A TEXT-BOOK ON PRACTICAL MATHEMATICS FOR ADVANCED TECHNICAL STUDENTS

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PREFACE

For the past nine years I have been lecturing in this subject to students taking courses in Mechanical and Electrical Engineering at the Woolwich Polytechnic, and this book is based on the work done by the senior students there. So as not to make the book too cumbersome for a text-book, a preliminary knowledge of the fundamental principles of Algebra, Trigonometry, and Mensuration, and the use of Logarithms and squared paper, has been assumed, this being well within the scope of the elementary student. The book is meant to cover a two- or three-years' course, and it is roughly divided into three sections :

(1) Algebra and Trigonometry.
(2) The Differential and Integral Calculus.
(3) The application of the subject-matter of the two previous sections to concrete examples.

The work in *Section I* has been carefully selected in such a way as to help the student with the later work in the Calculus. There is no doubt that after the idea of the Calculus has been thoroughly grasped, a great many of the so-called difficulties which arise out of the work are entirely due to a weakness in the knowledge of the fundamental principles of Algebra and Trigonometry. For instance, many students fail in the integration of $\sin^2 x$, not because they do not know how to integrate, but because they fail to see or fail to remember that

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

Again, many fail to integrate algebraic functions because they have such weak notions of partial fractions and simple substitutions. *Section I* has been written with the idea of removing this weakness.

The Calculus has been treated as thoroughly as the size of the book allows. It might be said that this part of the work has been elaborated too much for the practical side of Mathematics ; but it must be remembered that the Calculus cannot be successfully applied to the problems which occur in actual practice until the student has become thoroughly familiar with its under-lying principles and methods, and this familiarity can only be obtained by steady practice. It is unfair to a student to give him as a standard form

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

and then expect him to use it as a formula to integrate any function which might resemble it, or by some means reduce to it. This might be working along the line of least resistance, but it is not educational: neither is it to the best interests of the student. to whom sound work in Differentiation and Integration is an absolute necessity.

The work in *Section III* consists of the Mathematics involved in those problems more or less familiar to the technical student, and before this work should be attempted it is essential that the work in the two previous sections should be fully grasped.

I have devoted a chapter to the study of Interpolation and the best way of dealing with tabular values, and I have endeavoured to put this part of the subject in a reasonable form. The method of Harmonic Analysis given in Chapter XXII is the one I have found from experience to be best adapted to class work.

The examples are numerous, and have been chosen in accordance with the text. The answers are given, and these have been carefully checked; but it is possible, as may be expected when dealing with such a number, that errors might occur, and I should be grateful to any teacher or student drawing my attention to them if such is the case.

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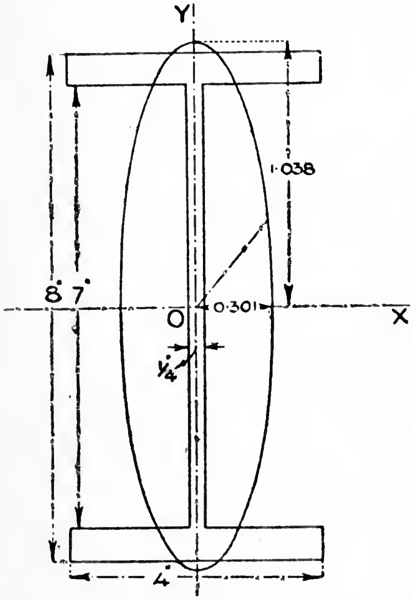
ERRATA

Page 36, Examples II, Question 8, *should read,*
'length of the diagonal AC.'

Page 142, Examples IX, Question 16, *should read,*
'where $x = 0$ and $x = 5$.'

Page 233. In Fig. 63, *the thickness of the web should be* '$\frac{1}{4}''$' *not* '$\frac{1}{2}''$.'



Page 468. Answer to Question 1, Examples VIII, *should be*
(1) $y_{max} = 35$ when $x = 1$, $y_{min} = 34$ when $x = 2$

Page 469. Answer to Question 5, Examples IX, *should be*
(5) $y = 1{\cdot}059x + 1{\cdot}059$, $y + 0{\cdot}945x = 5{\cdot}065$

Page 475. Answer to Question 29, Examples XIII, *should be*
(29) $\dfrac{x^3}{3}\tan^{-1}x - \dfrac{x^2}{6} + \dfrac{1}{6}\log_e(1+x^2)$

Page 477. Answer to Question 15, Examples XVIII, *should be*
(15) 2·546, 5·544, 0·6023

Page 480. Answer to Question 17, Examples XXI, *should be*
(17) $x = 0{\cdot}2686e^{-4{\cdot}025t}\sin(10{\cdot}22t + 1{\cdot}274)$, $T = 0{\cdot}6148$ sec.

Page 480. Answer to Question 25, Examples XXI, *should be*
(25) $v = 544{\cdot}4e^{-6250t}\sin(14525t + 1{\cdot}164)$

Mann's Practical Mathematics.

PRACTICAL MATHEMATICS

CHAPTER I

1. *The resolution of a quadratic expression into factors.*

Taking the most general form for an expression of the second degree

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a} + \frac{c}{a} - \frac{b^2}{4a}\right)$$

and the expression can be written as

$$a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right\}$$

in terms of the difference of two squares.

Hence the factors will be

$$a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

The nature of the factors depends upon the form taken by the expression $b^2 - 4ac$.

If $b^2 - 4ac$ is a perfect square, the factors are exact.

If $b^2 - 4ac$ is positive and not a perfect square, the expression can be split up into factors, but the numerical parts of each factor can only be given correct to as many significant figures as desired.

If $b^2 - 4ac$ is negative, then the factors can only b given in terms of complex quantities.

If $b^2 - 4ac = 0$, then the actual expression is itself a perfect square.

To find the factors of $8x^2 + 13x - 22$

$$\begin{aligned} 8x^2 + 13x - 22 &= 8(x^2 + 1{\cdot}625x - 2{\cdot}75) \\ &= 8\{x^2 + 1{\cdot}625x + ({\cdot}8125)^2 - 2{\cdot}75 - ({\cdot}8125)^2\} \\ &= 8\{(x + 0{\cdot}8125)^2 - 3{\cdot}410\} \\ &= 8\{(x + 0{\cdot}8125)^2 - (1{\cdot}847)^2\} \\ &= 8(x + 2{\cdot}659)(x - 1{\cdot}035) \end{aligned}$$

2. The previous method can be replaced by the solution of a quadratic equation; for if $x = \alpha_1$ makes the expression $x^2 + \frac{b}{a}x + \frac{c}{a}$ disappear, then $x - \alpha_1$ is a factor. Also if $x = \alpha_2$ makes the expression disappear, then $x - \alpha_2$ is a factor. Consequently $a(x - \alpha_1)(x - \alpha_2)$ will be the factors of $ax^2 + bx + c$ where α_1 and α_2 are the roots of the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

To find the factors of $5x^2 - 7x - 22$

$$5x^2 - 7x - 22 = 5(x^2 - 1{\cdot}4x - 4{\cdot}4)$$

Solving the equation

$$\begin{aligned} x^2 - 1{\cdot}4x - 4{\cdot}4 &= 0 \\ x^2 - 1{\cdot}4x + (0{\cdot}7)^2 &= 4{\cdot}89 \\ x - 0{\cdot}7 &= \pm\, 2{\cdot}211 \\ x &= 2{\cdot}911 \text{ or } -1{\cdot}511 \end{aligned}$$

The factors are $\quad 5(x - 2{\cdot}911)(x + 1{\cdot}511)$

3. *Partial Fractions.* For the integration of algebraic fractions it is necessary that the fraction must be expressed in its simplest and most convenient form for integration. For such purposes a fraction is much better dealt with when it is expressed as the sum or difference of simpler fractions. These simpler fractions are spoken of as "Partial Fractions," and the number of partial fractions which can represent a given fraction depends upon the number of factors, linear or otherwise, in the denominator of that fraction.

If, for example, the denominator contains three factors, then there will be three partial fractions, the respective denominators of which are the three factors taken in order. Thus, $\frac{3x + 2}{(x - 2)(x + 3)(2x - 5)}$ can be written as $\frac{A}{x - 2} + \frac{B}{x + 3} + \frac{C}{2x - 5}$ providing the necessary values of A, B, and C are found. Also $\frac{x^2}{(x + 1)(x^3 - 1)}$ or $\frac{x^2}{(x + 1)(x - 1)(x^2 + x + 1)}$ can be written as

$$\frac{A}{x + 1} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + x + 1}$$

Care must be taken that the numerator of any partial fraction shall always be of one degree less than that of its denominator.

4. Our work in partial fractions can be divided up into four different cases.

Case I. When the denominator of the fraction is the product of a certain number of different linear factors.

$$\frac{3x + 2}{(x - 2)(x + 3)(2x - 5)} = \frac{A}{x - 2} + \frac{B}{x + 3} + \frac{C}{2x - 5}$$

Then $A(x+3)(2x-5)+B(x-2)(2x-5)+C(x-2)(x+3)$
$=3x+2$ for all values of x,

when $x=2$ $\quad -5A \quad = 8 \quad A = -\frac{8}{5}$

when $x=-3$ $\quad +55B \quad = -7 \quad B = -\frac{7}{55}$

when $x=\frac{5}{2}$ $\quad \frac{11}{4}C \quad = \frac{19}{2} \quad C = \frac{38}{11}$

The partial fractions are $-\frac{8}{5(x-2)} - \frac{7}{55(x+3)} + \frac{38}{11(2x-5)}$

In this type o example, and in the other types when possible, it is much better to choose the values of x so that they make the factors disappear in turn.

Case II. When the denominator of the fraction consists of one factor raised to an integral power.

Taking $\frac{4x^2-2x+3}{(3x-2)^3}$ as an example,

put $a = 3x-2$, then $x = \frac{1}{3}(a+2)$

Then
$$4x^2-2x+3 = \frac{4}{9}(a+2)^2 - \frac{2}{3}(a+2)+3$$
$$= \frac{4}{9}a^2 - \frac{10}{9}a + \frac{31}{9}$$

The fraction becomes $\frac{4a^2-10a+31}{9a^3}$

This may be written as $\frac{4}{9a} - \frac{10}{9a^2} + \frac{31}{9a^3}$

Replacing a by $3x-2$, we get as the partial fractions
$$\frac{4}{9(3x-2)} - \frac{10}{9(3x-2)^2} + \frac{31}{9(3x-2)^3}$$

This result shows us how the partial fractions should be arranged when a certain factor in the denominator of a fraction is raised to a power.

For $\frac{x^3}{(x+1)(3x-2)^3}$ must be written as
$$\frac{A}{x+1} + \frac{B}{3x-2} + \frac{C}{(3x-2)^2} + \frac{D}{(3x-2)^3}$$

Case III. When the denominator of the fraction is the product of a certain number of linear factors, but one of these factors is raised to a power.
$$\frac{4x+3}{(x-2)^2(2x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{2x+1}$$

Then $A(x-2)(2x+1)+B(2x+1)+C(x-2)^2 = 4x+3$ for all values of x,

when $x = 2$ $\qquad 5B = 11 \qquad B = \frac{11}{5}$

when $x = -\frac{1}{2}$ $\qquad \frac{25}{4}C = 1 \qquad C = \frac{4}{25}$

when $x = 0$ $\qquad -2A + B + 4C = 3 \qquad A = -\frac{2}{25}$

The partial fractions are $-\dfrac{2}{25(x-2)} + \dfrac{11}{5(x-2)^2} + \dfrac{4}{25(2x+1)}$

Case IV. When the denominator of the fraction contains a quadratic factor which cannot be resolved into linear factors.

$$\frac{x^2}{x^3-1} = \frac{x^2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Then $A(x^2+x+1)+(Bx+C)(x-1) = x^2$ for all values of x,

when $x = 1$ $\qquad 3A = 1 \qquad A = \frac{1}{3}$

when $x = 0$ $\qquad A - C = 0 \qquad C = \frac{1}{3}$

when $x = 2$ $\qquad 7A + 2B + C = 4 \qquad B = \frac{2}{3}$

The partial fractions are $\dfrac{1}{3(x-1)} + \dfrac{2x+1}{3(x^2+x+1)}$

5. If the numerator is of higher degree than the denominator, then the denominator must be divided into the numerator, and the fraction whose numerator is the remainder must be split up into partial fractions.

Thus by division $\dfrac{x^4}{x^3-8} = x + \dfrac{8x}{x^3-8}$

Then $\dfrac{8x}{x^3-8} = \dfrac{8x}{(x-2)(x^2+2x+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+4}$

$A(x^2+2x+4)+(Bx+C)(x-2) = 8x$ for all values of x,

when $x = 2$ $\qquad 12A = 16 \qquad A = \frac{4}{3}$

when $x = 0$ $\qquad 4A - 2C = 0 \qquad C = \frac{8}{3}$

when $x = 1$ $\qquad 7A - B - C = 8 \qquad B = -\frac{4}{3}$

$$\frac{x^4}{x^3-8} = x + \frac{4}{3(x-2)} + \frac{8-4x}{3(x^2+2x+4)}$$

In general the best method of dealing with the work on partial fractions is to begin by selecting values of x which will make particular factors disappear; this can be done for all linear factors: then, if necessary, make x zero and use the results already found. After this, if the solution to the question is still incomplete, a further step must be taken, and this is best done by equating coefficients of like powers of x.

Sometimes we have to deal with an example which can only be done by equating coefficients of like powers of x and solving the resulting simultaneous equations for the quantities A, B, C, &c.

Taking $\frac{x^2}{x^4 + x^2 + 1}$ as an example of this type,

$$\frac{x^2}{x^4 + x^2 + 1} = \frac{x^2}{(x^2 + x + 1)(x^2 - x + 1)}$$
$$= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1}$$

Then $(Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1) = x^2$ for all values of x.

Equating coefficients of x^3 $\quad A + C = 0 \quad . \quad . \quad . \quad (1)$
Equating coefficients of x^2 $\quad -A + B + C + D = 1 \quad . \quad . \quad . \quad (2)$
Equating coefficients of x $\quad A - B + C + D = 0 \quad . \quad . \quad . \quad (3)$
When $x = 0$ $\quad B + D = 0 \quad . \quad . \quad . \quad (4)$
Adding (1) and (4) $\quad A + B + C + D = 0 \quad . \quad . \quad . \quad (5)$

Subtracting (2) from (5) $\quad 2A = -1 \qquad A = -\frac{1}{2}$

from (1) $\quad C = -A = \frac{1}{2}$

Subtracting (3) from (5) $\quad 2B = 0 \qquad B = 0$
from (4) $\quad D = -B = 0$

The partial fractions are $\quad -\frac{x}{2(x^2 + x + 1)} + \frac{x}{2(x^2 - x + 1)}$

6. *The Binomial Theorem.*

If $(x + a)^n = x^n + Aax^{n-1} + Ba^2x^{n-2} + Ca^3x^{n-3} + \ldots$
where A, B, C . . . are independent of x and a,
Then $(x + a)^{n+1} = x^{n+1} + \alpha ax^n + \beta a^2x^{n-1} + \gamma a^3x^{n-2} + \ldots$
where α, β, γ . . . are independent of x and a.

For $(x + a)^{n+1} = (x + a)(x + a)^n$

$$= (x + a)(x^n + Aax^{n-1} + Ba^2x^{n-2} + Ca^3x^{n-3} + \ldots)$$
$$= x^{n+1} + Aax^n + Ba^2x^{n-1} + Ca^3x^{n-2} \ldots$$
$$+ ax^n + Aa^2x^{n-1} + Ba^3x^{n-2} + \ldots$$
$$= x^{n+1} + (A + 1)ax^n + (B + A)a^2x^{n-1}$$
$$+ (C + B)a^3x^{n-2} + \ldots$$
$$= x^{n+1} + \alpha ax^n + \beta a^2x^{n-1} + \gamma a^3x^{n-2} + \ldots$$

and since α, β, γ . . . are derived only from A, B, C . . . they must be independent of x and a.

Thus $(x + a)^n$ can be written as a series of terms containing descending powers of x and ascending powers of a, such that in any one term the sum of the powers of x and a is always n.

If we can find how the coefficients A, B, C . . . depend upon n, then we can establish a general way of expanding $(x + a)^n$.

By multiplication

$$\begin{array}{llllllll}
 & & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \\
(x+a)^1 & = x & + \; a & & & & & \\
(x+a)^2 & = x^2 & + \; 2ax & + \; a^2 & & & & \\
(x+a)^3 & = x^3 & + \; 3ax^2 & + \; 3a^2x & + \; a^3 & & & \\
(x+a)^4 & = x^4 & + \; 4ax^3 & + \; 6a^2x^2 & + \; 4a^3x & + \; a^4 & & \\
(x+a)^5 & = x^5 & + \; 5ax^4 & + \; 10a^2x^3 & + \; 10a^3x^2 & + \; 5a^4x & + \; a^5 & \\
(x+a)^6 & = x^6 & + \; 6ax^5 & + \; 15a^2x^4 & + \; 20a^3x^3 & + \; 15a^4x^2 & + \; 6a^5x & + \; a^6
\end{array}$$

The coefficients in the second vertical column give the different values of A for corresponding values of n, and obviously $A = n$.

As A depends upon the first power of n, let B depend upon the second power of n, and the most general form that B can take is $b + cn + dn^2$ where b, c, and d are constants.

$$\begin{array}{llll}
\text{Let } B = b + cn + dn^2 & & & \\
\quad \text{when } n = 2 \quad B = 1 & b + 2c + 4d = 1 & \ldots & (1) \\
\quad \text{when } n = 3 \quad B = 3 & b + 3c + 9d = 3 & \ldots & (2) \\
\quad \text{when } n = 4 \quad B = 6 & b + 4c + 16d = 6 & \ldots & (3) \\
\text{Subtracting (1) from (2)} & c + 5d = 2 & \ldots & (4) \\
\text{Subtracting (2) from (3)} & c + 7d = 3 & \ldots & (5) \\
\text{Subtracting (4) from (5)} & 2d = 1 & d = \frac{1}{2} & \\
\qquad \text{from (4)} & c = 2 - 5d & c = -\frac{1}{2} & \\
\qquad \text{from (1)} & b = 1 - 2c - 4d & b = 0 &
\end{array}$$

Thus $$B = \frac{n^2}{2} - \frac{n}{2} = \frac{n(n-1)}{2}$$

If $$A = \frac{n}{1} \text{ and } B = \frac{n}{1} \times \frac{n-1}{2}$$

Then $$B = A \times \frac{n-1}{2}$$

That is, to obtain B from A, multiply A by the fraction whose numerator is " one less " and whose denominator is " one more " than that of A.

If this rule for obtaining a coefficient from the preceding coefficient is general, then to get C from B, C would be

$$B \times \frac{n-2}{3} \text{ or } A \times \frac{(n-1)(n-2)}{2 \cdot 3} \text{ or } \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

As B is of the second degree in n, C will be of the third degree in n, and its most general form will be $b + cn + dn^2 + hn^3$, where b, c, d, and h are constants.

$$
\begin{array}{llll}
\text{Let } C = b + cn + dn^2 + hn^3 & & & \\
\text{when } n = 3 \quad C = 1 & b + 3c + 9d + 27h & = 1 & \quad \ldots (1) \\
\text{when } n = 4 \quad C = 4 & b + 4c + 16d + 64h & = 4 & \quad \ldots (2) \\
\text{when } n = 5 \quad C = 10 & b + 5c + 25d + 125h & = 10 & \quad \ldots (3) \\
\text{when } n = 6 \quad C = 20 & b + 6c + 36d + 216h & = 20 & \quad \ldots (4) \\
\text{subtracting (1) from (2)} & c + 7d + 37h & = 3 & \quad \ldots (5) \\
\text{subtracting (2) from (3)} & c + 9d + 61h & = 6 & \quad \ldots (6) \\
\text{subtracting (3) from (4)} & c + 11d + 91h & = 10 & \quad \ldots (7) \\
\text{subtracting (5) from (6)} & 2d + 24h & = 3 & \quad \ldots (8) \\
\text{subtracting (6) from (7)} & 2d + 30h & = 4 & \quad \ldots (9) \\
\text{subtracting (8) from (9)} & 6h & = 1 & \quad h = \frac{1}{6}
\end{array}
$$

$$
\begin{array}{lll}
\text{from (8)} & 2d = 3 - 24h & d = -\frac{1}{2} \\
\text{from (5)} & c = 3 - 7d - 37h & c = \frac{1}{3} \\
\text{from (1)} & b = 1 - 3c - 9d - 27h & b = 0
\end{array}
$$

Thus

$$
\begin{aligned}
C &= \frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3} \\
&= \frac{n}{6}(n^2 - 3n + 2) \\
&= \frac{n(n-1)(n-2)}{6}
\end{aligned}
$$

This result agrees with the anticipated result for C. Hence

$$(x + a)^n = x^n + \mathrm{A}ax^{n-1} + \mathrm{B}a^2x^{n-2} + \mathrm{C}a^3x^{n-3} + \mathrm{D}a^4x^{n-4} + \ldots$$

where

$$
\begin{aligned}
\mathrm{A} &= \frac{n}{1} \\
\mathrm{B} &= \mathrm{A} \times \frac{n-1}{2} = \frac{n(n-1)}{1 \cdot 2} \\
\mathrm{C} &= \mathrm{B} \times \frac{n-2}{3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \\
\mathrm{D} &= \mathrm{C} \times \frac{n-3}{4} = \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}
\end{aligned}
$$

The expansion can also be written as

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\underline{|2}}a^2x^{n-2} + \frac{n(n-1)(n-2)}{\underline{|3}}a^3x^{n-3} + \ldots$$

Expand $\left(2x - \dfrac{1}{2x}\right)^7$. Here $n = 7$.

$$A = \frac{7}{1}$$

$$B = A \times \frac{6}{2} = 7 \times 3 = 21$$

$$C = B \times \frac{5}{3} = 21 \times \frac{5}{3} = 35$$

$$D = C \times \frac{4}{4} = 35$$

$$E = D \times \frac{3}{5} = 35 \times \frac{3}{5} = 21$$

$$F = E \times \frac{2}{6} = 21 \times \frac{1}{3} = 7$$

$$G = F \times \frac{1}{7} = 7 \times \frac{1}{7} = 1$$

Then $\left(2x - \dfrac{1}{2x}\right)^7 = \left\{2x + \left(-\dfrac{1}{2x}\right)\right\}^7$

$$= (2x)^7 + 7(2x)^6\left(-\frac{1}{2x}\right) + 21(2x)^5\left(-\frac{1}{2x}\right)^2$$

$$+ 35(2x)^4\left(-\frac{1}{2x}\right)^3 + 35(2x)^3\left(-\frac{1}{2x}\right)^4$$

$$+ 21(2x)^2\left(-\frac{1}{2x}\right)^5 + 7(2x)\left(-\frac{1}{2x}\right)^6$$

$$+ \left(-\frac{1}{2x}\right)^7$$

$$= 128x^7 - 224x^5 + 168x^3 - 70x + \frac{35}{2x} - \frac{21}{8x^3}$$

$$+ \frac{7}{32x^5} - \frac{1}{128x^7}$$

Expand $(1 - 2x)^{-3}$ to 5 terms. Here $n = -3$.

$$A = \frac{-3}{1} = -3$$

$$B = A \times \frac{-4}{2} = -3 \times -2 = 6$$

$$C = B \times \frac{-5}{3} = 6 \times -\frac{5}{3} = -10$$

$$D = C \times \frac{-6}{4} = -10 \times -\frac{3}{2} = 15$$

$$\text{Then}\quad (1-2x)^{-3} = \{1+(-2x)\}^{-3}$$
$$=1+(-3)(-2x)+6(-2x)^2+(-10)(-2x)^3 +15(-2x)^4+\ldots$$
$$=1+6x+24x^2+80x^3+240x^4+\ldots$$

Expand $(1-x)^{-\frac{3}{5}}$ to five terms. Here $n = -\dfrac{3}{5}$

$$A = \frac{-\frac{3}{5}}{1} = -\frac{3}{5}$$

$$B = A \times \frac{-\frac{8}{5}}{2} = -\frac{3}{5} \times -\frac{4}{5} = \frac{12}{25}$$

$$C = B \times \frac{-\frac{13}{5}}{3} = \frac{12}{25} \times -\frac{13}{15} = -\frac{52}{125}$$

$$D = C \times \frac{-\frac{18}{5}}{4} = -\frac{52}{125} \times -\frac{9}{10} = \frac{234}{625}$$

$$\text{Then}\quad (1-x)^{-\frac{3}{5}} = \{1+(-x)\}^{-\frac{3}{5}}$$
$$= 1+\left(-\frac{3}{5}\right)(-x)+\frac{12}{25}(-x)^2+\left(-\frac{52}{125}\right)(-x)^3 +\frac{234}{625}(-x)^4+\ldots$$
$$= 1+\frac{3}{5}x+\frac{12}{25}x^2+\frac{52}{125}x^3+\frac{234}{625}x^4+\ldots$$

7. *Approximations.*

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\underline{|2}}a^2x^{n-2} + \frac{n(n-1)(n-2)}{\underline{|3}}a^3x^{n-3} + \ldots$$

becomes $(1+a)^n = 1 + na + \dfrac{n(n-1)}{\underline{|2}}a^2 + n\dfrac{(n-1)(n-2)}{\underline{|3}}a^3 + \ldots$

when $x = 1$.

If a is small compared with 1 and the value of n is kept reasonably small, between the limits $+2$ and -2, the terms of the expansion for $(1+a)^n$ rapidly become smaller and smaller. By taking the first two terms of this expansion, we obtain a first approximation for $(1+a)^n$.

The first approximations are :

$$(1+a)^n \approx 1+na$$
$$(1-a)^n \approx 1-na$$
$$(1+a)^{-n} \approx 1-na$$
$$(1-a)^{-n} \approx 1+na$$

If a better result is required, the first three terms of the expansion must be taken, and this gives rise to the second approximations.

The second approximations are :

$$(1+a)^n \approx 1 + na + \frac{1}{2}n(n-1)a^2$$

$$(1-a)^n \approx 1 - na + \frac{1}{2}n(n-1)a^2$$

$$(1+a)^{-n} \approx 1 - na + \frac{1}{2}n(n+1)a^2$$

$$(1-a)^{-n} \approx 1 + na + \frac{1}{2}n(n+1)a^2$$

To find the first and second approximations of $\sqrt[3]{130}$.

$$\sqrt[3]{130} = (125+5)^{\frac{1}{3}} = 5\left(1+\frac{1}{25}\right)^{\frac{1}{3}}$$

Since $\quad n = \dfrac{1}{3} \qquad A = \dfrac{\frac{1}{3}}{1} = \dfrac{1}{3}$

$$B = A \times \frac{-\frac{2}{3}}{2} = \frac{1}{3} \times -\frac{1}{3} = -\frac{1}{9}$$

Then $\quad \sqrt[3]{130} = 5\left\{1 + \dfrac{1}{3}\cdot\dfrac{1}{25} - \dfrac{1}{9}\cdot\left(\dfrac{1}{25}\right)^3 \ldots\right\}$

The first approximation is $5\left(1+\dfrac{1}{75}\right) = 5{\cdot}0667$ given to four places of decimals.

The term to be subtracted from the first approximation to give the second approximation is $\dfrac{5}{9} \times \dfrac{1}{25^2} = 0{\cdot}0009$ given to four places of decimals.

The second approximation is 5·0658.

Also 5·066 is the result correct to four significant figures.

To find the first and second approximations of $\sqrt[5]{1000}$.

$$\sqrt[5]{1000} = (1024-24)^{\frac{1}{5}} = 4\left(1-\frac{3}{128}\right)^{\frac{1}{5}}$$

Since $\quad n = \dfrac{1}{5} \qquad A = \dfrac{\frac{1}{5}}{1} = \dfrac{1}{5}$

$$B = \frac{-\frac{4}{5}}{2} \times A = -\frac{2}{5} \times \frac{1}{5} = -\frac{2}{25}$$

Then $\quad \sqrt[5]{1000} = 4\left\{1 - \dfrac{1}{5}\cdot\dfrac{3}{128} - \dfrac{2}{25}\cdot\left(\dfrac{3}{128}\right)^2 \ldots\right\}$

The first approximation is $4\left(1-\dfrac{3}{640}\right) = 3{\cdot}98125$ given to five places of decimals.

The term to be subtracted from the first approximation to give the second approximation is $\frac{8}{25} \times \left(\frac{3}{128}\right)^2 = 0{\cdot}00018$ given to five places of decimals.

The second approximation is 3·98107.

Also 3·981 is the result correct to four significant figures.

8. *Applications of the approximate use of Binomial Theorem.*

(1) To find the position of the crosshead corresponding to a given angular position of the crank.

Let OA, Fig. 1, represent the crank, of length r, rotating about the fixed centre O, and let AB represent the connecting rod, of length l.

In the extreme position C of the crosshead, the crank and

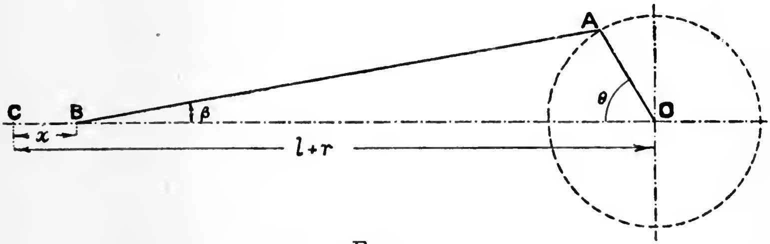


FIG. 1.

connecting rod are in the same straight line and the distance OC $= l + r$.

Let θ represent the angular position of the crank and x the distance of the crosshead from C.

Then $x + l\cos\beta + r\cos\theta = l + r$ where β is the angle the connecting rod makes with the line of centres.

Also $\qquad l\sin\beta = r\sin\theta$

but
$$\cos\beta = \sqrt{1 - \sin^2\beta}$$
$$= \left(1 - \frac{r^2}{l^2}\sin^2\theta\right)^{\frac{1}{2}}$$
$$= 1 - \frac{r^2}{2l^2}\sin^2\theta \text{ approximately,}$$

since $\frac{r^2}{l^2}\sin^2\theta$ is small compared with 1.

Then
$$x = r - r\cos\theta + l - l\cos\beta$$
$$= r(1 - \cos\theta) + \frac{r^2}{2l}\sin^2\theta$$
$$= r(1 - \cos\theta) + \frac{r^2}{4l}(1 - \cos 2\theta)$$

If the crank is making n revolutions per second, the angle turned through per second is $2\pi n$ radians, and if θ is the angle turned through in t seconds

Then $$\theta = 2\pi nt \text{ radians}$$

and $$x = r(1 - \cos 2\pi nt) + \frac{r^2}{4l}(1 - \cos 4\pi nt)$$

(2) The effect of heat on the pendulum.

The time of swing of a pendulum is given by $\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum in feet.

Let l_1 be the length of a pendulum making n_1 beats per second at a temperature of $T_1°$ C.

Then $$t_1 = \pi\sqrt{\frac{l_1}{g}} \quad \text{and } n_1 t_1 = 1$$

Let l_2 be the length of the same pendulum at a temperature of $T_2°$ C. and at this temperature it makes n_2 beats per second.

Then $$t_2 = \pi\sqrt{\frac{l_2}{g}} \quad \text{and } n_2 t_2 = 1$$

but $l_2 = l_1\{1 + \alpha(T_2 - T_1)\}$ where α is the coefficient of expansion of the pendulum rod.

Now $$\frac{t_1}{t_2} = \frac{1}{\sqrt{1 + \alpha(T_2 - T_1)}}$$

and $$\frac{n_2}{n_1} = \{1 + \alpha(T_2 - T_1)\}^{-\frac{1}{2}}$$

Then $\frac{n_2}{n_1} = 1 - \frac{\alpha}{2}(T_2 - T_1)$ to the first approximation and $\frac{n_2}{n_1} = 1 - \frac{\alpha}{2}(T_2 - T_1) + \frac{3\alpha^2}{8}(T_2 - T_1)^2$ to the second approximation, since $\alpha(T_2 - T_1)$ is small compared with 1.

Working to the first approximation

$$\frac{\alpha}{2}(T_2 - T_1) = 1 - \frac{n_2}{n_1} = \frac{n_1 - n_2}{n_1}$$

Then $n_1 - n_2 = \frac{1}{2}n_1\alpha(T_2 - T_1)$ giving the loss of the pendulum per second due to the temperature increasing from $T_1°$ to $T_2°$.

If the range of temperature is great, it might be necessary to take the second approximation.

Then $$\frac{\alpha}{2}(T_2 - T_1) - \frac{3\alpha^2}{8}(T_2 - T_1)^2 = 1 - \frac{n_2}{n_1} = \frac{n_1 - n_2}{n_1}$$

and $$n_1 - n_2 = n_1\left\{\frac{\alpha}{2}(T_2 - T_1) - \frac{3\alpha^2}{8}(T_2 - T_1)^2\right\}$$

$$= \frac{1}{2}n_1\alpha(T_2 - T_1)\left\{1 - \frac{3\alpha}{4}(T_2 - T_1)\right\}$$

If the pendulum actually beats seconds at the lower temperature, then we can put $n_1 = 1$ in the relation which gives the loss per second.

9. *The Exponential and Logarithmic Series.*

Putting $a = \frac{1}{n}$ in the expansion

$$(1+a)^n = 1 + na + \frac{n(n-1)}{\underline{|2}}a^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^3 + \ldots$$

we get

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{\underline{|2}}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{\underline{|3}}\left(\frac{1}{n}\right)^3 + \ldots$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \ldots$$

Making n infinitely great, all of the fractions having n for their denominators become infinitely small and can therefore be neglected. Thus in the limit the right hand side becomes

$$1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots$$

a quantity having a definite value, calling this quantity e

then $$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots$$

By evaluating the series the value of e can be found correct to as many significant figures as required.

To find e correct to six significant figures.

	1·000000
2	1·000000
3	·500000
4	·166667
5	·041667
6	·008333
7	·001389
8	·000198
9	·000025
	·000003
	2·718282

$e = 2{\cdot}71828$ correct to six significant figures.

Now $\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^n\right\}^x = e^x$ when n is made infinitely great, but $\left(1+\frac{1}{n}\right)^{nx}$ can be expanded by means of the Binomial Theorem.

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\left(\frac{1}{n}\right)+\frac{nx(nx-1)}{\underline{|2}}\left(\frac{1}{n}\right)^2+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\left(\frac{1}{n}\right)^3+\ldots$$

$$=1+\ x\quad+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}\quad+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\ \ldots$$

Making n infinitely great, we get

$$e^x=1+\ x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ \ldots$$

By giving x different values we can use this series to calculate the corresponding values of e^x. Thus when $x=\frac{1}{2}$

$$e^{\frac{1}{2}}=1+\frac{1}{2}+\frac{1}{2^2\underline{|2}}+\frac{1}{2^3\underline{|3}}+\ \ldots$$

2	1·000000
4	·500000
6	·125000
8	·020833
10	·002604
12	·000260
14	·000022
	·000002
	1·648721

Then $e^{\frac{1}{2}}=1{\cdot}64872$ correct to six significant figures.

If $e^{\frac{1}{2}}=1{\cdot}64872$, then $\log_e 1{\cdot}64872=\frac{1}{2}$, and therefore we can use the series for e^x as a means of calculating numbers if we are given their logarithms to the base e. We can thus consider "e" to be the base of a system of logarithms, and such a logarithm is called the Napierian or hyperbolic logarithm.

To change the base of a system of logarithms we may put $x=cy$.

Then $$e^x=e^{cy}=a^y$$

where $$e^c=a\ \text{ or }\ c=\log_e a$$

$$e^{cy}=1+cy+\frac{c^2y^2}{\underline{|2}}+\frac{c^3y^3}{\underline{|3}}+\ \ldots$$

and $$a^y=1+y\log_e a+\frac{y^2}{\underline{|2}}(\log_e a)^2+\frac{y^3}{\underline{|3}}(\log_e a)^3+\ \ldots$$

Replacing a by $1+z$

$$(1+z)^y=1+y\log_e(1+z)+\frac{y^2}{\underline{|2}}\{\log_e(1+z)\}^2+\ \ldots \qquad (1)$$

But $(1+z)^y$ can be expanded by means of the Binomial Theorem.

$$\therefore \quad (1+z)^y = 1 + yz + \frac{y(y-1)}{\underline{|2}}z^2 + \frac{y(y-1)(y-2)}{\underline{|3}}z^3 + \dots \quad (2)$$

Equating coefficients of y in (1) and (2)

$$\log_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \dots \quad (3)$$

This series can be conveniently used for values of z less than 1, and it can therefore be employed as a means of calculating the logarithms of numbers between 1 and 2.

Replacing z by $-z$ we have

$$\log_e(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \dots \quad (4)$$

a series which can be used for the calculation of the logarithms of numbers 0 and 1.

Subtracting (4) from (3) gives

$$\log_e(1+z) - \log_e(1-z) \text{ or } \log_e\frac{1+z}{1-z} = 2\left\{z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right\}$$

putting $\dfrac{1+z}{1-z} = \dfrac{n+1}{n}$ or $z = \dfrac{1}{2n+1}$ the series becomes

$$\log_e\frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}$$

a series in which the terms rapidly become smaller and smaller as n is made larger, and it can be used for the calculation of the logarithms of numbers greater than 2.

For $\log_e(n+1) - \log_e n \quad = 2\left\{\dfrac{1}{2n+1} + \dfrac{1}{3(2n+1)^3} + \dots\right\}$

when $\quad n = 1 \qquad \log_e 2 = 2\left\{\dfrac{1}{3} + \dfrac{1}{3\cdot3^3} + \dfrac{1}{5\cdot5^5} + \dots\right\}$

$= 0{\cdot}6931$

when $\quad n = 2 \qquad \log_e 3 - \log_e 2 = 2\left\{\dfrac{1}{5} + \dfrac{1}{3\cdot5^3} + \dfrac{1}{5\cdot5^5} \dots\right\}$

$= 0{\cdot}4055$

Then $\qquad \log_e 3 = 1{\cdot}0986$

Now $\qquad \log_e 4 = 2\log_e 2 = 1{\cdot}3862$

when $\quad n = 4 \qquad \log_e 5 - \log_e 4 = 2\left\{\dfrac{1}{9} + \dfrac{1}{3\cdot9^3} + \dfrac{1}{5\cdot9^5} \dots\right\}$

$= 0{\cdot}2231$

Then $\qquad \log_e 5 = 1{\cdot}6093$

10. In working with logarithms to the base 10, and logarithms to the base e, many students find difficulty in changing from one system to the other.

Let y be the common logarithm and x the Napierian logarithm of a certain number N.

Then $\log_{10}N = y$ and $N = 10^y$

Also $\log_e N = x$ and $N = e^x$

Hence $10^y = e^x$.

Taking common logarithms of both sides

$$y = x \log_{10} e = 0{\cdot}4343x$$

Thus common log $= 0{\cdot}4343$ Nap. log or $\dfrac{\text{Nap. log}}{2{\cdot}3026}$

and Nap. log $= \dfrac{\text{common log}}{0{\cdot}4343}$ or $2{\cdot}3026$ common log.

Examples I

(1) Working correct to four significant figures, find the factors of

(a) $x^2 + 3{\cdot}94x - 5{\cdot}62$
(b) $x^2 - 5{\cdot}72x + 4{\cdot}77$
(c) $x^2 - 8{\cdot}92x - 4{\cdot}86$

Find the partial fractions of

(2) $\dfrac{3x+5}{(x-3)(x+4)}$ (3) $\dfrac{x^3-2}{x^2-4}$

(4) $\dfrac{x-8}{(2x+3)(3x-2)}$ (5) $\dfrac{5x-6}{(5x-3)(3-x)}$

(6) $\dfrac{x^2-3x+1}{(x-1)(x+2)(x-3)}$ (7) $\dfrac{5x^2+7x+1}{(2x+1)(3x-2)(3x+1)}$

(8) $\dfrac{x^5-x+1}{(x^2-9)(x+2)}$ (9) $\dfrac{x^2-2}{(3-2x)(x+8)(5-3x)}$

(10) $\dfrac{5x^2+8x-12}{(x+4)^3}$ (11) $\dfrac{8x^3-7}{(x-5)^4}$

(12) $\dfrac{7x^2+9x+1}{(3x-2)^4}$ (13) $\dfrac{5x^3-3x+1}{(4x+3)^5}$

(14) $\dfrac{x^2}{x^3-1}$ (15) $\dfrac{x^4}{x^3-8}$

(16) $\dfrac{x^4+1}{x^3+1}$ (17) $\dfrac{x^2}{(x^3+8)(x+2)}$

(18) $\dfrac{3x^2 - 2x + 1}{(x-3)^2(x+1)}$ (19) $\dfrac{8x+3}{(x-2)^2(x+1)^2}$

(20) $\dfrac{x^2}{x^4 + x^2 + 1}$ (21) $\dfrac{3x+1}{x^2 + 3{\cdot}94x - 5{\cdot}62}$

(22) $\dfrac{2x - 5{\cdot}72}{x^2 - 5{\cdot}72x + 4{\cdot}77}$ (23) $\dfrac{1}{(x+3)(x^2 - 3{\cdot}5x + 4{\cdot}94)}$

(24) Using the Binomial Theorem, find the first five terms in the expansions for (*a*) $(1+2x)^7$, (*b*) $(1-x)^{-5}$, (*c*) $(1+2x)^{\frac{1}{2}}$, (*d*) $(1+x)^{-\frac{1}{3}}$, (*e*) $\dfrac{1}{1+x^2}$, (*f*) $\dfrac{1}{\sqrt{1-x^2}}$.

(25) Give to five places the first and second approximations of (*a*) $\sqrt{138}$, (*b*) $\sqrt{627}$, (*c*) $\sqrt[3]{734}$, (*d*) $\sqrt[3]{500}$, (*e*) $\sqrt[4]{4000}$, (*f*) $\sqrt[4]{630}$.

(26) Using the Binomial Theorem, find the first five terms in the expansion for $(1-x)^{-\frac{1}{2}}$. Use your result to calculate the value of $\dfrac{1}{\sqrt{0{\cdot}95}}$ correct to four places of decimals.

(27) If l is the length of connecting rod, r the length of crank, and x is the distance of the crosshead from the extreme dead point; then approximately $x = r(1 - \cos\theta) + \dfrac{r^2}{4l}(1 - \cos 2\theta)$ where θ is the angular position of the crank. Taking $r = 1$ and $l = 5$, calculate the approximate value of x when $\theta = 45°$. What is the true value, and what is the percentage error?

Taking $r = 1$, $l = 3$, and $\theta = 45°$, calculate the approximate and true values of x and find the percentage error.

(28) If a pendulum beats seconds at 15° C. and the rod is brass, find the number of seconds lost per day when the temperature increases to 35° C., and find the number of seconds gained per day when the temperature falls to 0° C. Linear coefficient of expansion of brass $= 1{\cdot}9 \times 10^{-5}$.

(29) Using the series $\log_e(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \ldots$, calculate $\log_e 1{\cdot}2$ correct to four places of decimals.

(30) Using the series $e^x = 1 + x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \ldots$, find the values of $\sqrt{e}$, $e^{\frac{1}{3}}$, and $\dfrac{1}{\sqrt{e}}$ correct to five significant figures.

(31) Using the series

$$\log_e(n+1) - \log_e n = \frac{1}{2}\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \ldots\right\}$$

calculate correct to five places of decimals $\log_e 2$, $\log_e 3$, $\log_e 4$, and $\log_e 5$.

(32) Find the Napierian logarithms of 1·492, 0·01964, and 326·1.

(33) Find the numbers whose Napierian logarithms are 0·1624, 2·9214, and −2·0732.

(34) If $\phi = \log_e \frac{T}{273}$ where $T = 273 + t$, find ϕ when t has the values 50, 100, 150, 200, 250, and 300.

(35) If $l = 3{\cdot}11 \times 10^{-4} u \log_e \frac{r_0}{r_1}$ and $u = 300$, $l = 0{\cdot}04$. Find r_0 when r_1 has the values 0·0115, 0·0120, 0·0125, and 0·0130.

CHAPTER II

11. *The Solution of Triangles.* It is possible to perform the solution of triangles without the aid of formulæ, and the work can easily be done from first principles; a knowledge of the definitions of the trigonometrical ratios and of the properties of the right-angled triangle is all that is necessary. The method affords an excellent example of the application of the trigonometrical ratios to practical work.

We have three cases to consider, viz.:

Case I. When the three sides are given.

Case II. When two sides and one angle are given.

Case III. When one side and two angles are given.

Case I. Here it is best to take the longest side as the base and find the angles at the base, for since the base is the longest side the angles at the base must be acute. Let the three sides be 18, 14, and 9.

Let h be the length of the perpendicular drawn from the vertex to the base, and let x and $(18 - x)$ be the segments into which the foot of the perpendicular divides the base.

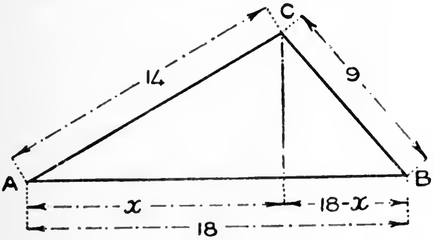


FIG. 2.

$$\begin{aligned} \text{Then} \qquad h^2 &= 14^2 - x^2 \\ \text{and} \qquad h^2 &= 9^2 - (18 - x)^2 \\ 196 - x^2 &= 81 - 324 + 36x - x^2 \\ 36x &= 439 \\ x &= 12{\cdot}19 \end{aligned}$$

Then $\cos A = \dfrac{x}{14} = \dfrac{12{\cdot}19}{14} = 0{\cdot}8709$ $\qquad A = 29° \, 27'$

Also $\cos B = \dfrac{18 - x}{9} = \dfrac{5{\cdot}81}{9} = 0{\cdot}6456$ $\qquad B = 49° \, 48'$

$C = 180 - (A + B)$ $\qquad C = 100° \, 45'$

Case II. This case gives rise to three distinct examples according to the position the given angle occupies with respect to the two given sides. This angle may be :

(*a*) Contained by the two sides.
(*b*) Opposite the smaller side.
(*c*) Opposite the larger side.

(*a*) Let the two sides be 31 and 22 and the included angle 62°. Take the longest side as the base.

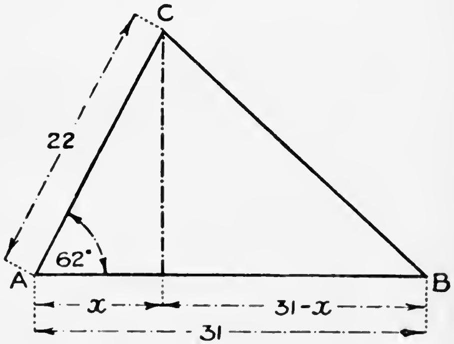


FIG. 3.

$$\frac{h}{22} = \sin 62° = 0{\cdot}8829$$

$$h = 0{\cdot}8829 \times 22 = 19{\cdot}42$$

$$\frac{x}{22} = \cos 62° = 0{\cdot}4695$$

$$x = 0{\cdot}4695 \times 22 = 10{\cdot}33$$

$$\text{CB} = \sqrt{h^2 + (31 - x)^2}$$

$$= \sqrt{19{\cdot}42^2 + 20{\cdot}67^2} = 28{\cdot}36$$

$$\text{Also } \tan \text{B} = \frac{h}{31 - x} = \frac{19{\cdot}42}{20{\cdot}67} = 0{\cdot}9393 \qquad \text{B} = 43°\ 12'$$

$$\text{C} = 180 - (\text{A} + \text{B}) \qquad \text{C} = 74°\ 48'$$

(*b*) Let the two sides be 9 and 8, and the angle opposite to the smaller side be 56°.

To draw the triangle, let AD be a line of indefinite length, make AB = 9 and $\widehat{\text{BAD}} = 56°$. With B as centre and radius equal to 8 draw an arc of a circle cutting AD in the points C_1 and C. The triangles ABC and ABC_1 satisfy the given conditions.

Since the triangle BCC_1 is isosceles, the perpendicular BE bisects the base.

Let $C_1E = CE = y$, and $AE = x$.

$$\frac{h}{9} = \sin 56° = 0{\cdot}8290$$

$$h = 0{\cdot}8290 \times 9 = 7{\cdot}461$$

$$\frac{x}{9} = \cos 56° = 0{\cdot}5592$$

$$x = 0{\cdot}5592 \times 9 = 5{\cdot}033$$

$$y = \sqrt{8^2 - h^2} = \sqrt{8^2 - 7{\cdot}461^2}$$

$$= \sqrt{15{\cdot}46 \times 0{\cdot}539} = 2{\cdot}887$$

$$\sin \theta = \frac{h}{8} = \frac{7{\cdot}461}{8} = 0{\cdot}9326$$

$$\theta = 68° \ 51'$$

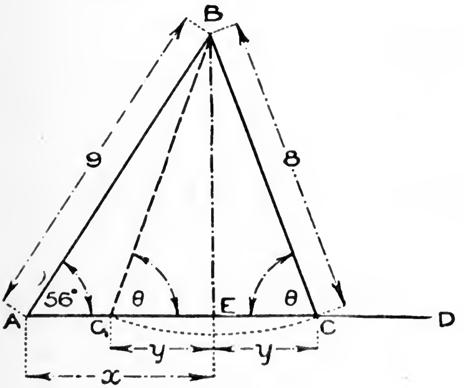


FIG. 4.

For the triangle ABC $\qquad AC = x + y = 7{\cdot}920$

$$C = \theta = 68° \ 51'$$

$$B = 180° - (A + C) = 55° \ 9'$$

For the triangle ABC_1 $\qquad AC_1 = x - y = 2{\cdot}146$

$$C_1 = 180° - \theta = 111° \ 9'$$

$$B = 180° - (A + C_1) = 12° \ 51'$$

It must be noticed that the nature of the triangle drawn to fulfil these conditions depends upon the relation which the smaller side bears to the height of the triangle. For if $BC > h$ the arc of the circle cuts the line AD in two points, thus giving rise to two triangles.

If $BC = h$ the arc touches the line AD, and a right-angled triangle results.

If BC $< h$ the arc does not cut the line at all, and it is impossible to draw the triangle.

Working the question in the above manner enables us to decide upon the particular form the question takes, for as we begin by finding the value of h, it is a simple matter to compare that value with the length of the smaller side.

(*c*) Let the two sides be 17 and 20 and the angle opposite to the larger side be 38°.

Make AB = 17 and $\widehat{BAD} = 38°$. With B as centre and radius equal to 20 draw an arc of a circle cutting AD in the points C_1

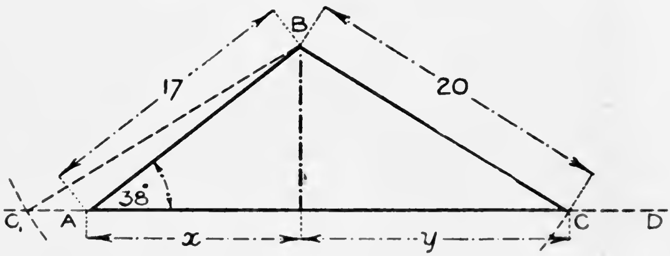


FIG. 5.

The triangle BAC_1 does not satisfy the given conditions, since the angle BAC_1 is the supplement of 38°.

$$\frac{h}{17} = \sin 38° = 0{\cdot}6157$$

$$h = 0{\cdot}6157 \times 17 = 10{\cdot}47$$

$$\frac{x}{17} = \cos 38° = 0{\cdot}7880$$

$$x = 0{\cdot}7880 \times 17 = 13{\cdot}40$$

$$y = \sqrt{20^2 - h^2} = \sqrt{20^2 - 10{\cdot}47^2}$$

$$= \sqrt{30{\cdot}47 \times 9{\cdot}53} = 17{\cdot}04$$

$$\sin C = \frac{h}{20} = \frac{10{\cdot}47}{20} = 0{\cdot}5235$$

$$C = 31° \ 34'$$

$$B = 180° - (A + C) = 110° \ 26'$$

$$AC = x + y = 30{\cdot}44$$

Case III. Let the base be 17 and the angles at the base 34° and 61°.

$$\text{Then} \quad \frac{h}{x} = \tan 34° = 0{\cdot}6745$$

$$h = 0{\cdot}6745x$$

$$\frac{h}{17 - x} = \tan 61° = 1{\cdot}804,$$

$$h = 30{\cdot}67 - 1{\cdot}804x$$

Then
$$0{\cdot}6745x = 30{\cdot}67 - 1{\cdot}804x$$
$$2{\cdot}479x = 30{\cdot}67$$
$$x = 12{\cdot}37$$

$$\frac{x}{\text{AB}} = \cos 34° = 0{\cdot}8290$$

$$\text{AB} = \frac{12{\cdot}37}{0{\cdot}8290} = 14{\cdot}92$$

$$\frac{17 - x}{\text{BC}} = \cos 61° = 0{\cdot}4848$$

$$\text{BC} = \frac{4{\cdot}63}{0{\cdot}4848} = 9{\cdot}550$$

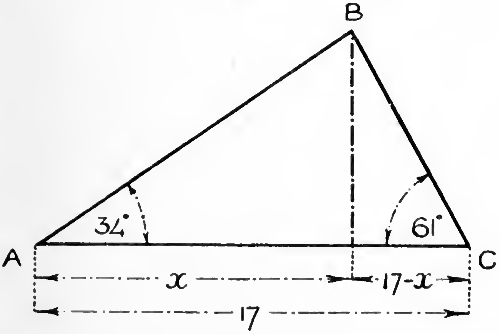


FIG. 6.

12. *Angles greater than* 90°. In order to find the trigonometrical ratios of a given acute angle, we make that angle the base angle of a right-angled triangle; but when we have to deal with an angle greater than 90°, we cannot treat it in the same way, since the base angle of a right-angled triangle must be acute.

If we look upon an angle as being measured by the amount of rotation of a line with reference to a fixed line, then we can consider one arm of an angle to occupy a fixed position, while the other arm can be taken as a movable arm rotating with reference to the fixed arm. If we take a point on the movable arm and from that point draw a perpendicular to the fixed arm, we shall obtain a right-angled triangle which in some way will enable us to determine the trigonometrical ratios of the angle between the two arms.

For angles between 0° and 360° there are four cases to be considered, depending upon the position of the moving arm.

Case I. When the angle lies between 0° and 90°.
Case II. When the angle lies between 90° and 180°.
Case III. When the angle lies between 180° and 270°.
Case IV. When the angle lies between 270° and 360°.

Let **P** (Fig. 7) be a point taken on the moving arm such that **OP** is the same length in each case. From **P** draw a perpendicular to the fixed arm. In Cases II and III this perpendicular meets the fixed arm produced. The triangle **OPR** will give us the trigonometrical ratios of the angle in each case.

If in each case the angle **POR** is made equal to α, then the

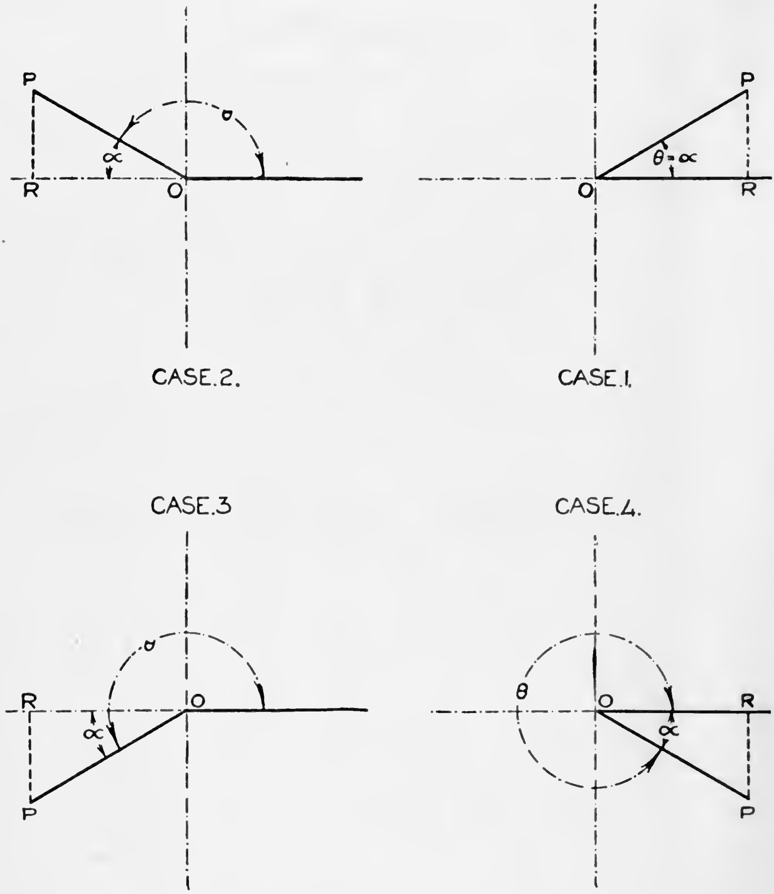


FIG. 7.

right-angled triangle **OPR** is equal in all respects, for four distinct positions of the moving arm **OP**. Thus the angles α, $180° - \alpha$, $180° + \alpha$, and $360° - \alpha$ have the common property that their trigonometrical ratios are all derived from the same right-angled triangle, and therefore the sines of these angles will all be numerically equal. This also holds for the other trigonometrical ratios.

If we take $\alpha = 50°$, then $\sin\alpha = 0{\cdot}7660$; but the sines of the angles 130°, 230°, and 310° will all have this same numerical value.

It must be noticed, however, that in each case the position of the right-angled triangle is different, and therefore we have to make allowance for that difference in position.

If we take a reference circle and draw two diameters inclined at an angle α to the horizontal diameter, then we can show the triangle POR placed in its different positions with reference to the circle; obviously there is one triangle in each quadrant. Let the radius of the circle always be positive. Taking the

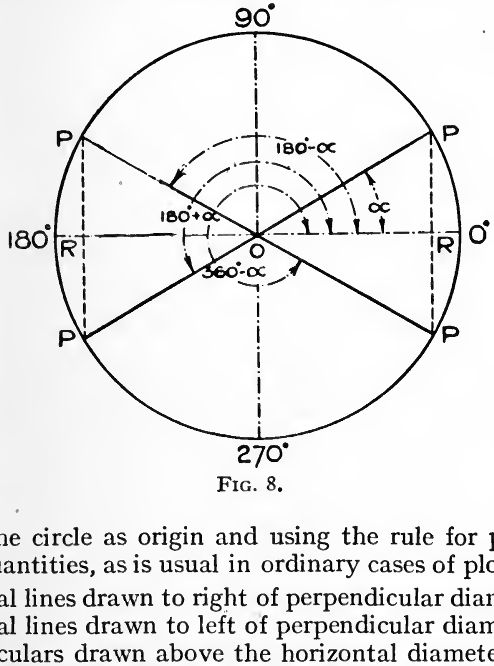


FIG. 8.

centre of the circle as origin and using the rule for positive and negative quantities, as is usual in ordinary cases of plotting, then:

Horizontal lines drawn to right of perpendicular diameter are +.
Horizontal lines drawn to left of perpendicular diameter are −.
Perpendiculars drawn above the horizontal diameter are +.
Perpendiculars drawn below the horizontal diameter are −.

We can now assign to the perpendicular and base of the right-angled triangle POR the algebraic sign according to the position with respect to the horizontal and vertical diameters.

Tabulating the results:

		α	$180° - \alpha$	$180° + \alpha$	$360° - \alpha$
Hypotenuse	OP	+	+	+	+
Perpendicular	PR	+	+	−	−
Base	OR	+	−	−	+
Sine		+	+	−	−
Cosine		+	−	−	+
Tangent		+	−	+	−

If an angle is such that the movable arm takes up a position in a certain quadrant, then the trigonometrical ratios of that angle must be given the algebraical signs peculiar to that quadrant.

To find sin 123° 32′, cos 209° 19′, and tan 324° 43′.

(*a*) 123° 32′ is an angle in the second quadrant.

$$\text{Then } \sin 123^\circ\, 32' \text{ is } +$$
$$\text{But } 123^\circ\, 32' = 180^\circ - \alpha$$
$$\alpha = 56^\circ\, 28'$$
$$\text{Hence } \sin 123^\circ\, 32' = +\sin 56^\circ\, 28' = +\,0{\cdot}8335$$

(*b*) 209° 19′ is an angle in the third quadrant.

$$\text{Then } \cos 209^\circ\, 19' \text{ is } -$$
$$\text{But } 209^\circ\, 19' = 180^\circ + \alpha$$
$$\alpha = 29^\circ\, 19'$$
$$\text{Hence } \cos 209^\circ\, 19' = -\cos 29^\circ\, 19' = -\,0{\cdot}8720$$

(*c*) 324° 43′ is an angle in the fourth quadrant.

$$\text{Then } \tan 324^\circ\, 43' \text{ is } -$$
$$\text{But } 324^\circ\, 43' = 360^\circ - \alpha$$
$$\alpha = 35^\circ\, 17'$$
$$\text{Hence } \tan 324^\circ\, 43' = -\tan 35^\circ\, 17' = -\,0{\cdot}7072$$

13. *Angles greater than* 360°. The rotation of the moving arm with reference to the fixed arm is not limited to one complete revolution: each revolution increases the magnitude of the angle by 360°: but for a certain angle the moving arm is sure to take up a position in one of the four quadrants. Thus an angle greater than 360° can be taken to be made up of two parts: an exact multiple of 360°, and an angle between 0° and 360°; and it is the second part which must be used in order to determine the trigonometrical ratios.

To find cos 829°.

$$829^\circ = 2 \times 360^\circ + 109^\circ$$
$$\text{Then } \cos 829^\circ = \cos 109^\circ$$
$$\text{But } 109^\circ \text{ is in the second quadrant}$$
$$\text{Then } \cos 109^\circ \text{ is } -$$
$$\text{But } 109^\circ = 180^\circ - \alpha$$
$$\alpha = 71^\circ$$
$$\text{Then } \cos 829^\circ = \cos 109^\circ = -\cos 71^\circ = -\,0{\cdot}3256$$

14. *Negative Angles*. Up to now we have taken the moving arm as rotating in anti-clockwise direction, but that arm can also rotate in the opposite direction—that is, in clockwise direction. In order to distinguish between these two directions we take an angle measured in clockwise direction as negative. Now an angle of $(-\alpha)$ represents an angle α measured in clockwise direction, and the moving arm takes up a certain position in one of the four quadrants; but that same position could be obtained

by measuring an angle $360° - \alpha$ or $n \times 360° - \alpha$ in anti-clockwise direction.

$$\text{Thus } -121° = 360° - 212° = 148°$$
$$-589° = 720° - 589° = 131°$$
$$-872° = 1080° - 872° = 108°$$

and the trigonometrical ratios of the corresponding positive angles can be found in the usual way.

15. *The Area of a Triangle.* If h and h_1 are the perpendiculars drawn from A and C to the opposite sides respectively—

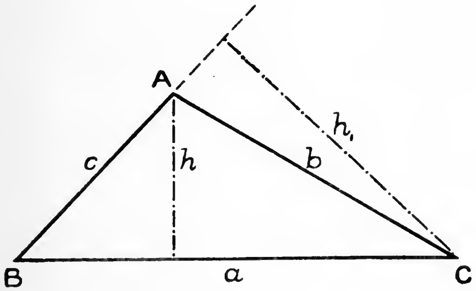


Fig. 9.

$$\text{Then area } = \frac{1}{2}ah \text{ or } \frac{1}{2}ch_1$$

$$\text{but} \quad \frac{h}{c} = \sin B \text{ or } h = c \sin B$$

$$\text{and} \quad \frac{h}{b} = \sin C \text{ or } h = b \sin C$$

$$\text{also} \quad \frac{h_1}{b} = \sin (180° - A) = \sin A \text{ or } h_1 = b \sin A$$

$$\text{Hence area } = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A$$

Putting these relations in words, the area of a triangle is half the product of two sides and the sine of the included angle.

Let h be the perpendicular drawn from A, and x and $(a - x)$ the segments into which the foot of the perpendicular divides the base.

$$\text{Then} \quad h^2 = c^2 - x^2$$
$$\text{and} \quad h^2 = b^2 - (a - x)^2$$
$$b^2 - a^2 + 2ax - x^2 = c^2 - x^2$$
$$2ax = a^2 + c^2 - b^2$$
$$x = \frac{a^2 + c^2 - b^2}{2a}$$

But $\quad h^2 = c^2 - x^2 = (c + x)(c - x)$

$$h = \sqrt{\left(\frac{2ac + a^2 + c^2 - b^2}{2a}\right)\left(\frac{2ac - a^2 - c^2 + b^2}{2a}\right)}$$

$$= \frac{1}{2a}\sqrt{\{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\}}$$

$$= \frac{1}{2a}\sqrt{(a + b + c)(a - b + c)(a + b - c)(b - a + c)}$$

$$= \frac{1}{2a}\sqrt{16s(s - a)(s - b)(s - c)}$$

where $\quad 2s = a + b + c$ the perimeter of the triangle

Then $\quad h = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}$

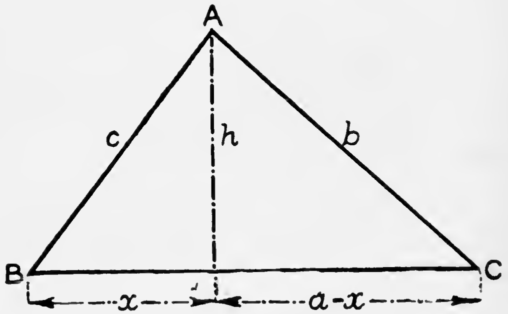


FIG. 10.

