to the energy exerted by the weight of 1800 lbs. falling through the height of 25 ft. The weight falls further through the height of 1 in. Hence,

The energy of the blow $= 1800 \times 25 = 45{,}000$ ft.-lbws.

The total energy spent in driving the pile $= 45{,}000 + \frac{1800}{12}$ $= 45{,}150$ ft.-lbws.

Let the total frictional force to be overcome be F_r, then the work done is $F_r \times \frac{1}{12}$ ft.-lbws. Hence, $F_r = 12 \times 45{,}150$ ft.-lbws.; the surface of the pile $= 20 \times 2{\cdot}356$ ft.^{2}

Hence, the frictional resistance per square foot of pile-surface $= \frac{12 \times 45{,}150}{20 \times 2{\cdot}356} = 11{,}500$ lbws. (*Ans.*).

(83) (I.C.E., Feb. 1904).—**In a steam hammer the diameter of the piston is 36 inches, the total weight of the hammer and piston is 20 tons, and the effective steam pressure is 40 lbs. per square inch. Find the acceleration with which the hammer descends, and its velocity after descending through a distance of 4 feet. If the hammer then comes into contact with the iron, and compresses it through a distance of 1 inch, find the mean force of compression.**

Solution.—The area of the piston $= 1017{\cdot}88$ inches^2; hence, the total effective pressure of the steam $= 18{\cdot}18$ tnws. The force moving the piston and hammer is therefore $18{\cdot}18 + 20 = 38{\cdot}18$ tnws.; the mass to be moved is $20/g$. Let a ft./s.^{2} be the acceleration with which the hammer descends, then

$$38{\cdot}18 = \frac{20}{g}a. \quad \text{Hence, } a = 61{\cdot}45 \text{ ft./s.}^2.$$

Let v ft./s. be the velocity of the piston and hammer after descending through a height of 4 ft., then

$$v = \sqrt{2 \times 61{\cdot}45 \times 4} = 22{\cdot}17 \text{ ft./s.}$$

The total energy spent in compressing the iron is equal to the energy exerted by the force of $38{\cdot}18$ tnws. travelling through the distance 4 ft. 1 in. If C be the mean force of compression in tnw., we have

$$38{\cdot}18 \times 4\tfrac{1}{12} = \tfrac{1}{12}\,C. \quad \text{Hence, } C = 1871 \text{ tnws. } (\textit{Ans.}).$$

(84) (I.C.E., Feb. 1907).—**Find the horse-power required to haul six 10-ton trucks up a gradient of 1 in 20 at $7\frac{1}{2}$ miles an hour. Calculate also how long an engine of this horse-power,**

weighing 25 tons and with the six trucks attached, would take to develop this speed on the level, the air and other resistances being 11 lbs. per ton in each case.

Solution.—The six trucks weigh together W tnw.; the speed is $u=11$ ft./s.; the weight of the engine is W_1tnw.; $\cos\alpha = 1/\sqrt{1+\tan^2\alpha} = 20/\sqrt{401}$; we can therefore take $\cos\alpha = 1$ and $\sin\alpha = 1:20$; the frictional resistances are therefore the same on the incline as on the level.

(1) *Up the gradient.*—The resistances to the motion are $2240W\sin\alpha$ and fW; the drawbar-pull, that is the effort, is therefore $F_e = 2240W\sin\alpha + fW$.

$$\text{Hence, H.P. required} = \frac{F_e u}{550} = \frac{(2240\sin\alpha + f)W}{550}u \quad . \quad . \quad (1)$$

or 147·7 H.P. (*Ans.*).

(2) *On the level.*—Let t be the required time. The mass to be removed is $2240(W+W_1)/g$; the resistance to the motion $=f(W+W_1)$ lbws.; we must assume that the indicator-diagram remains unchanged from the start until the train attains the speed $u=11$ ft./s. The effort F_e will thus be constant, and as the resistances are given to be constant, the acceleration will also remain constant. We have therefore $a=u/t$. Hence, $F_e u = 550\times 147{\cdot}6$, and the force-equation is

$$F_e = f(W+W_1) + \frac{2240(W+W_1)}{g}a$$

$$t = 10 \text{ secs. } (\textit{Ans.}).$$

(85) (I.C.E., Oct. 1906).—**Taking the resistance as 13 lbs. per ton, find the horse-power required to produce a speed of 40 miles per hour in a train weighing 300 tons in $3\frac{1}{2}$ minutes: (1) on the level; (2) down an incline of 1 in 320.**

Solution.—Let W be the weight of the train in tnw., u the speed in ft./s., t the time in secs., n the number of horse-power, and F_e the effort in lbws.

(1) *On the level.*—The frictional resistance is f lbws. per tnw. of W. Hence, the effective force is $F_e - fW$. As F_e is supposed to be constant the acceleration is also constant. Hence,

$$\frac{2240W}{g}a = F_e - fW;\ v = at;\ F_e v = 550n \quad . \quad . \quad (1)$$

Eliminating a and F_e between the three in (1), we get

$$n=\frac{fWv}{550}+\frac{2240Wv^2}{550gt} \quad . \quad . \quad . \quad (2)$$

Inserting the given values in (2) we have

$$n=1038 \text{ H.P. } (Ans.) \quad . \quad . \quad . \quad (3)$$

(2) *On the incline.*—As α is very small we may take $\cos\alpha=1$ and $\sin\alpha=\tan\alpha=1:320$. The frictional resistance is therefore the same as in the first case; but the effort to be developed by the engine is less by $2240W\sin\alpha$. Let n_1 be the number of horse-power developed by $2240W\sin\alpha$, then $2240W\sin\alpha\, v=550n_1$. Hence, the value of n in this case is equal to (3) *minus* n_1, or

$$n=1038-224=814 \text{ H.P. } (Ans.).$$

(86) (I.C.E., Oct. 1907).—**A train, weighing 150 tons and running at 30 miles an hour, has the steam cut off and the brakes applied at a certain point. The brakes would bring it to rest on the level in a distance of 300 yards, but it is on an incline of 1 in 100. At what distance would the train come to rest if running (*a*) up the incline, (*b*) down the incline?**

Solution.—Let W be the weight of the train in tnw.; u the speed in ft./s.; s_1 (known), s_2, and s_3 respectively the distances in feet on the level, up the incline and down the incline in which the train is brought to rest. The resistance F_r to the motion on the level consists of the resistance due to the brakes *plus* the frictional resistance. The latter is fW on the level and $fW\cos\alpha$ on the incline, but as α is very small we may take $\cos\alpha=1$ and $\sin\alpha=\tan\alpha=1:100$. Hence, F_r is the same in all three cases.

(1) *Level.*—The energy-equation is

$$0=F_r s_1+\frac{0-u^2}{2}\frac{W}{g}.$$

Hence,
$$F_r=\frac{Wu^2}{2gs_1}\text{tnw.}=5{\cdot}012 \text{ tnws.} \quad . \quad . \quad (1)$$

(2) *Up the incline.*—The resistance to the motion $=F_r+W\sin\alpha$. Hence, the energy-equation is

$$0=(F_r+W\sin\alpha)s_2-\frac{u^2W}{2g}, \quad \text{or} \quad s_2=\frac{Wu^2}{2g(F_r+W\sin\alpha)} \quad (2)$$

Hence,
$$s_2=693 \text{ ft. } (Ans.).$$

(3) *Down the incline.*—The resistance to the motion is F_r, but there is an effort $= W \sin \alpha$. Hence, the energy-equation is

$$W \sin \alpha \, s_3 = F_r s_3 - \frac{u^2}{2} \frac{W}{g}, \quad \text{or} \quad s_3 = \frac{W u^2}{2g(F_r - W \sin \alpha)} \qquad (3)$$

Hence, $s_3 = 1284$ ft. (*Ans.*).

(87) (Q. Nov. 1906).—**A smooth wedge weighing 5 lbs. can slide on a smooth horizontal plane. A weight of 1 lb. is placed on the sloping surface of the wedge, 1 foot from the bottom edge, and allowed to slide down. If the angle of the wedge is 30°, and if weight and wedge start from rest, prove that the weight reaches the bottom of the slope in about $\frac{1}{3}$ second.**

Solution.—The wedge will be displaced during the required interval of time t an amount $CD = s_3$ (fig. 88), and the weight

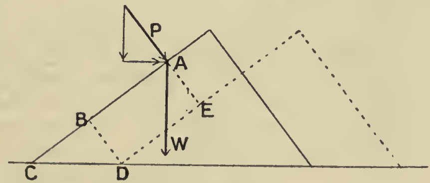


FIG. 88.

will describe the straight line AD. The latter may be resolved into displacement $AB = s_1$ along the sloping surface of the wedge and displacement $AE = s_2$ in the direction perpendicular to the latter surface. Let a_1, a_2, and a_3 denote respectively the accelerations in the directions of s_1, s_2, and s_3, then we have

$$s_1 = \tfrac{1}{2} a_1 t^2; \; s_2 = \tfrac{1}{2} a_2 t^2; \; s_3 = \tfrac{1}{2} a_3 t^2; \; s_2 = s_3 \sin \alpha \quad . \quad (1)$$

The components of the weight, W, are $W \sin \alpha$ and $W \cos \alpha$. A part of $W \cos \alpha$ is applied to produce the acceleration a_2, and the remainder produces a pressure P on the slope. Hence,

$$a_1 = g \sin \alpha; \; a_2 \frac{W}{g} + P = W \cos \alpha; \; a_3 M = P \sin \alpha; \; a_3 \sin \alpha = a_2 \quad (2)$$

where M is the mass of the wedge. Eliminating a_2 and P between the three last expressions in (2), we get

$$a_3 = \frac{gW \sin \alpha \cos \alpha}{Mg + W \sin^2 \alpha}, \quad \text{and} \quad s_3 = \frac{1}{2} \frac{gW \sin \alpha \cos \alpha}{Mg + W \sin^2 \alpha} t^2 \quad . \quad (3)$$

Let $AC = l$, then

$$s_1 = \tfrac{1}{2} g \sin\alpha\, t^2 = l - s_3 \cos\alpha = l - \frac{1}{2}\,\frac{gW \sin\alpha \cos^2\alpha}{Mg + W\sin^2\alpha} t^2,$$

and
$$t = \sqrt{\frac{2l}{g\sin\alpha}\,\frac{Mg + W\sin^2\alpha}{Mg + W}} \quad . \quad . \quad (4)$$

Inserting the given values in (4) we get

$$t = 0{\cdot}33\ (\textit{Ans.}).$$

(88) (I.C.E., Oct. 1898).—**A body of 50 lbs. is on an inclined plane of inclination 35°. The coefficient of friction is 0·16. A force x acts up the plane, making an angle of 10° with the plane (45° with the horizontal). Working either graphically or arithmetically, find x. First, when it just allows the body to slip down the plane; second, when it pulls the body up the plane (fig. 89).**

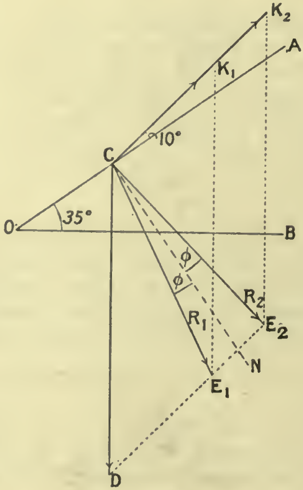


Fig. 89.

Graphical Solution. C D = the weight, W, of the body; C N is the normal, and CE_1E_2C is the cone of friction. Let the value of x in first case be x_1, and in the second case be x_2. The resultant of $x_1 = CK_1$ and W must fall along CE_1 and the resultant of $x_2 = CK_2$ and W must fall along CE_2. The values are

$$x_1 = 23{\cdot}1 \text{ lbws.} \quad \text{and} \quad x_2 = 34{\cdot}8 \text{ lbws.}$$

Analytical Solution.—Let angles A O B and K_2C A be α and β respectively. Resolve W and x respectively into W $\sin\alpha$, W $\cos\alpha$, $x\sin\beta$ and $x\cos\beta$. The force of friction is then $(W\cos\alpha - x\sin\beta)\tan\phi$.

First case.—The force-equation is

$$W\sin\alpha = x_1\cos\beta + W\cos\alpha\tan\phi - x_1\sin\beta\tan\phi.$$

Hence, $$x_1 = \frac{\sin(\alpha-\phi)}{\cos(\beta+\phi)}W, \quad \text{or} \quad x_1 = 23{\cdot}1 \text{ lbws.}$$

Second case.—The force-equation is

$$x_2\cos\beta = W\cos\alpha\tan\phi - x_2\sin\beta\tan\phi + W\sin\alpha.$$

Hence, $$x_2 = \frac{\sin(\alpha+\phi)}{\cos(\beta-\phi)}W, \quad \text{or} \quad x_2 = 34{\cdot}8 \text{ lbws.}$$

(89) (Q. Nov. 1907).—**A length, measuring 2 miles, of a track has varying gradients, but it is known that the point B is 100 feet above A. A car weighing 5 tons is driven along A B by a motor which exerts 40 H.P. uniformly, the speed is 60 miles an hour at A and 50 miles an hour at B, and the time occupied is 2½ minutes. Calculate approximately the amount of frictional resistance to the motion of the car, on the assumption that this is uniform.**

Solution.—If the horizontal projection of A B, call it d, were given, then (see [**155**]) the work done on friction would be = frictional resistance into d. But d is not given, and we must therefore assume that the gradient $\tan\alpha$, although varying, is always so small that $\cos\alpha$ may be considered equal to one. In these circumstances the work done on friction is = the frictional resistance into A B.

Let F_f be the total frictional resistance, then the work to be done is

(1) W_1 = the work done in lifting the car through a height of 100 ft. $= 5 \times 2240 \times 100 = 1{,}120{,}000$ ft.-lbws.

(2) W_2 = the work done on friction $= F_f \times 2 \times 5280 = 10{,}560\, F_f$ ft.-lbws.

The energy to our disposal is

(1) E_1 = difference of kinetic energy at A and B

$$= \frac{1}{2} \times \frac{5 \times 2240}{32}\left[\left(60 \times \frac{22}{15}\right)^2 - \left(50 \times \frac{22}{15}\right)^2\right] = 149{,}000 \text{ ft.-lbws.}$$

(2) E_2 = the energy developed by the motor $= 40 \times 33{,}000 \times 2{\cdot}5 = 3{,}300{,}000$ ft.-lbws.

Energy-equation : $E_1 + E_2 = W_1 + W_2$.

Hence, $F_f = 220$ lbws., or 44 lbws. per tnw. of car (*Ans.*).

(90) (I.C.E., Oct. 1906).—**State the principle of the conservation of energy. In a cable tramway, one car (weight 14 tons) is on a down-gradient of 1 in 50, and another (weight 11 tons) is on an up-gradient of 1 in 300. The cable connecting them weighs 8 tons, and is always equally distributed on the two slopes. Find the acceleration and the speed of the cars, due to gravity only, after running 400 feet from rest, if all frictional resistances are neglected.**

Solution.—The principle of conservation of energy states:

Energy cannot be destroyed nor created, but can change its form.

There are 4 tons of cable on each slope to be dealt with.

(1) *Up-gradient.*—The weight on the slope = 11 tnws. + 4 tnws. = 15 tnws. The force pulling the weight down the incline $= 15 \sin\alpha = 15/300$ tnws. The mass on the slope $= 15/g$.

(2) *Down-gradient.*—Weight on the slope = (14 + 4) tnws. = 18 tnws. The force pulling the weight down = 18/50 tnw. The mass on the slope $= 18/g$.

Hence, the force-equation is

$$\frac{18}{50} - \frac{15}{300} = a(15+18)/g \quad . \quad . \quad . \quad (1)$$

As the acceleration a in (1) is constant, the speed u after a run of 400 ft. from rest is, by article **163**, $u = \sqrt{2a400}$.

Hence, $a = 0{\cdot}3$ ft./s.^{2}; $u = 15{\cdot}5$ ft./s. (*Ans.*).

(91) (Q. Oct. 1909).—**A train of 350 tons is ascending an incline of 1 in 200, and the resistance to motion is 12 lbws. per ton. What is the acceleration of the train when its velocity is 15 miles per hour, if the horse-power then developed by the engine be 500?**

Solution.—The forces which resist the motion are: the force of friction $= F_f$ and $W \sin\alpha$. The effort is F_e. Hence, the effective force $= F_e - F_f - W \sin\alpha$. We have now

$$a\frac{W}{g} = F_e - F_f - W\sin\alpha,$$

therefore

$$a = \frac{(F_e - F_f - W\sin\alpha)g}{W} \quad . \quad . \quad . \quad (1)$$

Let the speed be u when the engine develops n H.P., then

$$F_e u = 550n \text{ ft.-lbws./s.}, \quad \text{or} \quad F_e = 550n/u. \qquad (2)$$

The value of F_e inserted in (1) gives

$$a = (550n/u - F_f - W \sin\alpha)g/W.$$

The numerical values are: $n = 500$; $u = 22$ ft./s.; $F_f = 12 \times 350 \cos\alpha = 4200$ lbws.; as α is very small we can take $\cos\alpha = 1$ and $\sin\alpha = \tan\alpha = 1 : 200$; $W = 350 \times 2240 = 784{,}000$ lbws.; mass of train $= W/g = 784{,}000/32{\cdot}2$ gravitation units.

Hence, $$a = 0{\cdot}13 \text{ ft./s.}^2 \ (\textit{Ans.}).$$

(92) (I.C.E., Feb. 1898).—**Show that, if a man can throw a ball straight up so that it is 6 seconds before he catches it again, he can throw it to a horizontal distance of 96 yards.**

Solution.—Let $2t$ be the time which elapses before the man catches the ball again,

then $$v_i = gt \qquad (1)$$

The maximum range, d, which can be attained when the man throws the ball at an angle α with the horizontal corresponds to $\alpha = 45°$ (article **179**). Let the time of flight be T, then

$$v_i \sin 45° = g\frac{T}{2}, \quad \text{and} \quad d = v_i \cos 45°\, T \qquad (2)$$

Eliminating v_i and T between the three equations in (1) and (2) we get

$$d = gt^2 \sin 90° = gt^2 \qquad (3)$$

We have $g = 32{\cdot}2$, $t = 3$.

Hence, $$d = 289{\cdot}8 \text{ ft.} = 96{\cdot}6 \text{ yds. } (\textit{Ans.}).$$

(93) (I.C.E., Feb. 1907).—**A gun mounted on a tower overlooking the sea is capable of discharging shells horizontally with a velocity of 600 feet per second. Determine the equation of motion of the shells, referred to vertical and horizontal axes having their origin at the muzzle, and find the range over the sea, which is 300 feet below.**

Solution.—Let v_i be the muzzle velocity, and the x-axis be vertical. At the end of time-interval t the shell will have travelled a horizontal distance y and a vertical distance x. The horizontal velocity is constant and equal to v_i, whereas the vertical motion is due to gravity.

Hence, $$y = v_i t, \quad \text{and} \quad x = \tfrac{1}{2}gt^2 \qquad (1)$$

By eliminating t between the two in (1) we obtain the equation of the motion,

$$y^2 = \frac{2v_i^2}{g}x \quad . \quad . \quad . \quad . \quad (2)$$

(2) is a parabola. Let r be the range over the sea to which corresponds a value of $x = 300$ ft.

Hence, $r^2 = \frac{2v_i^2}{g} \times 300$, but $v_i = 600$.

Hence, $r = 2590$ ft. (*Ans.*).

(94) (I.C.E., Feb. 1909).— **The radius of curvature of a trajectory at a point in the rising branch is 6·85 miles and is inclined at 30° to the vertical. Determine the velocity of the projectile at the point in question. How much higher will the projectile rise? Neglect the air resistance.**

Solution.—In fig. 90 A T is the tangent and A N the normal to

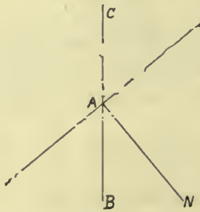


Fig. 90.

the given point A on the trajectory. C A B is the vertical line through point A. We have, $a_n = v^2/\rho$; hence, the required velocity $v = \sqrt{\rho a_n}$, but a_n is the resolute of the total acceleration (here retardation) in the direction of the normal. Let angle N A B $= \theta$, then $a_n = g \cos \theta$, and as ρ is given the velocity v can be determined.

The projectile will continue to rise until the vertical velocity becomes zero. The vertical velocity at point A is obviously $v_1 = v \sin \theta$, and the projectile will still rise a height $h_1 = v_1^2/2g$ from the given point.

The given numerical values are: $\rho = 6·85$ miles $= 36,168$ ft.; $\theta = 30°$.

Hence, $v = 1004$ ft.; $h_1 = 3915$ ft. (*Ans.*).

CHAPTER XXX.

ROTARY MOTION.

183. Rotation.

When the path described by a moving particle is a circle, the motion is called *rotation.* When a particle moves in a curved line it moves momentarily in the osculating circle, and curvilinear motion in general may therefore be considered as a rotary motion in which the radius of rotation changes continuously. By article **165** a curvilinear motion requires a normal force, and whatever the resultant force—which acts on the particle—may be, it consists of two components, the tangential force and the normal force.

If the tangential force be zero the speed remains unaltered, and when the normal force is zero the particle goes off at the tangent and its path is henceforth a straight line. The normal force is therefore a deviating force, and as it tends towards the centre of curvature it is sometimes called the *centripetal* force.

The deviating force may be produced in several ways, but it always acts by push on the mass, which in turn resists the push by an equal reaction. The latter reaction has been called the *centrifugal* force, because, superficially regarded, it seems as if the mass tries to leave the curved path in the direction of the normal, whereas the mass really tries to leave the curve in the tangential direction. Centrifugal force is a misleading and unscientific expression, which ought to be abandoned.

184. Rotating masses.

Let us consider a system of masses rotating about a fixed axis, *i.e.* about an axis in fixed bearings. The particles are supposed to be rigidly connected, mutually as well as to the axis. Each particle describes a circle with centre at the axis; the speed of rotation is supposed to be less than any speed which requires a deviating force greater than that which the connection between the particles can supply. Let there be a number of tangential forces acting on the system at different distances from the axis thus:

F_e', F_e'' . . . act respectively at distances ρ_1, ρ_2 . . .; and resistances F_r', F_r'' . . . act respectively at distances d_1, d_2 . . . .

Such a system may be exemplified by a main workshop shaft on which a number of pulleys are fixed. The effort is transmitted from the workshop engine by a moving belt, which passes over one of the pulleys. The energy is transmitted, travelling along the shaft, to the other pulleys. The latter are connected by belting with the machines which require the energy developed by the engine.

The energy exerted by all the efforts together while the system turns through an angle $\delta\alpha$ is,

$$F_e'\delta s_1 + F_e''\delta s_2 + \ . \ . \ . \ = F_e'\rho_1\delta\alpha + F_e''\rho_2\delta\alpha + \ . \ . \ .$$
$$= \delta\alpha(F_e'\rho_1 + F_e''\rho_2 + \ . \ . \ . \ ) = F_e\rho\,\delta\alpha \qquad (1)$$

where F_e is the effort, at a given distance ρ from the axis, which exerts the same amount of energy as all the efforts exert together while the system turns through the angle $\delta\alpha$. (1) gives

$$F_e\rho = F_e'\rho_1 + F_e''\rho_2 + \ . \ . \ . \qquad (2)$$

The product $F_e\rho$ is called the *torque* or *turning moment* of F_e. Hence, *the resultant torque of a system of tangential forces is equal to the algebraical sum of the torques of the several forces.*

In the same manner the resultant torque of the resistances is $F_r\rho$. If the kinetic energy of the system changes from E_k' to E_k'' during n revs., then the energy-equation is

$$\rho\int_0^{2\pi n} F_e\,d\alpha = \rho\int_0^{2\pi n} F_r\,d\alpha + E_k'' - E_k' \qquad (3)$$

F_e and F_r must be given as functions of α. The instantaneous energy-equation is

$$F_e\rho\,\delta\alpha = F_r\,\rho\,\delta\alpha + \delta E_k \qquad (4)$$

(*a*) The system is accelerating, and the kinetic energy is increasing when the effective torque $(F_e - F_r)\rho$ is positive.

(*b*) When the effective torque is negative the system is retarding, and is only kept in motion by losing kinetic energy.

(*c*) The motion is a uniform rotation when $\delta E_k = 0$, *i.e.* when $(F_e - F_r)\rho = 0$ or $F_e = F_r$.

185. Moment of inertia.

We will now find an expression for the kinetic energy of a system of rotating masses. Let m_1, m_2 . . . be the masses, r_1, r_2 . . . the radii of the circles described by the particles, and u_1, u_2 . . . the speeds of the respective particles. It is obvious that $u_n = u_1 r_n / r_1$. Hence, the kinetic energy accumulated in the system is

$$E_k = \tfrac{1}{2}[m_1 u_1^2 + m_2 u_2^2 + \ldots] = \frac{u_1^2}{2r_1^2}[m_1 r_1^2 + m_2 r_2^2 + \ldots] \qquad (1)$$

but u_1/r_1 is the angular velocity, ω, of the system.

Hence, $$E_k = \tfrac{1}{2}\omega^2 \sum mr^2 = \tfrac{1}{2}\omega^2 \mathbf{I} \quad . \quad . \quad . \quad [156]$$

$\sum mr^2$, which is denoted by **I**, is called the *moment of inertia*, or the *second mass-moment* of the system of rotating masses w.r.t. the axis of rotation. **I** depends not only on the amount of mass contained in the system, but also on the distribution of the mass about the axis.

If we compare [**156**] with [**148**] we shall find that [**156**] may be derived from [**148**] by substituting ω for u and **I** for $M = \sum m$.

We may consider a translation as a rotation about an axis at an infinite distance from the body, and we may therefore consider $r_1 = r_2 \ldots = r_n$, say equal to r.

Hence, $$E_k = \frac{\omega^2}{2}\sum mr^2 = \tfrac{1}{2}\frac{u^2}{r^2} r^2 \sum m = \tfrac{1}{2}u^2 M \quad . \quad . \quad (2)$$

The moment of inertia of a single mass m, describing a circle with radius r, is $\mathbf{I} = mr^2$. If therefore M be the sum of the masses of a rotating system, then $\mathbf{I} = \sum mr^2 = M R_g^2$, or

$$R_g = \sqrt{\frac{\mathbf{I}}{M}} \quad . \quad . \quad . \quad . \quad [157]$$

R_g is called the *radius of gyration*, or the *swing radius*, of the system. Hence, we may imagine all the masses of the rotating system to be concentrated into one point describing a circle with radius R_g without affecting the kinetic energy of the rotary motion.

We may also substitute a mass M_1, describing a circle with radius R_1, for a system of rotating masses whose moment of inertia is **I**. The condition is that

$$M_1 R_1^2 = \mathbf{I}; \text{ hence, } R_1 = \sqrt{\frac{\mathbf{I}}{M_1}} \text{ when } M_1 \text{ is given};$$

$$M_1 = \frac{\mathbf{I}}{R_1^2} \text{ when } R_1 \text{ is given} \quad . \quad . \quad [158]$$

The dimensions of **I** are $[M][L]^2$.

We may now write the energy-equation (3) in the preceding article thus,

$$\rho\int_0^{2\pi n} F_e\,da = \rho\int_0^{2\pi n} F_r\,da + \tfrac{1}{2}(\omega_f^2 - \omega_i^2)I \qquad [159]$$

where ω_f and ω_i are the final and initial angular velocities respectively; or, reducing the mass of the system to a point at distance ρ from the axis,

$$\rho\int_0^{2\pi n} F_e\,da = \rho\int_0^{2\pi n} F_r\,da + \tfrac{1}{2}(u_f^2 - u_i^2)M_1 \qquad [160]$$

where $M_1\rho^2 = I$, and u_f and u_i are respectively the final and initial speeds of a point at distance ρ from the axis.

186. Equilibrium of tangential forces.

A system of forces is said to be in equilibrium, or to be mutually balanced, when they, by their combined action, cannot produce acceleration. A system of forces which is in equilibrium cannot therefore produce motion, and if the system be moving the motion must be uniform.

Hence, δE_k in (4), article 184, must be zero, *i.e.* the rotation is uniform, when $F_e\rho - F_r\rho = 0$, or *a system of tangential forces is in equilibrium when the algebraical sum of the torques of the several forces w.r.t. the axis of rotation is zero.*

187. Instantaneous centre.

Let A and B (fig. 91) be any two points belonging to a plane

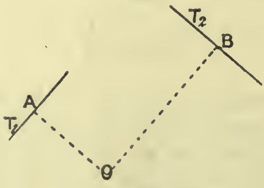


FIG. 91.

figure which is moving in its own plane. Let it also be known that the instantaneous motions of the two points take place respectively in the directions A T_1 and B T_2. Draw the normals to points A and B, meeting at point O. It can now be proved that point O is instantaneously at rest, or, in other words, that the given plane figure rotates instantaneously about point O.

If O were not at rest we could resolve its velocity along O A and O B, but points A and B move along the tangents to their respective paths, and their motions can have no components along their normals. Hence, point O is at rest.

Point O is called the *instantaneous centre* of motion of the plane figure; and the motion of any point C takes place along the perpendicular on O C through C. Let v_a, v_b, and v_c be the respective velocities of points A, B, and C, then $v_a : v_b : v_c :: OA : OB : OC$. Hence, if the velocity of one of the three points be known, the velocity of any other point of the plane figure may be found.

In fig. 29, p. 62, the instantaneous centre of motion of the sliding-rod is the point of intersection of the two perpendiculars, drawn through points A and B, on the x-axis and the y-axis respectively. The line drawn from the instantaneous centre through point C on the ellipse is the normal to the curve at point C. Hence, the tangent to the ellipse at the latter point is determined.

188. Rolling circle.

Fig. 92 represents a circle O B A, with centre at C and of radius $CO = r$, which is rolling without slipping on the straight line D O.

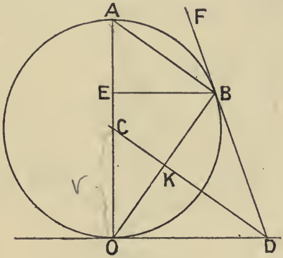


FIG. 92.

The point of contact O is obviously the instantaneous centre of motion; hence, the motion of any point B on the circle is perpendicular on O B and along the line B A—point A being the point on the circle which lies diametrically opposite to O. Draw tangent F B D to the circle at B, then draw C K D; the latter line bisects O B as well as angle O D B. Draw B E parallel to D O; A B bisects angle F B E.

Finally, resolve the velocity v_b of point B in the directions B F and B E. We have now that the two components of v_b are equal, say equal to x, because B A bisects angle F B E; $v_b : v :: OB : r$, v being the velocity of the centre C; $v_b = 2x \sin \alpha$, where α is angle D O K; $OB = 2r \sin \alpha$. Hence, $x = v$, or, in words, the motion of any point on the rolling circle can be resolved into two components, viz. *a translation* with a linear velocity v equal to that of the centre and parallel to the fixed straight line, *and a rotation about the centre* with an angular velocity $\omega = v/r$.

The velocity of point A is equal to $2v$ and is made up of the velocity v of translation and the velocity $v = r\omega$. The velocity of the instantaneous centre is zero, being the resultant of the velocity of translation v and the velocity *minus* $r\omega$.

189. Instantaneous axis.

Let the directions of the instantaneous motions of two points A and B belonging to a moving body be known. Draw the normal planes through A and B; the latter two planes intersect in a straight line which must be instantaneously at rest, and is therefore the *instantaneous axis* of motion of the body.

190. Rolling friction.

The rolling of a wheel on a surface is a rotation about the axis of the wheel combined with a translation of the axis. A rotary

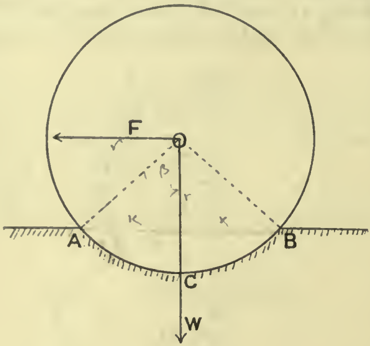


FIG. 93.

motion requires the application of a tangential force. In the case of the rolling wheel the tangential force is the force of friction between the two surfaces of contact. Hence, if there were no friction there would be no rolling, and the motion of the wheel would entirely be translatory.

Consider the wheel in fig. 93. Due to its weight, W, the wheel makes a groove in the road, which is not perfectly rigid, and it will therefore be necessary to apply a force, F, at the axis and parallel to the road, in order to bring the wheel out of the groove. Let $2k$ be the length of the chord A B, β = angle A O C, and r the radius of the wheel, then we have

$$Fr \cos \beta - Wr \sin \beta = 0,$$

but as β is very small we may take $\cos\beta = 1$; we have also $r\sin\beta = k$. Hence, the force of rolling friction is

$$F = \frac{k}{r}W = W\tan\theta \; . \qquad . \qquad . \qquad . \quad [161]$$

F is thus directly proportional to the chord of the groove and inversely proportional to the radius of the wheel. If the road be a plane of inclination α, then the pressure of the wheel is $W\cos\alpha$. Hence, in the latter case, $F = W\cos\alpha\tan\theta$; the wheel will be at rest on the plane when $W\sin\alpha = F = W\cos\alpha\tan\theta$, or when $\alpha = \theta$. Angle $\theta = \text{ang}\left(\tan = \frac{k}{r}\right)$ is therefore the *angle of repose of rolling friction* for a given sized wheel and the bearing-surfaces in a given condition. k is also called the *coefficient of rolling friction*, and is given in feet, metres, etc.

CHAPTER XXXI.

AREA-MOMENTS.

191. Moments.

In the case of certain physical and geometrical quantities, the product of the quantity and the n^{th} power of its distance from a given point, or a given straight line, or a given plane, is called *the n^{th} moment* of the quantity w.r.t. the given point, straight line, or plane. We have hitherto met with such expressions as second mass-moment and torque (the first moment of a force) w.r.t. an axis of rotation. In the following we shall have many opportunities to use the term moment.

192. Area-moments.

Let δA be the element of a plane area, r the distance of δA from a given point O (fig. 94) in the plane, then, by the preceding

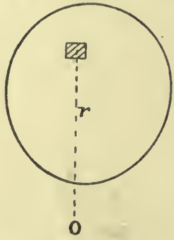


FIG. 94.

article, $\sum \delta A\, r^n$ is the n^{th} moment (the n^{th} *polar moment*), of the area w.r.t. the given point. Similarly, when x and y are the usual rectangular co-ordinates, then $\sum \delta A\, x^n$ and $\sum \delta A\, y^n$ are the n^{th} moments of the given area w.r.t. the y-axis and the x-axis respectively.

We shall only require the first and second area-moments, for which the following notations will be used :—

First moments:

$$\sum \delta A\, r = (a.m.)_p, \quad \sum \delta A\, x = (a.m.)_y, \quad \text{and} \quad \sum \delta A\, y = (a.m.)_x.$$

Second moments:

$$\sum \delta A\, r^2 = (A.M.)_p, \quad \sum \delta A\, x^2 = (A.M.)_y, \quad \text{and} \quad \sum \delta A\, y^2 = (A.M.)_x.$$

It is obvious that none of the second moments can be zero, whereas we may choose the co-ordinate axes so that one or all three of the first moments will be zero. The point in the plane w.r.t. which the first area-moments are zero is called the *centroid* of the area (*see* Chap. XXXV.).

If the origin of a set of rectangular axes be chosen as the pole, then $r^2 = x^2 + y^2$.

Hence, $$(A.M.)_p = (A.M.)_y + (A.M.)_x \quad . \quad . \quad . \quad [162]$$

Or *the sum of the second area-moments w.r.t. any pair of rectangular axes, having the same origin, is constant and equal to the second polar area-moment w.r.t. the given origin.*

Let the second area-moment, $(A.M.)'_x$, w.r.t. an axis $a'b'$ in the plane and through the centroid be known, and let it be required

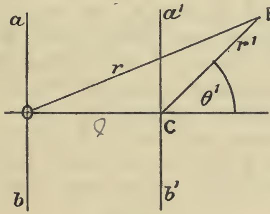


FIG. 95.

to find the second area-moment, $(A.M.)_x$, w.r.t. an axis ab parallel to and at a distance $OC = l$ from $a'b'$ (fig. 95). We have,

$$(A.M.)'_x = \sum \delta A\, y^2,$$

and $$(A.M.)_x = \sum \delta A (y + l)^2 = \sum \delta A\, y^2 + l^2 \sum \delta A + 2l \sum \delta A\, y;$$

but $$\sum \delta A = A \quad \text{and} \quad \sum \delta A\, y = (a.m.)'_x = 0$$

since $a'b'$ passes through the centroid.

Hence, $$(A.M.)_x = (A.M.)'_x + Al^2 \quad . \quad . \quad . \quad [163A]$$

Or in words: *The second moment of a plane figure w.r.t. a given axis in the plane is equal to the second moment of the plane figure* *w.r.t. an axis parallel to the given axis and passing through the centroid*, plus *the area of the plane figure into the square of the distance from the centroid to the given axis.*

Let the second polar moment $(A.M.)'_p$ w.r.t. the centroid, C, be known, and let it be required to find $(A.M.)_p$ w.r.t. a given point O (fig. 95), which is at a distance l from C.

$$(A.M.)'_p = \sum \delta A\, r'^2 ; \;\; (A.M.)_p = \sum \delta A\, r^2 ;$$

but $$r^2 = r'^2 + l^2 + 2r'l \cos \theta'.$$

Hence, $$(A.M.)_p = \sum \delta A\, r'^2 + l^2 \sum \delta A + 2l \sum \delta A\, r' \cos \theta',$$

but the latter moment is $= 2l(a.m.)'_y$, which is zero, since $a'b'$ passes through C.

Hence, $$(A.M.)_p = (A.M.)'_p + Al^2 \quad . \quad . \quad . \quad [163B]$$

Or in words: *The second polar moment of a plane figure w.r.t. a point O in its plane is equal to the second polar moment of the plane figure w.r.t. the centroid* plus *the area of the plane figure into the square of the distance from the centroid to the point O.*

The student will often meet with the expression "moment of inertia of an area," an expression used instead of the second area-moment. An area does not contain mass, and can therefore not possess inertia. The dimensions of the second area-moments are $[L]^4$, *whereas those of the moment of inertia are* $[M]\,[L]^2$. Second area-moments are expressed in inches^4, cm.^{4}, etc.

The n^{th} area-moment is evidently equal to the product of the n^{th} power of a length R and the area. R may conveniently be termed the n^{th} mean radius of the area w.r.t. the given point or given line. Thus,

$$(a.m.) = r_m A, \quad \text{and} \quad (A.M.) = R_m^2 A \quad . \quad . \quad [164]$$

The second mean radius, R_m in [164], is often erroneously called the radius of gyration of the area.

193. Second area-moments of a circle.

(i.) $(A.M.)_p$ *of a circle w.r.t. the centre.* For a circular ring of radius r and depth δr we have, $\delta(A.M.)_p = 2\pi r\, \delta r\, r^2$.

Hence, $$(A.M.)_p = 2\pi \int_0^R r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} . \quad . \quad . \quad (1)$$

where R is the radius and D the diameter of the circle. The second mean radius of the area of the circle w.r.t. the centre is

$$R_m = R/\sqrt{2}, \quad \text{because} \quad \pi R^2\, R_m^2 = \frac{\pi R^4}{2} \quad . \quad . \quad (2)$$

$(A.M.)_p$ of a circle-sector w.r.t. the centre will therefore be

$$(A.M.)_p \text{ of circle-sector} = \frac{\pi R^4}{2}\frac{\theta}{2\pi} = \frac{R^4\theta}{4} \quad . \quad . \quad (3)$$

where θ is the angle at the centre.

(ii.) $(A.M.)_p$ *of a circle w.r.t. a point* O *in the plane of the circle and at a distance* l *from the centre.*

By [163B] we have $(A.M.)_p$ w.r.t. point O

$$= \frac{\pi R^4}{2} + \pi R^2 l^2 = \frac{\pi R^2}{2}(R^2 + 2l^2)\,;\ R_m = \sqrt{\frac{R^2 + 2l^2}{2}} \quad . \quad (4)$$

(iii.) *The second area-moment of a circle w.r.t. a diameter.*

Since a diameter is an axis of symmetry, it follows that (A.M.) w.r.t. any diameter is

$$(A.M.)_y = (A.M.)_x = \frac{(A.M.)_p}{2}.$$

Hence,

$$(A.M.) \text{ of a circle w.r.t. a diameter} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}\,;\ R_m = \frac{R}{2} \quad (5)$$

194. General expression for $(A.M.)_p$.

Let O be the pole and O X (fig. 96) the initial axis of a system

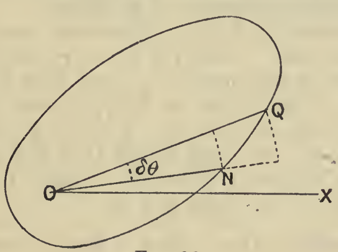


FIG. 96.

of polar co-ordinates. $\delta(A.M.)_p$ of area-element N O Q must lie between $\delta\theta\, r^4/4$ and $\delta\theta(r+\delta r)^4/4$ (see preceding article). Hence, in the limit

$$\frac{d(A.M.)_p}{d\theta} = \frac{r^4}{4}, \quad \text{and} \quad (A.M.)_p = \tfrac{1}{4}\int_0^{2\pi} r^4 d\theta \quad . \quad [165]$$

195. $(A.M.)_p$ of a regular polygon.

The sector of an n-sided regular polygon is an isosceles triangle whose angle at the vertex is $a = 2\pi/n$ radians. Let R be the radius of the inscribed circle, then the area of the triangle is $R^2 \tan\frac{\alpha}{2}$.

Let the median of an isosceles triangle be taken as the fixed axis and the vertex as the pole of a system of polar co-ordinates, then the equation of the base of the triangle is $r\cos\theta = h$, where h is the height of the triangle.

The $(\text{A.M.})_p$ of the isosceles triangle w.r.t. the vertex is, according to [165],

$$\frac{h^4}{2}\int_{\theta=0}^{\theta=\frac{\alpha}{2}}\frac{d\theta}{\cos^4\theta} = \frac{h^4}{2}\int_{\theta=0}^{\theta=\frac{\alpha}{2}}\tan^2\theta\, d\tan\theta + \frac{h^4}{2}\int_{\theta=0}^{\theta=\frac{\alpha}{2}}\frac{d\theta}{\cos^2\theta}$$

$$= \frac{h^4}{6}\left(3 + \tan^2\frac{\alpha}{2}\right)\tan\frac{\alpha}{2} \quad . \quad . \quad . \quad (1)$$

Hence, the second polar moment of an n-sided regular polygon w.r.t. the centre is

$$(\text{A.M.})_p = \frac{n\text{R}^4}{6}\left(3 + \tan^2\frac{\pi}{n}\right)\tan\frac{\pi}{n} = \text{A}\frac{\text{R}^2}{6}\left(3 + \tan^2\frac{\pi}{n}\right) \quad . \quad [166]$$

where A is the area of the polygon.

196. Second area-moments of an ellipse.

Let us determine $(\text{A.M.})_x$, *i.e.* the second moment of the ellipse w.r.t. the major axis. Draw the auxiliary circle (article **43**), and let the ordinate to the circle be y_1, and that of the ellipse be y, then $\delta(\text{A.M.})'_x$ of the circle is $2x\,\delta y_1\, y_1^2$, and that of the ellipse is $\delta(\text{A.M.})_x = 2x\,\delta y\, y^2$, but $y_1 : y :: a : b$; hence, $\delta(\text{A.M.})_x = b^3/a^3\delta(\text{A.M.})'_x$. But we have found that the second moment of a circle w.r.t. a diameter is $\pi\text{R}^4/4$. Hence, the second moment of the ellipse w.r.t. the major axis is,

$$(\text{A.M.})_x = \pi ab^3/4 = \text{A}b^2/4, \quad \text{and} \quad (\text{A.M.})_y = \pi a^3b/4 = \text{A}a^2/4 \qquad (1)$$

Hence, the second polar moment of an ellipse w.r.t. its centre is

$$(\text{A.M.})_p = \text{A}\frac{b^2}{4} + \text{A}\frac{a^2}{4} = \text{A}\frac{a^2+b^2}{4}. \quad . \quad . \quad (2)$$

197. Approximate determination of the second area-moments.

Let the problem be to find the second area-moments of the area enclosed by figure O B C D E O (fig. 97). $\text{O E} = l$.

In the limit, $\delta\text{A} = y\,\delta x$ is a rectangle; we must therefore determine the second area-moment of a rectangle w.r.t. its base. Let h be the height and b the base of the rectangle, then

(A.M.) of a rectangle w.r.t. the base

$$=\sum b\,\delta y\, y^2 = b\int_0^h y^2 dy = \frac{bh^3}{3} = \frac{Ah^2}{3} \quad . \quad . \quad [167]$$

Hence, the general formulæ for the second area-moments of an area, as fig. 97, are

$$(\text{A.M.})_x = \sum \frac{y^3}{3}\delta x = \frac{1}{3}\int_0^l y^3 dx \quad . \quad . \quad [168]$$

and $$(\text{A.M.})_y = \sum y\,\delta x\, x^2 = \int_0^l y\, x^2 dx \quad . \quad . \quad [169]$$

When [168] and [169] cannot be integrated we may determine the area-moments by Simpson's rule.

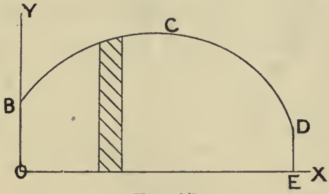


FIG. 97.

(i.) To find $(\text{A.M.})_x$ we must substitute $y^3/3$ for y in [99].

Hence, $$(\text{A.M.})_x = \frac{h}{9}\left[y_0{}^3 + y_{2n}^3 + 4\sum_{r=1}^{r=n} y_{2r-1}^3 + 2\sum_{r=1}^{r=n-1} y_{2r}^3\right] \quad . \quad (1)$$

(ii.) To find $(\text{A.M.})_y$ we must substitute yx^2 for y in [99]. The values for x are: $x=0$, $x=h$, $x=2h$ . . . , $x=2nh$. Hence,

$$(\text{A.M.})_y = \frac{h^3}{3}\left[(2n)^2 y_{2n} + 4\sum_{r=1}^{r=n}(2r-1)^2 y_{2r-1} + 2\sum_{r=1}^{r=n-1}(2r)^2 y_{2r}\right] \quad (2)$$

198. Second-radius ellipse.

Let $(\text{A.M.})_1$ and $(\text{A.M.})_2$ be the second moments of the area enclosed by a plane figure w.r.t. any pair of rectangular axes through a given point O in the plane of the figure. By [162] we have $(\text{A.M.})_p = (\text{A.M.})_1 + (\text{A.M.})_2$, or if A be the area, $(\text{A.M.})_p = \text{A}\rho^2 = \text{A}\,\text{R}_1^2 + \text{A}\,\text{R}_2^2$.

Hence, $$\text{R}_1^2 + \text{R}_2^2 = \rho^2 \quad . \quad . \quad . \quad . \quad (1)$$

where ρ is the second mean radius w.r.t. point O, R_1 and R_2 the second mean radii w.r.t. the axes of y and x respectively.

Let a and b be the maximum and minimum second mean radii w.r.t. point O, then

$$a^2 + b^2 = \rho^2 = R_1^2 + R_2^2 \quad . \quad . \quad . \quad (2)$$

Let O A_1 and O B_1 (fig. 98A), be equal to a and b in magnitude as well as in direction, then the rectangular axes X X and Y Y, which contain a and b respectively, are called the *principal axes* of the area through the given point O. Let now the problem be to

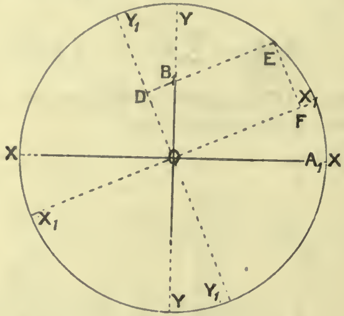


FIG. 98A.

determine the second mean radii O F = R_1 and O D = R_2 along a pair of rectangular axes, X_1X_1 and Y_1Y_1, inclined at an angle θ_1 to the initial axes X X and Y Y.

By article 5 $x_1 = x \cos\theta_1 + y \sin\theta_1$, and $y_1 = y\cos\theta_1 - x\sin\theta_1$. Hence, the second moments w.r.t. the new axes are

$$R_1^2 A = \sum \delta A\, x_1^2 = \sum \delta A(x^2\cos^2\theta_1 + y^2\sin^2\theta_1 + \sin 2\theta\, xy)$$

$$= Aa^2\cos^2\theta_1 + Ab^2\sin^2\theta_1 + \sin 2\theta_1 \sum \delta A\, xy \quad (3)$$

$$R_2^2 A = \sum \delta A\, y_1^2 = Aa^2\sin^2\theta_1 + Ab^2\cos^2\theta_1 - \sin 2\theta_1 \sum \delta A\, xy \quad (4)$$

Differentiating (3) w.r.t. θ_1 we get

$$A\frac{dR_1^2}{d\theta_1} = -2Aa^2\sin\theta_1\cos\theta_1 + 2Ab^2\sin\theta_1\cos\theta_1$$

$$+ 2\cos 2\theta_1 \sum \delta A\, xy \quad . \quad . \quad (5)$$

As maximum (A.M.) = Aa^2, then $\theta_1 = 0$ must make (5) equal to zero.

Hence, $\sum \delta A\, xy = 0$, and instead of (3) and (4) we get

$$R_1^2 = a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_1 \qquad . \quad . \quad . \quad (6)$$

$$R_2^2 = a^2 \sin^2 \theta_1 + b^2 \cos^2 \theta_1 \qquad . \quad . \quad . \quad (7)$$

Comparing (6) and (7) with the condition found in article **38**, it will be seen that the straight lines E D and E F are tangents to the ellipse whose semi-axes are a and b; the circle of radius ρ and with centre at point O is the director-circle of the ellipse. Hence, if the principal axes of a given plane figure w.r.t. a given point O be known, then—having constructed the ellipse whose semi-axes are a and b—the second mean radius through O in any given direction may be determined by drawing a tangent to the ellipse at right angles to the given direction, the length of the perpendicular drawn from O to the tangent is the length of the required mean radius.

The ellipse is known as the *ellipse of gyration*, or the *inertia-ellipse.* The first name implies a whirling of the plane figure about some axis, and the second name implies the presence of mass. Hence, both expressions are erroneous, and I have therefore called the ellipse the *second-radius-ellipse.*

The ellipse shows that the second moments w.r.t. axes through any point O in the plane of the area are symmetrical w.r.t. the axes of the ellipse, although the plane figure may have no axis of symmetry.

(i.) When the plane figure can be divided symmetrically by a straight line through point O, that line contains one of the axes of the second-radius-ellipse; because the second moments referred to any two straight lines, through point O and symmetrically situated w.r.t. the *axis of symmetry*, are equal.

(ii.) When the plane figure has *no axis of symmetry* through point O the principal axes may be determined by the following method:—

We commence by selecting two convenient rectangular axes X_1X_1, Y_1Y_1 and a third convenient axis X_2X_2 (fig. 98B). We then determine the area, A, of the plane figure, and the second moments referred to the three chosen axes.

Let the rectangular axes X X and Y Y be the principal axes; R_1, R_2, and R_3 the second mean radii along X_1X_1, Y_1Y_1, and X_2X_2 respectively, θ_1 = angle X O X_1 and θ_2 = angle X O X_2.

θ_1 and θ_2 are unknown, but $\theta_2 - \theta_1 = \gamma$ is known. We have now $R_1^2 + R_2^2 = \rho^2$; and by (6) we have

$$R_1^2 = a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_1 \qquad . \quad . \quad . \quad (8)$$

$$R_3^2 = a^2 \cos^2 \theta_2 + b^2 \sin^2 \theta_2 \qquad . \quad . \quad . \quad (9)$$

As $b^2 = \rho^2 - a^2$, we obtain by eliminating a^2 between (8) and (9)

$$\frac{\cos 2\theta_1}{\cos 2\theta_2} = \frac{2R_1^2 - \rho^2 + \rho^2 \cos 2\theta_1}{2R_3^2 - \rho^2 + \rho^2 \cos 2\theta_2} \quad . \quad . \quad . \quad (10)$$

and as $\theta_2 = \theta_1 + \gamma$ we have

$$\tan 2\theta_1 = \frac{(2R_1^2 - \rho^2)\cos 2\gamma - 2R_3^2 + \rho^2}{(2R_1^2 - \rho^2)\sin 2\gamma} \quad . \quad . \quad (11)$$

By (8) we have

$$a^2 = \frac{R_1^2 - \rho^2 \sin^2\theta_1}{\cos 2\theta_1}, \quad \text{and} \quad b^2 = \rho^2 - a^2 \quad . \quad . \quad (12)$$

Example.—*Find the principal axes through the centroid of the Z-shaped figure shown in fig.* 98c.

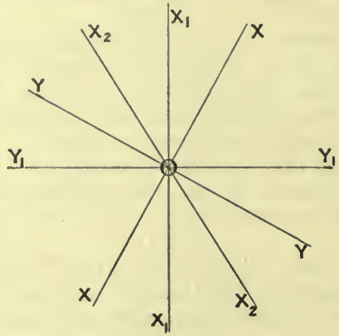


Fig. 98b.

The flanges are 7·5 ins. × 2 ins.; the width of the web = 1 in., and the height of the figure = 14 ins.

Solution.—The area of the figure will be found to be, A = 40 ins.²

$$\gamma = 45°, \sin 2\gamma = 1, \cos 2\gamma = 0.$$

$$R_1^2 = \frac{2\left(\frac{7·5 \times 7^3}{3} - \frac{6·5 \times 5^3}{3}\right)}{40} = 29\tfrac{1}{3} \text{ ins.}^2$$

$$R_2^2 = \frac{2\left(\frac{2 \times 7^3}{3} + \frac{5 \times (0·5)^3}{3}\right)}{40} = 11·444 \text{ ins.}^2$$

Hence, $$\rho^2 = R_1^2 + R_2^2 = 40·78 \text{ ins.}^2$$

The (A.M.) w.r.t. X_2X_2 is easily determined; we have to calculate the (A.M.)s of three isosceles right-angled triangles w.r.t. their respective hypotenuses. If h be the height of such a

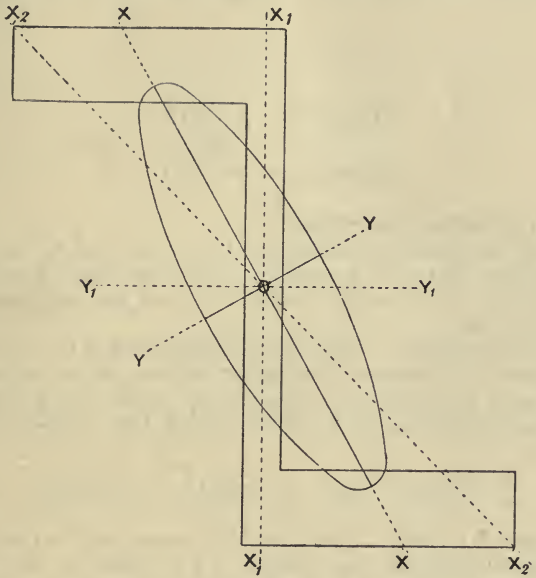


FIG. 98C.

triangle, the (A.M.) referred to the hypotenuse will be found to be $h^4/6$. We have now

$$R_3^2 = \rho^2 - \frac{2\left[\frac{(7\cdot5\sin 45^\circ)^4}{6} - \frac{(4\cdot5\sin 45^\circ)^4}{6} + \frac{(2\sin 45^\circ)^4}{6}\right]}{40}$$

$$= 40\cdot78 - 5\cdot77 = 35\cdot01 \text{ ins.}^2$$

$$\tan 2\theta_1 = \frac{-2R_3^2 + \rho^2}{2R_1^2 - \rho^2} = -58^\circ\ 32'. \quad \text{Hence, } \theta_1 = -29^\circ\ 16'.$$

$$a^2 = \frac{29\cdot333 - 40\cdot78\sin^2 29^\circ\ 16'}{\cos 58^\circ\ 32'} = 37\cdot52 \text{ ins.}^2;\ b^2 = \rho^2 - a^2 = 3\cdot26 \text{ ins.}^2$$

$a = 6\cdot13$ ins.; $b = 1\cdot81$ ins.; maximum (A.M.) $= Aa^2 = 1501$ ins.^{4}
minimum (A.M.) $= Ab^2 = 130\cdot4$ ins.^{4}

CHAPTER XXXII.

MOMENTS OF INERTIA.

199. I of bodies of revolution.

We will now proceed to determine the moment of inertia of a homogeneous body of revolution w.r.t. its axis. The mass contained in unit volume is denoted by m and the total mass of the body by M.

A.—Solid cylinder.—Let H be the height and R the radius of the cylinder. The mass of a hollow cylinder, whose inside and outside radii are r and $r+\delta r$ respectively, is $\delta \mathrm{M} = 2\pi r\,\delta r\,\mathrm{H}m$ and its moment of inertia is $\delta\mathbf{I} = \delta \mathrm{M} r^2 = 2\pi r^3\,\delta r\,\mathrm{H}m$. Hence, in the limit

$$\frac{d\mathbf{I}}{dr} = 2\pi\mathrm{H}mr^3. \quad \text{and} \quad \mathbf{I} = 2\pi\mathrm{H}m\int_0^{\mathrm{R}} r^3\,dr = \mathrm{M}\,\mathrm{R}^2/2\,. \quad [170]$$

B.—Hollow cylinder.—Let R_1 and R_2 respectively be the inside and outside radii of the cylinder. The mass of the body $=\pi(\mathrm{R}_2^2-\mathrm{R}_1^2)\mathrm{H}m=\mathrm{M}$, and $\mathbf{I}=$ the moment of inertia of the solid cylinder with the radius R_2 *minus* the moment of inertia of the solid cylinder with the radius R_1, or

$$\mathbf{I} = 0{\cdot}5\pi\mathrm{H}m(\mathrm{R}_2^4-\mathrm{R}_1^4) = 0{\cdot}5\pi\mathrm{H}m(\mathrm{R}_2^2-\mathrm{R}_1^2)(\mathrm{R}_2^2+\mathrm{R}_1^2)$$
$$= 0{\cdot}5\mathrm{M}(\mathrm{R}_2^2+\mathrm{R}_1^2) \quad . \quad . \quad . \quad [171]$$

The radius of gyration of the solid cylinder is $\mathrm{R}/\sqrt{2}$, and that of the hollow cylinder is $\sqrt{(\mathrm{R}_2^2+\mathrm{R}_1^2)/2}$.

C.—Any body of revolution.—Let the curve (fig. 99) whose equation is $y=f(x)$ be the meridian-curve of the body. Two planes, both at right angles to the axis and at a distance δx apart, cut off an elementary portion of the body which is bounded by two circles whose radii are y and $y+\delta y$ respectively. $\delta\mathbf{I}$ will lie between $0{\cdot}5\pi y^4 m\,\delta x$ and $0{\cdot}5\pi(y+\delta y)^4 m\,\delta x$, or in the limit

$$\frac{d\mathbf{I}}{dx} = 0{\cdot}5\pi y^4 m, \quad \text{and} \quad \mathbf{I} = 0{\cdot}5\pi m\int_0^{\mathrm{H}} y^4\,dx\,. \quad . \quad [172]$$

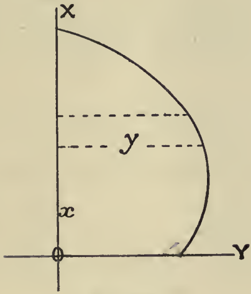


FIG. 99.

D.—Sphere.—The equation of the great circle is $x^2+y^2=\mathrm{R}^2$. Hence, by [172]

$$\mathbf{I}=0{\cdot}5\pi m\int_{-\mathrm{R}}^{+\mathrm{R}}y^4dx=\frac{8}{15}\pi m\mathrm{R}^5=0{\cdot}4\mathrm{M}\,\mathrm{R}^2 \quad . \quad . \quad [173]$$

E.—Spheroid.—See article **125**, the volume of the spheroid $=\frac{4}{3}\pi a^2b$; hence, $\mathrm{M}=\frac{4}{3}\pi a^2bm$.

$$\mathbf{I}=0{\cdot}5\pi m\int_{-b}^{+b}x^4dy=0{\cdot}5\pi m\frac{a^4}{b^4}\int_{-b}^{+b}(b^4+y^4-2b^2y^2)dy$$

$$=\frac{8}{15}\pi ma^4b=0{\cdot}4\mathrm{M}a^2.$$

F.—Cone.—Let H be the height of the cone and R the radius of the base. See article **120**, the volume of the cone $=\frac{1}{3}\pi\mathrm{R}^2\mathrm{H}$ and therefore $\mathrm{M}=\frac{1}{3}\pi\mathrm{R}^2\mathrm{H}m$. A plane parallel to the base and at a distance x from the apex cuts the cone in a circle whose radius is y.

But $y/\mathrm{R}=x/\mathrm{H}$; hence,

$$\mathbf{I}=0{\cdot}5\pi m\int_0^{\mathrm{H}}\frac{\mathrm{R}^4}{\mathrm{H}^4}x^4dx=0{\cdot}3\mathrm{M}\,\mathrm{R}^2 \quad . \quad . \quad [174]$$

G.—Approximate determination.—The integral in [172] can be determined approximately by Simpson's rule. Divide the height H into an even number of equal parts of length h. Measure the lengths of the ordinates $y_0, y_1 \ . \ . \ . \ y_{2n}$. Then,

$$\mathbf{I}=\tfrac{1}{6}\pi mh\left(y_0^4+y_{2n}^4+4\sum_{r=1}^{r=n}y_{2r-1}^4+2\sum_{r=1}^{r=n-1}y_{2r}^4\right) \quad . \quad [175]$$

EXAMPLE.—*A homogeneous body of revolution is* 10 *feet high. The lengths of the ordinates of the meridian-curve per foot run of the axis are given in the table below. Find* **I** *of the body w.r.t. its axis.*

y_0.	y_1.	y_2.	y_3.	y_4.	y_5.	y_6.	y_7.	y_8.	y_9.	y_{10}.
0	1·4142	2·0000	2·4495	2·8284	3·1623	3·4641	3·7417	4·0000	4·2426	4·4721

$h=1$. By [175] we have

$$\mathbf{I}=\tfrac{1}{6}\pi m(400+2640+960)=2094{\cdot}4m.$$

$$\mathrm{M}=\tfrac{1}{3}m\pi\left(y_0^2+y_{2n}^2+4\sum_{r=1}^{r=n}y_{2r-1}^2+2\sum_{r=1}^{r=n-1}y_{2r}^2\right)$$

$$=\tfrac{1}{3}\pi m(20+200+80)=315{\cdot}16m.$$

Hence, $\mathbf{I}=2094{\cdot}4m=6{\cdot}65\mathrm{M}$, and radius of gyration $=\sqrt{6{\cdot}65}$ $=2{\cdot}58$ ft.

If m is given in lbs. per cub. ft., then M will be expressed in lbs. and **I** will be expressed in the unit of the pound-foot-second system. If **I** is to be expressed in British engineering units, then we must divide m by $32{\cdot}187(g)$. Hence,

$\mathbf{I}=65{\cdot}1m$ British engineering units, but m is still expressed in lbs.

200. I of any right homogeneous cylinder.

Let the problem be to find the moment of inertia of a right homogeneous cylinder, of height H and with any plane base, w.r.t. an axis parallel to the generator of the cylinder.

Let fig. 94 (p. 246) be the base of the cylinder, and O the point at which the axis meets the plane of the base. If δA be the elementary area of the base and r the distance of δA from O, then the moment of inertia of the cylinder is

$$\mathbf{I}=\sum(\mathrm{H}\,\delta\mathrm{A}\,mr^2)=\mathrm{H}m\sum\delta\mathrm{A}\,r^2=\mathrm{H}m(\mathrm{A.M.})_p$$

$$=\mathrm{HA}m\mathrm{R}_m^2=\mathrm{M}\,\mathrm{R}_m^2 \quad . \quad . \quad . \quad [176]$$

where $(\mathrm{A.M.})_p$ is the second polar area-moment of the base w.r.t. point O. Hence, $\mathrm{R}_g=\mathrm{R}_m$. [176] applies also to any right prism with any base.

201. I of any right cone.

A cone is said to be *a right cone when the line connecting the apex and the centroid of the base*, the axis of the cone, *is perpendicular to the base.*

Let $(\mathrm{A.M.})_p$ be the second polar moment of the base w.r.t. the centroid, and $(\text{A.M.})'_p$ that of a section at distance x from the apex and parallel to the base. As the dimensions of area-moments are $[\text{L}]^4$, it follows that $(\text{A.M.})'_p : (\text{A.M.})_p : : x^4 : \text{H}^4$; but $\delta\mathbf{I} = (\text{A.M.})'_p\ \delta x\, m$;

hence, $$\mathbf{I} = m\int_0^{\text{H}} (\text{A.M.})'_p dx = m\ \frac{(\text{A.M.})_p}{\text{H}^4}\int_0^{\text{H}} x^4 dx$$

$$= 0{\cdot}2m\text{AR}_m^2\text{H} = 0{\cdot}6\text{MR}_m^2 \quad . \quad [177]$$

Example 1.—*The base of the right cone is an ellipse.* By article **196** $(\text{A.M.})_p$ of an ellipse w.r.t. the centre is $\text{A}(a^2 + b^2)/4$; hence, $\text{R}_m^2 = (a^2 + b^2)/4$, and

$$\mathbf{I} = 0{\cdot}15\text{M}(a^2 + b^2) \quad . \quad . \quad . \quad . \quad (1)$$

Example 2.—**I** *of a regular pyramid.* By article **195** $(\text{A.M.})_p$ of a regular polygon w.r.t. its centroid is

$$\text{A}\frac{\text{R}^2}{6}\left(3 + \tan^2\frac{\pi}{n}\right).$$

Hence, by [177]

$$\mathbf{I} = 0{\cdot}2m\text{A}\frac{\text{R}^2}{6}\left(3 + \tan^2\frac{\pi}{n}\right)\text{H} = 0{\cdot}1\left(3 + \tan^2\frac{\pi}{n}\right)\text{M R}^2 \quad . \quad [178]$$

202. Parallel axes.

Let the moment of inertia $\mathbf{I}_1$ of a solid about a certain axis be known, and let the problem be to find the moment of inertia **I** of the solid about a second axis which is parallel to and at a distance l from the first axis.

Fig. 95 represents a section through the solid at right angles to the two axes. C and O are respectively the points at which the first and second axes meet the section. An elementary cylinder, which is parallel to the axes, is cut by the plane of the section at B; the mass of the latter cylinder is δM. We have now $\delta\mathbf{I}_1 = \delta\text{M}\, r_1^2$ and $\delta\mathbf{I} = \delta\text{M}\, r^2$, but $r^2 = r_1^2 + l^2 + 2r_1 l \cos\theta_1$; hence, $\delta\mathbf{I} = \delta\text{M}\, r_1^2 + \delta\text{M}\, l^2 + 2l\, \delta\text{M}\, r_1 \cos\ \theta_1$,

or $$\mathbf{I} = \mathbf{I}_1 + \text{M}l^2 + 2l\sum(\delta\text{M}\, r_1 \cos\theta_1) \quad . \quad . \quad (1)$$

But $\sum(\delta\text{M}\, r_1 \cos\theta_1)$ is the first mass-moment of a solid w.r.t. a plane containing the first axis and intersecting OC at right angles. The point about which all the first mass-moments of a body are zero is called the *centroid*, or *mass-centre*, or the *inertia-centre* of the body (*see* Chap. XXXVI.). Hence, when the first of the given axes passes through the mass-centre, then

$$\mathbf{I} = \mathbf{I}_1 + \text{M}l^2 \quad . \quad . \quad . \quad . \quad [179]$$

or in words: *The moment of inertia of a solid w.r.t. any axis is equal to the moment of inertia of the solid w.r.t. the axis, which is parallel to the first one and which passes through the mass-centre,* plus *the mass of the solid into the square of the perpendicular distance between the axes.*

EXAMPLE 1.—*To find* **I** *of a pendulum.* The pendulum of a large clock consists of a long thin rod of length l suspended at the upper end and joined to a large sphere of radius R at the lower end. The total length of the pendulum is thus $l+2R$.

The moment of inertia **I** of the suspended pendulum is equal to the moment of inertia **I**′ of the rod w.r.t. the axis of suspension, *plus* the moment of inertia **I**″ of the sphere also w.r.t. the axis of suspension.

The elementary *moment of inertia*, at a distance x from the axis, *of a thin rod suspended at the one end* is $\delta \mathbf{I}' = a\, \delta x\, m' x^2$, where a is the area of the cross-section of the rod.

Hence, $$\mathbf{I}' = am'\int_0^l x^2\, dx = am'l^3/3 = M'l^2/3 \qquad [180]$$

The **I** of a sphere about a diameter is by [173] equal to $0{\cdot}4M''R^2$. The axis of suspension is at a distance $l+R$ from the centre of the sphere; hence, by [179] we have $\mathbf{I}'' = 0{\cdot}4M''R^2 + M''(l+R)^2$; and **I** of the pendulum $= M'l^2/3 + 0{\cdot}4M''R^2 + M''(l+R)^2$.

EXAMPLE 2.—*To find* **I** *of a homogeneous cylinder of revolution about an axis through the mass-centre and perpendicular to the axis of the cylinder.*

Two parallel planes, at right angles to the axis of the cylinder and at a distance δx apart, enclose a thin slice of the cylinder. The limiting value of the moment of inertia of this thin slice about the diameter which is parallel to the given axis is, by article **193**, $\delta \mathbf{I}_1 = \pi r^4 m\, \delta x/4$, where r is the radius of the base of the cylinder. But the slice is at a distance x from the given axis; hence, the moment of inertia of the slice w.r.t. the given axis is, $\delta \mathbf{I} = \pi r^4 m\ \delta x/4 + \pi r^2 \delta x\, m x^2$. Let H be the height of the cylinder, then the required moment of inertia is:

$$\mathbf{I} = \pi r^2 m \int_{-H/2}^{+H/2} (r^2/4 + x^2)dx = \pi r^2 H m (r^2/4 + H^2/12) = M\frac{3r^2 + H^2}{12}.$$

203. I of flywheel-rims.

A flywheel is a wheel of large diameter, having a rim of considerable mass, which moves at a great speed. The moving rim is the seat of a large quantity of kinetic energy, which can be augmented or diminished without causing an appreciable variation of angular velocity. The rim must therefore have a considerable moment of inertia.

Let fig. 100 represent the cross-section of the rim through the axis of rotation, and let O be the centroid of the section. We will choose a set of rectangular axes through O; O X being perpendicular to and O Y parallel to the axis of rotation. A plane at distance y from O X and perpendicular to the axis of the wheel cuts the rim in a circle-ring, whose outside and inside radii are $R+x$ and $R-x$ respectively, R being the radius of the circle which point O describes when the wheel rotates. The $(A.M.)_p$

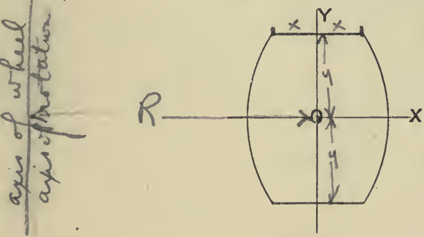


FIG. 100.

of the circle-ring is $\pi(R+x)^4/2-\pi(R-x)^4/2$ (see article **193**) or $(A.M.)_p=4\pi Rx(R^2+x^2)$, and $\delta I=(A.M.)_p\delta y\, m$. Hence, the moment of inertia of the flywheel-rim is

$$\mathbf{I}=8\pi Rm\int_0^w x(R^2+x^2)dy \quad . \quad . \quad . \quad [\mathbf{181}]$$

where $2w$ is the width of the rim measured along the y-axis. It is understood that the co-ordinate axes are axes of symmetry.

x in [**181**] is often very small in comparison with R, and x^2 may in such cases be neglected. The integral in [**181**] is then $R^2A/4$, A being the area of the cross-section, and we have $I=2\pi R^3Am$, but by article **128** $V=2\pi RA$, and therefore $M=2\pi RAm$.

Hence, $$\mathbf{I}=MR^2 \quad . \quad . \quad . \quad . \quad [\mathbf{182}]$$

or in words: *When half of the depth of the rim is very small compared with the radius of the circle described by the centroid of the cross-section of the rim, then we may approximately consider the mass of the rim as concentrated at the centroid.*

A.—The cross-section is a rectangle.—Let t be the depth of the rim, then x in [**181**] is constant and equal to $t/2$.

Hence, $$\mathbf{I} = \pi \mathrm{R}tm(4\mathrm{R}^2 + t^2)\int_0^w dy = \pi \mathrm{R}tm(4\mathrm{R}^2 + t^2)w.$$

But $\mathrm{V} = 2\pi\mathrm{R}2wt$; hence,

$$\mathbf{I} = 0{\cdot}25\mathrm{M}(4\mathrm{R}^2 + t^2),$$

and when t^2 is very small compared with $4\mathrm{R}^2$, then

$$\mathbf{I} = \mathrm{M\,R}^2.$$

We may also consider the rim as a hollow cylinder with radii r_1 and r_2. **I** of the rim is by article **199**

$$\mathbf{I} = 0{\cdot}5\mathrm{M}(r_1{}^2 + r_2{}^2).$$

The latter result will also be obtained by substituting $(r_1 + r_2)/2$ for R, and $(r_2 - r_1)$ for t in the above formulæ for V and **I**.

B.—The cross-section is a circle.—The equation of the circle is $x^2 + y^2 = r^2$, and therefore $w = r$.

Hence, $$\mathbf{I} = 8\pi\mathrm{R}m\int_0^r x(\mathrm{R}^2 + r^2 - y^2)dy$$

or, approximately,

$$\mathbf{I} = 8\pi\mathrm{R}m(\mathrm{R}^2 + r^2)\int_0^r x\,dy = \mathrm{M}(\mathrm{R}^2 + r^2).$$

C.—The cross-section is an ellipse.—When the major axis is parallel to the axis of the wheel, the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad \text{and} \quad x^2 = b^2\left(1 - \frac{y^2}{a^2}\right), \quad \text{and} \quad w = a.$$

Hence, $$\mathbf{I} = 8\pi\mathrm{R}m\int_0^a x\left(\mathrm{R}^2 + b^2 - \frac{b^2}{a^2}y^2\right)dy,$$

or, approximately,

$$\mathbf{I} = 8\pi\mathrm{R}m(\mathrm{R}^2 + b^2)\int_0^a x\,dy = 2\pi\mathrm{R}m(\mathrm{R}^2 + b^2)\mathrm{A} = \mathrm{M}(\mathrm{R}^2 + b^2),$$

where $\mathrm{A} = \pi ab$ is the area of the ellipse.

When the minor axis is parallel to the axis of the ring we must substitute a for b;

hence, $$\mathbf{I} = \mathrm{M}(\mathrm{R}^2 + a^2).$$

CHAPTER XXXIII.

EQUILIBRIUM OF FORCES.

204. Resultant of parallel forces.

The force-polygon of a system of parallel forces is a straight line parallel to the given forces. Hence, *the magnitude of the resultant of a system of parallel forces is equal to the algebraical sum of the forces.*

Two parallel forces and their resultant are obviously coplanar forces. In determining the resultant, we must consider separately whether the forces act in the same sense or in opposite senses.

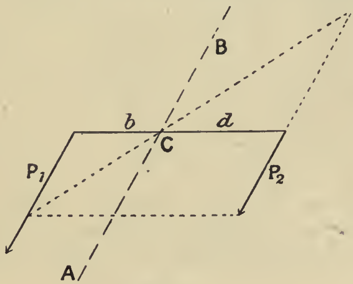


FIG. 101.

(i.) *The two parallel forces act in the same sense.*—Let P_1 and P_2 (fig. 101) be the two given forces, and A B the line of action of their resultant, R. The problem is to find the position of A B.

Take any convenient point 0 in the plane, and let r_1, r_2, and r be respectively the perpendicular distances of P_1, P_2, and A B from 0. The two given forces and $-R = -(P_1 + P_2)$ must balance. Hence, the algebraical sum of their torques about an axis through 0 and perpendicular on the plane must be zero (article **186**), or

$$P_1 r_1 + P_2 r_2 - Rr = 0, \quad \text{or} \quad P_1(r_1 - r) + P_2(r_2 - r) = 0.$$

Hence, $\frac{P_1}{r - r_2} = \frac{P_2}{r_1 - r} = \frac{R}{r_1 - r_2}$, or $\frac{P_1}{d} = \frac{P_2}{b} = \frac{R}{a}$, where $a = b + d$.

Hence, $$\frac{b}{d} = \frac{P_2}{P_1} \quad . \quad . \quad . \quad . \quad [183]$$

(ii.) *The two parallel forces act in opposite senses.*—Let $P_1 > P_2$ (fig. 102). As the sense of $-R$ is the same as that of P_2, we may consider P_1 as the resultant of $(-R)$ and P_2. Hence, by applying [183] we have

$$\frac{R}{a} = \frac{P_2}{b} = \frac{P_1}{d}, \text{ where } d = a + b.$$

Hence, $$\frac{a}{b} = \frac{R}{P_2} = \frac{P_1 - P_2}{P_2}, \quad \text{and} \quad b = \frac{P_2}{R} a \quad . \quad . \quad [184]$$

Let there be any number of parallel forces some of which act in one sense and others in the opposite sense. We may proceed

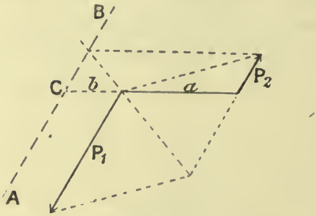


FIG. 102.

to determine the resultants R_1 and R_2 of the two groups by a successive application of [183]. Having found R_1 and R_2, we may find the resultant R of the given system of forces by applying [184].

205. Couples.

The resultant R will be zero when the numerical values of P_1 and P_2 in [184] are equal, say equal to P. The forces, however, are not balanced since $P(r_1 - r_2) = Pp$ is not zero. *A system of two equal and parallel forces acting in opposite senses but not in the same straight line is called a couple.* A couple produces, or tends to produce, a rotation of the body—upon which it acts—about an axis perpendicular to the plane of the couple, and through the centre of the straight line joining the points of application of the two given forces. The latter axis is called *the axis of the couple*, and *the perpendicular distance*, p, *between the two forces of the couple is called the arm, or the lever, of the couple.*

The torque Pp of the two forces of the couple is called *the torque of the couple,* and *is the product of one of the parallel forces into the lever of the couple.*

A couple is not determined by the positions and magnitudes of its two forces, but by its torque and the position and the direction of its axis.

It has been shown that the torque of the couple is P$p = P(r_1 - r_2)$, where r_1 and r_2 are respectively the levers of the two parallel forces w.r.t. any axis parallel to that of the couple. Hence, *the torque of the couple w.r.t. any axis parallel to the axis of the couple is constant.* Hence, it follows that:

(i.) *Couples are parallel when their axes are parallel;*

(ii.) *Couples are equal when their torques are equal;*

(iii.) *The resultant couple of any number of parallel couples is parallel to the given couples, and its magnitude* (torque) *is equal to the algebraical sum of the magnitudes* (torques) *of the given couples;*

(iv.) If the body on which the couple is applied can only turn about a fixed axis, the couple may be moved parallel to itself without disturbing the rotary motion. Hence, *when a system of forces, containing couples, is balanced, one or more of the couples may be moved parallel to themselves without disturbing the equilibrium* (*see* also article 222).

206. Resolution and composition of couples.

As a couple possesses magnitude (torque) and swing-direction

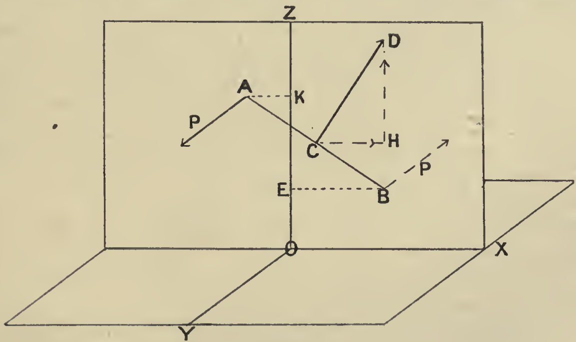


FIG. 103.

it may be represented by a straight line, whose length is proportional to the torque. It is conventional to draw the line from the centre of the lever along the axis of the couple and in such a sense that, looking from a point on the line towards the plane of the forces, the couple shall be seen to turn in the anti-clockwise direction.

In fig. 103 C D is the axis of a couple making an angle α with the z-axis of a system of rectangular space-co-ordinates. The axis of the couple is contained in the xz-plane, and the couple is turned so that its two forces, P, are perpendicular to the xz-plane.

It will be seen that the forces P produce a torque $\mathrm{P}(\mathrm{AK}+\mathrm{EB}) = \mathrm{P}p\cos\alpha$ about the z-axis, and another torque $\mathrm{P}\,\overline{\mathrm{KE}} = \mathrm{P}p\sin\alpha$ about the x-axis. But as $\mathrm{CD} \propto \mathrm{P}p$ we have, $\mathrm{HD} \propto \mathrm{P}p\cos\alpha$ and $\mathrm{CH} \propto \mathrm{P}p\sin\alpha$. Hence, the couples $\mathrm{P}p\cos\alpha$ and $\mathrm{P}p\sin\alpha$ are the components of $\mathrm{P}p$ in the directions of the axes of z and x respectively; and, conversely, $\mathrm{P}p$ is the resultant of $\mathrm{P}p\cos\alpha$ and $\mathrm{P}p\sin\alpha$. Hence, *couples as represented by their axes are compounded and resolved like vectors.* We have thus, the triangle, the parallelogram, the polygon, and the parallelopiped of couples.

207. Equilibrium of coplanar forces.

The resultant of any given system of coplanar forces may be found by the method described in article **153**. Let the number of the given forces be n and let R_1 be the resultant of the $n-1$ forces, then the resultant R of the whole system is the same as that of R_1 and P_n. There are four different cases to be considered, viz.:

(i.) R_1 *and* P_n *are equal and parallel forces acting in opposite senses but not in the same straight line.* The given system has no resultant, but is reduced to a couple whose torque is T. The given system will be balanced by adding a couple $(-\mathrm{T})$ whose axis is perpendicular on the plane of the system.

(ii.) R_1 *and* P_n *are equal and parallel forces which act in opposite senses and in the same straight line.* The given system of forces is balanced.

(iii.) R_1 *and* P_n *are either unequal and parallel forces, or equal and parallel forces acting in the same sense.* The system has a resultant parallel to P_n.

Equilibrium will be established by adding a force $(-\mathrm{R})$ in the line of action of R.

(iv.) R_1 *and* P_n *are not parallel.* The given system has a resultant, and balance is obtained as in (iii.).

We might also choose a pair of rectangular axes, as in article **154**, and resolve the given forces in the directions of the axes. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the positive angles which the respective forces form with the positive direction of the x-axis. The components of the given forces parallel to the x-axis are $\mathrm{P}_1\cos\alpha_1, \mathrm{P}_2\cos\alpha_2, \ldots \mathrm{P}_n\cos\alpha_n$; and those parallel to the y-axis are, $\mathrm{P}_1\sin\alpha_1, \mathrm{P}_2\sin\alpha_2, \ldots \mathrm{P}_n\sin\alpha_n$. The components are positive or negative according as $\cos\alpha$ and $\sin\alpha$ are positive or negative. Let there be an axis, the z-axis, through the origin O and perpendicular on the plane of the forces. Each force $P\cos\alpha$ produces a pressure on the z-axis and a couple $(-y\,P\cos\alpha)$ about the latter axis. Similarly, each force $P\sin\alpha$ produces a pressure $P\sin\alpha$ on, and a couple $xP\sin\alpha$ about, the z-axis. Hence,

the total pressure on the z-axis along the x-axis is $= \sum P\cos\alpha = X$,

and,

the total pressure on the z-axis along the y-axis is $= \sum P\sin\alpha = Y$.

The torque of the resultant couple about the z-axis is

$$\sum - yP\cos\alpha + \sum xP\sin\alpha = T.$$

(α) The given system of forces is balanced when $X=0$, $Y=0$, and $T=0$.

(β) $T=0$ *but* X *and* Y *are not zero.*—The system is reduced to a single resultant $R=\sqrt{X^2+Y^2}$ which passes through O, and forms an angle θ with the positive direction of the x-axis, $\tan\theta=\frac{Y}{X}$. To balance the system a force R through O and forming angle $(\theta-180)$ with the positive direction of the x-axis must be added.

(γ) $X=0$, $Y=0$ *but* T *not zero.*—The system will be balanced by adding a couple $-T$, whose axis is parallel to the z-axis.