This is the perpendicular drawn to the side a. It can be easily shown that the other two perpendiculars are

$$h_b = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$$

$$h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

The area of the triangle $= \frac{1}{2}ah$

$$= \sqrt{s(s - a)(s - b)(s - c)}$$

16. *The Compound Angle.* It is sometimes necessary to express the trigonometrical ratios of the sum or difference of two given angles in terms of the trigonometrical ratios of those angles.

Let $\widehat{POM} = A$ and $\widehat{POR} = B$, the angles being measured in anti-clockwise direction in the first case, thus producing the compound angle ROM = A + B, and in the second case B is measured in clockwise direction, thus producing the compound angle ROM = A – B.

P is a point taken on the line of separation of the angles A and B, and PR is the line drawn perpendicular to OP, meeting the arm of the angle (A + B) or (A – B) in R.

PN and RM are perpendiculars drawn to OX from P and R respectively, while PQ is drawn through P parallel to OX, meeting RM, or RM produced in Q.

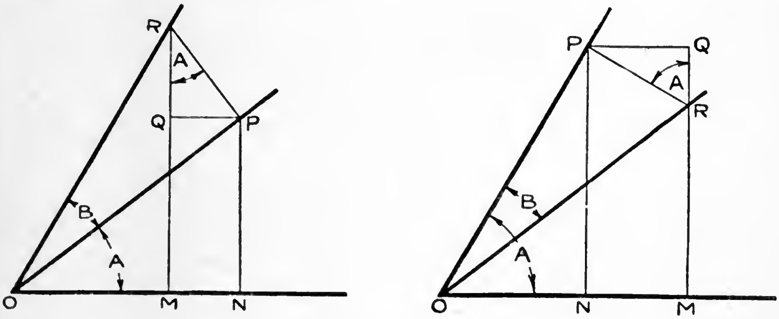


FIG. 11.

$$\text{Then } \sin(A+B) = \frac{RM}{OR} = \frac{RQ+QM}{OR} = \frac{RQ+PN}{OR}$$

$$= \frac{RQ}{OR} + \frac{PN}{OR} = \frac{RQ}{RP}\cdot\frac{RP}{OR} + \frac{PN}{OP}\cdot\frac{OP}{OR}$$

$$= \cos A \cdot \sin B + \sin A \cdot \cos B$$

$$\cos(A+B) = \frac{OM}{OR} = \frac{ON-MN}{OR} = \frac{ON-PQ}{OR}$$

$$= \frac{ON}{OR} - \frac{PQ}{OR} = \frac{ON}{OP}\cdot\frac{OP}{OR} - \frac{PQ}{RP}\cdot\frac{RP}{OR}$$

$$= \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\text{Also } \sin(A-B) = \frac{RM}{OR} = \frac{MQ-QR}{OR} = \frac{PN-QR}{OR}$$

$$= \frac{PN}{OR} - \frac{QR}{OR} = \frac{PN}{OP}\cdot\frac{OP}{OR} - \frac{QR}{RP}\cdot\frac{RP}{OR}$$

$$= \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$\cos(A-B) = \frac{OM}{OR} = \frac{ON+NM}{OR} = \frac{ON+PQ}{OR}$$

$$= \frac{ON}{OR} + \frac{PQ}{OR} = \frac{ON}{OP}\cdot\frac{OP}{OR} + \frac{PQ}{RP}\cdot\frac{RP}{OR}$$

$$= \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$\text{Tan}\,(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cdot\cos B+\cos A\cdot\sin B}{\cos A\cdot\cos B-\sin A\cdot\sin B}$$

$$=\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}$$ on dividing numerator and denominator by $\cos A \cdot \cos B$.

Also $$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$

$$=\frac{\sin A\cdot\cos B-\cos A\cdot\sin B}{\cos A\cdot\cos B+\sin A\cdot\sin B}$$

$$=\frac{\tan A-\tan B}{1+\tan A\cdot\tan B}$$ on dividing numerator and denominator by $\cos A \cdot \cos B$.

These relations for the trigonometrical ratios of the angles $(A + B)$ and $(A - B)$ are of utmost importance, and they should be treated as fundamental relations, since so much of the higher work in Trigonometry depends upon them.

17. *The expression* a *sin* θ + b *cos* θ. By comparing the expression $a \sin\theta + b\cos\theta$ with the relations

$$\sin(A+B)=\sin A\cdot\cos B+\cos A\cdot\sin B$$
$$\cos(A+B=\cos A\cdot\cos B-\sin A\cdot\sin B$$

we are enabled to put it as either a sine function or a cosine function, according as to whether the algebraic signs of a and b are alike or unlike.

(1) When the signs are alike

$$a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\left\{\sin\theta\cdot\frac{a}{\sqrt{a^2+b^2}}+\cos\theta\cdot\frac{b}{\sqrt{a^2+b^2}}\right\}$$

This converts the quantities a and b into fractions which are trigonometrical ratios of the angle β, the base angle of a right-angled triangle whose perpendicular is b and whose base is a. (Fig. 12.)

The expression thus becomes

$$\sqrt{a^2+b^2}\{\sin\theta\cdot\cos\beta+\cos\theta\cdot\sin\beta\}$$

and finally, $$\sqrt{a^2+b^2}\cdot\sin(\theta+\beta)\text{ where }\tan\beta=\frac{b}{a}$$

If both the signs are negative, then

$$-a\sin\theta-b\cos\theta=-(a\sin\theta+b\cos\theta)$$
$$=-\sqrt{a^2+b^2}\cdot\sin(\theta+\beta)\text{ where }\tan\beta=\frac{b}{a}$$

(2) When the signs are unlike

$$b\cos\theta-a\sin\theta=\sqrt{a^2+b^2}\left\{\cos\theta\cdot\frac{b}{\sqrt{a^2+b^2}}-\sin\theta\cdot\frac{a}{\sqrt{a^2+b^2}}\right\}$$

This converts the quantities a and b into fractions which are trigonometrical ratios of the angle α, the base angle of a right-angled triangle whose perpendicular is a and whose base is b. (Fig. 13.)

The expression thus becomes

$$\sqrt{a^2 + b^2} \cdot \{\cos \theta \cdot \cos \alpha - \sin \theta \,.\, \sin \alpha\}$$

and finally, $\sqrt{a^2 + b^2} \cdot \cos (\theta + \alpha)$ where $\tan \alpha = \dfrac{a}{b}$

If b is negative and a is positive, then

$$a \sin \theta - b \cos \theta = -(b \cos \theta - a \sin \theta)$$

$$= -\sqrt{a^2 + b^2} \cdot \cos (\theta + \alpha) \text{ where } \tan \alpha = \frac{a}{b}$$

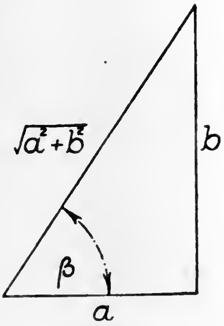


FIG. 12.

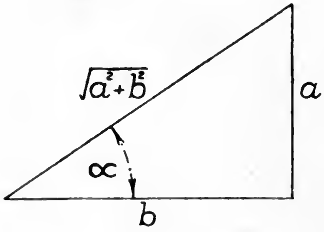


FIG. 13.

These results may be summarised as follows :

$$\pm (b \cos \theta + a \sin \theta) = \pm \sqrt{a^2 + b^2} \sin (\theta + \beta) \text{ where } \tan \beta = \frac{b}{a}$$

$$\pm (b \cos \theta - a \sin \theta) = \pm \sqrt{a^2 + b^2} \cos (\theta + \alpha) \text{ where } \tan \alpha = \frac{a}{b}$$

18. By taking the relation $y = a \sin \theta + b \cos \theta$ and putting it in the form $y = \sqrt{a^2 + b^2} \sin (\theta + \beta)$, where $\tan \beta = \dfrac{b}{a}$, we are enabled to very easily ascertain some of the impor ant features of the function. Since the sine of an angle is never greater than 1 and never less than -1, y will be greatest when $\sin (\theta + \beta) = 1$ —that is, when $\theta + \beta$ has the values $\dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$, etc., or when θ has the values $\dfrac{\pi}{2} - \beta, \dfrac{5\pi}{2} - \beta, \dfrac{9\pi}{2} - \beta$, etc.

Thus the greatest value of y is $\sqrt{a^2 + b^2}$, and this occurs when θ has the values $\dfrac{\pi}{2} - \beta, \dfrac{5\pi}{2} - \beta$, etc., at intervals of 2π.

Similarly the least value is $-\sqrt{a^2+b^2}$, and this occurs when θ has the values $\frac{3\pi}{2}-\beta$, $\frac{7\pi}{2}-\beta$, etc., at intervals of 2π.

Also y vanishes when $\sin(\theta+\beta)=0$.

That is, when $\theta+\beta$ has the values 0, π, 2π, etc., or when θ has the values $-\beta$, $\pi-\beta$, $2\pi-\beta$, etc., at intervals of π.

This method of treatment can be applied to the other three cases.

19. *The equation* a *sin* $\theta+$ b *cos* $\theta=\pm c$.

Dividing throughout by $\sqrt{a^2+b^2}$, the equation becomes

$$\sin\theta\cdot\frac{a}{\sqrt{a^2+b^2}}+\cos\theta\cdot\frac{b}{\sqrt{a^2+b^2}}=\pm\frac{c}{\sqrt{a^2+b^2}}$$

$$\text{or }\sin(\theta+\beta)=\pm\frac{c}{\sqrt{a^2+b^2}}\quad\ldots\ldots\quad(1)$$

$$\text{and }\tan\beta=\frac{b}{a}\quad\ldots\ldots\quad(2)$$

(1) gives the values of the angle $\theta+\beta$, (2) gives the value of the angle β, and the values of θ can be found by subtraction.

When the fraction $\frac{c}{\sqrt{a^2+b^2}}$ is positive, the angle $\theta+\beta$ has two values, one between $0°$ and $90°$ and the other between $90°$ and $180°$. When the fraction is negative, the angle $\theta+\beta$ has two values, one between $180°$ and $270°$ and the other between $270°$ and $360°$.

Solve the equation $8\sin\theta+11\cos\theta=-12$.

$$\text{Then }\sin\theta\cdot\frac{8}{\sqrt{185}}+\cos\theta\cdot\frac{11}{\sqrt{185}}=-\frac{12}{\sqrt{185}}$$

$$\begin{aligned}\sin\theta\,.\,\cos\beta+\cos\theta\,.\,\sin\beta&=-0{\cdot}8822\\ \sin(\theta+\beta)&=-0{\cdot}8822\\ \tan\beta&=\frac{11}{8}=1{\cdot}3750\\ \theta+\beta&=241°\ 55'\text{ or }298°\ 5'\\ \beta&=53°\ 59'\\ \theta&=187°\ 56'\text{ or }244°\ 6'\end{aligned}$$

The equation $b\cos\theta-a\sin\theta=\pm c$.

Dividing throughout by $\sqrt{a^2+b^2}$, the equation becomes

$$\cos\theta\,.\,\frac{b}{\sqrt{a^2+b^2}}-\sin\theta\,.\,\frac{a}{\sqrt{a^2+b^2}}=\pm\frac{c}{\sqrt{a^2+b^2}}$$

$$\text{or }\cos(\theta+\alpha)=\pm\frac{c}{\sqrt{a^2+b^2}}\quad\ldots\quad(1)$$

$$\text{and }\tan\alpha=\frac{a}{b}\quad\ldots\ldots\quad(2)$$

(1) gives the values of the angle $\theta+\alpha$.
(2) gives the value of the angle α.

When the fraction $\frac{c}{\sqrt{a^2 + b^2}}$ is positive, the angle $\theta + \alpha$ has two values, one between 0° and 90° and the other between 270° and 360°. When the fraction is negative, the angle $\theta + \alpha$ has two values, one between 90° and 180° and the other between 180° and 270°.

Solve the equation $9 \cos \theta - 14 \sin \theta = -15$.

$$\text{Then } \cos \theta \cdot \frac{9}{\sqrt{277}} - \sin \theta \cdot \frac{14}{\sqrt{277}} = -\frac{15}{\sqrt{277}}$$

$$\cos \theta \, . \cos \alpha - \sin \theta \, . \sin \alpha = -0{\cdot}9014$$

$$\cos (\theta + \alpha) = -0{\cdot}9014$$

$$\tan \alpha = \frac{14}{9} = 1{\cdot}5555$$

$$\theta + \alpha = 154° \, 20' \text{ or } 205° \, 40'$$

$$\alpha = 57° \, 16'$$

$$\theta = 97° \, 4' \text{ or } 148° \, 24'$$

It should be noticed that the nature of the fraction $\frac{c}{\sqrt{a^2 + b^2}}$ decides upon the possibilities of the equation, for since the value of the fraction represents the sine or cosine of an angle, when $c > \sqrt{a^2 + b^2}$, the fraction is greater than 1 and there is no solution to the equation, since the sine or cosine can never be greater than 1 or less than -1.

If $c < \sqrt{a^2 + b^2}$ the fraction is less than 1 and the equation has two roots.

If $c = \sqrt{a^2 + b^2}$ the fraction is equal to 1 and the equation has one root, for

(1) $\text{Sin } (\theta + \beta) = 1$ and $\theta = 90° - \beta$
(2) $\text{Sin } (\theta + \beta) = -1$ and $\theta = 270° - \beta$
(3) $\text{Cos } (\theta + \alpha) = 1$ and $\theta = 360° - \alpha$
(4) $\text{Cos } (\theta + \alpha) = -1$ and $\theta = 180° - \alpha$

When $c = 0$ the equation can be solved in a much simpler way, for if $a \sin \theta \pm b \cos \theta = 0$

then $a \sin \theta = \mp b \cos \theta$

and $\tan \theta = \mp \frac{b}{a}$, giving the values of θ at once.

20. *The Multiple Angles.* The relations for sin (A + B) and cos (A + B) can be used for expressing the trigonometrical ratios of multiple angles of a given angle in terms of the trigonometrical ratios of that angle.

$$\sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

When $A = B$ $\quad \sin 2A = 2 \sin A \cdot \cos A$

$$\cos (A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A \end{aligned}$$

$$\begin{aligned} \tan 2A &= \frac{2 \sin A \cdot \cos A}{\cos^2 A - \sin^2 A} \\ &= \frac{2 \tan A}{1 - \tan^2 A} \end{aligned}$$

Also $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

and $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

Putting $A = \frac{\theta}{2}$ $\quad \sin \frac{\theta}{2} = \sqrt{\frac{1}{2}(1 - \cos \theta)}$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1}{2}(1 + \cos \theta)}$$

$$\begin{aligned} \tan \frac{\theta}{2} &= \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \\ &= \sqrt{\frac{(1 - \cos \theta)^2}{1 - \cos^2 \theta}} \\ &= \frac{1 - \cos \theta}{\sin \theta} \end{aligned}$$

$$\begin{aligned} \sin 3A = \sin (2A + A) &= \sin 2A \cdot \cos A + \cos 2A \cdot \sin A \\ &= 2 \sin A \cdot \cos^2 A + \sin A (1 - 2 \sin^2 A) \\ &= \sin A \{2(1 - \sin^2 A) + 1 - 2 \sin^2 A\} \\ &= \sin A (3 - 4 \sin^2 A) \end{aligned}$$

The relations for cos 3A, sin 4A, etc., can be obtained in a similar manner, and they can be well left as exercises for the student.

Example. The general term of the series for $e^{ax} \sin bx$ is $(a^2 + b^2)^{\frac{n}{2}} \frac{x^n}{\underline{|n}} \sin n\alpha$ where $\tan \alpha = \frac{b}{a}\cdot$ Taking $a = 2$, $b = 1$, find the first five terms of the series.

Then $e^{ax} \sin bx = (a^2 + b^2)^{\frac{1}{2}} x \sin \alpha + (a^2 + b^2) \frac{x^2}{\underline{|2}} \sin 2\alpha + \ldots$

and $\quad e^{2x} \cdot \sin x = x\sqrt{5} \sin \alpha + \frac{x^2}{\underline{|2}} 5 \sin 2\alpha + \frac{x^3}{\underline{|3}} 5 \sqrt{5} \sin 3\alpha \ldots$

Thus to obtain the first five terms we have to find the values of $\sin \alpha$, $\sin 2\alpha$, $\sin 3\alpha$, $\sin 4\alpha$, and $\sin 5\alpha$, knowing that $\tan \alpha = \frac{1}{2}$.

If $\tan \alpha = \frac{1}{2}$, then $\sin \alpha = \frac{1}{\sqrt{5}}$ and $\cos \alpha = \frac{2}{\sqrt{5}}$

$$\sin 2\alpha = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}, \quad \cos 2\alpha = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$$

$$\sin 3\alpha = \frac{4}{5} \cdot \frac{2}{\sqrt{5}} + \frac{3}{5} \cdot \frac{1}{\sqrt{5}} = \frac{11}{5\sqrt{5}}, \ \cos 3\alpha = \frac{3}{5} \cdot \frac{2}{\sqrt{5}} - \frac{4}{5} \cdot \frac{1}{\sqrt{5}} = \frac{2}{5\sqrt{5}}$$

$$\sin 4\alpha = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}, \ \cos 4\alpha = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

$$\sin 5\alpha = \frac{24}{25} \cdot \frac{2}{\sqrt{5}} - \frac{7}{25} \cdot \frac{1}{\sqrt{5}} = \frac{41}{25\sqrt{5}},$$

$$\cos 5\alpha = -\frac{7}{25} \cdot \frac{2}{\sqrt{5}} - \frac{24}{25} \cdot \frac{1}{\sqrt{5}} = -\frac{38}{25\sqrt{5}}$$

$$\text{Then } e^{2x} \cdot \sin x = x\sqrt{5} \cdot \frac{1}{\sqrt{5}} + \frac{x^2}{\underline{|2}} \cdot 5 \cdot \frac{4}{5} + \frac{x^3}{\underline{|3}} \cdot 5\sqrt{5} \cdot \frac{11}{5\sqrt{5}} + \frac{x^4}{\underline{|4}} \cdot 25 \cdot \frac{24}{25} + \ldots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5 + \ldots$$

21. Now

$$\sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B \ldots (1)$$
$$\sin (A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B \ldots (2)$$
$$\cos (A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B \ldots (3)$$
$$\cos (A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B \ldots (4)$$

Adding (1) and (2)

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cdot \cos B \ldots (5)$$

Subtracting (2) from (1)

$$\sin (A + B) - \sin (A - B) = 2 \cos A \cdot \sin B \ldots (6)$$

Adding (3) and (4)

$$\text{os } (A + B) + \cos (A - B) = 2 \cos A \cdot \cos B \ldots (7)$$

Subtracting (3) from (4)

$$\cos (A - B) - \cos (A + B) = 2 \sin A \cdot \sin B \ldots (8)$$

In these relations A is taken as being greater than B.

The relations (5), (6), (7), and (8) can be used in two different ways:

(*a*) To express the sums or differences of sines or cosines as the products of sines and cosines.

(*b*) To express the products of sines and cosines as the sums or differences of sines or cosines.

(*a*) Comparing $\sin (x + h) - \sin x$ with relation (6).

Then $A + B = x + h$ and $A - B = x$

Hence $A = x + \frac{h}{2}$ and $B = \frac{h}{2}$

and $\sin (x + h) - \sin x = 2 \cos \left(x + \frac{h}{2}\right) \cdot \sin \frac{h}{2}$

Using relation (8), $\cos(x+h) - \cos x = -2\sin\left(x+\frac{h}{2}\right)\cdot\sin\frac{h}{2}$

These results will be used in the differentiation, from first principles, of sine and cosine functions.

(*b*) If $\quad A = pt$ and $B = pt - c$

Then $\quad A + B = 2pt - c$ and $A - B = c$

$$\text{From (5)}\ \sin pt\cdot\cos(pt-c) = \frac{1}{2}\{\sin(2pt-c)+\sin c\}$$

$$\text{From (6)}\ \cos pt\cdot\sin(pt-c) = \frac{1}{2}\{\sin(2pt-c)-\sin c\}$$

$$\text{From (7)}\ \cos pt\cdot\cos(pt-c) = \frac{1}{2}\{\cos(2pt-c)+\cos c\}$$

$$\text{From (8)}\ \sin pt\cdot\sin(pt-c) = \frac{1}{2}\{\cos c-\cos(2pt-c)\}$$

This method of transformation, and these results, will be found to be very useful in our subsequent work on Fourier's Series, and also to find the mean values of periodic functions.

Examples II

Solve the following triangles:

(1) (*a*) 3 sides 18, 14, 9.
(*b*) 3 sides 7·36, 5·72, 3·84.

(2) (*a*) Sides 22 and 31, included angle 62°.
(*b*) Sides 5·16 and 3·96, included angle 55°.

(3) (*a*) Sides 9 and 8, angle 56° opposite to the smaller side.
(*b*) Sides 3·72 and 2·25, angle 32° opposite to the smaller side.

(4) (*a*) Sides 17 and 20, angle 38° opposite to the larger side.
(*b*) Sides 3·92 and 5·72, angle 44° opposite to the larger side.

(5) (*a*) Base 17, angles at the base 29° and 44°.
(*b*) Base 2·96, angles at the base 34° and 61°.

(6) The sides of a triangle are 5·6, 4·4, and 2·8. Find the area, the angles, and the lengths of the perpendiculars.

(7) ABC is a triangle, right-angled at C; the angle ABC is 75°; the side AC is divided into four equal parts by points D, E, and F. Find the angles DBC, EBC, and FBC.

(8) ABCD is a quadrilateral. AB = 1·8″, BD = 2·4″, DC = 3·4″, DA = 2·6″, and BC = 3·2″. Find the area, the angles, and the length of the diagonal AD.

(9) Write down the values of sin 215°, cos 93°, tan 321°, cos 236°, sin 112°, tan 184°, sin 527°, cos 412°, tan 729°, sin (– 312°), cos (– 196°), and tan (– 521°).

(10) Write down the values of sin 101° 20′, cos 198° 32′, tan 278° 43′, sin 386° 22′, cos 557° 53′, tan 785° 39′, sin (– 221° 17′), cos (– 86° 19′), and tan (– 394° 49′).

(11) If $x = r(1 - \cos\theta) + \frac{r^2}{4l}(1 - \cos 2\theta)$ and $r = 1$, $l = 5$. Calculate the values of x when θ has the values 0, 30°, 60°, 90°, 120°, 150°, and 180°.

(12) If $y = e^{a \sin\theta}$ and $a = 2$, calculate the values of y when θ has the values $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, 0, -\frac{\pi}{6}, -\frac{\pi}{3}, -\frac{\pi}{2}$. Plot y and θ on squared paper, and use your graph to solve the equation $e^{2 \sin\theta} = 2$. Verify your result by calculation.

(13) Put $9 \sin\theta + 13 \cos\theta$ in the form $A \sin(\theta + \alpha)$, giving the values of A and α. From the result find the value of θ which causes the expression to vanish.

(14) Put $21 \cos\theta - 16 \sin\theta$ in the form $A \cos(\theta + \alpha)$, giving the values of A and α. From the result find the greatest and least values of the expression and the values of θ producing them.

Solve the equations :

(15) $3 \sin\theta + 7 \cos\theta = 7{\cdot}5$.
(16) $8 \sin\theta + 12 \cos\theta = -14{\cdot}1$.
(17) $13 \cos\theta - 8 \sin\theta = 15$.
(18) $11 \cos\theta - 15 \sin\theta = -18$.
(19) $35 \cos\theta - 12 \sin\theta = -37$.
(20) $5 \sin\theta + 12 \cos\theta = 13$.
(21) $8 \sin\theta + 11 \cos\theta = 0$.

(22) If $\tan A = \frac{1}{2}$, without using the tables find the trigonometrical ratios of the angles $\frac{A}{2}$, 2A, and 3A.

(23) If $x = \tan\frac{\theta}{2}$, show that $a \sin\theta + b \cos\theta = c$ can be put in the form $2ax + b(1 - x^2) = c(1 + x^2)$, and hence solve the equation $9 \sin\theta + 13 \cos\theta = 15$.

(24) From a circular disc of metal 8″ radius, a sector whose angle is 54° is cut away, and the remainder is formed into a conical vessel. Find the volume of that vessel.

Solve the equations, keeping r always positive :

(25) $r \cos\theta = 6$, $r \sin\theta = 11$.
(26) $r \cos\theta = -5$, $r \sin\theta = 13$.
(27) $r \cos\theta = -10$, $r \sin\theta = -15$.
(28) $r \cos\theta = 12$, $r \sin\theta = -7$.

CHAPTER III

22. The Complex Quantity. The quantity $\sqrt{-9}$ can be taken as $\sqrt{9 \times -1}$, which reduces to $3\sqrt{-1}$. Treating $\sqrt{-8}$ in the same way, it becomes $2{\cdot}828\sqrt{-1}$, the number being correct to four significant figures. Thus the square root of any negative quantity can be reduced to the form $b\sqrt{-1}$ or bi where b is a real number which can be exact, or given correct to as many significant figures as desired. The result of multiplying a real quantity by i or $\sqrt{-1}$ is to make the product imaginary.

Many quadratic equations are spoken of as having imaginary roots, but it is only in a few special cases for which the roots are wholly imaginary.

Taking the quadratic equation $x^2 - 16x + 100 = 0$

$$\text{Then } x^2 - 16x + 64 = -100 + 64 = -36$$

$$x - 8 = \pm 6i$$

$$\text{and} \qquad x = 8 + 6i \text{ or } 8 - 6i$$

We can thus have quantities consisting of two distinct parts, a real part and an imaginary part. Such quantities are spoken of as being complex.

A complex quantity can be expressed generally in the form $a + bi$, where a and b are numbers which can be exact or given correct to so many significant figures.

23. Two complex quantities can only be equal providing the real parts are equal and the imaginary parts are equal.

Thus if $a + bi = c + di$, then $a = c$ and $b = d$.

For, if not, suppose $a > c$, then $a = c + x$, where x is the difference between two real quantities and must therefore be real.

$$\text{Then } c + x + bi = c + di$$

$$x = di - bi$$

This makes x imaginary, because it is equal to the difference of two imaginary quantities, but x must be real, therefore a cannot be greater than c. In the same way it can be shown that a cannot be less than c. Hence a must be equal to c.

If $a = c$, then b must be equal to d.

24. *The Powers of* i. Positive Powers:

i	$= i$	$i^5 = i \times i^4 = i =$	i	$i^9 = i \times i^8 = i =$	i
i^2	$= -1$	$i^6 = i^2 \times i^4 = i^2 =$	-1	$i^{10} = i^2 \times i^8 = i^2 =$	-1
$i^3 = i \times i^2$	$= -i$	$i^7 = i^3 \times i^4 = i^3 =$	$-i$	$i^{11} = i^3 \times i^8 = i^3 =$	$-i$
$i^4 = (i^2)^2$	$= 1$	$i^8 = i^4 \times i^4 = i^4 =$	1	$i^{12} = i^4 \times i^8 = i^4 =$	1

Thus the first four powers of i give i, -1, $-i$, and 1; the next four powers give the same quantities in the same order, as also will the next four powers. It follows, then, that any positive integral power of i will give ± 1 when the power is even and $\pm i$ when the power is odd.

Negative Powers:

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i \qquad i^{-5} = i^{-1} \times i^{-4} = i^{-1} = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1 \qquad i^{-6} = i^{-2} \times i^{-4} = i^{-2} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = \frac{i}{-i^2} = i \qquad i^{-7} = i^{-3} \times i^{-4} = i^{-3} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1 \qquad i^{-8} = i^{-4} \times i^{-4} = i^{-4} = 1$$

Thus the negative integral powers of i will give ± 1 when the power is even and $\pm i$ when the power is odd. The complex quantity can be treated algebraically, provided the treatment is combined with a knowledge of the values of the different powers of i.

25. *Multiplication of Complex Quantities.* They can be multiplied algebraically and the value of i^2 put, where it occurs, in the result.

$$\begin{aligned} \text{Thus } (5-8i)(2+5i) &= 10 + 9i - 40i^2 \\ &= 50 + 9i \qquad \text{since } i^2 = -1 \\ \text{Also } (5-8i)(2+5i)(2-3i) &= (50+9i)(2-3i) \\ &= 100 - 132i - 27i^2 \\ &= 127 - 132i \end{aligned}$$

When two complex quantities are multiplied together, care should be taken to reduce the product to the form $a + bi$ before multiplying by a third complex quantity.

26. *Division of Complex Quantities.* If we consider the complex quantities $a + bi$ and $a - bi$, we notice that the product is $a^2 - b^2i^2$, which reduces to $a^2 + b^2$, and this provides us with a means of removing the imaginary term from a complex quantity.

Hence if we wish to divide $50 + 9i$ by $5 - 8i$ we can represent the process by the fraction $\frac{50+9i}{5-8i}$ and simplify the fraction. By multiplying numerator and denominator of the fraction by $5 + 8i$, we can make the denominator entirely real without altering the value of the fraction.

$$\begin{aligned} \text{Then } \frac{50+9i}{5-8i} \times \frac{5+8i}{5+8i} &= \frac{250 + 445i + 72i^2}{25 - 64i^2} \\ &= \frac{178 + 445i}{89} \\ &= 2 + 5i \end{aligned}$$

Example. Simplify $\dfrac{18+13i}{(2+3i)(5-6i)}$

$$\frac{18+13i}{(2+3i)(5-6i)} = \frac{18+13i}{10+3i-18i^2}$$
$$= \frac{18+13i}{28+3i}$$
$$= \frac{18+13i}{28+3i} \times \frac{28-3i}{28-3i}$$
$$= \frac{504+310i-39i^2}{784-9i^2}$$
$$= \frac{543+310i}{793}$$
$$= 0{\cdot}6847 + 0{\cdot}3909i$$

27. *Extraction of the Square Root of Complex Quantities.* This can be done by a purely algebraic process.

For if $\sqrt{a \pm bi} = x \pm yi$

Then $a \pm bi = x^2 \pm 2xyi + y^2i^2 = x^2 - y^2 \pm 2xyi$

Equating the real and imaginary parts, we get

$$x^2 - y^2 = a$$
$$2xy = b$$

a pair of simultaneous equations to be solved for x and y.

To find the square root of $21 - 16i$.

Then $\sqrt{21-16i} = x - yi$

and $21 - 16i = x^2 - y^2 - 2xyi$

Hence

$$\left.\begin{aligned} x^2 - y^2 &= 21 \\ 2xy &= 16 \end{aligned}\right\}$$
$$\left.\begin{aligned} x^4 - 2x^2y^2 + y^4 &= 441 \\ 4x^2y^2 &= 256 \end{aligned}\right\}$$
$$x^4 + 2x^2y^2 + y^4 = 697$$
$$\left.\begin{aligned} x^2 + y^2 &= 26{\cdot}40 \\ x^2 - y^2 &= 21 \end{aligned}\right\}$$
$$x^2 = 23{\cdot}70 \qquad x = 4{\cdot}868$$
$$y^2 = 2{\cdot}70 \qquad y = 1{\cdot}645$$

Then $\sqrt{21-16i} = 4{\cdot}868 - 1{\cdot}645i$

If, in the complex quantity of which we wish to extract the square root, the real part is negative, it is better to make it positive in the following manner :

$$\sqrt{-a \pm bi} = \sqrt{-1(a \mp bi)} = i\sqrt{a \mp bi}$$

and we can extract the square root of $a \mp bi$ and afterwards multiply the result by i.

Thus $\sqrt{-21+16i} = \sqrt{-1(21-16i)}$

$$= i\sqrt{21-16i}$$
$$= i\{4{\cdot}868 - 1{\cdot}645i\} \text{ from the previous example}$$
$$= 4{\cdot}868i - 1{\cdot}645i^2$$
$$= 1{\cdot}645 + 4{\cdot}868i$$

28. *The Trigonometrical Form of a Complex Quantity.* When we wish to raise a complex quantity to an integral or fractional power, the work is simpler and more definite if we transform it into its equivalent trigonometrical form.

Thus if $a + bi = r(\cos\theta + i\sin\theta)$

Equating the real and imaginary parts

$$\left.\begin{aligned} r\cos\theta &= a \\ r\sin\theta &= b \end{aligned}\right\}$$

Squaring and adding $\quad r^2(\cos^2\theta + \sin^2\theta) = r^2 = a^2 + b^2$

Dividing $\quad \tan\theta = \frac{b}{a}$

Hence $a + bi = r(\cos\theta + i\sin\theta)$ if $r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$.

It must be remembered that this transformation is to be done with the direct object of raising a complex quantity to a power or extracting the root of a complex quantity. Hence if we wished to find the nth root of $a + bi$, we should have to find the nth root of $r(\cos\theta + i\sin\theta)$, or the value of $r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$. Now if r is negative and n is even, we cannot find the value of $r^{\frac{1}{n}}$ without introducing a further imaginary quantity, but when n is odd we could extract the nth root. If r is positive, the value of $r^{\frac{1}{n}}$ is real whether n is odd or even. It is therefore advisable to perform the above transformation with the condition that r must always be positive, and so the value of the angle θ depends not only upon the values of a and b, but also upon the signs of a and b.

We will next consider the different cases depending upon the signs of a and b. Let A be the acute angle whose tangent is $\frac{b}{a}$.

Case I. When a and b are positive.

Then $r\sin\theta = b$ and $r\cos\theta = a$

The angle θ must be such that its sine and cosine are both positive: an angle between 0° and 90°. Thus $\theta = A$.

Therefore $a + bi = \sqrt{a^2+b^2}\,(\cos A + i\sin A)$ where $\tan A = \frac{b}{a}$.

Case II. When a is negative and b is positive.

Then $-a + bi = r(\cos\theta + i\sin\theta)$

and $r\cos\theta = -a$ and $r\sin\theta = b$

The angle θ must be such that its sine is positive and its cosine negative, an angle between 90° and 180°. Thus $\theta = 180° - A$.

Therefore $-a + bi = \sqrt{a^2 + b^2}\{\cos(180° - A) + i\sin(180° - A)\}$ where $\tan A = \frac{b}{a}$.

Case III. When a and b are both negative.

$$\text{Then } -a - bi = r(\cos\theta + i\sin\theta)$$
$$\text{and } r\cos\theta = -a \text{ and } r\sin\theta = -b$$

The angle θ must be such that its sine and cosine are both negative: an angle between 180° and 270°. Thus $\theta = 180° + A$.

Therefore $-a - bi = \sqrt{a^2 + b^2}\{\cos(180° + A) + i\sin(180° + A)\}$ where $\tan A = \frac{b}{a}$.

Case IV. When a is positive and b is negative.

$$\text{Then } a - bi = r(\cos\theta + i\sin\theta)$$
$$\text{and } r\cos\theta = a \text{ and } r\sin\theta = -b.$$

The angle θ must be such that its sine is negative and its cosine positive: an angle between 270° and 360°. Thus $\theta = 360° - A$.

Therefore $a - bi = \sqrt{a^2 + b^2}\{\cos(360° - A) + i\sin(360° - A)\}$ where $\tan A = \frac{b}{a}$.

Example 1. Express $-7 + 5i$ in the form $r(\cos\theta + i\sin\theta)$.

$$\text{Then } -7 + 5i = r(\cos\theta + i\sin\theta)$$
$$\text{and } r\cos\theta = -7,\ r\sin\theta = 5$$
$$r = \sqrt{49 + 25} = \sqrt{74} = 8{\cdot}602$$

Since $\sin\theta$ is $+$ and $\cos\theta$ is $-$

$$\text{Then } \theta = 180° - A, \text{ where } \tan A = \frac{5}{7} = 0{\cdot}7143$$
$$= 180° - 35°\ 32'$$
$$= 144°\ 28'$$
$$\text{and } -7 + 5i = 8{\cdot}602(\cos 144°\ 28' + i\sin 144°\ 28')$$

Example 2. Express $-8 - 11i$ in the form $r(\cos\theta + i\sin\theta)$.

$$\text{Then } -8 - 11i = r(\cos\theta + i\sin\theta)$$
$$\text{and } r\cos\theta = -8,\ r\sin\theta = -11$$
$$r = \sqrt{64 + 121} = \sqrt{185} = 13{\cdot}60$$

Since $\sin\theta$ is $-$ and $\cos\theta$ is $-$

$$\text{Then } \theta = 180° + A, \text{ where } \tan A = \frac{11}{8} = 1{\cdot}375$$
$$= 180° + 52°\ 59'$$
$$= 232°\ 59'$$
$$\text{and } -8 - 11i = 13{\cdot}60(\cos 232°\ 59' + i\sin 232°\ 59')$$

29. *Multiplication of Trigonometrical Complex Quantities.* In multiplication and division we are more concerned with the

behaviour of the portion $(\cos\theta + i\sin\theta)$, for r, being a positive number, is quite easily dealt with.

$$\begin{aligned}(\cos A + i\sin A)(\cos B + i\sin B) &= \cos A\cdot\cos B + i^2\sin A\cdot\sin B\\ &\quad + i(\sin A\cdot\cos B + \cos A\cdot\sin B)\\ &= (\cos A\cdot\cos B - \sin A\cdot\sin B)\\ &\quad + i(\sin A\cdot\cos B + \cos A\cdot\sin B)\\ &= \cos(A+B) + i\sin(A+B)\end{aligned}$$

Thus the product of two trigonometrical complex quantities gives a trigonometrical complex quantity, the angle of which is the *sum* of the two angles in the factors. This can be extended to the product of any number of factors.

For $(\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)$

$$\begin{aligned}&= \{\cos(A+B) + i\sin(A+B)\}(\cos C + i\sin C)\\ &= \cos(A+B+C) + i\sin(A+B+C)\end{aligned}$$

and in general $(\cos A + i\sin A)(\cos B + i\sin B)\ \ldots\ n$ factors

$$= \cos(A+B+\ldots\ n \text{ angles}) + i\sin(A+B+\ldots\ n \text{ angles}).$$

Division can be performed by representing the process by a fraction and then simplifying that fraction.

To divide $(\cos A + i\sin A)$ by $(\cos B + i\sin B)$ we must simplify the fraction $\dfrac{\cos A + i\sin A}{\cos B + i\sin B}$, and this can be done by multiplying numerator and denominator by $\cos B - i\sin B$.

Then $\dfrac{\cos A + i\sin A}{\cos B + i\sin B} = \dfrac{\cos A + i\sin A}{\cos B + i\sin B} \times \dfrac{\cos B - i\sin B}{\cos B - i\sin B}$

$$= \frac{\cos A\cdot\cos B - i^2\sin A\cdot\sin B + i\{\sin A\cdot\cos B - \cos A\cdot\sin B\}}{\cos^2 B - i^2\sin^2 B}$$

$$= \frac{(\cos A\cdot\cos B + \sin A\cdot\sin B) + i(\sin A\cdot\cos B - \cos A\cdot\sin B)}{\cos^2 B + \sin^2 B}$$

$$= \cos(A-B) + i\sin(A-B)$$

Thus division or the simplification of a fraction gives a trigonometrical complex quantity whose angle is the angle in the numerator *diminished* by the angle in the denominator. Thus, in general, if we have a fraction whose numerator is the product of factors of the form $(\cos A + i\sin A)$, and whose denominator is the product of factors of the form $(\cos\alpha + i\sin\alpha)$, the fraction reduces down to $(\cos\theta + i\sin\theta)$, where θ is the sum of the angles in the numerator diminished by the sum of the angles in the denominator.

$$\frac{(\cos A + i\sin A)(\cos B + i\sin B)\ \ldots\ n \text{ factors}}{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\ \ldots\ m \text{ factors}}$$

$$\begin{aligned}&= \frac{\cos(A+B+\ \ldots\ n \text{ angles}) + i\sin(A+B+\ \ldots\ n \text{ angles})}{\cos(\alpha+\beta+\ \ldots\ m \text{ angles}) + i\sin(\alpha+\beta+\ \ldots\ m \text{ angles})}\\ &= \cos\{(A+B\ \ldots\ n \text{ angles}) - (\alpha+\beta\ \ldots\ m \text{ angles})\}\\ &\quad + i\sin\{(A+B\ \ldots\ n \text{ angles}) - (\alpha+\beta\ \ldots\ m \text{ angles})\}\\ &= \cos\theta + i\sin\theta\end{aligned}$$

Example. Simplify $\dfrac{(\cos 20^\circ + i \sin 20^\circ)(\cos 110^\circ + i \sin 110^\circ)}{(\cos 30^\circ + i \sin 30^\circ)(\cos 50^\circ + i \sin 50^\circ)}$

This becomes $\dfrac{\cos 130^\circ + i \sin 130^\circ}{\cos 80^\circ + i \sin 80^\circ}$

and finally $\cos 50^\circ + i \sin 50^\circ$

30. *The Powers of* $(\cos\theta + i\sin\theta)$. It has already been shown that

$$(\cos A + i \sin A)(\cos B + i \sin B) \ldots n \text{ factors}$$
$$= \cos (A + B + \ldots n \text{ angles}) + i \sin (A + B + \ldots n \text{ angles})$$

If $A = B = \ldots \theta$, making all of the angles equal

Then $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

To raise a trigonometrical complex quantity to a power, multiply the angle by the power, and the product will give the angle of the resulting complex quantity.

So far this rule only applies when the power is a positive integer.

(*a*) If n is negative and this rule holds

Then
$$(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i \sin n\theta$$

and this can be proved

for
$$(\cos\theta + i\sin\theta)^{-n} = \frac{1}{(\cos\theta + i\sin\theta)^n}$$
$$= \frac{1}{\cos n\theta + i\sin n\theta}$$
$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$
$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$
$$= \cos n\theta - i\sin n\theta$$

Hence the rule can be applied when the power is a negative integer.

(*b*) If the power is a fraction and this rule holds

Then
$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

and this can be proved

for
$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q}$$
$$= \cos\theta + i\sin\theta$$

Hence
$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

and
$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p$$
$$= \cos\frac{p}{q}\cdot\theta + i\sin\frac{p}{q}\cdot\theta$$

Also $(\cos\theta + i\sin\theta)^{-\frac{p}{q}} = \cos\left(-\frac{p}{q}\cdot\theta\right) + i\sin\left(-\frac{p}{q}\cdot\theta\right)$ can be proved in the same way as $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ has been proved.

Thus in general $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, whatever form n may take: n may be positive or negative, integral or fractional.

Example 1. Reduce $\dfrac{(5-4i)^{\frac{1}{2}}(3i-2)^{\frac{2}{3}}}{(-8-3i)^{\frac{2}{5}}}$ to the form $a + bi$.

Now $\quad 5 - 4i = 41^{\frac{1}{2}}(\cos\theta + i\sin\theta)$

where $\theta = 360° - A$ and $\tan A = \dfrac{4}{5} = 0{\cdot}8$

$$A = 38°\ 40'$$
$$\theta = 321°\ 20'$$

Then $\quad (5-4i)^{\frac{1}{2}} = 41^{\frac{1}{4}}(\cos 321°\ 20' + i\sin 321°\ 20')^{\frac{1}{2}}$

$$= 41^{\frac{1}{4}}(\cos 160°\ 40' + i\sin 160°\ 40')$$

Next $\quad -2 + 3i = 13^{\frac{1}{2}}(\cos\theta + i\sin\theta)$

where $\theta = 180° - A$ and $\tan A = \dfrac{3}{2} = 1{\cdot}5$

$$A = 56°\ 19'$$
$$\theta = 123°\ 41'$$

Then $\quad (3i-2)^{\frac{2}{3}} = 13^{\frac{1}{3}}(\cos 123°\ 41' + i\sin 123°\ 41')^{\frac{2}{3}}$

$$= 13^{\frac{1}{3}}(\cos 82°\ 27' + i\sin 82°\ 27')$$

Next $\quad -8 - 3i = 73^{\frac{1}{2}}(\cos\theta + i\sin\theta)$

where $\theta = 180° + A$ and $\tan A = \dfrac{3}{8} = 0{\cdot}375$

$$A = 20°\ 32'$$
$$\theta = 200°\ 32'$$

Then $(-8-3i)^{\frac{2}{5}} = 73^{\frac{1}{5}}(\cos 200°\ 32' + i\sin 200°\ 32')^{\frac{2}{5}}$

$$= 73^{\frac{1}{5}}(\cos 80°\ 13' + i\sin 80°\ 13')$$

Hence $\dfrac{(5-4i)^{\frac{1}{2}}(3i-2)^{\frac{2}{3}}}{(-8-3i)^{\frac{2}{5}}}$

$$= \frac{41^{\frac{1}{4}} \times 13^{\frac{1}{3}}}{73^{\frac{1}{5}}}\left\{\frac{(\cos 160°\ 40' + i\sin 160°\ 40')(\cos 82°\ 27' + i\sin 82°\ 27')}{(\cos 80°\ 13' + i\sin 80°\ 13')}\right\}$$

$$= 2{\cdot}523\left\{\frac{\cos 243°\ 7' + i\sin 243°\ 7'}{\cos 80°\ 13' + i\sin 80°\ 13'}\right\}$$
$$= 2{\cdot}523(\cos 162°\ 54' + i\sin 162°\ 54')$$
$$= 2{\cdot}523(-\cos 17°\ 6' + i\sin 17°\ 6')$$
$$= 2{\cdot}523(-0{\cdot}9558 + 0{\cdot}2940i)$$
$$= -2{\cdot}411 + 0{\cdot}7415i$$

Example 2. Reduce $(3-7i)^{-3}$ to the form $a+bi$.

Now
$$3-7i = 58^{\frac{1}{2}}(\cos\theta + i\sin\theta)$$
$$\text{where } \theta = 360 - A \text{ and } \tan A = \frac{7}{3} = 2{\cdot}3333$$
$$A = 66^\circ\ 48'$$
$$\theta = 293^\circ\ 12'$$

Then
$$\begin{aligned}(3-7i)^{-3} &= 58^{-\frac{3}{2}}(\cos 293^\circ\ 12' + i\sin 293^\circ\ 12')^{-3}\\ &= 0{\cdot}002264\{\cos(-879^\circ\ 36') + i\sin(-879^\circ\ 36')\}\\ &= 0{\cdot}002264(\cos 200^\circ\ 24' + i\sin 200^\circ\ 24')\\ &= 0{\cdot}002264(-\cos 20^\circ\ 24' - i\sin 20^\circ\ 24')\\ &= 0{\cdot}002264(-0{\cdot}9373 - 0{\cdot}3486i)\\ &= -10^{-4}(21{\cdot}21 + 7{\cdot}894i)\end{aligned}$$

Examples III

Simplify the following expressions, giving each in the form $a+bi$, the values of a and b given correct to four significant figures.

(1) $(5+4i)(3+7i)(2-3i)$ (2) $(7-2i)(5i-3)(8+3i)$

(3) $\dfrac{(2-3i)(3+2i)}{(4-3i)}$ (4) $\dfrac{5-6i}{(2+3i)(3-5i)}$

(5) $\dfrac{(7i-5)(5-2i)}{(8+5i)(3-7i)}$ (6) $\dfrac{(2i-7)(3+10i)}{(8-i)(4+3i)}$

Extract the square roots of :

(7) $15+7i$ (8) $9+13i$

(9) $12-19i$ (10) $15-8i$

(11) $-8+13i$ (12) $-18+11i$

(13) $-14-19i$ (14) $-21-16i$

(15) i (16) $\dfrac{1}{i}$

Express the following complex quantities in the form $r(\cos\theta + i\sin\theta)$, always keeping r positive :

(17) $8+3i$ (18) $18+11i$

(19) $11-15i$ (20) $9-8i$

(21) $-7+5i$ (22) $-10+17i$

(23) $-14-9i$ (24) $-12-17i$

(25) Express $\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$ in the form $r(\cos\theta + i\sin\theta)$, giving the values of r and θ, and hence simplify $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^4$, $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{-3}$, $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{\frac{2}{3}}$, and $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{\frac{4}{3}}$

(26) Simplify $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{10}+\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{10}$ and $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{6}$ $-\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{6}$

(27) Simplify $(1+\sqrt{3}i)^5+(1-\sqrt{3}i)^5$

(28) Simplify $(\sqrt{3}+i)^4-(\sqrt{3}-i)^4$

Simplify the following expressions, giving each result in the form $a+bi$:

(29) $(5i-7)^{\frac{2}{3}}$

(30) $(7+10i)^{\frac{2}{5}}$

(31) $(8-5i)^{\frac{1}{4}}$

(32) $(-6-13i)^{\frac{1}{3}}$

(33) $\sqrt{\dfrac{5+6i}{8-3i}}$

(34) $\dfrac{(5-4i)^{\frac{1}{2}}}{(2+5i)^{\frac{1}{3}}}$

(35) $\dfrac{(5-7i)^{\frac{4}{5}}}{(2+7i)^{\frac{1}{3}}(3i-4)^{\frac{1}{2}}}$

(36) $\dfrac{(2+5i)^{\frac{1}{2}}(-4-9i)^{\frac{2}{3}}}{(8i-3)^{\frac{4}{5}}}$

CHAPTER IV

31. *The Graphical Representation of a Complex Quantity.*

Now $i = r(\cos\theta + i\sin\theta)$ where $\theta = 90^\circ$ and $r = 1$

$$= \cos 90^\circ + i\sin 90^\circ$$

Thus i can be represented as the radius, drawn vertically upwards, of a unit circle.

$$i^2 = (\cos 90^\circ + i\sin 90^\circ)^2$$
$$= \cos 180^\circ + i\sin 180^\circ$$

Then i^2 or -1 can be represented as the radius, drawn horizontally to the left, of a unit circle.

$$i^3 = (\cos 90^\circ + i\sin 90^\circ)^3$$
$$= \cos 270^\circ + i\sin 270^\circ$$

Then i^3 or $-i$ can be represented as the radius, drawn vertically downwards, of a unit circle.

$$i^4 = (\cos 90^\circ + i\sin 90^\circ)^4$$
$$= \cos 360^\circ + i\sin 360^\circ$$

Then i^4 or $+1$ can be represented as the radius, drawn horizontally to the right, of a unit circle.

We therefore see that each time we multiply by i we are turning the radius of a unit circle, in anti-clockwise direction, through a right angle. Also the odd powers of i always involve the angle 90° and odd multiples of 90°, while the even powers of i always involve the angle 180° and multiples of 180°. Now the odd powers of i give $\pm i$, and the even powers give ± 1. Thus, taking the horizontal direction as the direction of measurement for real quantities, we can take the vertical direction as the direction of measurement for imaginary quantities. The complex quantity $a + bi$ can be represented graphically as the sum of two magnitudes, a measured horizontally and b measured vertically, but in this representation we have also to take into consideration the algebraic signs of a and b.

Case I. Representation of $a + bi$.

OC = a measured to the right
BC = b measured vertically upwards
OB = $\sqrt{a^2 + b^2} = r$

$\widehat{BOX} = A = \theta$ in the relation

$$a + bi = r(\cos\theta + i\sin\theta)$$

Case II. Representation of $-a + bi$.

$OC = a$ measured to the left

$BC = b$ measured vertically upwards

$OB = \sqrt{a^2 + b^2} = r$

$\widehat{BOX} = \theta = 180° - A$ in the relation

$$-a + bi = r(\cos\theta + i\sin\theta)$$

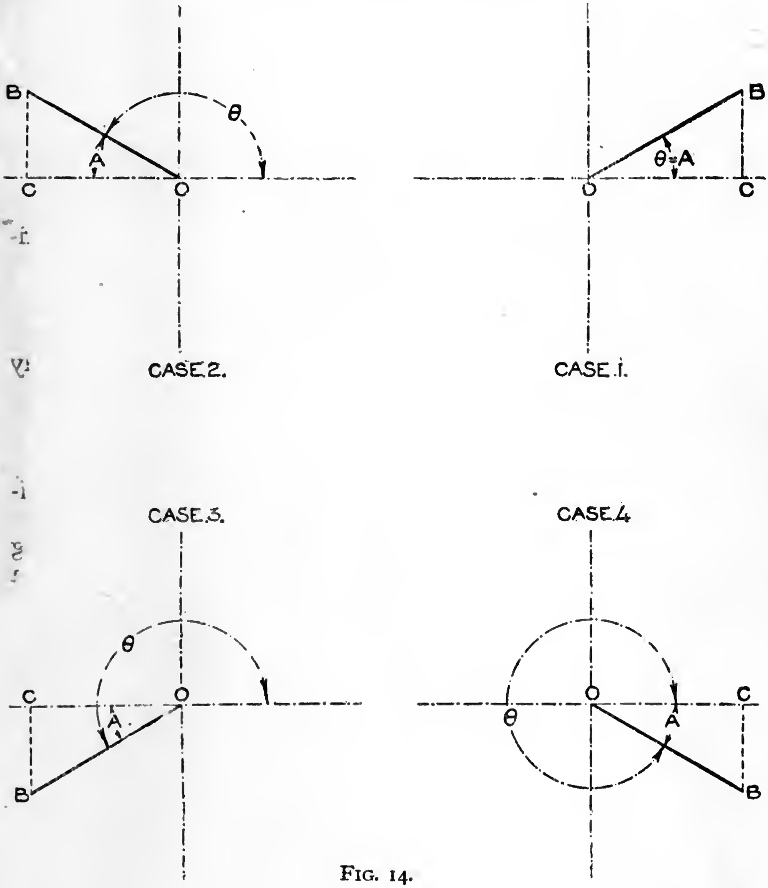


FIG. 14.

Case III. Representation of $-a - bi$.

$OC = a$ measured to the left

$BC = b$ measured vertically downwards

$OB = \sqrt{a^2 + b^2} = r$

$\widehat{BOX} = \theta = 180° + A$ in the relation

$$-a - bi = r(\cos\theta + i\sin\theta)$$

Case IV. Representation of $a - bi$.

$$OC = a \text{ measured to the right}$$
$$BC = b \text{ measured vertically downwards}$$
$$OB = \sqrt{a^2 + b^2} = r$$
$$\widehat{BOX} = \theta = 360° - A \text{ in the relation}$$
$$a - bi = r(\cos\theta + i\sin\theta)$$

It is evident from the above that a complex quantity can also be represented by the radial line OB, which makes an angle θ with the initial line OX. The real part is the projection of OB on the horizontal axis, and the imaginary part is the projection of OB on the vertical axis.

32. *The Use of* i *as an Operator.* It has already been shown that

$$\begin{aligned} i &= \cos 90° + i \sin 90° \\ i^2 &= \cos 180° + i \sin 180° \\ i^3 &= \cos 270° + i \sin 270° \\ i^4 &= \cos 360° + i \sin 360° \end{aligned}$$

Hence if we work with a circle of unit radius and commence with the perpendicular radius as the initial line, the effect of raising i to a power is equivalent to turning this initial radius through a certain number of right angles. This number is fixed by the power.

Also, if we commence with a horizontal radius in this circle, the process of multiplying by i is represented by turning that

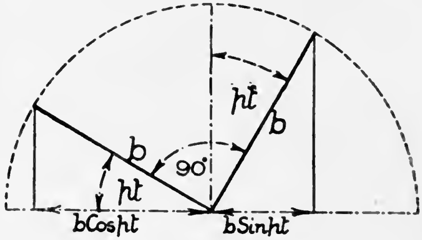


FIG. 15.

horizontal radius anti-clockwise to the vertical position; while multiplying by $-i$ would be represented by turning it clockwise to the vertical position. If we operate in the same way on a quantity a, then the result is expressed by turning a horizontal radius of length a, anti-clockwise or clockwise, to the vertical position.

The quantity $b \sin pt$ measured horizontally can be represented as the horizontal projection of a radial line of length b, inclined at an angle pt to the initial vertical line. Then operating on this by i would have the effect of turning this radial line through a right angle. The horizontal projection of the radial line in its new position is $b \cos pt$, and this is the real result of the operation.

If a complex quantity $a+bi$ operates on $\sin pt$, we have to consider two radial lines **OA** and **OB** each inclined at an angle pt to the initial vertical line.

$$\text{OA} = a \qquad \text{OD} = a \sin pt$$
$$\text{OB} = b \qquad \text{OE} = b \sin pt$$

OD and **OE** being the horizontal projections of **OA** and **OB** respectively. The effect of the operation is to turn the radius

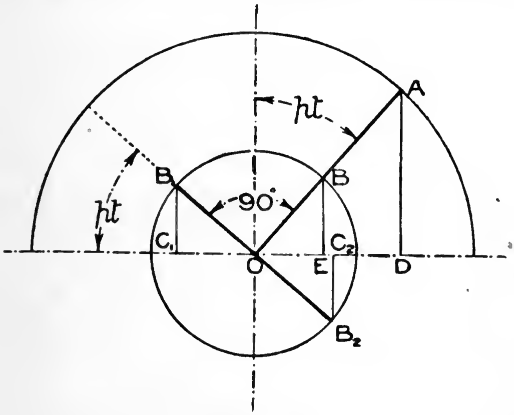


FIG. 16.

OB through a right angle, while the position of the radius **OA** remains unchanged.

$\mathbf{C_1D}$ is the horizontal projection of the radial lines after the operation has been performed.

But $\mathbf{C_1D} = a \sin pt + b \cos pt$, and this is the real result of the operation.

If $a - bi$ operates on $\sin pt$, the radius **OB** is turned through a right angle in clockwise direction, and $\mathrm{C_2D}$ is the horizontal projection of the radial lines after the operation has been performed.

Thus the result of operating with $a - bi$ on $\sin pt$ is

$$a \sin pt - b \cos pt$$

Now $\dfrac{1}{a+bi} = \dfrac{a-bi}{a^2-b^2i^2} = \dfrac{a-bi}{a^2+b^2}$. Then $\dfrac{1}{a+bi}$, operating on $\sin pt$, will give a result which can be obtained by dividing the result of operating with $a - bi$ on $\sin pt$ by a^2+b^2.

$$\frac{1}{a+bi} \text{ operating on } \sin pt \text{ gives } \frac{a \sin pt - b \cos pt}{a^2+b^2}$$

Also, since $\dfrac{1}{a-bi} = \dfrac{a+bi}{a^2-b^2i^2} = \dfrac{a+bi}{a^2+b^2}$

Then $\dfrac{1}{a-bi}$ operating on $\sin pt$ gives $\dfrac{a \sin pt + b \cos pt}{a^2+b^2}$

Classifying these results we get :

(1) $a+bi$ operating on $\sin pt$ gives $a \sin pt + b \cos pt$

or $$\sqrt{a^2+b^2} \cdot \sin(pt+\alpha)$$

(2) $a-bi$ operating on $\sin pt$ gives $a \sin pt - b \cos pt$

or $$\sqrt{a^2+b^2} \cdot \sin(pt-\alpha)$$

(3) $\dfrac{1}{a+bi}$ operating on $\sin pt$ gives $\dfrac{1}{a^2+b^2}\{a \sin pt - b \cos pt\}$

or $$\frac{1}{\sqrt{a^2+b^2}} \cdot \sin(pt-\alpha)$$

(4) $\dfrac{1}{a-bi}$ operating on $\sin pt$ gives $\dfrac{1}{a^2+b^2}\{a \sin pt + b \cos pt\}$

or $$\frac{1}{\sqrt{a^2+b^2}} \cdot \sin(pt+\alpha)$$

In each case $\tan \alpha = \dfrac{b}{a}$

Example. The voltage applied at the sending end of a long telephone line being $v_0 \sin qt$, the current entering the line is

$$v_0\sqrt{\frac{s+ikq}{r+ilq}} \cdot \sin qt$$

where per unit length of cable, r is resistance, l is inductance, s is leakance, and k is permittance or capacity.

If $r = 6$ ohms, $l = 3 \times 10^{-3}$ henries, $k = 5 \times 10^{-9}$ farads, $s = 3 \times 10^{-6}$ mho, and is $q = 6 \times 10^3$, find the current.

$$s + ikq = \sqrt{s^2+k^2q^2}\{\cos\alpha + i \sin\alpha\} \text{ where } \tan\alpha = \frac{kq}{s}$$

$$r + ilq = \sqrt{r^2+l^2q^2}\{\cos\beta + i \sin\beta\} \text{ where } \tan\beta = \frac{lq}{r}$$

$$\text{Then } v_0\sqrt{\frac{s+ikq}{r+ilq}} = v_0\left\{\frac{s^2+k^2q^2}{r^2+l^2q^2}\right\}^{\frac{1}{4}} \cdot \left\{\frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta}\right\}^{\frac{1}{2}}$$

$$= Av_0\{\cos(\alpha-\beta) + i\sin(\alpha-\beta)\}^{\frac{1}{2}} \text{ where } A = \left\{\frac{s^2+k^2q^2}{r^2+l^2q^2}\right\}^{\frac{1}{4}}$$

$$= Av_0\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + i\sin\left(\frac{\alpha-\beta}{2}\right)\right\}$$

$$\text{Current} = v_0\sqrt{\frac{s+ikq}{r+ilq}} \text{ operating on } \sin qt$$

$$= \mathrm{A}v_0\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + i\sin\left(\frac{\alpha-\beta}{2}\right)\right\} \text{ operating on } \sin qt$$

$$= \mathrm{A}v_0\left\{\cos\left(\frac{\alpha-\beta}{2}\right)\cdot\sin qt + \sin\left(\frac{\alpha-\beta}{2}\right).\cos qt\right\}$$

$$= \mathrm{A}v_0\sin\left\{\frac{1}{2}(\alpha-\beta) + qt\right\}$$

$$\mathrm{A} = \left[\frac{9\times10^{-12} + 25\times10^{-18}\cdot36\times10^{6}}{36 + 9\times10^{-6}\cdot36\times10^{6}}\right]^{\frac{1}{4}}$$

$$= \left[\frac{9\times10^{-12}(1+100)}{36(1+9)}\right]^{\frac{1}{4}}$$

$$= 10^{-3}\left(\frac{101}{40}\right)^{\frac{1}{4}} = 10^{-3}\cdot(2{\cdot}525)^{\frac{1}{4}} = 1{\cdot}261\times10^{-3}$$

$$\tan\alpha = \frac{5\times10^{-9}\cdot6\times10^{3}}{3\times10^{-6}} = 10 \qquad \alpha = 84^\circ\ 18'$$

$$\tan\beta = \frac{3\times10^{-3}\cdot6\times10^{3}}{6} = 3 \qquad \beta = 71^\circ\ 34'$$

$$\alpha-\beta = 12^\circ\ 44' \qquad \frac{1}{2}(\alpha-\beta) = 6^\circ\ 22' = 0{\cdot}1111 \text{ radians}$$

Then current $= 1{\cdot}261\times10^{-3}\, v_0\sin(6000t + 0{\cdot}1111)$
the angle being expressed in radians.

33. *The Exponential form of* $(\cos\theta + i\sin\theta)$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Taking θ to be 1 radian
Then $(\cos 1 + i\sin 1)^n = \cos n + i\sin n$
Putting $k = \cos 1 + i\sin 1$
Then $k^n = \cos n + i\sin n$, or $k^a = \cos\alpha + i\sin\alpha$, where n or α represents any angle *taken in radians.*

It must be remembered that k is a complex quantity of the form $a+bi$, for $\cos 1 = 0{\cdot}5403$, and $\sin 1 = 0{\cdot}8415$, and $k = 0{\cdot}5403 + 0{\cdot}8415i$.

If $k^{a} = \cos\alpha + i\sin\alpha$

$$k^{-a} = \frac{1}{\cos\alpha + i\sin\alpha} = \cos\alpha - i\sin\alpha$$

and $2i\sin\alpha = k^a - k^{-a}$ by subtraction

$$= 1 + \alpha\log_e k + \frac{\alpha^2}{\lfloor 2}(\log_e k)^2 + \ldots$$

$$-\left(1 - \alpha\log_e k + \frac{\alpha^2}{\lfloor 2}(\log_e k)^2 + \ldots\right)$$

$$= 2\left(\alpha\log_e k + \frac{\alpha^3}{\lfloor 3}(\log_e k)^3 + \ldots\right)$$

Then $i\dfrac{\sin\alpha}{\alpha} = \log_e k + \dfrac{\alpha^2}{\lfloor 3}(\log_e k)^3 + \ldots$

When α is made infinitely small $\dfrac{\sin\alpha}{\alpha} = 1$, and the terms involving the powers of α become negligibly small

and $i = \log_e k$, or $k = e^i$

Hence $\cos\alpha + i\sin\alpha = k^a = (e^i)^a = e^{ia}$

and $\cos\alpha - i\sin\alpha = k^{-a} = e^{-ia}$

It should be noted that since $k = e^i$

$$k = 1 + i + \frac{i^2}{\lfloor 2} + \frac{i^3}{\lfloor 3} + \frac{i^2}{\lfloor 4} + \ldots$$

$$= 1 + i - \frac{1}{\lfloor 2} - \frac{i}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots$$

$$= \left(1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 6} + \ldots\right) + i\left(1 - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} - \ldots\right)$$

$= 0{\cdot}5403 + 0{\cdot}8415i$, which agrees with the value given above.