(δ) *Neither* X, Y, *nor* T *are zero.*—The given system of forces can be reduced to a single resultant R which does not pass through O. To find the line of action, A B, of the resultant we proceed as follows: Let r be the length of the perpendicular let fall from O upon A B, then $rR=T$ or $r=T/R$. As we know the direction and sense of $R=\sqrt{X^2+Y^2}$, A B can be determined. The given system of forces will be balanced by adding the force $-R$ along A B.

208. Link-frames.

By a link-frame is understood a plane polygon consisting of straight bars connected by frictionless movable joints.

Fig. 104A represents a triangular link-frame with three forces P_1, P_2, and P_3 acting respectively at the three joints and in the plane of the frame.

The lettering of the frame and the forces may be done as shown in fig. 104A. It will be seen that every bar has a letter to its right and left, which is done for the purpose of naming it. The bars are named thus: O A, O B, and O C. As every force has also a letter on each side they may be named — when following the clockwise direction—A B, B C, and C A, instead of of P_1, P_3, and P_2. Finally, the joints are named respectively O A B, O B C, and O C A.

Resolve each of the forces P_1, P_2, and P_3 in the directions of its adjacent bars. The given forces will obviously be in equilibrium when

$$S_1 + S_4 = O\ ;\ S_2 + S_5 = O\ ;\ S_3 + S_6 = O \quad . \quad . \quad (1)$$

Let us choose any convenient point O (fig. 104B), and draw the three straight lines O A, O B, and O C respectively parallel to the bars O A, O B, and O C in fig. 104A. Next set off to scale $O A = S_4 = -S_1$; $O B = S_2 = -S_5$, and $O C = S_6 = -S_3$, then $A B = P_1$, $B C = P_3$, and $C A = P_2$.

To find whether three given coplanar forces P_1, P_2, and P_3 are balanced, we proceed by drawing the polygon of forces A B C A (fig. 104B), then choose any convenient point O and draw lines O A, O B, and O C. Then, through any point 1 in the line of action of P_1 (fig. 104C), draw a line 1–2 parallel to O A and meeting P_2 at point 2; through point 2 draw a line parallel to O C and meeting P_3 at point 3; then, through the latter point, draw a line parallel to O B. If the latter line meets P_1 at point 1, then the three given forces are balanced, as in fig. 104A. If, however, the straight line through point 3 and parallel to O B does not meet P_1 at point 1, as in fig. 104C, then the three forces are not balanced, but produce or tend to produce a rotary motion, whose torque is pP_1. By adding a force equal to and parallel to P_1 at point 4, and a force $-P_1$ at point 1, the system of forces will be balanced.

This method of examining a given system of forces may be extended to any number of coplanar forces. The system is called *graphic statics.*

The diagram (fig. 104B) is called the force-stress diagram, because it consists of the polygon of the external forces, and at the same time serves to determine the pull or push which take place at the ends of each bar.

It is obvious that if three coplanar forces are balanced they must meet at one point, as in fig. 104A. But if the three forces are not balanced they will not meet at one point, as in fig. 104C.

209. Reciprocal figures.

When n coplanar straight lines radiate from a point O, and an n-sided polygon be drawn, whose sides are parallel to or perpendicular to the corresponding lines through O, then the polygon is called the *reciprocal* for point O.

Thus the polygon of a number of balanced forces, passing through one point O, is the reciprocal for point O.

In fig. 104B, triangles O A B, O C A, and O B C, are reciprocals

Fig. 104A.

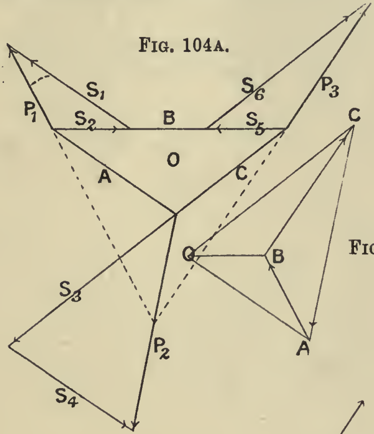


Fig 104B.

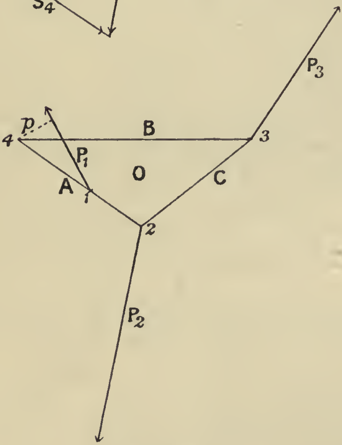


Fig. 104C.

for joints O A B, O C A, and O B C respectively; hence, the force-stress diagram (fig. 104B) is the *reciprocal force-diagram* for the frame in fig. 104A.

210. The method of sections.

If the external forces acting at the joints of the link-frame (fig.

105) D A, D B, and D C are balanced, equilibrium will not be disturbed by cutting the bars D B and D C, provided that the necessary external forces be added at the points where the bars are cut; the two latter forces and force B C must balance, but only B C is known. As, however, the resultant of any two of the three forces acts in the line of the third force, we may choose any convenient point, O, in the line of action of one of the unknown

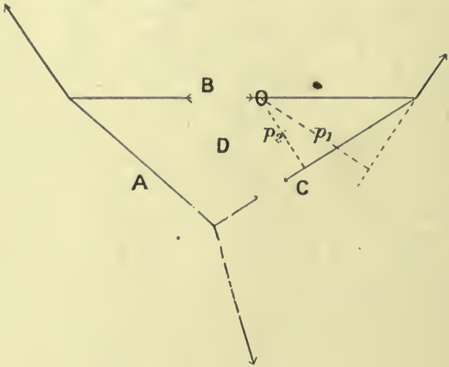


FIG. 105.

forces, say D B, as fulcrum, and measure the lengths of the perpendiculars, p_1 and p_2, let fall from point O on the forces B C and C D respectively. We have thus,

$$p_1 \times \text{force } BC = p_2 \times \text{force } CD.$$

Hence,
$$\text{force } CD = \frac{p_1}{p_2} \times \text{force } BC.$$

By this process, known as *the method of sections*, all the internal forces in the bars may be determined.

CHAPTER XXXIV.

MACHINES.

211. Definition.

A machine is a contrivance by which energy is transmitted from one point to another while undergoing a required modification.

212. Efficiency—Mechanical advantage—Velocity-ratio.

Any machine, however complicated, is a combination of one or both of the two elementary machines: The *lever* and the *inclined plane.* The pulley and the wheel and axle are examples of levers; the screw and the wedge are both modifications of the inclined plane.

The energy received by the engine or prime-mover is converted by the latter into energy of a suitable kind and then transmitted by machines to the point where it is to be utilised, whether it be in the lifting of weights, the working of materials, etc. The energy-equation for such a system of machinery is —

Energy received by prime-mover = energy available + energy lost in the machinery.

The ratio $\frac{\text{energy available}}{\text{energy received}}$ is called the *efficiency* of the system of machinery, and is always less than unity, as it is impossible to avoid losses however well the machinery may be designed. Each elementary machine receives energy from the preceding one and delivers energy to the following one. Let η_1, η_2, etc., be the efficiencies of the respective elements of a series of machines, and E the amount of energy received by the first element. The amounts of energy delivered by the successive elements are $\eta_1 E$; $\eta_2(\eta_1 E)$; $\eta_3(\eta_2\eta_1 E)$ . . . . Hence, the efficiency of the series is $\eta = \eta_1\eta_2\eta_3$. . . . or, *the efficiency of a series of machine-elements is equal to the product of the efficiencies of the respective elements.*

Let P be the effort applied at the receiving end of the machine, W the load or useful resistance to be overcome at the other end of the contrivance, H and h corresponding displacements of P and W respectively, then

$\frac{W}{P} = \beta$ is called *the mechanical advantage* of the machine;

$\frac{H}{h} = \alpha$ is called the *displacement-ratio*, or, more commonly, the *velocity-ratio* of the machine.

The *efficiency* of the machine is

$$\eta = \frac{Wh}{PH} = \frac{\beta}{\alpha}.$$

Let P_1 be the effort required to overcome W if there were no waste, then the efficiency would be

$$\eta_1 = \frac{Wh}{P_1H} = 1, \quad \text{or} \quad \frac{W}{P_1} = \alpha, \quad \text{and} \quad \frac{P_1}{P} = \frac{\beta}{\alpha} = \eta \quad . \quad [185]$$

Hence, the *theoretical mechanical advantage is equal to the velocity-ratio*, and *the efficiency* of the machine *is equal to the ratio of the theoretical effort and the actual effort.*

213. The screw.

The screw-line or helix is the curve of least distance between two points on the surface of a cylinder of revolution. Hence, if the cylinder be rolled off, the helix will appear as a straight line inclined towards the base of the cylinder. The angle of slope is called the *pitch-angle*, and the rise corresponding to the length of the circumference of the base is called the *pitch*.

When the screw is applied as a machine to lift weights, the screw-jack, we may take it that the load is equally distributed over the bearing-surface of the screw, and hence, we may consider the load as being concentrated at one point on the screw-surface, situated on the mean-helix. The screw as a machine may thus be considered as an inclined plane.

Let D and d be respectively the external and internal diameters of the screw, p the pitch, and θ the pitch-angle. The mean diameter corresponding to the mean helix is obviously $d_m = (D + d)/2$, and the slope of the inclined plane is $\tan\theta = p/\pi d_m$.

214. Pulley and belt.

To enable the belt to drive the pulley the tension, T_2, at the point where the belt leaves the pulley must be greater than the tension, T_1, at the point where the belt meets the pulley. The driving force of the belt is thus $T_2 - T_1$, and if the velocity of the belt be v, the driving power of the belt is obviously $(T_2 - T_1)v$.

If T_2 and T_1 be expressed in lbw. and v in feet per second, the *horse-power* transmitted by the belt is $(T_2 - T_1)v/550$.

If D and d be respectively the diameters of the pulleys on the driving and driven shafts and no slipping takes place, the angular velocity-ratio of the two shafts is d/D.

To prevent slipping the total force of friction between the belt and the face of the pulley-rim must never be less than $T_2 - T_1$. Fig. 106 represents a pulley and belt, A B C D being the arc embraced by the belt. A B is an arc subtending angle θ, and B C is an element of length δs subtending angle B O C $= \delta\theta$. If the total force of friction between the belt and the arc A B of the rim be $F = \mu p$,

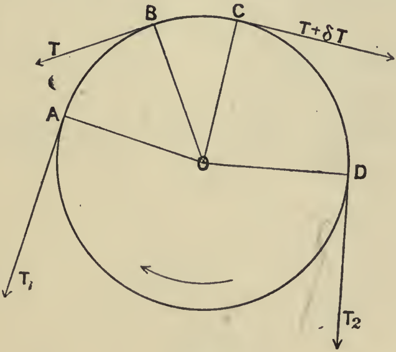


FIG. 106.

then slipping will just be prevented when $\delta T = \delta F = \mu\,\delta p$, where δp is the resultant of T and $T + \delta T$. If the tension at B and C were both T, δp would be $= 2T \sin(\delta\theta/2)$, and if the tensions at the two points were both $T + \delta T$, we should have $\delta p = 2(T + \delta T)\sin(\delta\theta/2)$. Hence, $2T\sin(\delta\theta/2) < \delta p < 2(T + \delta T)\sin(\delta\theta/2)$; hence, in the limit, $dp/d\theta = T$.

Let w be the weight of unit length of belt, r the radius of the pulley, and v the velocity of the belt. The belt-element B C will require a normal force $\frac{w}{g}\frac{v^2}{r}\,\delta s$, and the pressure between the belt-element and the pulley-rim will be reduced to

$$\delta p - \frac{w}{g}\frac{v^2}{r}\,\delta s = \delta p - \frac{w}{g}v^2\,\delta\theta, \quad \text{and} \quad \delta F = \delta T$$

$$= \mu\left(\delta p - \frac{w}{g}v^2\,\delta\theta\right) = \mu\left(T\,\delta\theta - \frac{w}{g}v^2\,\delta\theta\right).$$

Hence, in the limit, $$\frac{dT}{d\theta}=\mu\left(T-\frac{w}{g}v^2\right) \quad . \quad . \quad . \quad . \quad (1)$$

Integrating (1) we get

$$\int_{T_1}^{T_2}\frac{dT}{T-\frac{w}{g}v^2}=\mu\int_0^\gamma d\theta, \quad \text{or} \quad \frac{T_2-\frac{w}{g}v^2}{T_1-\frac{w}{g}v^2}=e^{\mu\gamma}. \quad . \quad (2)$$

where γ = angle A O D.

The belt will be able to drive if the resistance Q reduced to the pulley-rim is equal to or less than the friction between the belt and the rim, or

$$T_2-T_1=Q \quad . \quad . \quad . \quad . \quad (3)$$

By (2) and (3) we have

$$T_2=\frac{Qe^{\mu\gamma}}{e^{\mu\gamma}-1}+\frac{w}{g}v^2 \quad \text{and} \quad T_1=\frac{Q}{e^{\mu\gamma}-1}+\frac{w}{g}v^2 \quad . \ [186]$$

CHAPTER XXXV.

CENTROIDS OF PLANE FIGURES.

215. Centroid of a plane curve.

The first moment, $\delta(s.m.)$, of an arc-element w.r.t. the x-axis is $y\,\delta s$, and that w.r.t. the y-axis is $x\,\delta s$ (fig. 107).

Through a point C (α, β), in the plane of the given curve, draw two straight lines respectively parallel to the given axes.

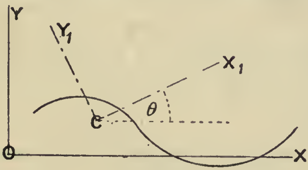


FIG. 107.

The first arc-moments about the former lines are $\sum(x-\alpha)\delta s$ and $\sum(y-\beta)\delta s$.

Let x_1 and y_1 be the co-ordinates referred to a pair of rectangular axes having their origin at C and the x_1-axis forming a positive angle θ with the given x-axis, then by article **5** we have

that $\quad x-\alpha = x_1\cos\theta - y_1\sin\theta, \quad$ and $\quad y-\beta = x_1\sin\theta + y_1\cos\theta.$

Hence, $\qquad \sum(x-\alpha)\delta s = \cos\theta\sum x_1\delta s - \sin\theta\sum y_1\delta s,$

and $\qquad \sum(y-\beta)\delta s = \sin\theta\sum x_1\delta s + \cos\theta\sum y_1\delta s.$

If we choose point C so that

$$\sum(x-\alpha)\delta s = 0 \quad \text{and} \quad \sum(y-\beta)\delta s = 0 \quad . \quad (1)$$

then we have also $\sum x_1\delta s = 0$ and $\sum y_1\delta s = 0$. Hence, *the first arc-moment about any straight line through point C is zero. The* *latter point is called the* **centroid** *of the plane curve;* and its co-ordinates, α and β, referred to the given system of co-ordinates are determined by (1).

Hence, $$\alpha = \frac{\sum x\,\delta s}{s}, \text{ and } \beta = \frac{\sum y\,\delta s}{s},$$

where s is the total length of the given curve.

As $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

we have,

$$\alpha = \frac{\int_{x_1}^{x_2} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}, \text{ and } \beta = \frac{\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx} \quad [187]$$

When the curve possesses an axis of symmetry the latter contains the centroid. Hence, if there are more than one axis of symmetry (circle, ellipse, etc.), the point at which the axes meet is the centroid.

A.—*Centroid of circular arc.*—Let the equation of the circle be $x^2 + y^2 = r^2$, and c the length of the chord. If the axes be drawn so that the y-axis bisects the given arc, then the abscissa, α, of the centroid is naught. We have that $dy/dx = -x/y$, and $x = \sqrt{r^2 - y^2}$.

Hence, $$\beta = \frac{1}{s}\int_{-\frac{c}{2}}^{+\frac{c}{2}} y\sqrt{1 + \frac{x^2}{y^2}}\,dx = \frac{rc}{s}.$$

B.—The student should now study article **134**.

216. Centroid of a plane area.

The first area-moments of the area-element of a plane figure w.r.t. to the axes of a system of rectangular co-ordinates are $\delta(a.m.)_x = y\,\delta A$, and $\delta(a.m.)_y = x\,\delta A$, where $\delta A = \delta x\,\delta y$ (fig. 107A).

By the same process of reasoning as we adopted in the preceding article we shall find that the plane area possesses a centroid whose co-ordinates are

$$\alpha = \frac{\sum x\,\delta A}{A} = \frac{\sum x\,\delta x\,\delta y}{A}, \text{ and } \beta = \frac{\sum y\,\delta A}{A} = \frac{\sum y\,\delta x\,\delta y}{A} \quad [188]$$

Any axis of symmetry contains the centroid. Hence, the meeting-point of two or more axes of symmetry is the centroid.

The student is advised not to use the *erroneous expressions mass-centre and centre of gravity of a curve or of a surface*, which are so often used instead of the term centroid. Neither a curve nor a surface possesses mass; hence, they cannot be subject to

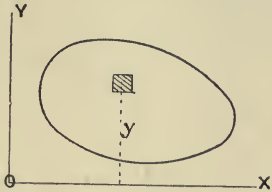


FIG. 107A.

gravitation. The dimensions of the first area-moment are $[L]^3$, those of the first mass-moment are [M] [L], and those of the first force-moment are $[M]\,[L]^2\,[T]^{-2}$.

A.—Centroid of a triangle.—The centroid must be situated on the median B M (fig. 108); we have therefore only to find the

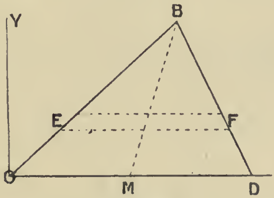


FIG. 108.

co-ordinate β. Let $b = \mathrm{O\,D}$ be the length of the base, h the height, and $\mathrm{E\,F} = \eta$. The elementary area of the triangle is, $\delta \mathrm{A} = \eta\,\delta y$; but $\eta : b :: (h-y) : h$.

Hence, $\quad \eta = b - \dfrac{b}{h}y,\ \delta(a.m.)_x = by\ \delta y - \dfrac{b}{h}y^2\,\delta y,$

$$(a.m.)_x = b\int_0^h y\ dy - \frac{b}{h}\int_0^h y^2 dy = \frac{bh^2}{6}, \text{ and } \mathrm{A} = \frac{bh}{2}.$$

Hence, $$\beta = \frac{(a.m.)_x}{\mathrm{A}} = \frac{h}{3},$$

or *the centroid is situated on the median B M at a distance of $\frac{2}{3}$ of the height from the vertex.*

B.—Centroid of a circle sector.—The radius O D $= r$ which bisects the angle at the centre of the given sector (fig. 109) contains the centroid. The centroid of an elementary sector must, according to the preceding proposition, be situated at a distance $\frac{2}{3}r$ from O. Hence, the circular arc F M L, whose radius is $\frac{2}{3}r$, is the locus of the centroids of all the elementary sectors

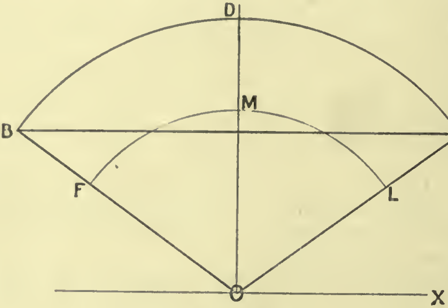


FIG. 109.

between O B and O E. Hence, the centroid of the given sector is the same as the centroid of the arc F M L, or, by the preceding article,

$$\beta = \frac{\overline{\mathrm{OF}}\ \overline{\mathrm{FL}}}{\mathrm{FML}} = \frac{\frac{2}{3}r\frac{2}{3}c}{\frac{2}{3}s} = \frac{2}{3}\frac{rc}{s}.$$

C.—Centroid of a circle segment.—Let the segment be B D E B (fig. 109). The centroid is situated on O D at a distance β from O.

Area of sector O B D E O $= 0{\cdot}5\ rs$; Area of triangle O B E O $= 0{\cdot}5\ cr \cos\frac{a}{2}$; where a is the angle at the centre.

$(a.m.)_x$ of sector $= r^2c/3$; $(a.m.)_x$ of triangle $= cr^2\cos^2\frac{a}{2}\Big/3$; $(a.m.)_x$ ...t $= (a.m.)_x$ of sector *minus* $(a.m.)_x$ of triangle $= c^3/12$.

$$\beta = \frac{c^3}{12\mathrm{A}},$$

... area of the segment.

...*d of a trapezoid.*—The centroid will be found on the median K L at a perpendicular distance β from B E (fig. 110). Let C D $=b$, B E $=a$, and area F H $=\delta A$ at a distance y from B E.

$$\delta A=\left[b+(a-b)\left(1-\frac{y}{h}\right)\right]\delta y, \quad \text{and} \quad \delta(a.m.)_x$$
$$=\delta A\, y=ay\ \delta y-\frac{a-b}{h}y^2\,\delta y.$$

Hence,

$$(a.m.)_x=a\int_0^h y\ dy-\frac{a-b}{h}\int_0^h y^2\,dy=\frac{a+2b}{6}h^2;\ A=\frac{a+b}{2}h.$$

Hence, $$\beta=\frac{(a.m.)_x}{A}=\frac{a+2b}{3(a+b)}h.$$

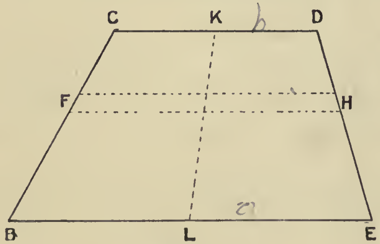


FIG. 110.

E.—Centroid of a parabolic sector.—The sector O B D O (fig. 111) is bounded by the parabolic arc O B, the abscissa O D $=d$, and the

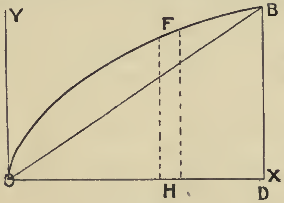


FIG. 111.

ordinate D B $=b$ to point B. Area F H $=\delta A$ at a distance x from O. Equation of the parabola is $y^2=4ax$. We have that

$$\delta A=y\,\delta x=2\sqrt{a}\,\sqrt{x}\,\delta x;\ \delta(a.m.)_y=\delta A\ x=2\sqrt{a}\,x^{\frac{3}{2}}\,\delta x.$$

The ordinate of the centroid of δA is $y/2$; hence, $\delta(a.m.)_x=\delta A\,y/2=2ax\,\delta x$. $A=\frac{2}{3}bd$; $b^2=4ad$.

Hence,

$$(a.m.)_x = 2a\int_0^d x\,dx = \frac{db^2}{4};\ (a.m.)_y = 2\sqrt{a}\int_0^d x^{\frac{3}{2}}\,dx = \tfrac{2}{5}bd^2.$$

Let the co-ordinates of the centroid be α and β, then

$$\alpha = \frac{(a.m.)_y}{A} = \frac{3}{5}d, \quad \text{and} \quad \beta = \frac{(a.m.)_x}{A} = \frac{3}{8}b.$$

F.—Centroid of a parabolic segment.—The segment is bounded by the parabolic arc O B and the chord O B (fig. 111).

For sector O B D O we have

$$(a.m.)_x = \frac{db^2}{4}, \quad \text{and} \quad (a.m.)_y = \tfrac{2}{5}bd^2.$$

For triangle O B D O we have,

$$(a.m.)_x = \frac{bd}{2}\frac{b}{3} = \frac{b^2d}{6}, \quad \text{and} \quad (a.m.)_y = \frac{bd}{2}\frac{2d}{3} = \frac{bd^2}{3}.$$

$(a.m.)_x$ of segment $= (a.m.)_x$ of sector *minus* $(a.m.)_x$ of triangle $= \dfrac{b^2d}{12}$.

$(a.m.)_y$ of segment $= (a.m.)_y$ of sector *minus* $(a.m.)_y$ of triangle $= \dfrac{bd^2}{15}$.

$$\text{Area of segment} = \frac{2bd}{3} - \frac{bd}{2} = \frac{bd}{6}.$$

Hence, $$\alpha = \frac{bd^2}{15}\,\frac{6}{bd} = \frac{2d}{5};\ \beta = \frac{b^2d}{12}\,\frac{6}{bd} = \frac{b}{2}.$$

G.—Approximate determination of the centroid of a plane figure.—When the equation of the curve bounding the plane figure is unknown the centroid may be found approximately by Simpson's rule [99]. We have that $(a.m.)_y = \sum \delta A\, x = \sum yx\,\delta x$, and $(a.m.)_x = \sum \delta A \frac{y}{2} = \frac{1}{2}\sum y^2\,\delta x$ (fig. 97). Hence, [99] may be applied to determine $(a.m.)_y$ and $(a.m.)_x$ by substituting yx, and $y^2/2$ for y respectively; the values of x being 0, h, $2h$, $3h$, . . . $2nh$.

Hence, $$(a.m.)_y = \frac{h^2}{3}\left[2ny_{2n} + 4\sum_{r=1}^{r=n}(2r-1)y_{2r-1} + 2\sum_{r=1}^{r=n-1}2ry_{2r}\right] \quad [189]$$

$$(a.m.)_x = \frac{h}{6}\left[y_0^2 + y_{2n}^2 + 4\sum_{r=1}^{r=n}y_{2r-1}^2 + 2\sum_{r=1}^{r=n-1}y_{2r}^2\right] \quad . \quad [190]$$

$\alpha = \dfrac{(a.m.)_y}{A}$, $\beta = \dfrac{(a.m.)_x}{A}$, A being determined by [99].

EXAMPLE.—*Determine the co-ordinates of the centroid of the plane figure which is bounded by the axes and the curve, whose equidistant ordinates are given below. The distance between consecutive ordinates is* $h=2$.

Suffix r	0	1	2	3	4	5	6	7	8
y_r	0	5·66	8·00	9·80	11·31	12·65	13·86	14·97	16
ry_r	0	5·66	16·00	29·40	45·24	63·25	83·16	104·79	128·00
y_r^2	0	32	64	96	128	160	192	224	256

By applying [189], [190], and [99] it will be found that

$$\alpha=9{\cdot}6 \quad \text{and} \quad \beta=6{\cdot}0.$$

H.—To determine the centroid of a plane area by graphical method.—Let P_1, P_2, and P_3 (fig. 112A), be three parallel

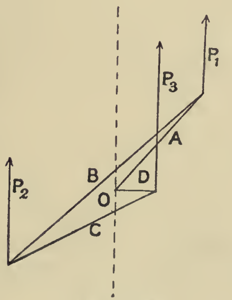


FIG. 112A.

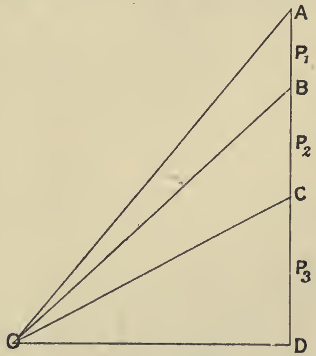


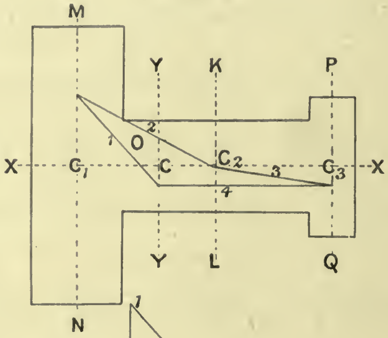
FIG. 112B.

coplanar forces, and let the problem be to find the line of action of their resultant R. Draw first the force-polygon which is the straight line DA (fig. 112B), choose any convenient point O as pole and draw the stress-polygon in the usual way, then draw the link-polygon (fig. 112A); then the line of action of R passes through joint OADO.

Let p_1, p_2, and p_3 be the perpendicular distances between R and P_1, P_2, and P_3 respectively, then $p_2P_2 = p_1P_1 + p_3P_3$.

The area of fig. 113A is made up of three parallelograms whose areas, from the left to the right, are respectively A_1, A_2, and A_3, and their respective centroids are the middle-points, C_1, C_2, and C_3. The problem is to determine the centroid C of the whole

FIG. 113A.



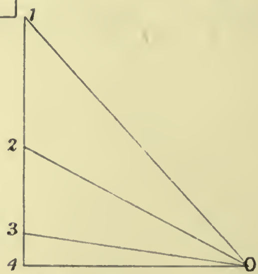


FIG. 113B.

area A. The first area-moment about Y Y must be zero, or $A_1\overline{C\,C_1} = A_2\overline{C\,C_2} + A_3\overline{C\,C_3}$. Hence, we may apply the construction shown in figs. 112A and 112B by substituting areas for forces, and thus determine a point on Y Y. In fig. 113B line 1–4 is drawn parallel to M N, and (1–2) : (2–3) : (3–4) : : $A_1 : A_2 : A_3$; take any convenient point O and draw the lines O–1, O–2, O–3, and O–4, then draw the link-frame (fig. 113A) in the usual way. The joint of bars O–1 and O–4 is a point on Y Y.

The student should now study article **128**.

CHAPTER XXXVI.

MASS-CENTRE.

217. Mass-centre.

The magnitude and the point of application, C, of the resultant, R, of the two given parallel forces, P_1 and P_2 (fig. 101), are obviously independent of the angle which the direction of the given forces makes with the line joining their points of application. Point C is therefore called the *centre* of the two parallel forces. Hence, a system of any number of parallel forces possesses a centre,

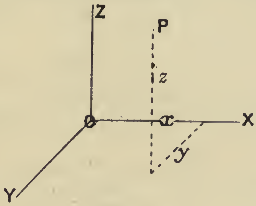


FIG. 114.

which is the point of application of the resultant of the forces, and is independent of the direction of the forces.

We have seen that the weight of a particle is equal to the mass of the particle into the acceleration of gravity, but the value and the direction of the latter varies with the position of the particle relative to other bodies, especially to the earth and the heavenly bodies. The weight of a physical body is the resultant of the weights of all the particles which constitute the body.

When the dimensions of a body are sufficiently small the acceleration of gravity may be taken to be the same for all the particles of the body, and we may consider the weights of the particles as a system of parallel forces. The body possesses then a *weight-centre*, generally called the *centre of gravity* (C.G.).

Take a system of three rectangular co-ordinates (fig. 114), and let the z-axis be vertical. We have then three planes of reference, viz. the xy-plane, the xz-plane, and the yz-plane; a point P in space is determined by (x, y, z). Let (α, β, γ) be the C.G. of a body, M the mass, and $W = Mg$ the weight of the body. The moment of W w.r.t. any of the three planes of reference is equal to the sum of the moments of the weights of the particles w.r.t. the same plane of reference. Hence,

$$\alpha W = \alpha Mg = \sum xmg\,;\ \beta W = \beta Mg = \sum ymg\,;\ \gamma W = \gamma Mg = \sum zmg, \quad \text{or}$$

$$\alpha = \frac{g\sum xm}{gM} = \frac{\sum xm}{M}, \text{ and similarly } \beta = \frac{\sum ym}{M},\ \gamma = \frac{\sum zm}{M} \qquad [191]$$

Hence, the position of the C.G. in the body is independent of gravity, and only depends on the distribution of the masses of the particles within the body. We may therefore use the term *mass-centre* (M.C.) instead of centre of gravity.[1]

When the dimensions of the body, such as a mountain, are large, then the magnitude as well as the direction of g vary with the position of each particle. Hence, *such a body possesses no centre of gravity.*

A body, however large, has always an M.C., see **[191]**, but it may not have a C.G. Hence, *the term "centre of gravity" is superfluous and misleading, and the term "mass-centre" only ought to be adopted.*

218. Mass-centre of a homogeneous body.

Let m be the mass of unit volume of a homogeneous body, then $m\,\delta V$ is the mass of a volume-element, $xm\,\delta V$ is the first mass-moment of an element of the body w.r.t. the yz-plane, and $mV = M$ is the mass of the body. Hence, the co-ordinates of the mass-centre of the body are

$$\alpha = \frac{\sum xm\,\delta V}{M} = \frac{m\sum x\,\delta V}{mV} = \frac{\sum x\,\delta V}{V}, \text{ and similarly}$$

$$\beta = \frac{\sum y\,\delta V}{V},\ \gamma = \frac{\sum z\,\delta V}{V} \quad . \quad . \quad (1)$$

or the mass-centre of a homogeneous body coincides with the centroid of its mathematical body.

[1] As the weight mg and the z-co-ordinate are both vertical, the moment zmg would be zero; hence, for the purpose of determining γ we imagine the direction of the weights horizontal; this will not alter the position of the C.G. as explained in the present article.

Hence, *when the centroid of a mathematical body has been determined, the mass-centre of the corresponding homogeneous physical body is also determined.*

Any plane that divides the mathematical body into two symmetrical portions contains the centroid. If there be two planes of symmetry their line of intersection contains the centroid, and if there be more than two planes of symmetry the centroid is the point at which the three planes meet. If a body has a centre (sphere, ellipsoid, etc.), that point is the centroid.

A.—Centroid of a right cylinder.—By a right cylinder is understood a cylinder whose generators are perpendicular on the ends.

The axis of a cylinder is the straight line joining the centroids of the end-surfaces.

The volume-element of the cylinder is $\delta V = h\,\delta A$, where A is the area of the end-surface and h the height of the cylinder. The first volume-moment about any plane containing the axis is $p\,\delta V = ph\,\delta A$, p being the perpendicular distance between δV and the plane. Hence, the distance ρ of the centroid from the plane is

$$\rho = \frac{\sum ph\,\delta A}{\sum h\,\delta A} = \frac{\sum p\,\delta A}{A};$$

but $\sum p\,\delta A$ is the area-moment of the end-surface about a line through its centroid; hence, $\rho = 0$, or *the centroid of the cylinder* lies in the axis, and *is* obviously *the middle-point of the axis of the cylinder.* The same rule holds for a right prism.

B.—Centroid of a body of revolution.—Any plane containing the axis of rotation divides the body symmetrically. Hence, the centroid of the body lies in the axis. Take any plane, Q, at right angles to the axis as the plane of reference. At a distance x from Q the sectional area of the body is A_x, the volume-element is $\delta V = A_x\,\delta x$, and the elementary first volume-moment is $xA_x\,\delta x$. Hence, the centroid of the body lies in the axis and at a distance

$$a = \frac{\sum x\,A_x\,\delta x}{V} \text{ from the plane of reference} \quad . \quad (2)$$

C.—Centroid of a cone or a pyramid.—The axis of the body is the straight line joining the apex and the centroid of the base. For the same reason, as explained in the present article, the centroid of the body lies in the axis.

By article **120** we have

$$A_x = \frac{x^2}{H^2}A\,;$$

hence, $$\sum x\,A_x\delta x = \frac{A}{H^2}\int_0^H x^3\,dx = \frac{A\,H^2}{4}\,;\ V = \frac{A\,H}{3}$$

Hence, $$a = \tfrac{3}{4}H \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or *the centroid of a cone or of a pyramid lies in the axis of the body at a distance equal to* $\frac{3}{4}$ *of the height from the apex.*

D.—Centroid of a spherical segment (see article **123**).

$$A_x = \pi y^2 = \pi(r^2 - x^2)\,;\ V = \frac{\pi h^2(3r-h)}{3}\,;\ x\,\delta V = \pi(r^2 - x^2)x\,\delta x.$$

Hence, $$a = \pi\int_{r-h}^{r} x(r^2 - x^2)\,dx \Big/ V = \frac{3}{4}\,\frac{(2r-h)^2}{3r-h} \quad . \quad . \quad (4)$$

E.—Centroid of a spherical sector.—The centroids of the spherical segment and spherical sector are situated in the axis.

First volume-moment of sector = first volume-moment of segment + first volume-moment of cone.

Volume of sector,

$$V = \frac{\pi h^2(3r-h)}{3} + \frac{\pi h(2r-h)(r-h)}{3} = \frac{2}{3}\pi r^2 h.$$

First volume-moment of sector

$$= \frac{\pi h^2(2r-h)^2}{4} + \frac{\pi h(2r-h)(r-h)}{3}\,\frac{3(r-h)}{4} = \frac{\pi h r^2(2r-h)}{4}.$$

Hence, $$a = \frac{\pi h r^2(2r-h)}{4}\,\frac{3}{2\pi r^2 h} = \frac{3}{8}(2r-h) \quad . \quad . \quad (5)$$

F.—Centroid of a paraboloid of revolution.—The paraboloid is generated by the rotation of the parabola about its axis. As plane of reference we will take the plane Q which is generated by the rotation of the y-axis. Let $y^2 = 4ax$ be the equation of the parabola and H the height of the body. At distance x from Q the sectional area of the body is $A_x = \pi y^2 = 4a\pi x\,;\ \delta V = 4a\pi x\,\delta x\,;$ $V = 4\pi a\int_0^H x\,dx = 2a\pi H^2\,;\ x\,\delta V = 4a\pi x^2\,\delta x.$

Hence, $$a = 4a\pi\int_0^H x^2 dx \Big/ 2a\pi H^2 = \tfrac{2}{3}H \quad . \quad . \quad . \quad (6)$$

G.—M.C. *of a body determined by experiment.*—The M.C. of a homogeneous disc with two flat end-faces may conveniently be determined by balancing the disc on a knife's edge, in two directions meeting at a point C on the lower face. The M.C. is the middle-point of the vertical axis through point C. When the two end-faces of the disc are parallel, point C is the centroid of the face.

CHAPTER XXXVII.

EXAMPLES.

(95) (Q. Nov. 1908).—**A circular pulley, moment of inertia I, and radius r, is mounted on frictionless bearings. Over the pulley hangs a light string carrying a weight P at one end and a weight Q at the other. The system is in motion, and there is no slipping of the string on the pulley. Find an expression for the kinetic energy of the system when the angular velocity of the wheel is ω.**

Find the values of the tension in the string when the weights are 8 lbs. and 11 lbs., and the pulley is a disc of uniform thickness weighing 10 lbs.

Solution.—Let $M = \mathbf{I}/r^2$ be the mass of the pulley reduced to radius r (article **185**), T_1 and T_2 the tensions in the string between P and the pulley and between the pulley and Q respectively. We may consider the system as consisting of three portions

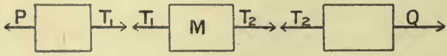


FIG. 115.

(fig. 115), moving together with the same acceleration. The whole system of the three masses is urged on by the effective tangential force Q − P, Q being greater than P, and the sense of the motion is therefore that of Q. Let a be the acceleration of the system, then

$$Q - P = a\left(\frac{Q}{g} + M + \frac{P}{g}\right).$$

Hence,
$$a = \frac{(Q - P)g}{Q + P + Mg} \quad . \quad . \quad . \quad . \quad (1)$$

The masses of P and Q are urged on by the effective forces $T_1 - P$ and $Q - T_2$ respectively.

Hence, $$T_1 - P = a\frac{P}{g}\,;\ Q - T_2 = a\frac{Q}{g}.$$

Hence, $$T_1 = \frac{P(2Q+Mg)}{Q+P+Mg} \text{ and } T_2 = \frac{Q(2P+Mg)}{Q+P+Mg} \quad . \quad . \quad (2)$$

When the pulley is a disc, *i.e.* a solid cylinder of revolution, and its weight is W, then

$$\mathbf{I} = \frac{Wr^2}{2g} \text{ (article 200).}$$

Hence, $$M = \frac{W}{2g},\ T_1 = \frac{P(4Q+W)}{2(P+Q)+W},\ \text{and } T_2 = \frac{Q(4P+W)}{2(P+Q)+W} \quad (3)$$

Inserting the given values in (3) we get

$$T_1 = 9 \text{ lbws.},\ T_2 = 9\tfrac{5}{8} \text{ lbws. } (\textit{Ans.}).$$

The three portions of the kinetic energy of the system are: E_k stored in the pulley $= 0{\cdot}5\omega^2\mathbf{I}$, E_k stored in $Q = 0{\cdot}5(r\omega)^2 Q/g$, and that in $P = 0.5(r\omega)^2 P/g$.
Hence,

$$\text{total } E_k = 0{\cdot}5\omega^2\left(\mathbf{I} + r^2\frac{Q+P}{g}\right) = \frac{\omega^2 r^2}{4g}[W + 2(Q+P)] \ (\textit{Ans.}) \quad (4)$$

(96) (Q. Nov. 1906).—**A cylindrical drum weighing 40 lbs. and having a radius of 1 foot and a radius of gyration of 9 inches can turn without friction about its axis, which is horizontal and in fixed bearings. A weight of 10 lbs. is attached to one end of a string which is coiled round the drum. The drum is held with the weight hanging freely, and is then let go so that the weight falls, causing the string to unwind and the drum to turn. Find the angle through which the drum turns in the first second after the drum is let go.**

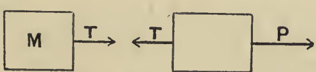


FIG. 116.

Solution.—Let W = weight of drum, r = radius of drum, M = mass of drum reduced to the circumference, T = tension in the string, and P = suspended weight. The weight of the string is neglected.

The whole system is urged on by the constant force P; hence, the acceleration is constant. M is urged on by force T, and the mass of P by force P − T (fig. 116). Hence, $T = aM$ and $P - T = aP/g$. But $Mr^2 = R_g^2 W/g$ (article 185).

Hence, $$a = \frac{Pr^2 g}{Pr^2 + W R_g^2} \quad \text{and} \quad T = \frac{P W R_g^2}{Pr^2 + W R_g^2} \quad . \quad (1)$$

Let s be the distance traversed by P and α the angle turned through in time t, then

$$s = \frac{1}{2}at^2 = \frac{Pr^2 g}{2(Pr^2 + W R_g^2)} t^2 \quad \text{and} \quad \alpha = \frac{s}{r} \text{ radians}$$

$$= \frac{Prg}{2(Pr^2 + W R_g^2)} t^2 \frac{180}{\pi} \text{ degrees} \quad . \quad (2)$$

The given numerical values are:

$P = 10$ lbws.; $W = 40$ lbws.; $r = 1$ ft.; $R_g = 0{\cdot}75$ ft.; $t = 1$ sec.

$T = 7$ lbws. (nearly); $\alpha = 283^\circ\ 45'$ (*Ans.*).

(97) (I.C.E., Oct. 1909).—A double-armed swing-bridge revolves upon a horizontal turn-table at the centre of its length, being actuated by a chain-drum 32 feet in diameter. The two main girders of the bridge, 180 feet in length, are spaced 30 feet apart, transversly (centre to centre) and each girder has the uniform weight of 10 cwts. per foot. Reduce the inertia of these revolving girder-masses to the driving-point.