34. *The Series for $\sin\alpha$ and $\cos\alpha$.*

$e^{ia} = \cos\alpha + i\sin\alpha \qquad e^{-ia} = \cos\alpha - i\sin\alpha$

Subtracting $\quad 2i\sin\alpha = e^{ia} - e^{-ia}$

and $\quad \sin\alpha = \dfrac{1}{2i}(e^{ia} - e^{-ia})$

Adding $\quad 2\cos\alpha = e^{ia} + e^{-ia}$

and $\quad \cos\alpha = \dfrac{1}{2}(e^{ia} + e^{-ia})$

From these relations we can readily find the series for $\sin\alpha$ and $\cos\alpha$.

For $\quad e^{ia} = 1 + i\alpha + \dfrac{i^2\alpha^2}{\lfloor 2} + \dfrac{i^3\alpha^3}{\lfloor 3} + \dfrac{i^4\alpha^4}{\lfloor 4} + \ldots$

$\quad e^{-ia} = 1 - i\alpha + \dfrac{i^2\alpha^2}{\lfloor 2} - \dfrac{i^3\alpha^3}{\lfloor 3} + \dfrac{i^4\alpha^4}{\lfloor 4} + \ldots$

Subtracting $\quad e^{ia} - e^{-ia} = 2\left(i\alpha + \dfrac{i^3\alpha^3}{\lfloor 3} + \dfrac{i^5\alpha^5}{\lfloor 5} + \ldots\right)$

$$= 2\left(i\alpha - \frac{i\alpha^3}{\lfloor 3} + \frac{i\alpha^5}{\lfloor 5} + \ldots\right)$$

and $\sin\alpha = \dfrac{1}{2i}(e^{ia} - e^{-ia}) = \alpha - \dfrac{\alpha^3}{\lfloor 3} + \dfrac{\alpha^5}{\lfloor 5} \ldots$

Adding $\quad e^{ia} + e^{-ia} = 2\left(1 + \dfrac{i^2\alpha^2}{\lfloor 2} + \dfrac{i^4\alpha^4}{\lfloor 4} + \ldots\right)$

$$= 2\left(1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} \ldots\right)$$

and $\cos\alpha = \dfrac{1}{2}(e^{ia} + e^{-ia}) = 1 - \dfrac{\alpha^2}{\lfloor 2} + \dfrac{\alpha^4}{\lfloor 4} \ldots$

The method of finding the relation which gives the approximate length of a circular arc is a good application of the use of the sine series.

Let ABC (Fig. 17) be the circular arc and AB the chord of the whole arc, BC the chord of the semi-arc.

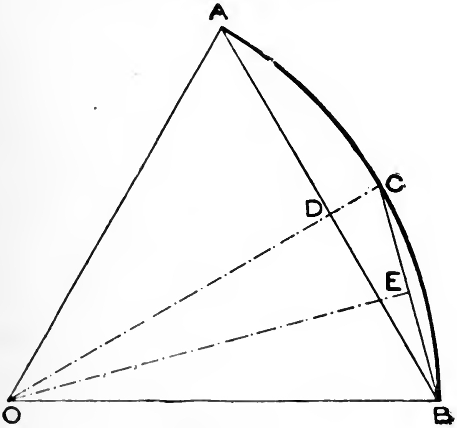


FIG. 17.

Let arc ABC $= l$, chord AB $= c$, chord BC $= h$, angle AOB $= \dfrac{l}{r}$ radians.

$$\text{Then } \frac{AD}{AO} = \sin\frac{l}{2r}$$

$$\text{or} \quad \frac{c}{2r} = \sin\frac{l}{2r}$$

$$\text{Also } \frac{CE}{CO} = \sin\frac{l}{4r} \quad \text{or } \frac{h}{2r} = \sin\frac{l}{4r}$$

$$\text{Then } \frac{c}{2r} = \sin\frac{l}{2r} = \frac{l}{2r} - \frac{l^3}{8\underline{|3}r^3} + \frac{l^5}{32\underline{|5}r^5} \ldots$$

$$\text{and } \frac{h}{2r} = \sin\frac{l}{4r} = \frac{l}{4r} - \frac{l^3}{64\underline{|3}r^3} + \frac{l^5}{1024\underline{|5}r^5} \ldots$$

$$c = l - \frac{l^3}{4\underline{|3}r^2} + \frac{l^5}{16\underline{|5}r^4} \ldots$$

$$8h = 4l - \frac{l^3}{4\underline{|3}r^2} + \frac{l^5}{64\underline{|5}r^4} \ldots$$

$$\text{By subtraction } \quad 8h - c = 3l - \frac{3l^5}{64\underline{|5}r^4} \ldots$$

$$\text{and} \qquad \frac{8h - c}{3} = l\left(1 - \frac{l^4}{7680r^4} \ldots\right)$$

If the fraction $\frac{l}{r}$ is less than 1, all terms on the right-hand side, except the first, may be neglected, and approximately

$$l = \frac{8h - c}{3}$$

35. If we put $x = \cos\alpha + i\sin\alpha$

then $$\frac{1}{x} = \cos\alpha - i\sin\alpha$$

and $$x + \frac{1}{x} = 2\cos\alpha \quad \ldots\ldots \quad (1)$$

$$x - \frac{1}{x} = 2i\sin\alpha \quad \ldots\ldots \quad (2)$$

Also $$x^n = (\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

$$\frac{1}{x^n} = \frac{1}{\cos n\alpha + i\sin n\alpha} = \cos n\alpha - i\sin n\alpha$$

and $$x^n + \frac{1}{x^n} = 2\cos n\alpha \quad \ldots\ldots \quad (3)$$

$$x^n - \frac{1}{x^n} = 2i\sin n\alpha \quad \ldots\ldots \quad (4)$$

These results are very useful for expressing powers of sines and cosines, and also products of powers of sines and cosines, in terms of sines and cosines of multiple angles.

(*a*) Working with $\sin^5\alpha$

$$(2i\sin\alpha)^5 = \left(x - \frac{1}{x}\right)^5$$

$$32i^5\sin^5\alpha = x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$$

$$32i\sin^5\alpha = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 5\alpha - 10i\sin 3\alpha + 20i\sin\alpha$$

and $$\sin^5\alpha = \frac{1}{16}\sin 5\alpha - \frac{5}{16}\sin 3\alpha + \frac{5}{8}\sin\alpha$$

(*b*) Working with $\cos^4\alpha$

$$(2\cos\alpha)^4 = \left(x + \frac{1}{x}\right)^4$$

$$16\cos^4\alpha = x^4 + 4x^2 + 6 + 4\frac{1}{x^2} + \frac{1}{x^4}$$

$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$= 2\cos 4\alpha + 8\cos 2\alpha + 6$$

and $$\cos^4\alpha = \frac{1}{8}\cos 4\alpha + \frac{1}{2}\cos 2\alpha + \frac{3}{8}$$

(*c*) Working with $\sin^6 \alpha \cdot \cos^4 \alpha$

$$(2i \sin \alpha)^6 (2 \cos \alpha)^4 = \left(x - \frac{1}{x}\right)^6 \left(x + \frac{1}{x}\right)^4$$

$$-64 \sin^6 \alpha \cdot 16 \cos^4 \alpha = \left(x^2 - \frac{1}{x^2}\right)^4 \left(x - \frac{1}{x}\right)^2$$

$$-1024 \sin^6 \alpha \cdot \cos^4 \alpha = \left(x^8 - 4x^4 + 6 - 4\frac{1}{x^4} + \frac{1}{x^8}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= x^{10} - 2x^8 - 3x^6 + 8x^4 + 2x^2 - 12 + 2\frac{1}{x^2}$$

$$+ 8\frac{1}{x^4} - 3\frac{1}{x^6} - 2\frac{1}{x^8} + \frac{1}{x^{10}}$$

$$= \left(x^{10} + \frac{1}{x^{10}}\right) - 2\left(x^8 + \frac{1}{x^8}\right) - 3\left(x^6 + \frac{1}{x^6}\right)$$

$$+ 8\left(x^4 + \frac{1}{x^4}\right) + 2\left(x^2 + \frac{1}{x^2}\right) - 12$$

$$= 2 \cos 10\alpha - 4 \cos 8\alpha - 6 \cos 6\alpha$$

$$+ 16 \cos 4\alpha + 4 \cos 2\alpha - 12$$

and $$\sin^6 \alpha \cos^4 \alpha = \frac{1}{512}\{6 - 2 \cos 2\alpha - 8 \cos 4\alpha + 3 \cos 6\alpha$$

$$+ 2 \cos 8\alpha - \cos 10\alpha\}$$

36. *The Hyperbolic Functions.* The expressions $\frac{1}{2}(e^a - e^{-a})$ and $\frac{1}{2}(e^a + e^{-a})$ are spoken of as the hyperbolic sine and cosine respectively, of the angle α

and symbolically $$\sinh \alpha = \frac{1}{2}(e^a - e^{-a})$$

$$\cosh \alpha = \frac{1}{2}(e^a + e^{-a})$$

also $$\tanh \alpha = \frac{e^a - e^{-a}}{e^a + e^{-a}}$$

(*a*) Then $$\cosh^2 \alpha - \sinh^2 \alpha = \frac{1}{4}\{(e^a + e^{-a})^2 - (e^a - e^{-a})^2\}$$

$$= \frac{1}{4}\{e^{2a} + 2 + e^{-2a} - e^{2a} + 2 - e^{-2a}\}$$

$$= 1$$

(*b*) $$2 \sinh \alpha \cdot \cosh \alpha = 2 \cdot \frac{1}{4}(e^a - e^{-a})(e^a + e^{-a})$$

$$= \frac{1}{2}(e^{2a} - e^{-2a})$$

$$= \sinh 2\alpha$$

(*c*) $$\cosh^2 \alpha + \sinh^2 \alpha = \frac{1}{4}\{(e^a + e^{-a})^2 + (e^a - e^{-a})^2\}$$
$$= \frac{1}{4}\{e^{2a} + 2 + e^{-2a} + e^{2a} - 2 + e^{-2a}\}$$
$$= \frac{1}{2}(e^{2a} + e^{-2a})$$
$$= \cosh 2\alpha$$

Then $$\cosh 2\alpha = 2\cosh^2 \alpha - 1$$

or $$\cosh 2\alpha = 1 + 2\sinh^2 \alpha$$

(*d*) $$\cosh^2 \alpha = \frac{1}{2}(\cosh 2\alpha + 1)$$
$$\sinh^2 \alpha = \frac{1}{2}(\cosh 2\alpha - 1)$$

Putting $\theta = 2\alpha$ Then $$\cosh\frac{\theta}{2} = \sqrt{\frac{\cosh\theta + 1}{2}}$$
$$\sinh\frac{\theta}{2} = \sqrt{\frac{\cosh\theta - 1}{2}}$$

37. *The Series for sinh α and cosh α.*

$$e^a = 1 + \alpha + \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^4}{\underline{|4}} + \ldots$$
$$e^{-a} = 1 - \alpha + \frac{\alpha^2}{\underline{|2}} - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^4}{\underline{|4}} + \ldots$$

Subtracting $$e^a - e^{-a} = 2\left(\alpha + \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} + \ldots\right)$$

and $$\sinh\alpha = \frac{1}{2}(e^a - e^{-a}) = \alpha + \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} + \ldots$$

Adding $$e^a + e^{-a} = 2\left(1 + \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} + \ldots\right)$$

and $$\cosh\alpha = \frac{1}{2}(e^a + e^{-a}) = 1 + \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^3}{\underline{|4}} \ldots$$

38. It is important that we should be able to find the angle when we are given one or other of its hyperbolic functions.

(*a*) If $\sinh\alpha = x$

Then $$\frac{1}{2}(e^a - e^{-a}) = x$$
$$e^{2a} - 1 = 2x \cdot e^a$$
$$e^{2a} - 2x \cdot e^a + x^2 = x^2 + 1$$
$$e^a - x = \sqrt{x^2 + 1}$$
$$e^a = x + \sqrt{x^2 + 1}$$

and $$\alpha = \log_e\{x + \sqrt{x^2 + 1}\}$$

(*b*) If $\cosh \alpha = x$

Then $$\frac{1}{2}(e^a + e^{-a}) = x$$
$$e^{2a} + 1 = 2x \cdot e^a$$
$$e^{2a} - 2x \cdot e^a + x^2 = x^2 - 1$$
$$e^a - x = \sqrt{x^2 - 1}$$
$$e^a = x + \sqrt{x^2 - 1}$$
and $$\alpha = \log_e\{x + \sqrt{x^2 - 1}\}$$

(*c*) If $\tanh \alpha = x$

Then $$\frac{e^a - e^{-a}}{e^a + e^{-a}} = x$$
$$e^a - e^{-a} = x \cdot e^a + x \cdot e^{-a}$$
$$e^a(1 - x) = e^{-a}(1 + x)$$
$$e^{2a} = \frac{1 + x}{1 - x}$$
Then $$\alpha = \frac{1}{2}\log_e \frac{1 + x}{1 - x}$$

39. The following table will give a means of comparing the circular functions with the hyperbolic functions :

Circular Functions.	*Hyperbolic Functions.*
$\sin \alpha = \frac{1}{2i}(e^{ia} - e^{-ia})$	$\sinh \alpha = \frac{1}{2}(e^a - e^{-a})$
$\cos \alpha = \frac{1}{2}(e^{ia} + e^{-ia})$	$\cosh \alpha = \frac{1}{2}(e^a + e^{-a})$
$\tan \alpha = \frac{e^{ia} - e^{-ia}}{i(e^{ia} + e^{-ia})}$	$\tanh \alpha = \frac{e^a - e^{-a}}{e^a + e^{-a}}$
$\sin \alpha = \alpha - \frac{\alpha^3}{\underline{\lvert 3}} + \frac{\alpha^5}{\underline{\lvert 5}} \ldots$	$\sinh \alpha = \alpha + \frac{\alpha^3}{\underline{\lvert 3}} + \frac{\alpha^5}{\underline{\lvert 5}} + \ldots$
$\cos \alpha = 1 - \frac{\alpha^2}{\underline{\lvert 2}} + \frac{\alpha^4}{\underline{\lvert 4}} \ldots$	$\cosh \alpha = 1 + \frac{\alpha^2}{\underline{\lvert 2}} + \frac{\alpha^4}{\underline{\lvert 4}} + \ldots$
$\sin^2 \alpha + \cos^2 \alpha = 1$	$\cosh^2 \alpha - \sinh^2 \alpha = 1$
$\sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha$	$\sinh 2\alpha = 2 \sinh \alpha \cdot \cosh \alpha$
$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$	$\cosh 2\alpha = \cosh^2 \alpha + \sinh^2 \alpha$
$= 1 - 2 \sin^2 \alpha$	$= 1 + 2 \sinh^2 \alpha$
$= 2 \cos^2 \alpha - 1$	$= 2 \cosh^2 \alpha - 1$
$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}$	$\sinh \frac{\theta}{2} = \sqrt{\frac{\cosh \theta - 1}{2}}$
$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$	$\cosh \frac{\theta}{2} = \sqrt{\frac{\cosh \theta + 1}{2}}$

Examples IV

Find the result of operating with each of the following complex quantities on $\sin qt$:

(1) $2 + 5i$ (2) $5 + 11i$

(3) $5 - 8i$ (4) $12 - 5i$

(5) $\dfrac{1}{5 + 6i}$ (6) $\dfrac{1}{11 + 7i}$

(7) $\dfrac{1}{7 - 5i}$ (8) $\dfrac{1}{5 - 13i}$

(9) $\sqrt{i}$ (10) $\dfrac{1}{\sqrt{i}}$

(11) $\sqrt{8 + 5i}$ (12) $\sqrt{12 - 7i}$

(13) $\sqrt{\dfrac{3 + 4i}{5 - 12i}}$ (14) $\sqrt{\dfrac{8 - 11i}{9 + 7i}}$

(15) If $x = 0{\cdot}82$, find $\sinh x$, $\cosh x$, and $\tanh x$.

(16) If $\sinh x = 0{\cdot}65$, find x.

(17) If $\cosh x = 2{\cdot}25$, find x.

(18) If $\tanh x = 0{\cdot}75$, find x.

(19) Express $\cosh \dfrac{1+i}{2}$ and $\sinh \dfrac{1+i}{3}$ each in the form $a + bi$, giving the values of a and b correct to four significant figures.

(20) If $\sinh \theta = \dfrac{x+6}{2\sqrt{3}}$, find θ in terms of x.

(21) If $\cosh \theta = \dfrac{x+8}{2\sqrt{7}}$, find θ in terms of x.

(22) In a telephone line of length l, where q is $2\pi f$, if f is frequency or pitch of a musical note; let r, the resistance per mile, be 88; let k, the capacity per mile, be 5×10^{-8}. Take $q = 5000$. Let $n = \sqrt{rkqi}$ where i means $\sqrt{-1}$. Let $l = 40$ miles. If $R = 100 + 0{\cdot}04qi$ be the resistance of the receiving telephone, the current through it is $C = \dfrac{2V_0}{\left(R + \dfrac{r}{n}\right)e^{ln}}$, where V_0 is $10 \sin 5000t$. C is of the form $a \sin (qt + b)$ where q is 5000, find a and b.

(B. of E., 1912.)

Express each of the following functions in terms of the sines or cosines of multiple angles:

(23) $\sin^3 \theta$ (24) $\sin^6 \theta$

(25) $\cos^5 \theta$ (26) $\cos^6 \theta$

(27) $\sin^3 \theta \cdot \cos^3 \theta$ (28) $\sin^4 \theta \cdot \cos^4 \theta$

(29) $\sin^3 \theta \cdot \cos^5 \theta$ (30) $\sin^4 \theta \cdot \cos^2 \theta$

CHAPTER V

40. In order to determine the position of a point in space, it is necessary to refer the point to three fixed planes. These three planes intersect at a point which is taken as the origin; while any pair of these planes intersect in a straight line which passes through the origin. Thus for the three planes of reference there will be three different pairs of planes, and therefore there will be three different lines of intersection, each one passing through the origin. These three lines of intersection are called " the axes of reference " or " the co-ordinate axes."

Generally the three planes of reference are rectangular—that is, one plane is at right angles to each of the other two. The

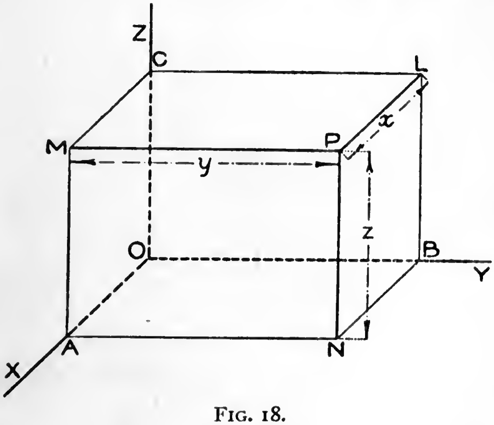


FIG. 18.

three co-ordinate axes will therefore be mutually perpendicular—that is, one axis will be perpendicular to each of the other two. This can be well illustrated by means of a cube, with its base horizontal. One corner of the cube can be taken as the origin; the three edges which radiate from this corner will be the three axes of reference, while these three edges are the lines of intersection of the three adjacent plane faces of the cube, two of these being vertical and the other horizontal.

The position of a point with reference to the three rectangular planes of reference is completely defined by the perpendicular distances of the point from these three planes.

If the co-ordinates of a point P are (x, y, z), then x is the perpendicular distance of the point from the plane YOZ, and the point must therefore lie on a plane parallel to YOZ, the distance between these planes being x. y is the perpendicular distance of the point from the plane ZOX, and the point must therefore lie on a plane parallel to ZOX, the distance between these planes being y. z is the perpendicular distance of the point from the plane XOY, and the point must therefore lie on a plane parallel to XOY, the distance between these planes being z.

The intersection of these three new planes will give the position of the point P, and these three planes combined with the three planes of reference produce a right rectangular prism one corner

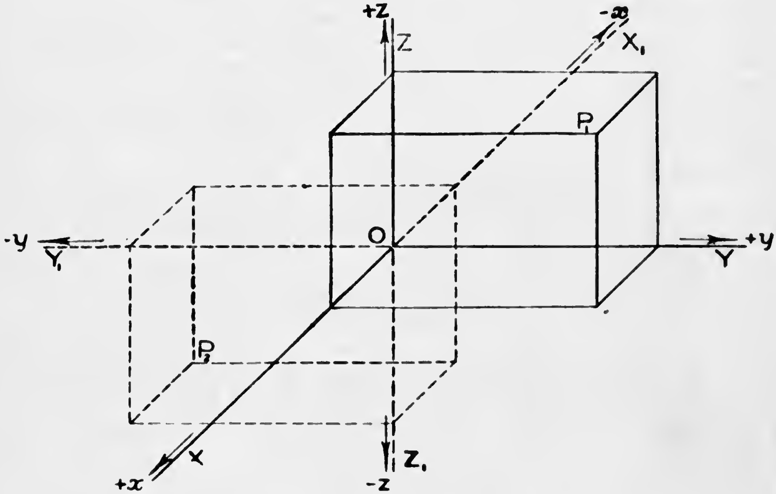


Fig. 19.

of which is the origin and the opposite corner the point **P**, while **OP** is a solid diagonal of the prism. The lengths of the three edges, **PL**, **PM**, and **PN**, meeting at the point **P**, are the co-ordinates of that point; while the lengths of the three edges **OA**, **OB**, and **OC**, meeting at the origin, are also the co-ordinates of the point **P**. Hence, to find the position of the point whose co-ordinates are (x, y, z), we have to measure $OA = x$, along **OX**, $OB = y$, along **OY**, and $OC = x$, along **OZ**. Take **OA**, **OB**, and **OC** to be the three adjacent edges of a right rectangular prism, and complete the prism. The required point will be the corner opposite to the origin.

41. The plane **ZOY** (Fig. 19) is taken as a front vertical plane, and any line drawn parallel to the axis **OX** will be perpendicular to that plane. The x co-ordinate of a point **P** may be positive or negative. It is positive when **P** lies in front of the plane **ZOY**,

and is therefore measured along OX; it is negative when P lies behind the plane ZOY, and is therefore measured along OX_1.

The plane ZOX is taken as a side vertical plane, and any line drawn parallel to the axis OY will be perpendicular to that plane. The y co-ordinate of a point P is positive when P lies to the right of the plane ZOX, and is therefore measured along OY. It is negative when P lies to the left of the plane ZOX, and is therefore measured along OY_1. The plane XOY is taken as a horizontal plane, and any line drawn parallel to the axis OZ will be perpendicular to that plane. The z co-ordinate of a point P is positive if P lies above the plane XOY, and is therefore measured

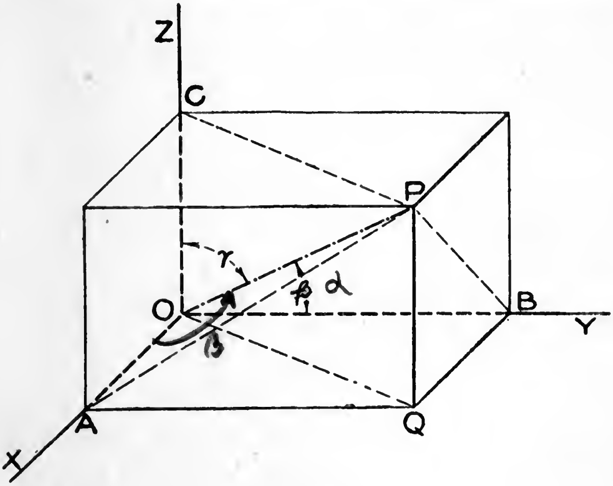


FIG. 20.

along OZ. It is negative when P lies below the plane XOY, and is therefore measured along OZ_1.

Fig. 19 is drawn to illustrate this, the co-ordinates of P_1 being (1, 3, 2), while the co-ordinates of P_2 are $(-1, -3, -2)$.

42. Let P (Fig. 20) be a point whose co-ordinates are (x, y, z).

Then $OA = x$, $OB = y$, and $OC = z$.

Since OC is perpendicular to the plane containing PC, the angle PCO is a right angle,

$$\begin{aligned}
\text{and} \quad OP^2 &= OC^2 + PC^2 \\
\text{But} \quad CP &= OQ \\
\text{and} \quad OQ^2 &= OB^2 + BQ^2 \\
&= OB^2 + OA^2 \\
\text{Hence } OP^2 &= OC^2 + OB^2 + OA^2 \\
&= z^2 + y^2 + x^2 \\
\text{or} \quad OP &= \sqrt{x^2 + y^2 + z^2}
\end{aligned}$$

This gives the distance of a point P from the origin in terms of the co-ordinates of that point.

Let α, β, γ be the angles which OP makes with the axes OX, OY, and OZ respectively.

(1) The triangle POA is such that the base angle POA $= \alpha$, and $\widehat{PAO} = 90°$ since OA is perpendicular to AP.

Hence
$$\cos\alpha = \frac{OA}{OP}$$
$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

(2) The triangle POB is such that the base angle POB $= \beta$, and $\widehat{PBO} = 90°$ since OB is perpendicular to BP.

Hence
$$\cos\beta = \frac{OB}{OP}$$
$$= \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

(3) The triangle POC is such that the base angle POC $= \gamma$, and $\widehat{PCO} = 90°$ since OC is perpendicular to CP.

Hence
$$\cos\gamma = \frac{OC}{OP}$$
$$= \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

The cosines of the angles α, β, and γ, which the line joining a point to the origin makes with the axes OX, OY, and OZ respectively, are spoken of as the "direction cosines" of that line, and are usually denoted by l, m, and n.

The sum of the squares of the three direction cosines is 1; for

$$l^2 + m^2 + n^2 = \frac{x^2}{x^2 + y^2 + z^2} + \frac{y^2}{x^2 + y^2 + z^2} + \frac{z^2}{x^2 + y^2 + z^2}$$
$$= 1$$

43. Let P_1 and P_2 be two points whose co-ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2). If the axes of reference are so chosen that P_1 is taken as the origin, then the co-ordinates of P_2 with reference to these axes will be $(x_2 - x_1)$, $(y_2 - y_1)$, and $(z_2 - z_1)$.

Then $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, thus giving the length of a line joining two points in terms of the co-ordinates of those points.

If α, β, and γ are the angles which the line P_1P_2 makes with the true axes of reference OX, OY, and OZ, they will also be the angles which the line makes with the parallel axes P_1X, P_1Y, and P_1Z.

Then
$$l = \cos \alpha = \frac{x_2 - x_1}{P_1P_2}$$
$$m = \cos \beta = \frac{y_2 - y_1}{P_1P_2}$$
$$n = \cos \gamma = \frac{z_2 - z_1}{P_1P_2}$$

thus giving the direction cosines of any line in terms of the co-ordinates of any two points taken on the line. It follows that if P is *any point* on a line, the co-ordinates of P being (x, y, z), and Q is a *given point* on the same line, the co-ordinates of Q being (a, b, c).

If r is the length of line between P and Q,

then $$l = \cos \alpha = \frac{x - a}{r}$$

or $$\frac{x - a}{l} = r$$

$$m = \cos \beta = \frac{y - b}{r}$$

or $$\frac{y - b}{m} = r$$

$$n = \cos \gamma = \frac{z - c}{r}$$

or $$\frac{z - c}{n} = r$$

Hence $$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} = r$$

This is known as the symmetrical equation to a straight line.

44. Let P_1, P_2, and P_3 be three points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

If P_1 be taken as the origin, the co-ordinates of P_2 will be $(x_2 - x_1)$, $(y_2 - y_1)$, and $(z_2 - z_1)$, while the co-ordinates of P_3 will be $(x_3 - x_1)$, $(y_3 - y_1)$, $(z_3 - z_1)$.

Hence $(P_1P_2)^2 = p_3^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

and $(P_1P_3)^2 = p_2^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2$

If P_2 be taken as the origin, the co-ordinates of P_3 will be $(x_3 - x_2)$, $(y_3 - y_2)$, and $(z_3 - z_2)$.

Hence $(P_2P_3)^2 = p_1^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2$

Thus the three sides of the triangle $P_1P_2P_3$ can be determined if the co-ordinates of the three angular points are known.

If Δ is the area of the triangle,

$$\Delta = \sqrt{s(s-p_1)(s-p_2)(s-p_3)}$$

where $$s = \frac{1}{2}(p_1 + p_2 + p_3)$$

If h_1, h_2, and h_3 are the lengths of the perpendiculars drawn from the points P_1, P_2, and P_3 to the opposite sides respectively,

then
$$\Delta = \frac{1}{2} p_1 h_1 \quad \text{or} \quad h_1 = \frac{2\Delta}{p_1}$$
$$\Delta = \frac{1}{2} p_2 h_2 \quad \text{or} \quad h_2 = \frac{2\Delta}{p_2}$$
$$\Delta = \frac{1}{2} p_3 h_3 \quad \text{or} \quad h_3 = \frac{2\Delta}{p_3}$$

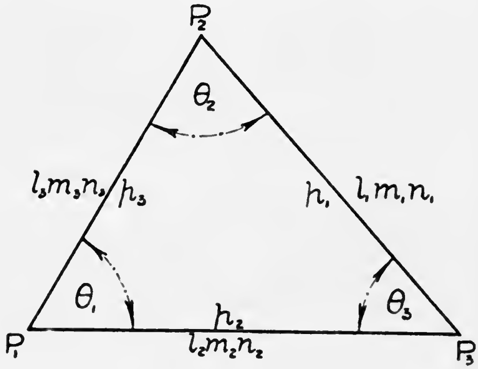


FIG. 21.

If θ_1, θ_2, and θ_3 are the angles of the triangle,

then
$$\Delta = \frac{1}{2} p_2 p_3 \sin\theta_1 \quad \text{or} \quad \sin\theta_1 = \frac{2\Delta}{p_2 p_3}$$
$$\Delta = \frac{1}{2} p_3 p_1 \sin\theta_2 \quad \text{or} \quad \sin\theta_2 = \frac{2\Delta}{p_3 p_1}$$
$$\Delta = \frac{1}{2} p_1 p_2 \sin\theta_3 \quad \text{or} \quad \sin\theta_3 = \frac{2\Delta}{p_1 p_2}$$

and from these relations the angle between two given lines can be determined.

Also the angles θ_1, θ_2, θ_3 can be found by means of the following relations:

$$\cos\theta_1 = \frac{p_2^2 + p_3^2 - p_1^2}{2p_2p_3}$$
$$\cos\theta_2 = \frac{p_3^2 - p_1^2 + p_2^2}{2p_3p_1}$$
$$\cos\theta_3 = \frac{p_1^2 + p_2^2 - p_3^2}{2p_1p_2}$$

Now $\cos\theta_1$

$$= \frac{1}{2p_2p_3}\{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (x_3 - x_2)^2 - (y_3 - y_2)^2 - (z_3 - z_2)^2\}$$

$$= \frac{1}{2p_2p_3}\{2(x_1^2 - x_3x_1 - x_2x_1 + x_3x_2) + 2(y_1^2 - y_3y_1 - y_2y_1 + y_3y_2) + 2(z_1^2 - z_3z_1 - z_2z_1 + z_3z_2)\}$$

$$= \frac{1}{p_2p_3}\{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) + (z_2 - z_1)(z_3 - z_1)\}$$

$$= \left(\frac{x_2 - x_1}{p_3}\right)\left(\frac{x_3 - x_1}{p_2}\right) + \left(\frac{y_2 - y_1}{p_3}\right)\left(\frac{y_3 - y_1}{p_2}\right) + \left(\frac{z_2 - z_1}{p_3}\right)\left(\frac{z_3 - z_1}{p_2}\right)$$

If α_3, β_3, and γ_3 are the angles which P_1P_2 makes with the axes OX, OY, and OZ respectively, and l_3, m_3, and n_3 are the corresponding direction cosines,

then
$$l_3 = \cos\alpha_3 = \frac{x_2 - x_1}{p_3}$$
$$m_3 = \cos\beta_3 = \frac{y_2 - y_1}{p_3}$$
$$n_3 = \cos\gamma_3 = \frac{z_2 - z_1}{p_3}$$

Also, if α_2, β_2, and γ_2 are the angles which P_1P_3 makes with the axes OX, OY, and OZ respectively, and l_2, m_2, and n_2 are the corresponding direction cosines,

then
$$l_2 = \cos\alpha_2 = \frac{x_3 - x_1}{p_2}$$
$$m_2 = \cos\beta_2 = \frac{y_3 - y_1}{p_2}$$
$$n_2 = \cos\gamma_2 = \frac{z_3 - z_1}{p_2}$$

Hence
$$\cos\theta_1 = l_3l_2 + m_3m_2 + n_3n_2$$

Similarly it can be proved that

$$\cos\theta_2 = l_3l_1 + m_3m_1 + n_3n_1$$

and
$$\cos\theta_3 = l_2l_1 + m_2m_1 + n_2n_1$$

thereby giving the cosine of an angle between two lines in terms of the direction cosines of those lines.

In general, if θ is the angle between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively,

then
$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

If the two lines are at right angles,

then
$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Example 1. The co-ordinates of two points P and Q are (3, 7, 5) and (5, 2, 8) respectively. Find the length of PQ, its direction cosines, and the angles it makes with the axes of reference.

Taking P as the origin, the co-ordinates of Q are (2, − 5, 3),

$$\text{and} \qquad PQ = \sqrt{2^2 + (-5)^2 + 3^2}$$
$$= 6{\cdot}164$$
$$l = \cos\alpha = \frac{2}{6{\cdot}164}$$
$$= 0{\cdot}3244$$
$$\alpha = 71^\circ\ 4'$$
$$m = \cos\beta = \frac{-5}{6{\cdot}164}$$
$$= -0{\cdot}8112$$
$$\beta = 144^\circ\ 12'$$
$$n = \cos\gamma = \frac{3}{6{\cdot}164}$$
$$= 0{\cdot}4866$$
$$\gamma = 60^\circ\ 53'$$

Example 2. The co-ordinates of three points P, Q, and R are (3, 6, 2), (5, 9, 7), and (8, 3, 9) respectively. Find the lengths of the lines joining these points, the angles between the lines, and the lengths of the perpendiculars drawn from each point to the opposite line.

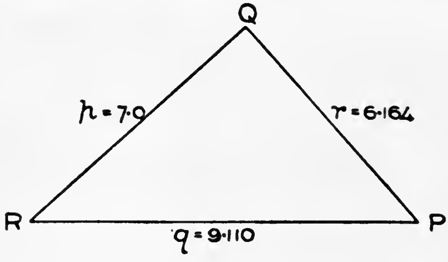


Fig. 22.

Taking P as the origin, the co-ordinates of Q are (2, 3, 5), and the co-ordinates of R are (5, − 3, 7).

$$\text{Then} \qquad PQ = \sqrt{2^2 + 3^2 + 5^2}$$
$$= 6{\cdot}164$$
$$PR = \sqrt{5^2 + (-3)^2 + 7^2}$$
$$= 9{\cdot}110$$

Taking Q as the origin, the co-ordinates of R are (3, − 6, 2).

$$QR = \sqrt{3^2 + (-6)^2 + 2^2}$$
$$= 7$$

Working with the triangle PQR and calling the sides p, q, and r, then $p = 7$, $q = 9{\cdot}110$, and $r = 6{\cdot}164$.

$$s = \frac{1}{2}(p + q + r)$$
$$= 11{\cdot}137$$
$$\text{Area} = \Delta = \sqrt{11{\cdot}137 \times 4{\cdot}137 \times 2{\cdot}027 \times 4{\cdot}973}$$
$$= 21{\cdot}55$$
$$\sin \mathrm{P} = \frac{43{\cdot}10}{9{\cdot}110 \times 6{\cdot}164}$$
$$= 0{\cdot}7676$$
$$\mathrm{P} = 50^\circ\ 8'$$
$$\sin \mathrm{Q} = \frac{43{\cdot}10}{7 \times 6{\cdot}164}$$
$$= 0{\cdot}9989$$
$$\mathrm{Q} = 87^\circ\ 18'$$
$$\sin \mathrm{R} = \frac{43{\cdot}10}{7 \times 9{\cdot}110}$$
$$= 0{\cdot}6759$$
$$\mathrm{R} = 42^\circ\ 32'$$

These are the angles between the lines.

If h_p, h_q, and h_r are the perpendiculars drawn from the points P, Q, and R respectively,

then
$$h_p = \frac{43{\cdot}10}{7} = 6{\cdot}157$$
$$h_q = \frac{43{\cdot}10}{9{\cdot}110} = 4{\cdot}732$$
$$h_r = \frac{43{\cdot}10}{6{\cdot}164} = 6{\cdot}988$$

The angles between the lines can also be found in the following manner:

$$\cos \mathrm{P} = \frac{q^2 + r^2 - p^2}{2qr}$$
$$= 0{\cdot}6410$$
$$\mathrm{P} = 50^\circ\ 8'$$
$$\cos \mathrm{Q} = \frac{p^2 + r^2 - q^2}{2pr}$$
$$= 0{\cdot}0463$$
$$\mathrm{Q} = 87^\circ\ 20'$$
$$\cos \mathrm{R} = \frac{p^2 + q^2 - r^2}{2pq}$$
$$= 0{\cdot}7370$$
$$\mathrm{R} = 42^\circ\ 31'$$

45. *The Plane.* The plane is represented by the general equation of the first degree in x, y, and z. Then the equation

$$Ax + By + Cz + D = 0$$

represents a plane, and the values of the constants A, B, C, and D can be given in terms of the intercepts the plane makes on the axes of reference. For if the intercepts are a, b, and c on the axes OX, OY, and OZ respectively

Then when $x = a$, $y = 0$, and $z = 0$ hence $Aa + D = 0$

when $y = b$, $x = 0$, and $z = 0$ hence $Bb + D = 0$

when $z = c$, $x = 0$, and $y = 0$ hence $Cc + D = 0$

Therefore $a = -\dfrac{D}{A}$, $b = -\dfrac{D}{B}$, and $c = -\dfrac{D}{C}$

But $$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z = -1$$

or $$\frac{x}{\dfrac{D}{A}} + \frac{y}{\dfrac{D}{B}} + \frac{z}{\dfrac{D}{C}} = -1$$

Then $$-\frac{x}{a} - \frac{y}{b} - \frac{z}{c} = -1$$

or $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This gives the equation of a plane in terms of the intercepts.

Let OP (Fig. 23) be the perpendicular drawn from the origin to the plane whose equation is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Let x_1, y_1, z_1 be the co-ordinates of P; and if α, β, and γ are the angles made by OP with the axes OX, OY, and OZ respectively, and l, m, n are the direction cosines of OP

Then $$l = \cos\alpha = \frac{x_1}{p}$$

$$m = \cos\beta = \frac{y_1}{p}$$

$$n = \cos\gamma = \frac{z_1}{p}$$

where p is the length of OP.

Since OP is perpendicular to the plane, it is perpendicular to the lines OA, OB, and OC in that plane, and therefore the triangles POA, POB, and POC are right angled.

Hence $$l = \cos\alpha = \frac{p}{a}$$

$$m = \cos\beta = \frac{p}{b}$$

$$n = \cos\gamma = \frac{p}{c}$$

but $$l^2 + m^2 + n^2 = 1$$

$$p^2\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right\} = 1$$

and $$p^2 = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

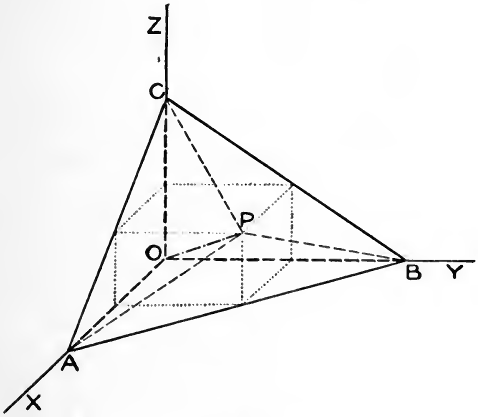


FIG. 23.

thus giving the length of the perpendicular drawn from the origin to the plane.

Next $$\cos\alpha = \frac{p}{a} = \frac{x_1}{p}$$

$$\cos\beta = \frac{p}{b} = \frac{y_1}{p}$$

$$\cos\gamma = \frac{p}{c} = \frac{z_1}{p}$$

Hence $$x_1 = \frac{p^2}{a}$$

$$y_1 = \frac{p^2}{b}$$

$$z_1 = \frac{p^2}{c}$$

thus giving the co-ordinates of the point P.

Now $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the equation to the plane.

Then
$$x\cdot\frac{p}{a}+y\cdot\frac{p}{b}+z\cdot\frac{p}{c}=p$$
or
$$x\cos\alpha+y\cos\beta+z\cos\gamma=p$$
$$lx+my+nz=p$$

thus expressing the equation to a plane in terms of the direction cosines and the length of the perpendicular drawn from the origin to the plane.

Also α, β, and γ are the angles made by the line **OP** with the axes **OX**, **OY**, and **OZ** respectively, but **OP** is perpendicular to the plane. Hence the angles the plane makes with these axes will be the complements of these angles, and therefore the plane makes angles $(90°-\alpha)$, $(90°-\beta)$, and $(90°-\gamma)$ with the axes **OX**, **OY**, and **OZ** respectively.

Example. For the plane $21x+35y+15z-105=0$. Find:

(1) The intercepts on the axes of reference.

(2) The length of **OP**, the perpendicular drawn from the origin to the plane.

(3) The angles which **OP** makes with the axes of reference.

(4) The angles which the plane makes with the axes of reference.

(5) The co-ordinates of **P**.

$$21x+35y+15z=105$$

Then
$$\frac{x}{5}+\frac{y}{3}+\frac{z}{7}=1$$

Hence the plane makes intercepts of 5, 3, and 7 on the axes **OX**, **OY**, and **OZ** respectively.

If p is the length of **OP** and α, β, and γ are the angles **OP** makes with the axes **OX**, **OY**, and **OZ**,

then
$$l=\cos\alpha=\frac{p}{5}$$
$$m=\cos\beta=\frac{p}{3}$$
$$n=\cos\gamma=\frac{p}{7}$$

But $l^2+m^2+n^2=1$

Hence $\frac{p^2}{25}+\frac{p^2}{9}+\frac{p^2}{49}=1$

and
$$p^2=\frac{9\times25\times49}{1891}$$
$$p=2{\cdot}414$$

$$\cos \alpha = \frac{2 \cdot 414}{5} = 0 \cdot 4828$$

$$\alpha = 61^\circ\ 8'$$

$$\cos \beta = \frac{2 \cdot 414}{3} = 0 \cdot 8047$$

$$\beta = 36^\circ\ 25'$$

$$\cos \gamma = \frac{2 \cdot 414}{7} = 0 \cdot 3449$$

$$\gamma = 69^\circ\ 50'$$

The plane makes angles 29° 52′, 53° 35′, and 20° 10′ with the axes OX, OY, and OZ respectively.

If x_1, y_1, z_1 are the co-ordinates of P

Then
$$x_1 = \frac{p^2}{5} = 1 \cdot 168$$

$$y_1 = \frac{p^2}{3} = 1 \cdot 943$$

$$z_1 = \frac{p^2}{7} = 0 \cdot 8328$$

46. To find the perpendicular distance of a given point from a given plane.

Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ be the equation to the plane. Then if p is the perpendicular distance from the origin to the plane, and l, m, and n are the direction cosines of that perpendicular

Then
$$l = \frac{p}{a},\ m = \frac{p}{b},\ \text{and}\ n = \frac{p}{c}$$

Hence $lx + my + nz = p$ will be the equation to the plane, and

$$p^2 = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

Let x_1, y_1, z_1 be the co-ordinates of the given point, and through this point let a plane be drawn parallel to the given plane. Since the two planes are parallel, the perpendiculars drawn to these planes will be parallel, and will therefore have the same direction cosines.

Hence $lx + my + nz = p_1$ will be the equation to the parallel plane, where p_1 is the perpendicular distance from the origin to the plane.

Also $lx_1 + my_1 + nz_1 = p_1$, since the plane passes through the point whose co-ordinates are (x_1, y_1, z_1). Now $p_1 - p$ is the distance between these two parallel planes, and this will also be the perpendicular distance of the given point from the given plane.

Then
$$p_1 - p = lx_1 + my_1 + nz_1 - p$$
$$= \left\{\frac{x_1}{b} + \frac{y_1}{a} + \frac{z_1}{c} - 1\right\}p$$
$$= \frac{\left\{\frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} - 1\right\}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Example. Find the perpendicular distance from the point whose co-ordinates are (8, 3, 7) to the plane whose equation is $15x - 9y + 5z = 45$.

Then $\frac{x}{3} - \frac{y}{5} + \frac{z}{9} = 1$ will be the equation to the plane where 3, -5, and 9 are the intercepts.

$$\text{The perpendicular distance} = \frac{\frac{8}{3} - \frac{3}{5} + \frac{7}{9} - 1}{\sqrt{\frac{1}{9} + \frac{1}{25} + \frac{1}{81}}}$$
$$= \frac{83}{\sqrt{331}}$$
$$= 4{\cdot}562$$

47. To find the angle between two planes.

Let $\frac{x}{a_1} + \frac{y}{b_1} + \frac{z}{c_1} = 1$ and $\frac{x}{a_2} + \frac{y}{b_2} + \frac{z}{c_2} = 1$ be the equations to the planes.

Then for the first plane $l_1x + m_1y + n_1z = p_1$, where
$$l_1 = \frac{p_1}{a_1}, \qquad m_1 = \frac{p_1}{b_1}, \qquad \text{and } n_1 = \frac{p_1}{c_1}$$

Also
$$p_1^2 = \frac{1}{\frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}}$$

These relations give the direction cosines of the perpendicular to the plane.

For the second plane $l_2x + m_2y + n_2z = p_2$, where
$$l_2 = \frac{p_2}{a_2}, \qquad m_2 = \frac{p_2}{b_2}, \qquad \text{and } n_2 = \frac{p_2}{c_2}$$

Also
$$p_2^2 = \frac{1}{\frac{1}{a_2^2} + \frac{1}{b_2^2} + \frac{1}{c_2^2}}$$

These relations give the direction cosines of the perpendicular to this plane.

If θ is the angle between the two planes, then this will also be the angle between the two perpendiculars,

and
$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= \frac{p_1 p_2}{a_1 a_2} + \frac{p_1 p_2}{b_1 b_2} + \frac{p_1 p_2}{c_1 c_2}$$
$$= p_1 p_2 \left\{ \frac{1}{a_1 a_2} + \frac{1}{b_1 b_2} + \frac{1}{c_1 c_2} \right\}$$

Example. Find the angle between the planes $21x + 35y + 15z = 105$ and $15x - 9y + 5z = 45$.

Then for the first plane $\frac{x}{5} + \frac{y}{3} + \frac{z}{7} = 1$

and
$$p_1^2 = \frac{1}{\frac{1}{25} + \frac{1}{9} + \frac{1}{49}}$$
$$p_1 = 2{\cdot}414$$

For the second plane $\frac{x}{3} - \frac{y}{5} + \frac{z}{9} = 1$

and
$$p_2^2 = \frac{1}{\frac{1}{9} + \frac{1}{25} + \frac{1}{81}}$$
$$p_2 = 2{\cdot}474$$

Then
$$\cos\theta = 2{\cdot}414 \times 2{\cdot}474 \left\{ \frac{1}{15} - \frac{1}{15} + \frac{1}{63} \right\}$$
$$= 0{\cdot}0948$$
$$\theta = 84^\circ\ 34'$$

48. *The Polar Co-ordinates of a Point.*

The polar co-ordinates of a point P are :

r, the distance the point is from the origin.

θ, the angle between the plane containing OP and the plane ZOX.

ϕ, the angle OP makes with the axis OZ.

Then CPO is a triangle, right angled at C

Hence $\quad OC = r\cos\phi$, and $PC = r\sin\phi$

But $\quad OQ = PC = r\sin\phi$

Also AOQ is a triangle, right angled at A

Then
$$\frac{AO}{OQ} = \cos\theta$$

and
$$AO = r\cos\theta \cdot \sin\phi$$

Also $$\frac{AQ}{OQ} = \sin\theta$$

and $$AQ = r\sin\theta \cdot \sin\phi$$

If x, y, z are the rectangular co-ordinates of the point P, they can be expressed in terms of the polar co-ordinates.

$$x = AO = r\cos\theta \cdot \sin\phi$$
$$y = AQ = r\sin\theta \cdot \sin\phi$$
$$z = OC = r\cos\phi$$

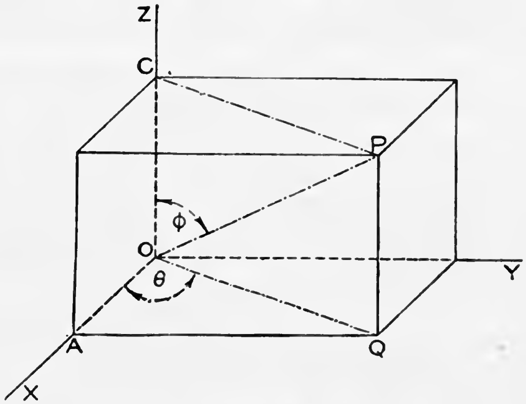


FIG. 24.

Conversely, the polar co-ordinates can be expressed in terms of the rectangular co-ordinates.

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

If α, β, and γ are the angles **OP** makes with the axes **OX**, **OY**, and **OZ** respectively, and l, m, and n are the corresponding direction cosines

$$l = \cos\alpha = \frac{x}{r}$$
$$= \cos\theta \cdot \sin\phi$$
$$m = \cos\beta = \frac{y}{r}$$
$$= \sin\theta \cdot \sin\phi$$
$$n = \cos\gamma = \frac{z}{r}$$
$$= \cos\phi$$

EXAMPLES V

(1) The co-ordinates of a point P are $(-3, 5, -7)$. If O is the origin, find the length of OP and the angles OP makes with the axes of reference.

(2) The co-ordinates of a point P are $(2{\cdot}1, 3{\cdot}4, 4{\cdot}7)$. If O is the origin, find the length of OP and the angles OP makes with the axes of reference.

(3) The co-ordinates of a point P are $(-3{\cdot}2, 5{\cdot}1, 3{\cdot}9)$. If O is the origin, find the length of OP and the angles OP makes with the axes of reference.

(4) The co-ordinates of a point P are $(2, 4, -6)$, and of a point Q $(3, -7, 5)$. Find the length of PQ and the angles PQ makes with the axes of reference.

(5) The co-ordinates of a point P are $(1{\cdot}8, 5{\cdot}3, 2{\cdot}9)$, and of a point Q $(3{\cdot}7, 2{\cdot}9, 5{\cdot}4)$. Find the length of PQ and the angles PQ makes with the axes of reference.

(6) P and Q are two points whose co-ordinates are $(3, 7, 2)$ and $(5, 3, 7)$ respectively. Find the length of the perpendicular drawn to the line PQ from the origin.

(7) The co-ordinates of three points P, Q, and R are $(3, 2, 4)$, $(5, 4, 7)$, and $(4, 7, 2)$ respectively. Find the lengths of the sides, the area, and the angles of the triangle PQR.

(8) The co-ordinates of three points P, Q, R are $(5, 2, 4)$, $(7, 5, 2)$, and $(9, 3, 7)$ respectively. Find the angle between the lines PQ and QR, and also the perpendicular distance from the point R to the line PQ.

(9) For each of the three planes

$$x - 2y + z - 2 = 0$$
$$12x + 15y - 10z - 60 = 0$$
and
$$6x + 4y + 3z - 12 = 0$$

Find (*a*) the intercepts on the axes of reference; (*b*) the length of OP, the perpendicular drawn from the origin to the plane; (*c*) the angles which OP makes with the axes of reference; (*d*) the angles which the plane makes with the axes of reference; (*e*) the co-ordinates of P.

(10) Find the angle between the two planes

$$3x + 2y + 6z - 6 = 0$$
and
$$4x + 5y + 10z - 20 = 0$$

(11) Find the direction cosines of the perpendiculars drawn from the origin to each of the planes

$$3x - 2y + 6z - 6 = 0$$
$$6x + 3y - 2z - 6 = 0$$

Do these two planes intersect at right angles?

(12) Find the length of the perpendicular drawn from the point whose co-ordinates are (8, 3, 5) to the plane whose equation is $21x - 35y + 15z - 105 = 0$

(13) The rectangular co-ordinates of a point are (1·9, 3·7, 2·4): find the polar co-ordinates.

(14) The polar co-ordinates of a point are ($r = 7{\cdot}2$, $\theta = 62°$, $\phi = 43°$): find the rectangular co-ordinates.

(15) The polar co-ordinates of a point P are ($r = 3{\cdot}2$, $\theta = 51°$, $\phi = 78°$), and of a point Q ($r = 6{\cdot}5$, $\theta = 69°$, $\phi = 38°$). If O is the origin, find the angle POQ.

CHAPTER VI

49. *The Slope of a Line.* The slope of a line could be measured directly by means of the angle it makes with the axis of x, but generally, in cases of plotting, the quantities plotted horizontally and vertically are not taken to the same scale, and therefore we do not get a true representation of the angle of slope. Now this angle can be given in terms of any of its trigonometrical ratios, and we have to consider which of these ratios can be most conveniently adapted to squared paper work. The tangent is given in terms of the quantities plotted vertically and horizontally, and therefore, if we take the line to form the hypotenuse of a right-angled triangle, then the perpendicular of this triangle can

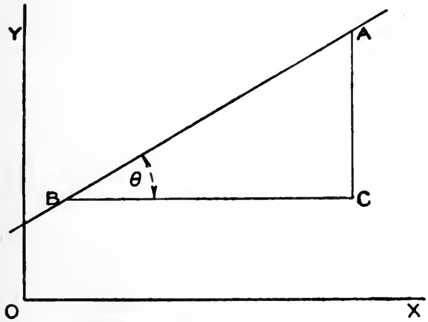


FIG. 25.

be measured by means of the vertical scale, and the base by means of the horizontal scale. Hence the angle of slope of a line can be obtained definitely by means of its tangent.

To find the slope of a line, take two points A and B (Fig. 25) on the line, as far removed as the limits of the question allow. Make AB the hypotenuse of the right-angled triangle ABC.

Then the slope of the line $= \tan\theta$

$$= \frac{AC}{BC}$$

where AC must be measured on the vertical scale and BC on the horizontal scale.

50. *The Slope of a Curve.* The slope of a curve at a given point may be approximately taken as the slope of a very small chord of the curve drawn from that point, and the smaller the chord is

made the more nearly correct does the approximation become. Thus, if we can find the slope of an infinitely small chord, one extremity of which is at the point, then we have found the actual slope of the curve at that point.

Let **P** be a point on a curve (Fig. 26), **PQ** a chord, and **PT** the tangent to the curve at the point **P**. Let **PT** make an angle θ with the axis of x and the chord **PQ** make an angle α with **PT**.

Then the slope of the chord **PQ** $= \tan(\theta + \alpha)$.

As the point **Q** approaches **P**, the chord **PQ** becomes smaller and smaller, and so does the angle α, and when the chord **PQ** is made infinitely small the angle α becomes negligible in comparison with θ.

Thus the slope of the infinitely small chord $= \tan\theta$.

It follows, therefore, that the slope of the infinitely small chord **PQ**, which gives the actual slope of the curve at the point **P**, is the same as that of the tangent to the curve at the point **P**.

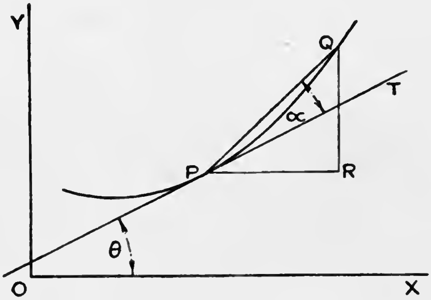


FIG. 26.

We can now take the slope of a curve at a certain point to be given by $\tan\theta$, where θ is the angle which the tangent to the curve, at that point, makes with the axis of x.

This provides us with a graphical way of finding the slope of a curve. We can draw the tangent to the curve at the required point, take two points on this line as far removed as the paper permits, make that part of the line between these two points the hypotenuse of a right-angled triangle, measure the perpendicular of this triangle to the vertical scale and the base to the horizontal scale, and

$$\text{the slope of the curve} = \tan\theta = \frac{\text{perpendicular}}{\text{base}}$$

We cannot obtain the true value of the slope of a curve in this way, because we have no definite construction for drawing the true tangent to the curve; we can only draw what appears to be the true tangent. If the supposed tangent is inclined to the axis of x at an angle slightly smaller than that of the true tangent, its effect is to make the perpendicular of the right-angled triangle slightly less, and at the same time to make the base slightly more. This makes the error in the fraction $\frac{\text{perpendicular}}{\text{base}}$ quite pronounced.

51. If we are given the law of a curve, then we are in a position to find the slope of the curve without using the graphical method, and the errors introduced by that method will not affect the result.

Let the co-ordinates of the point P (Fig. 26) be x, y, and the co-ordinates of the point Q $(x + \delta x)$, $(y + \delta y)$, where δx and δy are the increases in the values of x and y respectively.

δx and δy are also the base and perpendicular of the right-angled triangle **PQR**, whose hypotenuse is the chord **PQ**.

Then the slope of the chord **PQ** $= \tan(\theta + \alpha) = \frac{\delta y}{\delta x}$

When the chord **PQ** becomes infinitely small, the slope of the curve is the limiting value of the fraction $\frac{\delta y}{\delta x}$ when δx is made infinitely small, and this limiting value is represented by $\frac{dy}{dx}$.

Then the slope of the curve $= \frac{dy}{dx} = \tan\theta$.

As an example on the application of this method, let the law of a curve be $y = a + bx + cx^2$ where a, b, and c are constants.

Then at the point **P**, $\quad y = a + bx + cx^2$

at the point **Q**, $\quad y + \delta y = a + b(x + \delta x) + c(x + \delta x)^2$

$$= a + bx + b\,\delta x + cx^2 + 2cx\,\delta x + c(\delta x)^2$$

Subtracting $\quad \delta y = b\,\delta x + 2cx\,\delta x + c\,(\delta x)^2$

Slope of the chord **PQ**, $\quad \frac{\delta y}{\delta x} = b + 2cx + c\,\delta x$

making δx infinitely small.

Slope of the curve at the point **P**, $\quad \frac{dy}{dx} = b + 2cx$

Referring again to Fig. 26, since $\frac{\delta y}{\delta x} = \tan(\theta + \alpha)$, it necessarily follows that $\frac{\delta x}{\delta y} = \cot(\theta + \alpha)$.

In the limit when δx becomes infinitely small, $\frac{dy}{dx}$ becomes $\tan\theta$ and $\frac{dx}{dy}$ becomes $\cot\theta$. Because $\tan\theta$ and $\cot\theta$ are mutually reciprocal, $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are mutually reciprocal.

Then $$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}},\quad \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},\quad \text{or } \frac{dy}{dx}\times\frac{dx}{dy}=1$$

52. In general, if the law of the curve is $y=f(x)$,

then $$y+\delta y=f(x+\delta x)$$
$$\delta y=f(x+\delta x)-f(x)$$
$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

and then $\frac{dy}{dx}$ is the limiting value of the fraction $\frac{f(x+\delta x)-f(x)}{\delta x}$ when δx is made infinitely small.

This gives us a general method of determining the slope of a curve when the law of the curve is known, and this process of finding the slope is called *differentiation.*

In order to avoid going through this process each time we wish to work with a certain curve, we establish the results of differentiating well-known functions of x and use these as standard results.

$y=ax^n$ where a and n are constants.

If $$y=ax^n$$

then $$y+\delta y=a(x+\delta x)^n$$
$$=a\left\{x^n+nx^{n-1}\,\delta x+\frac{n(n-1)}{\underline{|2}}x^{n-2}(\delta x)^2\ \ldots\right\}$$
$$\delta y=a\left\{nx^{n-1}\,\delta x+\frac{n(n-1)}{\underline{|2}}x^{n-2}(\delta x)^2+\ \ldots\right\}$$
$$\frac{\delta y}{\delta x}=a\left\{nx^{n-1}+\frac{n(n-1)}{\underline{|2}}x^{n-2}\,\delta x+\ \ldots\right\}$$

When δx is made infinitely small, all of the terms on the right-hand side involving δx and powers of δx can be neglected

and $$\frac{dy}{dx}=anx^{n-1}$$

It should be noticed here that a and n, the constants of the curve, are constants of different types. a is a constant multiplier, and remains a multiplier during differentiation ; n is a constant power and differentiation diminishes it by unity, while the result is multiplied by n.

We can apply this result to differentiate any function of x of the form $y=ax^n$.

(1) If $$y = 8\sqrt{x}$$

then $$y = 8x^{\frac{1}{2}} \qquad \text{then } a = 8 \text{ and } n = \frac{1}{2}$$

$$\frac{dy}{dx} = 8 \times \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= 4x^{-\frac{1}{2}} = \frac{4}{\sqrt{x}}$$

(2) If $$y = \frac{12}{\sqrt{x^3}}$$

then $$y = \frac{12}{x^{\frac{3}{2}}} = 12x^{-\frac{3}{2}} \qquad a = 12 \text{ and } n = -\frac{3}{2}$$

$$\frac{dy}{dx} = 12\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$$

$$= -18x^{-\frac{5}{2}} = -\frac{18}{x^{2\cdot 5}}$$

53. If $y = a$. This represents a line drawn parallel to the axis of x, at a distance a from it. Since it is parallel to the axis of x, the angle $\theta = 0$, and $\tan\theta = 0$,

and so the slope, or $$\frac{dy}{dx} = \tan\theta = 0$$

Thus if $y = a$ $$\frac{dy}{dx} = 0$$

54. $y = e^{ax}$ where a is a constant and e is the base of Napierian logarithms.

If $$y = e^{ax}$$

then $$y + \delta y = e^{a(x+\delta x)} = e^{ax} \times e^{a\,\delta x}$$

$$\delta y = e^{ax} \times e^{a\,\delta x} - e^{ax}$$

$$= e^{ax}(e^{a\,\delta x} - 1)$$

$$= e^{ax}\left(1 + a\,\delta x + \frac{a^2}{\underline{|2}}(\delta x)^2 + \ldots - 1\right)$$

$$= e^{ax}\left(a\,\delta x + \frac{a^2}{\underline{|2}}(\delta x)^2 + \ldots\right)$$

$$\frac{\delta y}{\delta x} = e^{ax}\left(a + \frac{a^2}{\underline{|2}}\delta x + \ldots\right)$$

Making δx infinitely small,

then $$\frac{dy}{dx} = ae^{ax}$$

We can use this result to establish other standard forms.

(1) If $y = c^x$ where c is a constant,

$$y = e^{ax} = (e^a)^x = c^x$$

where $e^a = c$ or $a = \log_e c$

Then
$$\frac{dy}{dx} = ae^{ax}$$
$$= c^x \log_e c$$

(2) If $y = \log_e x$

then $x = e^y$

and
$$\frac{dx}{dy} = e^y = x$$

but
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x}$$

(3) If $y = \log_a x$ where a is a constant,

then $x = a^y$

and
$$\frac{dx}{dy} = a^y \log_e a = x \log_e a$$

but
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x \log_e a}$$

55. *The Trigonometrical Functions.*

(1) If $y = \sin(ax + b)$ where a and b are constants,

then $y + \delta y = \sin\{a(x + \delta x) + b\}$

$$\delta y = \sin(ax + b + a\,\delta x) - \sin(ax + b)$$
$$= 2\cos\left(ax + b + \frac{a\,\delta x}{2}\right)\sin\frac{a\,\delta x}{2}$$

$$\frac{\delta y}{\delta x} = a\cos\left(ax + b + \frac{a\,\delta x}{2}\right)\frac{\sin\frac{a\,\delta x}{2}}{\frac{a\,\delta x}{2}}$$

Making δx infinitely small,
$$\frac{\sin\frac{a\,\delta x}{2}}{\frac{a\,\delta x}{2}} = 1$$

and
$$\frac{dy}{dx} = a\cos(ax + b)$$

(2) If $y = \cos(ax + b)$ where a and b are constants,

then $y + \delta y = \cos\{a(x + \delta x) + b\}$

$$\delta y = \cos(ax + b + a\,\delta x) - \cos(ax + b)$$

$$= -2\sin\left(ax + b + \frac{a\,\delta x}{2}\right)\sin\frac{a\,\delta x}{2}$$

$$\frac{\delta y}{\delta x} = -a\sin\left(ax + b + \frac{a\,\delta x}{2}\right)\frac{\sin\frac{a\,\delta x}{2}}{\frac{a\,\delta x}{2}}$$

Making δx infinitely small, $\dfrac{\sin\frac{a\,\delta x}{2}}{\frac{a\,\delta x}{2}} = 1$

and $\dfrac{dy}{dx} = -a\sin(ax + b)$

(3) If $y = \tan(ax + b)$ where a and b are constants, then

$$y = \frac{\sin(ax + b)}{\cos(ax + b)}$$

and

$$y + \delta y = \frac{\sin\{a(x + \delta x) + b\}}{\cos\{a(x + \delta x) + b\}}$$

$$\delta y = \frac{\sin(ax + b + a\,\delta x)}{\cos(ax + b + a\,\delta x)} - \frac{\sin(ax + b)}{\cos(ax + b)}$$

$$= \frac{\sin(ax + b + a\,\delta x)\cos(ax + b) - \cos(ax + b + a\,\delta x)\sin(ax + b)}{\cos(ax + b + a\,\delta x)\cos(ax + b)}$$

$$= \frac{\sin a\,\delta x}{\cos(ax + b + a\,\delta x)\cos(ax + b)}$$

$$\frac{\delta y}{\delta x} = a\,\frac{\frac{\sin a\,\delta x}{a\,\delta x}}{\cos(ax + b + a\,\delta x)\cos(ax + b)}$$

Making δx infinitely small $\dfrac{\sin a\,\delta x}{a\,\delta x} = 1$

and $\dfrac{dy}{dx} = \dfrac{a}{\cos^2(ax + b)} = a\sec^2(ax + b)$

These results enable us to work with the trigonometrical functions of all forms of angles, by properly adjusting the constants a and b.