Solution.—Let w be the weight of each girder per foot run, D the diameter of the drum, d the distance between the centre-lines of the girders, and $2l$ the length of each girder.

Considering each girder as a thin rod of mass w/g per foot run, we may proceed to determine the moment of inertia of each w.r.t. a vertical axis through the centre, the mass-centre, of the girder. The mass of a length δx of the girder is

$$\frac{w}{g}\delta x, \text{ and } \delta \mathbf{I} = \frac{w}{g}\delta x\, x^2. \quad \text{Hence, } \mathbf{I} = 2\frac{w}{g}\int_0^l x^2 dx = \frac{2}{3}\frac{w}{g}l^3 \quad (1)$$

The total mass of each girder $= 2wl/g$. Hence, by article 202 the moment of inertia of the two girders w.r.t. the axis of the drum is

$$2 \times \frac{2}{3}\frac{w}{g}l^3 + 2\frac{w}{g}2l\frac{d^2}{4} = \frac{1}{3}\frac{w}{g}(4l^2 + 3d^2)\,l \quad . \quad . \quad (2)$$

Let M be the mass of the girders reduced to the driving-point, then

$$M\frac{D^2}{4} = \frac{1}{3}\frac{w}{g}(4l^2 + 3d^2)\,l, \text{ or } M = \frac{4}{3}\frac{w}{g}\frac{4l^2 + 3d^2}{D^2}\,l \text{ gravitation units} \quad (3)$$

$w = 0{\cdot}5$; $l = 90$; $d = 30$; $D = 32$.

Hence, $M = 2057$ tons (*Ans.*).

(98) (I.C.E., Feb. 1906).—**A flywheel weighing 10 tons, whose radius of gyration is 5 feet, rests on bearings 10 inches in diameter. If the coefficient of friction of the axle and bearings is 0·006, find the constant moment which must act upon the wheel to get up a speed of 20 revolutions per minute in one minute.**

Solution.—Let W be the weight of the flywheel, T the required constant torque or moment which must act on the wheel to get up a speed of n revs./s. in time t, and d the diameter of the bearings. The force of friction is $W\mu$ and its torque is $W\mu d/2$, which latter is also constant.

Reducing the tangential forces to a point at distance R_g from the axis we have as in a translating motion: the effort = the resistance *plus* the accelerating force. Multiplying the latter equation by R_g, we have: *the torque of the effort is equal to the torque of the resistance* plus *the torque of the accelerating tangential force.*

The accelerating force $= Ma$, where a is the linear acceleration at distance R_g from the axis and $M = W/g$. The angular acceleration $= \omega/t = 2\pi n/t$. Hence, $a = R_g 2\pi n/t$, and therefore

$$T = W\mu\frac{d}{2} + \frac{W}{g}R_g^2\frac{2\pi n}{t} \quad . \quad . \quad . \quad (1)$$

But $R_g^2 W/g$ is the moment of inertia. Hence, we have: *the torque of the effort is equal to the torque of the resistance* plus *the angular acceleration into the moment of inertia.*

Inserting the given values in (1), we get

$$T = 2240 \times 10 \times 0{\cdot}006 \times \frac{10}{12 \times 2} + \frac{2240 \times 10}{g} \times 25 \times \frac{2\pi \times 20}{60 \times 60}$$

$$= 663 \text{ lbws. and ft. } (\textit{Ans.}).$$

(99) (Q. June 1909).—**A uniform disc 6 inches in diameter weighing 10 lbs. can rotate freely about a horizontal axis. A mass of 2 lbs. is fixed at one end of a horizontal diameter and the disc is then released: neglecting friction, find the angular velocity of the disc when the 2 lb. mass reaches the lowest point of the swing.**

Also find the corresponding angular velocity if the disc be keyed to an axle 1 inch in diameter and if a 3 lb. weight be suspended from a string which is wound upon the axle as the disc rotates.

Solution.—Let W be the weight and D the diameter of the disc, P the mass in lbs. on the end of the horizontal diameter and g_1 the local value of the acceleration of gravity.

The weight of P is Pg_1/g lbws. and the energy exerted by the latter weight in descending to the lowest point is $PDg_1/2g$. Let **I** be the moment of inertia of the disc, then the total moment of inertia is $\mathbf{I}+\frac{P}{g}\frac{D^2}{4}$. Hence, the energy-equation at the lowest point is

$$\frac{Pg_1}{g}\frac{D}{2}=\frac{1}{2}\omega^2\left(\mathbf{I}+\frac{P}{g}\frac{D^2}{4}\right);\text{ but }\mathbf{I}=\frac{W}{g_1}\frac{D^2}{8}.$$

Hence,

$$\omega=\sqrt{\frac{8Pg_1^2}{(Wg+2Pg_1)D}} \quad . \quad . \quad . \quad (1)$$

We have given $P=2$ lbs.; $W=10$ lbws. Neglecting the difference between g and g_1, we get

Ans. to the first question: $\omega=8{\cdot}6$ radians/sec.

Let d be the diameter of the axle and Q the weight suspended by the string. While P descends to the lowest point, the disc makes 1/4 rev. Hence, Q is lifted an amount $\pi d/4$. The energy-equation is now

$$\frac{Pg_1}{g}\frac{D}{2}=\frac{\pi d}{4}Q+\frac{\omega^2}{2}\left(\frac{W}{g_1}\frac{D^2}{8}+\frac{P}{g}\frac{D^2}{4}\right)+\frac{Q}{g_1}\frac{v^2}{2},$$

where $v=$ the linear velocity of Q at the moment P passes the lowest point. But $v=\omega d/2$.

Hence,

$$\omega=\sqrt{\frac{4g_1(2g_1PD-\pi gdQ)}{gWD^2+2g_1PD^2+2gQd^2}} \quad . \quad . \quad (2)$$

Inserting the given numerical values in (2) and taking $g_1=g$, we get

Ans. to the second question: $\omega=6{\cdot}6$ radians/sec.

NOTE.—In the above formulæ g_1 is an acceleration but g is a pure number (see article **173**).

(100) (I.C.E., Feb. 1907).—An experimental flywheel has an external diameter of 18 inches, the rim is 2 inches wide and 1 inch thick. The arms and boss may be neglected. A weight of 100 lbs. is attached to a cord wound round the axis. Find (1) the velocity of a point on the outside of the rim if the weight falls 15 feet freely under gravity, neglecting friction; (2) how many revolutions the wheel will make before being brought to rest by a braking force of 10 lbs. applied to the rim. Weight of cast iron is 450 lbs. per cubic foot.

Solution.—Let D be the external diameter of the wheel, P the weight attached to the cord, and d the diameter of the shaft. By article **(203)** $\mathbf{I} = M(4R^2 + t^2)/4$ and $M = 2\pi Rbt\Delta$, where Δ is the density of the material and b the width of the rim. Hence, the numerical value of the moment of inertia is $\mathbf{I} = 0{\cdot}44$ gravitation units (foot, lbs./g, sec.).

(1) Let h be the height through which the weight P falls, v the velocity of a point on the face of the rim, and v_1 the velocity of P. We have $v_1 : v :: d : D$. Hence, the energy-equation is

$$Ph = \frac{P}{g}\frac{v_1^2}{2} + \frac{1}{2}\omega^2\mathbf{I} \quad . \quad . \quad . \quad . \quad (1)$$

but $\omega = 2v/D$. Hence, we can write (1) thus,

$$Ph = \frac{P}{2g}v^2\frac{d^2}{D^2} + 2\frac{v^2}{D^2}\mathbf{I}, \quad \text{or} \quad v = \sqrt{\frac{2gD^2Ph}{Pd^2 + 4g\mathbf{I}}} \quad . \quad (2)$$

Taking $d = 1$ inch, $v = 62$ ft./s. (nearly) (*Ans.*).

(2) Let n be the number of revolutions the wheel will make, and Q the braking force.

The distance travelled by a point on the face of the rim will be $n\pi D$, and P will fall a further distance $n\pi d$ before the wheel stops. Hence, at the moment the wheel is brought to rest, the energy-equation is

$$Pn\pi d + Ph = Qn\pi D, \quad \text{or} \quad n = \frac{Ph}{\pi(QD - Pd)}. \quad . \quad (3)$$

The wheel will be stopped if $QD > Pd$ or $d < QD/P$. Hence, we must take $d < 1{\cdot}8$ ins. If we take $d = 1$ in. as above, we get

$$n = 72 \text{ revs. } (\textit{Ans.}).$$

(101) (Q. June 1908).—**Two wheels, on spindles in fixed bearings, suddenly engage so that their angular velocities become inversely proportional to their radii and in opposite directions. One wheel, moment of inertia I_1 and radius a, has angular velocity ω initially; and the other wheel, moment of inertia I_2 and radius b, is initially at rest. Find the new angular velocities in terms of the quantities here given; explain briefly the dynamical principles involved in the calculations.**

Solution.—If there be no loss of energy due to shock, friction, or any other causes, the original energy accumulated in wheel No. 1 will be divided between the two wheels. We have

Original energy stored in No. 1 $= \frac{1}{2}\omega^2 \mathrm{I}_1$.
Final ,, ,, ,, $= \frac{1}{2}\omega_1^2 \mathrm{I}_1$.
Original ,, ,, No. 2 = zero.
Final ,, ,, ,, $= \frac{1}{2}\omega_2^2 \mathrm{I}_2$.

Hence, $\frac{1}{2}\omega^2\mathrm{I}_1 = \frac{1}{2}\omega_1^2\mathrm{I}_1 + \frac{1}{2}\omega_2^2\mathrm{I}_2$, or $\omega^2\mathrm{I}_1 = \omega_1^2\left(\mathrm{I}_1 + \frac{\omega_2^2}{\omega_1^2}\mathrm{I}_2\right)$. (1)

If there be no slip between the wheels, then

$$\left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{a^2}{b^2},$$

which inserted in (1) gives

$$\omega_1 = \omega b\sqrt{\frac{\mathrm{I}_1}{b^2\mathrm{I}_1 + a^2\mathrm{I}_2}};\ \ \omega_2 = \omega a\sqrt{\frac{\mathrm{I}_1}{b^2\mathrm{I}_1 + a^2\mathrm{I}_2}}\ (Ans.).$$

(102) (Q. June 1909).—**An equilateral triangle, each side 4 inches in length, is cut from sheet metal weighing 5 lbs. per square foot. Find the moment of inertia of the triangle about one side, measured in "lb.-ft.²" units.**

Solution.—Let A B (fig. 117) be a very thin homogeneous plate

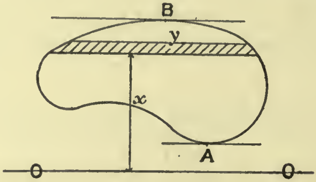


FIG. 117.

of uniform thickness and weighing w per unit area. It is required to find the moment of inertia **I** of the plate w.r.t. O – O.

Area-element $= y\,\delta x$; mass of the corresponding volume-element $= y\,\delta x\frac{w}{g}$; moment of inertia of mass of volume-element $= y\,\delta x\frac{w}{g}x^2$.

Hence $$\mathrm{I} = \frac{w}{g}\int_{x_1}^{x} yx^2\,dx \text{ gravitation units} \quad . \quad . \quad (1)$$

Let the length of the side of the triangle be d and the height h, then

$$\frac{y}{d} = \frac{h-x}{h} = 1 - \frac{x}{h}, \quad\text{or}\quad y = d\left(1 - \frac{x}{h}\right).$$

If m be the mass per unit area, then

$$\mathbf{I} = md\int_0^h \left(1 - \frac{x}{h}\right)x^2\,dx = m\frac{dh^3}{12} \quad . \quad . \quad . \quad (2)$$

I is required to be expressed in lb.-ft.² units and not in gravitation units. We have given $d = 1/3$ ft.; $h = d\sin 60°$ and we will take $m = 5$ lbs. per sq. foot.

$$\mathbf{I} = 0{\cdot}00334 \text{ lb.-ft.}^2 \text{ units } (Ans.).$$

(103) (I.C.E., Oct. 1908, first part).—**Find the position of the instantaneous centre of a motor-car wheel 30 inches in diameter when skidding, the car travelling at 10 miles an hour and the wheel revolving at a speed which would correspond to 4 miles per hour without slip.**

Solution.—Let d be the diameter of the wheel, u the speed of the axle, and ω the angular speed of the wheel.

The problem is to find the diameter, D, of a wheel, whose centre travels at a speed u, and which rotates at a rate of $2u/\mathrm{D} = \omega$. Hence, $\mathrm{D} = 2u/\omega$.

We have given $u = 10$, $\frac{d}{2}\omega = 4$; hence, $\mathrm{D} = 2{\cdot}5d = 75$ ins.

In fig. 118 A E is the road, A B the real wheel, O D the imaginary wheel. Hence, point O is the instantaneous centre.

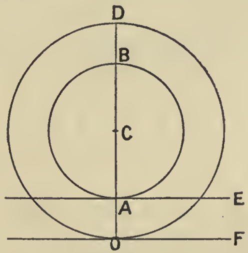


FIG. 118.

Ans.—The position of the instantaneous centre is vertically below the centre of the wheel and at a distance A O = 22·5 ins. below the road.

Point A in fig. 118 travels at a rate of 6 miles/hr., point C at 10 miles/hr., point B at 14 miles/hr., and point D at a rate of 20 miles/hr.

(104) (I.C.E., Oct. 1907).—Define "instantaneous centre." In the mechanism shown in fig. 119 a force, E, of 2000 lbs., acts on the slider C and moves it at a rate of 1 foot per second. Find the velocity of P in magnitude and direction. What weight hung vertically at P will balance the force E?

$$AB = 2{\cdot}5;\ AC = 3{\cdot}6;\ CB = 1{\cdot}5 \text{ and } BP = 0{\cdot}8.$$

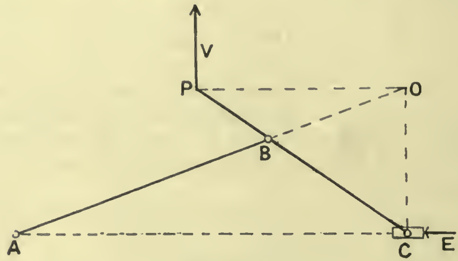


FIG. 119.

Solution.—As AB turns about A the motion of B will be perpendicular to AB and the instantaneous centre must lie somewhere in AB produced, C moves along AC; hence, the instantaneous centre must be situated in the perpendicular drawn through C on AC. Hence, point O is the instantaneous centre.

The line OP appears to be parallel to AC; hence, if the latter be horizontal, the weight at P will hang at right angles to OP. It will be found that the length of $OP = 1{\cdot}92$ and that of $OC = 1{\cdot}3$. Hence, the velocity V of P will be $V : 1 :: 1{\cdot}92 : 1{\cdot}3$, or

$$V = 1{\cdot}48 \text{ ft./s. } (Ans.).$$

Let the weight at P be W, then $1{\cdot}92W = 1{\cdot}3 \times 2000$, or

$$W = 1354 \text{ lbws. } (Ans.).$$

(105) (I.C.E., Feb. 1898).—Apply the principle of the conservation of energy to find the velocity of a thin hollow circular cylinder after rolling a distance of 12 feet down a plane inclined at a slope of 1 vertical in 5 horizontal.

Solution.—Let W be the weight of the drum, r its radius, and d the distance it rolls down the plane.

The M.C. of the drum falls through a height $d \sin \alpha$ while the drum rolls distance d down the plane. The kinetic energy of the drum is partly kinetic energy of translation E_k', and partly kinetic energy of rotation E_k'' (see article 188). We have

$$E_k' = \frac{W}{g}\frac{v^2}{2}; \; E_k'' = \tfrac{1}{2}\omega^2 I \quad . \quad . \quad . \quad (1)$$

where v is the linear velocity of the centre. But $\mathbf{I} = Wr^2/g$ and $\omega = v/r$. Hence, $E_k' = E_k'' = Wv^2/2g$. The energy-equation is therefore

$$W d \sin\alpha = \frac{W}{g} v^2.$$

Hence, $$v = \sqrt{gd\sin\alpha} \quad . \quad . \quad . \quad . \quad (2)$$

If the drum had fallen through the vertical height $d\sin\alpha$ without rotation the velocity would be $v = \sqrt{2gd\sin\alpha}$ (see article 176).

We have $$d = 12; \; \sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = 0{\cdot}196.$$

$$v = 8{\cdot}7 \text{ ft./s. } (Ans.).$$

(106) (I.C.E., Feb. 1909).—**A steel shaft 3 inches diameter has two flywheels keyed to it near the ends, the mass of each flywheel being 500 lbs. and the radius of gyration 1 foot. The ends of the shaft rests on two elevated horizontal rails along which the shaft can roll. A rope 1 inch in diameter is coiled round the portion of the shaft between the flywheels, one end being fixed to the shaft and on the free end a load of 500 lbs. is hung. Find the acceleration of the system along the rails.**

Solution.—Let d be the diameter of the shaft, t the diameter of the rope, M the masses of the two flywheels and the shaft, M_1 the latter masses reduced to radius of gyration R_g, P the weight suspended at the free end of the rope, v the velocity along the rails, and v_1 the velocity of P.

The portion of the rope between the centre of the rope and the shaft will be compressed and the other portion of the rope will be extended. Hence, if the rope be perfectly flexible and inelastic, we may consider that P is suspended at the free end of a rope of no thickness and coiled on a shaft whose diameter is $d+t$.

All frictional and other resistance are to be neglected. As the system is urged on by a constant force P the acceleration a along the rails is also constant. The motion of M is partly a translation and partly a rotation. We have now $v : v_1 :: d : d+t$, and when

P has dropped through a height h from rest, $2a_1h=v_1^2$, where a_1 is the acceleration of P.

Hence,
$$h=\frac{d+t}{d}\frac{v^2}{2a} \quad . \quad . \quad . \quad . \quad (1)$$

The kinetic energy of translation is

$$E_k'=M\frac{v^2}{2}+\frac{P}{g}\frac{v_1^2}{2}=\left[M+\left(\frac{d+t}{d}\right)^2\frac{P}{g}\right]\frac{v^2}{2} \quad . \quad . \quad (2)$$

The kinetic energy of rotation is

$$E_k''=\tfrac{1}{2}\omega^2 M_1 R_g^2=\frac{2v^2}{d^2}M_1R_g^2 \quad . \quad . \quad . \quad (3)$$

Hence, the energy-equation is

$$Ph=P\frac{d+t}{d}\frac{v^2}{2a}=\left[M+\left(\frac{d+t}{d}\right)^2\frac{P}{g}\right]\frac{v^2}{2}+\frac{2v^2}{d^2}M_1R_g^2 \quad . \quad (4)$$

Hence,
$$a=\frac{d(d+t)Pg}{d^2Mg+(d+t)^2P+4R_g^2M_1g} \quad . \quad . \quad (5)$$

If we neglect the mass of the shaft, then $M=M_1=1000/g$. We have also $d=0{\cdot}25$, $t=1/12$, $P=500$ and $R_g=1$.

$$a=0{\cdot}326 \text{ ft./s.}^2 \ (\textit{Ans.}).$$

If T be the tension in the rope; the mass of P is urged on by a force $P-T$ and with an acceleration a_1. But $a_1 : a :: d+t : d$.

Hence,
$$T=P-\frac{P}{g}\frac{d+t}{d}a=493 \text{ lbws.}$$

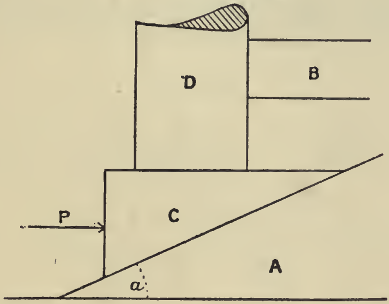
(107) (Q. Nov. 1907).—**The sketch (fig. 120A) shows a column D, of weight W, which is being raised vertically, with uniform velocity, by a force P applied to the wedge C. The inclined plane A and the support B are fixed, and the upper face of the wedge is horizontal. There is the same coefficient of friction between A and C as between C and D, and D and B. Draw a diagram of the forces acting on the wedge and on the pillar, and find the ratio P to W, when $\mu=\tan 18°$, and the inclination of the plane is 30°.**

Solution.—The motions of the wedge and of the column will be uniform when the forces acting on the three sets of surfaces of contact are respectively balanced, *i.e.* the resultants R_1, R_2, and R_3 must make an angle ϕ with the respective normals (article 181).

Let us first consider the forces acting between C and D; they are: (i.) W, which is vertical; (ii.) a horizontal force to overcome the force of friction F_2; the resultant of these two forces is R_2.

The forces between B and D are: (i.) A normal horizontal pressure, which is numerically equal to F_2; (ii.) the vertical force to overcome the force of friction F_3; their resultant is R_3.

FIG. 120A.



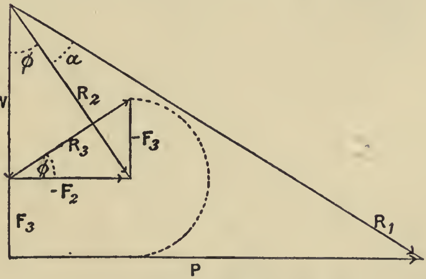


FIG. 120B.

Finally, the forces between A and C are: (i.) the vertical forces W and F_3; (ii.) the force P; their resultant is R_1.

Fig. 120B is the diagram of forces. We have, first, triangle W, $(-F_2)$, R_2, the angle between W and R_2 being ϕ. Secondly, triangle $(-F_2)$, $(-F_3)$, R_3, the angle between R_3 and F_2 is also ϕ. Finally, triangle $(W+F_3)$, P, R_1, the angle between R_1 and W being $\phi+\alpha$, where α is the inclination of A.

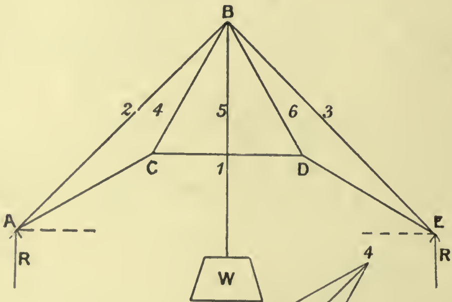
It will be seen that $P=(W+F_3)\tan(\phi+\alpha)$, and $F_3=F_2\tan\phi$ $=W\tan^2\phi$. Hence, $\frac{P}{W}=\sec^2\phi\tan(\phi+\alpha)$.

Inserting the given numerical values we get

$$\frac{P}{W}=1\cdot 23 \text{ (nearly) } (\textit{Ans.}).$$

(108) (I.C.E., Feb. 1907).—**State and prove the theorem known as the triangle of forces. Calculate, otherwise than graphically, the stresses in the members A B, B C, A C, and C D of the truss (fig. 121A), when a load of 500 lbs. is slung from the apex B, and state which are struts and which ties.**

FIG. 121A.



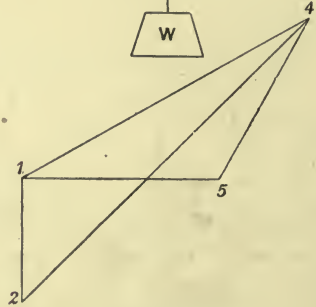


FIG. 121B.

Solution.—The author has added the figures 1 . . . 6. The required stresses may be determined by drawing: (i.) the triangle of the forces through point A; (ii.) the triangle of the forces through point C. This is done in fig. 121B.

Hence, $\text{stress }(4\text{–}1) = \dfrac{\sin 45^\circ}{\sin 15^\circ} \times 250 = 683 \text{ lbws.};$

$$\text{stress }(2\text{–}4) = \frac{\sin 120^\circ}{\sin 15^\circ} \times 250 = 836{\cdot}5 \text{ lbws.};$$

$$\text{stress }(4\text{–}5) = (4\text{–}1) \times \frac{\sin 30^\circ}{\sin 60^\circ} = \text{stress }(5\text{–}1) = 394{\cdot}3 \text{ lbws.}$$

Bars 4–1, 4–5, and 5–1 are ties; bar 2–4 is a strut.

(109) (Q. June 1908).—**A ladder on a horizontal floor leans against a vertical wall. Show that, if the centre of gravity is at its middle point, the greatest inclination to the wall which is consistent with equilibrium is $\tan^{-1}\dfrac{2\mu}{1-\mu^2}$, where μ is the coefficient of friction for the contact both with the floor and with the wall; and that, in this position, the lines of action of the resultant forces acting at the foot and at the top of the ladder are at right angles to one another.**

Solution.—In the case of a ladder standing on a horizontal floor and leaning against a vertical wall, we have three forces to deal with, viz.: (i.) the weight W of the ladder which is vertical,

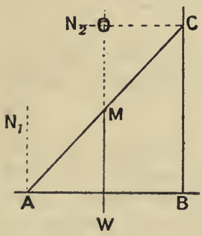


FIG. 122A.

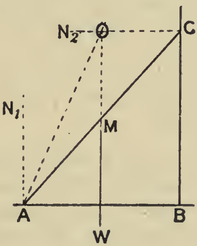


FIG. 122B.

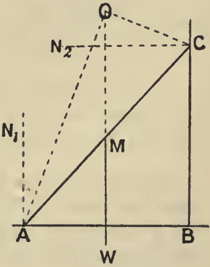


FIG. 122C.

and which, in the present case, passes through the middle point of the ladder; (ii.) the reaction R_1 of the floor; and (iii.) the reaction R_2 of the wall. But three forces cannot be mutually balanced unless they meet at one point. Hence, W, R_1, and R_2 must meet at the same point O.

In fig. 122A, both floor and wall are perfectly smooth; hence, the reactions R_1 and R_2 are both normal to their respective surfaces. R_2 acting along the normal $C\,N_2$ meets W at point O, but R_1 acting along the normal $A\,N_1$ cannot possibly pass through point O unless the whole ladder stands against the wall. Hence, equilibrium is impossible in any slanting position of the ladder.

It is therefore necessary that the floor shall be rough, but the wall may be smooth. This is shown in fig. 122B. The ladder will not slip as long as angle $OAN_1 \leqq \phi$. Point A will be furthest from the wall when angle $OAN_1 = \phi$.

The inclination of the ladder can be increased when the wall is rough also. This is shown in fig. 122C. Point A will be furthest from the wall when point O is at its highest position, and which corresponds with angle $OAN_1 = OCN_2 = \phi$. Let θ be the value of angle ACB in the latter case, h the height of point O above the floor, and l the length of the ladder. We have that

$$h = l\cos\theta + \frac{l}{2}\sin\theta\tan\phi,$$

but $\frac{l}{2}\sin\theta = h\tan\phi$, $l\cos\theta = BC$, and $AB = 2h\tan\phi$.

Hence, $$\tan\theta = \frac{2\tan\phi}{1-\tan^2\phi} = \frac{2\mu}{1-\mu^2} = \tan 2\phi.$$

Hence, $$\theta = 2\phi = \text{ang}\left(\tan = \frac{2\mu}{1-\mu^2}\right) \; (Ans.).$$

In quadrilateral AOCBA, angle $A = 90° - \phi$, angle $C = 90° + \phi$, and angle $B = 90°$. Hence, angle O is 90°, or R_1 and R_2 are at right angles to one another.

(110) (Q. Nov. 1908).—**A uniform ladder weighing 100 lbs. rests against a smooth vertical wall, making an angle of 60° with the ground. A man weighing 150 lbs. ascends the ladder; show that the ladder will slip before he reaches the top unless the coefficient of friction between the ladder and the ground be at least 0·46.**

Solution.—The conditions of wall and floor are the same as those in fig. 122B.

Let θ be the value of angle CAB and P the weight of the man.

While the man is ascending the ladder between A and M, the resultant of P and W will meet CN_2 at a point to the left of O; and when he reaches M the resultant will pass through point O. While he is moving between M and C the resultant will meet CN_2 at a point to the right of O; and on reaching the top of the ladder the resultant will meet $C N_2$ at a point D at a distance x from O. The value of x is determined thus,

$$(W+P) : \frac{l\cos\theta}{2} :: P : x; \text{ hence, } x = \frac{Pl\cos\theta}{2(W+P)}.$$

$$\tan DAN_1 = \left(\frac{AB}{2}+x\right)\Big/CB = \left[\frac{l\cos\theta}{2}+\frac{Pl\cos\theta}{2(W+P)}\right]\Big/l\sin\theta$$

$$= \frac{W+2P}{2(W+P)}\cot\theta \ngtr \tan\phi.$$

Inserting the given numerical values we get

$$\tan DAN_1 = 0{\cdot}462,$$

or the coefficient of friction must not be less than 0·462 (*Ans.*).

(111) (Q. June 1909).—**A heavy circular disc, of weight W lbs., stands on a rough inclined plane of inclination α, the plane of the disc being in the plane of the greatest slope. The disc is supported by a tangential force applied, by means of a string, at its highest point. Find the tension in the string and the magnitude of the frictional force at the point of contact.**

Solution.—Let T be the tangential force, r the radius of the disc, and P the pressure on the plane.

The force T, which is horizontal, acts as a brake to prevent the rolling of the disc. The inclined plane is not perfectly rigid.

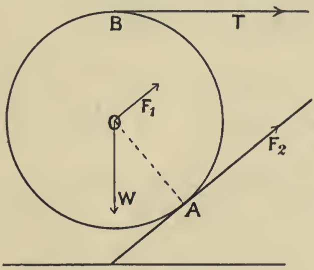


FIG. 123.

Hence, when $\alpha=\theta$, where θ is the angle of repose of rolling friction, then T = O. The force of rolling friction is $F_1 = P\tan\theta$ (see article 190).

We have also to prevent skidding; this is avoided when α does not exceed a certain value, which in this case is greater than the angle of repose, ϕ, of sliding friction. The force of sliding friction is $F_2 = P\tan\phi$.

We have obviously $P = W\cos\alpha + T\sin\alpha$. Hence,

$F_1 = (W\cos\alpha + T\sin\alpha)\tan\theta$, and $F_2 = (W\cos\alpha + T\sin\alpha)\tan\phi$.

The forces acting on the system are (fig. 123): T, acting at point B, is horizontal; W acting at O, is vertical; F_1, acting at O, is parallel to the inclined plane; and F_2 acting at A along the inclined plane.

Rolling will be prevented when the algebraical sum of the couples about the instantaneous centre, A, is zero, or

$$Wr\sin\alpha - Tr(1+\cos\alpha) - F_1 r = 0.$$

Hence,
$$T = \frac{\sin\alpha - \cos\alpha\tan\theta}{1+\cos\alpha+\sin\alpha\tan\theta}W = \frac{\sin(\alpha-\theta)}{\cos\theta+\cos(\alpha-\theta)}W$$

$$= \frac{\sin(\alpha-\theta)}{2\cos\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\theta\right)}W \ (Ans.).$$

$$F_1 = P\tan\theta = \frac{\cos\frac{\alpha}{2}\sin\theta}{\cos\left(\frac{\alpha}{2}-\theta\right)}W.$$

$$F_2 = W\left[\cos\alpha + \frac{\sin\alpha\sin(\alpha-\theta)}{\cos\theta+\cos(\alpha-\theta)}\right]\tan\phi$$

$$= W\frac{\cos\theta(1+\cos\alpha)}{2\cos\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\theta\right)}\tan\phi = W\frac{\cos\theta\cos\frac{\alpha}{2}}{\cos\left(\frac{\alpha}{2}-\theta\right)}\tan\phi \ (Ans.).$$

The components, parallel to the inclined plane, of W and T are respectively $W\sin\alpha$ and $T\cos\alpha$. Hence, skidding will just be prevented when $W\sin\alpha - T\cos\alpha - F_2 = 0$, or when

$$\sin\alpha - \frac{\sin(\alpha-\theta)\cos\alpha}{2\cos\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\theta\right)} - \frac{\cos\theta\cos\frac{\alpha}{2}}{\cos\left(\frac{\alpha}{2}-\alpha\right)}\tan\phi = 0,$$

or when
$$\sin(\alpha-\phi) = \sin(\phi-\theta)\sec\theta.$$

Hence, we must have

$$\alpha \leqq \phi + \text{ang}\,[\sin = \sin(\phi-\theta)\sec\theta].$$

(112) (I.C.E., Feb. 1906).—A flexible cord is carried by two pegs A and D in the same horizontal line and 24 inches apart. A weight of 8 lbs. hangs at B and an unknown weight at C, thereby causing the cord to assume the form shown in fig. 124A. Find by a graphical construction the magnitude of the unknown weight and the tensions in parts A B, B C and C D of the cord.

Solution.—The following dimensions in fig. 124A are given: A F = 6 ins., F E = 10 ins., E D = 8 ins., F B = 6 ins., and E C = 10 ins. The figures 0 . . . 3 have been added by the author.

FIG. 124A.

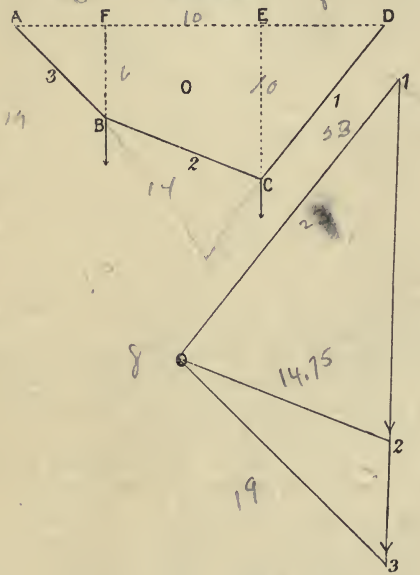


FIG. 124B.

Fig. 124B is the reciprocal force diagram and requires no explanation. It will be found that the tensions in A B, B C, and C D are respectively

19 lbws.; 14·75 lbws.; 23 lbws. (*Ans.*).

The weight at C will be found to be 23·6 lbws. (*Ans.*).

(113) (Q. Oct. 1909). — Two weights, each of 2 lbs., are attached to different points of a string which hangs from a

support. The upper weight is acted on by a horizontal force of 4 lbs. to the left and the lower weight by a horizontal force of 1 lb. to the right. Find by a graphical construction the tension of each portion of the string when in equilibrium, and the angle which each portion makes with the vertical.

Solution.—Fig. 125 is the force diagram. Force 1–2 is the resultant of a horizontal force of 4 lbws. and a vertical force of 2 lbws.; force 2–3 is the resultant of a horizontal force of 1 lbw. and a vertical force of 2 lbws.; complete the force polygon by drawing force 3–1.

The tension in the upper part of the string must be parallel

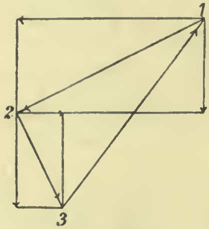


FIG. 125.

to and equal to 3–1, and the tension in the lower part must be parallel to and equal to 2–3.

The upper portion of the string will make an angle θ_1 with the vertical, $\tan \theta_1 = 0{\cdot}75$. Hence, $\theta_1 = 36° \ 52'$.

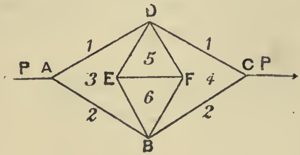
The lower portion of the string will make an angle θ_2 with the vertical, $\tan \theta_2 = 0{\cdot}5$. Hence, $\theta_2 = 26° \ 34'$.

(114) (Q. Nov. 1907).—**Draw the reciprocal force diagram to give the forces in the several bars of the frame (fig. 126A), and show that the tension in the bar E F is $\overline{EF}\ P/\overline{AC}$.**

Solution.—The figures 1 . . . 6 have been added by the author. Fig. 126B is the reciprocal force diagram. The reciprocal of joint A is 2–1–3–2, the external force P being 2–1. Similarly, the reciprocal of joint C is 1–2–4–1. The reciprocals of joints B and D are respectively 2–3–6–4–2 and 1–3–5–4–1; and those of joints E and F are respectively 3–5–6–3 and 4–6–5–4.

To find which bars are in tension and which are in compression. Imagine the bars at joint A being cut by a circle with centre at A, and at the same time external forces being added at the points where the bars are cut, and which balance the external force at A. The sense-circuit of these forces is 2–1–3–2 in fig. 126A. Hence, the external forces acting along bars 1–3 and 3–2 will respectively be in the directions and senses 1–3 and 3–2 (fig. 126B). Hence, both bars are being pulled. The same takes place at joint C. Take next joint D and cut the bars by a circle with centre at D, at the same time adding external forces at the points where the bars are cut and which are mutually balanced. As the bars 1–3 and 4–1 are in tension, the sense-circuit is 3–1–4–5–3. Hence, bars 5–3 and 4–5 are in compression. Similarly, at joint B where

FIG. 126A.



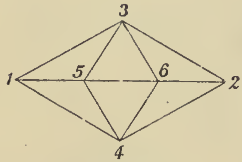


FIG. 126B.

bars 6–3 and 4–6 are in compression. W.r.t. joint E the sense-circuit is 3–5–6–3. . Hence, bar 5–6 is in tension.

The tension in bar E F (5–6) bears to force P (1–2) (fig. 126B) in the same ratio as the length E F to the length A C in fig. 126A.

Hence, the tension in bar $EF = \overline{EF}\ P/\overline{AC}$ (*Ans.*).

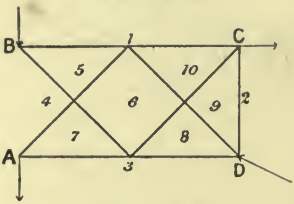
(115) (Q. Nov. 1906, second part).—**Draw a reciprocal force diagram for the frame shown in fig. 127A, pin-jointed at the points at which the lines of the figure meet; the frame being loaded with given unequal weights at A and B, and supported by a horizontal tie at C and a fixed pin at D.**

Solution.—The author has added figures 1 . . . 10 in fig. 127A.

In constructing the reciprocal force diagram we begin by setting off the two given vertical forces 3–4 and 4–1, then draw a horizontal line through point 1. The reciprocal of joint B is constructed by drawing lines 1–5 and 5–4 respectively parallel to bars 1–5 and 5–4. Similarly, the reciprocal figure of joint A is 3–4–7–3, lines 4–7 and 7–3 being respectively parallel to the bars of the same names. We then proceed to construct the reciprocal figures of joints 4–5–6–7 and 6–10–9–8. The vertical through 9 determines the reciprocal of joint D, and at the same time determines the magnitude of the external force at C. We can now draw line 2–3, which represents the external force at D in magnitude and direction. The four external forces will be seen to be balanced as they form a continuous circuit.

To find which bars are ties and which are struts.—Let us start at joint A by cutting the bars by a circle with centre at A, and add the necessary external forces at the points where the bars are cut. The latter forces must balance the given external force at

FIG. 127A.



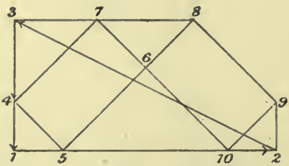


FIG. 127B.

A and thus form the sense-circuit 3–4–7–3 (fig. 127B). Hence, bar 7–3 will be pushed and bar 4–7 will be pulled. Repeat similar operations at joints B, C, and D. It will be found that: bars 4–7, 6–5, 1–5, 10–1, 9–10, and 8–6 are ties; and bars 7–3, 5–4, 6–7, 3–8, 2–9, 8–9, and 10–6 are struts.

(116).—A horizontal beam, 30 feet long, is supported at both ends and loaded with weights of 10 tnws. and 16 tnws. at distances of 10 feet and 20 feet respectively from one of the supports. Find the reactions of the supports, and draw the bending-moment diagram for the whole beam.

Solution.—Let A B C D (fig. 128A) represent the given beam supported at A and D. The letters indicate the points of application of the forces, and at the same time, being placed to the left of the forces, the letters will also be used to name the forces. Thus, force $AB = 10$ tnws.; force $BC = 16$ tnws.; $DA = R_a$, and $CD = R_d$.

(i.) *Calculation.*—The two weights and the two reactions must balance.

Hence, $$10+16+R_a+R_d=0 \quad . \quad . \quad . \quad (1)$$

But condition (1) is not sufficient; the forces must not form a couple. Hence, the algebraical sum of their torques about any axis perpendicular on their plane must be zero. By choosing the

Fig. 128a.

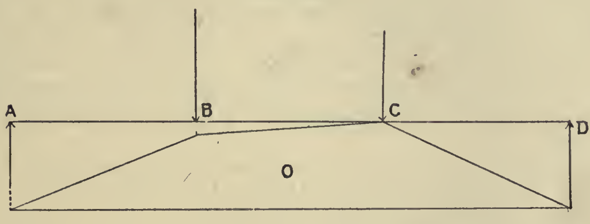


Fig. 128b.

axis through point A, we shall eliminate force R_a. Hence, by taking moments about A we get

$$-10\times 10-16\times 20+R_d\times 30=0, \quad \text{or} \quad R_d=14 \text{ tnws.}$$

As the moments of the weights are negative the moment of R_d will be positive. Hence, R_d is directed upwards, and we will reckon the sense of gravity as negative. Inserting the value of R_d in (1) we get $R_a=12$ tnws.

If we take moments about any point on the beam, then the torque of the forces to the left of the point must be numerically equal to the torque of the forces to the right of the point; but the two torques must have opposite signs as their algebraical sum must be zero. Hence, the forces acting on the beam produce a bending of the beam at any point on the beam. The resultant moment of the forces to the left or to the right of the point is called the *bending-moment*, B.M., at the given point.

B.M. *between* A *and* B.—Let us find B.M. at a point at a distance x from A, remembering that the sign of a lever is positive reckoned from the fulcrum towards the right and is negative in the opposite direction. Hence, B.M. $= -R_a x$; the equation of B.M. is a descending straight line starting at point A. At point B, B.M. $= -100$ tnws.-ft.

B.M. *between* B *and* C.—At distance x from point A we have, B.M. $= -R_a x + \text{force } \overline{AB} \times (x - AB)$, or B.M. $= -2x - 100$, which is also a descending straight line. At point C, B.M. $= -140$ tnws.-ft.

B.M. *between* C *and* D.—At distance x from A, B.M. $= -12x + 10(x - AB) + 16(x - AC)$, or B.M. $= 14x - 420$, which is an ascending straight line, ending at point D.

The bending-moment diagram consists thus of three straight lines; B.M. being zero at points A and D, and maximum at point C.

(ii.) *Graphical method.*—For the purpose of determining the values of R_a and R_d, we may construct a link-frame containing four bars, which shall be in equilibrium when each of the four forces R_a, R_d, A B, and B C act respectively at a joint of the frame.

We commence by drawing the reciprocal force diagram (fig. 128B). Draw the vertical line A C, setting off A B = 10 and B C = 16. Choose any convenient point O, and from the latter draw the straight lines O A, O B, and O C.