If $b = 0$ the angle becomes ax, the ordinary multiple angle.

If $b = 0$ and $a = 1$, the angle is simply x.

56. *The Inverse Trigonometrical Functions.*

(1) If $y = \sin^{-1} x$

Then $$x = \sin y$$

and $$\frac{dx}{dy} = \cos y = \sqrt{1 - x^2} \quad \text{(Fig. 27)}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1 - x^2}}$$

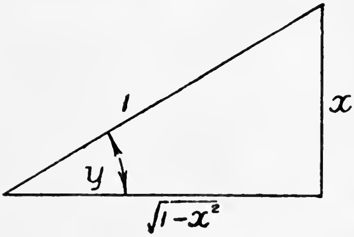


FIG. 27.

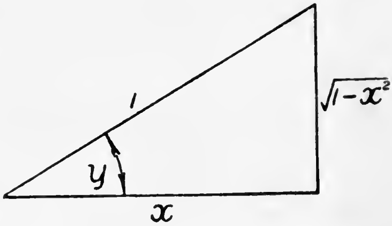


FIG. 28.

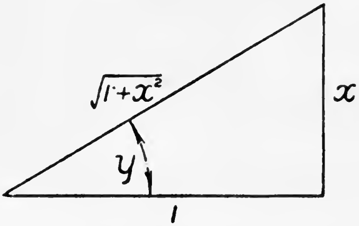


FIG. 29.

(2) If $y = \cos^{-1} x$

then $$x = \cos y$$

and $$\frac{dx}{dy} = -\sin y = -\sqrt{1 - x^2} \quad \text{(Fig. 28)}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{\sqrt{1 - x^2}}.$$

(3) If $y = \tan^{-1} x$

then $$x = \tan y$$

and $$\frac{dx}{dy} = \sec^2 y = 1 + x^2 \quad \text{(Fig. 29)}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1 + x^2}$$

57. *The Hyperbolic Functions.*

(1) If $y = \sinh x$

then $$y = \frac{1}{2}(e^x - e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}\{e^x - (-e^{-x})\}$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

(2) If $y = \cosh x$

then $$y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$= \sinh x.$$

58. *The Rules for Differentiation.*

(*a*) Now $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta x} \times \frac{\delta z}{\delta z}$ since $\frac{\delta y}{\delta x}$ is a fraction, and the value of a fraction remains the same when numerator and denominator are multiplied by the same quantity. Let δz represent the increase in some function z, which itself depends upon x, then when δx is made infinitely small δz becomes infinitely small.

Then $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \times \frac{\delta z}{\delta x}$$

Now $\frac{\delta y}{\delta z}$ is the slope of a chord of the curve obtained by plotting z horizontally and y vertically, and this becomes $\frac{dy}{dz}$, the actual slope of the curve when δx, and therefore δz, become infinitely small.

Similarly, when δx is made infinitely small $\frac{\delta z}{\delta x}$ becomes $\frac{dz}{dx}$, the actual slope of the curve obtained by plotting x horizontally and z vertically.

Thus, in the limit, when δx is made infinitely small,

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

The following examples will illustrate the use of this rule:

(1) To differentiate $\sin^n x$.

Then $$y = z^n \text{ where } z = \sin x$$

$$\frac{dy}{dz} = nz^{n-1} \text{ and } \frac{dz}{dx} = \cos x$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= nz^{n-1} \cos x$$
$$= n \sin^{n-1} x \cos x$$

(2) To differentiate $e^{\sin^{-1} x}$

Then $$y = e^z \text{ where } z = \sin^{-1} x$$
$$\frac{dy}{dz} = e^z \text{ and } \frac{dz}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= \frac{e^z}{\sqrt{1 - x^2}}$$
$$= \frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$$

(3) To differentiate $\tan^{-1} x^2$

Then $$y = \tan^{-1} z \text{ where } z = x^2$$
$$\frac{dy}{dz} = \frac{1}{1 + z^2} \text{ and } \frac{dz}{dx} = 2x$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= \frac{2x}{1 + z^2}$$
$$= \frac{2x}{1 + x^4}$$

(4) To differentiate $\log_e(a + bx + cx^2)$

Then $$y = \log_e z \text{ where } z = a + bx + cx^2$$
$$\frac{dy}{dz} = \frac{1}{z} \text{ and } \frac{dz}{dx} = b + 2cx$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= \frac{b + 2cx}{z}$$
$$= \frac{b + 2cx}{a + bx + cx^2}$$

When we differentiate the logarithm of any function of x, we get a fraction the denominator of which is that function of x and the numerator is the differential coefficient of the denominator.

For if $y = \log_e f(x)$,

then $$y = \log_e z \text{ where } z = f(x)$$

$$\frac{dy}{dz} = \frac{1}{z} \text{ and } \frac{dz}{dx} = \frac{d}{dx}\{f(x)\}$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= \frac{1}{z}\frac{d}{dx}\{f(x)\}$$

$$= \frac{\frac{d}{dx}\{f(x)\}}{f(x)}$$

(*b*) To differentiate the sum of a certain number of functions of x.

For if $y = u + v + w + \ldots$ where $u, v, w \ldots$ are functions of x,

then $$y + \delta y = (u + \delta u) + (v + \delta v) + (w + \delta w) + \ldots$$

and $$\delta y = \delta u + \delta v + \delta w + \ldots$$

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x} + \ldots$$

Making δx infinitely small,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots$$

Hence the differential coefficient of the sum of a certain number of functions of x is the sum of the differential coefficients of each function.

Thus if $y = a + b\sqrt{x} + \dfrac{c}{\sqrt{x}} + dx^3$

Then $$y = a + bx^{\frac{1}{2}} + cx^{-\frac{1}{2}} + dx^3$$

$$\frac{dy}{dx} = 0 + \frac{1}{2}bx^{\frac{1}{2}-1} - \frac{1}{2}cx^{-\frac{1}{2}-1} + 3dx^{3-1}$$

$$= \frac{1}{2}bx^{-\frac{1}{2}} - \frac{1}{2}cx^{-\frac{3}{2}} + 3dx^2$$

$$= \frac{b}{2\sqrt{x}} - \frac{c}{2\sqrt{x^3}} + 3dx^2$$

(c) To differentiate a product.

If $y = uv$ where u and v are functions of x,

$$\text{then} \qquad y + \delta y = (u + \delta u)(v + \delta v)$$
$$= uv + u\,\delta v + v\,\delta u + \delta u\,\delta v$$
$$\text{and} \qquad \delta y = u\,\delta v + v\,\delta u + \delta u\,\delta v$$
$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

Making δx infinitely small,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Since δu becomes infinitely small along with δx, the term $\delta u\,\dfrac{dv}{dx}$ becomes negligibly small when δx is made infinitely small.

To differentiate $x^n\,n^x$,

$$y = x^n\,n^x = u\,v$$
$$\text{Then} \qquad u = x^n \text{ and } \frac{du}{dx} = nx^{n-1}$$
$$v = n^x \text{ and } \frac{dv}{dx} = n^x \log_e n$$
$$\text{but} \qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x^n\,n^x \log_e n + n^x\,nx^{n-1}$$
$$= n^x\,x^{n-1}(x \log_e n + n)$$

If $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ when $y = uv$, dividing throughout by y we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$$

Sometimes this second form is more easily worked with than the first form. Also each term on the right-hand side is built up from one term of the product only. For example, the first term only contains u, the second term only contains v, and therefore we can extend the result to suit the case when the product contains any number of factors.

Thus if $y = u\,v\,w$,

$$\text{then} \qquad \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

(d) To differentiate a fraction.

$$\text{Let} \qquad y = \frac{u}{v}$$
$$\text{then} \qquad y + \delta y = \frac{u + \delta u}{v + \delta v}$$

and
$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$
$$= \frac{uv + v\,\delta u - uv - u\,\delta v}{v(v + \delta v)}$$
$$= \frac{v\,\delta u - u\,\delta v}{v(v + \delta v)}$$
$$\frac{\delta y}{\delta x} = \frac{v\,\frac{\delta u}{\delta x} - u\,\frac{\delta v}{\delta x}}{v(v + \delta v)}.$$

Making δx infinitely small,
$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

To differentiate $\dfrac{\log_e x}{x^2}$
$$y = \frac{\log_e x}{x^2} = \frac{u}{v}$$

Then $u = \log_e x$ and $\dfrac{du}{dx} = \dfrac{1}{x}$
$$v = x^2 \text{ and } \frac{dv}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{x^2\,\frac{1}{x} - 2x\log_e x}{x^4}$$
$$= \frac{x(1 - 2\log_e x)}{x^4}$$
$$= \frac{1 - 2\log_e x}{x^3}$$

59. *Logarithmic Differentiation.* A method which can be used to great advantage is that of taking logarithms of both sides and then differentiating throughout.

For example, let $y = \dfrac{u^a\,v^b}{w^c}$

Then $\log_e y = a\log_e u + b\log_e v - c\log_e w$

As u, v, and w are functions of x, the right-hand side can be easily differentiated, as we have a simple means of differentiating the logarithm of a function of x (see § 58 (a)); but we must also be able to differentiate $\log_e y$ with respect to x.

Then if $z = \log_e y$, $\dfrac{dz}{dx}$ is required,

but $$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

and $$\frac{dz}{dy} = \frac{1}{y}$$

Hence $$\frac{dz}{dx} = \frac{1}{y}\frac{dy}{dx}$$

That is, $\dfrac{1}{y}\dfrac{dy}{dx}$ is the result of differentiating $\log_e y$ with respect to x.

To differentiate $x^n e^{nx} \sin^n x$.

$$y = x^n e^{nx} \sin^n x$$

$$\log_e y = n \log_e x + nx + n \log_e \sin x$$

Differentiating $$\frac{1}{y}\frac{dy}{dx} = n\left\{\frac{1}{x} + 1 + \frac{\cos x}{\sin x}\right\}$$

$$= \frac{n}{x}\{1 + x + x \cot x\}$$

Multiplying throughout by y or $x^n e^{nx} \sin^n x$

$$\frac{dy}{dx} = nx^{n-1} e^{nx} \sin^n x\{1 + x + x \cot x\}$$

We can use logarithmic differentiation to establish the differential coefficients of $\cot(ax + b)$, $\sec(ax + b)$, and $\operatorname{cosec}(ax + b)$, and these results can be used as standard forms.

(1) $$y = \cot(ax + b) = \frac{\cos(ax + b)}{\sin(ax + b)}$$

Then $$\log_e y = \log_e \cos(ax + b) - \log_e \sin(ax + b)$$

Differentiating $$\frac{1}{y}\frac{dy}{dx} = \frac{-a \sin(ax + b)}{\cos(ax + b)} - \frac{a \cos(ax + b)}{\sin(ax + b)}$$

$$= -a\left\{\frac{\sin^2(ax + b) + \cos^2(ax + b)}{\sin(ax + b)\cos(ax + b)}\right\}$$

$$= -\frac{a}{\sin(ax + b)\cos(ax + b)}$$

Then $$\frac{dy}{dx} = -\frac{a}{\sin(ax + b)\cos(ax + b)} \times \frac{\cos(ax + b)}{\sin(ax + b)}$$

$$= -\frac{a}{\sin^2(ax + b)}$$

$$= -a \operatorname{cosec}^2(ax + b)$$

(2) $$y = \sec(ax + b) = \frac{1}{\cos(ax + b)}$$

Then $$\log_e y = -\log_e \cos(ax + b)$$

Differentiating $\frac{1}{y}\frac{dy}{dx} = -\left\{\frac{-a \sin (ax+b)}{\cos (ax+b)}\right\}$

$= a \tan (ax+b)$

and $\frac{dy}{dx} = a \tan (ax+b) \sec (ax+b)$

(3) $y = \operatorname{cosec} (ax+b) = \frac{1}{\sin (ax+b)}$

Then $\log_e y = -\log_e \sin (ax+b)$

Differentiating $\frac{1}{y}\frac{dy}{dx} = -\frac{a \cos (ax+b)}{\sin (ax+b)}$

$= -a \cot (ax+b)$

and $\frac{dy}{dx} = -a \cot (ax+b) \operatorname{cosec} (ax+b)$

TABLE OF STANDARD FORMS

y	$\frac{dy}{dx}$
a	0
ax^n	anx^{n-1}
e^{ax}	ae^{ax}
c^x	$c^x \log_e c$
$\log_e x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log_e a}$
$\sin (ax+b)$	$a \cos (ax+b)$
$\cos (ax+b)$	$-a \sin (ax+b)$
$\tan (ax+b)$	$a \sec^2 (ax+b)$
$\cot (ax+b)$	$-a \operatorname{cosec}^2 (ax+b)$
$\sec (ax+b)$	$a \sec (ax+b) \tan (ax+b)$
$\operatorname{cosec} (ax+b)$	$-a \operatorname{cosec} (ax+b) \cot (ax+b)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

Rules for Differentiation.

(1) $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ or $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

(2) $\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx}$

(3) If $y = u + v + w + \dots$ $\quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots$.

(4) If $y = u\,v$ $\quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

or $\frac{1}{y}\,\frac{dy}{dx} = \frac{1}{u}\,\frac{du}{dx} + \frac{1}{v}\,\frac{dv}{dx}$

(5) If $y = \frac{u}{v}$ $\quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(6) Differentiating $\log_e y$ with respect to x gives $\frac{1}{y}\,\frac{dy}{dx}$

60. The following examples will illustrate how the rules for differentiation can be applied and how the table of standard forms can be used :

Example 1. Differentiate $(1 - x)\sqrt{1 + x^2}$

(*a*) Treating it as a product and working with the rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Then $\quad u = 1 - x$ and $\frac{du}{dx} = -1$

$$v = (1 + x^2)^{\frac{1}{2}} = z^{\frac{1}{2}} \text{ where } z = 1 + x^2$$

$$\frac{dv}{dz} = \frac{1}{2}z^{-\frac{1}{2}} = \frac{1}{2\sqrt{1 + x^2}} \text{ and } \frac{dz}{dx} = 2x$$

Then $\quad \frac{dv}{dx} = \frac{dv}{dz}\,\frac{dz}{dx} = \frac{x}{\sqrt{1 + x^2}}$

$$\frac{dy}{dx} = (1 - x)\frac{x}{\sqrt{1 + x^2}} - \sqrt{1 + x^2}$$

$$= \frac{1}{\sqrt{1 + x^2}}\{x(1 - x) - (1 + x^2)\}$$

$$= \frac{-2x^2 + x - 1}{\sqrt{1 + x^2}}$$

(*b*) Treating it as a product and working with the rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$$

Then $u = 1 - x$

$$\frac{du}{dx} = -1 \text{ and } \frac{1}{u}\frac{du}{dx} = -\frac{1}{1-x}$$

Also $v = (1 + x^2)^{\frac{1}{2}}$

$$\frac{dv}{dx} = \frac{x}{(1+x^2)^{\frac{1}{2}}} \text{ and } \frac{1}{v}\frac{dv}{dx} = \frac{x}{1+x^2}$$

Then

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{1-x} + \frac{x}{1+x^2}$$

$$= \frac{-(1+x^2) + x(1-x)}{(1+x^2)(1-x)}$$

$$= \frac{-2x^2 + x - 1}{(1+x^2)(1-x)}$$

Multiplying across by y

$$\frac{dy}{dx} = \frac{-2x^2 + x - 1}{(1+x^2)(1-x)} \times (1-x)\sqrt{1+x^2}$$

$$= \frac{-2x^2 + x - 1}{\sqrt{1+x^2}}$$

(*c*) Working logarithmically

$$\log_e y = \log_e(1-x) + \frac{1}{2}\log_e(1+x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{1-x} + \frac{1}{2}\,\frac{2x}{1+x^2}$$

$$= -\frac{1}{1-x} + \frac{x}{1+x^2}$$

$$= \frac{-2x^2 + x - 1}{(1-x)(1+x^2)}$$

Multiplying across by y

$$\frac{dy}{dx} = \frac{-2x^2 + x - 1}{(1-x)(1+x^2)} \times (1-x)\sqrt{1+x^2}$$

$$= \frac{-2x^2 + x - 1}{\sqrt{1+x^2}}$$

Example 2. Differentiate $e^{ax} \sin bx$.

(*a*) Using the rule $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$u = e^{ax} \text{ and } \frac{du}{dx} = ae^{ax}$$

$$v = \sin bx \text{ and } \frac{dv}{dx} = b \cos bx$$

Then
$$\begin{aligned}\frac{dy}{dx} &= be^{a} \cos bx + ae^{ax} \sin bx \\ &= e^{ax}(b \cos bx + a \sin bx) \\ &= e^{ax} \sin bx(b \cot bx + a)\end{aligned}$$

(*b*) Using the rule $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{u}\dfrac{du}{dx} + \dfrac{1}{v}\dfrac{dv}{dx}$

$$u = e^{ax}, \quad \frac{du}{dx} = ae^{ax} \text{ and } \frac{1}{u}\frac{du}{dx} = a$$

$$v = \sin bx, \frac{dv}{dx} = b \cos bx, \text{ and } \frac{1}{v}\frac{dv}{dx} = \frac{b \cos bx}{\sin bx} = b \cot bx$$

Then
$$\frac{1}{y}\frac{dy}{dx} = a + b \cot bx$$

and
$$\frac{dy}{dx} = e^{ax} \sin bx(a + b \cot bx)$$

(*c*) Working logarithmically

$$\begin{aligned}\log_e y &= ax + \log_e \sin bx \\ \frac{1}{y}\frac{dy}{dx} &= a + \frac{b \cos bx}{\sin bx} \\ &= a + b \cot bx\end{aligned}$$

and
$$\frac{dy}{dx} = e^{ax} \sin bx\,(a + b \cot bx)$$

Example 3. Differentiate $\dfrac{1 + x^2}{\sqrt{1 - x^2}}$.

(*a*) Working logarithmically

$$\begin{aligned}\log_e y &= \log(1 + x^2) - \frac{1}{2}\log_e(1 - x^2) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{2x}{1 + x^2} - \frac{1}{2}\left\{\frac{-2x}{1 - x^2}\right\} \\ &= \frac{2x}{1 + x^2} + \frac{x}{1 - x^2} \\ &= \frac{2x(1 - x^2) + x(1 + x^2)}{(1 + x^2)(1 - x^2)} \\ &= \frac{x(3 - x^2)}{(1 + x^2)(1 - x^2)}\end{aligned}$$

and $$\frac{dy}{dx} = \frac{x(3-x^2)}{(1+x^2)(1-x^2)} \times \frac{1+x^2}{\sqrt{1-x^2}}$$
$$= \frac{x(3-x^2)}{(1-x^2)^{\frac{3}{2}}}$$

(*b*) Using the rule $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$u = 1 + x^2 \text{ and } \frac{du}{dx} = 2x$$

$$v = (1-x^2)^{\frac{1}{2}} \text{ and } \frac{dv}{dx} = \frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}\left\{2x\sqrt{1-x^2} + \frac{x}{\sqrt{1-x^2}}(1+x^2)\right\}$$
$$= \frac{1}{(1-x^2)^{\frac{3}{2}}}\{2x(1-x^2) + x(1+x^2)\}$$
$$= \frac{1}{(1-x^2)^{\frac{3}{2}}}(3x - x^3)$$
$$= \frac{x(3-x^2)}{(1-x^2)^{\frac{3}{2}}}$$

In general the most straightforward method to use for differentiation is to work logarithmically, and this should be used in all cases where it is possible.

EXAMPLES VI

Differentiate with respect to x.

(1) $e^{\sin x}$ (2) $e^{\sqrt{x}}$

(3) $\log_e \sin x$ (4) $x \sin x$

(5) $\dfrac{\sin x}{x}$ (6) $\dfrac{x}{\sin x}$

(7) $x^3 \sin x$ (8) $x^3 e^x$

(9) $x^n \log_e x$ (10) $x^n \log_a x$

(11) $x^4 \tan x$ (12) $x^n \log_e \sin x$

(13) $\dfrac{x^n}{\sin nx}$ (14) $\dfrac{\sin nx}{x^n}$

(15) $\dfrac{\sin nx}{e^{nx}}$ (16) $\dfrac{e^{nx}}{\sin nx}$

(17) $e^{nx} \cos nx$ (18) $x^n e^{nx} \sin nx$

(19) $e^{nx} \tan nx$ (20) $e^{nx} \sec nx$

(21) $e^{nx} \tan nx \sec nx$

(22) $\dfrac{x^3}{e^x \sin x}$

(23) $\dfrac{ax^2 + bx + c}{cx^2 + bx + a}$

(24) $(x + a)^n(x + b)^m$

(25) $\log_e \dfrac{1 + x}{1 - x}$

(26) $\log_e \dfrac{1 + x^2}{1 - x^2}$

(27) $\log_e \dfrac{x^2 + x + 1}{x^2 - x + 1}$

(28) $\log_e \dfrac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2} - x}$

(29) $e^x \log_e (x + a)$

(30) $a^x \log_e (x + a)$

(31) $\log_e (x + e^x)$

(32) $e^x + \log_e x$

(33) $\dfrac{e^x}{\log_e x}$

(34) $\log_e (xe^x)$

(35) $\cos x \cos 2x \cos 3x$

(36) $\dfrac{\sin x \sin 2x}{\sin 3x}$

(37) $e^{\sqrt{\sin x}}$

(38) $e^{\sin^{-1} x}$

(39) $\cos^{-1} \dfrac{1 - x^2}{1 + x^2}$

(40) $x \tan^{-1} x$

(41) $\log_e x \tan^{-1} x$

(42) $x \log_e (\tan^{-1} x)$

(43) $\tan x \sin^{-1} x$

(44) $\tan^{-1} x \sec x$

(45) $e^x \sin^{-1} x$

(46) $\sqrt{\dfrac{1 + x}{1 - x}}$

(47) $\sqrt{\dfrac{1 - x}{1 + x}}$

(48) $\sqrt{\dfrac{1 - x^2}{1 + x^2}}$

(49) $\sqrt{\dfrac{1 - x^3}{1 + x^3}}$

(50) $\dfrac{1 - x^2}{\sqrt{1 + x^2}}$

(51) $\sqrt{\dfrac{1 - x}{1 + x + x^2}}$

(52) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

(53) $\dfrac{1 - x}{\sqrt{1 + x^2}}$

(54) $\sin^{-1} \dfrac{1}{\sqrt{1 + x^2}}$

(55) $\sin^{-1} \dfrac{5 + 4x}{\sqrt{41}}$

CHAPTER VII

61. *Successive Differentiation.* When we are given y as a function of x and we find $\frac{dy}{dx}$, then $\frac{dy}{dx}$ is also expressed as a function of x, and therefore it is possible to differentiate again with respect to x. This process is represented symbolically by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$. In fact, with a few exceptions the process of differentiation can be performed as many times as we please, and the result is represented symbolically by $\frac{d^ny}{dx^n}$ where n gives the number of times the differentiation has been performed.

Thus if
$$y = x^n$$
$$\frac{dy}{dx} = nx^{n-1}$$
$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$
$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$$
and
$$\frac{d^ry}{dx^r} = n(n-1)(n-2)\ \ldots\ (n-r+1)x^{n-r}$$

while taking n as a positive integer and putting $r = n$
$$\frac{d^ny}{dx^n} = n(n-1)(n-2)\ \ldots\ 1$$
$$= \underline{|n}$$

If n is a positive integer, x^n can only be differentiated n times; but if n is a negative integer or a fraction, positive or negative, there is no limit to the number of times x^n can be differentiated. For successive differentiation it is convenient, if possible, to find the general result of differentiating n times, *i.e.* $\frac{d^ny}{dx^n}$, and then any differential coefficient can be found by giving n the required value in that general result. There are some functions for which the nth differential coefficient can be readily obtained, such as x^n, $\log_e x$, e^{ax}, a^x, $\sin(ax+b)$, $\cos(ax+b)$, $e^{ax}\sin bx$, $e^{ax}\cos bx$.

To find the nth differential coefficient of $e^{ax}\sin bx$.

$$y = e^{ax}\sin bx$$

$$\begin{aligned}\frac{dy}{dx} &= ae^{ax}\sin bx + be^{ax}\cos bx\\ &= e^{ax}(a\sin bx + b\cos bx)\\ &= \sqrt{a^2+b^2}\,e^{ax}\sin(bx+\alpha),\text{ where }\tan\alpha = \frac{b}{a}\end{aligned}$$

It appears in this example that differentiation is equivalent to multiplication by $\sqrt{a^2+b^2}$, and at the same time increasing the angle by α. If this is so, then

$$\frac{d^2y}{dx^2} = (a^2+b^2)\,e^{ax}\sin(bx+2\alpha)$$

$$\frac{d^3y}{dx^3} = (a^2+b^2)^{\frac{3}{2}}\,e^{ax}\sin(bx+3\alpha)$$

and $$\frac{d^ny}{dx^n} = (a^2+b^2)^{\frac{n}{2}}\,e^{ax}\sin(bx+n\alpha)$$

If this result is true for all integral values of n, it must be true for $(n+1)$, and $\frac{d^{n+1}y}{dx^{n+1}} = (a^2+b^2)^{\frac{n+1}{2}}\,e^{ax}\sin\{bx+(n+1)\alpha\}$, and this result can be established by differentiating $\frac{d^ny}{dx^n}$

and $$\begin{aligned}\frac{d^{n+1}y}{dx^{n+1}} &= (a^2+b^2)^{\frac{n}{2}}\{ae^{ax}\sin(bx+n\alpha)+be^{ax}\cos(bx+n\alpha)\}\\ &= (a^2+b^2)^{\frac{n}{2}}\,e^{ax}\{a\sin(bx+n\alpha)+b\cos(bx+n\alpha)\}\\ &= (a^2+b^2)^{\frac{n}{2}}\,e^{ax}\sqrt{a^2+b^2}\sin(bx+n\alpha+\alpha)\\ &= (a^2+b^2)^{\frac{n+1}{2}}\,e^{ax}\sin\{bx+(n+1)\alpha\}\end{aligned}$$

which agrees with the anticipated result, and therefore

$$\frac{d^ny}{dx^n} = (a^2+b^2)^{\frac{n}{2}}\,e^{ax}\sin(bx+n\alpha)\text{ where }\tan\alpha = \frac{b}{a}$$

for all integral values of n.

62. Other functions are of such a form that it is impossible to obtain the nth differential coefficient, but the work of successive differentiation can be made as simple as possible by working to some general rule. Functions such as $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, $e^x\sec x$, $e^x\operatorname{cosec} x$, can be treated in this way.

For example, to successively differentiate $\tan x$, we want as a general rule to be able to differentiate $\tan^n x$.

$$y = \tan^n x = z^n \text{ where } z = \tan x$$

$$\frac{dy}{dz} = nz^{n-1} \text{ and } \frac{dz}{dx} = \sec^2 x$$

$$\begin{aligned} \frac{dy}{dx} &= nz^{n-1} \sec^2 x \\ &= n \tan^{n-1} x(1 + \tan^2 x) \\ &= n \tan^{n-1} x + n \tan^{n+1} x \end{aligned}$$

If we give n any integral value, we are enabled to differentiate any integral power of $\tan x$.

When $n = 1$ $y = \tan x$ and $\frac{dy}{dx} = 1 + \tan^2 x$

$n = 2$ $y = \tan^2 x$ and $\frac{dy}{dx} = 2(\tan x + \tan^3 x)$

$n = 3$ $y = \tan^3 x$ and $\frac{dy}{dx} = 3(\tan^2 x + \tan^4 x)$

Successively differentiating $\tan x$

$$\begin{aligned} y &= \tan x \\ \frac{dy}{dx} &= 1 + \tan^2 x \\ \frac{d^2y}{dx^2} &= 2 \tan x + 2 \tan^3 x \\ \frac{d^3y}{dx^3} &= 2(1 + \tan^2 x) + 6(\tan^2 x + \tan^4 x) \\ &= 2 + 8 \tan^2 x + 6 \tan^4 x \\ \frac{d^4y}{dx^4} &= 16(\tan x + \tan^3 x) + 24(\tan^3 x + \tan^5 x) \\ &= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \\ \frac{d^5y}{dx^5} &= 16(1 + \tan^2 x) + 120(\tan^2 x + \tan^4 x) + 120(\tan^4 x + \tan^6 x) \\ &= 16 + 136 \tan^2 x + 240 \tan^4 x + 120 \tan^6 x \\ \frac{d^6y}{dx^6} &= 272(\tan x + \tan^3 x) + 960(\tan^3 x + \tan^5 x) + 720(\tan^5 x \\ &\qquad + \tan^7 x) \\ &= 272 \tan x + 1232 \tan^3 x + 1680 \tan^5 x + 720 \tan^7 x \end{aligned}$$

Similarly, if we find the differential coefficients of $\cot^n x$, $\sec x$, $\tan^n x$, and $\operatorname{cosec} x$, $\cot^n x$, we can use them to differentiate successively $\cot x$, $\sec x$, and $\operatorname{cosec} x$ respectively.

As a further application of the use of this method, let us find the first four differential coefficients of $e^x \sec x$.

To differentiate $e^x \sec x$ successively, we require to be able to differentiate $e^x \sec x \tan^n x$.

$$y = e^x \sec x \tan^n x$$

$$\log y = x + \log \sec x + n \log \tan x$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{\sec x \tan x}{\sec x} + n\frac{\sec^2 x}{\tan x}$$

$$= \frac{1}{\tan x}\{\tan x + \tan^2 x + n(1 + \tan^2 x)\}$$

$$\frac{dy}{dx} = e^x \sec x \tan^{n-1} x\{n + \tan x + (n+1)\tan^2 x\}$$

$$= e^x \sec x\{n \tan^{n-1} x + \tan^n x + (n+1)\tan^{n+1} x\}$$

when $n = 0$, $y = e^x \sec x$, $\frac{dy}{dx} = e^x \sec x\,(1 + \tan x)$

when $n = 1$, $y = e^x \sec x \tan x$, $\frac{dy}{dx} = e^x \sec x\,(1 + \tan x + 2\tan^2 x)$

when $n = 2$, $y = e^x \sec x \tan^2 x$, $\frac{dy}{dx} = e^x \sec x\,(2\tan x + \tan^2 x + 3\tan^3 x)$

when $n = 3$, $y = e^x \sec x \tan^3 x$, $\frac{dy}{dx} = e^x \sec x\,(3\tan^2 x + \tan^3 x + 4\tan^4 x)$

Thus if $y = e^x \sec x$

$$\frac{dy}{dx} = e^x \sec x\,(1 + \tan x)$$

$$\frac{d^2y}{dx^2} = e^x \sec x\{(1 + \tan x) + (1 + \tan x + 2\tan^2 x)\}$$

$$= e^x \sec x\,(2 + 2\tan x + 2\tan^2 x)$$

$$\frac{d^3y}{dx^3} = e^x \sec x\{2(1 + \tan x) + 2(1 + \tan x + 2\tan^2 x) + 2(2\tan x + \tan^2 x + 3\tan^3 x)\}$$

$$= e^x \sec x\,(4 + 8\tan x + 6\tan^2 x + 6\tan^3 x)$$

$$\frac{d^4y}{dx^4} = e^x \sec x\{4(1 + \tan x) + 8(1 + \tan x + 2\tan^2 x) + 6(2\tan x + \tan^2 x + 3\tan^3 x) + 6\,(3\tan^2 x + \tan^3 x + 4\tan^4 x)\}$$

$$= e^x \sec x\{12 + 24\tan x + 40\tan^2 x + 24\tan^3 x + 24\tan^4 x\}$$

63. *Development of Functions.* If $z = f(x+y)$ where x and y are independent variables,

Then $$\left(\frac{dz}{dy}\right)_{x\text{ const}} = \left(\frac{dz}{dx}\right)_{y\text{ const}}$$

for let $z = f(w)$ where $w = x + y$

Keeping x constant $$\frac{dz}{dy} = \frac{dz}{dw}\frac{dw}{dy}$$
$$= \frac{dz}{dw}, \quad \text{since } \left(\frac{dw}{dy}\right)_{x\text{ const}} = 1$$

Keeping y constant $$\frac{dz}{dx} = \frac{dz}{dw}\frac{dw}{dx}$$
$$= \frac{dz}{dw}, \quad \text{since } \left(\frac{dw}{dx}\right)_{y\text{ const}} = 1$$

Thus $$\left(\frac{dz}{dy}\right)_{x\text{ const}} = \left(\frac{dz}{dx}\right)_{y\text{ const}}$$

The function $z = f(x+y)$ can be expressed as a series of terms of descending powers of x, or ascending powers of y.

Then $z = f(x+y) = A + By + Cy^2 + Dy^3 + \ldots$

where the coefficients A, B, C . . . are functions of x but are independent of y.

Then $$\left(\frac{dz}{dy}\right)_{x\text{ const}} = B + 2Cy + 3Dy^2 + 4Ey^3 + \ldots$$

and $$\left(\frac{dz}{dx}\right)_{y\text{ const}} = \frac{dA}{dx} + y\frac{dB}{dx} + y^2\frac{dC}{dx} + y^3\frac{dD}{dx} + \ldots$$

Since these two expansions are equal, we can equate coefficients of like powers of y.

And $$B = \frac{dA}{dx}$$

$$2C = \frac{dB}{dx} = \frac{d^2A}{dx^2} \qquad C = \frac{1}{2}\frac{d^2A}{dx^2}$$

$$3D = \frac{dC}{dx} = \frac{1}{2}\frac{d^3A}{dx^3} \qquad D = \frac{1}{\lfloor 3}\frac{d^3A}{dx^3}$$

$$4E = \frac{dD}{dx} = \frac{1}{\lfloor 3}\frac{d^4A}{dx^4} \qquad E = \frac{1}{\lfloor 4}\frac{d^4A}{dx^4}$$

Therefore $$z = f(x+y) = A + y\frac{dA}{dx} + \frac{y^2}{\lfloor 2}\frac{d^2A}{dx^2} + \frac{y^3}{\lfloor 3}\frac{d^3A}{dx^3} + \ldots,$$

in which A can be easily expressed as a function of x by putting $y = 0$, for then $A = f(x)$. This expansion is known as Taylor's Expansion.

(*a*) To expand $(x+y)^n$

Then $z=(x+y)^n$; and when $y=0$, $z=x^n$, and this gives A as a function of x.

If $A=x^n$

$$\frac{d^rA}{dx^r}=n(n-1)(n-2)\ \ldots\ (n-r+1)x^{n-r}$$

when $\quad r=1 \quad \frac{dA}{dx}=nx^{n-1}$

$\quad r=2 \quad \frac{d^2A}{dx^2}=n(n-1)x^{n-2}$, and so on.

But $z=(x+y)^n$

$$=A+y\frac{dA}{dx}+\frac{y^2}{\underline{|2}}\frac{d^2A}{dx^2}+\frac{y^3}{\underline{|3}}\frac{d^3A}{dx^3}+\ldots$$

$$=x^n+y\,nx^{n-1}+\frac{y^2}{\underline{|2}}n(n-1)x^{n-2}+\frac{y^3}{\underline{|3}}n(n-1)(n-2)x^{n-3}+\ldots$$

$$=x^n+nx^{n-1}y+\frac{n(n-1)}{\underline{|2}}x^{n-2}y^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}y^3+\ldots$$

(*b*) To expand $\log_e(x+y)$

Then $z=\log_e(x+y)$; and when $y=0$, $z=\log_e x$, and this gives A as a function of x.

If $A=\log_e x$

$$\frac{dA}{dx}=\frac{1}{x}$$

$$\frac{d^2A}{dx^2}=-\frac{1}{x^2}$$

$$\frac{d^3A}{dx^3}=(-1)(-2)x^{-3}=\frac{\underline{|2}}{x^3}$$

$$\frac{d^4A}{dx^4}=(-1)(-2)(-3)x^{-4}=-\frac{\underline{|3}}{x^4}$$

$$\frac{d^nA}{dx^n}=(-1)^n\frac{\underline{|n-1}}{x^n}$$

But $z=\log_e(x+y)$

$$=A+y\frac{dA}{dx}+\frac{y^2}{\underline{|2}}\frac{d^2A}{dx^2}+\frac{y^3}{\underline{|3}}\frac{d^3A}{dx^3}+\ldots$$

$$=\log_e x+y\left(\frac{1}{x}\right)+\frac{y^2}{\underline{|2}}\left(-\frac{1}{x^2}\right)+\frac{y^3}{\underline{|3}}\left(\frac{\underline{|2}}{x^3}\right)+\frac{y^4}{\underline{|4}}\left(-\frac{\underline{|3}}{x^4}\right)\ldots$$

$$=\log_e x+\frac{y}{x}-\frac{y^2}{2x^2}+\frac{y^3}{3x^3}-\frac{y^4}{4x^4}\ldots$$

Putting $x = 1$, we get

$$\log_e(1 + y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots$$

64. *Application of Taylor's Expansion to the solution of equations.*

Now $f(x + y) = A + y\dfrac{dA}{dx} + \dfrac{y^2}{\underline{|2}}\dfrac{d^2A}{dx^2} + \dfrac{y^3}{\underline{|3}}\dfrac{d^3A}{dx^3} + \ldots$

But if y is taken as a small quantity

then $f(x + y) = A + y\dfrac{dA}{dx} + \dfrac{1}{2}y^2\dfrac{d^2A}{dx^2}$ approximately.

Thus if we are solving an equation and we find by trial that a root of the equation lies between two definite limits, say $x = x_1$ and $x = x_2$.

Then calling the equation $f(x) = 0$

and when $x = x_1,\ f(x_1) = a$

also when $x = x_2,\ f(x_2) = b$

Then, if a is nearer to 0 than b, the actual root of the equation can be $x_1 + h$ and h will be small.

But $f(x_1 + h) = 0$

and hence $A + h\dfrac{dA}{dx} + \dfrac{1}{2}h^2\dfrac{d^2A}{dx^2} = 0$

where $A = f(x)$ and A, $\dfrac{dA}{dx}$, and $\dfrac{d^2A}{dx^2}$ have the values when $x = x_1$.

This gives a quadratic equation for h.

Next, if b is nearer to 0 than a, the actual root of the equation can be $x_2 - k$, and k will be small.

But $f(x_2 - k) = 0$

and hence $A - k\dfrac{dA}{dx} + \dfrac{1}{2}k^2\dfrac{d^2A}{dx^2} = 0$

where $A = f(x)$ and A, $\dfrac{dA}{dx}$, and $\dfrac{d^2A}{dx^2}$ have the values when $x = x_2$.

This gives a quadratic equation for k.

Example 1. To find the root of the equation $x^3 - 10x^2 + 40x - 35 = 0$, knowing that it lies between 1 and 2.

Then A or $f(x) = x^3 - 10x^2 + 40x - 35$

when $x = 1, f(x) = -4$, and when $x = 2, f(x) = 13$.

Hence the root is nearer to 1 than it is to 2

and $A = x^3 - 10x^2 + 40x - 35 = -4$ when $x = 1$

$$\frac{dA}{dx} = 3x^2 - 20x + 40 = 23 \text{ when } x = 1$$

$$\frac{d^2A}{dx^2} = 6x - 20 = -14 \text{ when } x = 1$$

But $$A + h\frac{dA}{dx} + \frac{1}{2}h^2\frac{d^2A}{dx^2} = 0$$

$$-4 + 23h - 7h^2 = 0$$

$$h^2 - 3{\cdot}286h + 0{\cdot}571 = 0$$

$$h - 1{\cdot}643 = \pm 1{\cdot}459$$

$$h = 0{\cdot}184$$

The root lies between 1·18 and 1·19.
To get a better approximation for the root

$$A = x^3 - 10x^2 + 40x - 35 = -0{\cdot}08097 \text{ when } x = 1{\cdot}18$$

$$\frac{dA}{dx} = 3x^2 - 20x + 40 = 20{\cdot}5772 \text{ when } x = 1{\cdot}18$$

$$\frac{d^2A}{dx^2} = 6x - 20 = -12{\cdot}92 \text{ when } x = 1{\cdot}18$$

But $$A + h\frac{dA}{dx} + \frac{1}{2}h^2\frac{d^2A}{dx^2} = 0$$

$$-0{\cdot}08097 + 20{\cdot}5772h - 6{\cdot}46h^2 = 0$$

$$h^2 - 3{\cdot}18536h + 0{\cdot}01253 = 0$$

$$h - 1{\cdot}59268 = \pm 1{\cdot}58874$$

$$h = 0{\cdot}00394$$

The root will be 1·1839.

Example 2. Solve the equation $\frac{\sin x}{x} = \frac{3}{4}$.

The root is evidently between 1 and 2.

The equation is $\sin x - \frac{3}{4}x = 0$

and $$A = \sin x - 0{\cdot}75\,x = 0{\cdot}0915 \text{ when } x = 1$$

$$\frac{dA}{dx} = \cos x - 0{\cdot}75 = -0{\cdot}2097 \text{ when } x = 1$$

$$\frac{d^2A}{dx^2} = -\sin x = -0{\cdot}8415 \text{ when } x = 1$$

But $$f(x+h) = A + h\frac{dA}{dx} + \frac{1}{2}h^2\frac{d^2A}{dx^2}$$

and $$0{\cdot}0915 - 0{\cdot}2097h - 0{\cdot}4207h^2 = 0$$

$$h^2 + 0{\cdot}4985h - 0{\cdot}2175 = 0$$

$$h + 0{\cdot}2493 = \pm 0{\cdot}5287$$

$$h = 0{\cdot}2794$$

The root is between 1·27 and 1·28 and is nearer to 1·28. To get a nearer approximation take the root as $(1\cdot28 - k)$.

Then $A = \sin x - 0\cdot75x = -0\cdot0020$ when $x = 1\cdot28$

$$\frac{dA}{dx} = \cos x - 0\cdot75 = -0\cdot4633 \text{ when } x = 1\cdot28$$

$$\frac{d^2A}{dx^2} = -\sin x = -0\cdot9580 \text{ when } x = 1\cdot28$$

But
$$f(x-k) = A - k\frac{dA}{dx} + \frac{1}{2}k^2\frac{d^2A}{dx^2}$$

$$-0\cdot0020 + 0\cdot4633k - 0\cdot4790k^2 = 0$$

$$k^2 - 0\cdot967224\,k + 0\cdot0041754 = 0$$

$$k - 0\cdot483612 = \pm\, 0\cdot479276$$

$$k = 0\cdot004336$$

The root is $1\cdot28 - 0\cdot004336 = 1\cdot275664$.

Working to five significant figures the root is 1·2757.

65. *Maclaurin's Theorem.* If we start with Taylor's Expansion and put $x = 0$.

Then
$$f(x+y) = A + y\frac{dA}{dx} + \frac{y^2}{\underline{|2}}\frac{d^2A}{dx^2} + \frac{y^3}{\underline{|3}}\frac{d^2A}{dx^3} + \dots$$

and
$$f(y) = A_{x=0} + y\left(\frac{dA}{dx}\right)_{x=0} + \frac{y^2}{\underline{|2}}\left(\frac{d^2A}{dx^2}\right)_{x=0} + \frac{y^3}{\underline{|3}}\left(\frac{d^3A}{dx^3}\right)_{x=0} + \dots$$

where $A, \dfrac{dA}{dx}, \dfrac{d^2A}{dx^2} \dots$ have their respective values when $x = 0$, and these will be constant coefficients, since they are independent of x and y. The expansion may therefore be expressed as

$$f(x) = A_{x=0} + x\left(\frac{dA}{dx}\right)_{x=0} + \frac{x^2}{\underline{|2}}\left(\frac{d^2A}{dx^2}\right)_{x=0} + \frac{x^3}{\underline{|3}}\left(\frac{d^3A}{dx^3}\right)_{x=0} + \dots$$

where $A = f(x)$.

An alternative proof can be obtained in the following manner:

Let $y = f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots A_nx^n + \dots$

where A_0, A_1, A_2, etc., are constant coefficients.

Then when $x = 0,\quad A_0 = y_{x=0}$

$$\frac{dy}{dx} = A_1 + 2A_2x + 3A_3x^2 + \dots nA_nx^{n-1} + \dots$$

and when $x = 0,\ A_1 = \left(\dfrac{dy}{dx}\right)_{x=0}$

$$\frac{d^2y}{dx^2} = \underline{|2}\,A_2 + 3\times 2A_3x + \dots n(n-1)A_nx^{n-2} + \dots$$

and when $x = 0$, $\lfloor 2 A_2 = \left(\frac{d^2y}{dx^2}\right)_{x=0}$ and $A_2 = \frac{1}{\lfloor 2}\left(\frac{d^2y}{dx^2}\right)_{x=0}$

$$\frac{d^3y}{dx^3} = \lfloor 3 A_3 + \ldots \quad n(n-1)(n-2)A_n x^{n-3} + \ldots$$

and when $x = 0$, $\lfloor 3 A_3 = \left(\frac{d^3y}{dx^3}\right)_{x=0}$ and $A_3 = \frac{1}{\lfloor 3}\left(\frac{d^3y}{dx^3}\right)_{x=0}$

Differentiating n times and putting $x = 0$ we get

$$\left(\frac{d^ny}{dx^n}\right)_{x=0} = \lfloor n A_n \text{ and } A_n = \frac{1}{\lfloor n}\left(\frac{d^ny}{dx^n}\right)_{x=0}$$

Then $y = f(x) = y_{x=0} + x\left(\frac{dy}{dx}\right)_{x=0} + \frac{x^2}{\lfloor 2}\left(\frac{d^2y}{dx^2}\right)_{x=0} + \frac{x^3}{\lfloor 3}\left(\frac{d^3y}{dx^3}\right)_{x=0} + \ldots$

$$\frac{x^n}{\lfloor n}\left(\frac{d^ny}{dx^n}\right)_{x=0} + \ldots$$

We have therefore a means by which we can expand a function of x in a series of terms of ascending powers of x. The success in the working with this expansion depends upon the ease with which we can obtain the successive differential coefficients of $f(x)$. Thus any function whose general differential coefficient is readily obtained can be as readily expanded.

To expand $e^{ax} \sin bx$.

It has been already shown that the n^{th} differential coefficient of $e^{ax} \sin bx$ is $(a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin (bx + n\alpha)$, where $\tan \alpha = \frac{b}{a}$

Then
$$\left(\frac{d^ny}{dx^n}\right)_{x=0} = (a^2 + b^2)^{\frac{n}{2}} \sin n\alpha$$

$$\left(\frac{dy}{dx}\right)_{x=0} = (a^2 + b^2)^{\frac{1}{2}} \sin \alpha$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = (a^2 + b^2) \sin 2\alpha$$

but
$$y = f(x) = y_{x=0} + x\left(\frac{dy}{dx}\right)_{x=0} + \frac{x^2}{\lfloor 2}\left(\frac{d^2y}{dx^2}\right)_{x=0} + \ldots$$

$$\frac{x^n}{\lfloor n}\left(\frac{d^ny}{dx^n}\right)_{x=0} + \ldots$$

and $y = e^{ax} \sin bx = x(a^2 + b^2)^{\frac{1}{2}} \sin \alpha + \frac{x^2}{\lfloor 2}(a^2 + b^2) \sin 2\alpha + \ldots$

$$\frac{x^n}{\lfloor n}(a^2 + b^2)^{\frac{n}{2}} \sin n\alpha + \ldots \quad \text{where } \tan \alpha = \frac{b}{a}$$

66. We can also apply Maclaurin's Theorem to the case in which the process of finding the successive differential coefficients is rendered simple by working to a general rule. Functions such as $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, can be expanded in this manner.

Thus, to find the expansion for $\sec x$, we have to find the successive differential coefficients of $\sec x$, and to do this we shall require the differential coefficients of $\sec x \tan^n x$.

$$y = \sec x \tan^n x$$

$$\frac{dy}{dx} = n \sec x \tan^{n-1} x \sec^2 x + \tan^n x \sec x \tan x$$

$$= \sec x\{n \tan^{n-1} x(1 + \tan^2 x) + \tan^{n+1} x\}$$

$$= \sec x\{n \tan^{n-1} x + (n+1) \tan^{n+1} x\}$$

Thus when $n = 0$, $y = \sec x$ and $\dfrac{dy}{dx} = \sec x \tan x$

when $n = 1$, $y = \sec x \tan x$ and $\dfrac{dy}{dx} = \sec x\{1 + 2\tan^2 x\}$

when $n = 2$, $y = \sec x \tan^2 x$ and $\dfrac{dy}{dx} = \sec x\{2\tan x + 3\tan^3 x\}$

when $n = 3$, $y = \sec x \tan^3 x$ and $\dfrac{dy}{dx} = \sec x\{3\tan^2 x + 4\tan^4 x\}$

Using these results we can now successively differentiate $\sec x$.

$$y = \sec x$$

$$\frac{dy}{dx} = \sec x \tan x$$

$$\frac{d^2y}{dx^2} = \sec x\{1 + 2\tan^2 x\}$$

$$\frac{d^3y}{dx^3} = \sec x\{\tan x + 2(2\tan x + 3\tan^3 x)\}$$

$$= \sec x\{5\tan x + 6\tan^3 x\}$$

$$\frac{d^4y}{dx^4} = \sec x\{5(1 + 2\tan^2 x) + 6(3\tan^2 x + 4\tan^4 x)\}$$

$$= \sec x\{5 + 28\tan^2 x + 24\tan^4 x\}$$

$$\frac{d^5y}{dx^5} = \sec x\{5\tan x + 28(2\tan x + 3\tan^3 x) + 24(4\tan^3 x + 5\tan^5 x)\}$$

$$= \sec x\{61\tan x + 180\tan^3 x + 120\tan^5 x\}$$

$$\frac{d^6y}{dx^6} + \sec x\{61(1 + 2\tan^2 x) + 180(3\tan^2 x + 4\tan^4 x) + 120(5\tan^4 x + 6\tan^6 x)\}$$

$$= \sec x\{61 + 662\tan^2 x + 1320\tan^4 x + 720\tan^6 x\}$$

When $x = 0$, sec x and its successive differential coefficients become 1, 0, 1, 0, 5, 0, 61, and therefore there are no odd powers of x in the expansion.

$$y = \sec x = 1 + \frac{x^2}{\underline{|2}} + \frac{5x^4}{\underline{|4}} + \frac{61x^6}{\underline{|6}} + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots$$

67. There are some functions which can be expanded without the aid of Maclaurin's Theorem, and this remark applies more to the inverse trigonometrical functions.

To find an expansion for $\sin^{-1} x$.

$$y = \sin^{-1} x = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \ldots$$

and when $x = 0$, $y = 0$, and therefore $A_0 = 0$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + 5A_5 x^4 + \ldots$$

But $\dfrac{1}{\sqrt{1 - x^2}}$ or $(1 - x^2)^{-\frac{1}{2}}$ can be expanded with the aid of the Binomial Theorem and

$$(1 - x)^{-\frac{1}{2}} = 1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \frac{5}{16} x^6 + \frac{35}{128} x^8 + \ldots$$

Equating coefficients of like powers of x, it should be noticed that there are no terms in the second expansion involving odd powers of x, and therefore the coefficients of these terms in the first expansion must all be zero.

Then $A_2 = A_4 = A_6 = A_8 = \ldots 0$

Also $A_1 = 1$, $3A_3 = \frac{1}{2}$, $5A_5 = \frac{3}{8}$, $7A_7 = \frac{5}{16}$, $9A_9 = \frac{35}{128}$

Then $A_1 = 1$, $A_2 = \frac{1}{6}$, $A_5 = \frac{3}{40}$, $A_7 = \frac{5}{112}$, $A_9 = \frac{35}{1152}$

and $y = \sin^{-1} x = x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \dfrac{5x^7}{112} + \dfrac{35x^9}{1152} + \ldots$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\frac{x^9}{9} + \ldots$$

68. If we have a simple relation between the first and second differential coefficients, and this is combined with a knowledge of some of the initial conditions, we are enabled to express y as

a series of terms of ascending powers of x. In other words, we can solve a differential equation and give the result as a series.

Example. If $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, and when $x = 0$, $y = 0$, and $\frac{dy}{dx} = 1$, express y as a series of terms of ascending powers of x.

Then $y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots$

$$\frac{dy}{dx} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \ldots$$

$$\frac{d^2y}{dx^2} = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + \ldots$$

When $x = 0$, $y = 0$, then $A_0 = 0$; and when $x = 0$, $\frac{dy}{dx} = 1$, then $A_1 = 1$.

Also
$$\begin{aligned} \frac{d^2y}{dx^2} &= 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + 30A_6x^4 + \ldots \\ -2\frac{dy}{dx} &= -2A_1 - 4A_2x - 6A_3x^2 - 8A_4x^3 - 10A_5x^4 \ldots \\ 2y &= 2A_0 + 2A_1x + 2A_2x^2 + 2A_3x^3 + 2A_4x^4 + \ldots \end{aligned}$$

Since $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Then
$$\begin{array}{lll} 2A_2 - 2A_1 + 2A_0 = 0 & A_2 = 1 & \\ 6A_3 - 4A_2 + 2A_1 = 0 & 6A_3 = 2 & A_3 = \frac{1}{3} \\ 12A_4 - 6A_3 + 2A_2 = 0 & 12A_4 = 0 & A_4 = 0 \\ 20A_5 - 8A_4 + 2A_3 = 0 & 20A_5 = -\frac{2}{3} & A_5 = -\frac{1}{30} \\ 30A_6 - 10A_5 + 2A_4 = 0 & 30A_6 = -\frac{1}{3} & A_6 = -\frac{1}{90} \end{array}$$

Hence $y = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90}$ etc.

69. Sometimes, when it is necessary to expand a certain function of x, it is convenient to form a simple relation between the first and second differential coefficients and use this relation in the above way.

Thus if $y = e^{\sin^{-1}x}$

$$\log_e y = \sin^{-1} x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

and
$$\frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} = z$$

$$\log_e z = \sin^{-1} x - \frac{1}{2}\log_e(1-x^2)$$

$$\frac{1}{z}\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{1-x^2}$$

and
$$\frac{dz}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\left\{\frac{1}{\sqrt{1-x^2}} + \frac{x}{1-x^2}\right\}$$

Then
$$(1-x^2)\frac{d^2y}{dx^2} = e^{\sin^{-1}x} + \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$= y + x\frac{dy}{dx}$$

We can therefore use the relation $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$, combined with the condition that when $x = 0$, $y = 1$, to express y as a series of terms of ascending powers of x.

Examples VII

(1) If $x = 3e^{-t}\sin 3t$, prove that $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 10x = 0$.

(2) If $x = 10t\ e^{-2t}$, prove that $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = 0$.

(3) If $x = 4(e^{-2t} - e^{-4t})$, prove that $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$.

(4) If $x = 5(1 + 4t)e^{-2t}$, prove that $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = 0$.

(5) If $x = 4(3e^{-2t} - 2e^{-4t})$, prove that $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$.

(6) If $y = e^{\sin x}$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\cos x + y\sin x = 0$.

(7) If $y = e^{ax}\sin bx$, prove that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2 + b^2)y = 0$.

(8) If $y = e^{ax}\cos bx$, prove that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2 + b^2)y = 0$.

(9) If $y = e^x \tan x$, prove that $\frac{d^2y}{dx^2} - (1 + 2\tan x)\frac{dy}{dx} + y(\tan x - \cot x) = 0$.

(10) If $y = x e^x$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, and hence find $\frac{d^ny}{dx^n}$.

(11) Using Maclaurin's Expansion, find the first five terms in the expansions for $\sin nx$ and $\cos nx$.

(12) If $y = \tan(x + y)$, show that $\frac{dy}{dx} = -\frac{1+y^2}{y^2}$ and $\frac{d^2y}{dx^2} = -\frac{2}{y^3}\left(\frac{1+y^2}{y^2}\right)$.

(13) Find the first five terms in the expansion for $\tan^{-1} x$.

(14) If $y = e^x \tan^n x$, find $\frac{dy}{dx}$, and use the result to find the first five differential coefficients of $e^x \tan x$. Hence find the first five terms in the expansion for $e^x \tan x$.

(15) If $x = \frac{y+1}{y}$, prove that $y = (1-x)\frac{dy}{dx}$, $2\frac{dy}{dx} = (1-x)\frac{d^2y}{dx^2}$, $3\frac{d^2y}{dx^2} = (1-x)\frac{d^3y}{dx^3}$, and hence show that $\frac{d^ny}{dx^n} = \frac{\underline{|n}\, y}{(1-x)^n}$.

(16) If $y = \frac{x}{x+1}$, prove that $\frac{dy}{dx} = \frac{1-y}{1+x}$, $\frac{d^2y}{dx^2} = -\frac{2}{1+x}\frac{dy}{dx}$, $\frac{d^3y}{dx^3} = -\frac{3}{1+x}\frac{d^2y}{dx^2}$, and hence show that $\frac{d^ny}{dx^n} = (-1)^{n+1}\frac{\underline{|n}\,(1-y)}{(1+x)^n}$.

(17) If $x = \frac{y+1}{y-1}$, prove that $\frac{dy}{dx} = \frac{y-1}{1-x}$, $\frac{d^2y}{dx^2} = \frac{2}{1-x}\frac{dy}{dx}$, $\frac{d^3y}{dx^3} = \frac{3}{1-x}\frac{d^2y}{dx^2}$, and hence show that $\frac{d^ny}{dx^n} = \frac{\underline{|n}\,(y-1)}{(1-x)^n}$

(18) If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, giving each result in terms of θ.

(19) Using Taylor's Expansion, find the approximate root of the equation $\tan x = x$, knowing that the root lies between 4 and 4·5 radians.

(20) Using Taylor's Expansion, find the approximate root of the equation $3\sin\frac{x}{2} + \frac{2}{x} = 3{\cdot}7$, knowing that the root lies between 0·5 and 1·0 radians.

(21) Using Taylor's Expansion, find the approximate root of the equation $x^4 + 5x^3 - 7x^2 + 10x - 12 = 0$, knowing that the root lies between 1 and 2.

(22) Using Taylor's Expansion, find the approximate root of the equation $x^5 - 8x^3 + 12x - 185 = 0$, knowing that the root lies between 3 and 4.

(23) If $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$, express y as a series of ascending powers of x, knowing that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 2$.

(24) If $(1 + x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$, express y as a series of ascending powers of x, knowing that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$.

(25) If $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$, express y as a series of ascending powers of x, knowing that when $x = 0$, $y = 1$.

CHAPTER VIII

70. Curves may be divided into two classes according to the algebraic sign of the slope.

(1) If y increases as x increases, then a tangent drawn to such a curve will be inclined to the axis of x at an acute angle, and

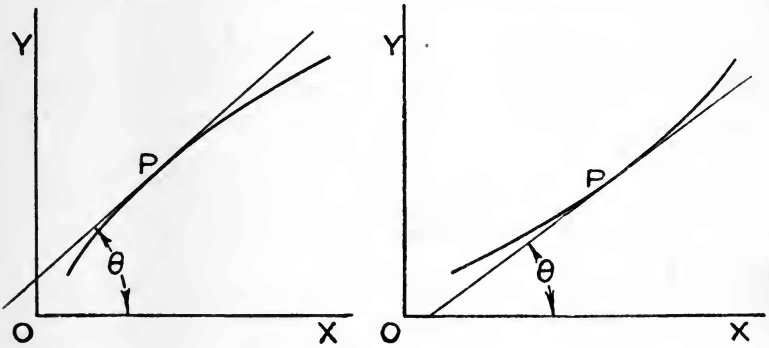


FIG. 30.

tan θ will be positive and therefore $\frac{dy}{dx}$ will be positive. Hence when $\frac{dy}{dx}$, or the slope of a curve, is positive y increases as x increases.

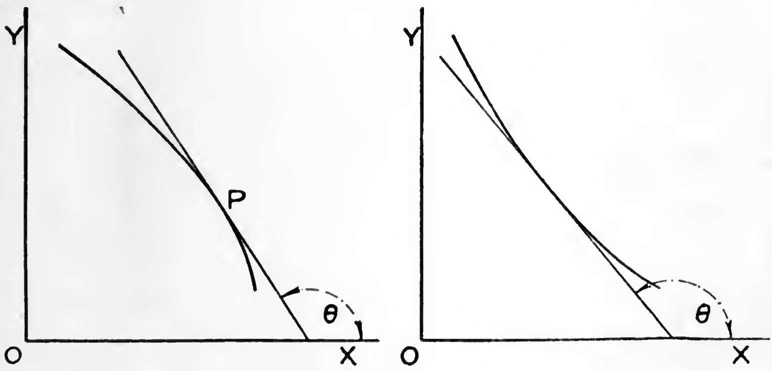


FIG. 31.

(2) If y decreases as x increases, then a tangent drawn to such a curve will be inclined to the axis of x at an obtuse angle, and

$\tan\theta$ will be negative and therefore $\frac{dy}{dx}$ will be negative. Hence when $\frac{dy}{dx}$, or the slope of a curve, is negative, y decreases as x increases.

When we wish to find the highest or lowest point of a curve, we do so by drawing the horizontal tangent to the curve and the point of contact is the required point. As the tangent is horizontal, it is parallel to the axis of x and therefore its slope, or $\frac{dy}{dx}$, is zero.

Thus the condition $\frac{dy}{dx} = 0$ gives a point on the curve at which the tangent is horizontal, and such a point may be either a maximum point, a minimum point, or a point of inflexion.

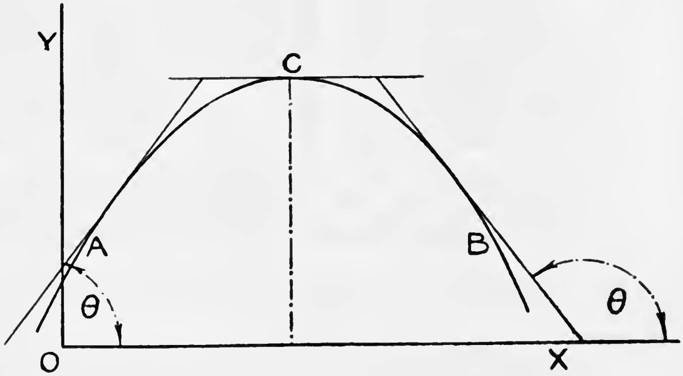


FIG. 32.

71. *Case I.* The maximum point. Let C be the highest point of a curve, and at C $\frac{dy}{dx} = 0$ (Fig. 32).

Moving along the curve from A to C, the angle θ is acute and decreases to $0°$. Then $\frac{dy}{dx}$ is positive and decreases to 0.

Moving along the curve from C to B, the angle θ is obtuse and decreases from $180°$. Then $\frac{dy}{dx}$ is negative and decreases from 0.

Therefore in the neighbourhood of a maximum point $\frac{dy}{dx}$ is always decreasing, and also $\frac{dy}{dx}$ changes in sign from positive, through zero, to negative.

If $\frac{dy}{dx}$ is decreasing as x increases, then the curve obtained by plotting x horizontally and $\frac{dy}{dx}$ vertically is such that its slope or $\frac{d^2y}{dx^2}$ is negative.

Thus at a maximum point on a curve we have the conditions

$$(1)\quad \frac{dy}{dx} = 0$$

$$(2)\quad \frac{d^2y}{dx^2} \text{ is negative.}$$

72. *Case II.* The minimum point. Let C be the lowest point of a curve, and at C $\quad \frac{dy}{dx} = 0$ (Fig. 33).

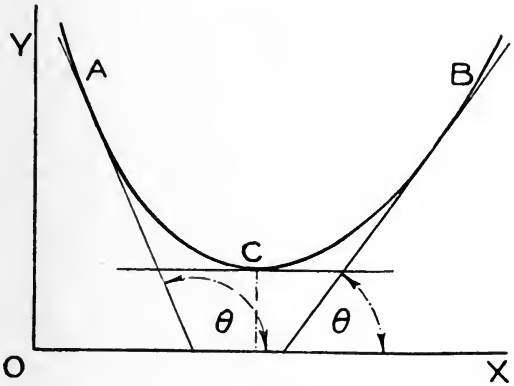


FIG. 33.

Moving along the curve from A to C, the angle θ is obtuse and increases to 180°.

Then $\frac{dy}{dx}$ is negative and increases to 0.

Moving along the curve from C to B, the angle θ is acute and increases from 0°.

Then $\frac{dy}{dx}$ is positive and increases from 0.

Therefore in the neighbourhood of a minimum point $\frac{dy}{dx}$ is always increasing, and also $\frac{dy}{dx}$ changes in sign from negative, through zero, to positive.

If $\frac{dy}{dx}$ is increasing as x increases, then the curve obtained by plotting x horizontally and $\frac{dy}{dx}$ vertically is such that its slope or $\frac{d^2y}{dx^2}$ is positive.

Thus at a minimum point on a curve we have the conditions

$$(1)\ \frac{dy}{dx} = 0$$

$$(2)\ \frac{d^2y}{dx^2} \text{ is positive.}$$

73. *Case III.* The point of inflexion. Here we only have to consider the case of the point of inflexion at which the tangent to the curve is horizontal.

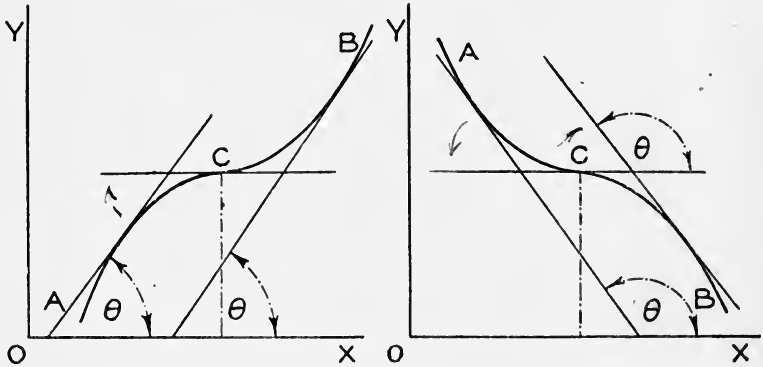


FIG. 34.

The horizontal line T_1CT_2 touches the upper branch of the curve at C and also the lower branch at C.

Then at C, $\frac{dy}{dx} = 0$ (Fig. 34).

(*a*) Moving along the curve from A to C, the angle θ is acute and decreases to 0°.

Then $\frac{dy}{dx}$ is positive and decreases to 0.

Moving along the curve from C to B the angle θ is acute and increases from 0°.

Then $\frac{dy}{dx}$ is positive and increases from 0.

Therefore in the neighbourhood of a point of inflexion $\frac{dy}{dx}$ is always positive and $\frac{dy}{dx}$ decreases to a zero value and then increases.

Since $\frac{dy}{dx}$ decreases to a zero value and then increases as x increases, the curve obtained by plotting x horizontally and $\frac{dy}{dx}$ vertically is of such a form that its lowest point occurs when $\frac{dy}{dx} = 0$; hence at this point the tangent to the curve must be horizontal, and so the slope or $\frac{d^2y}{dx^2}$ is zero.

Thus at a point of inflexion on a curve we have the conditions

$$(1) \quad \frac{dy}{dx} = 0$$

$$(2) \quad \frac{d^2y}{dx^2} = 0$$

(*b*) Moving along the curve from A to C, the angle θ is obtuse and increases to 180°.

Then $\frac{dy}{dx}$ is negative and increases to 0.

Moving along the curve from C to B the angle θ is obtuse and decreases from 180°.

Then $\frac{dy}{dx}$ is negative and decreases from 0.

Therefore in the neighbourhood of a point of inflexion $\frac{dy}{dx}$ is always negative, and $\frac{dy}{dx}$ increases to a zero value and then decreases.

Since $\frac{dy}{dx}$ increases to a zero value and then decreases as x increases, the curve obtained by plotting x horizontally and $\frac{dy}{dx}$ vertically is of such a form that its highest point occurs when $\frac{dy}{dx} = 0$; hence at this point the tangent to the curve must be horizontal, and so the slope or $\frac{d^2y}{dx^2}$ is zero.

Thus at a point of inflexion on a curve we have the conditions

$$(1) \quad \frac{dy}{dx} = 0$$

$$(2) \quad \frac{d^2y}{dx^2} = 0$$

74. If the law of a curve is $y = f(x)$, then $\frac{dy}{dx}$ will give the slope of the curve at any point, the condition for a horizontal tangent is obtained by putting $\frac{dy}{dx} = 0$. This is an equation to be solved for x, and the root will give a point on the curve at which the tangent is horizontal. This point may be a maximum point, a minimum point, or a point of inflexion. In order to decide which it happens to be, $\frac{d^2y}{dx^2}$ is found and the value of x obtained by solving the equation $\frac{dy}{dx} = 0$ is substituted in the resulting expression. If the result is negative, the point is a maximum point; if positive, a minimum point; if zero, a point of inflexion.

Example 1. State the nature of the points on the curve $y = 2x^3 - 9x^2 - 60x - 25$ at which the tangent is horizontal.

$$y = 2x^3 - 9x^2 - 60x - 25$$

$$\frac{dy}{dx} = 6x^2 - 18x - 60$$

The tangent is horizontal when $\frac{dy}{dx} = 0$

That is, when

$$6x^2 - 18x - 60 = 0$$

$$x^2 - 3x - 10 = 0$$

$$(x - 5)(x + 2) = 0$$

$$x = 5 \text{ and } x = -2$$

There are two points at which the tangent is horizontal.

Now

$$\frac{d^2y}{dx^2} = 12x - 18$$

When $x = 5$, $\frac{d^2y}{dx^2} = 42$, a positive value, and y is a minimum when $x = 5$.

When $x = -2$, $\frac{d^2y}{dx^2} = -42$, a negative value, and y is a maximum when $x = -2$.

Then the expression $2x^3 - 9x^2 - 60x - 25$ has its maximum value 43 when $x = -2$, and its minimum value -300 when $x = 5$.

Example 2. State the nature of the points on the curve $y = 3x^4 - 8x^3 - 24x^2 + 96x - 30$ at which the tangent to the curve is horizontal

$$y = 3x^4 - 8x^3 - 24x^2 + 96x - 30$$

$$\frac{dy}{dx} = 12x^3 - 24x^2 - 48x + 96$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

That is, when
$$12x^3 - 24x^2 - 48x + 96 = 0$$
or
$$x^3 - 2x^2 - 4x + 8 = 0$$
$$(x^2 - 4)(x - 2) = 0$$
$$x = 2 \text{ and } x = -2$$

There are two points at which the tangent is horizontal

Now
$$\frac{d^2y}{dx^2} = 36x^2 - 48x - 48$$

When $x = 2$, $\frac{d^2y}{dx^2} = 0$, and a point of inflexion occurs when $x = 2$,

When $x = -2$, $\frac{d^2y}{dx^2} = 192$, a positive value, and y is a minimum when $x = -2$.

Then the expression $3x^4 - 8x^3 - 24x^2 + 96x - 30$ has a minimum value -206 when $x = -2$, and there is a point of inflexion when $x = 2$.

75. *Example* 3. Find the dimensions of the cone of greatest volume which can be cut from a sphere of given radius.

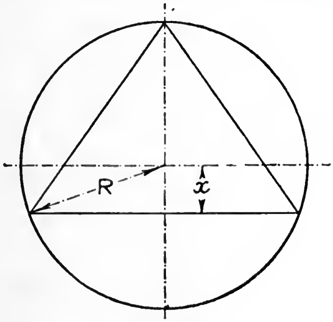


FIG. 35.

Let x be the perpendicular distance of the base of the cone from the centre of the sphere and let R be the radius of the sphere.

Height of cone $= \mathrm{R} + x$

Radius of base of cone $= \sqrt{\mathrm{R}^2 - x^2}$

Volume of cone $= \frac{\pi}{3}(\mathrm{R} + x)(\mathrm{R}^2 - x^2)$

$$v = \frac{\pi}{3}(\mathrm{R}^3 + \mathrm{R}^2x - \mathrm{R}x^2 - x^3)$$

$$\frac{dv}{dx} = \frac{\pi}{3}(\mathrm{R}^2 - 2\mathrm{R}x - 3x^2)$$

Now v is a maximum when $\frac{dv}{dx} = 0$

That is, when $\frac{\pi}{3}(R^2 - 2Rx - 3x^2) = 0$

$$R^2 - 2Rx - 3x^2 = 0$$
$$(R - 3x)(R + x) = 0$$
$$x = \frac{R}{3}$$

Height of cone $= R + x = \frac{4R}{3}$

Radius of base $= \sqrt{R^2 - x^2}$

$$= \sqrt{R^2 - \frac{R^2}{9}} = \frac{2\sqrt{2}}{3}R$$

Volume of cone $= \frac{\pi}{3}\,\frac{4R}{3}\,\frac{8R}{9}$

$$= \frac{32\pi R^3}{81}$$

76. *Example* 4. Find the dimensions of a cylindrical vessel of greatest capacity which can be made from a given amount of sheet metal—(1) When the vessel has no lid; (2) When the vessel has a lid.

Let S be the area of sheet metal used—(1) Without lid.

S = surface area of vessel $= 2\pi xy + \pi x^2$, where x = radius of base and y = height.

$$\text{Volume of vessel} = \pi x^2 y$$
$$v = \pi x^2\left\{\frac{S - \pi x^2}{2\pi x}\right\}$$
$$= \frac{1}{2}(Sx - \pi x^3)$$

$$\text{Then } \frac{dv}{dx} = \frac{1}{2}(S - 3\pi x^2)$$

Now v is a maximum when $\frac{dv}{dx} = 0$

That is, when $\frac{1}{2}(S - 3\pi x^2) = 0$

$$\text{Then} \quad x = \sqrt{\frac{S}{3\pi}}$$

Height of vessel $= y = \frac{S - \pi x^2}{2\pi x}$

$$= \frac{2S}{6\pi}\sqrt{\frac{3\pi}{S}}$$
$$= \sqrt{\frac{S}{3\pi}}$$

$$\text{Volume of vessel} = \pi x^2 y$$

$$= \pi \frac{S}{3\pi}\sqrt{\frac{S}{3\pi}}$$

$$= \frac{S}{3}\sqrt{\frac{S}{3\pi}}$$

(2) With lid.

S = surface area of vessel = $2\pi xy + 2\pi x^2$, where x = radius of base, and y = height.

$$\text{Volume of vessel} = \pi x^2 y$$

$$v = \pi x^2\left\{\frac{S - 2\pi x^2}{2\pi x}\right\}$$

$$= \frac{1}{2}\{Sx - 2\pi x^3\}$$

Then $$\frac{dv}{dx} = \frac{1}{2}(S - 6\pi x^2)$$

Now v is a maximum when $\dfrac{dv}{dx} = 0$

That is, when $$\frac{1}{2}(S - 6\pi x^2) = 0$$

Then $$x = \sqrt{\frac{S}{6\pi}}$$

$$\text{Height of vessel} = y = \frac{S - 2\pi x^2}{2\pi x}$$

$$= \frac{2S}{6\pi}\sqrt{\frac{6\pi}{S}}$$

$$= 2\sqrt{\frac{S}{6\pi}}$$

$$\text{Volume of vessel} = \pi x^2 y$$

$$= \pi \frac{S}{6\pi}\, 2\sqrt{\frac{S}{6\pi}}$$

$$= \frac{S}{3}\sqrt{\frac{S}{6\pi}}$$

77. *Example* 5. A sector is removed from a circular disc of sheet metal of given radius, and the remainder is formed into a conical vessel. Find the angle of the sector, so that the volume of the conical vessel shall be greatest.

Let θ be the angle of the sector ACB. Then $2\pi - \theta$ is the angle of the sector which has to be removed.

Length of arc ACB = $R\theta$, where R is the radius of the circular disc.

Now the circumference of the base of the conical vessel must be the same as the length of the arc ACB.

Circumference of base $= R\theta$

Radius of base $= \dfrac{R\theta}{2\pi}$

Length of slant side $= R$

$$\text{Height} = \sqrt{R^2 - \frac{R^2\theta^2}{4\pi^2}}$$

$$\text{Volume of conical vessel} = \frac{\pi}{3}\,\frac{R^2\theta^2}{4\pi^2}\sqrt{R^2 - \frac{R^2\theta^2}{4\pi^2}}$$

$$v = \frac{R^3}{12\pi}\sqrt{\theta^4 - \frac{\theta^6}{4\pi^2}}$$

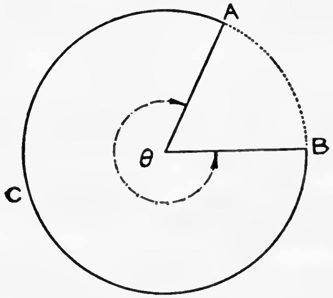


FIG. 36.

Now v will be greatest when $\sqrt{\theta^4 - \dfrac{\theta^6}{4\pi^2}}$ is greatest: that is, when $\theta^4 - \dfrac{\theta^6}{4\pi^2}$ is a maximum.

$$\theta^4 - \frac{\theta^6}{4\pi^2} \text{ is a maximum when } \quad 4\theta^3 - \frac{6\theta^5}{4\pi^2} = 0$$

That is, when $$\theta^2 = \frac{2}{3}\,4\pi^2$$

or $$\theta = 2\pi\sqrt{\frac{2}{3}}$$

The angle of the sector $= 2\pi - \theta$

$$= 2\pi\left(1 - \sqrt{\frac{2}{3}}\right)$$

$$= 1{\cdot}153 \text{ radians or } 66^\circ\ 6'.$$

The dimensions of the conical vessel can now be given.

$$\text{Radius} \quad = \frac{R\theta}{2\pi} = \sqrt{\frac{2}{3}}R$$

$$\text{Height} \quad = \sqrt{R^2 - \frac{2}{3}R^2} = \sqrt{\frac{1}{3}}\,R$$

$$\text{Volume} \quad = \frac{\pi}{3}\,\frac{2R^2}{3}\sqrt{\frac{1}{3}}\,R$$

$$= \frac{2\pi R^3}{9\sqrt{3}}$$

78. *Example* 6. The cost of a ship per hour is £c where $c = a + bs^n$ and a, b, and n are constants and s is the speed in knots. Find the speed of the ship so that it will travel a passage of m nautical miles at a minimum total cost.

$$\text{Time of passage} \quad = \frac{m}{s}\text{ hours}$$

$$\text{Total cost of passage} = \frac{m}{s}(a + bs^n)\text{ pounds} = y$$

$$y = m\left(\frac{a}{s} + bs^{n-1}\right)$$

$$y \text{ will be a minimum when } \frac{dy}{ds} = 0$$

$$\frac{dy}{ds} = m\{-as^{-2} + b(n-1)s^{n-2}\}$$

and $$\frac{dy}{ds} = 0 \quad \text{when } b(n-1)s^{n-2} = \frac{a}{s^2}$$

$$s^n = \frac{a}{b(n-1)}$$

$$s = \left\{\frac{a}{b(n-1)}\right\}^{\frac{1}{n}}$$

giving the speed in terms of the known constants a, b, and n.

79. The function $y = ae^{-kt}\sin(pt - c)$ affords a striking example of alternating maximum and minimum values.

a, k, p, and c are constants.

Since e^{-kt} is never zero, $y = 0$ when $\sin(pt - c) = 0$. That is, when $(pt - c)$ has the values $0, \pi, 2\pi, 3\pi \ldots n\pi \ldots$ or when t has the values $\frac{c}{p}, \frac{c}{p} + \frac{\pi}{p}, \frac{c}{p} + \frac{2\pi}{p} \ldots \frac{c}{p} + \frac{n\pi}{p} \ldots$

Now $\dfrac{dy}{dt} = a\{-ke^{-kt}\sin(pt-c) + pe^{-kt}\cos(pt-c)\}$

and $\dfrac{dy}{dt} = 0$ when $-ke^{-kt}\sin(pt-c) + pe^{-kt}\cos(pt-c) = 0$

$$k\sin(pt-c) = p\cos(pt-c)$$

$$\tan(pt-c) = \frac{p}{k}$$

If $\alpha = \tan^{-1}\dfrac{p}{k}$, then $\tan(pt-c) = \tan\alpha$

and $pt - c = \alpha,\ \pi+\alpha,\ 2\pi+\alpha,\ \ldots\ n\pi+\alpha$

Hence $\dfrac{dy}{dt} = 0$ when t has the values

$$\frac{\alpha+c}{p},\quad \frac{\alpha+c}{p}+\frac{\pi}{p},\quad \frac{\alpha+c}{p}+\frac{2\pi}{p},\ \ldots\ \frac{\alpha+c}{p}+\frac{n\pi}{p}\ \ldots$$

Some of these values of t will give maximum values of y, while the other values of t will give minimum values of y.

Taking the general value of t, $t = \dfrac{\alpha+c}{p} + \dfrac{n\pi}{p}$

Then $pt - c = n\pi + \alpha$

and $\sin(pt-c) = \sin(n\pi+\alpha)$

When n is an even integer, $\sin(pt-c)$ is positive, and hence maximum values of y occur when t has the values

$$\frac{\alpha+c}{p},\quad \frac{\alpha+c}{p}+\frac{2\pi}{p},\quad \frac{\alpha+c}{p}+\frac{4\pi}{p},\ \text{etc.}$$

When n is an odd integer, $\sin(pt-c)$ is negative, and hence minimum values of y occur when t has the values

$$\frac{\alpha+c}{p}+\frac{\pi}{p},\quad \frac{\alpha+c}{p}+\frac{3\pi}{p},\quad \frac{\alpha+c}{p}+\frac{5\pi}{p},\ \text{etc.}$$

Putting $\dfrac{\alpha+c}{p} = b.$

Then when $t = b$ $\qquad y = ae^{-bk}\sin\alpha = \text{M}$

when $t = b + \dfrac{2\pi}{p}$ $\qquad y = ae^{-k\left(b+\frac{2\pi}{p}\right)}\sin\alpha$

$$= \text{M}e^{-\frac{2\pi k}{p}}$$

when $t = b + \dfrac{4\pi}{p}$ $\qquad y = ae^{-k\left(b+\frac{4\pi}{p}\right)}\sin\alpha$

$$= \text{M}e^{-\frac{4\pi k}{p}}$$

$$\text{when } t = b + \frac{6\pi}{p} \qquad y = ae^{-k\left(b+\frac{6\pi}{p}\right)} \sin \alpha$$

$$= \mathrm{M}e^{-\frac{6\pi k}{p}}$$

Thus the successive maximum values of y are in geometrical progression, the common ratio of which is $e^{-\frac{2\pi k}{p}}$

$$\text{Also when } t = b + \frac{\pi}{p} \qquad y = -ae^{-k\left(b+\frac{\pi}{p}\right)} \sin \alpha = \mathrm{N}$$

$$\text{when } t = b + \frac{3\pi}{p} \qquad y = -ae^{-k\left(b+\frac{3\pi}{p}\right)} \sin \alpha$$

$$= \mathrm{N}e^{-\frac{2\pi k}{p}}$$

$$\text{when } t = b + \frac{5\pi}{p} \qquad y = -ae^{-k\left(b+\frac{5\pi}{p}\right)} \sin \alpha$$

$$= \mathrm{N}e^{-\frac{4\pi k}{p}}$$

Thus the successive minimum values of y are in geometrical progression, the common ratio of which is $e^{-\frac{2\pi k}{p}}$

EXAMPLES VIII

Find the maximum and minimum values of y and the values of x, producing them in each of the following examples:

(1) $y = 2x^3 - 9x^2 + 12x + 30$.
(2) $y = x^3 - 75x + 24$.
(3) $y = 3x^4 + 4x^3 - 24x^2 - 48x + 64$.

(4) Find the minimum value of $\frac{3}{x^2} + 5x^3$, and the value of x which produces it.

In each of the following examples find the maximum value of x, and the value of t which produces it.

(5) $x = 3e^{-t} \sin 3t$.
(6) $x = \sqrt{10}\, e^{-t} \sin (3t + 1{\cdot}249)$.
(7) $x = 5e^{-t} \sin (3t + 0{\cdot}6428)$.
(8) $x = 10t\, e^{-2t}$.
(9) $x = 5\,(1 + 2t)\, e^{-2t}$.
(10) $x = 5\,(1 + 4t)e^{-2t}$.
(11) $x = 4\,(e^{-2t} - e^{-4t})$.
(12) $x = 4\,(2e^{-2t} - e^{-4t})$.
(13) $x = 4\,(3e^{-2t} - 2e^{-4t})$.

(14) If $y = x^3e^{-x}$, find the maximum value of y, and the value of x producing it.

(15) Find the dimensions of the cylinder of greatest volume which can be inscribed in a sphere of 10 inches radius.

(16) Find the dimensions of the cylinder of greatest curved surface which can be inscribed in a sphere of 10 inches radius.

(17) Find the dimensions of the cylinder of greatest total surface which can be inscribed in a sphere of 10 inches radius.

(18) Find the dimensions of a conical tent of greatest capacity, the area of canvas used being 500 square yards.

(19) If $y = x^{2n} - x^{1+n}$ where $n = 0{\cdot}885$, for what value of x is y a maximum, and what is the maximum value?

(20) The cost of a ship per hour is £c where $c = 8{\cdot}21 + \dfrac{s^3}{1200}$, s being the speed in knots. Express the total cost of a passage of 3400 nautical miles in terms of s. What value of s will make this total cost a minimum? Show that at speeds 10 per cent. greater or less than this, the total cost is not very much greater than what it is at the best speed. (B. of E., 1911.)

(21) The cost of a ship per hour is £c where $c = 4 + \dfrac{s^3}{1000}$, s being the speed in knots relatively to the water. Going up a river whose current is 5 knots, what is the speed which causes least total cost of passage? (B. of E., 1905.)

(22) In Q. 21, if the ship is going down the river, what is the speed which causes least total cost of passage?

(23) From a rectangular sheet of tin $12'' \times 10''$ equal squares are cut from each corner, and the remainder is formed into a rectangular vessel. Find the length of the side of the square so that the volume of the vessel shall be greatest.

(24) ABCD is a sheet of tin 10″ square. From the corners A and B squares of x'' side are cut away, and from the corners C and D rectangles of breadth x'' are cut away. The remainder is formed into a rectangular vessel with a lid. Find the dimension x so that the volume of this vessel shall be greatest.

(25) The stiffness of a beam of rectangular cross-section varies as bh^3 where b is the breadth and h is the depth of a cross-section. Find the dimensions of the stiffest beam which can be cut from a cylindrical log of 24 inches diameter.

(26) If $y = ae^{-kt} \sin (pt - c)$, and $c = 0{\cdot}135$, $a = 4$, $k = 300$, $p = 500$, find the first maximum and the first minimum value of y, and the values of t which produce them.

(27) Find the dimensions of the cylinder of greatest volume which can be inscribed in a cone 5″ high, radius of base 2″.

(28) Find the dimensions of the cylinder of greatest curved surface which can be inscribed in a cone 5″ high, radius of base 2″.

(29) Find the dimensions of the cylinder of greatest total surface which can be inscribed in a cone 5″ high, radius of base 2″.

(30) The annual cost of giving a certain amount of electric light to a certain town, the voltage being V and the candle-power of each lamp C, is found to be

$$A = a + \frac{b}{V} \text{ for electric energy}$$

and $$B = \frac{m}{C} + n\frac{V^{\frac{2}{3}}}{C^{\frac{5}{3}}} \text{ for lamp renewals.}$$

V	100	200
A	1500	1200
B	300	500

The given values are known when C = 10. Find *a*, *b*, *m*, and *n*. If C = 20, what value of V will give minimum total cost? (B. of E., 1907.)

(31) If a crank is at the angle θ from a dead point, and $\theta = qt$ where q is the angular velocity; if x feet is the distance of the piston from the end of its stroke, r feet is the length of the crank, and l feet the length of the connecting rod; then

$$x = r(1 - \cos\theta) + \frac{r^2}{4l}(1 - \cos 2\theta)$$

Taking $r = 1$ and $l = 5$, what is the angular position of the crank when the piston is moving with greatest velocity, and what is the distance of the piston from the end of its stroke at that instant?

CHAPTER IX

80. *The Equation of the Tangent to a Given Curve.* The equation of a line is $y = mx + c$ where m is the tangent of the angle of slope, or if the line is inclined to the axis of x at an angle θ, then $m = \tan\theta$.

The tangent to a curve is a straight line which passes through a given point on the curve and also has the same slope as the curve at that point.

Let the co-ordinates of a point on a curve be h, k. Then the slope of the curve at that point is the value of $\dfrac{dy}{dx}$ when $x = h$, and this must be the slope of the tangent.

The equation of the tangent is $y = x\left(\dfrac{dy}{dx}\right)_{x=h} + c$.

But this line also passes through the point,

and
$$k = h\left(\frac{dy}{dx}\right)_{x=h} + c$$
$$c = k - h\left(\frac{dy}{dx}\right)_{x=h}$$

Therefore
$$y = x\left(\frac{dy}{dx}\right)_{x=h} + k - h\left(\frac{dy}{dx}\right)_{x=h}$$
$$= k + (x - h)\left(\frac{dy}{dx}\right)_{x=h}$$

The normal to a curve is the line drawn perpendicular to the tangent through the point of contact.

If the tangent to a curve is inclined to the axis of x at an angle θ, then the corresponding normal is inclined to the axis of x at an angle $(90° + \theta)$.

$$\text{Slope of the normal} = \tan(90° + \theta)$$
$$= -\cot\theta$$
$$= -\frac{1}{m}, \text{ where } m = \tan\theta$$

Then the slope of the normal to the curve at the given point is

$$-\frac{1}{\left(\dfrac{dy}{dx}\right)_{x=h}} \quad \text{or} -\left(\frac{dx}{dy}\right)_{x=h}$$

The equation to the normal is $y = -x\left(\frac{dx}{dy}\right)_{x=h} + c_1$

But this line passes through the given point,

and $$k = -h\left(\frac{dx}{dy}\right)_{x=h} + c_1$$

$$c_1 = k + h\left(\frac{dx}{dy}\right)_{x=h}$$

Therefore $$y = -x\left(\frac{dx}{dy}\right)_{x=h} + k + h\left(\frac{dx}{dy}\right)_{x=h}$$

$$= k - (x - h)\left(\frac{dx}{dy}\right)_{x=h}$$

Example 1. To find the equations of the tangent and normal to the curve $y^2 = 4x$ at the point where $x = 4$.

Then $$y = 2x^{\frac{1}{2}} \text{ and } \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

when $$x = 4,\ y = 4, \text{ and } \frac{dy}{dx} = \frac{1}{2}$$

Equation of the tangent is $y = \frac{1}{2}x + c$

but $$4 = \frac{1}{2} \times 4 + c$$

and $$c = 2$$

Then $$y = \frac{1}{2}x + 2, \text{ or } 2y = x + 4$$

Equation of the normal is $y = -2x + c_1$

but $$4 = -2 \times 4 + c_1$$

and $$c_1 = 12$$

Then $$y = -2x + 12, \text{ or } 2x + y = 12.$$

Example 2. To find the equations of the tangent and normal to the curve $y = e^x \sin x$.

When $$x = 1,\ y = e \sin 1 = 0{\cdot}8415e$$

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$= e^x (\sin x + \cos x)$$

When $$x = 1,\ \frac{dy}{dx} = e(0{\cdot}8415 + 0{\cdot}5403) = 1{\cdot}3818e$$

Equation of the tangent is $y = mx + c$, or $y = x\ 1{\cdot}3818e + c$

but $\quad 0{\cdot}8415e = 1{\cdot}3818e + c$

and $\quad c = -0{\cdot}5403e$

and
$$y = 1{\cdot}3818ex - 0{\cdot}5403e$$
$$= 3{\cdot}758x - 1{\cdot}469$$

Equation of the normal is $y = -\dfrac{1}{m}x + c_1$ or $y = -\dfrac{x}{1{\cdot}3818e} + c_1$

but
$$0{\cdot}8415e = -\frac{1}{1{\cdot}3818e} + c_1$$
$$c_1 = 2{\cdot}554$$

and
$$y = -\frac{x}{1{\cdot}3818e} + 2{\cdot}554$$
$$= -0{\cdot}2663x + 2{\cdot}554.$$

81. *The Angle between a Line and a Curve at their Point of Intersection.* In this case the required angle is the angle between the line and the tangent to the curve at the point of intersection.

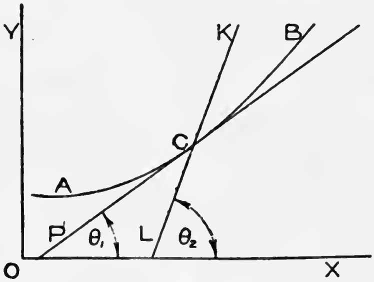


FIG. 37.

Let ABC (Fig. 37) be the curve and KL the line. C is the point of intersection, and PC is the tangent to the curve at the point C.

Then the required angle is $\widehat{PCL}$.

If $y = f(x)$ be the equation to the curve, and $y = mx + c$ the equation of the line, the co-ordinates of the point C are the values of x and y which satisfy these two equations simultaneously.

Let these values be $x = h$ and $y = k$.

Then the slope of the tangent $= \left(\dfrac{dy}{dx}\right)_{x=h}$

and hence
$$\left(\frac{dy}{dx}\right)_{x=h} = \tan\theta_1$$

This gives the inclination of the tangent PC

The slope of the line = m, and hence $m = \tan \theta_2$. This gives the inclination of the line KL.

$$\text{Then } \widehat{PCL} = \theta_2 - \theta_1$$

Example. Find the angle between the curve $xy = 1$ and the straight line $y = 2x + 1$ at the point of intersection.

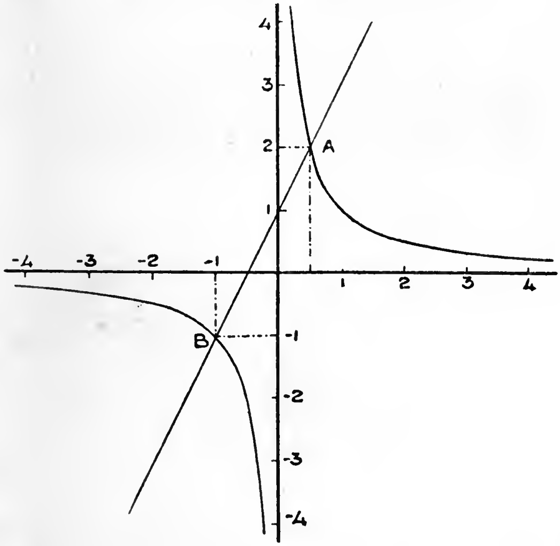


FIG. 38.

$$xy = 1$$
$$x(2x + 1) = 1$$
$$2x^2 + x - 1 = 0$$
$$(2x - 1)(x + 1) = 0$$
$$\text{or } x = \frac{1}{2} \text{ and } x = -1$$

There are two points of intersection: A, whose co-ordinates are $\frac{1}{2}$, 2; B, whose co-ordinates are -1, -1. The slope of the line is 2, and hence the inclination of the line is $\tan^{-1} 2 = 63° 26'$.

For the curve $\quad y = x^{-1}$

and $\quad \dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

Inclination of the tangent to the curve is $\tan^{-1}\left(-\dfrac{1}{x^2}\right)$

At the point A. Inclination of the tangent $= \tan^{-1}(-4)$

$= 104° 2'$

Required angle $= 104° 2' - 63° 26' = 40° 36'$

At the point B. Inclination of the tangent $= \tan^{-1}(-1)$

$= 135°$

Required angle $= 135° - 63° 26' = 71° 34'$

82. *The Change in Direction along a Curve between two Given Points.* This is the angle between the tangents to the curve at the given points. Let P and Q (Fig. 39) be the two points whose co-ordinates are x_1, y_1 and x_2, y_2 respectively.

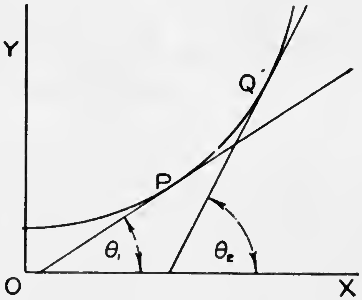


FIG. 39.

Let θ_1 be the inclination of the tangent at P and θ_2 the inclination of the tangent at Q.

But $\tan\theta_1 = \frac{dy}{dx}$ when $x = x_1$, thus giving the value of θ_1

and $\tan\theta_2 = \frac{dy}{dx}$ then $x = x_2$, thus giving the value of θ_2

The change in direction $= \theta_2 - \theta_1$

83. *Curvature.* Curvature is the rate at which the direction changes with the arc. Let PQ (Fig. 40) represent the small arc of a curve, and let θ and $\theta + \delta\theta$ be the inclinations of the tangents to the curve at P and Q respectively. Then the change in direction is $\delta\theta$, and this occurs over a length of arc δs.

The average curvature of the arc PQ $= \frac{\delta\theta}{\delta s}$, and, making δs infinitely small, this becomes the actual curvature at the point P.

$$\text{Actual curvature at P} = \frac{d\theta}{ds}$$

If δs be taken as a very small arc, it can be approximately taken as the small arc of a circle, the radius of which is R and the angle at the centre is $\delta\theta$, since the radii must be at right angles to the tangents at P and Q respectively.

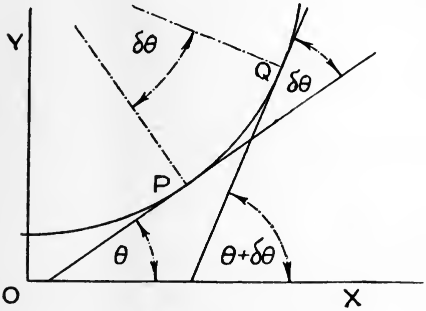


FIG. 40.

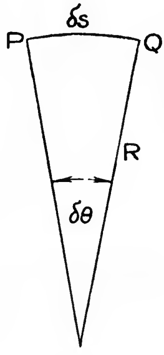


FIG. 41.

Then $$\delta\theta = \frac{\delta s}{R}$$

or $$\frac{\delta\theta}{\delta s} = \frac{1}{R}$$

The smaller the length of arc taken, the more nearly correct does this approximation become, and, making δs infinitely small,

$$\frac{d\theta}{ds} = \frac{1}{R}$$

The circle whose radius is R, the value of which is given by $\frac{ds}{d\theta}$, is spoken of as the Circle of Curvature, and R is the Radius of Curvature.

In order to find $\frac{d\theta}{ds}$, we want a relation giving us θ in terms of s, but if we are given the law of the curve as $y = f(x)$ (or y in terms of x), it is a very difficult matter to obtain such a relation, and therefore it is better to find the expression which is equivalent to $\frac{d\theta}{ds}$ in terms of x and y.

If δs is a very small arc, it may be taken as the hypotenuse of a right-angled triangle whose base angle is θ, and whose perpendicular and base are δy and δx respectively.

When δs is made infinitely small,

$$\frac{dy}{dx} = \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{dx}{ds} = \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking the first relation and differentiating both sides with respect to s,

$$\frac{d^2y}{dx^2}\,\frac{dx}{ds} = \sec^2\theta\,\frac{d\theta}{ds}$$

$$\frac{d^2y}{dx^2}\cos\theta = \sec^2\theta\,\frac{d\theta}{ds}$$

$$\frac{d\theta}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\sec^3\theta}$$

$$= \frac{\dfrac{d^2y}{dx^2}}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$= \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

and since

$$\frac{d\theta}{ds} = \frac{1}{\mathrm{R}}$$

$$\mathrm{R} = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

For a very flat curve $\dfrac{dy}{dx}$ is very small compared with 1

and

$$\frac{d\theta}{ds} = \frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2} \text{ approximately}$$

a result which is used in the consideration of the deflection of beams.

84. *The Co-ordinates of the Centre of Curvature.* Let P be a point on a curve, and let R be the radius of curvature at that point (Fig. 42). The co-ordinates of P are x_1, y_1. Let O_1 be the centre of curvature and x, y its co-ordinates.

Then $$x = x_1 - \mathrm{CP}$$
$$= x_1 - \mathrm{R} \sin \theta$$
and $$y = y_1 + \mathrm{O_1C}$$
$$= y_1 + \mathrm{R} \cos \theta$$
where θ is the angle whose tangent is $\frac{dy}{dx}$ when $x = x_1$.

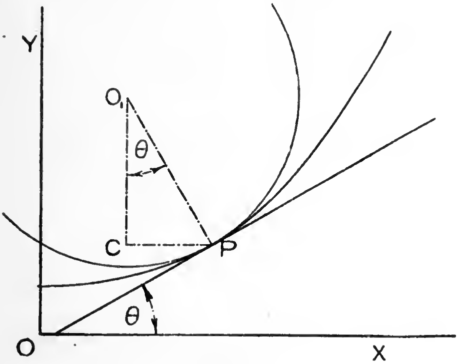


FIG. 42.

Example. Find the radius of curvature of the cycloid
$$x = a(\alpha - \sin \alpha),\ y = a(1 - \cos \alpha)$$
Then $$\frac{dx}{d\alpha} = a(1 - \cos \alpha)$$
$$\frac{dy}{d\alpha} = a \sin \alpha$$
and $$\frac{dy}{dx} = \frac{dy}{d\alpha} \times \frac{d\alpha}{dx} = \frac{\sin \alpha}{1 - \cos \alpha} = z$$
Then $$\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dz}{d\alpha} \times \frac{d\alpha}{dx}$$
To find $\frac{dz}{d\alpha}$ $$\log_e z = \log_e \sin \alpha - \log_e (1 - \cos \alpha)$$
$$\frac{1}{z}\frac{dz}{d\alpha} = \frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{1 - \cos \alpha}$$
$$= \frac{\cos \alpha - \cos^2 \alpha - \sin^2 \alpha}{\sin \alpha(1 - \cos \alpha)}$$
$$= \frac{\cos \alpha - 1}{\sin \alpha(1 - \cos \alpha)} = -\frac{1}{\sin \alpha}$$
and $$\frac{dz}{d\alpha} = -\frac{1}{\sin \alpha}\frac{\sin \alpha}{1 - \cos \alpha}$$
$$= -\frac{1}{1 - \cos \alpha}$$

Then $$\frac{d^2y}{dx^2} = -\frac{1}{a(1-\cos\alpha)^2}$$

Also $$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \left\{1+\frac{\sin^2\alpha}{(1-\cos\alpha)^2}\right\}^{\frac{3}{2}}$$

$$= \left\{\frac{1-2\cos\alpha+\cos^2\alpha+\sin^2\alpha}{(1-\cos\alpha)^2}\right\}^{\frac{3}{2}}$$

$$= \left\{\frac{2(1-\cos\alpha)}{(1-\cos\alpha)^2}\right\}^{\frac{3}{2}}$$

$$= \frac{2\sqrt{2}}{(1-\cos\alpha)^{\frac{3}{2}}}$$

Then $$R = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= -\frac{2\sqrt{2}\,a(1-\cos\alpha)^2}{(1-\cos\alpha)^{\frac{3}{2}}}$$

$$= -2a\sqrt{2(1-\cos\alpha)}$$

When $\alpha = 180°$, that is, at the highest point of the curve, $R = -4a$, or twice the diameter of the rolling circle.

Example. Find the radius of curvature and the co-ordinates of the centre of curvature of the curve $y^2 = 8x$ at the points where $x = 0$, $x = 2$, and $x = 8$.

$$y = 2\sqrt{2}\,x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \sqrt{2}\,x^{-\frac{1}{2}} = \sqrt{\frac{2}{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{2}}{2}x^{-\frac{3}{2}} = -\sqrt{\frac{1}{2x^3}}$$

$$R = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= -\sqrt{2}\,x^{\frac{3}{2}}\left(1+\frac{2}{x}\right)^{\frac{3}{2}}$$

$$= -\sqrt{2}\,(x+2)^{\frac{3}{2}}$$

When $x_1 = 0$, $R_1 = -\sqrt{2} \times 2^{\frac{3}{2}} = -4$

$x_2 = 2$, $R_2 = -\sqrt{2} \times 4^{\frac{3}{2}} = -8\sqrt{2} = -11{\cdot}13$

$x_3 = 8$, $R_3 = -\sqrt{2} \times 10^{\frac{3}{2}} = -20\sqrt{5} = -44{\cdot}72$

When $x_1 = 0$, $y_1 = 0$, $\dfrac{dy}{dx} = \infty$, $\tan \theta_1 = \infty$, and $\theta_1 = 90°$

$$\left.\begin{aligned} x &= x_1 - R_1 \sin \theta_1 = 0 + 4 \sin 90° = 4 \\ y &= y_1 + R_1 \cos \theta_1 = 0 - 4 \cos 90° = 0 \end{aligned}\right\}$$

When $x_2 = 2$, $y_2 = 4$, $\dfrac{dy}{dx} = 1$, $\tan \theta_2 = 1$, and $\theta_2 = 45°$

$$\left.\begin{aligned} x &= x_2 - R_2 \sin \theta_2 = 2 + \frac{8\sqrt{2}}{\sqrt{2}} = 10 \\ y &= y_2 + R_2 \cos \theta_2 = 4 - \frac{8\sqrt{2}}{\sqrt{2}} = -4 \end{aligned}\right\}$$

When $x_3 = 8$, $y_3 = 8$, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Then $\tan \theta_3 = \dfrac{1}{2}$, $\sin \theta_3 = \dfrac{1}{\sqrt{5}}$, $\cos \theta_3 = \dfrac{2}{\sqrt{5}}$

$$\left.\begin{aligned} x &= x_3 - R_3 \sin \theta_3 = 8 + \frac{20\sqrt{5}}{\sqrt{5}} = 28 \\ y &= y_3 + R_3 \cos \theta_3 = 8 - \frac{40\sqrt{5}}{\sqrt{5}} = -32 \end{aligned}\right\}$$

When $x = 0$, $R = -4$.	Co-ordinates of	centre of	curvature	$4, 0$
$x = 2$, $R = -11{\cdot}13$	”	”	”	$10, -4$
$x = 8$, $R = -44{\cdot}72$	”	”	”	$28, -32$

85. *The Point of Inflexion.* In the previous chapter we considered the case of a point of inflexion on a curve at which the tangent is horizontal, but a point of inflexion can occur when the tangent is not horizontal.

Let C (Fig. 43) be such a point, and let the tangent to the curve at this point be inclined to the axis of x at an angle α.

Case I. When the angle is acute. Let $\tan \alpha = m$. Moving along the curve from A to C, the angle θ is acute, and decreases to α.

Then $\dfrac{dy}{dx}$ is positive, and decreases to m.

Moving along the curve from C to B, the angle θ is acute, and increases from α.

Then $\dfrac{dy}{dx}$ is positive, and increases from m.

Thus $\frac{dy}{dx}$ is evidently a minimum at the point C, and if $\frac{dy}{dx}$ is a minimum, then $\frac{d^2y}{dx^2} = 0$.

Case II. When the angle α *is obtuse.* Let $\tan \alpha = -m$. Moving along the curve from A to C, the angle θ is obtuse, and increases to α.

Then $\frac{dy}{dx}$ is negative and increases to $-m$.

Moving along the curve from C to B, the angle θ is obtuse, and decreases from α.

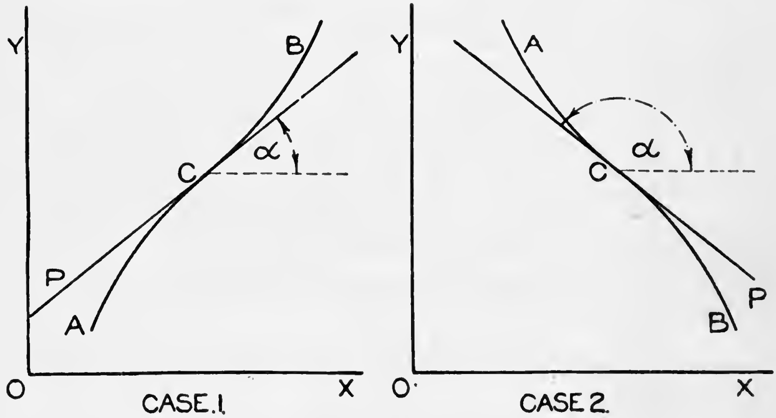


Fig. 43.

Then $\frac{dy}{dx}$ is negative and decreases from $-m$.

Thus $\frac{dy}{dx}$ is evidently a maximum at the point C, and if $\frac{dy}{dx}$ is a maximum, then $\frac{d^2y}{dx^2} = 0$.

In general a point of inflexion may be defined as a point on a curve at which the slope is greatest or least, while its position is given by the relation $\frac{d^2y}{dx^2} = 0$.

Example. Find the points of inflexion of the curve

$$y = x^4 + 2x^3 - 36x^2 + 48x - 52$$

$$\frac{dy}{dx} = 4x^3 + 6x^2 - 72x + 48$$

$$\frac{d^2y}{dx^2} = 12x^2 + 12x - 72$$

A point of inflexion occurs when $\dfrac{d^2y}{dx^2} = 0$

That is, when $\quad 12x^2 + 12x - 72 = 0$

or $\quad x^2 + x - 6 = 0$

$$(x + 3)(x - 2) = 0$$

at points where $\quad x = -3$ and $x = 2$

EXAMPLES IX

(1) Find the values of the slope of the curve $y = x^3 - 3x + 5$ at the points where $x = 1{\cdot}5$ and $x = 2{\cdot}0$. Find the equations of the tangents to the curve at these points. What is the angle between these tangents ?

(2) Find the value of the slope of the curve $y = 3x^2 - 4x + 3$ at the point where $x = 2$. Find the equations of the tangent and normal to the curve at that point.

(3) The curve $y = x^2 - 1$ is cut by the line $y = x + 5$. Find the co-ordinates of the points of intersection. Find the angles between the line and the curve at these points.

(4) Find the equations of the tangent and normal to the curve $y = 4x^3$ at the point where $x = 2$.

(5) Find the equations of the tangent and normal to the curve $y^3 = 8x^2$ at the point where $x = 2$.

(6) Find the co-ordinates of the point of intersection of the curves $x^2 + y^2 = 5$ and $x^2 - y^2 = 2$, and find the angle between the curves at that point.

(7) The two curves $xy = 1$ and $x^2 - y^2 = 4$ intersect at a point P. Find the co-ordinates of P and the angle between the two curves at that point.

(8) The curve $xy = 4$ is cut by the line $10y = 7x + 4$. Find the co-ordinates of the points of intersection and the angles between the line and the curve at these points.

(9) The curve $y = ax^n$ passes through the points (3, 10) and (6, 17). Find a and n. Find the value of the slope of the curve at a point P where $x = 2$. A second point Q is taken on the curve, and this point can be on either side of P. Find the co-ordinates of the two positions of Q, so that the angle turned through in moving along the curve from P to Q is 7°.

(10) The two curves $y^2 = 8x$ and $x^2 = 8y$ intersect at a point P, other than the origin. Find the co-ordinates of P and the angle between the curves at that point.

(11) The curve $y = ae^{bx}$ passes through the points (1, 3·5) and (10, 12·6). Find a and b. Find the value of the slope of the curve at the point where $x = 5$. Find the equations of the tangent and normal to the curve at that point.

(12) Find the values of the radius of curvature of the curve $y = 3{\cdot}2 + 1{\cdot}73x^{1{\cdot}75}$ at the points where $x = 1$, $x = 2$, and $x = 3$.

(13) The curve $y = a + bx^{1{\cdot}5}$ passes through the points (1, 1·82) and (4, 5·32). Find a and b. Find the value of the radius of curvature at the point where $x = 2$; find also the co-ordinates of the centre of curvature for that point.

(14) The curve $y = a + bc^x$ passes through the points (0, 28·62), (1, 35·70), and (2, 49·81). Find a, b, and c. Find the value of the radius of curvature at the point where $x = 1$.

(15) The curve $y = 10x^{\frac{1}{2}}$ is cut by the line $y = 2x - 10$. Find the co-ordinates of that point of intersection for which y is positive. Find the angle between the curve and the line at that point. (B. of E., 1913.)

(16) Find the values of the radius of curvature of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the points where $x = 0$ and $x = 4$.

(17) Find the value of the radius of curvature of the curve $xy = 4$ at the point where $x = 2$. Find also the co-ordinates of the corresponding centre of curvature.

(18) Find the co-ordinates of the point of inflexion on the curve $4y = 6x^2 - x^3$.

(19) Find the co-ordinates of the point of inflexion on the curve $y = e^{-x^2}$.

CHAPTER X

86. *Integration.* Integration is the converse of differentiation. If we differentiate a certain function with respect to x, the effect of integrating the result with respect to x will be to produce the original function. For example, if $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$, and integrating nax^{n-1} with respect to x will produce ax^n.

The process of integration is denoted by the symbol $\int$, and if x is the variable, the expression to be integrated is terminated by dx. This at once distinguishes the variable from the constants in the expression to be integrated.

Thus $\int y\,dx$ means that y must be integrated with respect to x, and this can be done provided we know the relation which gives y in terms of x. Also $\int x\,dy$ means that x must be integrated with respect to y, and to do this we must know the relation which gives x in terms of y.

To integrate ax^n, or to find $\int ax^n\,dx$.

$$\text{If} \quad y = ax^m, \text{ then } \frac{dy}{dx} = max^{m-1}$$

Since integration is the converse of differentiation,

$$m\int ax^{m-1}\,dx = ax^m$$

or

$$\int ax^{m-1}\,dx = \frac{ax^m}{m}$$

Replacing $m - 1$ by n,

Then

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1}$$

When the constant $a = 1$, then $\int x^n\,dx = \frac{x^{n+1}}{n+1}$. This result holds for all values of the power n, except the case when $n = -1$.

To integrate x^{-1} or to find $\int \frac{dx}{x}$

If $$y = \log_e x \quad \text{then } \frac{dy}{dx} = \frac{1}{x}$$

and conversely $$\int \frac{dx}{x} = \log_e x$$

This result can be used in a more general sense, for if we consider the integral $\int \frac{(2ax + b)\ dx}{ax^2 + bx + c}$ as an example on the use of it,

by putting $$y = ax^2 + bx + c$$

then $$\frac{dy}{dx} = 2ax + b$$

and dy can replace $(2ax + b)\ dx$ in the integral.

The integral then becomes $$\int \frac{dy}{y} = \log_e y$$
$$= \log_e (ax^2 + bx + c)$$

It should be noticed that the fraction $\frac{2ax + b}{ax^2 + bx + c}$ belongs to a particular type in which the numerator is the differential coefficient of the denominator, and the above method of treatment will do for all fractions belonging to this class. In general, if we integrate a fraction whose numerator is the differential coefficient of the denominator, the result will be the Napierian logarithm of the denominator.

We can now use as standard integrals

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} \quad \ldots \ldots \quad (1)$$

$$\int \frac{dx}{x} = \log_e x \quad \ldots \ldots \quad (2)$$

and $$\int \frac{\text{diff. coeff. of denominator}}{\text{denominator}} = \log_e (\text{denominator}) \quad \ldots \quad (3)$$

and employ them to integrate expressions which resemble them, or expressions which can ultimately be reduced down to resemble them.

87. The fraction whose numerator is the differential coefficient of the denominator or of some part of the denominator, can be readily integrated. The following examples will illustrate this

(*a*) $$\int \cot\ \theta\ d\theta = \int \frac{\cos\theta\ d\theta}{\sin\theta}$$
$$= \log_e \sin\theta$$

(*b*) $\displaystyle\int\frac{\cos\theta\, d\theta}{\sin^4\theta}$. Let $x = \sin\theta$

and $\dfrac{dx}{d\theta} = \cos\theta$

or $dx = \cos\theta\, d\theta$

The integral becomes $\displaystyle\int\frac{dx}{x^4} = \int x^{-4}\, dx$

$$= -\frac{x^{-3}}{3}$$

$$= -\frac{1}{3\sin^3\theta}$$

(*c*) $\displaystyle\int\frac{(10x-7)\, dx}{\sqrt{5x^2-7x+12}}$. Let $y = 5x^2 - 7x + 12$

and $\dfrac{dy}{dx} = 10x - 7$

or $dy = (10x - 7)\, dx$

The integral becomes $\displaystyle\int\frac{dy}{\sqrt{y}} = \int y^{-\frac{1}{2}}\, dy$

$$= \frac{y^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= 2\sqrt{5x^2 - 7x + 12}$$

(*d*) $\displaystyle\int\frac{\sec^2 x\, dx}{\tan^2 x}$. Let $y = \tan x$

and $\dfrac{dy}{dx} = \sec^2 x$

or $dy = \sec^2 x\, dx$

The integral becomes $\displaystyle\int\frac{dy}{y^2} = \int y^{-2}\, dy$

$$= -y^{-1}$$

$$= -\frac{1}{\tan x}$$

88. *The Integration of Algebraic Fractions whose Denominators split up into Linear Factors.* In this case the fraction can be split up into its partial fractions, and then each partial fraction can be integrated separately.

Case I. When the denominator is the product of unlike linear factors.

(*a*) To integrate $\dfrac{5x+4}{(x-2)(x+5)}$

Now $$\frac{5x+4}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

and $$A(x+5) + B(x-2) = 5x+4$$

When $x = 2$ $\quad 7A = 14 \qquad A = 2$

When $x = 5$ $\quad -7B = -21 \qquad B = 3$

Then $$\int\frac{(5x+4)\,dx}{(x-2)(x+5)} = 2\int\frac{dx}{x-2} + 3\int\frac{dx}{x+5}$$
$$= 2\log_e(x-2) + 3\log_e(x+5)$$

(*b*) To integrate $\dfrac{x+8}{(5x-3)(4-3x)}$

Now $$\frac{x+8}{(5x-3)(4-3x)} = \frac{A}{5x-3} + \frac{B}{4-3x}$$

and $$A(4-3x) + B(5x-3) = x+8$$

When $x = \frac{3}{5}$ $\quad \frac{11}{5}A = \frac{43}{5} \qquad A = \frac{43}{11}$

When $x = \frac{4}{3}$ $\quad \frac{11}{3}B = \frac{28}{3} \qquad B = \frac{28}{11}$

Then $$\int\frac{(x+8)\,dx}{(5x-3)(4-3x)} = \frac{43}{11}\int\frac{dx}{5x-3} + \frac{28}{11}\int\frac{dx}{4-3x}$$
$$= \frac{43}{55}\int\frac{5\,dx}{5x-3} - \frac{28}{33}\int\frac{-3\,dx}{4-3x}$$
$$= \frac{43}{55}\log_e(5x-3) - \frac{28}{33}\log_e(4-3x)$$

It should be noticed in this example that when integrating the partial fractions, the numerators are not the differential coefficients of the denominators, but by multiplying and dividing the first fraction by 5 and the second fraction by -3 this relation is at once made to hold.

Case II. When the denominator of the fraction consists of one linear factor raised to a power.

To integrate $\dfrac{5x^2-8x+2}{(2x-3)^3}$

Putting $$y = 2x-3, \text{ then } x = \frac{1}{2}(y+3)$$

and $$5x^2 - 8x + 2 = \frac{5}{4}y^2 + \frac{7}{2}y + \frac{5}{4}$$

Also $\qquad \dfrac{dy}{dx} = 2$ and $dx = \dfrac{1}{2}dy$

$$\text{Then} \int \frac{(5x^2 - 8x + 2)}{(2x - 3)^3}\, dx = \frac{1}{2}\int \frac{\left(\frac{5}{4}y^2 + \frac{7}{2}y + \frac{5}{4}\; dy\right)}{y^3}$$

$$= \frac{1}{2}\left[\frac{5}{4}\int\frac{dy}{y} + \frac{7}{2}\int\frac{dy}{y^2} + \frac{5}{4}\int\frac{dy}{y^3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{4}\int\frac{dy}{y} + \frac{7}{2}\int y^{-2}\, dy + \frac{5}{4}\int y^{-3}\, dy\right]$$

$$= \frac{1}{2}\left[\frac{5}{4}\log_e y + \frac{7}{2}\,\frac{y^{-1}}{-1} + \frac{5}{4}\,\frac{y^{-2}}{-2}\right]$$

$$= \frac{5}{8}\log_e(2x - 3) - \frac{7}{4(2x - 3)} - \frac{5}{16(2x - 3)^2}$$

Knowing the standard form $\int x^n\, dx = \dfrac{x^{n+1}}{n+1}$, we are in a position to evaluate $\int y^{-2}\, dy$ and $\int y^{-3}\, dy$.

Case III. When the denominator of the fraction contains unlike linear factors, some of which are raised to powers.

To integrate $\qquad \dfrac{x^2 + 4}{(x^2 - 4)(x + 2)}$

$$\text{Now } \frac{x^2 + 4}{(x^2 - 4)(x + 2)} = \frac{x^2 + 4}{(x + 2)^2(x - 2)} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{x - 2}$$

and $\quad A(x + 2)(x - 2) + B(x - 2) + C(x + 2)^2 = x^2 + 4$

When $x = -2 \qquad -4B = 8 \qquad B = -2$

When $x = 2 \qquad 16C = 8 \qquad C = \dfrac{1}{2}$

When $x = 0 \qquad -4A - 2B + 4C = 4 \qquad A = \dfrac{1}{2}$

$$\text{Then} \int\frac{(x^2 + 4)\, dx}{(x^2 - 4)(x + 2)} = \frac{1}{2}\int\frac{dx}{x + 2} - 2\int\frac{dx}{(x + 2)^2} + \frac{1}{2}\int\frac{dx}{x - 2}$$

$$= \frac{1}{2}\log_e(x + 2) + \frac{2}{x + 2} + \frac{1}{2}\log_e(x - 2)$$

$$= \frac{1}{2}\log_e(x^2 - 4) + \frac{2}{x + 2}$$

The first and third integrals are such that the numerator is the differential coefficient of the denominator, while the second, on

putting $y = x + 2$, becomes $\int \frac{dy}{y^2}$, since $\frac{dy}{dx} = 1$ and $\int \frac{dy}{y^2} = \int y^{-2}\, dy$
$= \frac{y^{-1}}{-1} = -\frac{1}{y} = -\frac{1}{x+2}$

Case IV. When the numerator is of higher degree than the denominator.

To integrate $$\frac{x^5}{(x^2-1)(x+1)}$$

By division $$\frac{x^5}{(x^2-1)(x+1)} = x^2 - x + 2 + \frac{2 + x - 2x^2}{(x^2-1)(x+1)}$$

but $$\frac{2 + x - 2x^2}{(x^2-1)(x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

and $A(x+1)(x-1) + B(x-1) + C(x+1)^2 = 2 + x - 2x^2$

when $x = 1$ $\quad 4C = 1 \quad C = \frac{1}{4}$

when $x = -1$ $\quad -2B = -1 \quad B = \frac{1}{2}$

when $x = 0$ $\quad -A - B + C = 2 \quad A = -\frac{9}{4}$

Hence $\int \frac{x^5\, dx}{(x^2-1)(x+1)}$

$$= \int x^2\, dx - \int x\, dx + 2\int dx - \frac{9}{4}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{x-1}$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x - \frac{9}{4}\log_e(x+1) - \frac{1}{2(x+1)} + \frac{1}{4}\log_e(x-1)$$

$$= \frac{x}{6}(2x^2 - 3x + 12) - \frac{1}{2(x+1)} - \frac{1}{4}\{9\log_e(x+1) - \log_e(x-1)\}$$

89. We next have to consider the integration of algebraic fractions the denominators of which are of the second degree but cannot be resolved into linear factors. In this case the method of integration depends upon the nature of the denominator.

For $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left\{\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)\right\} \text{ or}$$

$$a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right\}$$

according as $4ac$ is greater or less than b^2.

Hence $\quad ax^2 + bx + c = a(X^2 + A^2)$ or $a(X^2 - A^2)$

where $\quad X = x + \dfrac{b}{2a}$ and $A^2 = \dfrac{4ac - b^2}{4a^2}$ or $\dfrac{b^2 - 4ac}{4a^2}$

Thus a quadratic expression which will not factor may be expressed as the difference or the sum of two squares.

Case I. When the denominator of the fraction reduces to the form $a(X^2 - A^2)$.

(*a*) To integrate $\quad \dfrac{1}{x^2 + 10x + 13}$

Then
$$\begin{aligned} x^2 + 10x + 13 &= x^2 + 10x + 25 - 12 \\ &= (x+5)^2 - 12 \\ &= X^2 - A^2 \end{aligned}$$

where $\quad X = x + 5$, $A^2 = 12$, and also $dX = dx$.

Then
$$\int \frac{dx}{x^2 + 10x + 13} = \int \frac{dX}{X^2 - A^2}$$

But
$$\frac{1}{X^2 - A^2} = \frac{\alpha}{X + A} + \frac{\beta}{X - A}$$

and $\quad \alpha(X - A) + \beta(X + A) = 1$

when $X = A$ $\qquad 2A\beta = 1 \qquad \beta = \dfrac{1}{2A}$

when $X = -A$ $\qquad -2A\alpha = 1 \qquad \alpha = -\dfrac{1}{2A}$

Thus
$$\begin{aligned} \int \frac{dX}{X^2 - A^2} &= \frac{1}{2A}\left\{\int \frac{dX}{X - A} - \int \frac{dX}{X + A}\right\} \\ &= \frac{1}{2A}\{\log_e(X - A) - \log_e(X + A)\} \\ &= \frac{1}{2A}\log_e \frac{X - A}{X + A} \end{aligned}$$

and
$$\int \frac{dx}{x^2 + 10x + 13} = \frac{1}{4\sqrt{3}} \log_e \frac{x + 5 - 2\sqrt{3}}{x + 5 + 2\sqrt{3}}$$

(*b*) To integrate $\quad \dfrac{5x - 4}{x^2 + 12x + 15}$

Then
$$\frac{5x - 4}{x^2 + 12x + 15} = \frac{5}{2}\,\frac{2x + 12}{x^2 + 12x + 15} - \frac{34}{x^2 + 12x + 15}$$

The first fraction is one obtained by making the numerator the differential coefficient of the denominator; the multiplier $\frac{5}{2}$ is so chosen that full account is taken of the part $5x$ in the numerator of the original fraction. This has the effect of replacing the fraction to be integrated by two fractions, the first of which can be integrated at once, while the second can be integrated by the method of the previous example.

$$\int \frac{dx}{x^2 + 12x + 15} = \int \frac{dx}{(x+6)^2 - 21}$$

$$= \int \frac{dx}{X^2 - A^2} \text{ where } X = x + 6 \text{ and } A^2 = 21$$

$$= \frac{1}{2A} \log_e \frac{X - A}{X + A}$$

$$= \frac{1}{2\sqrt{21}} \log_e \frac{x + 6 - \sqrt{21}}{x + 6 + \sqrt{21}}$$

Then $\displaystyle\int \frac{(5x - 4)\,dx}{x^2+12x+15} = \frac{5}{2}\int \frac{(2x + 12)dx}{x^2 + 12x + 15} - 34\int \frac{dx}{x^2 + 12x + 15}$

$$= \frac{5}{2} \log_e (x^2 + 12x + 15) - \frac{17}{\sqrt{21}} \log_e \frac{x + 6 - \sqrt{21}}{x + 6 + \sqrt{21}}$$

(*c*) To integrate $\displaystyle\frac{8x^3 - 15}{4x^2 + 28x - 7}$

In this example the numerator is of higher degree than the denominator, and the first step then is to divide the numerator by the denominator.

Then $\displaystyle\frac{8x^3 - 15}{4x^2 + 28x - 7} = 2x - 14 + \frac{406x - 113}{4x^2 + 28x - 7}$

$$= 2x - 14 + \frac{1}{4}\left\{\frac{406x - 113}{x^2 + 7x - \frac{7}{4}}\right\}$$

$$= 2x - 14 + \frac{1}{4}\left\{\frac{203(2x + 7)}{x^2 + 7x - \frac{7}{4}} - \frac{1534}{x^2 + 7x - \frac{7}{4}}\right\}$$

Also $\displaystyle\int \frac{dx}{x^2 + 7x - \frac{7}{4}} = \int \frac{dx}{\left(x + \frac{7}{2}\right)^2 - 14}$

$$= \int \frac{dX}{X^2 - A^2} \quad \text{where } X = x + \frac{7}{2} \text{ and } A^2 = 14$$

$$= \frac{1}{2A} \log_e \frac{X - A}{X + A}$$

$$= \frac{1}{2\sqrt{14}} \log_e \frac{x + \frac{7}{2} + \sqrt{14}}{x + \frac{7}{2} - \sqrt{14}}$$

$$= \frac{1}{2\sqrt{14}} \log_e \frac{2x + 7 - 2\sqrt{14}}{2x + 7 + 2\sqrt{14}}$$

Then $\int \frac{(8x^3 - 15)\,dx}{4x^2 + 28x - 7}$

$$= 2\int x\,dx - 14\int dx + \frac{203}{4}\int \frac{(2x+7)dx}{x^2 + 7x - \frac{7}{4}} - \frac{767}{2}\int \frac{dx}{x^2 + 7x - \frac{7}{4}}$$

$$= x^2 - 14x + \frac{203}{4}\log_e\left(x^2 + 7x - \frac{7}{4}\right) - \frac{767}{4\sqrt{14}}\log_e \frac{2x+7-2\sqrt{14}}{2x+7+2\sqrt{14}}$$

(*d*) Care should be taken when the term involving x^2 in the denominator is negative, for then the denominator reduces to the form $a(A^2 - X^2)$.