Take any point in D A (fig. 128A) and draw line O A parallel to O A of fig. 128B, and likewise draw O B and O C in fig. 128A parallel to O B and O C of fig. 128B. Close the link-frame by drawing O D, and finally draw line O D in fig. 128B parallel to O D of fig. 128A; then C D $= R_d$ and D A $= R_a$.

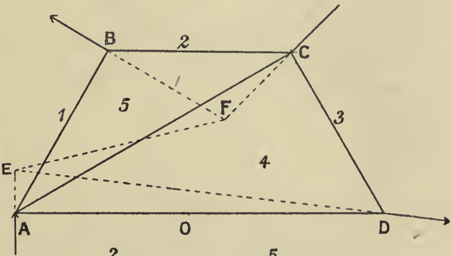
A portion of the vertical line through point B (fig. 128A) is intercepted by the link-frame. Call the latter intercept H, and let h be the perpendicular from O on A C (fig. 128B) then, H : length $\overline{AB}$ (fig. 128A) :: D A (fig. 128B) : h; as h is constant it follows that, $H \propto R_a \times \text{lever } \overline{AB}$. Hence, the intercept of the vertical line through any point in the beam is proportional to the B.M. at that point. The numerical value of H is 100 tnws.-ft. Hence, if we construct a scale so that a length of H/100 represents 1 tnw.-foot, then the link-frame (fig. 128A) may be used as a B.M. diagram.

(117) (I.C.E., Oct. 1907).—**A simple frame (see fig. 129A) is loaded at B and C as shown. It is supported by a vertical reaction at A, and by a reaction at D. Find the magnitude of the reaction at A, the magnitude and direction of the reaction at D, and the forces in the bars, stating which are struts and which are ties.**

Solution.—It is given that the force at B is 3 tnws., and forms an angle of 30° with the horizontal line; the force at C is 4 tnws., and forms an angle of 45° with the horizontal line; A B = C D = 2 ft., and A D = 4 ft. The figures 0 . . . 5, the letters F and E, and the dotted lines have been added by the author.

Fig. 129B is the reciprocal force diagram. The force 1–3 is the resultant of 1–2 and 2–3. We have therefore only to do with

Fig. 129A.



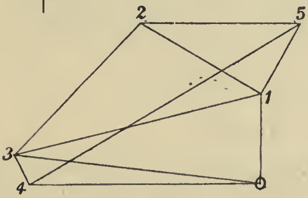


Fig. 129B.

forces 1–3, 0–1, and 3–0 which must meet at one point E. Hence, we obtain the direction of the reaction 3–0. By fig. 129B we determine the magnitude of 3–0 and 0–1.

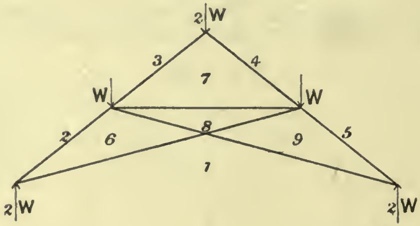
By fig. 129B we obtain the magnitudes of the following forces in tnws.: 0–1 = 1·94, 1–5 = 1·71, 2–5 = 3·45, 3–4 = 0·695, 3–0 = 5·38, 0–4 = 5·05, 4–5 = 6·78.

We have now to determine which of the bars are struts and which are ties. Let us begin at A. Cut the bars by a circle with centre at A; as equilibrium must not be disturbed we must add the necessary external forces at the points where the bars are cut. These forces must be in sense-circuit with 0–1. Hence, the circuit is 0–1–5–4–0. Hence, bar 1–5 will be pulled; bar 5–4 will be pushed; bar 4–0 will be pulled. Performing similar operations at B, C, and D, we shall find that bars 1–5, 4–0, 2–5, 3–4 are ties; and bar 5–4 is a strut.

(118) (Q. May 1907).—**Draw a reciprocal force diagram for the frame (fig. 130A), loaded and supported in the manner indicated.**

Solution.—Figures 1 . . . 9 have been added by the author.

FIG. 130A.



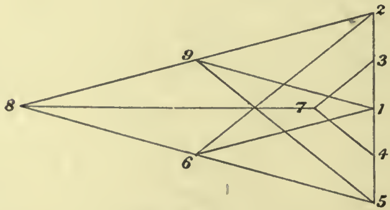


FIG. 130B.

Fig. 130B is the reciprocal force diagram. It has been constructed by setting off the vertical loads 2–3 = W, 3–4 = 2W and 4–5 = W. The two reactions 1–2 and 5–1 must be equal on account of symmetry, and their sum must be equal to 4W. We may then construct the reciprocals of joints 1–2–6–1 and 1–9–5–1, then that of 3–4–7–3, and finally the reciprocals of joints 2–3–7–8–6–2 and 4–5–9–8–7–4.

Cutting the bars by circles with centres at the respective joints, we shall find that bars 6–1, 1–9, 8–6, 9–8 are ties; bars 2–6, 9–5, 3–7, 7–8, 4–7 are struts.

(119) (I.C.E., Oct. 1906).—**The diameter of the pulleys of a Weston pulley-block (fig. 131) are 10 inches and 9 inches. Find**

the pull P required to raise 2000 lbs. if the efficiency for that load is 65 per cent.

Solution.—Let D be the diameter of the larger and d that of the smaller of the two concentric pulleys.

The tension in each of the two parts of the chain carrying W is obviously W/2 when we consider their directions to be very nearly vertical. While P moves a distance πD the tension W/2 to the right of the load will also travel a length πD, but upwards,

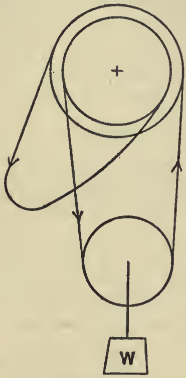


FIG. 131.

whereas the tension W/2 to the left will only have moved through a distance πd downwards. Neglecting friction, the energy-equation is

$$P_1 \pi D + \frac{W}{2}\pi d = \frac{W}{2}\pi D.$$

Hence, $$P_1 = \frac{D-d}{2D}W, \quad \text{and} \quad P = \frac{D-d}{2D}\frac{W}{\eta} \quad . \quad . \quad (1)$$

We have given: $D=10$, $d=9$, $W=2000$, $\eta=0{\cdot}65$.

Hence, $P=154$ lbws. (nearly) (*Ans.*).

By article 212 we have,

$$\frac{W}{P}\frac{1}{\alpha}=\eta. \quad \text{Hence,} \quad \alpha=\frac{W}{P_1}=\frac{2D}{D-d} \quad . \quad . \quad (2)$$

$$\textit{The mechanical advantage} = \frac{W}{P} = \frac{2D}{D-d}\eta \quad . \quad . \quad . \quad (3)$$

In the present case: $\alpha=20$; the mechanical advantage $=13$.

(120) (I.C.E., Oct. 1902).—**In experimenting with a Weston differential pulley-block, it is found that a force of 10 lbs. is required to raise a load of 50 lbs. and that a force of 18 lbs. will raise 100 lbs. The curve obtained by plotting these and other results with the forces as abscissæ and the loads as ordinates is found to be a straight line. Determine the constants in the equation of this line, and state what may be inferred as to the amount of friction in the contrivance. If the circumference of the larger wheel is 18 inches, and the velocity ratio of the pulley is 16, what is the circumference of the smaller wheel?**

Solution.—The equation of the straight line is $W = mP + c$. The constants m and c are determined by substituting for W and P the given values of the load and effort. We have

$$50 = 10m + c, \quad \text{and} \quad 100 = 18m + c \qquad . \qquad . \qquad (1)$$

Eliminating m and c between the two in (1) and the equation of the straight line, we obtain the required equation, viz.:

$$W = 6{\cdot}25P - 12{\cdot}5 \qquad . \qquad . \qquad . \qquad . \qquad (2)$$

We may measure the force of friction as a load F; the theoretical mechanical advantage will be $(W + F)/P = \alpha$. Hence, $F = (\alpha - 6{\cdot}25)P + 12{\cdot}5$, or as α is given to be 16,

$$F = 9{\cdot}75P + 12{\cdot}5 \text{ lbws.} \qquad . \qquad . \qquad . \qquad (3)$$

By the preceding problem we have $\alpha = 2D/(D - d)$, or

$$d = \frac{\alpha - 2}{\alpha}D \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (4)$$

$$d = 15{\cdot}75 \text{ ins. } (\textit{Ans.}).$$

(121) (Q. Nov. 1907).—**In a crane the lifting force is applied to a handle which rotates in a circle of 3 feet diameter, and it is found that when the handle makes five complete revolutions the load is lifted 1·46 feet. It is also found that with loads of 1 cwt., 3 cwts., and 5 cwts. respectively the lifting forces required are 17·5 lbs., 26 lbs., and 35 lbs. Find the efficiency of each of these loads.**

Solution.—Let d be the diameter of the circle in which the handle rotates, P the lifting force, and Q the load.

While the handle makes n revs. the load is lifted through a height h. Hence, $\alpha = \frac{\pi d n}{h}$; the mechanical advantage, $\beta = \frac{Q}{P}$; and the efficiency, $\eta = \frac{Qh}{P\pi d n} = \frac{\beta}{\alpha}$.

We have given: $h=1{\cdot}46$, $d=3$, and $n=5$.
Ans.:

α.	P.	Q.	β.	η.
32·3	17·5	112	6·4	20 per cent.
,,	26	336	12·92	40 ,,
,,	35	560	16·0	50 ,,

(122) (I.C.E., Oct. 1904).—**An iron wedge used for splitting a tree is struck so as to be subject to a vertical force of 4 tons. The taper of the wedge is 1 inch per foot, and the coefficient of friction against the tree is 0·1. Find the horizontal splitting force, and find the efficiency.**

Solution.—Let 2P be the driving force, 2θ the wedge-angle, and H the horizontal splitting force.

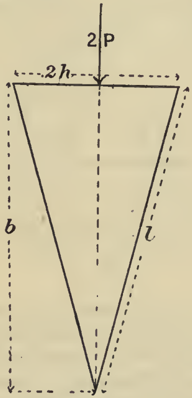


Fig. 132.

We may consider the wedge as consisting of two equal inclined planes, each of height h, length l, and base b; the angle of slope being θ (fig. 132).

While the wedge is driven in an amount x, the energy exerted by P is Px. H will move in the same interval of time a distance y, and the force of friction F will move a distance z. Hence, the energy-equation is

$$Px = Hy + Fz \quad . \quad . \quad . \quad . \quad (1)$$

We may take the normal pressure on the sides of the wedge as H. Hence, we may write (1), $Px = Hy + zH\tan\phi = H(y + x\tan\phi)$, or

$$H = \frac{Px}{y + z\tan\phi} = \frac{P}{\frac{y}{x} + \frac{z}{x}\tan\phi} = \frac{P}{\frac{h}{b} + \frac{l}{b}\tan\phi} = \frac{P\cos\theta}{\sin\theta + \tan\phi} \quad . \quad (2)$$

$$\text{The efficiency, } \eta = \frac{Hh}{Pb} = \frac{H}{P}\tan\theta \quad . \quad . \quad (3)$$

We have given: $P = 2$, $\tan\theta = 1 : 24$; we may take $\sin\theta = \tan\theta$, and $\cos\theta = 1$, $\tan\phi = 0{\cdot}1$.

$$H = 14{\cdot}12 \text{ tnws.}; \; \eta = 29{\cdot}4 \text{ per cent. } (Ans.).$$

$$\text{Velocity-ratio, } \alpha = \frac{b}{h} = 24; \text{ mechanical advantage, } \beta = \frac{H}{P} = 7{\cdot}06.$$

(123) (I.C.E., October 1908).—The sketch (fig. 133) shows diagrammatically the arrangement of levers in a platform weigh-

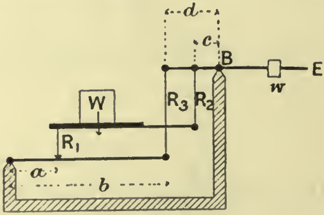


FIG. 133.

bridge. Investigate the connection between the lengths of *a*, *b*, *c*, and *d*, in order that the balance may be independent of the position of the weight W on the platform.

Solution.—Let x be the distance of W from R_1, l the distance between R_1 and R_2, and k the moment of w and the weight of the lever B E about B.

To find R_1 and R_2:

$$R_1 + R_2 = W; \; -Wx + R_2 l = 0.$$

Hence,
$$R_1 = W\frac{l - x}{l}, \; R_2 = W\frac{x}{l}.$$

The relation between R_1 and R_3 is:

$$-R_1a + R_3b = 0 \quad . \quad . \quad . \quad . \quad (1)$$

The relation between R_2 and R_3 is:

$$R_3d + R_2c = k \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating R_3 between (1) and (2) we get

$$-Wa + W\frac{x}{l}a + k\frac{b}{d} - W\frac{x}{l}\frac{c}{d}b = 0 \quad . \quad . \quad (3)$$

(3) must be independent of x.

Hence, $$W\frac{x}{l}a = W\frac{x}{l}\frac{c}{d}b, \text{ and } Wa = k\frac{b}{d} \quad . \quad . \quad (4)$$

From the first condition in (4) we get

$$a : b :: c : d \text{ (Ans.)}.$$

The second condition in (4) gives $W = k/c$. Hence, as k depends on the distance of w from B, the lever B E can be made to indicate the weights of W.

(124) (I.C.E., Feb. 1908).—**A lever system is shown diagrammatically in fig. 134, A and B being fixed fulcrums. The system**

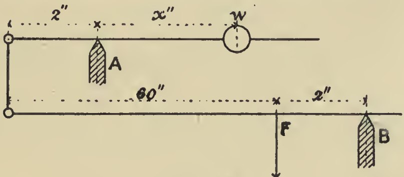


FIG. 134.

is balanced when the weight w, 100 lbs., is 4 inches to the right of A. Find the position of w when a force F of 20,000 lbs. is applied as shown.

Solution.—Remove w and F and add a vertical force q_1, at the vertical link, which just keeps the levers in a horizontal position. We may now consider the system of levers weightless; and we must now increase the vertical force at the vertical link by q_2 so that $q_1 + q_2 = Q$ can balance w about A. It was shown in article 186 that tangential forces are in equilibrium when the algebraical sum of their moments about the fulcrum is zero. Hence, Q is determined by $2Q = 4 \times 100$, or $Q = 200$ lbws.

When force F is applied a vertical force P must be added at the vertical link if the levers are to be kept in a horizontal position. P is determined by the equation, $62P = 2 \times 20{,}000$, or $P = 645{\cdot}16$ lbws. w must be moved to a distance x from A in order to balance $Q + P$. x is determined by

$$100x = 845{\cdot}16 \times 2, \quad \text{or} \quad x = 16{\cdot}9 \text{ ins. } (Ans.).$$

(125) (I.C.E., Feb. 1907).—**A screw-jack has a mean diameter of 2·25 inches and a pitch of 0·75 inch. It is turned by applying a force P tangentially at the end of a horizontal lever, 25 inches from the centre of the screw. Find (1) the magnitude of P when the jack is lifting 3 tons, neglecting friction; (2) the coefficient of friction between the threads that just suffices to prevent the screw running back under the 3-ton load.**

Solution.—The efficiency of the jack. Let F_e be the effort reduced to the mean radius r and l the length of the lever, then $Pl = F_e r$. The energy-equation for one revolution of the screw is

$$2\pi r F_e = Wp + \text{work done against friction} \quad . \quad (1)$$

Hence,
$$\eta = \frac{Wp}{2\pi r F_e} = \frac{Wp}{2\pi l P} \quad . \quad . \quad . \quad . \quad (2)$$

In fig. 135A $OB = 2\pi r$, $BA = p$, $CD = GF = F_e$, $CG = W$,

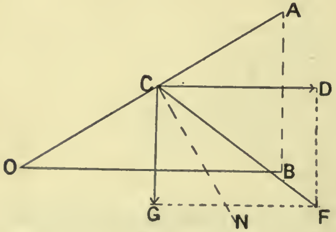


Fig. 135A.

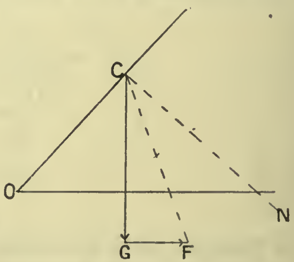


Fig. 135B.

CN is the normal, angle $FCN = \phi$. Hence, $F_e = W \tan(\theta + \phi)$. Let F_e' be the value of F_e when we neglect friction; we have then

$$F_e' = W \tan\theta \quad \text{and} \quad \eta_1 = \frac{Wp}{2\pi r F_e'} = 1 \quad . \quad . \quad (3)$$

In the case of friction we have

$$\eta = \frac{Wp}{2\pi r F_e} = \frac{Wp}{2\pi r F_e} \frac{2\pi r F_e'}{Wp} = \frac{F_e'}{F_e} = \frac{\tan\theta}{\tan(\theta + \phi)} \quad . \quad (4)$$

Question (1).

$$P = \frac{3 \times 0{\cdot}75 \times 2240}{2\pi \times 25} = 32 \text{ lbws. } (\textit{Ans.}).$$

Question (2).—In this case we must have $\theta = \phi$, the angle of repose. Hence, $\eta = \frac{\tan\phi}{\tan 2\phi} = \frac{1 - \tan^2\phi}{2}$; $\tan\theta = \frac{p}{2\pi r} = \frac{0{\cdot}75}{2{\cdot}25\pi}$ $= 0{\cdot}106 = \tan\phi$; hence, $\eta = 0{\cdot}494$; $P = \frac{32}{0{\cdot}494} = 65$ lbws.; velocity-ratio $= \frac{2\pi l}{p} = 209{\cdot}4$; mechanical advantage $= \frac{W}{P} = 103{\cdot}4$;

$$\tan\phi = 0{\cdot}106 \ (\textit{Ans.}).$$

When θ is greater than ϕ the "screw and nut" machine may be reversible, *i.e.* a pressure, W, applied on the nut and parallel to the axis of the screw, may turn the screw.

In fig. 135B the same letters refer to the same lines as in fig. 135A; angle $FCG = \theta - \phi$. $F_e = W\tan(\theta - \phi)$, and the torque required to turn the screw is $= F_e r = Wr\tan(\theta - \phi)$. The efficiency of the reversed "screw and nut" is

$$\eta = \frac{2\pi r F_e}{Wp} = \frac{\tan(\theta - \phi)}{\tan\theta} \quad . \quad . \quad . \quad (5)$$

This principle is used in some hand-drills. It is necessary that $Wr\tan(\theta - \phi)$ is equal to or greater than the moment of the force of friction between the drill and the material to be drilled; θ must therefore be considerably greater than ϕ.

(126) (I.C.E., Feb. 1906, first part).—**State the rule for finding the moment of inertia of a beam section which admits of being split up into a number of rectangular areas.**

Solution.—A beam section is a plane figure and can therefore have no moment of inertia, but it has a second area-moment w.r.t. a given straight line. If the latter line is parallel to one set of the sides of the rectangles, as in fig. 136, we may split the figure up into rectangles, all of which have one side on the line O–O. The second area-moment of these rectangles are all of the form $\frac{bh^3}{3}$, where b is the width and h the height of any one of the rectangles. The required (A.M.) is then obtained by a proper addition and subtraction of the various (A.M.)s.

In fig. 136 let b_1 and t_1 be the width and depth of the top-flange, b_2 and t_2 those of the bottom-flange, and b_3 and h the width and height of the web. The (A.M.) of the figure w.r.t. O–O is

$$(\text{A.M.}) = \frac{b_1(t_1+h+t_2)^3}{3} - \frac{(b_1-b_3)(h+t_2)^3}{3} + \frac{(b_2-b_3)t_2^3}{3}.$$

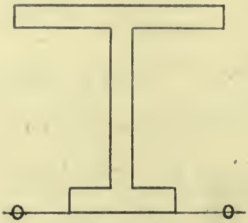


FIG. 136.

(127) (I.C.E., Feb. 1910).—**In a single-cylinder engine, intended to run at 120 revolutions per minute, the work done at each stroke is 2200 foot-lbs. while the work of acceleration is 400 foot-lbs. Then, for a flywheel 4 feet in diameter, what must be the weight of the rim if the speed is not to fluctuate beyond the limits of 119 and 121 revolutions? Neglect the mass of the wheel-arms.**

Solution.—If the tangential effort, which produces the rotary motion of a shaft, turning at its normal speed of revolution, is variable, while the tangential resistance to be overcome is constant, there will be a period during each revolution of the shaft when the energy exerted by the effort is greater than that consumed by the resistance. The excess of energy exerted will be accumulated in the flywheel as kinetic energy, and the angular velocity of the wheel will be increased from ω_1 to ω_2. This period will be followed by another one, during which the energy consumed by the resistance is greater than that exerted by the effort; the flywheel will give off the energy it received during the first period, and the angular velocity of the wheel will fall from ω_2 to ω_1.

Let W denote the energy which causes the irregularity of speed, R_g the radius of gyration, and M the mass of the wheel reduced to a distance R_g from the axis, then

$$W = 0{\cdot}5(\omega_2{}^2 - \omega_1{}^2)R_g^2M,$$

or

$$\omega_2 - \omega_1 = \frac{2W}{(\omega_2+\omega_1)R_g^2M}.$$

As $\omega_2 - \omega_1$ must be small, the mean angular velocity of the wheel will be $\omega = 0{\cdot}5(\omega_2 + \omega_1)$.

Hence, $$\frac{\omega_2 - \omega_1}{\omega} = \frac{W}{\omega^2 R_g^2 M} = k,$$

where k is a constant which must be the smaller the steadier the rotary motion is required to be.

If the shaft is to make n revs. per min., then

$$\omega = \frac{2\pi n}{60}; \quad \omega_1 = \frac{2\pi n_1}{60}; \quad \omega_2 = \frac{2\pi n_2}{60}.$$

Hence, $$\frac{\omega_2 - \omega_1}{\omega} = \frac{n_2 - n_1}{n} = k,$$

and $$M = \left(\frac{30}{\pi}\right)^2 \frac{W}{n^2 R_g^2 k}.$$

If W be given in foot-lbws. and R_g in feet, then

$$M = \left(\frac{30}{\pi}\right)^2 \frac{32{\cdot}19W}{n^2 R_g^2 k} = 2935 \frac{W}{n^2 R_g^2 k} \text{ lbs.}$$

The to-and-fro motion of the piston of a reciprocating engine is converted into rotary motion by the crank-connecting rod mechanism. In engines of this kind the crank-effort is never constant during a revolution of the shaft. Hence, a flywheel is required to store and return the energy which otherwise would cause too great a variation of the speed of rotation.

In the present problem $W = 400$ ft.-lbws., $n_1 = 119$, $n_2 = 121$, $n = 120$, $R_g = 4$ ft. Hence, the mass of the flywheel is required to be

$$M = 2935 \times \frac{400 \times 60}{16 \times 120^2} = 306 \text{ lbs. } (\textit{Ans.}).$$

(128) (Q. June 1910).—**Two drums of diameters 5 and 3 feet respectively are set with their axes parallel and their centres at a distance 7 feet apart. Find the necessary length of open belt to pass round the drums and be taut.**

Solution.—Let D be the diameter of the large pulley, d that of the smaller pulley, and l the distance between centres.

(i.) *Open belt* (fig. 137).—The length of the belt is

$$L = 2(ab + bc + ce) = 2 \times \frac{d}{2}\left(\frac{\pi}{2} - \phi\right) + 2l\cos\phi + 2 \times \frac{D}{2}\left(\frac{\pi}{2} + \phi\right)$$

$$= 2l\cos\phi + (D + d)\frac{\pi}{2} + (D - d)\phi; \quad \text{and} \quad \sin\phi = \frac{D - d}{2l}.$$

Inserting the given numerical values we get

$$\phi = 8^\circ\ 13' = 0{\cdot}1434 \text{ radians, and}$$
$$L = 13{\cdot}86 + 12{\cdot}57 + 0{\cdot}29 = 26{\cdot}72 \text{ ft. } (Ans.).$$

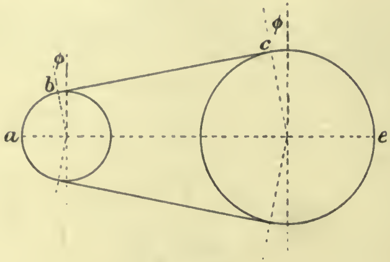


FIG. 137.

(ii.) *Crossed belt* (fig. 138).—The length of the belt is

$$L = 2(ab + bc + ce) = 2\left[\frac{d}{2}\left(\frac{\pi}{2} + \phi\right) + l\cos\phi + \frac{D}{2}\left(\frac{\pi}{2} + \phi\right)\right]$$
$$= 2l\cos\phi + (D + d)\left(\frac{\pi}{2} + \phi\right); \ \sin\phi = \frac{D + d}{2l}.$$

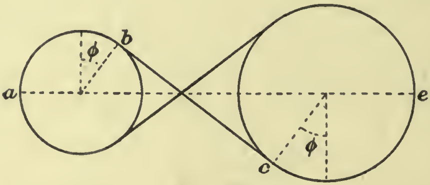


FIG. 138.

Inserting the same numerical values as given in the question we get

$$\phi = 34^\circ\ 51' = 0{\cdot}6082 \text{ radians, and}$$
$$L = 11{\cdot}49 + 17{\cdot}43 = 28{\cdot}6 \text{ ft. } (Ans.).$$

(129) (Q. June 1910).—**A uniform plank weighing 200 lbs. and 20 feet long is placed with one end upon the ground and rests upon a smooth cylinder of 6 feet diameter lying upon the ground. If the end upon the ground is 15 feet from the line of contact of the cylinder with the ground, determine graphically**

or otherwise the least horizontal force applied at the end upon the ground which will push the plank further up on the cylinder. The coefficient of friction between the ground and the plank is 0·2.

Solution.

(i.) *Analytical method.*—Let point B (fig. 139) be the mass-centre, l the length, and W the weight of the plank, and let r be the radius of the cylinder.

As the cylinder is smooth its axis must be fixed, or else it will roll off from underneath the plank.

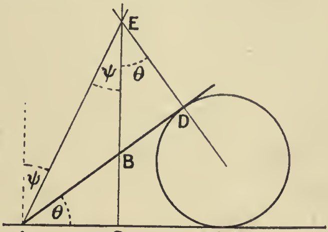


FIG. 139.

W acts along the vertical EBC, the reaction at D passes through the centre of the circle as the cylinder is smooth. Hence, the reaction, R, at A must act along the line AE. We have now

$$R = \frac{\sin\theta}{\sin(\theta+\psi)} W.$$

If H be the horizontal component of R, and F the force of friction at A, then the least horizontal force required at A to push the plank further up on the cylinder is

$$Q = F + H = R\cos\psi\tan\phi + R\sin\psi = \frac{\sin\theta\sin(\phi+\psi)}{\sin(\theta+\psi)\cos\phi} W$$

$$= \frac{\tan\phi + \tan\psi}{1 + \cot\theta\tan\psi} W.$$

To find θ and ψ.

$$\tan\frac{\theta}{2} = \frac{r}{AK}; \quad \tan\psi = \frac{AC}{EB + BC} = \frac{l\sin 2\theta}{2(2\overline{BD} + l\sin^2\theta)}.$$

Inserting the given values: $r = 3$ ft., $AK = 15$ ft., $l = 20$ ft.,

$BD = AK - \frac{l}{2} = 5$ ft., $\tan\phi = 0{\cdot}2$, and $W = 200$ lbws., it will be found that $\cot\theta = 2{\cdot}4$; $\tan\psi = 0{\cdot}548$.

Hence, $$Q = 65 \text{ lbws. } (\textit{Ans.}).$$

(ii.) *Graphical method.*—Draw fig. 139 to scale; resolve W along E A and E D; resolve R along A K and the perpendicular through A on A K, the former component is H; draw the cone of friction about the perpendicular through A and construct $R \cos\psi \tan\phi = F$, then

$$Q = H + F.$$

CHAPTER XXXVIII.

MOTION OF THE M.C.

219. Pressure on an axis due to revolving masses.

Let us consider a system of revolving masses rigidly connected to an axis O Z (fig. 140), which turns frictionless in fixed bearings. Let the plane of reference, which is perpendicular to O Z, contain two rectangular co-ordinate axes.

The inertia-pressure on the bearings due to a revolving mass m is $p = mr\omega^2$ acting in the direction of r and perpendicular

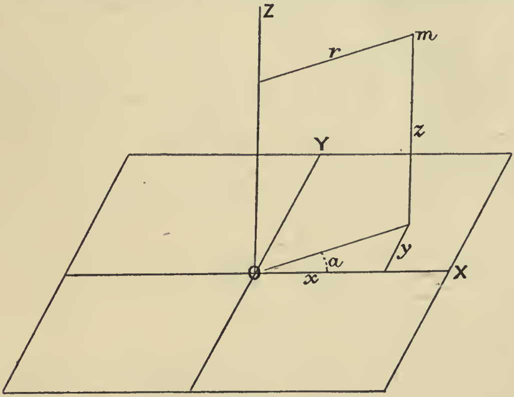


FIG. 140.

to O Z. p's components along the two axes in the plane of reference are $p \cos \alpha = m\omega^2 r \cos \alpha = \omega^2 mx$, and $m\omega^2 r \sin \alpha = \omega^2 my$. Let M be the total revolving mass, α and β the co-ordinates of the mass-centre, then the total inertia-pressure along the x-axis is $X = \omega^2 \sum mx = \omega^2 M\alpha$, and that along the y-axis is $Y = \omega^2 \sum my = \omega^2 M\beta$. Hence, the resultant inertia-pressure is

$$P = \omega^2 M \sqrt{\alpha^2 + \beta^2} = \omega^2 M r_g \quad . \quad . \quad . \quad (1)$$

or, *the resultant inertia-pressure due to a system of revolving masses is the same as if all the revolving masses were concentrated at their common mass-centre.*

The resultant inertia-pressure, P in (1), *is zero when the axis*, O Z, *passes through the mass-centre*, as then $r_g = 0$.

We have, however, seen that a system of forces need not be in equilibrium because their resultant is zero; it may be that the forces form a couple. Let us choose the x-axis so that the xz-plane (not the z-axis) passes through the mass-centre. Let R_1 and R_2 be the reactions respectively on the two bearings, R_1 acting in the plane of reference; θ_1 and θ_2, the angles which R_1 and R_2 form with the x-axis, and γ the distance of the mass-centre from the plane of reference. We have, when l is the distance between the bearings,

$$\text{the couple about the } y\text{-axis} = \omega^2 \sum mxz = \omega^2 M r_g \gamma = l R_2 \cos\theta_2 \quad (2)$$

$$\text{the couple about the } x\text{-axis} = \omega^2 \sum myz = l R_2 \sin\theta_2 \quad . \quad . \quad (3)$$

$$X = \omega^2 \sum mx = R_1 \cos\theta_1 + R_2 \cos\theta_2 = \omega^2 M r_g \quad . \quad (4)$$

$$Y = \omega^2 \sum my = R_1 \sin\theta_1 + R_2 \sin\theta_2 = 0 \quad . \quad . \quad (5)$$

The values of R_1, R_2, θ_1 and θ_2 may be determined by (2) . . . (5).

There will be no tendency to displace the z-axis when $R_1 = R_2 = 0$, which happens when

$$\sum mxz = 0, \quad \sum myz = 0 \quad \text{and} \quad r_g = 0 \quad . \quad [192]$$

We may take as an example a homogeneous cylinder of revolution.

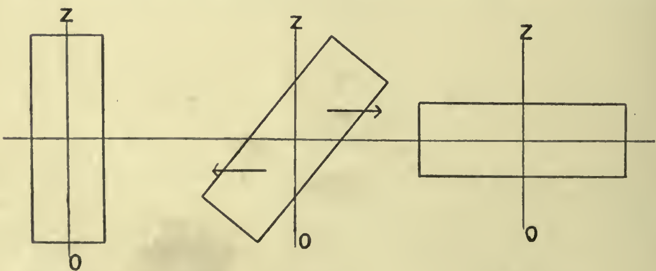


FIG. 141A. FIG. 141B. FIG. 141C.

In fig. 141 the axis of rotation, O Z, passes through the mass-centre of the cylinder. As long as O Z coincides with the axis of the cylinder, as in fig. 141A, [192] is satisfied; hence, $R_1 = R_2 = 0$. The equilibrium of the inertia-forces, however, is unstable, because the slightest deviation of O Z—although still passing through the M.C.—from the axis of the cylinder will cause the inertia-forces to form a couple, as shown in fig. 141B, and which tends to turn the body into the position shown in fig. 141C. Hence, there will be a pressure on each bearing. In fig. 141C, [192] is again satisfied, and the equilibrium of the inertia-forces is stable.

It is of great importance in machinery that the rotating masses should be so distributed round the axis of rotation as to produce no pressure on the bearings.

220. Moment of momentum.

The linear velocity of a rotating mass is $r\omega$; hence, the momentum of the mass is $mr\omega$, and the moment of momentum about the axis of rotation is $mr\omega r = \omega mr^2$. The total *moment of momentum* of a system of revolving masses about the axis of rotation is thus

$$\omega \sum mr^2 = \omega \mathbf{I} = \omega \mathrm{M} \mathrm{R}_g^2 \quad . \quad . \quad . \quad [193]$$

The moment of momentum is also called *angular momentum.*

The rate of change of the moment of momentum is

$$\frac{d\omega}{dt} \sum mr^2 = \frac{d\omega}{dt} \mathrm{M} \mathrm{R}_g^2,$$

but $r_1 \dfrac{d\omega}{dt}$ is the linear acceleration of mass m_1; hence,

$$m_1 r_1 \frac{d\omega}{dt} = \mathrm{F}_1, \text{ and } \frac{d\omega}{dt} \sum mr^2 = \sum \mathrm{F}r = \frac{d\omega}{dt} \mathrm{M} \mathrm{R}_g^2 = \frac{d\omega}{dt} \mathbf{I} \quad (1)$$

Hence, *the rate of change of angular momentum of a system of masses rotating about an axis is equal to the algebraical sum of the moments of the forces acting on the system* (compare article **166**).

221. Velocity of the mass-centre.

Fig. 142 represents a body which is acted upon by forces parallel to the plane of reference. Let $C(\xi, \eta)$ be the M.C. and M the mass of the body. We have

$$\mathrm{M}\xi = \sum mx, \text{ and } \mathrm{M}\eta = \sum my \quad . \quad . \quad (1)$$

By differentiating (1) w.r.t. time t we get (article **158**),

$$\mathrm{M} \frac{d\xi}{dt} = \sum m \frac{dx}{dt}, \text{ and } \mathrm{M} \frac{d\eta}{dt} = \sum m \frac{dy}{dt} \quad . \quad . \quad (2)$$

where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the velocities of a particle of mass m, in the directions of the x-axis and the y-axis respectively. Hence, *the velocity,* $\sqrt{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2}$, *of the* M.C. *of the body is equal to the sum of the momenta of the particles of the body, divided by the mass of the body;* also, *the momentum of the body is the same as if the masses of the particles were collected in the* M.C.

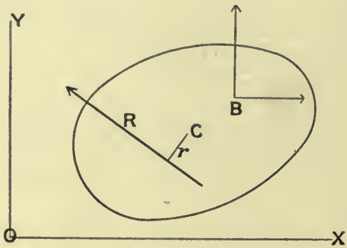


FIG. 142.

222. Acceleration of the mass-centre.

By differentiating (2) in the preceding article (see article **161**), we get

$$X = M\frac{d^2\xi}{dt^2} = \sum m\frac{d^2x}{dt^2}, \quad \text{and} \quad Y = M\frac{d^2\eta}{dt^2} = \sum m\frac{d^2y}{dt^2}. \qquad (1)$$

where $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ are the accelerations of the particle in the directions of the x-axis and the y-axis respectively. Hence, the acceleration of the M.C. is

$$\sqrt{\left(\frac{d^2\xi}{dt^2}\right)^2+\left(\frac{d^2\eta}{dt^2}\right)^2},$$

and the resultant of the forces is

$$R = \sqrt{X^2+Y^2} = M\sqrt{\left(\frac{d^2\xi}{dt^2}\right)^2+\left(\frac{d^2\eta}{dt^2}\right)^2},$$

or *the motion of the* M.C. *is the same as if all the forces were moved to that point parallel to their original directions.*

The motion of the body consists of two simultaneous motions, viz:

(i.) *A translation of the* M.C. *with an acceleration*, R/M, *in the direction of* R, and

(ii.) *A rotation with a torque*, Rr, *about an axis through the* M.C. *perpendicular to the plane of reference.*

If the forces which act on the body are not parallel to the same plane we must resolve them along three fixed rectangular co-ordinate axes and proceed as in the preceding and present articles.

As neither the rotation about the M.C., nor the magnitude and the direction of the resultant, R, are changed by moving a couple, which acts on the body, parallel to itself, it follows that: *A couple may be moved parallel to itself without disturbing the motion of the body* (compare article **205**).

223. Conservation of momentum.

When the forces, in the two preceding articles, *are mutually balanced their resultant is zero, and the* M.C. *will move in a straight line with uniform velocity.* This theorem is known as the principle of the *conservation of the motion of the* M.C. As *the total momentum of the body*, or the system of particles, *remains the same*, the theorem is also referred to as the principle of the *conservation of momentum.*

The forces which act on the body may be *external forces*, *i.e.* forces which are due to external causes; or the forces may be *internal forces*, *i.e.* mutual actions between the particles of the system, such as the attraction between heavenly bodies or as stresses between the particles of an elastic body, for instance, when the body is being extended, due to rise of temperature. Internal forces are always balanced, because one particle attracts or repels another one by a force which is equal to the force by which the latter particle attracts or repels the former. Hence, *the motion of the* M.C. *is unaffected by any mutual action between the particles of the system.*

The motion of the M.C. of a shell containing an explosive will thus be unaffected by the bursting of the shell, because the action and the reaction between the gases and the pieces of the bursting shell are equal, and are therefore balanced.

The present chapter has taught us that *we may regard the motion of a body in space as if the mass of the body were concentrated at the* M.C., *i.e.* as if the body were reduced to a single point, the M.C., with a mass equal to that of the body.

CHAPTER XXXIX.

GRAVITATION.

224. Law of gravitation.

This law states that *every particle in nature attracts every other particle with a mutual force which varies directly as the product of the masses of the particles and inversely as the square of the distance between them.* Hence, if the particles, whose masses are m_1 and m_2, are placed at a distance d apart, the force of attraction is

$$F_g = K\frac{m_1 m_2}{d^2} \quad . \quad . \quad . \quad . \quad [194]$$

where K is a constant called the gravitation constant; its dimensions are evidently $[L]^3[M]^{-1}[T]^{-2}$.

As the attraction between the particles is a mutual action it follows, by article 223, that the motion of the particles towards each other cannot affect the motion of their common M.C. Hence, the two masses are attracted towards their common M.C. Thus, both a falling body and the earth fall towards their common M.C. which, however, is practically at the M.C. of the earth.

The acceleration with which m_1 approaches m_2 is $\frac{F_g}{m_1} = K\frac{m_2}{d^2}$, and similarly the mass m_2 approaches m_1 with an acceleration $K\frac{m_1}{d^2}$. Hence, the apparent acceleration with which one particle approaches the other, viewed from either of them, is $K\frac{(m_1 + m_2)}{d^2}$.

If m_1 were the mass of the earth and m_2 that of a falling body, the acceleration of gravity would change with m_2. But m_2 is very small compared with the mass of the earth and may therefore be neglected; the acceleration of gravity may therefore be taken to be the same for all bodies.

225. Attraction of a homogeneous spherical shell.

Consider a very thin shell consisting of one layer of particles uniformly distributed over the surface of the shell, and let all the particles be similar in every respect. Let the problem be to determine the force of mutual attraction between the shell and a particle of mass m.

(i.) *The particle is outside the shell.*—The force of attraction (fig. 143) between a particle of the shell and m is $f_g = \mathrm{K}\frac{m_1 m}{\eta^2}$, and its two components are $\mathrm{X} = f_g \cos\alpha$ and $\mathrm{Y} = f_g \sin\alpha$. The total

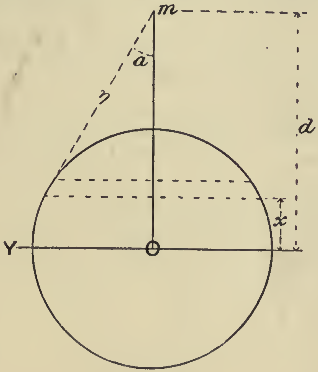


FIG. 143.

attractive force due to an elementary zone is $\delta \mathrm{F}_g = \sum \mathrm{X} = \sum f_g \cos\alpha$, $\sum \mathrm{Y}$ being obviously equal to zero.

By article **133** the area of the elementary zone is $2\pi r\,\delta x$, and if there be n particles per unit area, then the mass of the elementary zone is $2\pi r n m_1\,\delta x$. We have also

$$\cos\alpha = \frac{d-x}{\eta}, \quad \text{and} \quad \eta = \sqrt{(d-x)^2 + y^2}.$$

Hence,
$$\delta \mathrm{F}_g = 2\pi r \mathrm{K} n m_1 m \frac{d-x}{(d^2 - 2xd + r^2)^{\frac{3}{2}}}\,\delta x,$$

and
$$\mathrm{F}_g = 2\pi r \mathrm{K} n m_1 m \int_{-r}^{+r} \frac{d-x}{(d^2 - 2xd + r^2)^{\frac{3}{2}}}\,dx \qquad . \qquad . \qquad (1)$$

(1) may be integrated by substitution, thus

$$z^2 = d^2 - 2xd + r^2;$$

hence,
$$\frac{dx}{dz} = -\frac{\sqrt{d^2 - 2xd + r^2}}{d}, \quad \text{and} \quad x = \frac{d^2 + r^2 - z^2}{2d}.$$

Hence, $$F_g = 4\pi r^2 K \frac{nm_1 m}{d^2};$$

but $4\pi r^2 nm_1$ is the mass, M, of the shell.

Hence, $$F_g = K\frac{Mm}{d^2} \quad . \quad . \quad . \quad . \quad [195]$$

or the mutual attractive force between the shell and the particle is the same as if the mass of the shell were concentrated at the centre of the shell. As a solid sphere and a spherical shell of any thickness consists of a number of thin concentric shells, it follows that *the mutual attractive force of a solid homogeneous sphere or of a homogeneous spherical shell and a particle outside the sphere is the same as if the mass of the sphere or of the shell were concentrated at the centre of the sphere.*

The same law will, of course, hold if the sphere or the shell consists of homogeneous concentric shells.

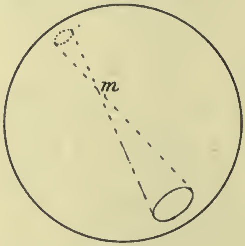


FIG. 144.

If we consider that the earth consists of homogeneous concentric shells, the weight of a mass m at the pole is

$$mg = K\frac{Mm}{R^2}, \quad \text{or} \quad K = \frac{gR^2}{M} \quad . \quad . \quad . \quad (2)$$

where M is the mass of the earth, R its polar radius, and g the acceleration of gravity at the pole. M has been determined by Cavendish and others, and as g and R are both known, the value of K can be determined.

$$K = 666 \times 10^{-10} \text{ C.G.S. units} \quad . \quad . \quad . \quad (3)$$

(ii.) *The particle is inside the shell.*—Describe, with the particle as apex, a cone with a very small apex-angle. The cone cuts out of the shell two portions a_1 and a_2 at distances r_1 and r_2 from the particle (fig. 144). If n_1 and n_2 be the number of particles contained in the two portions respectively, then $n_1 : n_2 :: r_1^2 : r_2^2$. Let F_g' and F_g'' be the forces of gravitation due to m and the particles of a_1 and a_2, then by [194]

$$F_g' = K\frac{n_1 m_1 m}{r_1^2} \quad \text{and} \quad F_g'' = K\frac{n_2 m_1 m}{r_2^2}, \quad \text{or} \quad \frac{F_g'}{F_g''} = \frac{n_1 r_2^2}{n_2 r_1^2} = 1,$$

or rather -1, as the two forces act in opposite senses. Hence, *the mutual attractive force of a homogeneous spherical shell and a particle inside the shell is zero.*

The same law holds for spherical shells of any thickness and also for shells consisting of homogeneous concentric spherical shells.

226. Mutual attraction of a solid homogeneous sphere and a particle in the interior of the sphere.

Let R be the radius of the sphere and r the distance of the particle from the centre of the sphere. By the preceding article it follows that the particle is not attracted by the shell whose radii are R and r, but is only attracted by the sphere whose radius is r. The mass of the latter sphere is $M = \frac{4}{3}\pi r^3 m_1$, and the force of mutual attraction between M and m is

$$F_g = K\frac{Mm}{r^2} = \frac{4}{3}\pi K m_1 m r \quad . \quad . \quad . \quad [196]$$

or *the mutual attractive force of a solid homogeneous sphere and a particle within the sphere varies directly as the distance of the particle from the centre of the sphere.* Hence, at the centre the force is zero.

CHAPTER XL.

THE PENDULUM.

227. Simple harmonic motion.

DEFINITION.—*When a particle moves in a straight line, so that its acceleration is always directed through, and varies as its distance from, a fixed point in the straight line, the particle is said to move with simple harmonic motion.*

In fig. 145 a particle N moves in a circle with constant speed u in the anti-clockwise direction. Another particle P moves so that it is always at the foot of the perpendicular from N on a

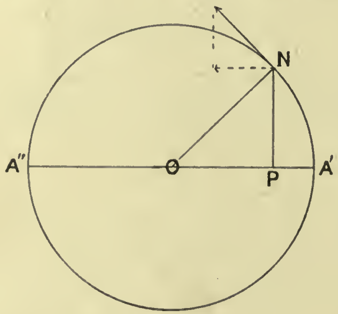


FIG. 145.

fixed diameter A′A″. Hence, P oscillates between A′ and A″, while N rotates at constant speed in the circle.

Let v be the velocity, at any instant, of P, r the radius of the circle, θ = angle P O N, O P = x, and P N = y.

(i.) *To find the time, t, in which the particle P describes the distance A′P.* t is also the time in which N describes arc A′N.

Hence, $\quad ut = r\theta, \quad \text{or} \quad t = \dfrac{1}{\omega}\text{ang}\left(\cos = \dfrac{x}{r}\right).$

The time, T, in which N describes the circle is called the *periodic time* of the motion. It is obvious that $2\pi r = u\text{T}$.

Hence, $$\text{T} = \frac{2\pi}{\omega} \quad \text{and} \quad \omega = \frac{2\pi}{\text{T}} \qquad . \quad . \quad . \qquad (1)$$

The periodic time is thus independent of r which is called the *amplitude* of the oscillation. The motion of P from A′ to A″ and back to A′ is called a *complete oscillation, or a complete vibration.*

The time in which P describes the distance between its present position and the nearest resting-place (A′ or A″) is thus

$$t = \frac{\text{T}}{2\pi}\text{ang}\left(\cos = \frac{x}{r}\right) \qquad . \quad . \quad . \qquad (2)$$

and the time in which P describes the distance between its present position and the furthest resting-place is

$$t = \frac{\text{T}}{2} - \frac{\text{T}}{2\pi}\text{ ang}\left(\cos = \frac{x}{r}\right) = \frac{\text{T}}{2}\left[1 - \frac{1}{\pi}\text{ang}\left(\cos = \frac{x}{r}\right)\right] \quad . \qquad (3)$$

(ii.) *To find the velocity, v, of the particle P.*—It will be seen that $v = -u\sin\theta = -u\frac{y}{r} = -\omega y$, or

$$v = -\frac{2\pi}{\text{T}}y, \quad \text{where} \quad y = \pm\sqrt{r^2 - x^2} \quad . \quad . \qquad (4)$$

v changes sense when y changes sign. v is a maximum at point O where $y = r$; v is zero at points A′ and A″ where $y = 0$; A′ and A″ are therefore resting-places.

(iii.) *To find the acceleration, a, of the particle P.*—The only force acting on particle N is the deviating force which produces an acceleration, $\omega^2 r$, always directed towards the centre O. The component of $\omega^2 r$ in the direction A′A″ is equal to the acceleration of P.

Hence, $$a = \omega^2 r\cos\theta = \frac{4\pi^2}{\text{T}^2}x \qquad . \quad . \quad . \qquad (5)$$

Hence, the acceleration of P is always directed towards, and varies as the distance from, point O, the position of equilibrium. *The motion of particle P is therefore a simple harmonic motion,* and the relations expressed in (1) . . . (5) are those characteristic of the latter kind of motion. The particle P is accelerated or retarded, according as the numerical value of x is decreasing or increasing.

By (5) the periodic time of a simple harmonic motion is

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} \quad . \quad . \quad (6)$$

228. Conical pendulum.

DEFINITION.—*A particle of mass, m, suspended from a fixed point by a weightless inextensible string and revolving in a horizontal circle is called a* conical pendulum.

To find the time, T, of a revolution, or the periodic time of a conical pendulum.

In fig. 146 $SQ = l$ is the length of the pendulum, $OQ = r$ the

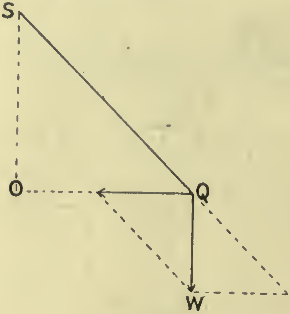


FIG. 146.

radius of the circle described by the particle Q, $W = mg$ the weight of the particle, $\phi =$ angle OSQ, and $h =$ the vertical height OS.

Resolve W into two components, one in the direction of SQ and another in the direction of r. The latter force is the deviating force F_n. If u be the constant speed of the particle, then

$$F_n = mg \tan\phi = mg\frac{r}{h} = m\frac{u^2}{r}, \quad \text{but} \quad u = \frac{2\pi r}{T}.$$

Hence, $\dfrac{g}{h} = \dfrac{4\pi^2}{T^2}$, and $T = 2\pi\sqrt{\dfrac{h}{g}} = 2\pi\sqrt{\dfrac{l\cos\phi}{g}}$. [197]

When ϕ is very small we may take $\cos\phi = 1$, and

$$T = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad [198]$$

which is independent of angle ϕ, or the *oscillations are isochronous.*

229. Simple pendulum.

DEFINITION.—*A particle of mass, m, suspended from a fixed point by a weightless inextensible string and oscillating in a vertical plane through the fixed point is called a* simple pendulum.

To find the periodic time of a simple pendulum when the angle of oscillation is very small.

In fig. 147 let $SP = l$ be the length of the pendulum, S the point of suspension, ϕ = angle OSP, d = DP the distance of the particle from the vertical OS.

At any position SP of the pendulum the effort $F_e = mg \sin \phi$ acts in the direction of the tangent to the arc which the particle describes, $\sin \phi = d/l$, therefore $F_e = mgd/l$. When the angle ϕ is very small, arc OP may be considered a straight line along which

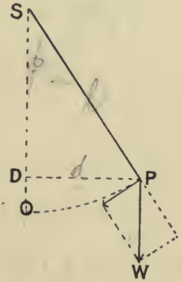


FIG. 147.

F_e acts, and as the latter is $\propto d$, the motion of the particle is a simple harmonic motion.

Let the circle in fig. 145 be horizontal, and let there be a point S vertically above O, and let point S be the point of suspension of a conical as well as of a simple pendulum, both of the same length l. The particle Q of the conical pendulum rotates at constant speed in the horizontal circle, and the particle P of the simple pendulum oscillates in the vertical plane along diameter A′A″. When angle ϕ is very small, the periodic times of the two pendulums are equal, viz. $T = 2\pi/\omega$. Hence, by [198] in the preceding article, the periodic time of a simple pendulum when the displacement angle is very small is

$$T = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad [199]$$

or the oscillations of a simple pendulum are *isochronous* when ϕ is very small.

230. Compound pendulum.

DEFINITION.—*A compound pendulum is a body which can oscillate about a fixed horizontal axis. The length of a compound pendulum is equal to the length of the simple pendulum whose periodic time is equal to that of the compound pendulum.*

Fig. 148 represents a body which can oscillate about a horizontal axis through point S, C is the M.C., S C = k, S O = l, S A the vertical through S, W = Mg is the weight of the body, ϕ = angle A S O.

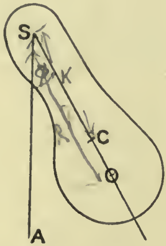


FIG. 148.

The energy exerted by W while the pendulum turns through angle ϕ is

$$Mgk(1 - \cos\phi) = 0{\cdot}5\,\omega^2 I \quad . \quad . \quad . \quad (1)$$

where **I** is the moment of inertia of the body about the axis of suspension, and ω is the angular velocity of the pendulum when S O passes the vertical position.

The energy exerted while the simple pendulum turns through angle ϕ is

$$mgl(1 - \cos\phi) = 0{\cdot}5\,\omega^2 ml^2 \quad . \quad . \quad . \quad (2)$$

(1) and (2) give

$$l = \frac{I}{Mk} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, the periodic time of a compound pendulum when ϕ is very small is,

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{I}{Mgk}} = 2\pi\sqrt{\frac{I}{Wk}} \quad . \quad . \quad [200]$$

Point O in the compound pendulum is called the *centre of oscillation.*

If $\mathbf{I}' = M R_g^2$ be the moment of inertia about an axis through the M.C. and parallel to the axis of suspension, then

$$\mathbf{I} = M R_g^2 + Mk^2 = M(R_g^2 + k^2);$$

hence,
$$l = \frac{M(R_g^2 + k^2)}{Mk} = \frac{R_g^2}{k} + k, \text{ i.e. } l > k.$$

If the pendulum be suspended from an axis through S, then

$$l = \frac{R_g^2 + k^2}{k}.$$

If the pendulum be suspended from an axis through O, then

$$l_1 = \frac{R_g^2 + (l-k)^2}{l-k}.$$

Hence,
$$l - l_1 = l - \frac{R_g^2 + (l-k)^2}{l-k} = lk - (R_g^2 + k^2) = 0, \text{ or } l = l_1.$$

Hence, *the centres of oscillation and suspension are convertible.*

If we take any straight line through the M.C., and take any two points S and S_1 at the same distance, d, from the M.C., the periodic times of oscillation about the two points will be the same, but $S S_1$ will not be equal to l unless $2d = \frac{R_g^2 + d^2}{d}$, or $d = R_g$. Hence, on any straight line through the M.C. there are four points S, O, S_1 and O_1 about which the times of oscillation are equal.

231. Kater's pendulum.

This pendulum is a bar which can oscillate about either of two parallel knife-edges whose plane passes through the M.C. By varying the position of the knife-edges the periodic times of small oscillations about either of them may be made equal. To be sure that the distance between the edges is equal to l, the edges should not be equidistant from the M.C.

By measuring l and T the value of g may be determined most accurately, thus

$$T = 2\pi\sqrt{\frac{l}{g}}; \text{ hence, } g = 4\pi^2\frac{l}{T^2} \qquad . \qquad . \quad [201]$$

232. Ballistic pendulum.

The velocity of a bullet may be determined by letting it strike against a large body suspended like a pendulum, called a *ballistic pendulum.*

Let the bullet strike the body in a horizontal direction in a plane perpendicular to the axis of suspension.

By article 220 we have that the moment of momentum of a body is equal to ωI, and also that the rate of change of moment of momentum is equal to the algebraical sum of the moments of the external forces. Hence, $I\,d\omega/dt = Fx$, where x is the lever of the instantaneous force w.r.t. the axis of suspension.

During the very short time, t, that F acts on the body the angular velocity changes from zero to ω. Let $p = \int_0^t F\,dt$ be the impulse due to the impact, then

$$px = \omega I \qquad . \quad . \quad . \quad . \quad (1)$$

The kinetic energy received by the body is $0{\cdot}5\,\omega^2 I$, causing the M.C. to be raised through a height h.

Hence, $$Mgh = 0{\cdot}5\,\omega^2 I \quad . \quad . \quad . \quad . \quad (2)$$

233. Centre of percussion.

The axis of suspension of a ballistic pendulum will receive a shock due to the impact of the bullet. Let R be the reaction of the bearings due to the latter shock.

The velocity of a particle of the body directly after impact is $r\omega$; hence, the change of momentum of the body is $\sum mr\omega = \omega \sum mr = \omega Mk$, but this change is equal to the sum of the impulses due to F and R,

or $$p + p_1 = \omega Mk \quad \text{where} \quad p_1 = \int_0^t R\,dt \quad . \quad . \quad (1)$$

By (1) in the preceding article we have

$$\frac{\omega I}{x} + p_1 = \omega Mk, \quad \text{or} \quad p_1 = \omega\left(Mk - \frac{I}{x}\right) \quad . \quad . \quad (2)$$

p_1 is zero when $x = \frac{I}{Mk}$, *i.e.* there will be no shock at the bearings of the axis if the velocity of the bullet be directed through the centre of oscillation of the pendulum. The latter point is therefore also called the *centre of percussion*.

Let θ be the angle through which the pendulum is turned by the impulse, then

$$h = k(1 - \cos\theta) = 2k\sin^2\frac{\theta}{2}, \quad \text{and} \quad 4Mgk\sin^2\frac{\theta}{2} = \omega^2 I \qquad (3)$$

As $$T = 2\pi\sqrt{\frac{I}{Mgk}},$$

we have, by eliminating ω and **I** between (1), (2), and (3),

$$px = \frac{MgkT}{\pi} \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (4)$$

If the bullet comes to rest in the pendulum, and m and v respectively are the mass and velocity of the bullet, then $p = mv$.

Hence,

$$v = \frac{MgkT}{\pi x m} \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad [202]$$

CHAPTER XLI.

EXAMPLES.

(130).—To find the volume of a right cylinder whose top end is cut off by a plane inclined at angle ϕ to the base.

F B D E F (fig. 149) is an elevation of the body and the curved figure is the base of the cylinder. The straight line in which the plane B D intersects the base has been chosen as axis of reference.

Let V be the volume of the body, then $\delta V = Z\,\delta A$. But as $Z = x\tan\phi$, $\delta V = x\,\delta A\tan\phi = \delta(a.m.)\tan\phi$. Hence, $V = (a.m.)\tan\phi$,

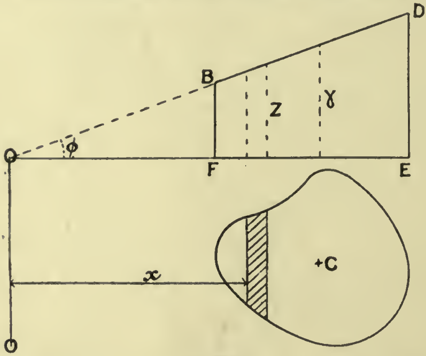


FIG. 149.

where $(a.m.)$ is the first area-moment of the base w.r.t. O – O.

Let α be the distance from the centroid, C, of the base to the axis of reference, and γ the height through C, then

$$V = \alpha A\tan\phi = \gamma A \qquad . \quad . \quad . \quad [203]$$

or, *the volume of the given body is equal to the volume of a right cylinder whose base is that of the body and whose height is the height of the body through the centroid of the base.*

Let ξ be the distance between the axis of reference and the foot of the perpendicular on the base through the centroid of the body, then

$$\xi V = \sum x\,\delta V = \tan\phi \sum x^2\,\delta A = (\text{A.M.})\tan\phi.$$

Let R_m be the second mean radius of the area of the base w.r.t. an axis through C and parallel to the axis of reference, then $(\text{A.M.}) = R_m^2 A + \alpha^2 A$. Hence,

$$\xi = \frac{R_m^2}{\alpha} + \alpha \quad . \quad . \quad . \quad . \quad [204]$$

If the heights, Z, represent intensities of pressure, so that the minimum intensity of pressure is $p = k\,\overline{FB}$, and the maximum intensity of pressure is $q = k\,\overline{ED}$, then $\tan\phi = \frac{q-p}{kb}$, where $b = FE$. The total pressure on the area is

$$P = kV = k\gamma A = k\alpha A\tan\phi = \alpha A\frac{q-p}{b} \quad . \quad . \quad [205]$$

The line of action of P passes through the centroid of the body and intersects the area at a point, *the centre of pressure*, whose distance from the axis of reference is ξ. The latter axis is, in this case, called the *neutral axis*.

By article **230** the length of a compound pendulum is

$$l = \frac{I}{M\alpha} = \frac{M\,R_m^2 + M\alpha^2}{M\alpha} = \frac{R_m^2}{\alpha} + \alpha.$$

Hence, ξ in [**204**] corresponds to the length of a pendulum whose bob is a very thin disc of the shape of the base of the above cylinder, and oscillating about the axis of reference.

(131) (I.C.E., Feb. 1904).—**A uniform circular plate, 1 foot in diameter and weighing 4 lbs., is hung in a horizontal plane by three fine parallel cords from the ceiling, and when set into small torsional oscillations about a vertical axis is found to have a period of 3 seconds. A body, whose moment of inertia is required, is laid diametrically across it, and the period is then found to be 5 seconds, the weight of the body being 6 lbs. Find the moment of inertia of the body about the axis of oscillation.**

Solution.—Let l be the length of each of the three parallel cords, W the weight of the circular plate, and r its radius.

Turning the plate through a small angle about its vertical axis and setting it free, it will overshoot its natural position and commence a series of isochronous oscillations. Each of the three wires may be considered a simple pendulum of length l, and suspending together a mass m having an acceleration whose vertical component is a. Hence, the periodic time of the oscillating plate is $T=2\pi\sqrt{\frac{l}{a}}$.

At the moment the plate swings through its natural position the energy-equation is

$$0{\cdot}5\,\omega_1^2 I = Wl(1-\cos\theta) \quad . \quad . \quad . \quad (1)$$

θ is the displacement angle of the wire, and **I** the moment of inertia of the plate about its vertical axis. As m is the mass of the plate reduced to its circumference we have also

$$0{\cdot}5\,\omega_2^2 l^2 m = mal(1-\cos\theta) \quad . \quad . \quad . \quad (2)$$

But $m=\mathbf{I}/r^2$ and $l\omega_2 = r\omega_1$. Hence, we have $\mathbf{I}/r^2 = W/a$, or $a=r^2 W/\mathbf{I}$. The periodic time of the oscillating plate is thus

$$T=2\pi\sqrt{\frac{l\mathbf{I}}{Wr^2}} \quad . \quad . \quad . \quad . \quad (3)$$

The data of the two given experiments are

(i.) $T=3$ secs.; $\mathbf{I}=M\frac{r^2}{2}=0{\cdot}5$ lb.-ft.^{2}; $W=4$.

(ii.) $T=5$ secs.; $\mathbf{I}=0{\cdot}5+\mathbf{I}_1$; $W=10$.

Hence, $\mathbf{I}_1=3$ lb.-ft.^{2} (nearly) (*Ans.*).

(132) (Q. Nov. 1906).—**Simple harmonic motion being defined as derived from uniform motion in a circle, investigate the law of acceleration.**

A particle performs 150 complete simple harmonic oscillations a minute and its greatest acceleration is 10 feet per second per second; find (1) its greatest velocity, (2) its mean velocity during the motion from one extreme position to the other.

Solution.—The answer to the first portion of the present question will be found in article 227. By the same article we have

$$\text{Maximum acceleration} = \frac{4\pi^2 r}{T^2}, \text{ and maximum velocity} = \frac{2\pi r}{T}.$$

The frequency of the oscillations $= 150/60 = 2{\cdot}5$; hence, $T = \frac{1}{2{\cdot}5}$ $= 0{\cdot}4$ sec. As the maximum value of $a = 10'/\text{sec.}^2$, we have

$r = \frac{10 \times (0{\cdot}4)^2}{4\pi^2} = 0{\cdot}04$ ft.; hence, max. velocity $= 0{\cdot}628$ ft./sec. (*Ans.*).

The mean velocity of the particle $= \frac{4r}{T} = 0{\cdot}4$ ft./sec. (*Ans.*).

(133) (I.C.E., Feb. 1901, second part). — **Prove that if a frictionless straight tunnel tube were constructed from London to Paris, and air-resistance prevented, parcels could be delivered between the two places, under gravity alone, in about 42 minutes.**

Solution.—Let arc L P (fig. 150) be a portion of the great circle

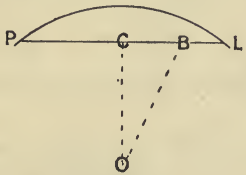


FIG. 150.

through London and Paris, chord L P the tunnel tube, O the centre of the earth, and θ angle C O B.

The acceleration at any point B in the tunnel is $g \sin\theta$, and if θ be very small we may take θ for $\sin\theta$. Hence, the parcel will move as the particle of a simple pendulum whose length is equal to the radius of the earth. Let the latter be R, then the time taken by the parcel is

$$t = \pi\sqrt{\frac{R}{g}}.$$

$R = 20{,}923{,}600$ ft. Hence,

$$t = 42{\cdot}2 \text{ mins. } (\textit{Ans.}).$$

(134) (I.C.E., Oct. 1908). — **In order to measure the instantaneous velocity of a flywheel 6 feet in diameter, a pendulum is employed consisting of a solid sphere 1 foot in diameter hanging on a wire $2\frac{1}{2}$ feet long. A pencil attached to the sphere at its horizontal diameter makes a continuous mark on the circumference of the wheel when the pendulum is at rest.**

The pendulum is deflected through 60° in the plane perpendicular to the flywheel and set free, and the new mark made by the pencil on the periphery intersects the former line at 45°. Determine the angular velocity of the flywheel in revolutions per minute.

Solution.—Let v be the velocity of the pencil when it touches the periphery of the wheel, and V the velocity of the latter. The centre of the sphere may be taken as the M.C. of the pendulum. Hence, the pencil as well as the M.C. are 3 feet from the suspension. The height through which the M.C. of the pendulum falls $= 3 - 3\cos 60° = 1{\cdot}5$ ft.

As the mark made by the moving pencil forms an angle of 45° with the mark made by the pencil when at rest, v must be equal to V.

Hence, $$v = V = \sqrt{2g \times 1{\cdot}5} = 9{\cdot}83 \text{ ft./sec.}$$

Radius of the wheel = 3 ft.; hence,

$$3\omega = V, \quad \text{or} \quad \omega = \frac{V}{3} = 3{\cdot}28 \text{ radians/sec.} = 31{\cdot}3 \text{ revs./min. } (\textit{Ans.}).$$

(135) (I.C.E., Feb. 1909).—**An impact tester for testing steel specimens consists of a compound pendulum made out of a bar 6 feet long of mass 40 lbs., pivoted at the top end. The bottom end of the bar strikes against the specimen to be tested, which is fixed in a vice immediately below the end of the pendulum in the natural position of rest.**

The pendulum is deflected through 60° from the vertical, is then set free, and after breaking the specimen it comes to rest at a deflection of 30° on the opposite side of the vertical. Find the velocity of the blow and the energy absorbed by the specimen.

Solution.—The M.C. of the bar is 3 ft. from the point of suspension. The height through which the M.C. falls $= 3 - 3\cos 60° = 1{\cdot}5$ ft. Hence, the velocity of the blow $= \sqrt{2g \times 1{\cdot}5} = 9{\cdot}83$ ft./sec.

After the blow the M.C. rises $(3 - 3\cos 30°) = 0{\cdot}402$ ft. Hence, the energy absorbed by the specimen

$$= 40\,(1{\cdot}5 - 0{\cdot}402) = 44 \text{ ft.-lbws. (nearly) } (\textit{Ans.}).$$

(136) (Q. Nov. 1907).—**A point is moving in a straight line with simple harmonic motion. Its velocity has the values 3 feet per second and 2 feet per second when its distances from the**

mean position are 1 foot and 2 feet respectively. Find the length of its path and the period of its motion.

Find also, correct to the third significant figure, what fraction of the period is occupied in passing between the specified points.

Solution.—By the moving point is meant a particle, because without mass no vibratory nor any other motion can take place.

By article 227

$$v=\frac{2\pi}{T}\sqrt{r^2-x^2}, \quad \text{or} \quad \frac{v^2T^2}{4\pi^2}+x^2=r^2.$$

It is given that, when $x=1'$, $v=3'/\text{sec.}$; and when $x=2'$, $v=2'/\text{sec.}$

Hence, $$\frac{9}{4\pi^2}T^2+1=r^2, \quad \text{and} \quad \frac{4}{4\pi^2}T^2+4=r^2 \qquad (1)$$

The simultaneous equations in (1) give

$$T=\pi\sqrt{\frac{12}{5}}=4\cdot9 \text{ secs.};$$

the length of the path $=2r=5\cdot06$ ft. (*Ans.*).

The time occupied in passing the specified points is

$$t=\frac{T}{2\pi}\text{ang}\left(\cos=\frac{1}{r}\right)-\frac{T}{2\pi}\text{ang}\left(\cos=\frac{2}{r}\right)$$
$$=\frac{T}{2\pi}\left[\text{ang}\left(\cos=\frac{1}{r}\right)-\text{ang}\left(\cos=\frac{2}{r}\right)\right].$$

The accurate value of r is $\sqrt{6\cdot4}$;

hence, $$\frac{1}{r}=\frac{\sqrt{10}}{8};\ \frac{2}{r}=\frac{\sqrt{10}}{4}.$$

Let $\cos\alpha=\frac{1}{r}$ and $\cos\beta=\frac{2}{r}$, then $\alpha-\beta=28°\ 57'\ 18''=0\cdot50536$ radians.

Hence, $$\frac{t}{T}=\frac{0\cdot50536}{2\pi}=0\cdot080 \text{ (*Ans.*)}.$$

(137) (Q. Oct. 1909).—**A weight of 2 lbs. at the end of an elastic string describes simple harmonic motion in a vertical line and passes from its highest position to its lowest, a distance of 6 inches, in $\frac{1}{4}$ second. Find the tension of the string when the particle is at its lowest point, and draw a curve to any suitable scale showing the velocity for any position of the weight.**

Solution.—$T = 2 \times 0{\cdot}25 = 0{\cdot}5$ secs. The acceleration is maximum in the lowest position and is

$$= \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 0{\cdot}25}{(0{\cdot}5)^2} = 39{\cdot}48 \text{ ft./secs.}^2;$$

the latter acceleration acts in the opposite sense of that of gravity. Hence, the tension in the string at the lowest point is

$$\frac{2}{g}\left(g + 39{\cdot}48\right) = \frac{2}{g} \times 71{\cdot}67 = 4{\cdot}45 \text{ lbws. } (\textit{Ans.}).$$

By article **227**

$$v = \frac{2\pi}{T}\sqrt{r^2 - x^2}, \quad \text{or} \quad v^2 = \frac{4\pi^2}{T^2}(r^2 - x^2); \quad \text{but} \quad \frac{4\pi^2}{T^2} = 160.$$

Hence, the relation between x and v is

$$\frac{x^2}{r^2} + \frac{v^2}{160r^2} = 1,$$

which is an ellipse with centre at the point of equilibrium and semi-axes $r = 0{\cdot}25$ and $4r\sqrt{10} = 3{\cdot}16$; the former is vertical and the latter is horizontal.

(138) (Q. Nov. 1907).—**A rigid body which can turn about a fixed axis receives a given impulse; explain how the angular velocity thus communicated to it may be calculated.**

A rifle is fixed to a heavy block which can swing about a fixed horizontal axis, the line of the barrel being at right angles to the axis. The discharge of the rifle produces such a recoil that the block swings through an angle θ from its equilibrium position. If in a series of experiments the same bullet is used but different charges of powder, prove that the muzzle velocity of the bullet is proportional to $\sin\frac{\theta}{2}$.

Solution.—W.r.t. the first part of the problem, see article **232**.
The block-and-rifle is a ballistic pendulum (see article **233**). The only force we have to deal with is the mutual action between the bullet and the block-and-rifle. Hence, the sum of the momenta remains unaltered. In this case the latter sum $= 0$, or the momentum of the bullet is equal to the impulse, p, received by the block-and-rifle. Hence, the velocity of the bullet, see [**202**], is $v \propto \sin\frac{\theta}{2}$.

(139) (Q. Nov. 1908). — **A water-jet issues from a nozzle 4 inches in diameter with a velocity of 50 ft.-secs. and strikes a plane surface perpendicularly. If a cubic foot of water weighs 62·5 lbs., find the pressure on the plane and find the horse-power of an engine which could deliver such a jet at a height of 40 feet above the tank from which the water is obtained; neglect friction in the pipe.**

Solution.—Let d be the internal diameter of the nozzle, v the velocity of the water, and h the height above the tank.

Quantity of water delivered per sec. $= \pi\frac{d^2}{4}v = \text{Q ft.}^3$

Weight " " " $= \text{Q}w = \text{W lbws.}$

Mass " " " $= \frac{\text{W}}{g} = \text{M}$ engineering units.

The power spent in delivering the jet at a height $h = \frac{\text{W}h}{550}$ H.P.

The total pressure on the plane $=$ P lbws. $=$ rate of change of momentum of the water $= \frac{\text{M}v - \text{M}v_1}{t}$.

If the plane surface remains stationary and the water does not rebound, then we may take $v_1 = 0$; as $t = 1$ sec. we have $\text{P} = \text{M}v$.

$\text{P} = 19{\cdot}8$ lbws.; power of engine $= 424$ H.P. (*Ans.*).

(140) (Q. Nov. 1908).—**Show how to find the maximum velocity in a simple harmonic motion when the periodic time and the amplitude are given.**

A mass of 5 lbs. is hung on to a light spring and is found to stretch it 4 inches; it is then pulled down a further 2 inches and released. Find the time of a complete oscillation and calculate the kinetic energy as the mass passes through the position of equilibrium.

Solution.—W.r.t. the first part of the problem, see article 227.

In fig. 151 points A and O are the positions of equilibrium of the lower end of the spring when respectively no weight and weight W is hung on to the end. The spring is further stretched to point B.

Let A O $= l_1$, O B $= l_2$, and let the horizontal ordinates to line A C represent the elastic force of the spring when the end of the latter is pulled down to the level of the ordinates. Hence, O C represents W.

W describes a simple harmonic motion, l_2 being the amplitude.

Let the force, which is required to stretch the spring from O to B, be P, then, if line O D be drawn parallel to A C, ordinate B D represents P. The work done while stretching the spring from O to B is represented by the area of triangle O D B O, and is

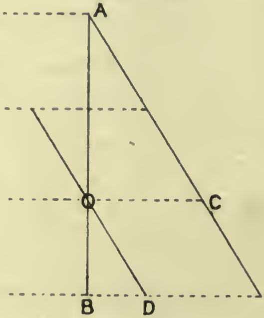


FIG. 151.

equal to $0{\cdot}5l_2\text{P}$ = kinetic energy accumulated in the mass, W/g, as it passes through point O. We have

$$\frac{\text{P}}{\text{W}}=\frac{l_2}{l_1}, \quad \text{or} \quad \text{P}=\frac{l_2}{l_1}\text{W}.$$

If v be the maximum velocity of the oscillating weight W, then

$$\frac{1}{2}v^2\frac{\text{W}}{g}=\frac{1}{2}l_2\text{P}, \quad \text{or} \quad v=l_2\sqrt{\frac{g}{l_1}}.$$

We have also $$v=\frac{2\pi l_2}{\text{T}}, \quad \text{or} \quad \text{T}=2\pi\sqrt{\frac{l_1}{g}},$$

or, *the periodic time of the oscillating weight is equal to that of a simple pendulum of length* l_1. Inserting the given numerical values in the expressions for T and the energy, we get

$\text{T}=0{\cdot}64$ secs.; kinetic energy $=2{\cdot}5$ ft.-lbws. (*Ans.*).

(141) (I.C.E., Oct. 1908).—What do you understand by the term moment of momentum of a body rotating about an axis?

A bullet of mass 1 oz. and velocity 1000 feet per second is fired into, and remains embedded in, a heavy door standing open, the line of fire being perpendicular to the door. The door is of oak, of uniform thickness throughout, 6 feet high by 3 feet

broad, and has a mass of 240 lbs. The bullet strikes the door 2 feet from the axis of the hinges. Find the angular velocity of the door if the hinges are frictionless. Determine the line of fire which would produce no reaction on the hinges.

Solution.—W.r.t. the first part of the question, see article 220.

Let W and v be respectively the weight and velocity of the bullet. We have $r\frac{W}{g}v = \omega \mathbf{I}$, where r is the distance of the point, at which the bullet strikes the door, from the axis of the hinges, and **I** is the moment of inertia of the door w.r.t. the latter axis. Hence, $\omega = \frac{rWv}{g\mathbf{I}}$.

To find **I**.—Let h, b, t, and M be respectively the height, width, thickness, and mass of the door. The horizontal section of the door is a rectangle whose sides are b and t. $(A.M.)'_p$ of the rectangle w.r.t. the centroid is

$$\frac{b^2 A}{12} + \frac{t^2 A}{12} = (b^2 + t^2)\frac{A}{12};$$

the $(A.M.)_p$ of the rectangle w.r.t. the axis of the hinges is therefore

$$(b^2 + t^2)\frac{A}{12} + \frac{b^2}{4}A = (4b^2 + t^2)\frac{A}{12}.$$

Volume of door $= hbt$, and mass of door $= mhbt = M$.

Hence, $$\mathbf{I} = (A.M.)_p hm = \frac{4b^2 + t^2}{12} M.$$

As t is not given, we may assume that t^2 is very small compared with $4b^2$; we may therefore take $\mathbf{I} = \frac{b^2}{3} M$.

Let l be the distance of the centre of percussion from the axis of the hinges, then $l = \frac{\mathbf{I}}{M\frac{b}{2}}$ (see article 233).

By inserting the given numerical values in the expressions for ω and l we get

$$\omega = \frac{2 \times \frac{1}{16} \times 1000 \times 3}{g \times 9 \times \frac{240}{g}} = 0{\cdot}174 \text{ radians/sec.}; \; l = \frac{2}{3} b = 2 \text{ ft. } (\textit{Ans.}).$$

(142) (Q. Nov. 1909).—**A block of wood, weighing 10 lbs., is supported by strings so that it is free to move as a simple pendulum of length 8 feet. A bullet, weighing 0·5 oz., is fired into the wood and becomes embedded in it, and the maximum horizontal deflection of the block is then observed to be 2·5 feet. What was the velocity of the bullet before hitting the wood?**

Solution.—Let l be the length of the pendulum, d its horizontal deflection, W_1 and W_2 the weights of the block and the bullet respectively.

Let us assume that the bullet strikes the block in a horizontal direction; the block and the bullet will then both be raised through a height h. The energy-equation is

$$\frac{W_2}{2g}v^2 = (W_1 + W_2)h,$$

or the velocity of the bullet just before striking the block was

$$v = \sqrt{2gh}\sqrt{\frac{W_1 + W_2}{W_2}} \quad . \quad . \quad . \quad (1)$$

where

$$h = l - \sqrt{l^2 - d^2} \quad . \quad . \quad . \quad . \quad (2)$$

Inserting the given values in (1) and (2) we shall find that

$$v = 91{\cdot}03 \text{ ft./sec. } (\textit{Ans.}).$$

(143) (Q. June 1910).—**State the second law of motion. Buckets are fixed to the circumference of a wheel of 16 feet diameter which is rotating at 100 revolutions per minute about a vertical axis. Water is caused to fall vertically into the buckets at a uniform rate so that, at the end of 2 minutes, 500 gallons are in the buckets. Find the extra couple necessary to be exerted upon the wheel to keep up its velocity, neglecting the increased friction at the bearings; determine also what extra horse-power is required.**

Solution.—Let r be the radius of the wheel, ω its angular velocity, and $F\rho$ the extra couple.

While the wheel turns through an angle $\delta\phi$ the mass of water poured into the buckets is δm. Hence, the energy-equation is

$$F\rho\,\delta\phi = \tfrac{1}{2}\omega^2 r^2\,\delta m,$$

or

$$F\rho = \frac{1}{2}\omega^2 r^2 \frac{\delta m}{\delta\phi} \quad . \quad . \quad . \quad . \quad (1)$$

Taking British engineering units, the extra horse-power equation is

$$\frac{1}{550}\,F\rho\frac{\delta\phi}{\delta t}=\frac{1}{2}\,\frac{1}{550}\,\omega^2r^2\,\frac{\delta m}{\delta t} \quad . \quad . \quad . \quad (2)$$

It is given that: $r=8$ ft.; $\omega=2\pi\frac{100}{60}=10{\cdot}472$ radians per sec.; total mass of water $=\frac{5000}{32{\cdot}19}$ which is poured into the buckets during 120 secs.

Hence, $\frac{\delta m}{\delta\phi}=\frac{5000}{32{\cdot}19\times 120\times 10{\cdot}472}$, and $\frac{\delta m}{\delta t}=\frac{5000}{32{\cdot}19\times 120}$,

$F\rho=432{\cdot}75$ lbws. and ft., and extra power $=8{\cdot}22$ H.P. (*Ans.*).

CHAPTER XLII.

HYDROMECHANICS.

234. Introduction.

In nature matter is found in three states of aggregation, thus: (i.) *solid*, (ii.) *liquid*, (iii.) *gaseous*. Matter in either of the latter two states is said to be *fluid*.

The different substances, in either of the three states in which they are found, differ so much in their physical properties and internal structure, that for our present purpose it is convenient to consider three imaginary classes of bodies; a body of either class is usually termed a perfect body. I think the latter expression is misleading, and prefer to use the term *imaginary body*. Hence, we will consider that:

(*A*) *An imaginary solid is a body whose shape and volume cannot be changed by the action of any force or forces.* It is absolutely *incompressible* and *rigid.*

(*B*) *An imaginary fluid is a body which makes no resistance whatever to change of shape.*

(*a*) *An imaginary liquid is an imaginary fluid whose volume cannot be changed by the action of any force or forces.* It is absolutely *incompressible.*

(*b*) *An imaginary gas is an imaginary fluid whose volume can be changed to any degree by the action of suitable forces.*

The mechanics for liquids is called *hydromechanics*, and its practical application is called *hydraulics.* The mechanics for gases is called *pneumatics.*

235. Transmission of pressure by fluid.

Let a vessel (fig. 152) be entirely filled with a fluid, whose particles are mutually at rest, and let there be two cylinders A and B let into the vessel at any convenient part of the vessel. Each cylinder is fitted with an airtight frictionless piston. Push the piston, A, in a small distance with a pressure whose intensity is p. If the fluid be a liquid the volume will be unaltered, but if it be a gas a compression of the gas will take place; but we will assume that the intensity of the pressure, q, on the piston, B, is not greater than it will allow the gas to expand to its original volume. In both cases, therefore, the volume remains unchanged; hence, the volume, V, swept by piston B, is equal to the volume swept by piston A; and as no energy is spent in changing the shape of the fluid we have, by article **170**, that

$$pV = qV, \quad \text{or} \quad p = q \quad . \quad . \quad . \quad (1)$$

Hence, *a fluid transmits pressure equally in all directions.* An imaginary solid can only transmit pressure in the direction of the

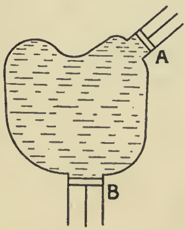


FIG. 152.

pressure which it receives, but not at all sideways, because the volume and the shape of the solid remains unaltered.

As a fluid cannot resist shearing it follows that the surface of a fluid must be at right angles to the direction of the resultant force acting on the fluid. Hence, the surface of a liquid contained in a vessel must be horizontal, *i.e.* at right angles to the direction of gravity.

236. Pressure of a liquid due to its weight.

The pressure of a fluid on an immersed plane surface is due to the weight of the fluid above the plane surface *plus* the pressure on the surface of the fluid.

We must, however, distinguish between a liquid and a gas. An imaginary liquid is incompressible; hence, the density of a liquid remains the same throughout the liquid. The density of a gas depends on the pressure to which it is subjected. As a lower layer of a gas carries the weight of the gas above it, it follows that the density of a gas increases from the surface towards the base of the vessel in which it is contained.

Let C (fig. 153) be an indefinitely small cylinder containing a piston and immersed in a liquid, say water. Let the piston sweep through a volume δV against the pressure of the liquid. The

exerted is $p\,\delta V$, where p is the intensity of the pressure of
iid on the piston. The piston displaces a weight of liquid
$w\,\delta V$, which must be raised to the surface. If the piston be at a depth, h, below the surface, we have that

$$p\,\delta V = w\,\delta V\,h, \quad \text{or} \quad p = wh. \qquad . \quad . \quad [206]$$

Hence, *the intensity of the pressure due to the weight of a liquid, at any point in the liquid, is equal to the weight of a column of the liquid whose base is unity and whose height is the depth of the liquid at the given point.*

The same law holds in the case of a gas, only w in [206] varies with the height of the gas above the given point, and the pressure cannot be readily calculated, but must be measured by certain instruments called manometers.

Let B D (fig. 154) be a plane surface immersed in a liquid, such

FIG. 153.