To integrate $\frac{1}{8 - 12x - x^2}$

Now $8 - 12x - x^2 = 44 - (x^2 + 12x + 36)$

$= 44 - (x + 6)^2$

Then $\int \frac{dx}{8 - 12x - x^2} = \int \frac{dx}{44 - (x + 6)^2}$

$= \int \frac{dX}{A^2 - X^2}$ where $A^2 = 44$ and $X = x + 6$

Now $\frac{1}{A^2 - X^2} = \frac{\alpha}{A + X} + \frac{\beta}{A - X}$

and $\alpha(A - x) + \beta(A + X) = 1$

when $X = A \qquad 2A\beta = 1 \qquad \beta = \frac{1}{2A}$

when $X = -A \qquad 2A\alpha = 1 \qquad \alpha = \frac{1}{2A}$

Then
$$\int \frac{dX}{A^2 - X^2} = \frac{1}{2A}\left\{\int \frac{dX}{A + X} + \int \frac{dX}{A - X}\right\}$$
$$= \frac{1}{2A}\left\{\int \frac{dX}{A + X} - \int \frac{-dX}{A - X}\right\}$$
$$= \frac{1}{2A}\{\log_e(A + X) - \log_e(A - X)\}$$
$$= \frac{1}{2A}\log_e \frac{A + X}{A - X}$$
$$= \frac{1}{4\sqrt{11}}\log_e \frac{2\sqrt{11} + x + 6}{2\sqrt{11} - x - 6}$$

In dealing with $\int \frac{dX}{A - X}$ we notice that the numerator is not the differential coefficient of the denominator, but making the integral negative and the numerator negative such is the case.

Thus $\int \frac{dX}{A - X} = -\int \frac{-dX}{A - X} = -\log_e(A - X)$

90. *Case II.* When the denominator can be expressed as the sum of two squares—that is, it reduces to the form $a(X^2 + A^2)$

(*a*) To integrate $$\frac{1}{x^2 + 14x + 60}$$

Then $$x^2 + 14x + 60 = x^2 + 14x + 49 + 11$$
$$= (x + 7)^2 + 11$$

Then $$\int \frac{dx}{x^2 + 14x + 60} = \int \frac{dx}{(x + 7)^2 + 11}$$
$$= \int \frac{dX}{X^2 + A^2} \text{ where } X = x + 7 \text{ and } A^2 = 11$$

put $$X^2 = A^2 \tan^2 \theta \qquad \text{or } X = A \tan \theta$$
Then $$X^2 + A^2 = A^2(\tan^2 \theta + 1) = A^2 \sec^2 \theta$$
also $$\frac{dX}{d\theta} = A \sec^2 \theta \qquad \text{or } dX = A \sec^2 \theta \, d\theta$$

Hence $$\int \frac{dX}{X^2 + A^2} = \int \frac{A \sec^2 \theta \, d\theta}{A^2 \sec^2 \theta}$$
$$= \frac{1}{A} \int d\theta$$
$$= \frac{1}{A} \theta$$
$$= \frac{1}{A} \tan^{-1} \frac{X}{A}$$
$$= \frac{1}{\sqrt{11}} \tan^{-1} \frac{x + 7}{\sqrt{11}}$$

(*b*) When the numerator of the fraction is of the first degree in x, the fraction should be expressed as the sum or difference of two fractions, the first being formed so that its numerator is the differential coefficient of the denominator.

To integrate $$\frac{7x + 3}{x^2 + 12x + 52}$$

Now $$\frac{7x - 3}{x^2 + 12x + 52} = \frac{7}{2} \frac{2x + 12}{x^2 + 12x + 52} - \frac{45}{x^2 + 12x + 52}$$

Also $$\int \frac{dx}{x^2 + 12x + 52} = \int \frac{dx}{(x + 6)^2 + 16}$$
$$= \int \frac{dX}{X^2 + A^2} \text{ where } X = x + 6 \text{ and } A^2 = 16$$
$$= \frac{1}{A} \tan^{-1} \frac{X}{A}$$
$$= \frac{1}{4} \tan^{-1} \frac{x + 6}{4}$$

Then $\displaystyle\int\frac{(7x-3)dx}{x^2+12x+52}=\frac{7}{2}\int\frac{(2x+12)dx}{x^2+12x+52}-45\int\frac{dx}{x^2+12x+52}$

$$=\frac{7}{2}\log_e(x^2+12x+52)-\frac{45}{4}\tan^{-1}\frac{x+6}{4}$$

(*c*) When the numerator of the fraction is of the same, or higher, degree than the denominator, before proceeding to integration the denominator should be divided into the numerator.

To integrate $\displaystyle\frac{2x^3-1}{2x^2-6x+11}$

Then $\displaystyle\frac{2x^3-1}{2x^2-6x+11}=x+3+\frac{7x-34}{2x^2-6x+11}$

$$=x+3+\frac{1}{2}\left\{\frac{7x-34}{x^2-3x+\frac{11}{2}}\right\}$$

$$=x+3+\frac{1}{2}\left\{\frac{7}{2}\,\frac{2x-3}{x^2-3x+\frac{11}{2}}-\frac{\frac{47}{2}}{x^2-3x+\frac{11}{2}}\right\}$$

But $\displaystyle\int\frac{dx}{x^2-3x+\frac{11}{2}}=\int\frac{dx}{\left(x-\frac{3}{2}\right)^2+\frac{13}{4}}$

$$=\int\frac{dX}{X^2+A^2}\text{ where }X=x-\frac{3}{2}\text{ and }A^2=\frac{13}{4}$$

$$=\frac{1}{A}\tan^{-1}\frac{X}{A}$$

$$=\frac{2}{\sqrt{13}}\tan^{-1}\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}$$

$$=\frac{2}{\sqrt{13}}\tan^{-1}\frac{2x-3}{\sqrt{13}}\quad\text{and}$$

$$\int\frac{(2x^3-1)\,dx}{2x^2-6x+11}=\int x\,dx+3\int dx+\frac{7}{4}\int\frac{(2x-3)\,dx}{x^2-3x+\frac{11}{2}}-\frac{47}{4}\int\frac{dx}{x^2-3x+\frac{11}{2}}$$

$$=\frac{1}{2}x^2+3x+\frac{7}{4}\log_e\left(x^2-3x+\frac{11}{2}\right)-\frac{47}{2\sqrt{13}}\tan^{-1}\frac{2x-3}{\sqrt{13}}$$

91. When the fraction is of such a form that the denominator is the product of linear and quadratic factors, the method of

integration must be a combination of the methods used in Case I. and Case II.

To integrate $$\frac{x^4}{x^3 - 8}$$

By division $$\frac{x^4}{x^3 - 8} = x + \frac{8x}{x^3 - 8}$$

But $$\frac{8x}{x^3 - 8} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4}$$

and $$A(x^2 + 2x + 4) + (Bx + C)(x - 2) = 8x$$

when $x = 2$ $\quad 12A = 16 \quad A = \frac{4}{3}$

when $x = 0$ $\quad 4A - 2C = 0 \quad C = \frac{8}{3}$

when $x = 1$ $\quad 7A - B - C = 8 \quad B = -\frac{4}{3}$

Then
$$\frac{x^4}{x^3 - 8} = x + \frac{4}{3(x-2)} - \frac{4x - 8}{3(x^2 + 2x + 4)}$$
$$= x + \frac{4}{3(x-2)} - \frac{4}{3}\left\{\frac{x - 2}{x^2 + 2x + 4}\right\}$$
$$= x + \frac{4}{3(x-2)} - \frac{4}{3}\left\{\frac{1}{2}\frac{2x + 2}{x^2 + 2x + 4} - \frac{3}{x^2 + 2x + 4}\right\}$$
$$= x + \frac{4}{3(x-2)} - \frac{2}{3}\frac{2x + 2}{x^2 + 2x + 4} + \frac{4}{x^2 + 2x + 4}$$

The fraction is now split up so that each part can be integrated separately.

$$\int\frac{dx}{x^2 + 2x + 4} = \int\frac{dx}{(x + 1)^2 + 3}$$
$$= \int\frac{dX}{X^2 + A^2} \text{ where } X = x + 1 \text{ and } A^2 = 3$$
$$= \frac{1}{A}\tan^{-1}\frac{X}{A}$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x + 1}{\sqrt{3}}$$

Then
$$\int\frac{x^4\,dx}{x^3 - 8} = \int x\,dx + \frac{4}{3}\int\frac{dx}{x - 2} - \frac{2}{3}\int\frac{(2x + 2)dx}{x^2 + 2x + 4} + 4\int\frac{dx}{x^2 + 2x + 4}$$
$$= \frac{1}{2}x^2 + \frac{4}{3}\log_e(x - 2) - \frac{2}{3}\log_e(x^2 + 2x + 4) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x + 1}{\sqrt{3}}$$
$$= \frac{1}{2}x^2 + \frac{2}{3}\log_e\frac{x^2 - 4x + 4}{x^2 + 2x + 4} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x + 1}{\sqrt{3}}$$

Examples X

Solve the following integrals :

(1) $\displaystyle\int\frac{(3x+2)\,dx}{3x^2+4x+7}$ (2) $\displaystyle\int\frac{(3x+2)\,dx}{\sqrt{3x^2+4x-7}}$

(3) $\displaystyle\int\frac{dx}{x\log_e x}$ (4) $\displaystyle\int\frac{dx}{\sqrt{1-x^2}\,\sin^{-1}x}$

(5) $\displaystyle\int\frac{dx}{(1+x^2)\tan^{-1}x}$ (6) $\displaystyle\int\frac{(e^x-e^{-x})\,dx}{e^x+e^{-x}}$

(7) $\displaystyle\int\tanh x\,dx$ (8) $\displaystyle\int\frac{\tan^{-1}x\,dx}{1+x^2}$

(9) $\displaystyle\int\frac{\sin^{-1}x\,dx}{\sqrt{1-x^2}}$ (10) $\displaystyle\int\frac{\sec^2 x\,dx}{\sqrt{\tan x}}$

(11) $\displaystyle\int\frac{\sin x\,dx}{\sqrt{\cos x}}$ (12) $\displaystyle\int\frac{x\,dx}{\sqrt{x^2-7}}$

(13) $\displaystyle\int\frac{x^2\,dx}{x^3-1}$ (14) $\displaystyle\int(4x-3)\sqrt{2x^2-3x+1}\,dx$

(15) $\displaystyle\int\frac{(2x-1)\,dx}{(x-3)(x-5)}$ (16) $\displaystyle\int\frac{x^2\,dx}{x^2-4}$

(17) $\displaystyle\int\frac{(5x-3)\,dx}{9-x^2}$ (18) $\displaystyle\int\frac{dx}{(3x-2)(4x-3)}$

(19) $\displaystyle\int\frac{(2x+3)\,dx}{(x-4)(5x+2)}$ (20) $\displaystyle\int\frac{(5x-2)\,dx}{(3-4x)(x+2)}$

(21) $\displaystyle\int\frac{x^2\,dx}{(x-1)(x+2)(x-3)}$ (22) $\displaystyle\int\frac{(x^2+3)\,dx}{(3x+1)(x-2)(3-2x)}$

(23) $\displaystyle\int\frac{(5x^3-3)\,dx}{(x-1)^4}$ (24) $\displaystyle\int\frac{x^2\,dx}{(x+3)^3}$

(25) $\displaystyle\int\frac{(x^3-3x+4)\,dx}{(2x+3)^4}$ (26) $\displaystyle\int\frac{(5x^2-7x+3)\,dx}{(3x-4)^3}$

(27) $\displaystyle\int\frac{(x+2)\,dx}{(x^2-1)(x+1)}$ (28) $\displaystyle\int\frac{x^2\,dx}{(x^2-4)(x-2)}$

(29) $\displaystyle\int\frac{dx}{(x^2-1)^2}$ (30) $\displaystyle\int\frac{(x+4)\,dx}{(x^2-4)(x-2)^2}$

(31) $\displaystyle\int\frac{dx}{x^2+1}$ (32) $\displaystyle\int\frac{dx}{x^2+5}$

(33) $\displaystyle\int\frac{(x+1)\,dx}{x^2+1}$ (34) $\displaystyle\int\frac{(x+5)\,dx}{x^2+5}$

(35) $\displaystyle\int\frac{dx}{x^2+8x+41}$ (36) $\displaystyle\int\frac{x\,dx}{x^2+6x+25}$

(37) $\int \frac{dx}{x^2 + 10x + 112}$

(38) $\int \frac{(x - 8)\,dx}{x^2 + 4x + 16}$

(39) $\int \frac{(x^2 - 1)\,dx}{x^2 + 2x + 4}$

(40) $\int \frac{dx}{x^2 - 7}$

(41) $\int \frac{(x + 3)\,dx}{x^2 - 8}$

(42) $\int \frac{dx}{3 - x^2}$

(43) $\int \frac{x^2\,dx}{5 - x^2}$

(44) $\int \frac{dx}{x^2 + 4x + 5}$

(45) $\int \frac{(x - 4)\,dx}{x^2 - 6x + 6}$

(46) $\int \frac{x^2\,dx}{x^2 + x + 1}$

(47) $\int \frac{dx}{6 - 4x - x^2}$

(48) $\int \frac{(2x - 1)\,dx}{12 - 10x - x^2}$

(49) $\int \frac{dx}{x^3 - 1}$

(50) $\int \frac{x\,dx}{1 - x^3}$

(51) $\int \frac{(x - 1)\,dx}{x^3 + 1}$

(52) $\int \frac{(x + 4)\,dx}{x^3 + 8}$

(53) $\int \frac{x^3\,dx}{8 - x^3}$

(54) $\int \frac{dx}{x^4 - 1}$

(55) $\int \frac{x\,dx}{x^4 - 1}$

(56) $\int \frac{x^2\,dx}{x^4 - 1}$

(57) $\int \frac{x^3\,dx}{x^4 - 1}$

(58) $\int \frac{(x + 4)\,dx}{3x^2 - 6x + 14}$

(59) $\int \frac{x\,dx}{10 - 4x - 3x^2}$

(60) $\int \frac{x\,dx}{x^4 + x^2 + 1}$

CHAPTER XI

92. The standard forms for $\int \sin(ax+b)\,dx$ and $\int \cos(ax+b)\,dx$.

If $y = \cos(ax+b)$, then $\dfrac{dy}{dx} = -a\sin(ax+b)$

Hence $$-\int a\sin(ax+b)\,dx = \cos(ax+b)$$

and $$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also if $y = \sin(ax+b)$, then $\dfrac{dy}{dx} = a\cos(ax+b)$

Hence $$\int a\cos(ax+b)\,dx = \sin(ax+b)$$

and $$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

93. *The Hyperbolic Functions.*

If $y = \cosh(ax+b)$, then $\dfrac{dy}{dx} = a\sinh(ax+b)$

Hence $$\int a\sinh(ax+b)\,dx = \cosh(ax+b)$$

and $$\int \sinh(ax+b)\,dx = \frac{\cosh(ax+b)}{a} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also if $y = \sinh(ax+b)$, then $\dfrac{dy}{dx} = a\cosh(ax+b)$

Hence $$\int a\cosh(ax+b)\,dx = \sinh(ax+b)$$

and $$\int \cosh(ax+b)\,dx = \frac{\sinh(ax+b)}{a} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

94. These results may be applied to the integration of algebraic fractions, the denominators of which consist of the square root of a quadratic expression.

It has been shown in the previous chapter that an expression of the form $ax^2 + bx + c$ reduces down to one of the three forms $a(\mathrm{A}^2 - \mathrm{X}^2)$, $a(\mathrm{X}^2 + \mathrm{A}^2)$, or $a(\mathrm{X}^2 - \mathrm{A}^2)$, where X is a linear function of x and the consideration of these integrals depends upon the particular form the denominator takes.

Case I. (*a*) When the denominator reduces to the form $\sqrt{A^2 - X^2}$. This is obviously the case when the term involving x^2 in the quadratic expression is negative.

(*a*) To integrate $\dfrac{1}{\sqrt{8 - 12x - x^2}}$

Then
$$8 - 12x - x^2 = 44 - (x^2 + 12x + 36)$$
$$= 44 - (x + 6)^2$$

Hence
$$\int \frac{dx}{\sqrt{8 - 12x - x^2}} = \int \frac{dx}{\sqrt{44 - (x + 6)^2}}$$
$$= \int \frac{dX}{\sqrt{A^2 - X^2}} \text{ where } X = x + 6 \text{ and } A^2 = 44$$

put
$$X^2 = A^2 \sin^2 \theta$$

Then
$$\sqrt{A^2 - X^2} = A\sqrt{1 - \sin^2 \theta} = A \cos \theta$$

and
$$X = A \sin \theta, \ \frac{dX}{d\theta} = A \cos \theta, \text{ and } dX = A \cos \theta \, d\theta$$

Therefore
$$\int \frac{dX}{\sqrt{A^2 - X^2}} = \int \frac{A \cos \theta \, d\theta}{A \cos \theta}$$
$$= \int d\theta$$
$$= \theta$$
$$= \sin^{-1} \frac{X}{A}$$

and finally,
$$\int \frac{dx}{\sqrt{8 - 12x - x^2}} = \sin^{-1} \frac{x + 6}{2\sqrt{11}}$$

(*b*) When the fraction has for a numerator a linear function of x, before proceeding to integration the fraction must be split up into two fractions, the first of which must have for its numerator the differential coefficient of the quadratic expression under the square root.

(*b*) To integrate $\dfrac{8x - 9}{\sqrt{15 - 7x - x^2}}$

Then
$$\frac{8x - 9}{\sqrt{15 - 7x - x^2}} = -4 \frac{-2x - 7}{\sqrt{15 - 7x - x^2}} - \frac{37}{\sqrt{15 - 7x - x^2}}$$

Now
$$\int \frac{(-2x - 7)\, dx}{\sqrt{15 - 7x - x^2}} = \int \frac{dy}{\sqrt{y}} \text{ where } y = 15 - 7x + x^2$$
$$= 2\sqrt{y}$$
$$= 2\sqrt{15 - 7x - x^2}$$

Also $\displaystyle\int \frac{dx}{\sqrt{15-7x-x^2}} = \int \frac{dx}{\sqrt{\frac{109}{4}-\left(x+\frac{7}{2}\right)^2}}$

$$= \int \frac{dX}{\sqrt{A^2-X^2}} \text{ where } X = x + \frac{7}{2} \text{ and } A^2 = \frac{109}{4}$$

$$= \sin^{-1}\frac{X}{A}$$

$$= \sin^{-1}\frac{x+\frac{7}{2}}{\frac{\sqrt{109}}{2}}$$

$$= \sin^{-1}\frac{2x+7}{\sqrt{109}}$$

Then $\displaystyle\int \frac{(8x-9)\,dx}{\sqrt{15-7x-x^2}} = -4\int \frac{(-2x-7)\,dx}{\sqrt{15-7x-x^2}} - 37\int \frac{dx}{\sqrt{15-7x-x^2}}$

$$= -8\sqrt{15-7x-x^2} - 37\sin^{-1}\frac{2x+7}{\sqrt{109}}$$

(*c*) When the square root of the quadratic expression appears in the numerator, the same substitution can be used for X, but a different integral is the result.

(*c*) To integrate $\sqrt{32+18x-x^2}$.

$$= \int \sqrt{32+18x-x^2}\,dx = \int \sqrt{113-(x-9)^2}\,dx$$

$$= \int \sqrt{A^2-X^2}\,dX \text{ where } X = x-9$$
and $A^2 = 113$

put $$X^2 = A^2\sin^2\theta$$

Then $$\sqrt{A^2-X^2} = A\sqrt{1-\sin^2\theta} = A\cos\theta$$

also $X = A\sin\theta$, $\dfrac{d\theta}{dX} = A\cos\theta$, and $dX = A\cos\theta\,d\theta$

Then $\displaystyle\int \sqrt{A^2-X^2}\,dX = A^2\int \cos^2\theta\,d\theta$

$$= \frac{A^2}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{A^2}{2}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}$$

$$= \frac{A^2}{2}\{\theta + \sin\theta\cos\theta\}$$

$$= \frac{A^2}{2}\left\{\sin^{-1}\frac{X}{A} + \frac{X\sqrt{A^2-X^2}}{A^2}\right\}$$

Finally,

$$\int\sqrt{32+18x-x^2}\,dx=\frac{113}{2}\left\{\sin^{-1}\frac{x-9}{\sqrt{113}}+\frac{(x-9)\sqrt{32+18x-x^2}}{113}\right\}$$

(*d*) When the numerator of the fraction is of the second degree; before proceeding to integration the fraction must be split up into three fractions, the first of which can be obtained by division.

(*d*) To integrate $\dfrac{2x^2-8x+9}{\sqrt{17-14x-x^2}}$

Then $\dfrac{2x^2-8x+9}{\sqrt{17-14x-x^2}}$

$$=\frac{-2(17-14x-x^2)}{\sqrt{17-14x-x^2}}-\frac{36x-43}{\sqrt{17-14x-x^2}}$$

$$=-2\sqrt{17-14x-x^2}-\left\{\frac{-18(-2x-14)}{\sqrt{17-14x-x^2}}-\frac{295}{\sqrt{17-14x-x^2}}\right\}$$

$$=-2\sqrt{17-14x-x^2}+\frac{18(-2x-14)}{\sqrt{17-14x-x^2}}+\frac{295}{\sqrt{17-14x-x^2}}$$

and each of these expressions can be integrated

$$\int\sqrt{17-14x-x^2}\,dx=\int\sqrt{66-(x+7)^2}\,dx$$

$$=\int\sqrt{A^2-X^2}\,dx \text{ where } X=x+7 \text{ and } A^2=66$$

$$=A^2\int\cos^2\theta\,d\theta \quad \text{where } X^2=A^2\sin^2\theta$$

$$=\frac{A^2}{2}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{A^2}{2}\left\{\theta+\frac{1}{2}\sin 2\theta\right\}$$

$$=\frac{A^2}{2}\left\{\sin^{-1}\frac{X}{A}+\frac{X\sqrt{A^2-X^2}}{A^2}\right\}$$

$$=33\sin^{-1}\frac{x+7}{\sqrt{66}}+\frac{1}{2}(x+7)\sqrt{17-14x-x^2}$$

$$\int\frac{(-2x-14)\,dx}{\sqrt{17-14x-x^2}}=\int\frac{dy}{\sqrt{y}} \quad \text{where } y=17-14x-x^2$$

$$=2\sqrt{y}$$

$$=2\sqrt{17-14x-x^2}$$

$$\int\frac{dx}{\sqrt{17-14x-x^2}}=\int\frac{dx}{\sqrt{66-(x+7)^2}}$$

$$=\int\frac{dX}{\sqrt{A^2-X^2}}\quad\text{where } X=x+7 \text{ and } A^2=66$$

$$=\sin^{-1}\frac{X}{A}$$

$$=\sin^{-1}\frac{x+7}{\sqrt{66}}$$

Finally $\displaystyle\int\frac{2x^2-8x+9}{\sqrt{17-14x-x^2}}\,dx = -2\int\sqrt{17-14x-x^2}\,dx$

$$+18\int\frac{(-2x-14)\,dx}{\sqrt{17-14x-x^2}}+295\int\frac{dx}{\sqrt{17-14x-x^2}}$$

$$=-66\sin^{-1}\frac{x+7}{\sqrt{66}}-(x+7)\sqrt{17-14x-x^2}+36\sqrt{17-14x-x^2}$$

$$+295\sin^{-1}\frac{x+7}{\sqrt{66}}$$

$$=(29-x)\sqrt{17-14x-x^2}+229\sin^{-1}\frac{x+7}{\sqrt{66}}$$

It is evident that integrals of this type depend upon two standard forms.

(1) $$\int\frac{dX}{\sqrt{A^2-X^2}}=\sin^{-1}\frac{X}{A}$$

(2) $$\int\sqrt{A^2-X^2}\,dX=\frac{A^2}{2}\left\{\sin^{-1}\frac{X}{A}+\frac{X\sqrt{A^2-X^2}}{A^2}\right\}$$

95. *Case II.* (*a*) When the denominator of the fraction reduces to the form $\sqrt{X^2+A^2}$, for this type we have to use hyperbolic functions.

(*a*) To integrate $\dfrac{1}{\sqrt{x^2+12x+48}}$

Now $\quad x^2+12x+48 = x^2+12x+36+12$

$$=(x+6)^2+12$$

Then $\displaystyle\int\frac{dx}{\sqrt{x^2+12x+48}}=\int\frac{dx}{\sqrt{(x+6)^2+12}}$

$$=\int\frac{dX}{\sqrt{X^2+A^2}}\quad\text{where } X=x+6 \text{ and } A^2=12$$

put $\quad X^2 = A\sinh^2\theta$

then $\sqrt{A^2 + X^2} = A\sqrt{1 + \sinh^2\theta} = A\cosh\theta$

and $X = A\sinh\theta,\ \frac{dX}{d\theta} = A\cosh\theta$, or $dX = A\cosh\theta\, d\theta$

$$\int\frac{dX}{\sqrt{X^2 + A^2}} = \int\frac{A\cosh\theta\, d\theta}{A\cosh\theta}$$
$$= \int d\theta$$
$$= \theta$$
$$= \sinh^{-1}\frac{X}{A}$$

Finally, $\displaystyle\int\frac{dx}{\sqrt{x^2 + 12x + 48}} = \sinh^{-1}\frac{x+6}{2\sqrt{3}}$

We also know that $\sinh\theta = \frac{X}{A}$

or $\frac{1}{2}(e^\theta - e^{-\theta}) = \frac{X}{A}$

$$e^{2\theta} - \frac{2X}{A}e^\theta = 1$$

$$e^{2\theta} - \frac{2X}{A}e^\theta + \frac{X^2}{A^2} = 1 + \frac{X^2}{A^2} = \frac{X^2 + A^2}{A^2}$$

$$e^\theta - \frac{X}{A} = \frac{\sqrt{X^2 + A^2}}{A}$$

$$e^\theta = \frac{X + \sqrt{X^2 + A^2}}{A}$$

and $\theta = \log_e\left\{\frac{X + \sqrt{X^2 + A^2}}{A}\right\}$

Thus $\displaystyle\int\frac{dx}{\sqrt{x^2 + 12x + 48}} = \log_e\left\{\frac{(x+6) + \sqrt{x^2 + 12x + 48}}{2\sqrt{3}}\right\}$

The integral $\displaystyle\int\frac{dX}{\sqrt{X^2 + A^2}}$ where X is a linear function of x, will give (1) an angle expressed in terms of its hyperbolic sine, or (2) a logarithmic function, and the results can be used as standard forms.

$$\int\frac{dX}{\sqrt{X^2 + A^2}} = \sinh^{-1}\frac{X}{A} \quad \ldots \quad (1)$$

or
$$\int\frac{dX}{\sqrt{X^2 + A^2}} = \log_e\left\{\frac{X + \sqrt{X^2 + A^2}}{A}\right\} \quad \ldots \quad (2)$$

(*b*) When the fraction has for a numerator a linear function of x, before proceeding to integration, the fraction must be split up into two fractions, the first of which must have for its numerator the differential coefficient of the quadratic expression under the square root.

(*b*) To integrate $\frac{4x-5}{\sqrt{x^2-18x+106}}$

Then $$\frac{4x-5}{\sqrt{x^2-18x+106}} = \frac{2(2x-18)}{\sqrt{x^2-18x+106}} + \frac{31}{\sqrt{x^2-18x+106}}$$

$$\int \frac{(2x-18)\,dx}{\sqrt{x^2-18x+106}} = \int \frac{dy}{\sqrt{y}} \text{ where } y = x^2-18x+106$$
$$= 2\sqrt{y}$$
$$= 2\sqrt{x^2-18x+106}$$

Also $$\int \frac{dx}{\sqrt{x^2-18x+106}} = \int \frac{dx}{\sqrt{(x-9)^2+25}}$$
$$= \int \frac{dX}{\sqrt{X^2+A^2}} \text{ where } X = x-9 \text{ and } A^2 = 25$$
$$= \sinh^{-1}\frac{X}{A} \text{ or } \log_e\left\{\frac{X+\sqrt{X^2+A^2}}{A}\right\}$$
$$= \sinh^{-1}\frac{x-9}{5} \text{ or } \log_e\left\{\frac{(x-9)+\sqrt{x^2-18x+106}}{5}\right\}$$

Then
$$\int \frac{(4x-5)\,dx}{\sqrt{x^2-18x+106}} = 2\int \frac{(2x-18)\,dx}{\sqrt{x^2-18x+106}} + 31\int \frac{dx}{\sqrt{x^2-18x+106}}$$
$$= 4\sqrt{x^2-18x+106} + 31\sinh^{-1}\frac{x-9}{5}$$
$$\text{or } 4\sqrt{x^2-18x+106} + 31\log_e\left\{\frac{(x-9)+\sqrt{x^2-18x+106}}{5}\right\}$$

(*c*) When the square root of the quadratic expression appears in the numerator, the same substitution can be used for X, but a different integral is the result.

(*c*) To integrate $\sqrt{x^2+24x+244}$

$$\int \sqrt{x^2+24x+244}\,dx = \int \sqrt{(x+12)^2+100}\,dx$$
$$= \int \sqrt{X^2+A^2}\,dX \quad \text{where } X = x+12$$
$$\text{and } A^2 = 100$$

put $$X^2 = A^2\sinh^2\theta$$

Then $\sqrt{X^2 + A^2} = A\sqrt{1 + \sinh^2 \theta} = A \cosh \theta$

also $X = A \sinh \theta, \ \frac{dX}{d\theta} = A \cosh \theta, \quad \text{or } dX = A \cosh \theta \, d\theta$

Then $\int \sqrt{X^2 + A^2}\, dX$

$$= A^2 \int \cosh^2 \theta \, d\theta$$

$$= \frac{A^2}{2}\int (1 + \cosh 2\theta)\, d\theta, \quad \text{since } \cosh 2\theta = 2\cosh^2 \theta - 1$$

$$= \frac{A^2}{2}\left\{\theta + \frac{1}{2}\sinh 2\theta\right\}$$

$$= \frac{A^2}{2}\left\{\theta + \sinh\theta \cosh\theta\right\}$$

$$= \frac{A^2}{2}\left\{\sinh^{-1}\frac{X}{A} + \frac{X\sqrt{X^2 + A^2}}{A^2}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

or $$\frac{A^2}{2}\left\{\log_e\left(\frac{X + \sqrt{X^2 + A^2}}{A}\right) + \frac{X\sqrt{X^2 + A^2}}{A^2}\right\} \quad \ldots\ldots \quad (2)$$

$$\int \sqrt{x^2 + 24x + 244}\, dx$$

$$= 50\left\{\sinh^{-1}\frac{x + 12}{10} + \frac{(x + 12)\sqrt{x^2 + 24x + 244}}{100}\right\} \quad \ldots \quad (1)$$

or $$50\left\{\log_e\left(\frac{x + 12 + \sqrt{x^2 + 24x + 244}}{10}\right)\right.$$

$$\left. + \frac{(x + 12)\sqrt{x^2 + 24x + 244}}{100}\right\} \quad \ldots\ldots\ldots \quad (2)$$

The integral $\int \sqrt{X^2 + A^2}\, dX$, where X is a linear function of x, can be solved by means of two standard forms.

(1) $$\int \sqrt{X^2 + A^2}\, dX = \frac{A^2}{2}\left\{\sinh^{-1}\frac{X}{A} + \frac{X\sqrt{X^2 + A^2}}{A^2}\right\}$$

(2) $$\int \sqrt{X^2 + A^2}\, dX = \frac{A^2}{2}\left\{\log_e\left(\frac{X + \sqrt{X^2 + A^2}}{A}\right) + \frac{X\sqrt{X^2 + A^2}}{A^2}\right\}.$$

(*d*) When the numerator of the fraction is of the second degree, the fraction must be split up into three fractions.

(*d*) To integrate $\dfrac{x^2 + 5x - 7}{\sqrt{x^2 + 4x + 12}}$

Then $\dfrac{x^2+5x-7}{\sqrt{x^2+4x+12}}$

$$= \frac{x^2+4x+12}{\sqrt{x^2+4x+12}} + \frac{x-19}{\sqrt{x^2+4x+12}}$$

$$= \sqrt{x^2+4x+12} + \frac{1}{2}\,\frac{2x+4}{\sqrt{x^2+4x+12}} - \frac{21}{\sqrt{x^2+4x+12}}$$

and each term can be integrated separately.

$$\int \sqrt{x^2+4x+12}\,dx = \int \sqrt{(x+2)^2+8}\,dx$$

$$= \int \sqrt{A^2+X^2}\,dX \quad \text{where } X = x+2 \text{ and } A^2 = 8$$

$$= \frac{A^2}{2}\left\{\sinh^{-1}\frac{X}{A} + \frac{X\sqrt{X^2+A^2}}{A^2}\right\}$$

$$= 4\left\{\sinh^{-1}\frac{x+2}{2\sqrt{2}} + \frac{(x+2)\sqrt{x^2+4x+12}}{8}\right\}$$

$$\int \frac{(2x+4)\,dx}{\sqrt{x^2+4x+12}} = \int \frac{dy}{\sqrt{y}} \quad \text{where } y = x^2+4x+12$$

$$= 2\sqrt{y}$$

$$= 2\sqrt{x^2+4x+12}$$

$$\int \frac{dx}{\sqrt{x^2+4x+12}} = \int \frac{dx}{\sqrt{(x+2)^2+8}}$$

$$= \int \frac{dX}{\sqrt{X^2+A^2}} \quad \text{where } X = x+2 \text{ and } A^2 = 8$$

$$= \sinh^{-1}\frac{X}{A}$$

$$= \sinh^{-1}\frac{x+2}{2\sqrt{2}}$$

Then $\displaystyle\int \frac{(x^2+5x-7)\,dx}{\sqrt{x^2+4x+12}}$

$$= \int \sqrt{x^2+4x+12}\,dx + \frac{1}{2}\int \frac{(2x+4)\,dx}{\sqrt{x^2+4x+12}} - 21\int \frac{dx}{\sqrt{x^2+4x+12}}$$

$$= 4\left\{\sinh^{-1}\frac{x+2}{2\sqrt{2}} + \frac{(x+2)\sqrt{x^2+4x+12}}{8}\right\} + \sqrt{x^2+4x+12}$$

$$- 21\sinh^{-1}\frac{x+2}{2\sqrt{2}}$$

$$= -17\sinh^{-1}\frac{x+2}{2\sqrt{2}} + \sqrt{x^2+4x+12}\left\{1+\frac{x+2}{2}\right\}$$

$$= -17\sinh^{-1}\frac{x+2}{2\sqrt{2}} + \frac{1}{2}(x+4)\sqrt{x^2+4x+12}$$

The result can also be expressed as

$$\int\frac{(x^2+5x-7)\,dx}{\sqrt{x^2+4x+12}}$$

$$= -17\log_e\left\{\frac{x+2+\sqrt{x^2+4x+12}}{2\sqrt{2}}\right\} + \frac{1}{2}(x+4)\sqrt{x^2+4x+12}$$

96. *Case III.* (*a*) When the denominator of the fraction reduces to the form $\sqrt{X^2-A^2}$, and in this case hyperbolic functions must be used.

(*a*) To integrate $\dfrac{1}{\sqrt{x^2+16x+36}}$

Now
$$x^2+16x+36 = x^2+16x+64-28$$
$$= (x+8)^2-28$$

Then
$$\int\frac{dx}{\sqrt{x^2+16x+36}} = \int\frac{dx}{\sqrt{(x+8)^2-28}}$$
$$= \int\frac{dx}{\sqrt{X^2-A^2}} \text{ where } X = x+8 \text{ and } A^2 = 28$$

put
$$X^2 = A^2\cosh^2\theta$$

then $\sqrt{X^2-A^2} = A\sqrt{\cosh^2\theta-1} = A\sinh\theta$

and
$$X = A\cosh\theta,\ \frac{dX}{d\theta} = A\sinh\theta, \text{ or } dX = A\sinh\theta\,d\theta$$

$$\int\frac{dX}{\sqrt{X^2-A^2}} = \int\frac{A\sinh\theta\,d\theta}{A\sinh\theta}$$
$$= \int d\theta$$
$$= \theta$$
$$= \cosh^{-1}\frac{X}{A}$$

and
$$\int\frac{dx}{\sqrt{x^2+16x+36}} = \cosh^{-1}\frac{x+8}{2\sqrt{7}}$$

We also know that $\cosh\theta = \dfrac{X}{A}$

or
$$\frac{1}{2}(e^\theta + e^{-\theta}) = \frac{X}{A}$$

$$e^{2\theta} - \frac{2X}{A}e^\theta = -1$$

$$e^{2\theta} - \frac{2X}{A}e^\theta + \frac{X^2}{A^2} = \frac{X^2}{A^2} - 1 = \frac{X^2 - A^2}{A^2}$$

$$e^\theta - \frac{X}{A} = \frac{\sqrt{X^2 - A^2}}{A}$$

$$e^\theta = \frac{X + \sqrt{X^2 - A^2}}{A}$$

and
$$\theta = \log_e\left\{\frac{X + \sqrt{X^2 - A^2}}{A}\right\}$$

Thus
$$\int \frac{dx}{\sqrt{x^2 + 16x + 36}} = \log_e\left\{\frac{(x+8) + \sqrt{x^2 + 16x + 36}}{2\sqrt{7}}\right\}$$

The integral $\int \frac{dX}{\sqrt{X^2 - A^2}}$ where X is a linear function of x, will give (1) an angle expressed in terms of its hyperbolic cosine or (2) a logarithmic function, and the results can be used as standard forms.

$$\int \frac{dX}{\sqrt{X^2 - A^2}} = \cosh^{-1}\frac{X}{A} \quad \ldots\ldots\ldots \quad (1)$$

or
$$\int \frac{dX}{\sqrt{X^2 - A^2}} = \log_e\left\{\frac{X + \sqrt{X^2 - A^2}}{A}\right\} \quad \ldots \quad (2)$$

(*b*) When the fraction has for a numerator a linear function of x, before proceeding to integration the fraction must be split up into two fractions, the first of which must have for its numerator the differential coefficient of the quadratic expression under the square root.

(*b*) To integrate $\frac{8x - 7}{\sqrt{3x^2 + 12x - 10}}$

Now
$$\frac{8x - 7}{\sqrt{3x^2 + 12x - 10}} = \frac{4}{3}\,\frac{6x + 12}{\sqrt{3x^2 + 12x - 10}} - \frac{23}{\sqrt{3x^2 + 12x - 10}}$$

$$\int \frac{(6x + 12)dx}{\sqrt{3x^2 + 12x - 10}} = \int \frac{dy}{\sqrt{y}} \text{ where } y = 3x^2 + 12x - 10$$

$$= 2\sqrt{y}$$
$$= 2\sqrt{3x^2 + 12x - 10}$$

Also $$\int\frac{dx}{\sqrt{3x^2+12x-10}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2+4x-\frac{10}{3}}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x+2)^2-\frac{22}{3}}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dX}{\sqrt{X^2-A^2}}\quad\text{where } X=x+2$$

$$\text{and } A^2=\frac{22}{3}$$

$$=\frac{1}{\sqrt{3}}\cosh^{-1}\frac{X}{A}$$

$$=\frac{1}{\sqrt{3}}\cosh^{-1}\frac{x+2}{\sqrt{\frac{22}{3}}}$$

$$=\frac{1}{\sqrt{3}}\cosh^{-1}\frac{\sqrt{3}(x+2)}{\sqrt{22}}$$

Or, expressing the result as a logarithmic function,

$$\int\frac{dx}{\sqrt{3x^2+12x-10}}=\frac{1}{\sqrt{3}}\int\frac{dX}{\sqrt{X^2-A^2}}$$

$$=\frac{1}{\sqrt{3}}\log_e\left\{\frac{X+\sqrt{X^2-A^2}}{A}\right\}$$

$$=\frac{1}{\sqrt{3}}\log_e\left\{\frac{x+2+\sqrt{x^2+4x-\frac{10}{3}}}{\sqrt{\frac{22}{3}}}\right\}$$

$$=\frac{1}{\sqrt{3}}\log_e\left\{\frac{\sqrt{3}(x+2)+\sqrt{3x^2+12x-10}}{\sqrt{22}}\right\}$$

Hence $$\int\frac{(8x-7)\,dx}{\sqrt{3x^2+12x-10}}$$

$$=\frac{4}{3}\int\frac{(6x+12)\,dx}{\sqrt{3x^2+12x-10}}-23\int\frac{dx}{\sqrt{3x^2+12x-10}}$$

$$=\frac{8}{3}\sqrt{3x^2+12x-10}-\frac{23}{\sqrt{3}}\cosh^{-1}\frac{\sqrt{3}(x+2)}{\sqrt{22}}$$

or $$\frac{8}{3}\sqrt{3x^2+12x-10}-\frac{23}{\sqrt{3}}\log_e\left\{\frac{\sqrt{3}(x+2)+\sqrt{3x^2+12x-10}}{\sqrt{22}}\right\}$$

(*c*) When the square root of the quadratic expression appears in the numerator.

(*c*) To integrate $\sqrt{x^2 - 10x + 20}$

$$\int \sqrt{x^2 - 10x + 20}\, dx = \int \sqrt{(x-5)^2 - 5}\, dx$$

$$= \int \sqrt{X^2 - A^2}\, dX \text{ where } X = x - 5 \text{ and } A^2 = 5$$

put $\qquad X^2 = A^2 \cosh^2 \theta$

Then $\qquad \sqrt{X^2 - A^2} = A\sqrt{\cosh^2 \theta - 1} = A \sinh \theta$

also $\qquad X = A \cosh \theta,\ \frac{dX}{d\theta} = A \sinh \theta, \text{ or } dX = A \sinh \theta\, d\theta$

Then $\qquad \int \sqrt{X^2 - A^2}\, dX = A^2 \int \sinh^2 \theta\, d\theta$

$$= \frac{A^2}{2} \int (\cosh 2\theta - 1)\, d\theta, \quad \text{since } \cosh 2\theta = 2 \sinh^2 \theta + 1$$

$$= \frac{A^2}{2}\left\{\frac{1}{2} \sinh 2\theta - \theta\right\}$$

$$= \frac{A^2}{2}\{\sinh \theta \cosh \theta - \theta\}$$

$$= \frac{A^2}{2}\left\{\frac{X\sqrt{X^2 - A^2}}{A^2} - \cosh^{-1} \frac{X}{A}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{A^2}{2}\left\{\frac{X\sqrt{X^2 - A^2}}{A^2} - \log_e \frac{(X + \sqrt{X^2 - A^2})}{A}\right\} \quad . \quad . \quad (2)$$

$$\int \sqrt{x^2 - 10x + 20}\, dx$$

$$= \frac{5}{2}\left\{\frac{(x-5)\sqrt{x^2 - 10x + 20}}{5} - \cosh^{-1} \frac{x-5}{\sqrt{5}}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{5}{2}\left\{\frac{(x-5)\sqrt{x^2 - 10x + 20}}{5} - \log_e\left(\frac{x - 5 + \sqrt{x^2 - 10x + 20}}{\sqrt{5}}\right)\right\} \quad (2)$$

The integral $\int \sqrt{X^2 - A^2}\, dX$ can therefore be solved by means of two standard forms.

(1) $\int \sqrt{X^2 - A^2}\, dX = \frac{A^2}{2}\left\{\frac{X\sqrt{X^2 - A^2}}{A^2} - \cosh^{-1} \frac{X}{A}\right\}$

(2) $\int \sqrt{X^2 - A^2}\, dX = \frac{A^2}{2}\left\{\frac{X\sqrt{X^2 - A^2}}{A^2} - \log_e\left(\frac{X + \sqrt{X^2 - A^2}}{A}\right)\right\}$

(*d*) When the numerator of the fraction is of the second degree, the fraction must be split up into three fractions.

(*d*) To integrate $\dfrac{2x^2 - 7x + 10}{\sqrt{x^2 - 6x + 6}}$

Then $\dfrac{2x^2 - 7x + 10}{\sqrt{x^2 - 6x + 6}}$

$$= \frac{2(x^2 - 6x + 6)}{\sqrt{x^2 - 6x + 6}} + \frac{5x - 2}{\sqrt{x^2 - 6x + 6}}$$

$$= 2\sqrt{x^2 - 6x + 6} + \frac{5}{2}\,\frac{2x - 6}{\sqrt{x^2 - 6x + 6}} + \frac{13}{\sqrt{x^2 - 6x + 6}}$$

and each term can be integrated separately.

$$\int\sqrt{x^2 - 6x + 6}\,dx = \int\sqrt{(x - 3)^2 - 3}\,dx$$

$$= \int\sqrt{X^2 - A^2}\,dX \text{ where } X = x - 3 \text{ and } A^2 = 3$$

$$= \frac{A^2}{2}\left\{\frac{X\sqrt{X^2 - A^2}}{A^2} - \cosh^{-1}\frac{X}{A}\right\}$$

$$= \frac{3}{2}\left\{\frac{(x - 3)\sqrt{x^2 - 6x + 6}}{3} - \cosh^{-1}\frac{x - 3}{\sqrt{3}}\right\}$$

$$\int\frac{(2x - 6)\,dx}{\sqrt{x^2 - 6x + 6}} = \int\frac{dy}{\sqrt{y}} \quad \text{where } y = x^2 - 6x + 6$$

$$= 2\sqrt{y}$$

$$= 2\sqrt{x^2 - 6x + 6}$$

$$\int\frac{dx}{\sqrt{x^2 - 6x + 6}} = \int\frac{dx}{\sqrt{(x - 3)^2 - 3}}$$

$$= \int\frac{dX}{\sqrt{X^2 - A^2}} \quad \text{where } X = x - 3 \text{ and } A^2 = 3$$

$$= \cosh^{-1}\frac{X}{A}$$

$$= \cosh^{-1}\frac{x - 3}{\sqrt{3}}$$

Then $\displaystyle\int\frac{(2x^2 - 7x + 10)\,dx}{\sqrt{x^2 - 6x + 6}}$

$$= 2\int\sqrt{x^2 - 6x + 6}\,dx + \frac{5}{2}\int\frac{(2x - 6)\,dx}{\sqrt{x^2 - 6x + 6}} + 13\int\frac{dx}{\sqrt{x^2 - 6x + 6}}$$

$$= 3\left\{\frac{(x-3)\sqrt{x^2-6x+6}}{3} - \cosh^{-1}\frac{x-3}{\sqrt{3}}\right\} + 5\sqrt{x^2-6x+6}$$

$$+ 13\cosh^{-1}\frac{x-3}{\sqrt{3}}$$

$$= 10\cosh^{-1}\frac{x-3}{\sqrt{3}} + (x-3+5)\sqrt{x^2-6x+6}$$

$$= 10\cosh^{-1}\frac{x-3}{\sqrt{3}} + (x+2)\sqrt{x^2-6x+6}$$

The result can also be expressed as

$$\int\frac{(2x^2-7x+10)\,dx}{\sqrt{x^2-6x+6}} = 10\log_e\left\{\frac{x-3+\sqrt{x^2-6x+6}}{\sqrt{3}}\right\}$$
$$+ (x+2)\sqrt{x^2-6x+6}$$

Examples XI

Solve the following integrals:

(1) $\int\frac{dx}{\sqrt{9-x^2}}$

(2) $\int\frac{(3x-2)\,dx}{\sqrt{9-x^2}}$

(3) $\int\sqrt{9-x^2}\,dx$

(4) $\int\frac{(3x^2-7x+3)\,dx}{\sqrt{9-x^2}}$

(5) $\int\frac{dx}{\sqrt{6x-x^2}}$

(6) $\int\frac{(4x-1)\,dx}{\sqrt{6x-x^2}}$

(7) $\int\sqrt{6x-x^2}\,dx$

(8) $\int\frac{(x^2+2x-1)\,dx}{\sqrt{6x-x^2}}$

(9) $\int\frac{dx}{\sqrt{9+8x-x^2}}$

(10) $\int\frac{(2x-3)\,dx}{\sqrt{9+8x-x^2}}$

(11) $\int\sqrt{9+8x-x^2}\,dx$

(12) $\int\frac{(2x^2-5)\,dx}{\sqrt{9+8x-x^2}}$

(13) $\int\frac{dx}{\sqrt{10+6x-3x^2}}$

(14) $\int\frac{(4-5x)\,dx}{\sqrt{10+6x-3x^2}}$

(15) $\int\sqrt{10+6x-3x^2}\,dx$

(16) $\int\frac{(5x^2+12)\,dx}{\sqrt{10+6x-3x^2}}$

(17) $\int\frac{dx}{\sqrt{x^2+25}}$

(18) $\int\frac{(2x-7)\,dx}{\sqrt{x^2+25}}$

(19) $\int\sqrt{x^2+25}\,dx$

(20) $\int\frac{(3x^2+4x+2)\,dx}{\sqrt{x^2+25}}$

(21) $\displaystyle\int\frac{dx}{\sqrt{x^2-12x+52}}$ (22) $\displaystyle\int\frac{(6x-5)\,dx}{\sqrt{x^2-12x+52}}$

(23) $\displaystyle\int\sqrt{x^2-12x+52}\,dx$ (24) $\displaystyle\int\frac{(x^2-3x+4)\,dx}{\sqrt{x^2-12x+52}}$

(25) $\displaystyle\int\frac{dx}{\sqrt{2x^2+6x+7}}$ (26) $\displaystyle\int\frac{(3x-7)\,dx}{\sqrt{2x^2+6x+7}}$

(27) $\displaystyle\int\sqrt{2x^2+6x+7}\,dx$ (28) $\displaystyle\int\frac{(4x^2-3)\,dx}{\sqrt{2x^2+6x+7}}$

(29) $\displaystyle\int\frac{dx}{\sqrt{x^2-16}}$ (30) $\displaystyle\int\frac{(5x-12)\,dx}{\sqrt{x^2-16}}$

(31) $\displaystyle\int\sqrt{x^2-16}\,dx$ (32) $\displaystyle\int\frac{(3x^2-8x+1)\,dx}{\sqrt{x^2-16}}$

(33) $\displaystyle\int\frac{dx}{\sqrt{x^2+10x}}$ (34) $\displaystyle\int\frac{(6x-5)\,dx}{\sqrt{x^2+10x}}$

(35) $\displaystyle\int\sqrt{x^2+10x}\,dx$ (36) $\displaystyle\int\frac{(x^2+12x-7)\,dx}{\sqrt{x^2+10x}}$

(37) $\displaystyle\int\frac{dx}{\sqrt{x^2-4x-21}}$ (38) $\displaystyle\int\frac{(3x-7)\,dx}{\sqrt{x^2-4x-21}}$

(39) $\displaystyle\int\sqrt{x^2-4x-21}\,dx$ (40) $\displaystyle\int\frac{(3x^2-7x+13)\,dx}{\sqrt{x^2-4x-21}}$

(41) $\displaystyle\int\frac{dx}{\sqrt{5x^2+10x-16}}$ (42) $\displaystyle\int\frac{(15x-8)\,dx}{\sqrt{5x^2+10x-16}}$

(43) $\displaystyle\int\sqrt{5x^2+10x-16}\,dx$ (44) $\displaystyle\int\frac{(12x^2+7)\,dx}{\sqrt{5x^2+10x-16}}$

CHAPTER XII

97. The work of this chapter is devoted to some of the different methods involved in the integration of well-known trigonometrical functions.

(a)
$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x}$$
$$= -\int \frac{-\sin x\, dx}{\cos x}$$
$$= -\log_e \cos x$$
$$= \log_e \sec x$$

(b)
$$\int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x}$$
$$= \log_e \sin x$$

(c)
$$\int \operatorname{cosec} x\, dx = \int \frac{dx}{\sin x}$$
$$= \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$
$$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}\, dx}{\tan \frac{x}{2}}$$

on dividing numerator and denominator by $\cos^2 \frac{x}{2}$.

Put $y = \tan \frac{x}{2}$. Then $dy = \frac{1}{2} \sec^2 \frac{x}{2}\, dx$

and
$$\int \operatorname{cosec} x\, dx = \int \frac{dy}{y}$$
$$= \log_e y$$
$$= \log_e \tan \frac{x}{2}$$

(*d*)
$$\int \sec x\, dx = \int \frac{dx}{\cos x}$$
$$= \int \frac{dx}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}$$
$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{1 - \tan^2 \frac{x}{2}}$$

on dividing numerator and denominator by $\cos^2 \frac{x}{2}$.

Put $y = \tan \frac{x}{2}$. Then $dy = \frac{1}{2} \sec^2 \frac{x}{2}\, dx$.

Then
$$\int \sec x\, dx = 2\int \frac{dy}{1 - y^2}$$

but
$$\frac{2}{1 - y^2} = \frac{A}{1 + y} + \frac{B}{1 - y}$$
$$A(1 - y) + B(1 + y) = 2$$

when $y = 1$ $\quad 2B = 2 \quad B = 1$

when $y = -1$ $\quad 2A = 2 \quad A = 1$

Hence
$$\int \sec x\, dx = \int \frac{dy}{1 + y} + \int \frac{dy}{1 - y}$$
$$= \log_e(1 + y) - \log_e(1 - y)$$
$$= \log_e \frac{1 + y}{1 - y}$$
$$= \log_e \left\{ \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right\}$$
$$= \log_e \left\{ \frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{x}{2}} \right\} \text{ since } \tan \frac{\pi}{4} = 1$$
$$= \log_e \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

98. To integrate $\tan^n x$, where n is an even or odd integer.

By putting $z = \tan x$, $\frac{dz}{dx} = \sec^2 x = 1 + z^2$

$$\int \tan^n x\, dx \text{ becomes } \int \frac{z^n\, dz}{1 + z^2}$$

By division

$$\int \tan^n x\,dx = \int z^{n-2}\,dz - \int z^{n-4}\,dz + \ldots \pm \int dz \mp \int \frac{dz}{1+z^2}$$

when n is even and

$$\int \tan^n x\,dx = \int z^{n-2}\,dz - \int z^{n-4}\,dz \ldots \mp \int z\,dz \pm \int \frac{z\,dz}{1+z^2}$$

when n is odd.

(*a*) When n is an even integer.

To integrate $\tan^6 x$

$$\int \tan^6 x\,dx = \int \frac{z^6\,dz}{1+z^2} \text{ where } z = \tan x$$

$$= \int z^4\,dz - \int z^2\,dz + \int dz - \int \frac{dz}{1+z^2}$$

$$= \frac{1}{5}z^5 - \frac{1}{3}z^3 + z - \tan^{-1} z$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$$

(*b*) When n is an odd integer.

To integrate $\tan^7 x$

$$\int \tan^7 x\,dx = \int \frac{z^7}{1+z^2} \text{ where } z = \tan x$$

$$= \int z^5\,dz - \int z^3\,dz + \int z\,dz - \int \frac{z\,dz}{z^2+1}$$

$$= \frac{1}{6}z^6 - \frac{1}{4}z^4 + \frac{1}{2}z^2 - \frac{1}{2}\log_e (z^2+1)$$

$$= \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \log_e \sec x$$

99. (*a*) To integrate $\dfrac{1}{a\sin^2 x \pm b\cos^2 x}$

(1) $$\int \frac{dx}{a\sin^2 x + b\cos^2 x} = \int \frac{\sec^2 x\,dx}{a\tan^2 x + b}$$

$$= \int \frac{dy}{ay^2 + b} \text{ where } y = \tan x$$

$$= \frac{1}{a}\int \frac{dy}{y^2 + \frac{b}{a}}$$

$$= \frac{1}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}\,y$$

$$= \frac{1}{\sqrt{ab}}\tan^{-1}\left\{\sqrt{\frac{b}{a}}\tan x\right\}$$

(2) $$\int\frac{dx}{a\sin^2 x-b\cos^2 x}=\int\frac{\sec^2 x\,dx}{a\tan^2 x-b}$$

$$=\int\frac{dy}{ay^2-b}\text{ where } y=\tan x$$

$$=\frac{1}{a}\int\frac{dy}{y^2-\frac{b}{a}}$$

$$=\frac{1}{2a}\sqrt{\frac{a}{b}}\left\{\int\frac{dy}{y-\sqrt{\frac{b}{a}}}-\int\frac{dy}{y+\sqrt{\frac{b}{a}}}\right\}$$

$$=\frac{1}{2\sqrt{ab}}\left\{\log_e\left(y-\sqrt{\frac{b}{a}}\right)-\log_e\left(y+\sqrt{\frac{b}{a}}\right)\right\}$$

$$=\frac{1}{2\sqrt{ab}}\log_e\frac{y\sqrt{a}-\sqrt{b}}{y\sqrt{a}+\sqrt{b}}$$

$$=\frac{1}{2\sqrt{ab}}\log_e\frac{\sqrt{a}\tan x-\sqrt{b}}{\sqrt{a}\tan x+\sqrt{b}}$$

(3) $$\int\frac{dx}{(a\sin x+b\cos x)^2}=\int\frac{dx}{a^2\sin^2 x+2ab\sin x\cos x+b^2\cos^2 x}$$

$$=\int\frac{\sec^2 x\,dx}{a^2\tan^2 x+2ab\tan x+b^2}$$

$$=\int\frac{dy}{a^2y^2+2aby+b^2}\text{ where } y=\tan x$$

$$=\int\frac{dy}{(ay+b)^2}$$

$$=\frac{1}{a}\int\frac{dz}{z^2}\text{ where } z=ay+b$$

$$=-\frac{1}{a}z^{-1}$$

$$=-\frac{1}{a}\frac{1}{ay+b}$$

$$=-\frac{1}{a(a\tan x+b)}$$

100. $$\int\frac{dx}{a\sin x+b\cos x}=\frac{1}{\sqrt{a^2+b^2}}\int\frac{dx}{\sin(x+\alpha)}\text{ where }\tan\alpha=\frac{b}{a}$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log_e\tan\frac{1}{2}(x+\alpha)$$

This integral can be taken another way, only the method entails more work.

$$\int \frac{dx}{a \sin x + b \cos x} = \int \frac{dx}{2a \sin \frac{x}{2} \cos \frac{x}{2} + b\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)}$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{2a \tan \frac{x}{2} + b - b \tan^2 \frac{x}{2}}$$

$$= 2\int \frac{dy}{b + 2ay - by^2} \quad \text{where } y = \tan \frac{x}{2}$$

$$= \frac{2}{b} \int \frac{dy}{1 + \frac{2a}{b} y - y^2}$$

$$= \frac{2}{b} \int \frac{dy}{1 + \frac{a^2}{b^2} - \left(y^2 - \frac{2a}{b} y + \frac{a^2}{b^2}\right)}$$

$$= \frac{2}{b} \int \frac{dy}{\frac{a^2 + b^2}{b^2} - \left(y - \frac{a}{b}\right)^2}$$

$$= \frac{2}{b} \int \frac{dX}{A^2 - X^2} \quad \text{where } X = y - \frac{a}{b} \text{ and } A^2 = \frac{a^2 + b^2}{b^2}$$

$$= \frac{1}{b} \frac{b}{\sqrt{a^2 + b^2}} \left\{\int \frac{dX}{A + X} + \int \frac{dX}{A - X}\right\}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \{\log_e (A + X) - \log_e (A - X)\}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \log_e \frac{A + X}{A - X}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \log_e \frac{\sqrt{a^2 + b^2} + by - a}{\sqrt{a^2 + b^2} - by + a}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \log_e \frac{\sqrt{a^2 + b^2} - a + b \tan \frac{x}{2}}{\sqrt{a^2 + b^2} + a - b \tan \frac{x}{2}}$$

101. The integration of $\sin^n \theta$ when n is an integer

(1) When n is odd,

$$\int \sin^7 \theta \, d\theta = \int \sin^6 \theta \sin \theta \, d\theta$$

Put $\qquad x = \cos \theta$, then $dx = -\sin \theta \, d\theta$

also $\qquad \sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2$

$$\text{Then} \quad \int \sin^7 \theta \, d\theta = -\int (1 - x^2)^3 \, dx$$

$$= -\int (1 - 3x^2 + 3x^4 - x^6) \, dx$$

$$= \frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x$$

$$= \frac{1}{7}\cos^7 \theta - \frac{3}{5}\cos^5 \theta + \cos^3 \theta - \cos \theta$$

(2) When n is even, this method fails, for by putting $\sin^n \theta = \sin^{n-1} \theta \sin \theta$ and making the substitution $x = \cos \theta$

$$\int \sin^n \theta \, d\theta = \int \sin^{n-1} \theta \sin \theta \, d\theta$$

$$= -\int (1 - x^2)^{\frac{n-1}{2}} \, dx$$

and $(1 - x^2)$ is raised to a fractional power, since $(n - 1)$ is odd. This will not give a definite expansion, but a series of an infinite number of terms. It is necessary to work in an entirely different manner and deal with the multiple angles of θ.

To integrate $\sin^6 \theta$.

Now if $\qquad x = \cos \theta + i \sin \theta$

$$\frac{1}{x} = \cos \theta - i \sin \theta$$

and $\qquad 2 \cos \theta = x + \dfrac{1}{x}, \quad 2i \sin \theta = x - \dfrac{1}{x}$

Also $\qquad x^n = \cos n\theta + i \sin n\theta$

$$\frac{1}{x^n} = \cos n\theta - i \sin n\theta$$

and $\qquad 2 \cos n\theta = x^n + \dfrac{1}{x^n}, \quad 2i \sin n\theta = x^n - \dfrac{1}{x^n}$

Then $\quad (2i \sin \theta)^6 = \left(x - \frac{1}{x}\right)^6$

$$-64 \sin^6 \theta = x^6 - 6x^4 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$\sin^6 \theta = -\frac{1}{64}\{2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\}$$

Therefore $\displaystyle\int \sin^6 \theta \, d\theta = \frac{1}{64}\int (20 - 30\cos 2\theta + 12\cos 4\theta - 2\cos 6\theta)\, d\theta$

$$= \frac{1}{64}\left\{20\theta - 15\sin 2\theta + 3\sin 4\theta - \frac{1}{3}\sin 6\theta\right\}$$

$$= \frac{1}{192}\{60\theta - 45\sin 2\theta + 9\sin 4\theta - \sin 6\theta\}$$

102. The integration of $\cos^n \theta$ when n is an integer

(1) When n is odd,

$$\int \cos^5 \theta \, d\theta = \int \cos^4 \theta \cos \theta \, d\theta$$

Put $\quad x = \sin \theta$, then $dx = \cos \theta \, d\theta$

Also $\quad \cos^2 \theta = 1 - \sin^2 \theta = 1 - x^2$

Then $\quad \displaystyle\int \cos^5 \theta \, d\theta = \int (1 - x^2)^2 \, dx$

$$= \int (1 - 2x^2 + x^4)\, dx$$

$$= x - \frac{2}{3}x^3 + \frac{1}{5}x^5$$

$$= \sin \theta - \frac{2}{3}\sin^3 \theta + \frac{1}{5}\sin^5 \theta$$

(2) When n is even, this method fails for the same reason that it does in the case of $\sin^n \theta$, but a result can be obtained by working in terms of the multiple angles of θ.

To integrate $\cos^4 \theta$

$$(2\cos \theta)^4 = \left(x + \frac{1}{x}\right)^4$$

$$16\cos^4 \theta = x^4 + 4x^2 + 6 + 4\frac{1}{x^2} + \frac{1}{x^4}$$

$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$\cos^4 \theta = \frac{1}{16}\{2\cos 4\theta + 8\cos 2\theta + 6\}$$

$$\text{Therefore} \int \cos^4\theta \; d\theta = \frac{1}{16}\int (2\cos 4\theta + 8\cos 2\theta + 6)\, d\theta$$

$$= \frac{1}{16}\left(\frac{1}{2}\sin 4\theta + 4\sin 2\theta + 6\theta\right)$$

$$= \frac{1}{32}\{\sin 4\theta + 8\sin 2\theta + 12\theta\}$$

103. The integration of $\sin^n \theta \cos^m \theta$ where n and m are integers. The substitutions $x = \sin\theta$, or $x = \cos\theta$ will enable us to integrate this expression, except the case when n and m are both even.

(1)
$$\int \sin^3\theta \cos^3\theta \, d\theta = \int \sin^3\theta \cos^2\theta \cos\theta \, d\theta$$

$$= \int x^3(1 - x^2)\, dx, \quad \text{when } x = \sin\theta$$

$$= \frac{1}{4}x^4 - \frac{1}{6}x^6$$

$$= \frac{1}{4}\sin^4\theta - \frac{1}{6}\sin^6\theta$$

(2)
$$\int \sin^4\theta \cos^3\theta \, d\theta = \int \sin^4\theta \cos^2\theta \cos\theta \, d\theta$$

$$= \int x^4(1 - x^2)\, dx, \quad \text{when } x = \sin\theta$$

$$= \frac{1}{5}x^5 - \frac{1}{7}x^7$$

$$= \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta$$

(3)
$$\int \sin^3\theta \cos^4\theta \, d\theta = \int \sin^2\theta \cos^4\theta \sin\theta \, d\theta$$

$$= -\int x^4(1 - x^2)\, dx, \quad \text{when } x = \cos\theta$$

$$= \frac{1}{7}x^7 - \frac{1}{5}x^5$$

$$= \frac{1}{7}\cos^7\theta - \frac{1}{5}\cos^5\theta$$

(4) When n and m are both even we must use the application of De Moivre's Theorem.