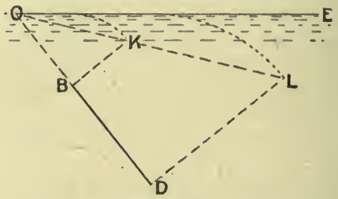


FIG. 154.

as water, and let the problem be to find the pressure of the water on either side of B D. Let the distance from the axis of reference, through O, to area-element δA be x, then the pressure on δA will be $wx \sin\phi\,\delta A$, ϕ being angle E O D. Hence, by example (130), Chap. XLI., *the total pressure on the plane surface is* P = the weight of the prism of water B K L D $= wAH$, *where A is the area of the immersed plane surface and H the depth of the liquid at the centroid of the area.*

The pressure, P, acts at a point, *the centre of pressure*, whose distance from O, see example (130), is

$$\xi = \frac{R_m^2}{a} + a \qquad . \quad . \quad . \quad . \quad (1)$$

The liquid-pressures will be in equilibrium if the liquid extends on both sides of the plane surface, as in fig. 154. Besides the pressure of the liquid there is the pressure of the atmosphere on the surface of the liquid; but this need not, as a rule, be taken into account, as it acts on both sides of the plane surface.

The horizontal and vertical components of P are respectively, $P_h = wAH\sin\phi = wHA_v$ and $P_v = wAH\cos\phi = wHA_h$, where A_v and A_h are the vertical and horizontal projections of A.

237. Weight of a solid immersed in a fluid.

Let a solid be immersed in a fluid (fig. 155) whose particles are mutually at rest; and let B D be a vertical cylindrical portion of the solid whose sectional area is δA. Let p be the intensity of the vertical pressure at B, and q that at D, then $(q-p)\delta A$ is obviously equal to the weight of the volume of fluid displaced by the portion B D of the solid. Hence, the difference between the vertical pressure, which tends to push the solid out of the fluid, and the vertical pressure, which tends to push the solid further down into the fluid, is equal to the weight of the volume of fluid displaced by the solid or, *a solid immersed in a fluid loses in weight an amount equal to the weight of the displaced fluid.*

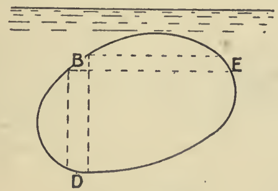


FIG. 155.

The horizontal pressures at B and E on the horizontal portion B E of the body are obviously equal, as their A_v are equal, but they act in opposite senses. Hence, the resultant horizontal pressure on the body is zero, and the body can therefore have no tendency to move in any horizontal direction.

238. Pressure on the walls of a vessel containing a liquid.

Let us next consider a vessel of any shape (fig. 156) containing a liquid. The horizontal pressure due to the weight of the liquid on a surface-element, N, of the vessel at depth x is $\delta P_h = wx\delta A_v$.

Whether the horizontal pressures at B and at E (fig. 155) are both external or both internal their resultant will be zero. Hence, the resultant horizontal pressure on the walls of the vessel is zero, and equilibrium will remain undisturbed until a hole is made in the side of the vessel, in which case the horizontal pressure opposite the hole will be unbalanced. The vessel will therefore be pushed by a force whose direction is horizontal, and which acts in the opposite sense in which the water flows out. This principle is made use of in the construction of reaction-turbines.

The vertical pressure on the surface-element, N, is $\delta P_v = wx\,\delta A_h$, and the vertical pressure on the surface-element, L, vertically below N is equal to the reaction of element N due to δP_v *plus* the weight of the column of liquid, whose base is δA_h and whose height is $y - x$. Hence, the total vertical pressure on surface-element L is wyA_h, or equal to the pressure due to the depth of liquid

FIG. 156.

at L. *The pressure on any part of the sides and the base of the vessel is* therefore *independent of the shape of the other part of the vessel.* Hence, *the total pressure on the base of a given vessel, containing a liquid, is the same as it would be if the vessel were a vertical cylinder on the given base, and filled with liquid to the given level.*

239. Water-pressure machine.

The machine shown diagrammatically in fig. 157 consists of a piston B, the *ram*, which can move watertight up and down in cylinder N. Water may be forced into the cylinder through the communicating pipe by a small plunger-pump worked by a vertical force F which acts at the end E of lever O E.

When the plunger-piston is raised the inlet-valve at the bottom of the pump-cylinder will open and the water from tank D will flow into the pump. When the plunger-piston is forced down, the inlet-valve shuts and the delivery-valve opens and water is forced into cylinder N.

Let A be the cross-sectional area of B, a that of the plunger, p the intensity of the water pressure in the machine, L and l respectively lengths O E and O C.

The upward pressure on B is $P = pA$, and that on the plunger is $= pa$; we have also $FL = pal$, or $F = pal/L$. Hence, the theoretical mechanical advantage of the machine is

$$\frac{P}{F} = \frac{AL}{al} \quad . \quad . \quad . \quad . \quad (1)$$

If piston B is to be raised through a height h, point E must be moved through a height d. The volume of water which must be forced into cylinder N to lift B is equal to Ah, and the plunger-piston must therefore move downwards an amount equal to Ah/a, but $d : L :: Ah/a : l$, or

$$\text{the velocity ratio} = \frac{d}{h} = \frac{AL}{al} \quad . \quad . \quad . \quad (2)$$

Hence, energy is transmitted through the machine according to the same law which we found for solid connections (see article 212).

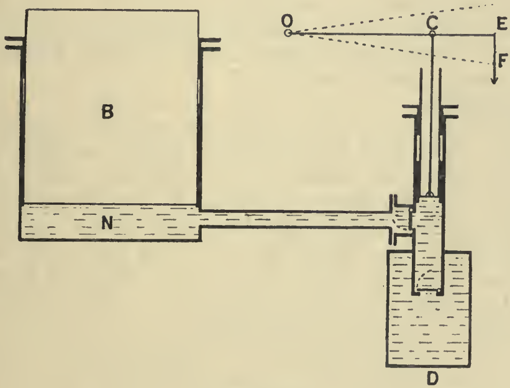


FIG. 157.

The machine (fig. 157) appears in three principal forms, thus :

(i.) The ram may be employed as the moving part of a press. The head of the ram is then a strong flat plate of larger diameter than the ram. The plate supports the substance to be pressed and moves up against a similar plate which is fixed by uprights to the frame of the machine. In this form the machine is called *hydraulic press* or *Bramah press.*

(ii.) The ram may be employed to raise a heavy load through a short height. The machine is then called *hydraulic jack.*

(iii.) The ram and cylinder N may be long and of comparatively small diameters; and B is heavily loaded with weights. Let W be the total weight of the loaded ram, then the intensity of the water pressure is $p = W/A$, which remains constant.

A portion of the high-pressure water in the machine is utilised in driving hydraulic machinery, such as hydraulic cranes, hydraulic riveters, etc. The pump, which is driven by an engine, must replenish the water used by the motors; but as the latter do not work continuously the pump may at times force more water into the machine than the motors require. The loaded ram will then be lifted and thus gain potential energy, which it gives off again by descending when the motors use water at a higher rate than the pump can supply. The loaded ram acts, therefore, in the manner of a flywheel on a rotating shaft by accumulating energy when the supply is greater than the demand, and giving energy off when the demand is greater than the supply. The loaded piston and its cyclinder in this form is called *hydraulic accumulator*.

The pump-engine is automatically stopped when the accumulator-ram is at its maximum height, and is started automatically when the ram begins to descend.

240. Buoyancy.

Let W_1 be the weight of a solid immersed in a fluid, and W_2 the weight of the fluid displaced by the solid. Both W_1 and W_2 are vertical forces, the former acting at the M.C. of the solid and in the same sense as gravity; whereas W_2 acts at the M.C.—the centre of buoyancy—of the fluid displaced, and in the opposite sense of gravity. The line of buoyancy—the line of action of W_2—must pass through the M.C. of the solid; otherwise W_1 and W_2 will form a couple, which will turn the solid in the fluid until the two M.C.s lie in the same vertical line.

The solid will thus be pushed by a vertical force, $W_1 - W_2$, and will sink in the fluid when $W_1 > W_2$; it will be at rest in any part of the fluid when $W_1 = W_2$, and will rise in the fluid when $W_1 < W_2$. The pushing force will remain constant as long as the solid is wholly immersed, but when $W_1 < W_2$ the force will diminish when the solid has reached the surface of the fluid, because then the volume of fluid displaced will decrease as long as the solid is rising out of the fluid. The solid will continue to rise until the weight of fluid displaced is equal to the weight of the solid.

When the solid is wholly immersed in the fluid the ratio W_1/W_2 is equal to the relative specific density of the solid substance, because W_1/W_2 is equal to the ratio of the masses of equal volumes of the solid substance and of the fluid.

241. Metacentre.

A homogeneous cylinder of revolution, a portion of which is cut off by a plane parallel to its axis, is floating on a liquid (figs. 158). It is evident that the line of buoyancy will pass through the axis as long as the solid does not lurch so much that a portion of the plane surface is immersed in the liquid.

The point at which two consecutive lines of buoyancy meet is called the *metacentre*. In the positions (figs. 158A and 158B) of the solid the metacentre is the middle point of the axis and the M.C., C, is below the axis. It is obvious that as long as the metacentre is vertically above the M.C. the floating body is in *stable equilibrium*; because the couple consisting of the weight of the body and the force of buoyancy (fig. 158B) will turn the body back to its position (fig. 158A). The same will take place if the body be standing on a horizontal plane surface. In the latter case, the line of buoyancy is replaced by the line of reaction of the plane surface; if the body be tilted to one side and let go, it will oscillate like a cradle about its position of stable equilibrium

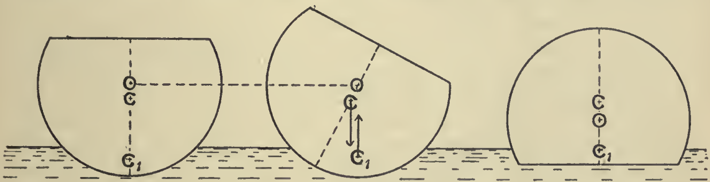


FIG. 158A. FIG. 158B. FIG. 158C.

corresponding to fig. 158A. An oscillating body has a metacentre; the metacentre of a pendulum lies in the axis of suspension.

The floating body will be in *neutral equilibrium* when the metacentre and the M.C. coincide; thus a long homogeneous cylinder of revolution and a homogeneous sphere will both float in neutral equilibrium.

In fig. 158C the floating body is in equilibrium, but it will be seen that the metacentre is below the M.C.; hence, when the body lurches in the least degree, it will heel over, and after a few oscillations about position fig. 158A, will remain in the latter position. The body in position fig. 150C is in *unstable equilibrium*.

The resistance of a ship against rolling is thus proportional to the distance between its metacentre and its M.C., the latter point always being lower than the former. It is generally necessary to ballast the ship to bring the M.C. sufficiently below the metacentre.

CHAPTER XLIII.

PNEUMATICS.

242. Properties of gases.

We have said that the volume of imaginary gases can be changed to any extent, *i.e.* the gas can be expanded or be compressed to any degree by the application of suitable forces.

Some gases in nature, such as atmospheric air, oxygen, hydrogen, nitrogen, etc., may be expanded or compressed to a great extent without changing their nature. The reason for this is that these gases are far from their fluid state of aggregation, and therefore remain gases for a long range of compression. Whereas carbonic acid gas can be condensed to a fluid by compressing it at a pressure of 36 atmospheres.

The former gases may therefore be said to follow, within certain limits, approximately our definition of imaginary gases, and we will therefore proceed to study some of their properties.

The volume occupied by a certain portion of a gas depends upon its temperature and the pressure to which it is subjected. If the pressure of the gas be diminished, the volume will increase, *i.e.* the gas will expand; the same effect is caused by raising the temperature of the gas.

243. Boyle's law.

Assume that we have a certain portion of a gas enclosed within a cylinder, with an airtight fitting piston. We may then compress the gas by placing weights on the top of the piston, in which case we shall find that the temperature of the gas will rise, but still by conveying a sufficient amount of heat from the gas we can keep the temperature constant. We could otherwise diminish the weights on the piston, and the gas would then expand again and at the same time the temperature would fall, but by adding the necessary quantity of heat to the gas, we can keep the temperature constant in this case also.

Let now $V_0, V_1, V_2, V_3 \dots$ denote the volumes occupied by the gas, corresponding to the intensities of pressure $P_0, P_1, P_2, P_3 \dots$

then we shall find by keeping the temperature of the gas constant that

$$\frac{V_0}{V_1}=\frac{P_1}{P_0};\ \frac{V_1}{V_2}=\frac{P_2}{P_1};\ \frac{V_2}{V_3}=\frac{P_3}{P_2} \quad . \quad . \quad . \quad (1)$$

or

$$V_0P_0=V_1P_1=V_2P_2=V_3P_3=C \quad . \quad . \quad [207]$$

This is the *first law of expansion* of gases, and may be stated as follows: *The product of the volume occupied by a portion of a gas at constant temperature into its intensity of pressure is constant.*

This law was first discovered by Robert Boyle in 1662, and verified by Mariotte in 1715.

The constant C in [207] varies with the temperature, everything else remaining the same.

244. The law of Charles.

The *second law of expansion* of gases states that, *the increase of volume of a portion of a gas at constant intensity of pressure is proportional to the temperature.*

It has been found that the increase of volume of a portion of a gas by heating it from the ice-point to the boiling-point is 0·3665 of its volume at the ice-point. The expansion of a gas by increase of temperature is therefore for one degree C. equal to 0·003665; this number is called the *coefficient of expansion* of gases.

Let V_0, V_1, and V_2 be the volumes of the same portion of a gas at 0 degrees, t_1 degrees, and t_2 degrees C. respectively, then

$$V_1=[1+0{\cdot}003665t_1]V_0 \quad . \quad . \quad . \quad (1)$$

$$V_2=[1+0{\cdot}003665t_2]V_0 \quad . \quad . \quad . \quad (2)$$

or

$$\frac{V_1}{V_2}=\frac{1+0{\cdot}003665t_1}{1+0{\cdot}003665t_2} \quad . \quad . \quad . \quad [208]$$

The second law of expansion of gases is also called the law of Charles, or the law of Gay-Lussac.

The two laws of expansion can be joined together in one formula, stating the relation between volume, temperature, and pressure of a gas.

Let V_0 and V' be volumes of the same portion of a gas at 0 degrees C. and under the intensities of pressures P_0 and P_1, then by [207]

$$\frac{V_0}{V'}=\frac{P_1}{P_0}. \quad . \quad . \quad . \quad . \quad (3)$$

and let V_1 be the volume of the gas at t_1 degrees C. and at pressure P_1, then

$$\frac{V'}{V_1}=\frac{1}{1+0\cdot003665t_1} \quad . \quad . \quad . \quad (4)$$

Now multiply (3) by (4), then

$$V_1=V_0(1+0\cdot003665t_1)\frac{P_0}{P_1} \quad . \quad . \quad . \quad (5)$$

If the temperature and the intensity of pressure be changed to t_2 and P_2 respectively, then the volume of the gas will be

$$V_2=V_0(1+0\cdot003665t_2)\frac{P_0}{P_2} \quad . \quad . \quad . \quad (6)$$

Now divide (5) by (6), then

$$\frac{V_1P_1}{1+0\cdot003665t_1}=\frac{V_2P_2}{1+0\cdot003665t_2}=R \quad . \quad . \quad (7)$$

where R is a constant, depending upon the nature and portion of the gas. Formula (7) expresses in symbols the two laws of expansion of gases.

245. Absolute temperature of gases.

The absolute temperature of a gas is a theoretical consequence of the law of Charles, by assuming that it is possible to continue the cooling of a gas until its volume is diminished to naught. This agrees with our definition of an imaginary gas. The temperature T_0 at which the volume of the gas should be naught must satisfy the following equation:

$$0=1+0\cdot003665T_0 \quad . \quad . \quad . \quad (1)$$

and we find

$$T_0=-272\cdot9 \text{ say } \textit{minus } 273^\circ \text{ C.} \quad . \quad . \quad [209]$$

The temperature T_0 is the absolute zero for imaginary gases, and the temperature of the gas reckoned from this zero is the absolute temperature of the gas. Thus, if the temperature of an imaginary gas is t degrees C., its absolute temperature is $T=273+t$. A degree on the absolute temperature scale is, of course, equal to a degree C.; we have only lowered zero 273° C.

Applying [209] we may write (7) in the preceding article thus:

$$\frac{V_1P_1}{T_1}=\frac{V_2P_2}{T_2}=R \quad . \quad . \quad . \quad . \quad [210]$$

where R is constant, but not equal to that in (7) in the preceding article.

[210] expresses the relation which exists between volume,

intensity of pressure, and absolute temperature of an imaginary gas.

By [**210**] we have $V P = R T$, but by article **170**, $V P$ is the work done on the gas by expanding it from volume $= 0$ to volume $= V$ under a constant intensity of pressure P, or we may say that it is the energy exerted while heating the gas from absolute zero to the absolute temperature, T, against a constant intensity of pressure, P. R is therefore the work done per degree of temperature.

246. The atmosphere.

Our earth is completely covered by an ocean of atmosphere consisting of such gases as are necessary for upholding life. The height of the atmosphere varies according to a number of circumstances, and at no time can the height be measured. We can, however, measure the intensity of its pressure on a surface by balancing it with a column of liquid on the top of which the air has been removed. Take, for instance, a glass tube closed at one end and open at the other, fill it with mercury, and boil the latter while in the tube, the open end being uppermost. All the air which might be in the mercury will thus escape during boiling; put the finger against the open end and let it down into a basin containing mercury. It will be seen that the mercury will not run out of the tube altogether, but will leave a column whose height depends on the condition of the atmosphere; on an average, it is 760 mm. $= 29{\cdot}94$ ins. near the sea-level. Above the column of mercury is left a space which is very nearly empty, *i.e.* it is as near as possible a perfect vacuum.

The vertical column of air above the surface of the mercury in the basin and the column of mercury in the tube form, as it were, two communicating tubes. Hence, the intensity of the pressure due to the air column is equal to the intensity of the pressure due to the mercury column on the surface of the mercury in the basin. Now a cubic cm. of water weighs 1 grw., and the specific density of mercury is 13·6; hence, a column of mercury whose height is 76 cm. and whose base is 1 cm.^{2} weighs: $76 \times 13{\cdot}6 \times 0{\cdot}001 = 1{\cdot}0336$ kgw., or 14·7 lbw. per inch^2. This is the *pressure of one atmosphere*. In the metric system the pressure of one atmosphere is often taken as 1 kgw. per mm.^{2}

The apparatus, just described, consisting of the basin and tube containing mercury, is called a *barometer*. The space above the mercury in the tube is not a perfect vacuum; it will always contain some mercury vapour, and it is impossible to remove every trace of air from the mercury. The pressure due to the remaining air and the mercury vapours, however, is very small in a well-made barometer, and is, according to some authorities, only equivalent to a column of mercury of 0·0013 mm. height, and is thus inappreciable.

The barometer measures the atmospheric pressure in columns of mercury, but the actual pressure, in kgw. per $mm.^2$ or in lbw. per $inch^2$, depends on the weight of unit volume of mercury. It is only at the sea-level and latitude of Paris that a cubic cm. of mercury weighs 13·6 gmw. The weight of unit volume of mercury depends on gravity and temperature.

A barometer may also be constructed on the principle of the spring-balance; it consists then of a cylindrical metal box closed at top and bottom by two corrugated diaphragms, which act as springs. The box is exhausted and the atmospheric pressure on the diaphragms is balanced by the elastic force of the diaphragms, whose deflection is magnified by levers and is indicated on a scale. A barometer of this kind is portable and is called an *aneroid barometer*.

The two kinds of barometer measure two different things. The mercury barometer measures the mass of the atmosphere by balancing it against the mass of the column of mercury, like a pair of scales. Whereas the aneroid barometer measures the weight of the atmosphere by balancing it against a force. Although the latter barometer is calibrated to indicate the atmospheric pressure in columns of mercury, it ought really to indicate the pressure in kgw. per $mm.^2$ or in lbw. per $inch^2$. Hence, the indications of the two kinds of barometer will not always agree accurately.

247. The siphon.

When a liquid, or a portion of it, is to be transferred from a

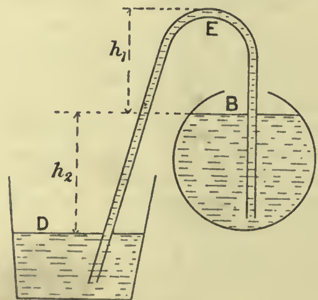


FIG. 159.

vessel B to another vessel D, the arrangement shown in fig. 159 may be employed.

E is a bent tube which must be completely filled with liquid; this may be done by sucking liquid through the tube from the free end. Let b be the height of the column of liquid corresponding to the atmospheric pressure, then the lifting pressures on the liquid in the tube at the surface-levels in B and D are respectively due to heights $(b-h_1)$ and $[b-(h_1+h_2)]$. Hence, the liquid runs out at the free end in D with a velocity due to height h_2. The tube E is called a *siphon*. It is obvious that h_1 must be less than b; hence, a siphon cannot work in a vacuum. The siphon will continue to work until B is empty or until the level of the liquid in B has fallen to that in D.

248. Raising water.

The mean height of the column of water which the atmosphere can support is equal to $29{\cdot}94 \times 13{\cdot}6$ ins. $= 33{\cdot}9$ ft. The atmospheric pressure may thus be utilised in raising water from one level to a higher one. For this purpose a vertical pipe is let into the water below, while a vacuum is being produced at the other end of the pipe.

Steam may be used to expel the air from the pipe, and then produce a vacuum by condensing the steam. The contrivance generally employed to exhaust the pipe is a pump, which consists of a cylinder with a reciprocating piston. The pipe, the suction-pipe, which is to be exhausted dips into the well below, and the top end of the pipe is connected to the pump-cylinder (fig. 160A).

In starting the pump there is no water above the piston, and the pump acts as an air-pump while exhausting the suction-pipe. When the pump-piston is at the end of its down-stroke there is still a small space, the clearance, left between the piston and the lower end of the pump cylinder, which is filled with air at the atmospheric pressure. Let V_1 be the clearance volume and V_2 the volume swept by the piston in one stroke then, provided there be no leakage and the temperature remains constant, by [207], $P_1V_1 = P_2(V_2+V_1)$,

or
$$P_2 = \frac{V_1}{V_1+V_2}P_1 \quad . \quad . \quad . \quad . \quad (1)$$

P_1 is the intensity of the atmospheric pressure, which also acts on the surface of the water in the well. Hence, the intensity of the effective pressure by which the water is lifted through the suction-pipe is

$$P_e = P_1 - P_2 = \frac{V_2}{V_1+V_2}P_1 \quad . \quad . \quad . \quad (2)$$

Let b and h be the heights of water which can be supported by P_1 and P_e respectively, then

$$h : b :: P_e : P_1, \quad \text{or} \quad h = \frac{V_2}{V_1 + V_2} b \quad . \quad . \quad (3)$$

V_1 is rarely smaller than $0{\cdot}05\ V_2$, and as the average value of b is about 34 ft., we get $h = 32{\cdot}4$ ft.

The pump is not working properly unless the water from the suction-pipe follows the piston during its up-stroke. Let l be the

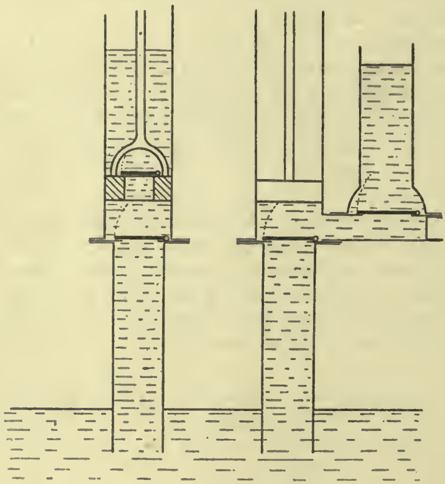


FIG. 160A. FIG. 160B.

length of the stroke and h_1 that of the suction-pipe, then the suction height is

$$h = h_1 + l = \frac{V_2}{V_1 + V_2} b . \quad . \quad . \quad . \ [211]$$

The atmospheric pressure may be much less than 30 ins. of mercury, the pump may leak, and the valves do not close instantaneously; for these reasons we cannot rely on a suction-height of more than 20 ft. to 25 ft.

The second function of the pump is to raise the water through a height h_2 from the suction-valve to the level of the spout through which the water is discharged.

In fig. 160A the water passes through the piston-valve during the down-stroke into the delivery-pipe, which is a continuation of the pump-cylinder. The water level is thus lifted through a height l in each up-stroke until it reaches the spout.

The piston in fig. 160B forces the water through the delivery-valve and raises the level of the water through a height l in each down-stroke. If A be the area of the piston, the volume of water discharged by either pump in a double-stroke of the piston is Al. The pump (fig. 160A) is called a *lift-pump*, and that in fig. 160B is called a *force-pump*.

To find the work done in raising the water from the level of the well to the level of the discharge spout.

When the piston (fig. 160A) is at a distance x from the bottom end of the cylinder in the up-stroke, the downward pressure on the piston is $= w\mathrm{A}(b + h_2 - x)$, and the upward pressure is $= w\mathrm{A}[b - (h_1 + x)]$. Hence, the piston is to be lifted against a pressure $= \mathrm{A}w(h_1 + h_2)$, and the work done during the up-stroke $= w\mathrm{A}l(h_1 + h_2)$ energy-units. The piston does not overcome any pressure during the down-stroke.

The pressure on the piston (fig. 160B) against the motion, in the down-stroke, is $w\mathrm{A}(b + h_2 - x - b) = w\mathrm{A}(h_2 - x)$, and the work done during the down-stroke is

$$= w\mathrm{A}h_2\int_0^l dx - w\mathrm{A}\int_0^l x\,dx = w\mathrm{A}lh_2 - w\mathrm{A}\frac{l^2}{2}.$$

In the up-stroke the pressure on the piston to be overcome is

$$= w\mathrm{A}[b - (b - h_1 - x)] = \mathrm{A}w(h_1 + x),$$

and the work done during the up-stroke is

$$= w\mathrm{A}h_1\int_0^l dx + w\mathrm{A}\int_0^l x\,dx = w\mathrm{A}lh_1 + w\mathrm{A}\frac{l^2}{2}.$$

Hence, the total work done during a double-stroke

$$= w\mathrm{A}l(h_1 + h_2).$$

The work done by either of the two pumps during a double-stroke of the piston is

$$w\mathrm{A}l(h_1 + h_2) = w\mathrm{A}l\mathrm{H} \quad . \quad . \quad . \quad (4)$$

H is the vertical height between the two water-levels.

The actual volume of water delivered by a pump is somewhat less than that we have just found. The valves do not shut at once at the moment the motion of the piston is reversed. Hence, some water will be forced out of the cylinder through the valves.

Other losses are due to leakage, air in the water, etc. Let η be the efficiency of the pump, then the volume of water delivered by the pump in a double-stroke is

$$\eta A l \text{ volume-units} \quad . \quad . \quad . \quad . \quad (5)$$

η may be taken between 0·9 and 0·98.

249. Air-vessel.

While the piston (fig. 160B) is in its up-stroke the column of water remains at rest in the delivery-pipe and must be started again at the moment the piston begins to move downwards. At that moment the pressure on the piston is equal to the weight of the column of water above the piston *plus* the inertia-pressure of the water. To ease the piston and piston-rod of the extra pressure, an air-vessel (fig. 161) may be placed between the vertical

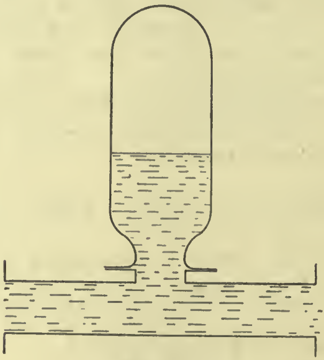


FIG. 161.

delivery-pipe and the pump. At the beginning of the stroke the piston forces water into the air-vessel, the air will be compressed, and the compressed air will in due time expand and assist the piston in starting the water on the other side of the vessel. The air in the vessel acts like a buffer of a railway carriage, it assists in the start and prevents any shock due to sudden changes of velocity. When the air-vessel is of sufficient size the flow of water in the delivery-pipe will be continuous.

250. Air-pump.

A pump which is used to extract the air from a vessel is called an *air-pump*.

A pump like the one shown in fig. 160A may be used for the purpose by connecting the suction-pipe with the vessel, the receiver, which is to be exhausted.

By (1) in article **248** the limiting intensity of pressure in the receiver would be $\frac{V_1}{V_1+V_2}P_1$. Hence, if we imagine a perfect air-pump, *i.e.* a pump with no clearance, the limiting pressure would apparently be zero. Let us, however, consider this case more fully.

Let P_0 be the initial intensity of pressure in the receiver, V its volume, and $P_1, P_2 \ldots P_n$ the intensities of pressure in the receiver at the end of the first, second . . . . n^{th} out-stroke of the pump, then

$$P_0V = P_1(V+V_2), \quad \text{or} \quad P_1 = \frac{V}{V+V_2}P_0;$$

and
$$P_1V = P_2(V+V_2), \quad \text{or} \quad P_2 = \left(\frac{V}{V+V_2}\right)^2 P_0.$$

Hence,
$$P_n = \left(\frac{V}{V+V_2}\right)^n P_0 \quad . \quad . \quad . \quad (1)$$

P_n can only be zero when $n=\infty$. Hence, it is impossible to obtain a perfect vacuum even with a perfect pump.

In order to hasten the exhaustion we may use a double-pump, *i.e.* two pumps working together, the two pistons moving always in opposite senses.

251. Air-compressor.

A pump which is used to increase the pressure of the air in a vessel, the receiver, is called an *air compressor.*

When a gas is being compressed its temperature rises, unless the heat which is generated during the process of compression is dissipated by cooling the gas. In the following we will assume that the temperature of the gas remains unaltered.

In fig. 162 D is the delivery-pipe which connects the barrel of the compressor with the receiver; the delivery-valve opens into the pipe. In the out-stroke of the piston the delivery-valve is shut and the piston-valve is open, and the pump takes air in at an intensity of pressure P. During the in-stroke the piston-valve is shut and the air is being compressed until the pressure has reached that of the air in the receiver; the delivery-valve opens and the air in the cylinder is forced into the receiver. Let P′ be the limiting intensity of pressure of the air in the receiver which can be obtained by the compressor, then

$$P'V_1 = P(V_1+V_2), \quad \text{or} \quad P' = \frac{V_1+V_2}{V_1}P \quad . \quad . \quad (1)$$

Hence, when $V_1 = O$, *i.e.* a perfect compressor, P′ will apparently be equal to ∞. Let us, however, also consider this case more fully.

During each in-stroke a volume V_2 of air at atmospheric pressure is forced into the receiver, whose volume is V. Hence, after n in-strokes the volume of air, at atmospheric pressure, contained in the receiver will be $nV_2 + VP_0$, where P_0 is the initial intensity of pressure of the air, in atmospheres, in the

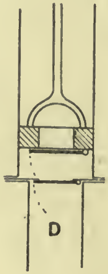


FIG. 162.

receiver. Hence, the intensity of pressure P_n, in atmospheres, of the air in the receiver after n in-strokes is

$$P_n = \frac{nV_2 + VP_0}{V} \quad . \quad . \quad . \quad . \quad (2)$$

P_n cannot be ∞ as it requires $n = \infty$.

252. Fluid friction.

If it were possible to produce an imaginary fluid and we poured the fluid into a vessel whose inside surface is a surface of revolution and then caused the vessel to be rotated about its axis, we should find that the fluid would not be affected at all by the rotation, but would remain at rest in the vessel.

If, however, the vessel contains a natural fluid, then every particle of the fluid will gradually take part in the rotary motion of the vessel. This shows us two things:—

(i.) That there is a force which resists the sliding of the fluid on the surface of the vessel. This force is very great indeed when the fluid adheres to the vessel.

(ii.) That natural fluids possess something—called *viscosity*—which is analogous to friction.

The force of friction between two neighbouring surfaces of a fluid is very different from the force of friction between the surfaces of two solid bodies. The latter force is proportional to the pressure between the bearing surfaces, and is independent of their areas and their relative velocity; and the friction of rest is greater than the sliding friction.

The force of friction between two fluid surfaces is—(1) independent of the pressure; (2) proportional to the area of the surfaces; (3) proportional to the relative velocity of the surfaces; and hence, (4) the friction at rest is equal to or approaches zero.

The viscosity of a liquid diminishes rapidly with rise of temperature, whereas the viscosity of gases has been found to increase with the temperature.

When a liquid flows through a straight capillary tube whose length is many times greater than its bore, the paths of the particles will be straight lines parallel to the axis of the tube. The particles of the liquid next to the wall of the tube will be at rest, hence, the liquid will flow as if the tube itself were made of the liquid. B F in fig. 163 is a horizontal capillary glass-

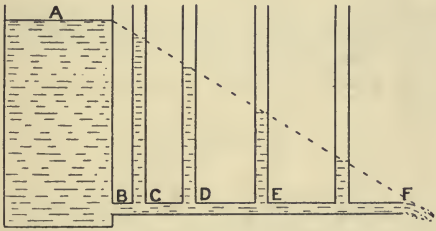


Fig. 163.

tube communicating at one end with a vessel A filled with a liquid. When the free end of B F is closed the liquid in the vertical tubes, which are open at the tops, will be found to stand at the same level as in A. This shows that, *when the liquid is at rest the internal friction is zero.*

When the free end of B F is open the liquid will flow with a velocity whose magnitude will depend on the level in A and on the viscosity of the liquid. If we keep the level in A constant, the velocity of the liquid will be uniform and the levels in the vertical tubes will arrange themselves in the dotted straight line, the steepness of which will increase in the same proportion as the velocity of the liquid in B F.

As the heights of the liquid in the vertical tubes measure the pressure of the flowing liquid in B F, we may deduce the following from the experiment:—

(i.) As the difference of pressure between any two points in

B F is proportional to the distance beween them, *the force of friction is proportional to the area of the sliding surfaces.*

(ii.) As the steepness of the dotted line increases in the same proportion as the velocity of the liquid, it follows that *the friction is proportional to the relative velocity of the particles.*

(iii.) As the pressure on the liquid in B F varies along the tube, *the force of friction is independent of the pressure.*

Let v be the velocity of the liquid, r the internal radius of the tube, p the difference of intensity of pressure between the ends of a length l of the tube and η a coefficient, then, according to Poiseuille,

$$v=\frac{r^2}{8\eta}\frac{p}{l} \quad . \quad . \quad . \quad . \quad (1)$$

As v and η are inversely proportional, the latter may be regarded as something analogous to a coefficient of friction. η is called the *coefficient of viscosity*, and its dimensions are evidently $[M]\ [L]^{-1}\ [T]^{-1}$, whereas the coefficient of friction between two solid bodies is a pure number.

If V be the volume of liquid passing through the tube per second, then

$$V=\pi r^2 v=\frac{\pi r^4}{8\eta}\frac{p}{l} \quad . \quad . \quad . \quad . \quad (2)$$

(2) has been found to hold also for gases when the volume V is measured at an intensity of pressure $(p_1+p_2)/2$, where p_1 and p_2 are the intensities of pressure at the ends of the tube. Hence, for gases

$$V=\frac{\pi r^4}{8\eta}\frac{p_1-p_2}{l} \quad . \quad . \quad . \quad . \quad (3)$$

l in (3) is the length of the tube. η is also called the *coefficient of transpiration.*

CHAPTER XLIV.

IMPACT.

253. Volume-elasticity.

In fig. 164 is shown a cylinder with an airtight fitting piston, which can move in the cylinder without friction. Both cylinder and piston are supposed to be made of a material which is a perfect insulator of heat, so that they can neither absorb nor conduct heat.

The cylinder contains a portion of an imaginary gas whose intensity of pressure, volume, and temperature are P_1, V_1, and T_1

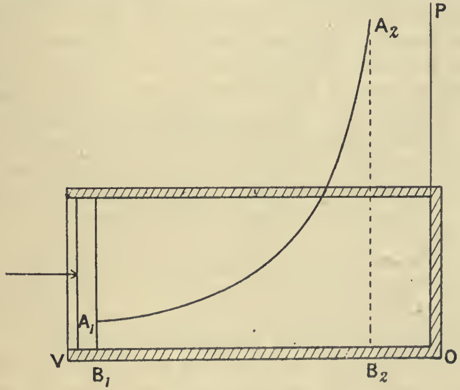


Fig. 164.

when the piston is in the position shown in the figure. The piston is now pushed gradually towards the bottom of the cylinder—compressing the gas—until it reaches position B_2A_2. Let the values of P, V, and T at the latter position of the piston be P_2, V_2, and T_2, then we have by [210]

$$P_1V_1/T_1 = P_2V_2/T_2 = R, \quad \text{or} \quad P_1V_1 = R\,T_1, \quad \text{and} \quad P_2V_2 = R\,T_2.$$

Hence,
$$P_2V_2 - P_1V_1 = R(T_2 - T_1) \quad . \quad . \quad . \quad (1)$$

By article **245** the left-hand side of (1) is the work done on the gas while increasing the intensity of pressure from P_1 to P_2; and the right-hand side of (1) is the work done in raising the temperature of the gas from T_1 to T_2. Hence, the work done on the gas during compression is turned into heat, and is represented by area $B_1A_1A_2B_2B_1$. The relation between P and V is represented by the curve A_1A_2, which is called an *adiabatic curve*, and the compression is called *adiabatic compression.* As no heat can escape from or be added to the gas, no loss of energy can take place during the process of adiabatic compression.

We may now gradually diminish the external pressure on the piston, and the gas will expand adiabatically driving the piston forward. When the piston reaches its initial position, B_1A_1, the total energy represented by area $B_2A_2A_1B_1B_2$ will have been spent, and the values of P, V, and T will again be P_1, V_1, and T_1.

As PV/T is always equal to R, the different stages of the gas during compression will thus be repeated during expansion in the reversed order. The gas is said to possess *perfect volume-elasticity.*

The forces which are to be overcome, in the case of compression, are due to mutual actions between the molecules of the gas, and may be termed *elastic forces.* If we consider a substance which possesses internal resistances, which are not elastic, then part of the energy spent in compressing the substance will be wasted, and the substance will not resume its original shape by a subsequent expansion. We may even go so far as to consider a substance—such as an imaginary liquid—which is void of any elastic forces. Such a substance would have no tendency to recover its initial shape; it would be *perfectly inelastic.*

In nature we find neither perfectly elastic nor perfectly inelastic materials, but some approach the first limit, as ivory, others approach the second limit, as butter, hard clay, etc.

254. Impact of two smooth homogeneous spheres.

When two bodies strike against one another they are said to impinge on one another, and the collision is termed an *impact.* A sphere is said to impinge *directly* on a fixed plane, if the direction of the velocity of its centre, just before impact, is normal. The impact of two spheres is said to be *direct* if the centres of the spheres, just before impact, are moving along the same straight line.

When two bodies collide their velocities will be altered, and their mutual action will cause their shape to be changed; according to the preceding article the latter deformation requires energy, hence the bodies will lose some of their kinetic energy.

255. Direct impact.

The only forces acting on the two spheres during direct impact are those due to the mutual action between the bodies. Hence, by Chap. XXXVIII., the motion of the common M.C. of the spheres is unaffected by the impact and it will continue to move with uniform velocity; also, the sum of the momenta of the two bodies remains constant.

There are three distinct stages during impact which must be considered, viz.:

(i.) The impact begins at the moment the two spheres touch. At this moment their velocities are the same as during approach. Let M and V_i be the mass and the velocity of sphere I., and m and v_i be those of sphere II. The velocity of approach will thus be $V_i - v_i$; the velocities being reckoned positive in the direction of V_i.

(ii.) The second stage is when the deformation of the bodies has reached its maximum. At this moment the two spheres will move instantaneously together with velocity V, and as the sum of the momenta remains unchanged, we have $MV_i + mv_i = (M+m)V$.

$$V = \frac{MV_i + mv_i}{M+m} \quad . \quad . \quad . \quad [212]$$

(iii.) The third stage is reached when the bodies have recovered as much as is possible of their original shape. At this moment they will separate with velocities V_f and v_f respectively. The velocity of separation is thus $V_f - v_f$.

When the bodies are perfectly elastic sphere I. will lose velocity $V_i - V$ between stages (i.) and (ii.); and as the bodies recover their shapes perfectly the action between stages (ii.) and (iii.) will be exactly equal, but opposite, to the action between stages (i.) and (ii.); sphere I. will therefore again lose velocity $V_i - V$.

Hence, $$V_f = V_i - 2(V_i - V) \quad . \quad . \quad . \quad (1)$$

Similarly, sphere II. will gain velocity $2(V - v_i)$.

Hence, $$v_f = v_i + 2(V - v_i) \quad . \quad . \quad . \quad . \quad (2)$$

As bodies in nature only recover partially we must take

$$V_f = V_i - K(V_i - V), \quad \text{and} \quad v_f = v_i + K(V - v_i) \quad . \quad (3)$$

Where $K < 2$ is a constant which depends on the material of which the spheres are made. The two equations in (3) give $V_f - v_f = -(K-1)(V_i - v_i)$, or

$$\frac{V_f - v_f}{V_i - v_i} = -(K-1) = -e \quad . \quad . \quad . \quad [213]$$

e is a constant and is known as the *coefficient of restitution* or sometimes the *coefficient of elasticity* Hence, *the ratio of the velocity of separation to the velocity of approach is constant, and depends on the material only, and not on the masses of the bodies.*

For perfectly inelastic bodies, $K = 1$, $e = 0$.

For perfectly elastic bodies, $K = 2$, $e = 1$.

For bodies in nature, $1 < K < 2$, or $0 < e < 1$.

By eliminating V between [212] and the two expressions in (3) we obtain

$$V_f = \frac{M V_i + mv_i - em(V_i - v_i)}{M + m},$$

and

$$v_f = \frac{M V_i + mv_i + eM(V_i - v_i)}{M + m} \quad . \quad . \quad [214]$$

When the spheres are perfectly inelastic bodies we have $e = 0$. Hence, $V_f = v_f = V$, or the bodies do not separate, but move together with velocity V.

When the spheres are perfectly elastic, or $e = 1$, we have

$$V_f = \frac{(M - m)V_i + 2mv_i}{M + m}, \quad \text{and} \quad v_f = \frac{2MV_i - (M - m)v_i}{M + m} \quad [215]$$

When $M = m$ in [215], we get $V_f = v_i$ and $v_f = V_i$, or the two bodies interchange velocities.

When M is very large compared with m and $V_i = 0$, we get $V_f = 0$, and $v_f = -v_i$, or the small body rebounds with a velocity equal to its velocity of approach, but in the opposite sense.

256. Kinetic energy lost during impact.

Let the loss of energy be L, then

$$L = \tfrac{1}{2}(MV_i^2 + mv_i^2) - \tfrac{1}{2}(MV_f^2 + mv_f^2),$$

or

$$2L = M(V_i^2 - V_f^2) + m(v_i^2 - v_f^2).$$

By (3) in the preceding article, we get

$$V_i^2 - V_f^2 = (V_i - V_f)(V_i + V_f) = (1 - e^2)(V_i - V)^2 \quad . \quad (1)$$

$$v_i^2 - v_f^2 = (v_i - v_f)(v_i + v_f) = (1 - e^2)(V - v_i)^2 \quad . \quad (2)$$

Hence, $$2L = (1 - e^2)[M(V_i - V)^2 + m(V - v_i)^2]$$

or $$L = \frac{1 - e^2}{2} \frac{Mm}{M + m}(V_i - v_i)^2 \quad . \quad . \quad . [216]$$

(i.) *Perfectly elastic bodies.* $e = 1$, hence $L = 0$, which was also shown in article 253.

(ii.) *Perfectly inelastic bodies.* $e = 0$.

$$V_f = v_f = V = \frac{M V_i + mv_i}{M + m};\ L = \frac{1}{2} \frac{Mm}{M + m}(V_i - v_i)^2 \quad . \quad (3)$$

(a) If M is very large compared with m and $V_i = 0$, *i.e.* M is at rest, then $L = 0{\cdot}5\ mv_i^2$, or the whole of the kinetic energy of the impinging body is lost. This case may be examplified by a lead bullet striking a target or a wall. The whole energy of the bullet will be converted into heat, which may cause the lead to melt.

(b) Let M be the mass of a water-wheel, and m the mass of a water-particle which impinges directly on the bucket of the wheel. The bucket moves with a constant speed V, *i.e.* the impinging of m on the bucket does not alter the speed of the wheel, but will cause the wheel to keep up its speed. We may now find the energy transmitted to the wheel by the impulse of each particle, assuming that both the water and the bucket are perfectly inelastic.

As M is very large compared with m and $v_i > V$, the energy lost by m is $0{\cdot}5\ m(v_i^2 - V^2)$. By (3) the total energy lost during impact is $0{\cdot}5\ m(v_i - V)^2$. Hence, the energy transmitted to the wheel is

$$0{\cdot}5\ m(v_i^2 - V^2) - 0{\cdot}5\ m(v_i - V)^2 = mV(v_i - V) \quad . \quad (4)$$

Let W be the weight of the water impinging on the wheel per second, then the power transmitted to the wheel per second is

$$\frac{W}{g}V(v_i - V), \quad \text{or} \quad \frac{W}{550g}V(v_i - V)\text{H.P.} \quad . \quad . \quad (5)$$

when the quantities are given in British engineering units. (5) is maximum when $v_i = 2V$. Hence, the maximum theoretical power transmitted to the wheel by the impulse of the water is

$$\frac{Wv_i^2}{2200g}\text{H.P.} \quad . \quad . \quad . \quad . \quad (6)$$

257. Oblique impact of two smooth homogeneous spheres.

Let V_i, v_i, V_f, and v_f (fig. 165) make angles α, β, ϕ, and θ respectively with the centre-line of the spheres when they touch.

Hence,
$$V_i \sin\alpha = V_f \sin\phi \qquad (1)$$
$$v_i \sin\beta = v_f \sin\theta \qquad (2)$$

The velocities along the centre-line are,

$$\text{before impact } V_i\cos\alpha \text{ and } v_i\cos\beta \qquad (3)$$
and
$$\text{after impact } V_f\cos\phi \text{ and } V_f\cos\theta \qquad (4)$$

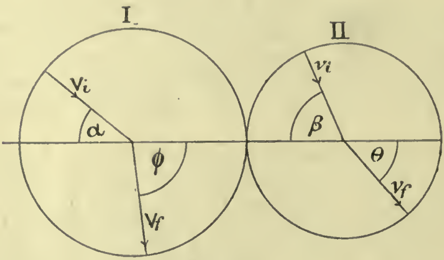


Fig. 165.

As the only forces which act on the spheres during impact are those due to the mutual action of the bodies, we may use [**214**] to determine (4), by substituting the values in (3) for V_i and v_i in [**214**].

Hence,
$$V_f\cos\phi = \frac{(M - em)V_i\cos\alpha + m(1+e)v_i\cos\beta}{M+m} \qquad (5)$$
$$v_f\cos\theta = \frac{(1+e)M\,V_i\cos\alpha - (eM - m)v_i\cos\beta}{M+m} \qquad (6)$$

Hence,
$$\tan\phi = \frac{(1)}{(5)},\ \tan\theta = \frac{(2)}{(6)} \qquad (7)$$

and
$$V_f = \frac{\sin\alpha}{\sin\phi}V_i,\ v_f = \frac{\sin\beta}{\sin\theta}v_i \qquad (8)$$

CHAPTER XLV.

EXAMPLES.

(144) (I.C.E., Oct. 1901).—**A vessel of water is weighed on a parcel spring-balance, the reading of which shows that the vessel and water weigh 11 lbs. A 7-lb. iron-weight is suspended by a fine wire from the hook of an ordinary spring-balance, and is lowered into the water until it is completely immersed. Under these conditions find (i.) the reading of the spring-balance from which the weight is suspended; (ii.) the reading of the parcel spring-balance on which the vessel stands. Give the reason for any change in the readings of the balances. Specific gravity of iron = 7·5; 1 cubic foot of water weighs 62·5 lbs.**

Solution.—Suppose that the spring-balance and iron-weight, together with the vessel and water, be all mounted on the parcel spring-balance. It is obvious that the reading of the latter balance would remain the same whether the iron-weight is out of the water or in the water. But the weight of the iron-weight, when immersed in the water, is diminished by the weight of the water displaced.

Hence, when the iron-weight is immersed in the water, the weight of the vessel and water will be increased by the weight of the water displaced, and the reading on the spring-balance will be diminished by an amount equal to the latter weight.

The weight of water displaced when the iron-weight is completely immersed is $W = 7/7\cdot5$ lbws. $= 14/15$ lbws. Hence, the reading of the parcel spring-balance $= 11 + 14/15 = 12$ lbws. (nearly); and the reading of the spring-balance $= 7 - 14/15 =$ just over 6 lbws.

(145) (I.C.E., Oct. 1897, second part).—**A bulb weighing 12 oz. is found to weigh 8 oz. when immersed in water, and 7 oz. when immersed in another liquid. What is the specific gravity of the liquid?**

Solution.—Let V and W_1 be the volume and weight of the bulb, W_2 its weight in water, and W_3 its weight in the liquid, then

The weight of volume V of water $= W_1 - W_2$.
The weight of volume V of liquid $= W_1 - W_3$.

Hence,

the specific density of the liquid $= \dfrac{W_1 - W_3}{W_1 - W_2} = \dfrac{12-7}{12-8} = 1 \cdot 25$ (*Ans.*).

(146) (Q. June 1909).—**A raft is formed by placing planks across two equal hollow circular cylinders each 20 feet long and 3 feet in diameter. The raft floats with one quarter of the vertical diameters of the cylinders immersed, and the water in which it floats weighs 1 ton to 35 cubic feet. Find what additional load will just completely immerse the cylinders.**

Solution.—Let d and l be respectively the diameter and length in feet of the cylinders, and A the immersed area of the cylinder-ends when 0·25 of the diameter is immersed.

The weight of the water displaced when the cylinders are completely immersed is

$$W_1 = \frac{2\pi d^2/4 \times l}{35} \text{tnws.} \quad . \quad . \quad . \quad (1)$$

and the weight of water displaced when 0·25 of the vertical diameters are immersed is

$$W_2 = \frac{2Al}{35} \text{tnws.} \quad . \quad . \quad . \quad . \quad (2)$$

Hence, the additional load which will just completely immerse the cylinders is

$$W = W_1 - W_2 = \frac{\pi d^2/4 - A}{35} 2l \text{ tnws.} \quad . \quad . \quad (3)$$

The angle at the centre subtending the immersed arc, when 0·25 of the diameter is immersed, is 120°. Hence, $A = \pi d^2/12 - dc/8$, c being the length of the chord. But $c = d \sin 60°$.

Hence, $A = 1 \cdot 38$ ft.^{2} and $W = 6 \cdot 5$ tnws. (*Ans.*).

(147) (Q. May 1907).—**A cylinder 6 inches long, having a cross-section of 1 square inch, floats totally submerged with its axis vertical in a vessel containing water and oil on the top of the water. If the cylinder weighs 0·2 lb., find how much of its length is below the surface of separation of the oil and the water. The specific gravity of the oil is 0·7, and the weight of 1 cubic inch of water is 0·036 lb.**

Solution.—Let a, h, and w_1 be respectively the cross-sectional area, the height, and the weight per unit volume of the cylinder, w the weight of unit volume of water, and h_1 the portion of the length of the cylinder below the surface of separation.

Weight of cylinder, $W_1 = ahw_1$.
Weight of water displaced, $W = ah_1w$.
Weight of oil displaced, $W_2 = a(h - h_1) \times 0{\cdot}7w$.

Hence, $\qquad W_1 = W + W_2, \quad \text{and} \quad \dfrac{h_1}{h} = \dfrac{w_1 - 0{\cdot}7w}{0{\cdot}3w},$

or, $\qquad h_1 = 4{\cdot}52$ ins. (*Ans.*).

(148) (I.C.E., Feb. 1898, second part).—**Prove that the oscillation under gravity of a liquid in a U-tube follows the simple harmonic law.**

Solution.—In fig. 166 O–O is the level of the liquid when it is at rest. The bore of the tube is assumed to be uniform, the area

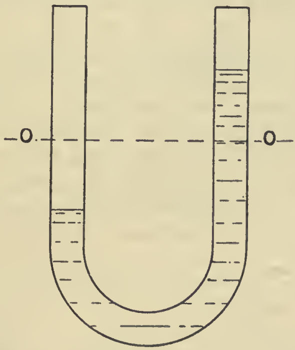


FIG. 166.

being A. The two levels of the liquid are always equidistant from O–O. Hence, at the moment either level is at a vertical distance x from O–O the force, which produces the motion of the liquid, is equal to $2xAw$, and if M be the total mass of liquid in the tube, the acceleration of each particle of liquid is $a = \dfrac{2wA}{M}x$, *i.e.* the acceleration is proportional to the distance from the position of equilibrium. Hence, the oscillation of the liquid follows the simple harmonic law.

Let maximum $x = h$, then, by article 227, maximum acceleration

$= \omega^2 h = \frac{2wA}{M} h$, or ω is independent of h. On account of friction between the liquid and the walls of the tube, and other resistances, the amplitude h will gradually diminish to zero. By article **227** the periodic time is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{2wA}},$$

which is independent of the amplitude. Hence, the oscillations of the liquid are *isochronous.*

Let l be the length of the liquid, *i.e.* the distance between the two surfaces of the liquid measured along the tube, then lA = volume of liquid, wlA = the weight of the liquid $= Mg$.

Hence,
$$T = 2\pi\sqrt{\frac{l}{2g}};$$

or, *the periodic time of the oscillations is equal to that of a simple pendulum whose length = half the length of the liquid.*

(149) (I.C.E., Feb. 1908).—**A U-tube of uniform section contains mercury. In one branch is also some water which fills 3 inches of the tube above the mercury in that branch. Find the difference in level of the surfaces of mercury in the two branches, (*a*) when both ends of the tube are exposed to the atmosphere; (*b*) when the end containing mercury only is exposed to a pressure of 1 lb. per square inch, the other end being exposed to the atmosphere. The specific gravity of mercury is 13·6.**

Solution.—Let h_2 be the difference in level of the surfaces of mercury in the two branches, h_1 the height of water above the mercury in the one branch, and n the specific density of mercury.

(*a*) There will be equilibrium when a column of water, whose height is h_1 and whose base is unity, is equal to a column of mercury whose height is h_2 and whose base is also unity, or when $h_1 = nh_2$.

Hence,
$$h_2 = \frac{h_1}{n} = \frac{3}{13 \cdot 6} = 0 \cdot 22 \text{ in. } (\textit{Ans.}).$$

(*b*) Take the intensity of the atmospheric pressure as 15 lbws. per inch^2; the difference of pressure on the surfaces in the two branches will therefore be 14 lbws. per inch.^{2} Hence, the weight of the column of mercury, whose height is h_2 and whose base is unity, is equal to the weight of a column of mercury, whose base is also unity and whose height is 0·22 in. *plus* the height of mercury corresponding to a weight of 14 lbws. per inch.^{2}

The weight of a column of water whose base is 1 ft.^{2} and whose height is H ft. is $= 62{\cdot}4$H lbws. If this column is to balance a pressure of 14 lbws. per inch^2, then

$$62{\cdot}4\text{H} = 14 \times 144, \quad \text{or} \quad \text{H} = \frac{14 \times 144}{62{\cdot}4} \text{ ft.}$$

A column of mercury which can balance the latter column of water must be

$$= \frac{14 \times 1728}{62{\cdot}4 \times 13{\cdot}6} = 28{\cdot}5 \text{ ins.}$$

Hence, $h_2 = 0{\cdot}22 + 28{\cdot}5 = 28{\cdot}72$ ins. (*Ans.*).

(150) (I.C.E., Feb. 1906).—**A ship sinks 2 inches on entering a river and then rises 1·5 inches on discharging 40 tons of cargo. Find its original displacement. Specific gravity of sea-water = 1·025.**

Solution.—40 tnws. of river-water corresponds to 1·5 ins., and x tnws. of river-water corresponds to 2 ins. Hence, $x : 40 :: 2 : 1{\cdot}5$, or $x = 160/3$ tnws.

Let the volume of the original displacement be V ft.^{3}, then V ft.^{3} of sea-water weigh $\text{V} \times 62{\cdot}4 \times 1{\cdot}025$ lbws. $= \text{W}_1$; and V ft.^{3} of fresh water weigh $\text{V} \times 62{\cdot}4$ lbws. $= \text{W}_2$.

Hence, $$\text{W}_1 - \text{W}_2 = 2240\, x.$$

Hence, $$\text{V} = \frac{160 \times 2240}{3 \times 62{\cdot}4 \times 0{\cdot}025} \text{ ft.}^3$$

and the

$$\text{original displacement} = \frac{62{\cdot}4 \times 1{\cdot}025\text{V}}{2240} \text{ tnws.} = 2187 \text{ tnws. } (\textit{Ans.}).$$

(151) (I.C.E., Oct. 1907).—**A diving bell is in the form of a hollow circular cylinder 8 feet high inside, open at the bottom and closed at the top. When the axis is vertical and the bottom just touches the water surface, it is filled with air at atmospheric pressure, 15 lbs. per square inch, and at a temperature of 60° F. It is lowered gradually into the water, the axis remaining vertical, until its bottom end is 20 feet below the surface. Find the height to which the water rises inside the bell, and the pressure of air inside, the temperature remaining the same. Find also the air pressure in lbs. per square inch which must be supplied to the bell to keep the water out.**

Solution.—Let h_i be the height inside the bell when the bottom just touches the water, p the intensity of the pressure of the

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atmosphere, D the depth to which the bottom end is lowered, P the intensity of the pressure, and h_f the height of the air inside the bell at depth D; finally, let w be the weight of unit volume of water.

$$P = p + Dw - (h_i - h_f)w \quad . \quad . \quad . \quad (1)$$

also

$$P : p :: h_i : h_f \quad . \quad . \quad . \quad . \quad (2)$$

P and h_f can be determined by the two simultaneous equations (1) and (2).

Inserting the given numerical values in (1) and (2), we get

$$P = 15 + \frac{20 \times 62{\cdot}4}{144} - \frac{8 \times 62{\cdot}4}{144} + \frac{62{\cdot}4}{144} h_f \quad . \quad . \quad (3)$$

$$Ph_f = 15 \times 8 = 120 \quad . \quad . \quad . \quad . \quad (4)$$

Hence, $h_f = 5{\cdot}33$ ft.; the water will rise inside the bell to a height $h_i - h_f = 2{\cdot}67$ ft. = 2 ft. 8 ins.; P = 22·5 lbws. per inch² (*Ans.*).

(152) (Q. Nov. 1906).—**A sphere, 1 foot in diameter, floats in a cylindrical vessel of water having a cross-section of 2 square feet. The sphere is of such weight that it is half immersed when floating freely. If a vertical force of 10 lbs. is applied to the top of the sphere so as to press it down, find how much the water-level rises. A cubic foot of water weighs 62·5 lbs.**

Solution.—In fig. 167 1–1, 2–2, and 3–3 are respectively

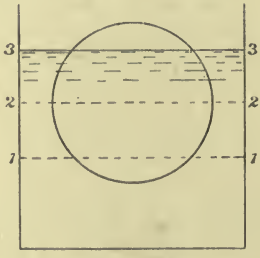

FIG. 167.

original water-level, the level when the sphere floats on the water, and the level when the vertical force is applied. The corresponding depths of the water are h_1, h_2, and h_3 respectively

Let A be the cross-sectional area of the vessel, d the diameter of the sphere, and P the vertical force.

When the sphere floats freely we have

$$h_2 A = h_1 A + \frac{2}{3}\pi\frac{d^3}{8}, \quad \text{or} \quad h_2 - h_1 = \frac{\pi d^3}{12A}.$$

When force P is applied we have

$$h_3A = h_2A + V, \quad \text{where} \quad P = wV.$$

Hence, $$h_3 - h_2 = \frac{P}{wA}.$$

We have given that $d = 1$ ft., $A = 2$ ft.², $P = 10$ lbws., and $w = 62{\cdot}5$ lbws.

Hence, $h_2 - h_1 = 0{\cdot}131$ ft.; $h_3 - h_2 = 0{\cdot}08$ ft. (*Ans.*).

(153) (Q. June 1909).—**The envelope, car, passengers, etc., of a balloon weigh 600 lbs.; its capacity is 20,000 cubic feet of coal-gas, the density of which is 0·45 that of air. Assuming that, under the atmospheric conditions prevailing at the time of the ascent, 1 cubic foot of air at the earth's surface would weigh 0·08 lb., find the acceleration with which the balloon will begin to rise.**

Solution.—Weight of air displaced by balloon is

$$W_1 = 20{,}000 \times 0{\cdot}08 \text{ lbws.}$$

Weight of gas contained in balloon, $W_2 = 20{,}000 \times 0{\cdot}45 \times 0{\cdot}08$ lbws.

Ascending effort, $F_e = W_1 - W_2$.

Let W be the weight of envelope, car, passengers, etc., then ascending effective force $F = F_e - W = (W_1 - W_2 - W)$ lbws. Mass to be moved by F is $M = (W + W_2)/g$. Hence, the balloon will begin to rise with an acceleration

$$a = F/M = \frac{W_1 - W_2 - W}{W + W_2} g = 6{\cdot}83 \text{ ft./sec.}^2 \text{ (}Ans.\text{)}.$$

(154) (Q. Nov. 1908).—**Two vertical cylinders, whose cross-sectional areas are A and a respectively, stand side by side and communicate by a pipe near the bottom. Water stands at the same level in the two. The water is now pushed from cylinder A into the other by a piston. Find the thrust on the piston when it has lowered the level in A by the distance x, and from the value of the thrust deduce an expression for the work done up to the point. Verify your result by finding the work done against gravity in lifting the water.**

Solution.—When the piston has lowered the level in A by a distance x, the water-level in the other cylinder has risen to a height y above the original level, thus $ya = xA$.

The thrust on the piston is

$$P = A(y+x)w = \frac{A(A+a)}{2a} wx \ (Ans.).$$

The work done while moving the piston through distance x is

$$\frac{A(A+a)}{a} w \int_0^x x\,dx = \frac{A(A+a)}{2a} wx^2 \ (Ans.).$$

The latter expression may also be determined by considering the work which must be done against gravity in lifting the water in the other cylinder.

While pushing the piston a distance δx downwards you must lift a column of water $= (y+x)a$ through a height δy. Hence, work done against gravity $= (y+x)aw\,\delta y$; but $y = \frac{A}{a}x$ and therefore $\delta y = \frac{A}{a}\delta x$. Hence,

$$\text{the work done against gravity} = \frac{A(A+a)}{a} w \int_0^x x\,dx = \frac{A(A+a)}{2a} wx^2,$$

which is the same expression as we found above.

(155) (I.C.E., Oct. 1909).—**A cubical box 2 feet by 2 feet by 2 feet, partly filled with water, is made to rotate about a vertical axis, passing through the centre of its base, at a uniform speed of 60 revolutions per minute. At the centre the water stands 3 inches above the floor of the box. To what height does it rise at each of the four corners? Trace the curve of the water-surface upon either of the four vertical walls, writing its equation.**

Solution.—Let D be the depth of the water at the centre of the box and take as plane of reference (X – Y plane) the horizontal plane at distance D above the floor of the box. The z-axis is the axis of rotation (fig. 168).

Let Q be a water-particle in the X – Z plane. There will be equilibrium if the direction of the resultant of the forces, acting on Q, is along the normal, Q N, to the surface of the water. The weight, mg, of the particle must therefore supply the deviating force, $mx\omega^2$; the other component of mg being normal to the surface of the water. If ϕ be the angle which the tangent T Q makes with the z-axis, then

$$\tan\phi = \frac{dx}{dz} = \frac{mg}{mx\omega^2} = \frac{g}{x\omega^2}.$$

By integration we get

$$\int_0^x x\,dx = \frac{g}{\omega^2}\int_0^z dz, \quad \text{or} \quad x^2 = \frac{2g}{\omega^2}z \qquad . \quad . \quad (1)$$

The equation (1) is a parabola with vertex at 0 and axis along the z-axis. Hence, the surface of the water, when the box is rotated about the z-axis, is a paraboloid, which can be produced

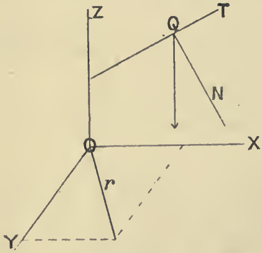

FIG. 168.

by the revolution of parabola (1) about its axis. At distance r from the z-axis we have

$$r^2 = \frac{2g}{\omega^2}z, \quad \text{but} \quad r^2 = x^2 + y^2.$$

Hence, the equation of the paraboloid w.r.t the given rectangular space-co-ordinates is

$$x^2 + y^2 = \frac{2g}{\omega^2}z \qquad . \quad . \quad . \quad . \quad (2)$$

Let a be the length of the edge of the box, and let the axes of x and of y each be perpendicular to one pair of vertical walls. The x-co-ordinate to any point on either of the two walls, parallel to the y-axis, is equal to $a/2$. Hence, the equation of the wall is $x = a/2$. By eliminating x between the latter equation and that of the paraboloid, we get

$$y^2 = \frac{2g}{\omega^2}\left(z - \frac{a^2\omega^2}{8g}\right) . \quad . \quad . \quad . \quad (3)$$

which is the equation of the curve of intersection between the wall and the surface of the water. The curve is a parabola whose axis is vertical, and whose vertex is at a height $\frac{a^2\omega^2}{2g}$ above the plane of reference.

Let H be the height above the floor of the box to which the water rises at each of the four corners, then $z = \text{H} - \text{D}$ and $y = a/2$ are the co-ordinates of the highest water-particles.

Hence, $$\frac{a^2}{4} = \frac{2g}{\omega^2}\left(\text{H} - \text{D} - \frac{a^2\omega^2}{8g}\right), \quad \text{or} \quad \text{H} = \text{D} + \frac{a^2\omega^2}{4g} \quad . \quad (4)$$

To prevent any overflowing of water we must have

$$\text{H} \not> a, \quad \text{or} \quad a \not< \text{D} + \frac{a^2\omega^2}{4g}.$$

Hence, $$\text{maximum } \omega = \frac{2}{a}\sqrt{g(a - \text{D})}.$$

We have given $a = 2$ ft., $\text{D} = 0{\cdot}25$ ft., $\omega = 2\pi$ radians/sec. Hence, $\text{H} = 1$ ft. 5·7 ins. ; equation of paraboloid, $x^2 + y^2 = 1{\cdot}63\,z$; equation of the parabola upon either of the four walls, $y^2 = 1{\cdot}63\,z - 1$; maximum $\omega = 7{\cdot}5$ radians/sec. $= 71{\cdot}67$ revs. per min. (*Ans.*).

(156) (Q. Nov. 1906). — **A prism whose section is a right-angled triangle A B C (B being the right angle) forms one side of a trough of water. The base B C of the prism rests on the bottom of the trough which is horizontal and the prism is hinged at the edge C so that it can turn about the edge, the vertical face A B being presented to the water. A B is three times B C, and the specific gravity of the material of which the prism is made is 2. Find (in terms of the height A B) the greatest depth of water which can be poured into the trough without the prism overturning. Assume that no water leaks under the prism.**

Solution.—In fig. 169 $\text{O B} = \text{B E} = h$ is the depth of water, W = the weight of the prism, and P = the total pressure of the water on the face A B. As the surface on which P acts is a rectangle

$$\text{P} = w0{\cdot}5hhl = 0{\cdot}5\,wlh^2 \quad . \quad . \quad . \quad (1)$$

l is the length of the trough.

$$\text{W} = 0{\cdot}5nw\overline{\text{A B}}\;\overline{\text{B C}}l = 0{\cdot}5\,nwq\overline{\text{B C}}^2 l \quad . \quad . \quad (2)$$

where n is the specific density of the prism, and $\text{A B} = q\overline{\text{B C}}$.

P acts at a point on A B which lies at a distance $\xi = \frac{\text{R}_m^2}{a} + a$ from O. The second area-moment about an axis parallel to l and through the centroid of the rectangle is $(\text{A.M.}) = R_m^2\,A = h^3l/12$. Hence, $R_m^2 = h^2/12$, and as $a = \frac{h}{2}$, we have $\xi = \frac{2}{3}h$ which is also the distance of the centroid of the triangle O E B from O.

W acts at the M.C. of prism A B C which is at a distance $2\overline{CB}/3$ from C. W and P meet at point L, and their resultant must act along L C when the maximum depth of water is poured into the

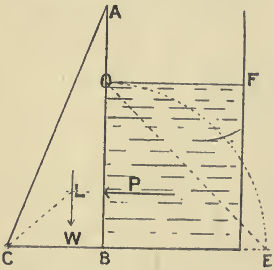

FIG. 169.

trough. Hence, the moment of P about C must be equal to the moment of W about C, or

$$\frac{1}{3}Ph = \frac{2}{3}W\,\overline{CB}, \quad \text{or} \quad h = 2\frac{W}{P}\overline{CB} = AB\sqrt[3]{\frac{2n}{q^2}} \quad . \quad (3)$$

$$h = 0{\cdot}753\,\overline{AB}\ (\textit{Ans.}).$$

(157) (Q. June 1908).—**A river embankment has its face vertical for a depth of 5 feet below the surface of the water, and inclined at 45° to the horizontal for a further depth of 10 feet, measured vertically. Find the magnitude and line of action of the resultant thrust on the inclined portion of the face. Take one foot-length of the embankment. A cubic foot of water weighs 62·4 lbs.**

Solution.—In fig. 170 F A B is the face of the embankment, O M the water-line, A E $= h =$ the depth at A, B D $=$ H $=$ the depth at B. A B inclines at an angle ϕ to the horizontal, and l is the length of the embankment.

Hence, O A $= h \operatorname{cosec}\phi$, A B $= (H - h)\operatorname{cosec}\phi$, $a =$ O A $+ 0{\cdot}5$ A B $= 0{\cdot}5\,(H + h)\operatorname{cosec}\phi$. (A.M.) of a rectangle whose sides are a and b w.r.t. an axis parallel to b and through the centroid is $= a^3b/12$, and

$R_m^2 = a^3b/12ab = a^2/12$. In the present case $a = AB$. Hence, the thrust, P, on the inclined portion of the embankment is

$$P = a \sin\phi \, \overline{AB} lw = 0{\cdot}5\, wl(H^2 - h^2) \operatorname{cosec}\phi \quad . \quad (1)$$

and $$\xi = \frac{(H-h)^2}{12 \sin^2\phi} \frac{\sin\phi}{0{\cdot}5(H+h)} + 0{\cdot}5(H+h) \operatorname{cosec}\phi$$

$$= \frac{2(H^2 + Hh + h^2)}{3(H+h)} \operatorname{cosec}\phi \quad . \quad . \quad . \quad (2)$$

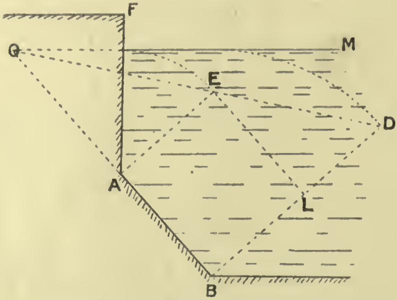

Fig. 170.

Inserting the given numerical values we get

$$P = 0{\cdot}5 \times 62{\cdot}4 \times 1 \times (100 - 25) \operatorname{cosec} 45^\circ = 3310 \text{ lbws. } (Ans.).$$

$$\xi = 11 \text{ feet } (Ans.).$$

(158) (Q. Nov. 1907).—The figure (fig. 171) shows a close-fitting paddle A B by which water is being lifted out of a rectangular channel and discharged over a cill at C. The channel is 30 feet wide, the length A B is 5 feet, and C is 2·5 feet below A. Considering only the water in front of the paddle, find the resultant thrust on the paddle, its line of action and its moment about A (i.) when A B is vertical; (ii.) when A B makes 30° with the vertical, as at A B_1.

Solution.—Let $AB = l$, $AD = l_1$, width of paddle $= a$, and angle $BAB_1 = \phi$.

(i.) The submersed portion of the paddle is a rectangle whose sides are $b = l - l_1$ and a. As the water-prism, whose weight is equal to the resultant thrust P, is a right triangular prism, whose height is a and whose base is an isosceles right-angled triangle of equal sides b, we have

$$P = wa\frac{b^2}{2} = wa\frac{(l-l_1)^2}{2}; \quad \xi = \frac{2}{3}b = \frac{2}{3}(l-l_1);$$

the moment of P about point A $= P(\xi + l_1)$.

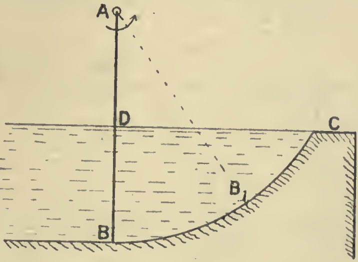

FIG. 171.

Taking $w = 62{\cdot}4$, and inserting the given values, we get

P = 1950 lbws.; ξ = 1 ft. 8 ins.;

the moment of P = 8125 lbws.-ft.

(ii.) The sides of the submersed rectangle are a and $(l - l_1/\cos\phi)$. The height of water above the centroid of the rectangle is $0{\cdot}5(l - l_1/\cos\phi)\cos\phi$.

Hence, $$P = wa\left(l - \frac{l_1}{\cos\phi}\right)\left(l - \frac{l_1}{\cos\phi}\right)\frac{\cos\phi}{2}; \quad \xi = \frac{2}{3}\left(l - \frac{l_1}{\cos\phi}\right);$$

$$\text{the moment of } P = P\left(\xi + \frac{l_1}{\cos\phi}\right).$$

Inserting the given values we get

P = 1206 lbws.; ξ = 1 ft. 5 ins.;

the moment of P = 5182 lbws.-ft. (*Ans.*).

(159) (I.C.E., Feb. 1904).—**A uniform raft of rectangular section which, when floating freely, is immersed to two-thirds its depth, has one end stranded so that the lower edge is in the plane of flotation, as in fig. 172. If the ends be assumed vertical in the position of equilibrium, show that the upper edge of the sea end is also in the plane of flotation, and that the pressure on the wall end is W/4, where W is the weight of the raft.**

Solution.—Let l be the distance between the ends of the raft, W the weight of the raft, W_2 the weight of water displaced, and R the reaction on the wall end.

W_2 acts at the mass-centre of the submersed portion of the raft. Taking moments about the lower edge at the wall we have

$$W\frac{l}{2}=\frac{2}{3}lW_2, \quad \text{and also} \quad R+W-W_2=0.$$

Hence, $\quad W_2=0{\cdot}75W, \quad$ and $\quad R=-0{\cdot}25\,W$ (*Ans.*).

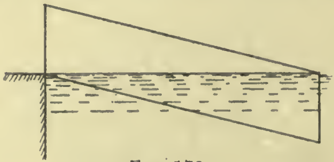

FIG. 172.

Let V be the volume of the submersed portion of the raft, and V_1 the volume of the raft, then

$$V:\tfrac{2}{3}V_1::W_2:W. \quad \text{Hence, } V=0{\cdot}5\,V_1 \; (\textit{Ans.}).$$

(160) (I.C.E., Oct. 1908).—**A vertical lock-gate has an aperture in it in the form of an equilateral triangle of 6 feet side, the lowest side being horizontal. The vertex of the triangle is 6 feet below the surface of the water. Find the total pressure and also the centre of pressure on a door closing the aperture.**

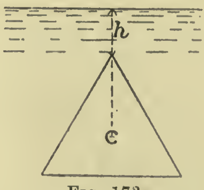

FIG. 173.

Solution.—Let l be the length of the side of the triangle (fig. 173). In this case b is the height of the triangle. We have

$$a=h+\tfrac{2}{3}b=h+\tfrac{2}{3}l\sin 60^\circ;\; A=\frac{l^2\sin 60^\circ}{2};$$

Let $(A.M.)_1$ be the second area-moment of the triangle w.r.t. the horizontal side, and (A.M.) be the second area-moment of the triangle w.r.t. the horizontal axis through C, then

$$(A.M.)=(A.M.)_1-A\frac{b^2}{9};\; (A.M.)_1=\frac{l}{b}\int_0^b(b-x)x^2dx=\frac{l^4\sin^3 60^\circ}{12}.$$

Hence,

$$R_m^2 = \frac{(\text{A.M.})}{\text{A}} = \frac{l^2 \sin^2 60^\circ}{18}, \text{ and } \xi = \frac{l^2 \sin^2 60^\circ}{18(h + \frac{2}{3}l \sin 60^\circ)} + h + \tfrac{2}{3}l \sin 60^\circ.$$

The total pressure on the door is

$$P = w\alpha A = w(h + \tfrac{2}{3}l \sin 60^\circ)\frac{l^2 \sin 60^\circ}{2}.$$

Inserting the given numerical values we get

$$P = 9206 \text{ lbws.} = 4{\cdot}1 \text{ tnws. } (\textit{Ans.}).$$

$$\xi = 9 \text{ ft. } 7{\cdot}5 \text{ ins. } (\textit{Ans.}).$$

(161) (I.C.E., Feb. 1902).—**The ram of a hydraulic accumulator is 10 inches diameter, and has a stroke of 11 feet. When fully loaded, the water-pressure is 800 lbs. per square inch. If the whole energy of the accumulator water could be used in 3 minutes, what horse-power would it exert?**

Solution.—See article 239.

$A = 78{\cdot}54$ ins.^{2} Take the pressure of the atmosphere as 15 lbws. per inch^2, then the weight of the fully loaded ram is $W = 78{\cdot}54(800 - 15) = 61{,}654$ lbws.

W descends through a height of 11 ft. Hence, the energy exerted during the descent $= 61{,}654 \times 11$ foot-lbws. As this energy is exerted during three minutes, the power exerted is

$$= \frac{61{,}654 \times 11}{3 \times 33{,}000} = 6{\cdot}85 \text{ H.P. } (\textit{Ans.}).$$

(162) (I.C.E., Feb. 1907).—**A tidal flap, uniform in section and weighing 2000 lbs., is 4 feet wide and 6 feet deep. It hangs on a horizontal hinge through the top edge, at an angle of 60° with the horizontal. Assuming that the water-level on the outer side is 0·26 feet, measured vertically, below the hinge, find the water-level on the inner side, relative to the hinge, when the flap is just about to open.**

Solution.—In fig. 174 O C is the flap and O the hinge. We will take it that the given water-level passes through point E to the left of O C, and the level of the tide through point F to the right of O C.

The water-pressure which tends to open the flap is

$P_1 = 0{\cdot}5\ \overline{CD}\ \overline{EC} \times 4 \times 62{\cdot}4$; but $\overline{OC} \sin 60^\circ = \overline{CD} + 0{\cdot}26$, and $\overline{EC} = \overline{CD} \cos 60^\circ$. Hence, $\overline{CD} = 4{\cdot}946$ ft., $\overline{EC} = 5{\cdot}71$ ft., and $P_1 = 3525$ lbws.; the latter acts at a distance $\overline{OE} + \frac{2}{3}\overline{EC} = 4{\cdot}1$ ft. from O. Hence, the moment of P_1 about $O = 14{,}452$ lbws. and ft.

The moment of the weight of the flap about point O $= -3 \times 2000 \times \cos 60^\circ = -3000$ lbws. and ft.

If the pressure on the flap due to the tide be P_2, then the moment of P_2 about O must be $= -(14{,}452 - 3000) = -11{,}452$ lbws. and ft.

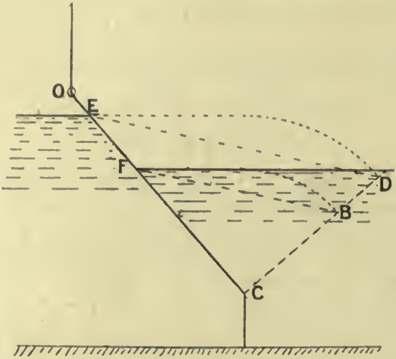

FIG. 174.

$P_2 = 0{\cdot}5 \times 4 \times 62{\cdot}4\ \ \overline{FC}\ \ \overline{CB}$; but $\overline{FC} \sin 60^\circ = CB$. Hence,

$P_2 = 144{\cdot}1\ \overline{CB}^2$. P_2 acts at a distance $\overline{OF} + \frac{2}{3}\overline{FC} = d$ from O; but $\overline{OF} = 6 - \overline{FC} = 6 - \overline{CB} \operatorname{cosec} 60^\circ$, and $d = 6 - \frac{1}{3}\overline{CB} \operatorname{cosec} 60^\circ$.

Hence, the moment of P_2 about the hinge $= 6 \times 144{\cdot}1\ \overline{CB}^2$

$- \frac{1}{3} \times 144{\cdot}1\ \overline{CB}^3 \operatorname{cosec} 60^\circ = 11{,}452$, or $\overline{CB}^3 - 15{\cdot}6\ \overline{CB}^2 + 206{\cdot}5 = 0$.

By trial we find that $\overline{CB} = 4{\cdot}27$ ft. Hence, the *water-level* at F is $(6 \sin 60^\circ - 4{\cdot}27) =$ just over 11 ins. below the hinge (*Ans.*).

(163) (I.C.E., Feb. 1909).—**A ball of steel falls 38·82 feet upon a horizontal slab of the same material. Assuming the coefficient of restitution (or resilience) to be 0·7, how high will the ball rise after impact? What is the ratio of kinetic energy just before and just after impact?**

Solution.—See articles 255 and 256.

$$V_f = V_i = 0\,;\ v_i = \sqrt{2g \times 38{\cdot}82}\,;\ \frac{m}{M} = 0\,;$$

hence,
$$v_f = -e v_i\,;\ L = \frac{1 - e^2}{2} m v_i^2.$$

Hence, energy just before impact $= \frac{1}{2}mv_i^2$;

$$\text{energy just after impact} = \tfrac{1}{2}mv_i^2 - \frac{1-e^2}{2}mv_i^2 = \frac{e^2}{2}mv_i^2;$$

Hence, the required ratio $= \dfrac{1}{e^2} = 2{\cdot}04$.

The ball will rise a height h after impact;

$$h = \frac{v_f^2}{2g} = \frac{e^2v_i^2}{2g} = 38{\cdot}82e^2 = 19 \text{ ft. } (Ans.).$$

(164) (Q. Nov. 1909).—**A cylinder 6 inches in diameter and 12 inches long, with plane ends, floats with its axis vertical in water contained in a cylindrical vessel 9 inches in diameter. If the cylinder weighs 7 lbs., plot to scale curves showing the relation between the force required to depress the cylinder and the distance through which it is depressed, and the relation between the work done and the depression.**

Solution.—Let D be the diameter of the vessel; l, d, and W the length, diameter, and weight of the cylinder respectively. Take as plane of reference the water-level when the cylinder is floating freely.

When the cylinder is depressed a distance δx the water-level will rise to a height δy.

$$\text{Hence,} \quad \frac{\pi d^2}{4}\delta x = \left(\frac{\pi D^2}{4} - \frac{\pi d^2}{4}\right)\delta y, \quad \text{or} \quad \delta y = \frac{d^2}{D^2 - d^2}\delta x.$$

If a force δF be required to depress the cylinder through a distance δx, then

$$\delta F = w\frac{\pi d^2}{4}(\delta x + \delta y) = \frac{\pi}{4}\,\frac{D^2 d^2}{D^2 - d^2}w\,\delta x.$$

Hence, by integration

$$F = \frac{\pi}{4}\,\frac{D^2 d^2}{D^2 - d^2}wx \quad . \quad . \quad . \quad . \quad (1)$$

(1) is a straight line through the origin. Maximum F is the force which is required to submerse the entire cylinder.

$$\text{Hence,} \qquad \text{maximum } F = \frac{\pi d^2}{4}lw - W \quad . \quad . \quad . \quad (2)$$

Equating (1) and (2) we obtain

$$\text{maximum } x = \frac{(\pi d^2 lw - 4W)(D^2 - d^2)}{\pi D^2 d^2 w} \quad . \quad . \quad (3)$$

The elementary energy-equation is

$$\delta E = F\,\delta x = \frac{\pi}{4}\frac{D^2d^2}{D^2 - d^2}wx\,\delta x \quad . \quad . \quad . \quad (4)$$

Hence, by integrating (4) we get

$$E = \frac{\pi}{8}\frac{D^2d^2}{D^2 - d^2}wx^2 \quad . \quad . \quad . \quad . \quad (5)$$

(5) is a parabola with vertex at the origin. Its axis lies on the positive branch of the y-axis.

We have given $D = 0{\cdot}75$, $d = 0{\cdot}5$, $l = 1$, $w = 62{\cdot}4$, $W = 7$.

Hence, $F = 22{\cdot}05x$; $E = 11{\cdot}03x^2$ (*Ans.*).

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