To integrate $\sin^2\theta\cos^4\theta$

Now $(2i\sin\theta)^2(2\cos\theta)^4 = \left(x-\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^4$

$$-64\sin^2\theta\cos^4\theta = \left(x^2-\frac{1}{x^2}\right)^2\left(x+\frac{1}{x}\right)^2$$

$$= \left(x^4-2+\frac{1}{x^4}\right)\left(x^2+2+\frac{1}{x^2}\right)$$

$$= \left(x^6+\frac{1}{x^6}\right)+2\left(x^4+\frac{1}{x^4}\right)-\left(x^2+\frac{1}{x^2}\right)-4$$

$$\sin^2\theta\cos^4\theta = -\frac{1}{64}\{2\cos 6\theta+4\cos 4\theta-2\cos 2\theta-4\}$$

Therefore $\int\sin^2\theta\cos^4\theta\,d\theta$

$$= \frac{1}{64}\int(4+2\cos 2\theta-4\cos 4\theta-2\cos 6\theta)\,d\theta$$

$$= \frac{1}{64}\left\{4\theta+\sin 2\theta-\sin 4\theta-\frac{1}{3}\sin 6\theta\right\}$$

$$= \frac{1}{192}\left\{12\theta+3\sin 2\theta-3\sin 4\theta-\sin 6\theta\right\}$$

Examples XII

Solve the following integrals:

(1) $\int\sec^2 x\,dx$ (2) $\int\operatorname{cosec}^2 x\,dx$

(3) $\int\frac{\sin^2 x\,dx}{\cos^4 x}$ (4) $\int\frac{\cos^2 x\,dx}{\sin^4 x}$

(5) $\int\tan(ax+b)\,dx$ (6) $\int\cot(ax+b)\,dx$

(7) $\int\tan^5 x\,dx$ (8) $\int\tan^6 x\,dx$

(9) $\int\cot^3 x\,dx$ (10) $\int\cot^4 x\,dx$

(11) $\int\frac{dx}{3\sin x+4\cos x}$ (12) $\int\frac{dx}{4\sin x-3\cos x}$

(13) $\int\frac{dx}{\sin x+\cos x}$ (14) $\int\frac{dx}{\sin x-\cos x}$

(15) $\int\frac{dx}{3\sin^2 x+4\cos^2 x}$ (16) $\int\frac{dx}{3\sin^2 x-4\cos^2 x}$

(17) $\int \frac{\tan^2 x\,dx}{\cos^2 x - \sin^2 x}$

(18) $\int \frac{dx}{3 + 4\cos x}$

(19) $\int \frac{dx}{3 + 4\sin x}$

(20) $\int \frac{dx}{4 + 3\sin x}$

(21) $\int \frac{dx}{4 + 3\cos x}$

(22) $\int \sin^3 x\,dx$

(23) $\int \sin^4 x\,dx$

(24) $\int \cos^5 x\,dx$

(25) $\int \cos^6 x\,dx$

(26) $\int \sin^3 x \cos^2 x\,dx$

(27) $\int \sin^4 x \cos^3 x\,dx$

(28) $\int \sin^5 x \cos^5 x\,dx$

(29) $\int \sin^4 x \cos^4 x\,dx$

(30) $\int \cos^2 x \sin^4 x\,dx$

(31) $\int \sin^2 x \cos^5 x\,dx$

CHAPTER XIII

104. *Integration by Parts.*

If $$y = uv$$

then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating throughout with respect to x,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

or $$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or symbolically $$\int u\,dv = uv - \int v\,du$$

This rule enables us, in many cases, to integrate the product of two different functions of x, for we can represent one of the functions by u and the other by dv in the integral $\int u\,dv$. In order to build up the right-hand side we want the terms corresponding to du and v. The first of these is obtained by differentiating the function denoted by u, and the second by integrating the function denoted by dv. It must not be assumed that the application of the rule of integration by parts will enable us to integrate a product straightway, for a consideration of the right-hand side shows us that the method takes an integral and splits it up into two parts, and the second part is an integral. The success of the rule depends upon whether this second integral is more easily dealt with than the original integral.

The well-known integrals, to which the rule of integration by parts can be successfully applied, can be divided into four distinct classes:

(1) $\int x^n \log_e x\,dx$

(2) $\int e^{ax} x^n\,dx$ and $\int x^n(\log_e x)^m\,dx$

(3) $\int x^n \sin ax\,dx$ and $\int x^n \cos ax\,dx$

(4) $\int e^{ax} \sin bx\,dx$ and $\int e^{ax} \cos bx\,dx$ while with these can be included $\int e^{ax} \sin^n x\,dx$ and $\int e^{ax} \cos^n x\,dx$.

105. (1) To integrate $x^n \log_e x$.

$$\text{Now} \int x^n \log_e x \, dx = uv - \int v \, du$$

Here we can put $u = x^n$ or $dv = x^n \, dx$, but we must differentiate x^n in the first case and integrate x^n in the second case. As we can easily do both operations, it is perfectly immaterial which selection we make. We can also put $u = \log_e x$ or $dv = \log_e x \, dx$, but we find that although we can easily differentiate $\log_e x$, we shall have great difficulty in integrating $\log_e x$, and in consequence we are led to make the selection $u = \log_e x$.

$$\text{Thus } u = \log_e x \text{ and } du = \frac{1}{x} \, dx$$

$$dv = x^n \, dx \text{ and } v = \int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\begin{aligned} \text{Then} \int x^n \log_e x \, dx &= \frac{x^{n+1}}{n+1} \log_e x - \int \frac{x^{n+1}}{n+1} \frac{1}{x} \, dx \\ &= \frac{x^{n+1} \log_e x}{n+1} - \frac{1}{n+1} \int x^n \, dx \\ &= \frac{x^{n+1} \log_e x}{n+1} - \frac{x^{n+1}}{(n+1)^2} \\ &= \frac{x^{n+1}}{n+1} \left\{ \log_e x - \frac{1}{n+1} \right\} \end{aligned}$$

When $n = 0$ the integral becomes $\int \log_e x \, dx$, and $\int \log_e x \, dx = x(\log_e x - 1)$.

(2) To integrate $x^n e^{ax}$

$$\text{Now} \qquad \int x^n e^{ax} \, dx = uv - \int v \, du$$

If $\qquad u = x^n$, then $du = nx^{n-1} \, dx$

$$dv = e^{ax} \, dx, \text{ then } v = \int e^{ax} \, dx = \frac{e^{ax}}{a}$$

$$\text{Therefore} \qquad \int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$$

In this case the result of applying the rule is to produce on the right-hand side an integral of the same form as the original integral, but in which the power of x has been diminished by 1. We can, however, use the above result as a standard form in which n and a can be given their assigned values.

To integrate x^4e^{3x}

$$\int x^4e^{3x}\,dx = \frac{1}{3}x^4e^{3x} - \frac{4}{3}\int x^3e^{3x}\,dx$$

$$= \frac{1}{3}x^4e^{3x} - \frac{4}{3}\left\{\frac{1}{3}x^3e^{3x} - \int x^2e^{3x}\,dx\right\}$$

$$= \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{3}\left\{\frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx\right\}$$

$$= \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{9}x^2e^{3x} - \frac{8}{9}\left\{\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx\right\}$$

$$= \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{9}x^2e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{81}e^{3x}$$

$$= \frac{e^{3x}}{3}\left\{x^4 - \frac{4}{3}x^3 + \frac{4}{3}x^2 - \frac{8}{9}x + \frac{8}{27}\right\}$$

To integrate $x^n(\log_e x)^m$, put $\log_e x = y$

Then $$x = e^y \text{ and } dx = e^y\,dy$$

$$\int x^n(\log_e x)^m\,dx = \int e^{ny}\,y^m\,e^y\,dy$$

$$= \int y^m\,e^{(n+1)y}\,dy$$

The integral thus reducing down to a form similar to $\int x^ne^{ax}\,dx$.

Thus, to integrate $x^4(\log_e x)^2$, put $\log_e x = y$.

Then $$x = e^y \text{ and } dx = e^y\,dy$$

and $$\int x^4(\log_e x)^2\,dx = \int e^{4y}\,y^2\,e^y\,dy$$

$$= \int y^2e^{5y}\,dy$$

$$= \frac{1}{5}y^2e^{5y} - \frac{2}{5}\int ye^{5y}\,dy$$

$$= \frac{1}{5}y^2e^{5y} - \frac{2}{5}\left\{\frac{1}{5}ye^{5y} - \frac{1}{5}\int e^{5y}\,dy\right\}$$

$$= \frac{1}{5}y^2e^{5y} - \frac{2}{25}ye^{5y} + \frac{2}{125}e^{5y}$$

$$= \frac{e^{5y}}{5}\left\{y^2 - \frac{2}{5}y + \frac{2}{25}\right\}$$

$$= \frac{x^5}{5}\left\{(\log_e x)^2 - \frac{2}{5}\log_e x + \frac{2}{25}\right\}$$

106. (3) To integrate $x^n \sin ax$ and $x^n \cos ax$. From the previous example it is obvious that by putting $u = x^n$ the integral on the right-hand side will contain x^{n-1}, that is, by applying the rule once we diminish the power of x by unity in the resulting integral. Hence to completely evaluate $\int x^n \sin ax\, dx$ or $\int x^n \cos ax\, dx$, we should have to apply the rule n times. We can make the work more methodical by applying the rule once to each of these integrals and using the results as standard forms for integral values of the power n.

Now $$\int x^n \sin ax\, dx = uv - \int v\, du$$

where $$u = x^n,\ du = nx^{n-1}\, dx$$

and $$dv = \sin ax\, dx,\ v = \int \sin ax\, dx = -\frac{1}{a}\cos ax$$

Then $$\int x^n \sin ax\, dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\, dx \quad . \ . \ . \quad (1)$$

Also $$\int x^n \cos ax\, dx = uv - \int v\, du$$

where $$u = x^n,\ du = nx^{n-1}\, dx$$

and $$dv = \cos ax\, dx,\ v = \int \cos ax\, dx = \frac{1}{a}\sin ax$$

Then $$\int x^n \cos ax\, dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax\, dx \quad . \ . \ . \quad (2)$$

As an example, let us apply these results to integrate $x^5 \sin 2x$.

Then $$\int x^n \sin 2x\, dx = -\frac{1}{2}x^n \cos 2x + \frac{n}{2}\int x^{n-1}\cos 2x\, dx$$

and $$\int x^n \cos 2x\, dx = \frac{1}{2}x^n \sin 2x - \frac{n}{2}\int x^{n-1}\sin 2x\, dx$$

$$\int x^5 \sin 2x\, dx$$

$$= -\frac{1}{2}x^5 \cos 2x + \frac{5}{2}\int x^4 \cos 2x\, dx$$

$$= -\frac{x^5}{2}\cos 2x + \frac{5}{2}\left\{\frac{1}{2}x^4 \sin 2x - 2\int x^3 \sin 2x\, dx\right\}$$

$$= -\frac{x^5}{2}\cos 2x + \frac{5x^4}{4}\sin 2x - 5\left\{-\frac{1}{2}x^3 \cos 2x + \frac{3}{2}\int x^2 \cos 2x\, dx\right\}$$

$$= \left(-\frac{x^5}{2}+\frac{5x^3}{2}\right)\cos 2x + \frac{5x^4}{4}\sin 2x - \frac{15}{2}\left\{\frac{1}{2}x^2 \sin 2x - \int x \sin 2x\, dx\right\}$$

$$= \left(-\frac{x^5}{2}+\frac{5x^3}{2}\right)\cos 2x + \left(\frac{5x^4}{4}-\frac{15x^2}{4}\right)\sin 2x + \frac{15}{2}\left\{-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx\right\}$$

$$= \left(-\frac{x^5}{2}+\frac{5x^3}{2}-\frac{15x}{4}\right)\cos 2x + \left(\frac{5x^4}{4}-\frac{15x^2}{4}+\frac{15}{8}\right)\sin 2x$$

107. (4) To integrate $e^{ax}\sin bx$ and $e^{ax}\cos bx$.

Now $\int e^{ax}\sin bx\, dx = uv - \int v\, du$

where $u = \sin bx, \quad du = b\cos bx\, dx$

and $dv = e^{ax}\, dx, \quad v = \int e^{ax}\, dx = \frac{e^{ax}}{a}$

Then $\int e^{ax}\sin bx\, dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\, dx$, and denoting $\int e^{ax}\sin bx\, dx$ by X and $\int e^{ax}\cos bx\, dx$ by Y,

$$X = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}Y \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also $\int e^{ax}\cos bx\, dx = uv - \int v\, du$

where $u = \cos bx, \quad du = -b\sin bx\, dx$

and $dv = e^{ax}\, dx, \quad v = \int e^{ax}\, dx = \frac{e^{ax}}{a}$

Then $\int e^{ax}\cos bx\, dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\, dx$

or
$$Y = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}X \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus giving a pair of simultaneous equations to be solved for X and Y.

Solving for X, $X = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}X$

$$X(a^2+b^2) = e^{ax}\{a\sin bx - b\cos bx\}$$

$$X = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

or $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx-\alpha)$ where $\tan\alpha = \frac{b}{a}$

Solving for Y, $\mathrm{Y} = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2}\mathrm{Y}$

$$\mathrm{Y}(a^2 + b^2) = e^{ax}(a \cos bx + b \sin bx)$$

$$\mathrm{Y} = \frac{e^{ax}}{a^2 + b^2}\{a \cos bx + b \sin bx\}$$

or $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{\sqrt{a^2+b^2}} \cos (bx - \alpha)$ where $\tan \alpha = \frac{b}{a}$

108. For the integration of $e^{ax} \sin^n x$ or $e^{ax} \cos^n x$, $\sin^n x$ and $\cos^n x$ must be expressed in terms of the sines or cosines of the multiple angles of x. Then each of the integrals will split up into a number of integrals each of the form $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$, and these can be integrated by the previous method. In this case though, because it will be necessary to work with $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$ for various numerical values of b, it is best to establish the results working with a and b, and use them as standard forms.

(a) To integrate $e^{2x} \sin^5 x$.

Now $\quad (2i \sin x)^5 = \left(y - \frac{1}{y}\right)^5$, if $y = \cos x + i \sin x$

$$32i \sin^5 x = y^5 - 5y^3 + 10y - 10\frac{1}{y} + 5\frac{1}{y^3} - \frac{1}{y^5}$$

$$= \left(y^5 - \frac{1}{y^5}\right) - 5\left(y^3 - \frac{1}{y^3}\right) + 10\left(y - \frac{1}{y}\right)$$

$$= 2i \sin 5x - 10i \sin 3x + 20i \sin x$$

and $\sin^5 x = \frac{1}{16} \sin 5x - \frac{5}{16} \sin 3x + \frac{5}{8} \sin x$

Then $\int e^{2x} \sin^5 x\, dx = \frac{1}{16}\int e^{2x} \sin 5x\, dx - \frac{5}{16}\int e^{2x} \sin 3x\, dx + \frac{5}{8}\int e^{2x} \sin x\, dx$ and each of these integrals can be determined by using as a standard form

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin (bx - \alpha), \text{ where } \tan \alpha = \frac{b}{a}$$

Hence $\int e^{2x} \sin^5 x\, dx = \frac{e^{2x}}{16\sqrt{29}} \sin (5x - \alpha) - \frac{5e^{2x}}{16\sqrt{13}} \sin (3x - \beta) + \frac{5e^{2x}}{8\sqrt{5}} \sin (x - \gamma)$, where $\tan \alpha = \frac{5}{2}$, $\tan \beta = \frac{3}{2}$ and $\tan \gamma = \frac{1}{2}$

Then $\int e^{2x} \sin^5 x \, dx = \frac{e^{2x}}{16}\left\{\frac{1}{\sqrt{29}} \sin\left(5x - \tan^{-1}\frac{5}{2}\right) - \frac{5}{\sqrt{13}} \sin\left(3x\right.\right.$
$\left.\left. - \tan^{-1}\frac{3}{2}\right) + \frac{10}{\sqrt{5}} \sin\left(x - \tan^{-1}\frac{1}{2}\right)\right\}$

(*b*) To integrate $e^{3x} \cos^4 x$.

Now
$$(2 \cos x)^4 = \left(y + \frac{1}{y}\right)^4, \text{ if } y = \cos x + i \sin x$$
$$16 \cos^4 x = y^4 + 4y^2 + 6 + 4\frac{1}{y^2} + \frac{1}{y^4}$$
$$= \left(y^4 + \frac{1}{y^4}\right) + 4\left(y^2 + \frac{1}{y^2}\right) + 6$$
$$= 2 \cos 4x + 8 \cos 2x + 6$$
$$\cos^4 x = \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}$$

Then
$$\int e^{3x} \cos^4 x \, dx = \frac{1}{8}\int e^{3x} \cos 4\, x \, dx + \frac{1}{2}\int e^{3x} \cos 2x \, dx$$
$+ \frac{3}{8}\int e^{3x} \, dx$ and using $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos (bx - \alpha)$
where $\tan \alpha = \frac{b}{a}$, as a standard form.

$$\int e^{3x} \cos^4 x \, dx = \frac{e^{3x}}{40} \cos (4x - \alpha) + \frac{e^{3x}}{2\sqrt{13}} \cos (2x - \beta) + \frac{e^{3x}}{8}$$

where
$$\tan \alpha = \frac{4}{3} \text{ and } \tan \beta = \frac{2}{3}$$

Then
$$\int e^{3x} \cos^4 x \, dx = \frac{e^{3x}}{8}\left\{\frac{1}{5} \cos\left(4x - \tan^{-1}\frac{4}{3}\right)\right.$$
$$\left. + \frac{4}{\sqrt{13}} \cos\left(2x - \tan^{-1}\frac{2}{3}\right) + 1\right\}$$

(*c*) To integrate $e^{4x} \sin^2 x \cos^3 x$.

Now $(2i \sin x)^2(2 \cos x)^3 = \left(y - \frac{1}{y}\right)^2\left(y + \frac{1}{y}\right)^3$, if $y = \cos x + i \sin x$

$$-32 \sin^2 x \cos^3 x = \left(y^2 - \frac{1}{y^2}\right)^2\left(y + \frac{1}{y}\right)$$
$$= \left(y^4 - 2 + \frac{1}{y^4}\right)\left(y + \frac{1}{y}\right)$$
$$= \left(y^5 + \frac{1}{y^5}\right) + \left(y^3 + \frac{1}{y^3}\right) - 2\left(y + \frac{1}{y}\right)$$
$$= 2 \cos 5x + 2 \cos 3x - 4 \cos x$$
$$\sin^2 x \cos^3 x = \frac{1}{8} \cos x - \frac{1}{16} \cos 3x - \frac{1}{16} \cos 5x.$$

Then $\int e^{4x} \sin^2 x \cos^3 x \, dx$

$$= \frac{1}{8}\int e^{4x} \cos x \, dx - \frac{1}{16}\int e^{4x} \cos 3x \, dx - \frac{1}{16}\int e^{4x} \cos 5x$$

$$= \frac{e^{4x}}{8\sqrt{17}} \cos (x - \alpha) - \frac{e^{4x}}{80} \cos (3x - \beta) - \frac{e^{4x}}{16\sqrt{41}} \cos (5x -$$

where $\tan \alpha = \frac{1}{4}$, $\tan \beta = \frac{3}{4}$, and $\tan \gamma = \frac{5}{4}$.

Hence $\int e^{4x} \sin^2 x \cos^3 x \, dx = \frac{e^{4x}}{16}\left\{\frac{2}{\sqrt{17}} \cos \left(x - \tan^{-1} \frac{1}{4}\right)\right.$

$$\left. - \frac{1}{5} \cos \left(3x - \tan^{-1} \frac{3}{4}\right) - \frac{1}{\sqrt{41}} \cos \left(5x - \tan^{-1} \frac{5}{4}\right)\right\}$$

EXAMPLES XIII

Solve the following integrals:

(1) $\int x^7 \log_e x \, dx$ (2) $\int \sqrt{x} \log_e x \, dx$

(3) $\int \frac{\log_e x \, dx}{x^5}$ (4) $\int \frac{\log_e x \, dx}{\sqrt{x}}$

(5) $\int x^3 (\log_e x)^2 \, dx$ (6) $\int \sqrt{x} (\log_e x)^3 \, dx$

(7) $\int x^3 e^{2x} \, dx$ (8) $\int x^2 e^{-2x} \, dx$

(9) $\int e^{\sin x} \cos^3 x \, dx$ (put $\sin x = z$) (10) $\int e^{\sin x} \sin 2x \, dx$ (put $\sin x = z$)

(11) $\int x^2 \sin 2x \, dx$ (12) $\int x^3 \cos 3x \, dx$

(13) $\int x^4 \sin 2x \, dx$ (14) $\int x^2 \sin^2 x \, dx$

(15) $\int x^3 \cos^2 x \, dx$ (16) $\int x^2 \sin^3 x \, dx$

(17) $\int x^2 \cos^3 x \, dx$ (18) $\int e^{2x} \sin 3x \, dx$

(19) $\int e^{2x} \cos 3x \, dx$ (20) $\int e^{-3x} \sin 2x \, dx$

$\int e^{-3x} \cos 2x \, dx$ (22) $\int e^{2x} \sin^2 x \, dx$

$\int e^{2x} \cos^2 x \, dx$ (24) $\int e^{3x} \sin^4 x \, dx$

$\int e^{3x} \cos^4 x \, dx$ (26) $\int e^{-2x} \sin^2 x \, dx$

) $\int e^{-2x} \cos^2 x \, dx$ (28) $\int \tan^{-1} x \, dx$

) $\int x^2 \tan^{-1} x \, dx$ (30) $\int \sin^{-1} x \, dx$

CHAPTER XIV

109. *The Meaning of an Integral.*

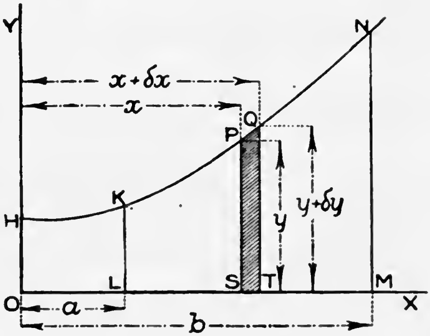


FIG. 44.

Let **P**, **Q** be two points taken on the curve $y = f(x)$, Fig. 44, and let the co-ordinates of **P** be x, y, and of **Q** $x + dx$, $y + dy$. When the ordinates are drawn to **P** and **Q**, the strip **PQTS** is produced, the top part of this strip being bounded by the arc **PQ** of the curve. If the breadth δx is small, the strip may be approximately taken as a trapezium.

$$\text{Area of the strip} = \frac{1}{2}(2y + \delta y)\,\delta x$$

$$= \left(y + \frac{1}{2}\,\delta y\right)\delta x$$

If δx is taken as being very small, then δy is also very small and can be neglected in comparison with y.

Hence the area of the strip or $\delta A = y\,\delta x$.

$$\text{Then} \qquad y = \frac{\delta A}{\delta x}$$

In the limit when δx is made infinitely small

$$y = \frac{dA}{dx}$$

Therefore A, the area under the curve, is a function of x, which, when differentiated with respect to x, will give y. Now as integration is the converse of differentiation, it follows therefore, that in order to obtain A, y must be integrated with respect to x.

Then
$$A = \int y\,dx$$

Considering the area KLMN, which is bounded by the axis of x, the ordinates at $x = a$ and $x = b$, and the portion KN of the curve. Let this area be divided up into a number of thin strips, the breadth of each strip being δx.

The number of strips will be $\dfrac{b - a}{\delta x}$.

The area of each strip will be approximately $y\,\delta x$.

Now, when δx is made very small, the number of strips taken is considerably increased, the area of each strip is considerably decreased, but at the same time the quantity $y\,\delta x$ represents more nearly the true area of each strip.

In the limit, when δx is made infinitely small, the area of each strip becomes infinitely small, but in order to find the area KLMN an infinitely great number of these strips must be added together.

Hence A, the area, is the sum of all such terms of the form $y\,\delta x$ when δx is made infinitely small, or $A = \Sigma y\,\delta x$ when δx is infinitely small. But it has already been shown that $A = \int y\,dx$; therefore $\int y\,dx = \Sigma y\,\delta x$ when δx is infinitely small. In other words, an integral is the limiting value of the sum of an infinitely great number of infinitely small terms.

110. *The Definite Integral.* It has been shown that the area under a curve is given by $\int y\,dx$, and if the law of the curve is known—that is, y is given as a function of x—this integral can be determined and the area expressed as a function of x.

We have now to consider what we really have when the area is expressed as a function of x, for at present we only know that it represents the area under some part of the curve. Before the value of this area can be found, we have to fix upon its actual position with respect to the axes of reference, and this can be done by fixing upon the initial and final ordinates.

Thus, if we erect ordinates at $x = a$ and $x = b$ (Fig. 44), we decide upon the actual position of the area and also fix upon the breadth, or the length of the base line. Therefore the values $x = a$ and $x = b$ are values of x which actually decide what the area under the curve will be.

Taking the relation Area $= \int y\,dx$, and after the integration has been performed, x is given the value a, the result will give the value of the area HOLK, the area which is bounded by the ordinates at $x = 0$ and $x = a$.

Also taking the same relation Area $= \int y\,dx$, and after the integration has been performed, x is given the value b, the result will give the value of the area HOMN, the area which is bounded by the ordinates at $x = 0$ and $x = b$.

Now area KLMN = area HOMN − area HOLK

$$= \int y\,dx \text{ (when } x = b) - \int y\,dx \text{ (when } x = a)$$

or area KLMN $= \int_a^b y\,dx$, where, after the integration has been performed, x is given the values b and a respectively and the difference of the two results is taken.

We notice now that integration can be performed with respect to x over a definite range, or between two definite values of x. These two limiting values of x are spoken of as the *superior* and *inferior* limits respectively, and the result obtained by replacing x by the value of the inferior limit must be subtracted from the result obtained by replacing x by the value of the superior limit. It must be clearly understood, however, that before we can substitute the values of the limits we must have performed the necessary integration.

111. *Areas.* Let P and Q be two points on a curve (Fig. 45), the co-ordinates of P being (a, h) and the co-ordinates of Q being (b, k). Let A be the area bounded by the curve, the axis of x, and the ordinates h and k. Let B be the area bounded by the curve, the axis of y, and the abscissæ a and b. Then A can be taken as the sum of vertical strips of area $y\,\delta x$ taken from $x = a$ to $x = b$, while B is the sum of horizontal strips of area $x\,\delta y$ taken from $y = h$ to $y = k$,

$$\text{and} \quad A = \int_a^b y\,dx$$

$$\text{while } B = \int_h^k x\,dy$$

A study of the figure will show that

$$\int_a^b y\,dx + \int_h^k x\,dy + ah = bk$$

As an example, let the law of the curve be $y = cx^n$, where c and n are constants.

$$\text{Then area A} = \int_a^b y\,dx$$

$$= c\int_a^b x^n\,dx$$

$$= \frac{c}{n+1}\Big[x^{n+1}\Big]_a^b$$

$$= \frac{c}{n+1}\{b^{n+1} - a^{n+1}\}$$

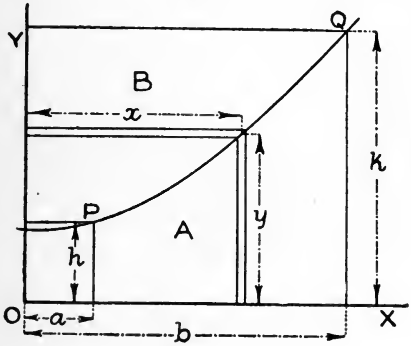


FIG. 45.

Now $h = ca^n$ and $k = cb^n$

Then $\qquad$ area $\text{B} = bk - ah -$ area A

$$= cb^{n+1} - ca^{n+1} - \frac{c}{n+1}\{b^{n+1} - a^{n+1}\}$$

$$= c\left\{ b^{n+1}\left(1 - \frac{1}{n+1}\right) - a^{n+1}\left(1 - \frac{1}{n+1}\right)\right\}$$

$$= \frac{nc}{n+1}\{b^{n+1} - a^{n+1}\}$$

The area B can be determined independently.

$$\text{For area B} = \int_h^k x\,dy$$

$$= \frac{1}{c^{\frac{1}{n}}}\int_{ca^n}^{cb^n} y^{\frac{1}{n}}\,dy$$

This is an awkward integral to evaluate, more particularly on account of the nature of the limits, and the work is rendered more simple by expressing the integral in terms of x.

Since $$y = cx^n$$
$$dy = ncx^{n-1}\,dx$$
and $$x\,dy = ncx^n\,dx$$
Therefore $$\text{area B} = \int x\,dy$$
$$= nc\int_a^b x^n\,dx$$
$$= \frac{nc}{n+1}\Big[x^{n+1}\Big]_a^b$$
$$= \frac{nc}{n+1}\{b^{n+1} - a^{n+1}\}$$

A comparison of the results for the areas A and B shows that if the law of the curve is of the form $y = cx^n$, then the area B is n times the area A.

112. *Example* 1. Working between the limits $x = 2$ and $x = 3$ for the curve $y = 3x^{\frac{3}{2}}$. Find (1) the area bounded by the ordinates at $x = 2$ and $x = 3$, and (2) the area bounded by the abscissæ which correspond to the ordinates at $x = 2$ and $x = 3$.

Then $$\text{area (1)} = \int_2^3 y\,dx$$
$$= 3\int_2^3 x^{\frac{3}{2}}\,dx$$
$$= \frac{6}{5}\Big[x^{\frac{5}{2}}\Big]_2^3$$
$$= 1{\cdot}2\{3^{\frac{5}{2}} - 2^{\frac{5}{2}}\}$$
$$= 1{\cdot}2\{9\sqrt{3} - 4\sqrt{2}\}$$
$$= 11{\cdot}92$$
and $$\text{area (2)} = \int x\,dy$$
$$= \frac{9}{2}\int_2^3 x^{\frac{3}{2}}\,dx, \text{ since } dy = \frac{9}{2}x^{\frac{1}{2}}\,dx$$
$$= \frac{9}{5}\Big[x^{\frac{5}{2}}\Big]_2^3$$
$$= 1{\cdot}8\{3^{\frac{5}{2}} - 2^{\frac{5}{2}}\}$$
$$= 1{\cdot}8\{9\sqrt{3} - 4\sqrt{2}\}$$
$$= 17{\cdot}88$$

Example 2. Find the first two points at which the curve $y = e^{-x} \sin x$ crosses the axis of x, and then find the area bounded by the curve and the axis of x between these points.

Now $\sin x = 0$ when x has the values 0, π, 2π, etc., and therefore $y = 0$ when x has these values.

The first two points at which the curve crosses the axis of x occur when $x = 0$ and when $x = \pi$.

$$\text{Then the area} = \int_0^\pi y\,dx$$
$$= \int_0^\pi e^{-x} \sin x\,dx.$$

To integrate $e^{-x} \sin x$ we must integrate by parts.

$$\int e^{-x} \sin x\,dx = uv - \int v\,du$$
$$\text{where } u = \sin x, \qquad du = \cos x\,dx$$
$$\text{and } dv = e^{-x}\,dx, \quad v = -e^{-x}$$

$$\text{Then} \int e^{-x} \sin x\,dx = -e^{-x} \sin x + \int e^{-x} \cos x\,dx \;\ldots\; (1)$$

$$\text{Also} \int e^{-x} \cos x\,dx = uv - \int v\,du$$
$$\text{where } u = \cos x, \qquad du = -\sin x\,dx$$
$$\text{and } dv = e^{-x}\,dx, \qquad v = -e^{-x}$$

$$\text{Then} \quad \int e^{-x} \cos x\,dx = -e^{-x} \cos x - \int e^{-x} \sin x\,dx \;\ldots\; (2)$$

Solving (1) and (2) for $\int e^{-x} \sin x\,dx$

$$\int e^{-x} \sin x\,dx = -e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \sin x\,dx$$

$$\text{and} \int e^{-x} \sin x\,dx = -\frac{1}{2}\{e^{-x} \sin x + e^{-x} \cos x\}$$

$$\text{Then area} = -\frac{1}{2}\Big[e^{-x} \sin x + e^{-x} \cos x\Big]_0^\pi$$
$$= -\frac{1}{2}\{e^{-\pi} \cos \pi - 1\}$$
$$= -\frac{1}{2}\{-e^{-\pi} - 1\}$$
$$= \frac{1}{2}(e^{-\pi} + 1)$$
$$= 0{\cdot}5216$$

Example 3. Find the area enclosed by the curve $xy = 1$ and the straight line $4y + 3x = 11$.

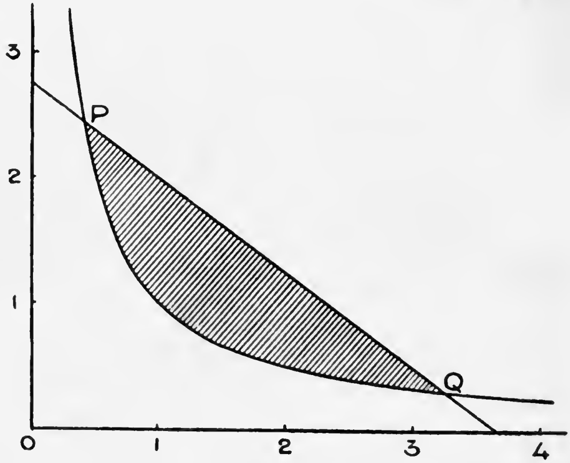


FIG. 46.

The points of intersection P and Q have for their co-ordinates the values of x and y, which satisfy the equations $xy = 1$ and $4y + 3x = 11$.

Then
$$\frac{x}{4}(11 - 3x) = 1$$
$$11x - 3x^2 = 4$$
$$3x^2 - 11x + 4 = 0$$
and
$$x = 0{\cdot}408, \quad x = 3{\cdot}259$$

Area $= \int y\, dx$ where y is the length of ordinate between the line and the curve.

Hence area $= \int \left\{\frac{1}{4}(11 - 3x) - \frac{1}{x}\right\} dx$

$$= \left[\frac{11}{4}x - \frac{3}{8}x^2 - \log_e x\right]_{0{\cdot}408}^{3{\cdot}259}$$

$$= \frac{11}{4}(3{\cdot}259 - 0{\cdot}408) - \frac{3}{8}(3{\cdot}259^2 - 0{\cdot}408^2) - \log_e \frac{3{\cdot}259}{0{\cdot}408}$$

$$= 7{\cdot}840 - 3{\cdot}291 - 2{\cdot}078$$

$$= 1{\cdot}841$$

113. *Surfaces of Revolution.* When an area rotates about an axis in its plane, it describes a surface of revolution, and the property of such a surface is that any section taken perpendicular to the axis of revolution is circular.

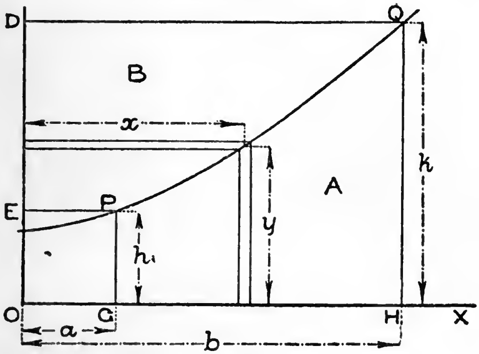


FIG. 47.

If the area under a curve (Fig. 47), bounded on the left and right by the ordinates at $x = a$ and $x = b$ respectively, be made to rotate about the axis of x, it describes a surface of revolution, and any section of this surface taken perpendicular to the axis of x will be circular.

Hence if this area is divided into a very large number of thin strips, each of breadth δx, each strip will describe a thin circular disc.

The volume of an elementary disc $= \pi y^2 \, \delta x$.

The total volume will be obtained by taking the sum of all these elementary discs between the limits $x = a$ and $x = b$.

$$\text{Total volume} = \pi \sum_{x=a}^{x=b} y^2 \, dx,$$

and, when δx is made infinitely small,

$$\text{for the area A,} \qquad V_{\text{OX}} = \pi \int_a^b y^2 \, dx$$

If the area B, that bounded by the abscissæ which correspond to the ordinates at $x = a$ and $x = b$, be made to rotate about the axis of y, another surface of revolution is described, and the section of this surface taken perpendicular to the axis of y will be circular.

Hence if this area is divided into a very large number of thin strips, each of breadth δy, each strip will describe a thin circular disc.

The volume of an elementary disc $= \pi x^2 \, \delta y$.

The total volume will be obtained by taking the sum of all these elementary discs between the limits $y = h$ and $y = k$.

$$\text{Total volume} = \pi \sum_{y=h}^{y=k} x^2\, \delta y$$

and, when δy is made infinitely small,

$$\text{for the area B,} \qquad V_{OY} = \pi \int_h^k x^2\, dy$$

When the area A rotates about the axis of y, the surfaces of revolution described by the rectangle DQHO, the rectangle EPGO and the area EPQD must be considered.

The rectangle DQHO describes a cylinder of volume $\pi b^2 k$, the rectangle EPGO describes a cylinder of volume $\pi a^2 h$, and the area EPQD describes a surface of revolution of volume $\pi \int_h^k x^2\, dy$.

$$\text{Hence for the area A, } V_{OY} = \pi b^2 k - \pi a^2 h - \pi \int_h^k x^2\, dy$$

$$= \pi \left\{ b^2 k - a^2 h - \int_h^k x^2\, dy \right\}$$

When the area B rotates about the axis of x, the surfaces of revolution described by the rectangle DQHO, the rectangle EPGO and the area PQHG must be considered.

The rectangle DQHO describes a cylinder of volume $\pi k^2 b$, the rectangle EPGO describes a cylinder of volume $\pi h^2 a$, and the area PQHG describes a surface of revolution of volume $\pi \int_a^b y^2\, dx$.

$$\text{Hence for the area B, } V_{OX} = \pi k^2 b - \pi h^2 a - \pi \int_a^b y^2\, dx$$

$$= \pi \left\{ k^2 b - h^2 a - \int_a^b y^2\, dx \right\}$$

As an example, let the law of the curve be $y = cx^n$ where c and n are constants.

$$\text{Then for the area A, } V_{OX} = \pi \int_a^b y^2\, dx$$

$$= \pi c^2 \int_a^b x^{2n}\, dx$$

$$= \frac{\pi c^2}{2n+1} \left[x^{2n+1} \right]_a^b$$

$$= \frac{\pi c^2}{2n+1} \{ b^{2n+1} - a^{2n+1} \}$$

Now $h = ca^n$ and $k = cb^n$

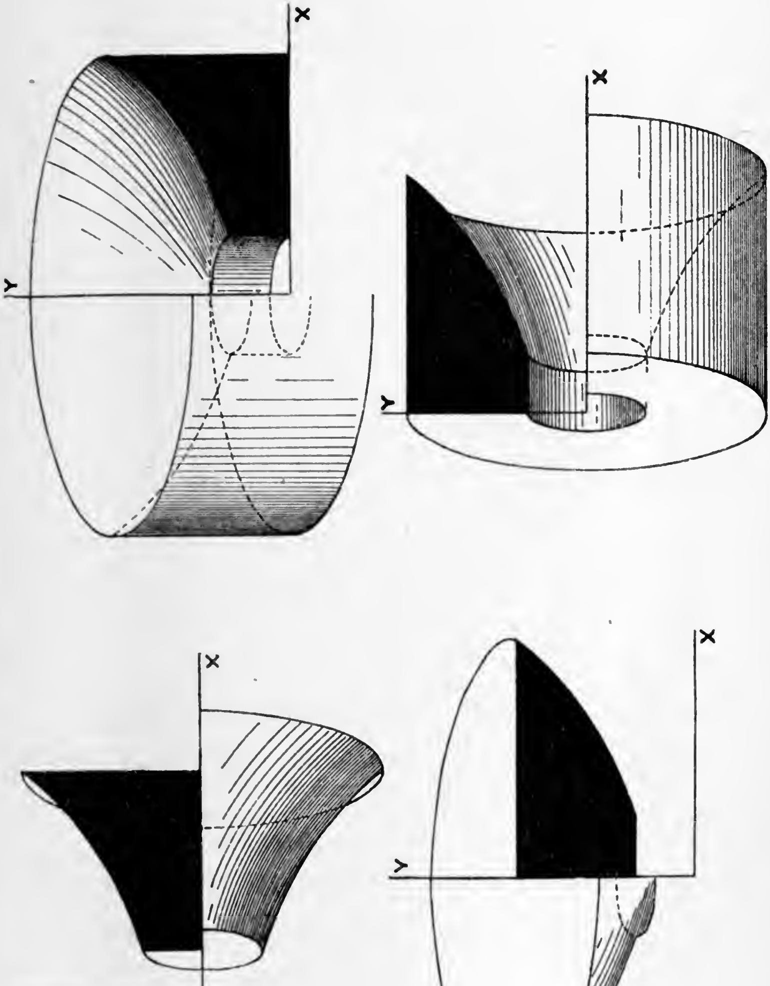


FIG. 48.

For the area B, $V_{OY} = \pi \int_h^k x^2\, dy$

$$= \pi \int_a^b ncx^{n+1}\, dx, \text{ since } dy = ncx^{n-1}\, dx$$

$$= \frac{\pi nc}{n+2}\Big[x^{n+2}\Big]_a^b$$

$$= \frac{\pi nc}{n+2}\{b^{n+2} - a^{n+2}\}$$

For the area A, $V_{OY} = \pi\left\{b^2k - a^2h - \int_h^k x^2\, dy\right\}$

$$= \pi\left\{cb^{n+2} - ca^{n+2} - \frac{nc}{n+2}(b^{n+2} - a^{n+2})\right\}$$

$$= c\pi\left[b^{n+2}\left(1 - \frac{n}{n+2}\right) - a^{n+2}\left(1 - \frac{n}{n+2}\right)\right]$$

$$= \frac{2c\pi}{n+2}\{b^{n+2} - a^{n+2}\}.$$

For the area B, $V_{OX} = \pi\left\{k^2b - h^2a - \int_a^b y^2\, dx\right\}$

$$= \pi\left\{c^2b^{2n+1} - c^2a^{2n+1} - \frac{c^2}{2n+1}(b^{2n+1} - a^{2n+1})\right\}$$

$$= \pi c^2\left[b^{2n+1}\left(1 - \frac{1}{2n+1}\right) - a^{2n+1}\left(1 - \frac{1}{2n+1}\right)\right]$$

$$= \frac{2n\pi c^2}{2n+1}\{b^{2n+1} - a^{2n+1}\}$$

Fig. 48 shows these four different surfaces of revolution.

114. *Example* 1. In the curve $y = 5x^2$, taking A as the area bounded by the ordinates at $x = 1$ and $x = 2$, and B as the area bounded by the abscissæ corresponding to the ordinates at $x = 1$ and $x = 2$, find the volumes of the surfaces of revolution:

(1) when the area A rotates about the axes of x and y respectively;

(2) when the area B rotates about the axes of x and y respectively.

For the area A, $V_{OX} = \pi \int_1^2 y^2\, dx$

$$= 25\pi \int_1^2 x^4\, dx$$

$$= 5\pi\Big[x^5\Big]_1^2$$

$$= 5\pi\{2^5 - 1\}$$

$$= 155\pi = 486{\cdot}8$$

For the area B, $$V_{OY} = \pi \int_5^{20} x^2\,dy$$

$$= \frac{\pi}{5}\int_5^{20} y\,dy$$

$$= \frac{\pi}{10}\Big[y^2\Big]_5^{20}$$

$$= \frac{\pi}{10}\{20^2 - 5^2\}$$

$$= 37{\cdot}5\pi = 117{\cdot}8$$

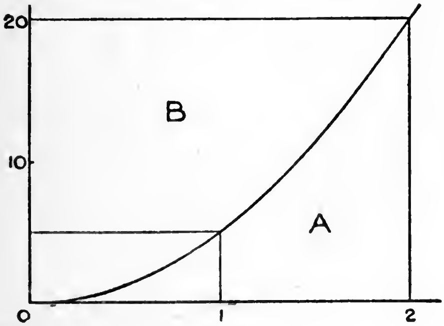


FIG. 49.

For the area A, $$V_{OY} = \pi\left\{20(2)^2 - 5(1)^2 - \int_5^{20} x^2\,dy\right\}$$

$$= \pi\{80 - 5 - 37{\cdot}5\}$$

$$= 37{\cdot}5\pi = 117{\cdot}8$$

For the area B, $$V_{OX} = \pi\left\{2(20)^2 - 1(5)^2 - \int_1^2 y^2\,dx\right\}$$

$$= \pi\{800 - 25 - 155\}$$

$$= 620\pi = 1947$$

Example 2. In the curve $y = e^{-x}$, taking A as the area bounded by the ordinates at $x = 1$ and $x = 2$, and B as the area bounded by the abscissæ corresponding to the ordinates at $x = 1$ and $x = 2$, find the volumes of the surfaces of revolution :

(1) when the area A rotates about the axes of x and y respectively ;

(2) when the area B rotates about the axes of x and y respectively.

For the area A, $$V_{OX} = \pi \int_1^2 y^2 \, dx$$
$$= \pi \int_1^2 e^{-x} \, dx$$
$$= -\pi \Big[e^{-x} \Big]_1^2$$
$$= -\pi \{e^{-2} - e^{-1}\}$$
$$= 0{\cdot}7307$$

For the area B, $$V_{OY} = \pi \int x^2 \, dy$$
$$= -\pi \int_2^1 x^2 e^{-x} \, dx, \text{ since } dy = -e^{-x} \, dx$$

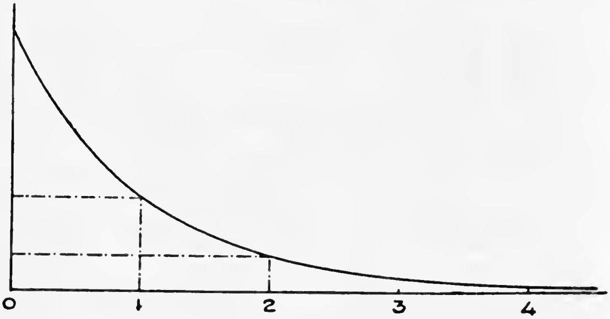


FIG. 50.

To integrate $x^2 e^{-x}$.

$$\int x^2 e^{-x} \, dx = -e^{-x} x^2 + 2 \int x e^{-x} \, dx$$
$$= -e^{-x} x^2 + 2\left\{ -x e^{-x} + \int e^{-x} \, dx \right\}$$
$$= -e^{-x} x^2 - 2x e^{-x} - 2e^{-x}$$
$$= -e^{-x}(x^2 + 2x + 2)$$

Hence for area B, $$V_{OY} = \pi \Big[e^{-x}(x^2 + 2x + 2) \Big]_2^1$$
$$= \pi \{5e^{-1} - 10e^{-2}\}$$
$$= 1{\cdot}528$$

For the area A, $$V_{OY} = \pi \left\{ e^{-2}(2)^2 + \int x^2 \, dy - e^{-1}(1)^2 \right\}$$
$$= \pi \{4e^{-2} + 5e^{-1} - 10e^{-2} - e^{-1}\}$$
$$= \pi \{4e^{-1} - 6e^{-2}\}$$
$$= 2{\cdot}072$$

For the area B, $V_{ox} = \pi\left\{(e^{-1})^2 + \int y^2\,dx - 2(e^{-2})^2\right\}$

$$= \pi\{e^{-2} + e^{-1} - e^{-2} - 2e^{-4}\}$$
$$= \pi(e^{-1} - 2e^{-4})$$
$$= 0{\cdot}9255$$

Examples XIV

Find the values of the following definite integrals:

(1) $\int_0^h (a + bx + cx^2)\,dx$ (2) $\int_{-h}^h (a + bx + cx^2)\,dx$

(3) $\int_0^2 e^x\,dx$ (4) $\int_0^2 e^{-x}\,dx$

(5) $\int_a^{ae} \frac{dx}{x}$ (6) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

(7) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ (8) $\int_0^{\pi} e^x \sin x\,dx$

(9) $\int_0^{\pi} e^x \cos x\,dx$ (10) $\int_0^{\pi} x \sin x\,dx$

(11) $\int_0^{\pi} x^2 \sin x\,dx$ (12) $\int_0^2 (a + bx^{\frac{3}{2}})^2\,dx$

(13) $\int_0^2 (a + be^x)\,dx$ (14) $\int_0^4 \frac{dx}{\sqrt{16 - x^2}}$

(15) $\int_0^4 \sqrt{16 - x^2}\,dx$ (16) $\int_0^3 \frac{dx}{\sqrt{x^2 + 9}}$

(17) $\int_0^3 \sqrt{x^2 + 9}\,dx$ (18) $\int_5^{10} \frac{dx}{\sqrt{x^2 - 25}}$

(19) $\int_5^{10} \sqrt{x^2 - 25}\,dx$ (20) $\int_2^3 \frac{dx}{x^2 - 1}$

(21) $\int_2^3 \frac{dx}{x^2 + 1}$ (22) $\int_0^2 x\sqrt{x^2 + 2}\,dx$

(23) The curve $y = a + bx^{1{\cdot}5}$ passes through the points (1, 1·82) and (4, 5·32). Find a and b. Find the area under the curve between $x = 2$ and $x = 4$.

(24) The curve $y = ae^{bx}$ passes through the points (2, 4·2) and (8, 10·4). Find a and b. Find the area under the curve between $x = 4$ and $x = 8$.

(25) Find the area enclosed by the line $2x + y = 8$ and the curve $xy = 4$.

(26) The curve $y = a + bc^x$ passes through the points (0, 26·62), (1, 35·70), and (2, 49·81). Find the values of a, b, and c. What is the value of the area under the curve between $x = 0$ and $x = 1$?

(27) The curve $y = x^2 - 1$ is cut by the line $y = x + 5$. Find the co-ordinates of the points of intersection and the area between the line and the curve.

(28) Find the area enclosed by the axes of reference and the curve $y = 3x^2 - 8x + 4$. Find also the area enclosed by the curve and the axis of x.

(29) Find the area enclosed by the two curves $y^2 = 4x$ and $x^2 = 4y$.

(30) Find the first two points at which the curve $y = e^x \sin x$ crosses the axis of x, and then find the area bounded by the curve and the axis of x between these points.

(31) Find the area of the loop of the curve $y^2 = x^2(x+1)$.

(32) Find the area enclosed by the axes of reference and the curve $x = y^2 - 9y + 18$. Find also the area enclosed by the curve and the axis of y.

(33) Find the area of the loop of the curve $y^2 = x^2(x + 4)$.

(34) Find the area enclosed by the curve $y = 10x^2 - 41x + 21$ and the axes of reference. Find also the area enclosed by the curve and the axis of x.

(35) The curve $y = 10\sqrt{x}$ rotates about the axis of x, generating a surface of revolution. Find the volume between the sections at $x = 1$ and $x = 9$. (B. of E., 1908.)

(36) The curve $y = 1 + 0{\cdot}2x^2$ rotates round the axis of x, generating a surface of revolution. What is the volume between the sections at $x = 0$ and $x = 10$ (B. of E., 1912.)

(37) If the same part of the curve in Question 36 rotates about the axis of y, what is the volume of the surface of revolution generated?

(38) The curve $y = ax^n$ passes through the points (2, 7·46), (4, 22·72). Find a and n. Let A be the area bounded by the curve, the axis of x and the ordinates at $x = 1$ and $x = 3$, and let B be the area bounded by the curve, the axis of y and the abscissæ corresponding to the ordinates at $x = 1$ and $x = 3$. Find the volumes of the surfaces of revolution generated by the area A rotating about the axes of x and y respectively, and by the area B rotating about the axes of x and y respectively.

(39) The curve $y = a + bx^{1{\cdot}5}$ passes through the points (1, 1·82) and (4, 5·32). Find a and b. Let the curve rotate about the axis of x describing a surface of revolution. Find the volume between the sections at $x = 1$ and $x = 4$.

(40) The curve $y = ae^{bx}$ passes through the points (1, 3·5) and (10, 12·6). Find a and b. The curve rotates about the axis of x, describing a surface of revolution. Find the volume between the sections at $x = 1$ and $x = 10$. (B. of E., 1913.)

(41) Find the volume of the surface of revolution formed by that part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$, rotating about the axis of x.

(42) That part of the line $y + 2x = 3$ intercepted between the axes of reference rotates about the axes of x and y respectively. Find the volumes of the two cones thus formed.

(43) That part of the circle $x^2 + y^2 = 25$ between $x = 1$ and $x = 5$ rotates about the axis of x. Find the volume of the zone of the sphere thus produced.

(44) That part of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ between $x = 0$ and $x = 5$ rotates about the axis of x. Find the volume of the surface of revolution generated. What would be the volume of the surface of revolution generated if the same part of the curve rotates about the axis of y?

(45) That part of the curve $y = x^2 - 8x + 15$ intercepted between the axes of reference, rotates about the axes of x and y respectively. Find the volume of the surface of revolution generated in each case.

CHAPTER XV

115. *The Centre of Gravity or Centroid.*

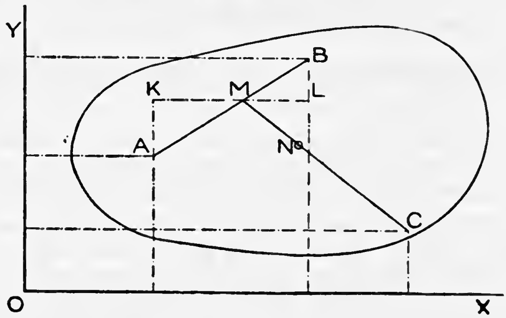


FIG. 51.

Let an irregular area (Fig. 51) be so placed with reference to the axes of x and y, that the co-ordinates of its centroid **P** are $\bar{x}$, $\bar{y}$. Let this area be divided up into a very large number of small areas, a_1, a_2, a_3 . . ., situated at the points A, B, C . . . respectively, the co-ordinates of these points being (x_1, y_1), (x_2, y_2), (x_3, y_3) . . .

The centroid of the two small areas a_1 and a_2 can be taken to be a point M on the line AB, such that

$$a_1\,\mathrm{AM} = a_2\,\mathrm{BM}$$

Let the co-ordinates of this point be $\bar{x}_1$, $\bar{y}_1$.

Since the triangles AKM and BLM are similar

$$\frac{\mathrm{AM}}{\mathrm{BM}} = \frac{\mathrm{KM}}{\mathrm{ML}}$$

$$= \frac{\bar{x}_1 - x_1}{x_2 - \bar{x}_1}$$

But
$$\frac{\mathrm{AM}}{\mathrm{BM}} = \frac{a_2}{a_1}$$

Then
$$\frac{a_2}{a_1} = \frac{\bar{x}_1 - x_1}{x_2 - \bar{x}_1}$$

$$a_2(x_2 - \bar{x}_1) = a_1(\bar{x}_1 - x_1)$$

$$\bar{x}_1(a_1 + a_2) = a_1x_1 + a_2x_2$$

$$\bar{x}_1 = \frac{a_1x_1 + a_2x_2}{a_1 + a_2}$$

The two areas a_1 and a_2 can now be replaced by a single area $(a_1 + a_2)$ situated at the point M, and this can now be combined with the third area a_3, the centroid of this system can be taken to be a point N on the line MC, and if the co-ordinates of N are $\bar{x}_2$, $\bar{y}_2$.

Then
$$\bar{x}_2 = \frac{(a_1 + a_2)\bar{x}_1 + a_3x_3}{(a_1 + a_2) + a_3}$$
$$= \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

This fraction is such that the numerator is the sum of the moments of the areas about the axis OY and the denominator is the sum of the areas. The effect of introducing to the system another area a_4, situated at a point whose co-ordinates are x_4, y_4, is to increase the numerator by the moment of that area, and at the same time to increase the denominator by that area.

This process must be continued until all the small areas which make up the total area are taken into account

and
$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + \ . \ . \ .}{a_1 + a_2 + a_3 + \ . \ . \ .}$$

or
$$A\bar{x} = \Sigma ax.$$

Working with the ordinates y_1, y_2, y_3, etc., it can be shown in a similar manner that
$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + \ . \ . \ .}{a_1 + a_2 + a_3 + \ . \ . \ .}$$

or
$$A\bar{y} = \Sigma ay$$

The relations $A\bar{x} = \Sigma ax$, and $A\bar{y} = \Sigma ay$ enable us to determine the position of the centroid P of the whole area.

116. Let P be the position of the centroid of the irregular area (Fig. 52) the co-ordinates of P being $\bar{x}$, $\bar{y}$.

Let this area rotate about the axis of x, describing a surface of revolution.

Taking the whole area to be built up from a system of small areas a_1, a_2, a_3, a_4, etc., situated at points A, B, C, D, the co-ordinates of these points being (x_1y_1), (x_2y_2), (x_3y_3), (x_4y_4), then each of these small areas will describe an elementary ring, the area a_1 describing a ring of volume $2\pi a_1y_1$, the area a_2 one of volume $2\pi a_2y_2$, and so on. The volume of the whole surface of revolution will be the sum of the volumes of all of these elementary rings.

Then
$$V_{OX} = 2\pi a_1y_1 + 2\pi a_2y_2 + 2\pi a_3y_3 + \ . \ . \ .$$
$$= 2\pi\{a_1y_1 + a_2y_2 + a_3y_3 + \ . \ . \ .\}$$
$$= 2\pi A\bar{y}$$

If the area rotates about the axis of y, another surface of revolution is described, which can be taken as the sum of a large number of elementary rings whose volumes will be $2\pi a_1 x_1$, $2\pi a_2 x_2$, $2\pi a_3 x_3$, and so on.

$$\text{Then } V_{OY} = 2\pi a_1 x_1 + 2\pi a_2 x_2 + 2\pi a_3 x_3 + \ldots$$
$$= 2\pi\{a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots\}$$
$$= 2\pi A\bar{x}$$

These results can be expressed in this way: "That if an area rotates about an axis in its plane, the volume of the surface of revolution generated will be given by the area multiplied by the circumferential distance travelled by the centroid in one revolution."

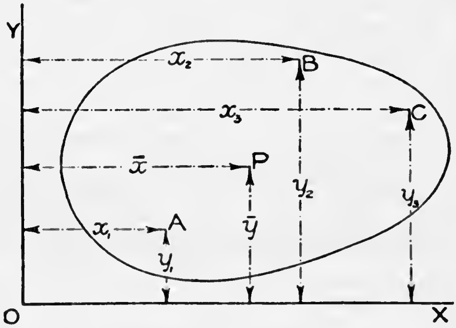


FIG. 52.

Also if $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centroid of a given area

$$\bar{x} = \frac{V_{OY}}{2\pi A}$$

and
$$\bar{y} = \frac{V_{OX}}{2\pi A}$$

These relations enable us to find the position of the centroid, providing we know the area and the volumes of the surfaces of revolution described as that area rotates about the two axes of reference.

117. *Example.* The curve $y = ax^n$ passes through the points (2, 5) and (4, 11). Find the values of the constants a and n. Taking A as the area bounded by the ordinates at $x = 2$ and $x = 4$, and B as the area bounded by the abscissæ corresponding to the ordinates at $x = 2$ and $x = 4$, find the positions of the centroids of the areas A and B.

Now $$5 = a2^n$$
$$11 = a4^n$$
$$2{\cdot}2 = 2^n, \text{ and } n = 1{\cdot}137$$

Also $$a = \frac{5}{2 \cdot 2} = 2 \cdot 273$$

Then $$y = 2 \cdot 273 x^{1 \cdot 137}$$

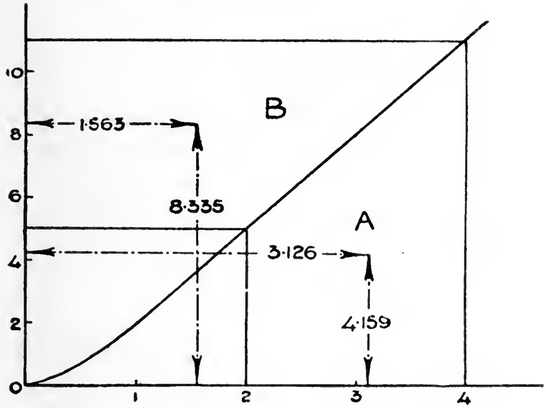


FIG. 53.

$$\text{Area A} = \int_2^4 y \, dx$$

$$= a \int_2^4 x^n \, dx$$

$$= \frac{a}{n+1} \left[x^{n+1} \right]_2^4$$

$$= \frac{2 \cdot 273}{2 \cdot 137} \{ 4^{2 \cdot 137} - 2^{2 \cdot 137} \}$$

$$= 15 \cdot 92$$

$$\text{Area B} = 44 - 10 - 15 \cdot 92$$

$$= 18 \cdot 08$$

For the area A, $$V_{\text{OX}} = \pi \int_2^4 y^2 \, dx$$

$$= \pi a^2 \int_2^4 x^{2n} \, dx$$

$$= \frac{\pi a^2}{2n+1} \left[x^{2n+1} \right]_2^4$$

$$= \frac{\pi (2 \cdot 273)^2}{3 \cdot 274} \{ 4^{3 \cdot 274} - 2^{3 \cdot 274} \}$$

$$= 415 \cdot 9$$

For the area B, $V_{OY} = \pi \int x^2\, dy$

$$= \pi n a \int_2^4 x^{n+1}\, dx, \quad \text{since } dy = n a x^{n+1}\, dx$$

$$= \frac{\pi n a}{n+2}\Big[x^{n+2}\Big]_2^4$$

$$= \frac{\pi \times 1{\cdot}137 \times 2{\cdot}273}{3{\cdot}137}\{4^{3{\cdot}137} - 2^{3{\cdot}137}\}$$

$$= 177{\cdot}5$$

For the area A, $V_{OY} = \pi \times 11 \times 4^2 - \pi \times 5 \times 2^2 - \pi \int x^2\, dy$

$$= 176\pi - 20\pi - 177{\cdot}5$$

$$= 312{\cdot}6$$

For the area B, $V_{OX} = \pi \times 4 \times 11^2 - \pi \times 2 \times 5^2 - \pi \int y^2\, dx$

$$= 484\pi - 50\pi - 415{\cdot}9$$

$$= 947{\cdot}1$$

Finally for the area A, $\bar{x} = \dfrac{V_{OY}}{2\pi A}$

$$= \frac{312{\cdot}6}{2\pi \times 15{\cdot}92}$$

$$= 3{\cdot}126$$

$$\bar{y} = \frac{V_{OX}}{2\pi A}$$

$$= \frac{415{\cdot}9}{2\pi \times 15{\cdot}92}$$

$$= 4{\cdot}159$$

and for the area B, $\bar{x} = \dfrac{V_{OY}}{2\pi A}$

$$= \frac{177{\cdot}5}{2\pi \times 18{\cdot}08}$$

$$= 1{\cdot}563$$

$$\bar{y} = \frac{V_{OX}}{2\pi A}$$

$$= \frac{947{\cdot}1}{2\pi \times 18{\cdot}08}$$

$$= 8{\cdot}335$$

Tabulating these results:

	Area	V_{OX}	V_{OY}	$\bar{x}$	$\bar{y}$
A	15·92	415·9	312·6	3·126	4·159
B	18·08	947·1	177·5	1·563	8·335

118. The position of the centroid of a given area may be found by direct integration. It has already been shown that if $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centroid, then

$$\bar{x} = \frac{\Sigma ax}{A}$$

and $$\bar{y} = \frac{\Sigma ay}{A}$$

or, in other words,

$A\bar{x}$ = sum of the moments of all the elementary areas taken with respect to the axis of y,

and $A\bar{y}$ = sum of the moments of all the elementary areas taken with respect to the axis of x.

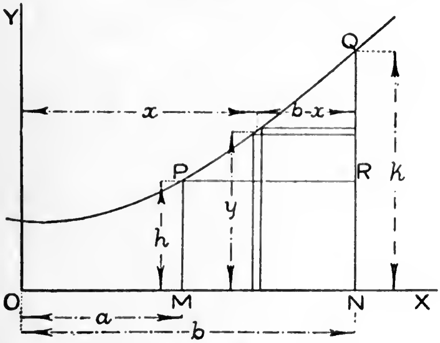


FIG. 54.

Considering the area PMNQ (Fig. 54) bounded by the arc PQ of a curve, the axis of x, and the ordinates at $x = a$ and $x = b$. Let this area be divided into a large number of thin strips each of breadth δx.

Then area of one strip = $y\ \delta x$

Moment of the strip = $xy\ \delta x$, since x is the perpendicular distance of the strip from the axis OY.

Then for the whole area,

$$\text{Total moment} = \sum_{x=a}^{x=b} xy\ \delta x$$

and in the limit when δx is made infinitely small,

$$\text{Total moment} = \int_a^b xy\, dx$$

Hence $$\mathrm{A}\bar{x} = \int_a^b xy\, dx, \quad \text{where } \mathrm{A} = \int_a^b y\, dx$$

If the whole area is divided into a very large number of thin strips each of breadth δy

$$\text{Then area of one strip} = (b - x)\ \delta y$$
$$\text{Moment of the strip} = (b - x)y\ \delta y$$

since y is the perpendicular distance of the strip from the axis **OX**.

Then for the area **PQR**,

$$\text{Total moment} = \sum_{y=h}^{y=k} (b - x)y\ \delta y$$

and in the limit when δy is made infinitely small,

$$\text{Total moment} = \int_h^k (b - x)y\, dy$$

This only gives the moment of the area **PQR**, and to this must be added the moment of the rectangle **PRNM**, in order to obtain the moment of the whole area **PMNQ**.

$$\text{Area of rectangle} = h(b - a)$$

$$\text{Moment of rectangle} = \frac{1}{2}\, h^2(b - a)$$

Hence $$\mathrm{A}\bar{y} = \int_h^k (b - x)y\, dy + \frac{1}{2}h^2(b - a), \quad \text{where } \mathrm{A} = \int_a^b y\, dx$$

The expression for $\bar{y}$ could also be obtained by taking the area as being divided up into thin vertical strips, each of breadth δx.

$$\text{Area of one strip} = y\ \delta x$$

$$\text{Moment of strip} = \frac{1}{2}y^2\ \delta x$$

since $\frac{1}{2}y$ is the perpendicular distance of the centroid of this strip from the axis of x.

Then for the whole area **PMNQ**,

$$\text{Total moment} = \frac{1}{2}\sum_{x=a}^{x=b} y^2\ \delta x$$

and in the limit when δx is made infinitely small,

$$\text{Total moment} = \frac{1}{2}\int_a^b y^2\, dx$$

Hence $$\mathrm{A}\bar{y} = \frac{1}{2}\int_a^b y^2\, dx, \quad \text{where } \mathrm{A} = \int_a^b y\, dx$$

119. Considering the area **PLKQ** (Fig. 55) bounded by the arc **PQ** of the curve, the axis of y, and the abscissæ which correspond to the ordinates at $x = a$ and $x = b$; let this area be divided into a large number of thin strips each of breadth δy.

$$\text{Then area of one strip} = x\,\delta y$$
$$\text{Moment of the strip} = xy\,\delta y$$

since y is the perpendicular distance of the strip from the axis **OX**.

Then for the whole area,

$$\text{Total moment} = \sum_{y=h}^{y=k} xy\,\delta y$$

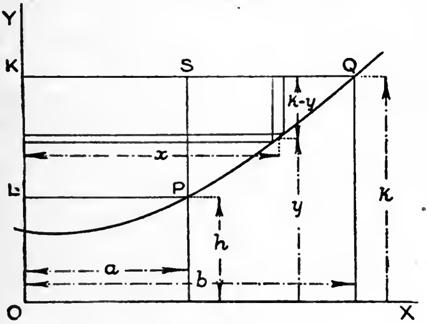


FIG. 55.

and in the limit when δy is made infinitely small,

$$\text{Total moment} = \int_h^k xy\,dy$$

Hence $$B\bar{y} = \int_h^k xy\,dy, \text{ where } B = \int_h^k x\,dy$$

If the whole area is divided into a very large number of thin strips each of breadth δx,

$$\text{Then area of one strip} = (k - y)\,\delta x$$
$$\text{Moment of the strip} = (k - y)x\,\delta x$$

since x is the perpendicular distance of the strip from the axis **OY**.

Then for the area **PQS**,

$$\text{Total moment} = \sum_{x=a}^{x=b} (k - y)x\,dx$$

and in the limit when δx is made infinitely small,

$$\text{Total moment} = \int_a^b (k - y)x\,dx$$

This only gives the moment of the area PQS, and to this must be added the moment of the rectangle PLKS, in order to obtain the moment of the whole area PLKQ.

$$\text{Area of rectangle} = a(k-h)$$

$$\text{Moment of rectangle} = \frac{1}{2}a^2(k-h)$$

Hence $\quad B\bar{x} = \int_a^b (k-y)x\,dx + \frac{1}{2}a^2(k-h)$, where $B = \int_h^k x\,dy$

The expression for $\bar{x}$ could also be obtained by taking the area as being divided into thin horizontal strips each of breadth δy.

$$\text{Area of one strip} = x\,\delta y$$

$$\text{Moment of strip} = \frac{1}{2}x^2\,\delta y$$

since $\frac{1}{2}x$ is the perpendicular distance of the centroid of this strip from the axis of y.

Then for the whole area PLKQ,

$$\text{Total moment} = \frac{1}{2}\sum_{y=h}^{y=k} x^2\,\delta y$$

and in the limit when δy is made infinitely small,

$$\text{Total moment} = \frac{1}{2}\int_h^k x^2\,dy$$

Hence $\quad B\bar{x} = \frac{1}{2}\int_h^k x^2\,dy,\quad$ where $B = \int_h^k x\,dy$

As an illustration of the application of this method of finding the position of the centroid of an area, let us take the curve of the previous example, $y = 2{\cdot}273x^{1{\cdot}137}$, and work, as before, with $x = 2$ and $x = 4$ as the limits for x, and $y = 5$ and $y = 11$ for the corresponding limits of y.

It has already been shown that area A = 15·92 and area B = 18·08.

(1) For the area A, and taking vertical strips each of breadth δx.

$$\text{Moment of strip} = xy\,\delta x$$

$$\begin{aligned}\text{Total moment} &= \int_2^4 xy\,dx \\ &= a\int_2^4 x^{n+1}\,dx \\ &= \frac{a}{n+2}\Big[x^{n+2}\Big]_2^4 \\ &= \frac{2{\cdot}273}{3{\cdot}137}\{4^{3{\cdot}137} - 2^{3{\cdot}137}\} \\ &= 49{\cdot}72\end{aligned}$$

Then $$\bar{x} = \frac{49{\cdot}72}{15{\cdot}92}$$
$$= 3{\cdot}123$$

Taking horizontal strips each of breadth δy,

$$\text{Moment of strip} = (4 - x)y\ \delta y$$

$$\text{Total moment of area PQR} = \int_5^{11} (4 - x)y\, dy$$
$$= 4\int_5^{11} y\, dy - na^2 \int_2^4 x^{2n}\, dx$$
$$= 2\Big[y^2\Big]_5^{11} - \frac{na^2}{2n+1}\Big[x^{2n+1}\Big]_2^4$$

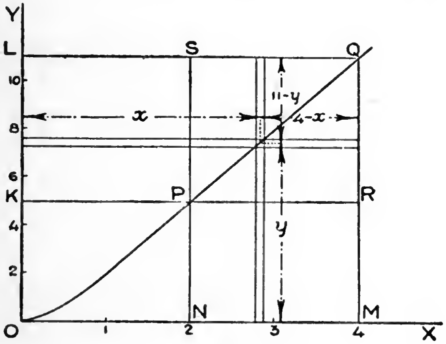


FIG. 56.

$$= 2\{11^2 - 5^2\} - \frac{1{\cdot}137 \times 2{\cdot}273^2}{3{\cdot}274}\{4^{3{\cdot}274} - 2^{3{\cdot}274}\}$$
$$= 192 - 150{\cdot}7$$
$$= 41{\cdot}3$$

$$\text{Moment of rectangle PRMN} = 25$$

For the whole area,

$$\text{Total moment} = 41{\cdot}3 + 25 = 66{\cdot}3.$$

Then $$\bar{y} = \frac{66{\cdot}3}{15{\cdot}92}$$
$$= 4{\cdot}164$$

Or by taking vertical strips,

$$\text{Moment of strip} = \frac{1}{2}\, y^2\ \delta x$$

$$\text{Total moment} = \frac{1}{2}\int_2^4 y^2\,dx$$

$$= \frac{1}{2}a^2\int_2^4 x^{2n}\,dx$$

$$= \frac{a^2}{2(2n+1)}\Big[x^{2n+1}\Big]_2^4$$

$$= \frac{2{\cdot}273^2}{2 \times 3{\cdot}274}\{4^{3\cdot274} - 2^{3\cdot274}\}$$

$$= 66{\cdot}24$$

Then $$\bar{y} = \frac{66{\cdot}24}{15{\cdot}92}$$

$$= 4{\cdot}160$$

(2) For the area B, and taking horizontal strips each of breadth δy.

$$\text{Moment of strip} = xy\,\delta y$$

$$\text{Total moment} = \int_5^{11} xy\,dy$$

$$= na^2\int_2^4 x^{2n}\,dx$$

$$= \frac{na^2}{2n+1}\Big[x^{2n+1}\Big]_2^4$$

$$= \frac{1{\cdot}137 \times 2{\cdot}273^2}{3{\cdot}274}\{4^{3\cdot274} - 2^{3\cdot274}\}$$

$$= 150{\cdot}7$$

Then $$\bar{y} = \frac{150{\cdot}7}{18{\cdot}08}$$

$$= 8{\cdot}335$$

Taking vertical strips each of breadth δx,

$$\text{Moment of strip} = (11 - y)x\,\delta x$$

$$\text{Total moment of area PQS} = \int_2^4 (11 - y)x\,dx$$

$$= 11\int_2^4 x\,dx - a\int_2^4 x^{n+1}\,dx$$

$$= \frac{11}{2}\Big[x^2\Big]_2^4 - \frac{a}{n+2}\Big[x^{n+2}\Big]_2^4$$

$$=\frac{11}{2}\{4^2-2^2\}-\frac{2 \cdot 273}{3 \cdot 137}\{4^{3 \cdot 137}-2^{3 \cdot 137}\}$$

$$= 66 - 49 \cdot 72$$

$$= 16 \cdot 28$$

Moment of rectangle PQLK = 12

For the whole area,

Total moment = 16·28 + 12 = 28·28

Then $$\bar{x} = \frac{28 \cdot 28}{18 \cdot 08}$$

$$= 1 \cdot 564$$

Or by taking horizontal strips,

$$\text{Moment of strip} = \frac{1}{2}x^2\,\delta y$$

$$\text{Total moment} = \frac{1}{2}\int_5^{11} x^2\,dy$$

$$= \frac{na}{2}\int_2^4 x^{n+1}\,dx$$

$$= \frac{na}{2(n+2)}\Big[x^{n+2}\Big]_2^4$$

$$= \frac{1 \cdot 137 \times 2 \cdot 273}{2 \times 3 \cdot 137}\{3^{3 \cdot 137} - 2^{3 \cdot 137}\}$$

$$= 28 \cdot 26$$

Then $$\bar{x} = \frac{28 \cdot 26}{18 \cdot 08}$$

$$= 1 \cdot 563$$

Example. Find the height of the centroid of the area bounded by the curve $y = \sin x$ and the axis of x, between the limits $x = 0$ and $x = \pi$.

$$\text{Area} = \int_0^\pi y\,dx$$

$$= \int_0^\pi \sin x\,dx$$

$$= -\Big[\cos x\Big]_0^\pi$$

$$= -\{\cos \pi - 1\}$$

$$= 2$$

Taking a vertical strip of breadth δx,

$$\text{Moment of strip} = \frac{1}{2} y^2 \, \delta x$$

$$\begin{aligned}
\text{Total moment} &= \frac{1}{2}\int_0^\pi y^2 \, dx \\
&= \frac{1}{2}\int_0^\pi \sin^2 x \, dx \\
&= \frac{1}{4}\int_0^\pi (1 - \cos 2x) \, dx \\
&= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi \\
&= \frac{1}{4}\{\pi - \frac{1}{2}(\sin 2\pi - \sin 0)\} \\
&= \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
\text{Height of centroid} &= \frac{\text{total moment}}{\text{area}} \\
&= \frac{\pi}{8}
\end{aligned}$$

Examples XV

(1) ABCD is a square, 5 inches side; E is a point taken in AD such that AE = 3 inches; F is a point taken in CD such that CF = 1 inch. Find the co-ordinates, with reference to the sides AB and BC as axes, of the centroid of the figure ABCFE.

(2) Prove that the centroid of the trapezium, whose parallel sides are a and b respectively and whose height is h, is situated above the side b at a height of $\frac{h}{3}\,\frac{2a+b}{a+b}$.

(3) Find the height above the base, of a semicircular area of radius a.

(4) Show that the centroid of a hemisphere of radius a is situated on its axis at a distance $\frac{3a}{8}$ from the base.

(5) Show that the centroid of a cone is situated on its axis at a distance $\frac{h}{4}$ from the base, where h is the height of the cone.

(6) A body is composed of a cylinder with a hemispherical base, the radii of the cylinder and hemisphere each being 5 inches. Find the height of the cylinder so that the centroid of the body shall lie in the surface common to the cylinder and hemisphere.

(7) A body is composed of a cone with a hemispherical base, the radii of the base of the cone and the hemisphere each being 5 inches. Find the height of the cone so that the centroid of the body shall lie in the surface common to the cone and hemisphere.

(8) Find the co-ordinates of the centroid of the section (Fig. 73, No. 1) with reference to the axes OX and OY.

(9) Find the height of the centroid of the section (Fig. 73, No. 2) above the base AB.

(10) Find the perpendicular distance of the centroid of the section (Fig. 73, No. 4) from the side AB.

(11) Find the co-ordinates of the centroid of the section (Fig. 73, No. 6) with reference to the axes OX and OY. (The full depth is 6″.)

(12) Find the perpendicular distance of the centroid of the section (Fig. 73, No. 7) from the side AB.

(13) Find the distance from the centre, of the centroid of a quadrant of a circle of radius 4 inches.

(14) ABCD is a square, 8 inches side. From the corner D a quadrant of a circle, 4 inches radius, is cut away. Find the co-ordinates of the centroid of the remainder, with reference to the sides AB and BC as axes.

(15) Find the co-ordinates of the centroid of the area bounded by the curve $y = 3x^2$, the axis of x, and the ordinates at $x = 0$ and $x = 3$.

(16) Find the co-ordinates of the centroid of the area bounded by the curve $y = x^2 - 9x + 18$ and the axes of reference.

(17) Find the co-ordinates of the centroid of the area bounded by the curve $y = x^2 - 9x + 18$ and the axis of x.

(18) Find the co-ordinates of the centroid of the area enclosed by the two curves $y^2 = 8x$ and $x^2 = 8y$.

(19) Find the co-ordinates of the centroid of the area bounded by the curve $y = 5\sqrt{x}$, the axis of x, and the ordinates at $x = 2$ and $x = 4$.

(20) Find the co-ordinates of the centroid of the quadrant of an ellipse, the equation of the ellipse being $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

(21) The curve $y = ax^n$ passes through the points (2, 5·37) and (5, 28·62). Find a and n. Find the co-ordinates of the centroid of the area bounded by the curve, the axis of x, and the ordinates at $x = 2$ and $x = 5$.

(22) Find the first two points at which the curve $y = e^x \sin x$ crosses the axis of x. Find the height above the axis of x, of the centroid of the area bounded by the curve and the axis of x between these points.

(23) The curve $y = x^2 + 5$ is cut by the line $y = 4x + 5$. Find the co-ordinates of the centroid of the area enclosed by the curve and the line.

CHAPTER XVI

120. The moment of inertia of a body about an axis is the sum of the products of the elementary masses which make up the whole body, and the squares of the perpendicular distances of these elementary masses from the given axis.

The moment of inertia of a lamina could be found in the same way by considering the elementary areas which make up the whole area and the squares of the perpendicular distances of these elementary areas from the given axis.

Thus, if there is a system of elementary masses $m_1, m_2, m_3, \ldots$ whose perpendicular distances from a given axis are $x_1, x_2, x_3 \ldots$ respectively, the moment of inertia of that system about the given axis is $m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + \ldots$ or $\Sigma\, mx^2$. Or if there is an area which is made up of a number of elementary areas a_1, $a_2, a_3 \ldots$ whose perpendicular distances from a given axis are $x_1, x_2, x_3 \ldots$ respectively, the moment of inertia of that area about the given axis is $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + \ldots$ or Σax^2.

The two following examples will illustrate how the expression for the moment of inertia is introduced in actual problems:

(1) A body is rotating about a fixed axis with a uniform angular velocity of w radians per second. Taking a small elementary mass m_1 situated at a perpendicular distance x_1 feet from the axis; in 1 second this mass turns through an angle of w radians, and therefore describes a circumferential distance of wx_1 feet.

The circumferential velocity of the mass $m_1 = wx_1$ ft. per sec.

$$\text{Kinetic energy of rotation} = \frac{1}{2}m_1(wx_1)^2 \text{ ft. pdls.}$$

The kinetic energy of rotation of the body will be the sum of the kinetic energies of all the elementary masses $m_1, m_2, m_3 \ldots$ situated at distances $x_1, x_2, x_3 \ldots$ respectively from the axis.

Hence the total kinetic energy

$$= \frac{1}{2}m_1w^2x_1^2 + \frac{1}{2}m_2wx_2^2 + \frac{1}{2}m_3wx_3^2 + \ldots$$

$$= \frac{1}{2}w^2\{m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + \ldots\}$$

$$= \frac{1}{2}\mathrm{I}w^2 \text{ ft. pdls.}$$

where **I** is the moment of inertia of the body about the axis of rotation.

If the whole mass of the body was considered to be concentrated at a point situated at a distance k from the axis, the circumferential velocity of this point $= kw$ ft. per second.

$$\text{Hence the total kinetic energy} = \frac{1}{2}\mathrm{M}k^2w^2 \text{ ft. pdls.}$$

Then $$\mathrm{I} = \mathrm{M}k^2$$

and k is defined as the "radius of gyration" of the body.

(2) Considering the case of a lamina immersed in a liquid to a certain depth. Then the pressure at any depth is ρx, where x is the depth and ρ is the weight of unit volume of the liquid.

Let the whole area be made up of a large number of small elementary areas, $a_1, a_2, a_3 \ldots$ situated at depths $x_1, x_2, x_3 \ldots$ respectively.

$$\text{Pressure at the depth } x_1 = \rho x_1$$

$$\text{Thrust on the area } a_1 = \rho a_1 x_1$$

The resultant thrust on the whole area will be the sum of all of the thrusts on the elementary areas.

$$\begin{aligned}\text{Resultant thrust} &= \rho\{a_1x_1 + a_2x_2 + a_3x_3 + \ldots\} \\ &= \rho\mathrm{A}\bar{x}\end{aligned}$$

where $\bar{x}$ is the depth of the centroid of the area below the surface.

$$\text{Moment of the thrust on the area } a_1 = \rho a_1 x_1^2.$$

The moment of the resultant thrust on the whole area will be the sum of the moments of all the thrusts on the elementary areas

$$\begin{aligned}\text{Total moment} &= \rho\{a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + \ldots\} \\ &= \rho\mathrm{I}\end{aligned}$$

where **I** is the moment of inertia of the area about the free surface.

If $\bar{z}$ is the depth of the "centre of pressure," that is the point at which the resultant thrust acts

Then $$\begin{aligned}\bar{z} &= \frac{\text{Total moment}}{\text{Resultant thrust}} \\ &= \frac{\rho\mathrm{I}}{\rho\mathrm{A}\bar{x}} \\ &= \frac{\mathrm{I}}{\mathrm{A}\bar{x}} \\ &= \frac{k^2}{\bar{x}}\end{aligned}$$

where k is the radius of gyration of the area about the free surface.

121. Let P be a point on a lamina about which there is a small elementary area a. (Fig. 57.)

Let G be the centroid of the lamina, and GX, GY, be two axes at right angles to one another, drawn in the plane of the figure. The co-ordinates of P with reference to the axes GX and GY are x and y.

Let O be any point in the lamina, and the axes OX and OY be drawn parallel to the axes GX and GY respectively.

The moment of inertia of the elementary area a about the axis GY is ax^2, then the moment of inertia of the whole area about that axis is Σax^2,

or $$\mathrm{I}_{GY} = \Sigma ax^2$$

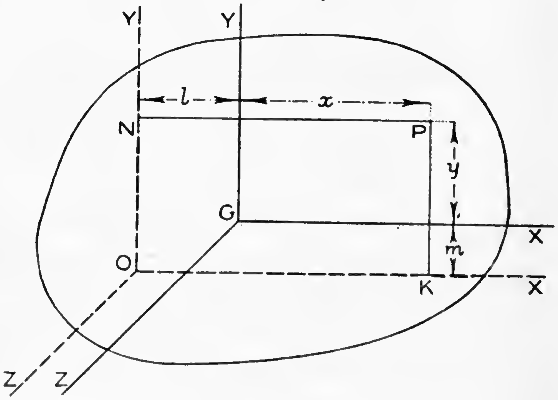


FIG. 57.

Also the moment of inertia of the elementary area a about the axis OY is $a(x + l)^2$ where l is the distance between the two parallel axes GY and OY.

Hence
$$\begin{aligned}
\mathrm{I}_{OY} &= \Sigma a(x + l)^2 \\
&= \Sigma ax^2 + 2\Sigma alx + \Sigma al^2 \\
&= \Sigma ax^2 + 2l\Sigma ax + l^2\Sigma a \\
&= \mathrm{I}_{GY} + 2l\,\mathrm{A}\bar{x} + l^2\mathrm{A} \\
&= \mathrm{I}_{GY} + \mathrm{A}l^2 \quad \ldots \ldots \quad (1)
\end{aligned}$$

Since $\mathrm{A}\bar{x} = 0$ as the axis GY passes through the centroid.

Similarly if m is the distance between the axes OX and GX,

then $$\mathrm{I}_{OX} = \mathrm{I}_{GX} + \mathrm{A}m^2 \quad \ldots \ldots \quad (2)$$

Let GZ and OZ be axes drawn perpendicular to the plane of the lamina through the points G and O respectively.

Then $$I_{OZ} = \Sigma a\, OP^2$$
$$= \Sigma a(OK^2 + ON^2)$$

But $$\Sigma a\, OK^2 = \Sigma ax^2 + Al^2$$

and $$\Sigma a\, ON^2 = \Sigma ay^2 + Am^2$$

Hence $$I_{OZ} = \Sigma a(x^2 + y^2) + A(l^2 + m^2)$$
$$= \Sigma a\, GP^2 + An^2$$
$$= I_{GZ} + An^2 \quad \ldots \ldots \ldots \quad (3)$$

where n is the distance between the axes GZ and OZ.

It is evident from relations 1, 2, and 3, that if the moment of inertia of an area about an axis passing through the centroid is known, the moment of inertia about any parallel axis can be found by adding the term Ad^2 where A is the area and d is the perpendicular distance between the two parallel axes.

Referring again to Fig. 57, we see that

$$I_{GZ} = \Sigma a\, GP^2$$
$$= \Sigma a(x^2 + y^2)$$
$$= \Sigma ax^2 + \Sigma ay^2$$
$$= I_{GY} + I_{GX}$$

Hence, if two axes are drawn at right angles to each other through the centroid and in the plane of the lamina, the sum of the moments of inertia about these axes will give the moment of inertia of the lamina about an axis drawn perpendicular to the plane of the figure and passing through the centroid.

Now, any number of pairs of rectangular axes can be drawn passing through the centroid, in the plane of the figure, but for any one pair the sum of the moments of inertia is constant. Thus, if the moment of inertia about one of these axes is a maximum, then the moment of inertia about the axis at right angles to it must be a minimum.

In dealing with questions on moments of inertia, we can therefore work with three well-defined axes. These axes are mutually perpendicular and pass through the centroid. Two of these axes must be drawn in the plane of the lamina and are such that the moments of inertia about them are greatest and least respectively. These axes are spoken of as the "Principal Axes of Inertia."

A principal axis can also be an axis of symmetry, and if an area has one axis of symmetry, this will give one principal axis, and the other principal axis can be determined by drawing it through the centroid and perpendicular to the axis of symmetry.

122. Let OX and OY be two rectangular axes drawn through the centroid in the plane of a given lamina (Fig. 58), and let

P

OP and OQ be another pair of rectangular axes inclined to the first pair at an angle α.

If G represents the position of an elementary area a, and the co-ordinates of G are x and y with reference to the axes OX and OY, then the moment of inertia of the lamina about OX $= \Sigma a y^2 = X$, and also $I_{OY} = \Sigma a x^2 = Y$

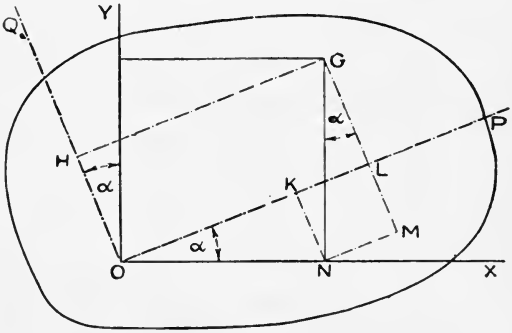


FIG. 58.

With reference to the axes OP and OQ,

$$I_{OP} = \Sigma a \, GL^2 = P$$

and $$I_{OQ} = \Sigma a \, GH^2 = Q$$

Now
$$\begin{aligned} GL &= GM - ML \\ &= GM - KN \\ &= y \cos\alpha - x \sin\alpha \end{aligned}$$

and
$$\begin{aligned} GH &= OK + KL \\ &= OK + NM \\ &= x \cos\alpha + y \sin\alpha \end{aligned}$$

Then
$$\begin{aligned} GL^2 &= (y \cos\alpha - x \sin\alpha)^2 \\ &= y^2 \cos^2\alpha + x^2 \sin^2\alpha - 2\,xy \sin\alpha \cos\alpha \\ &= y^2 \cos^2\alpha + x^2 \sin^2\alpha - xy \sin 2\alpha \end{aligned}$$

Also
$$\begin{aligned} GH^2 &= (x \cos\alpha + y \sin\alpha)^2 \\ &= x^2 \cos^2\alpha + y^2 \sin^2\alpha + xy \sin 2\alpha \end{aligned}$$

Hence
$$\begin{aligned} P = \Sigma a \, GL^2 &= \cos^2\alpha \Sigma a y^2 + \sin^2\alpha \Sigma a x^2 - \sin 2\alpha \Sigma a x y \\ &= X \cos^2\alpha + Y \sin^2\alpha - Z \sin 2\alpha \end{aligned}$$

where $Z = \Sigma a x y$

Also
$$\begin{aligned} Q = \Sigma a \, GH^2 &= \sin^2\alpha \Sigma a y^2 + \cos^2\alpha \Sigma a x^2 + \sin 2\alpha \Sigma a x y \\ &= X \sin^2\alpha + Y \cos^2\alpha + Z \sin 2\alpha \end{aligned}$$

Then
$$\begin{aligned} P - Q &= X(\cos^2\alpha - \sin^2\alpha) - Y(\cos^2\alpha - \sin^2\alpha) - 2Z \sin 2\alpha \\ &= (X - Y) \cos 2\alpha - 2Z \sin 2\alpha \end{aligned}$$

If the axes OP and OQ are so chosen that they are the principal axes of inertia, then, if P is a maximum, Q must be a minimum, and P − Q will be a maximum.

For P − Q to be a maximum $\dfrac{d(P-Q)}{d\alpha} = 0$

That is, $-2(X-Y)\sin 2\alpha - 4Z\cos 2\alpha = 0$

or $Z = -\dfrac{1}{2}(X-Y)\tan 2\alpha$

Thus, if P and Q are the principal moments of inertia,

$$P - Q = (X-Y)\cos 2\alpha + (X-Y)\sin 2\alpha \tan 2\alpha$$

$$(P-Q)\cos 2\alpha = (X-Y)(\cos^2 2\alpha + \sin^2 2\alpha)$$
$$= (X-Y)$$

Then $X - Y = (P-Q)\cos 2\alpha$

Also $X + Y = P + Q$

$$X = P\left\{\frac{1+\cos 2\alpha}{2}\right\} + Q\left\{\frac{1-\cos 2\alpha}{2}\right\}$$
$$= P\cos^2\alpha + Q\sin^2\alpha$$

Also $Y = P\left\{\dfrac{1-\cos 2\alpha}{2}\right\} + Q\left\{\dfrac{1+\cos 2\alpha}{2}\right\}$

$$= P\sin^2\alpha + Q\cos^2\alpha$$

Therefore if P and Q are the greatest and least moments of inertia respectively, taken about a pair of rectangular axes which pass through the centroid, the moment of inertia I about any other axis passing through the centroid is given by the relation $I = P\cos^2\alpha + Q\sin^2\alpha$, where α is the angle between that axis and the axis of greatest moment of inertia.

123. *The Momental Ellipse.* Let OX and OY be the principal axes of a plane figure and P and Q be the principal moments of inertia, P being greater than Q. Let OR be any axis making an angle θ with OX. If I is the moment of inertia of the figure about OR,

Then $I = P\cos^2\theta + Q\sin^2\theta$

Let the lengths OP, OR, and OQ be measured, to the same scale, along the axes OX, OR, and OY respectively, such that

$$OP = p = \sqrt{\frac{A}{P}}$$

$$OR = r = \sqrt{\frac{A}{I}}$$

and $OQ = q = \sqrt{\dfrac{A}{Q}}$

A being the area of the figure.

Since $$r = \sqrt{\frac{A}{I}}$$

$$\frac{1}{r^2} = \frac{I}{A}$$

$$= \frac{P}{A}\cos^2\theta + \frac{Q}{A}\sin^2\theta$$

$$= \frac{\cos^2\theta}{p^2} + \frac{\sin^2\theta}{q^2}$$

Then $$\frac{r^2\cos^2\theta}{p^2} + \frac{r^2\sin^2\theta}{q^2} = 1$$

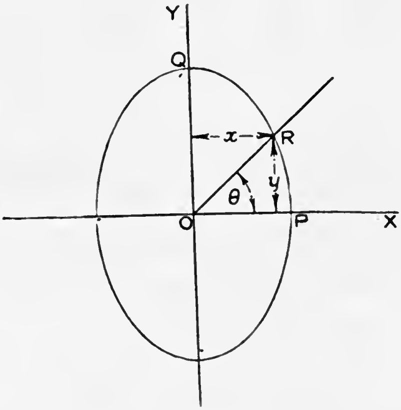


FIG. 59.

Let the co-ordinates of the point R, with reference to the axes OX and OY, be x and y.

Then $$r\cos\theta = x \quad \text{and } r\sin\theta = y$$

Therefore $$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1.$$

This is the equation to an ellipse whose semi-axes are p and q, and the point R must lie on this ellipse. It follows, therefore, that if P and Q, the principal moments of inertia, are known, then $p = \sqrt{\frac{A}{P}}$ and $q = \sqrt{\frac{A}{Q}}$, the semi-axes of the momental ellipse can be calculated. This ellipse can then be drawn, and if any radius be drawn and its length measured to the same scale as p and q, the moment of inertia about that radius $= \frac{A}{r^2}$.

If the principal moments of inertia are equal, then $P = Q$, and consequently $p = q$. The momental ellipse becomes a circle of radius p, and therefore the moment of inertia about any axis drawn through the centroid in the plane of the figure is constant.

124. *The Rectangle.* Let OX and OY be two axes drawn parallel to the sides and passing through the centroid. These are axes of symmetry, and are consequently principal axes of inertia.

Consider an elementary strip of breadth δy drawn parallel to the axis OX and at a distance y from it.

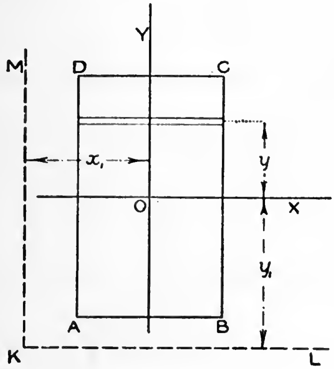


FIG. 60.

The moment of inertia of this strip about $OX = ay^2 \ \delta y$ where a is the breadth of the rectangle.

$$\text{Moment of inertia of rectangle about OX} = \sum_{y=-\frac{b}{2}}^{y=\frac{b}{2}} ay^2 \ \delta y.$$

$$\text{Then } I_{OX} = a \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 \, dy$$

$$= \frac{a}{3}\Big[y^3 \Big]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$= \frac{a}{3} \frac{b^3}{4}$$

$$= \frac{ab^3}{12}$$

By dividing the rectangle into strips of breadth δx, drawn parallel to the axis OY, it can be shown in a similar manner that $I_{OY} = \dfrac{ba^3}{12}$

Let KL be another axis drawn parallel to OX and at a distance y_1 from it.

Then
$$I_{KL} = \frac{ab^3}{12} + ab\, y_1^2$$
$$= ab\left\{\frac{b^2}{12} + y_1^2\right\}$$

Putting $y_1 = \frac{b}{2}$, this becomes the moment of inertia about the side AB, and
$$I_{AB} = ab\left\{\frac{b^2}{12} + \frac{b^2}{4}\right\}$$
$$= \frac{ab^3}{3}$$

Let KM be an axis drawn parallel to OY and at a distance x_1 from it.

Then
$$I_{KM} = \frac{ba^3}{12} + ab\, x_1^2$$
$$= ab\left\{\frac{a^2}{12} + x_1^2\right\}$$

Putting $x_1 = \frac{a}{2}$, this becomes the moment of inertia about the side AD, and
$$I_{AD} = ab\left\{\frac{a^2}{12} + \frac{a^2}{4}\right\}$$
$$= \frac{ba^3}{3}$$

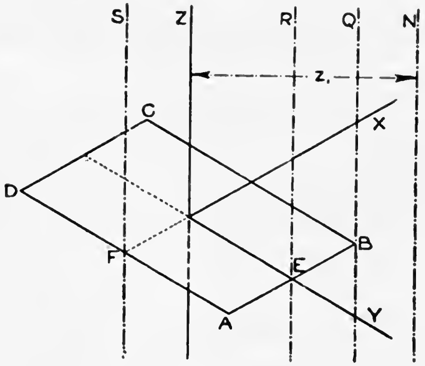


FIG. 61.

The figure shows the rectangle placed in isometric projection, OZ being the axis drawn through the centroid perpendicular to the plane of the figure.

Then
$$I_{OZ} = I_{OX} + I_{OY}$$
$$= \frac{ab^3}{12} + \frac{ba^3}{12}$$
$$= \frac{ab}{12}\{a^2 + b^2\}$$

Let PN be an axis drawn parallel to OZ at a distance z_1 from it.

Then
$$I_{PN} = I_{OZ} + ab\, z_1^2$$
$$= ab\left\{\frac{a^2}{12} + \frac{b^2}{12} + z_1^2\right\}$$

Putting $z_1 = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$, this becomes the moment of inertia about an axis BQ which is perpendicular to the plane of the figure and passes through a corner.

$$I_{BQ} = ab\left\{\frac{a^2}{12} + \frac{b^2}{12} + \frac{a^2}{4} + \frac{b^2}{4}\right\}$$
$$= \frac{ab}{3}\{a^2 + b^2\}$$

If E and F are the mid points of the sides AB and AD respectively, and ER and FS are axes drawn perpendicular to the plane of the figure through these points:

Putting $z_1 = \frac{b}{2}$, $\quad I_{ER} = ab\left\{\frac{a^2}{12} + \frac{b^2}{12} + \frac{b^2}{4}\right\}$
$$= \frac{ab}{12}\{a^2 + 4b^2\}$$

Putting $z_1 = \frac{a}{2}$, $\quad I_{FS} = ab\left\{\frac{a^2}{12} + \frac{b^2}{12} + \frac{a^2}{4}\right\}$
$$= \frac{ab}{12}\{4a^2 + b^2\}$$

125. *Example* 1. Draw the momental ellipse for a rectangle 2″ × 5″ and use it to find the moments of inertia about axes which pass through the centroid and make angles of 30°, 45°, and 60° respectively, with the smaller side of the rectangle.

Then
$$P = I_{OX} = \frac{1}{12} \times 2 \times 5^3$$
$$= 20{\cdot}83$$

and
$$Q = I_{OY} = \frac{1}{12} \times 5 \times 2^3$$
$$= 3{\cdot}333$$

Also $$p = \sqrt{\frac{\mathrm{A}}{\mathrm{P}}} = 0{\cdot}6928$$

and $$q = \sqrt{\frac{\mathrm{A}}{\mathrm{Q}}} = 1{\cdot}732$$

These are the semi-minor and semi-major axes of the momental ellipse.

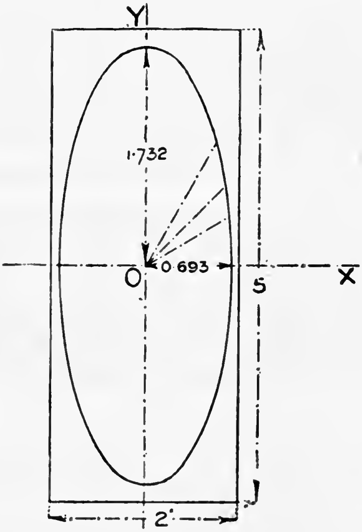


FIG. 62.

(1) $r_{30°} = 0{\cdot}78 \qquad \mathrm{I}_{30°} = \dfrac{10}{(0{\cdot}78)^2} = 16{\cdot}44$

(2) $r_{45°} = 0{\cdot}91 \qquad \mathrm{I}_{45°} = \dfrac{10}{(0{\cdot}91)^2} = 12{\cdot}08$

(3) $r_{60°} = 1{\cdot}14 \qquad \mathrm{I}_{60°} = \dfrac{10}{(1{\cdot}14)^2} = 7{\cdot}69$

These results may be verified by calculation for

$$\mathrm{I}_\alpha = \mathrm{P}\cos^2\alpha + \mathrm{Q}\sin^2\alpha$$

$$\mathrm{I}_{30°} = 20{\cdot}83\cos^2 30° + 3{\cdot}333\sin^2 30° = 16{\cdot}45$$

$$\mathrm{I}_{45°} = 20{\cdot}83\cos^2 45° + 3{\cdot}333\sin^2 45° = 12{\cdot}09$$

$$\mathrm{I}_{60°} = 20{\cdot}83\cos^2 60° + 3{\cdot}333\sin^2 60° = 7{\cdot}718$$

Example 2. For the given section find the greatest and least moments of inertia. Draw the momental ellipse for the section and use it to find the moment of inertia about the axis OR.

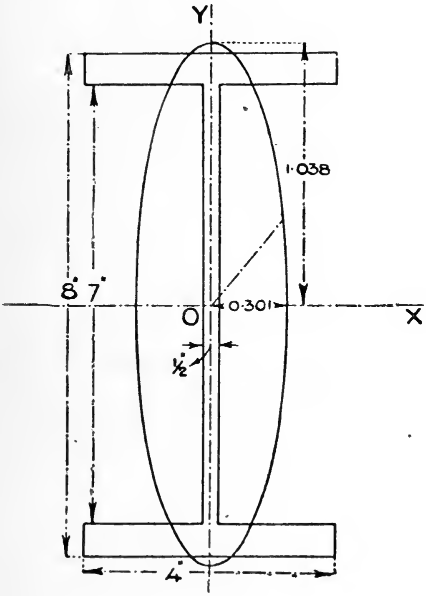


FIG. 63.

$$I_{OY} = 2 \times \frac{1}{12} \times \frac{1}{2} \times 4^3 + \frac{1}{12} \times 7 \times \left(\frac{1}{4}\right)^3$$

$$= 5{\cdot}342 = Q$$

$$I_{OX} = \frac{1}{12} \times \frac{1}{4} \times 7^3 + 2\left\{\frac{1}{12} \times 4 \times \left(\frac{1}{2}\right)^3 + 2 \times \left(\frac{15}{4}\right)^2\right\}$$

$$= 63{\cdot}48 = P$$

The area of the section = 5·75

and $$p = \sqrt{\frac{5{\cdot}75}{63{\cdot}48}} = 0{\cdot}301$$

$$q = \sqrt{\frac{5{\cdot}75}{5{\cdot}342}} = 1{\cdot}038$$

Also $$r_{50^\circ} = 0{\cdot}445, \quad I_{50^\circ} = \frac{5{\cdot}75}{0{\cdot}445^2} = 29{\cdot}03$$

126. *The Parallelogram and the Triangle.*

(1) Let a be the base and h the height of a parallelogram, and let OX be an axis drawn parallel to the base through the centroid.

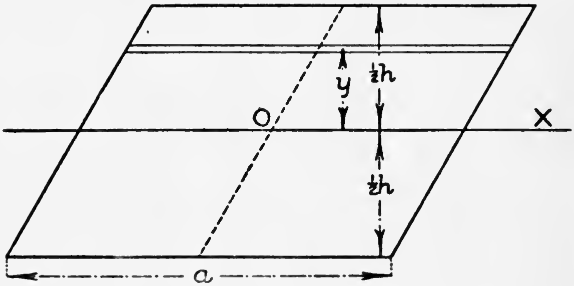


FIG. 64.

Consider an elementary strip of breadth δy drawn parallel to the axis OX and at a distance y from it.

The moment of inertia of this strip about OX $= ay^2\,\delta y$.

The moment of inertia of the parallelogram about OX

$$= \sum_{y=-\frac{h}{2}}^{y=\frac{h}{2}} ay^2\,dy.$$

$$I_{OX} = a\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\,dy$$

$$= \frac{a}{3}\Big[y^3\Big]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{ah^3}{12}$$

(2) Let a be the base and h the height of a triangle.

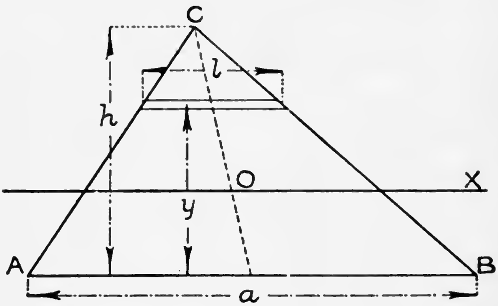


FIG. 65.

Consider an elementary strip of breadth δy drawn parallel to the base at a distance y from it.

If l is the length of this strip,

$$\frac{l}{h-y} = \frac{a}{h}$$

and
$$l = \frac{a}{h}(h-y)$$

$$\text{Area of strip} = l\ \delta y = \frac{a}{h}(h-y)\ \delta y$$

Moment of inertia of the strip about AB $= \dfrac{a}{h}(h-y)y^2\ \delta y$

For the whole triangle
$$I_{AB} = \frac{a}{h}\int_0^h (h-y)y^2\, dy$$
$$= \frac{a}{h}\left[\frac{1}{3}hy^3 - \frac{1}{4}y^4\right]_0^h$$
$$= \frac{a}{h}\,\frac{h^4}{12}$$
$$= \frac{ah^3}{12}$$

Let OX be an axis drawn parallel to the base and passing through the centroid of the triangle.

Then
$$I_{AB} = I_{OX} + \frac{ah}{2}\left(\frac{h}{3}\right)^2$$
$$I_{OX} = \frac{ah^3}{12} - \frac{ah^3}{18}$$
$$= \frac{ah^3}{36}$$

Example. Find the principal moments of inertia of the section of an angle iron $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$, and draw the momental ellipse for the section.

Now
$$\frac{13}{4}\bar{y} = \frac{7}{4} \times \frac{1}{4} + \frac{3}{2} \times 2 = 3\frac{7}{16}$$

Then
$$\bar{y} = \frac{55}{16} \times \frac{4}{13} = \frac{55}{52} = 1{\cdot}058''$$

The centroid O is evidently situated on the axis OX, and its perpendicular distances from the sides AB and BC are each $1{\cdot}058''$.

$$\text{Now } I_{AB} = \frac{1}{12} \times 3 \times \left(\frac{1}{2}\right)^3 + \frac{3}{2} \times \left(\frac{1}{4}\right)^2 + \frac{1}{12} \times \frac{1}{2} \times \left(\frac{7}{2}\right)^3 + \frac{7}{4} \times \left(\frac{7}{4}\right)^2$$

$$= 7{\cdot}271$$

$$I_{OL} = 7{\cdot}271 - 3{\cdot}25 \times (1{\cdot}058)^2$$

$$= 3{\cdot}632$$

Evidently $\qquad I_{OK} = I_{OL} = 3{\cdot}632$

The axis OX, being an axis of symmetry, is a principal axis, and by means of lines drawn parallel to OX the section can be divided into triangles and parallelograms.

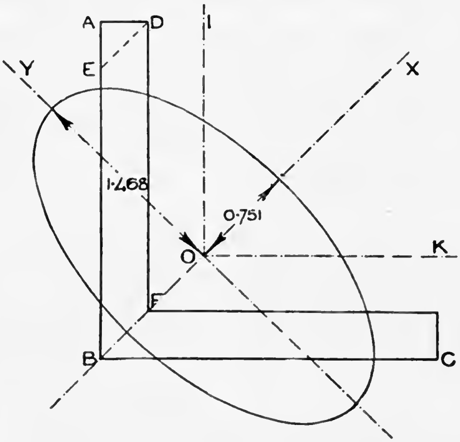


FIG. 66.

For the triangle ADE,

$$\text{Base} = \frac{1}{\sqrt{2}}, \quad \text{height} = \frac{1}{2\sqrt{2}}, \quad \text{area} = \frac{1}{8}$$

$$\text{Distance of centroid from axis OX} = \frac{3}{\sqrt{2}} + \frac{1}{6\sqrt{2}} = \frac{19}{6\sqrt{2}}$$

$$I_{OX} = \frac{1}{36} \times \frac{1}{\sqrt{2}} \times \left(\frac{1}{2\sqrt{2}}\right)^3 + \frac{1}{8} \times \left(\frac{19}{6\sqrt{2}}\right)^2$$

$$= \frac{1}{1152} + \frac{361}{576}$$

$$= 0{\cdot}628$$

For the parallelogram EDFB

$$\text{Base} = \frac{1}{\sqrt{2}}, \quad \text{height} = \frac{3}{\sqrt{2}}, \quad \text{area} = \frac{3}{2}$$

Distance of centroid from axis OX $= \dfrac{3}{2\sqrt{2}}$

$$I_{OX} = \frac{1}{12} \times \frac{1}{\sqrt{2}} \times \left(\frac{3}{\sqrt{2}}\right)^3 + \frac{3}{2} \times \left(\frac{3}{2\sqrt{2}}\right)^2$$
$$= \frac{9}{16} + \frac{27}{16}$$
$$= 2{\cdot}25$$

For the whole section $\quad I_{OX} = 2\{2{\cdot}25 + 0{\cdot}628\}$
$$= 5{\cdot}756$$

Let the axis OY be drawn perpendicular to OX.

Then $\quad I_{OY} + I_{OX} = I_{OL} + I_{OK}$

and $\quad I_{OY} = 7{\cdot}264 - 5{\cdot}756$
$$= 1{\cdot}508$$

Hence the principal moments of inertia are 5·756 and 1·508.
To draw the momental ellipse,

$$p = \sqrt{\frac{3{\cdot}25}{5{\cdot}756}} = 0{\cdot}751$$

$$q = \sqrt{\frac{3{\cdot}25}{1{\cdot}508}} = 1{\cdot}468$$

and these are the semi-minor and semi-major axes to be measured along OX and OY respectively.

127. *The Circle.* Consider an elementary ring of width δx bounded by concentric circles of radii x and $(x + \delta x)$ respectively.

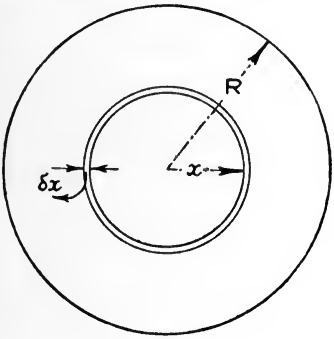


FIG. 67.

Area of the ring $= 2\pi x\ \delta x$.

Moment of inertia of the ring about an axis OZ which is perpendicular to the plane of the circle and passes through the centre $= 2\pi x^3\ \delta x$.

For the whole circle $I_{OZ} = 2\pi\int_0^R x^3\,dx$

$$= \frac{1}{2}\pi R^4$$

$$= \frac{1}{2}AR^2, \quad \text{where A is the area.}$$

For a circular ring, if R_2 and R_1 be the external and internal radii respectively, $I_{OZ} = 2\pi\int_{R_1}^{R_2} x^3\,dx$

$$= \frac{1}{2}\pi\{R_2^4 - R_1^4\}$$

$$= \frac{1}{2}A(R_2^2 + R_1^2), \quad \text{where A is the area.}$$

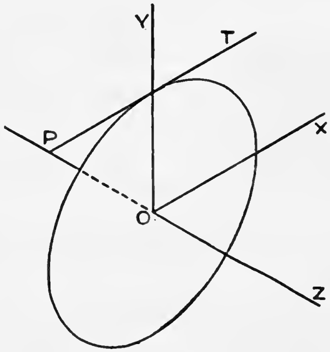


FIG. 68.

Fig. 68 shows the circle placed in isometric projection, OX and OY being two axes drawn at right angles to one another in the plane of the figure. Because the circle is symmetrical about any diameter, the moment of inertia is the same for all diameters.

Hence $I_{OX} = I_{OY}$

But $I_{OZ} = I_{OX} + I_{OY}$

$$= 2I_{OX} \quad \text{or } 2I_{OY}$$

Therefore $I_{OX} = I_{OY} = \frac{1}{2} I_{OZ}$

$$= \frac{1}{4}\pi R^4$$

$$= \frac{1}{4}AR^2$$

Let PT be a tangent to the circle.

Then $$I_{PT} = I_{OX} + AR^2$$
$$= \frac{5}{4} AR^2$$

For a circular ring, $$I_{OX} = I_{OY} = \frac{1}{2} I_{OZ}$$
$$= \frac{1}{4} \pi(R_2^4 - R_1^4)$$
$$= \frac{1}{4} A(R_2^2 + R_1^2)$$

128. *The Rectangular Prism.* Consider an elementary slice of thickness δy, cut by planes parallel to the base.

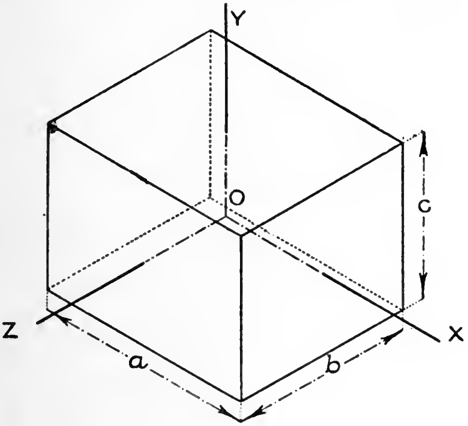


FIG. 69.

Mass of slice $= mab\ \delta y$ where m is the mass of unit volume.
Moment of inertia of the slice about the axis OY

$$= \frac{1}{12} mab\ (a^2 + b^2)\ \delta y$$

For the whole prism $$I_{OY} = \frac{1}{12} mab(a^2 + b^2) \int_0^c dy$$
$$= \frac{1}{12} mabc(a^2 + b^2)$$
$$= \frac{1}{12} M(a^2 + b^2),$$ where M is the mass of the prism.

It is evident that $$I_{OX} = \frac{1}{12} M(b^2 + c^2)$$

and $$I_{OZ} = \frac{1}{12} M(a^2 + c^2)$$

129. *The Cylinder.* Consider an elementary slice of thickness δx, cut by planes parallel to the base.

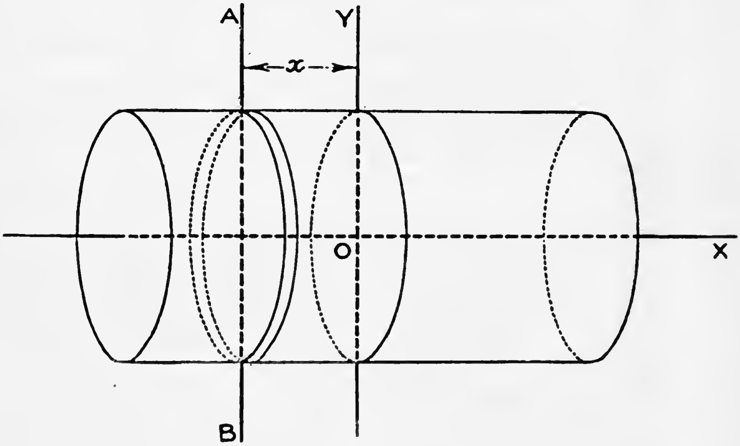


FIG. 70.

Mass of slice $= \pi m R^2 \, \delta x$

Moment of inertia of the slice about the axis OX $= \frac{1}{2} \pi m R^4 \, \delta x$

For the whole cylinder $$I_{OX} = \frac{1}{2} \pi m R^4 \int_{-\frac{l}{2}}^{\frac{l}{2}} dx$$

$$= \frac{1}{2} \pi m R^4 \Big[x \Big]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{1}{2} \pi m R^4 l$$

$$= \frac{1}{2} M R^2$$ where M is the mass.

If the cylinder is hollow, R_2 and R_1 being the external and internal radii respectively.

$$I_{OX} = \frac{1}{2} M \left(R_2^2 + R_1^2 \right)$$

The moment of inertia o' the elementary disc about the axis AB $= \frac{1}{4} \pi m R^4 \, \delta x$.

Hence the moment of inertia of the elementary disc about the axis OY $= \frac{1}{4}\pi m R^4\,\delta x + \pi m R^2 x^2\,\delta x$.

For the whole cylinder $I_{OY} = \frac{1}{4}\pi m R^4 \int_{-\frac{l}{2}}^{\frac{l}{2}} dx + \pi m R^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\,dx$

$$= \frac{1}{4}\pi m R^4 l + \pi m R^2 \left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{1}{4}\pi m R^4 l + \frac{1}{12}\pi m R^2 l^3$$

$$= M\left\{\frac{R^2}{4} + \frac{l^2}{12}\right\}$$

130. *The Cone.* Considering an elementary slice of thickness δy situated at a distance y from the base of the cone.

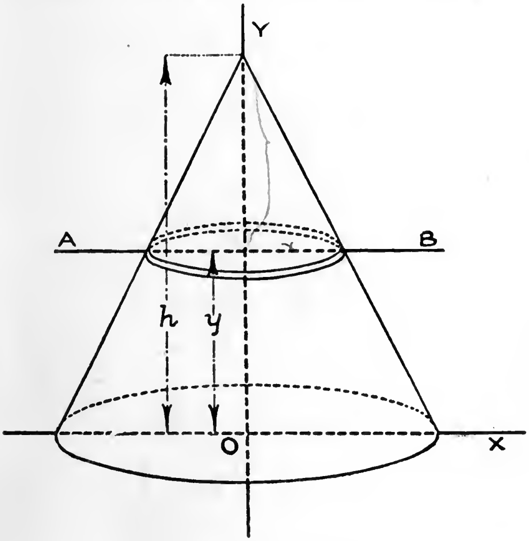


FIG. 71.

If x is the radius of the slice,

Then $$\frac{x}{h-y} = \frac{R}{h}$$

and $$x = \frac{R}{h}(h-y)$$

Mass of the slice $= \pi m x^2\,\delta y$.

Moment of inertia of the slice about OY $= \frac{1}{2}\pi m x^4\,\delta y$

For the whole cone

$$I_{OY} = \frac{1}{2}\pi m \int_0^h x^4\,dy$$

$$= \frac{1}{2}\pi m \frac{R^4}{h^4}\int_0^h (h-y)^4\,dy$$

$$= \frac{\pi m R^4}{2h^4}\left[-\frac{(h-y)^5}{5}\right]_0^h$$

$$= \frac{\pi m R^4}{2h^4} \times \frac{h^5}{5}$$

$$= \frac{1}{10}\pi m R^4 h$$

$$= \frac{3}{10} MR^2, \text{ where M is the mass of the cone.}$$

Moment of inertia of the slice about the axis AB $= \frac{1}{4}\pi m x^4\,\delta y$

Moment of inertia of the slice about the axis OX $= \frac{\pi}{4} m x^4\,\delta y + \pi m x^2 y^2\,\delta y$.

For the whole cone

$$I_{OX} = \frac{\pi m}{4}\int_0^h x^4\,dy + \pi m \int_0^h x^2 y^2\,dy$$

$$= \frac{\pi m R^4}{4h^4}\int_0^h (h-y)^4\,dy + \frac{\pi m R^2}{h^2}\int_0^h y^2(h-y)^2\,dy$$

$$= \frac{\pi m R^4}{4h^4}\int_0^h (h-y)^4\,dy + \frac{\pi m R^2}{h^2}\int_0^h (h^2y^2 - 2hy^3 + y^4)dy$$

$$= \frac{\pi m R^4}{4h^4}\left[-\frac{(h-y)^5}{5}\right]_0^h + \frac{\pi m R^2}{h^2}\left[\frac{1}{3}h^2y^3 - \frac{1}{2}hy^4 + \frac{1}{5}y^5\right]_0^h$$

$$= \frac{\pi m R^4}{4h^4} \times \frac{h^5}{5} + \frac{\pi m R^2}{h^2} \times \frac{h^5}{30}$$

$$= \frac{\pi m R^2 h}{10}\left\{\frac{R^2}{2} + \frac{h^2}{3}\right\}$$

$$= \frac{3}{10} M \left\{\frac{R^2}{2} + \frac{h^2}{3}\right\}$$

131. *The Sphere.* Considering an elementary slice of thickness δy, situated at a distance y from a horizontal plane passing through the centre of the sphere.

If x is the radius of this slice,
Then $x = \sqrt{R^2 - y^2}$ where R is the radius of the sphere.
Mass of the slice $= \pi m x^2 \, \delta y$.

Moment of inertia of the slice about the axis OY $= \frac{1}{2} \pi m x^4 \, dy$

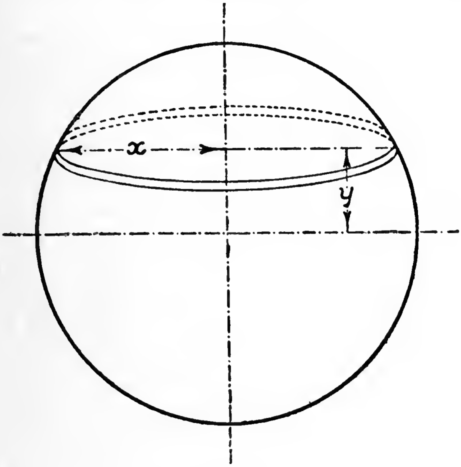


FIG. 72.

For the whole sphere $I_{OY} = \frac{\pi m}{2} \int_{-R}^{R} x^4 \, dy$

$$= \frac{\pi m}{2} \int_{-R}^{R} (R^2 - y^2)^2 \, dy$$

$$= \frac{\pi m}{2} \left[R^4 y - \frac{2}{3} R^2 y^3 + \frac{1}{5} y^5 \right]_{-R}^{R}$$

$$= \frac{\pi m}{2} \left[2R^5 - \frac{4}{3} R^5 + \frac{2}{5} R^5 \right]$$

$$= \frac{\pi m}{2} \times \frac{16R^5}{15}$$

$$= \frac{8}{15} \pi m R^5$$

$= \frac{2}{5} MR^2$, where M is the mass of the sphere.

If the sphere is hollow, R_2 and R_1 being the external and internal radii respectively. The moment of inertia can be taken as the difference between the moments of inertia of the external and internal spheres.

$$I_{OY} = \frac{2}{5}M_2R_2^2 - \frac{2}{5}M_1R_1^2$$

EXAMPLES XVI

(1) A rectangular lamina 5 ft. × 7 ft. is immersed in water with its plane vertical and the smaller edge horizontal; the centroid of the lamina is 12 ft. below the surface of the water. Find the depth of the centre of pressure.

(2) A circular lamina, 3 ft. radius, is immersed in water with its plane vertical and its centre 7 ft. below the surface of the water. Find the depth of the centre of pressure.

(3) Find the moment of inertia of a trapezium whose parallel sides are a and b respectively and whose height is h.

(1) About the side b

(2) About an axis parallel to the side b and passing through the centroid.

(4) A trapezoidal lamina whose parallel sides are 8 ft. and 5 ft. respectively and whose height is 6 ft. is immersed in water with its plane vertical and its parallel sides horizontal. The larger of the two parallel sides is situated at a depth of 10 ft. below the surface of the water. Find the depth of the centre of pressure when the smaller of the two parallel sides is situated (1) below the larger, and (2) above the larger.

(5) Find the moments of inertia of the section, Fig. 73, No. 1, about axes parallel to OX and OY respectively and passing through the centroid.

(6) Find the greatest and least moments of inertia of the section, Fig. 73, No. 2.

(7) Find the principal moments of inertia of the section, Fig. 73, No. 3, and find the lengths of the major and minor axes of the momental ellipse.

(8) Find the greatest and least moments of inertia of the section, Fig. 73, No. 4, and then find the moment of inertia about an axis which passes through the centroid and makes an angle of 30° with the side AB.

(9) Find the greatest and least radii of gyration of the section, Fig. 73, No. 5.

(10) Find the moments of inertia of the section, Fig. 73, No. 6, about the axes OX and OY, and hence find the moments of inertia about axes parallel to OX and OY respectively and passing through the centroid. (The full depth is 6″.)

(11) Find the moment of inertia of the section, Fig. 73, No. 7, about the side AB, and hence find the moment of inertia about an axis parallel to AB and passing through the centroid.

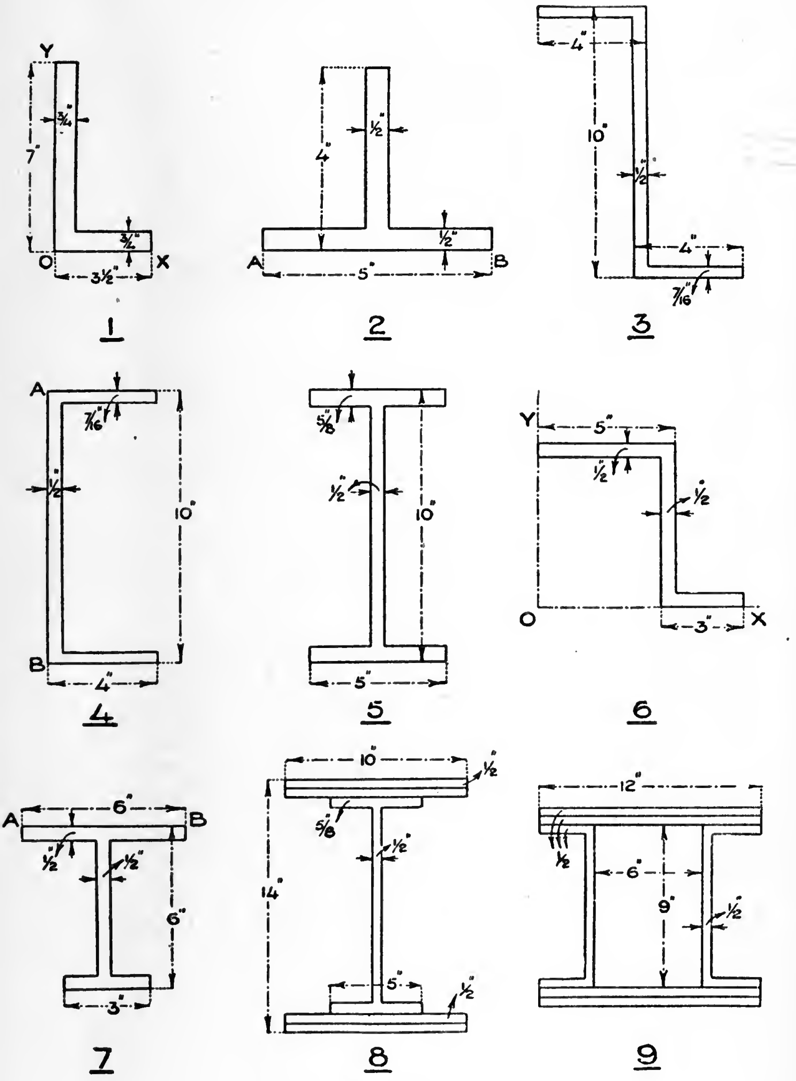

Y
¾"
7"
¾"
O
X
3½"
1
4"
½"
½"
A
5"
B
2
4"
10"
½"
4"
7/16"
3
A
7/16"
½"
10"
B
4"
4
5/8"
½"
10"
5"
5
Y
5"
½"
½"
O
3"
X
6
A
6"
B
½"
½"
6"
3"
7
10"
½"
5/8"
½"
14"
5"
½"
8
12"
½"
6"
9"
½"
9

FIG. 73.

(12) Find the greatest and least radii of gyration of the section, Fig. 73, No. 8.

(13) Find the principal moments of inertia of the section, Fig. 73, No. 9, and find the lengths of the major and minor axes of the momental ellipse.

(14) Find the moment of inertia of the section, Fig. 73, No. 1, about an axis passing through the centroid and making an angle of 45° with OX.

(15) A circular lamina of mass 5 lbs. and radius 5 ft. rotates uniformly about an axis perpendicular to its plane and just touching its circumference. Find the kinetic energy of rotation if the lamina makes 50 revolutions per minute.

(16) If the lamina in Question 15 rotates uniformly about a tangent and makes 50 revolutions per minute, what will be the kinetic energy of rotation ?

(17) Find the radius of gyration, about an axis passing through the centre and perpendicular to its plane, of a circular lamina of radius a, when the density d is such that $d = kx$ where k is a constant and x is the distance from the centre.

(18) In Question 17, if $d = k(a - x)$ where x is the distance from the centre, what will be the radius of gyration about the same axis ?

(19) Find the moment of inertia of the frustum of a cone about its axis, the height being 8 inches, radius of the top 3 inches, and the radius of the bottom 5 inches. A cubic inch of the material weighs 0·26 lb.

(20) ABCDE is a figure made up of a square ABCD, 5 inches side, and an equilateral triangle ADE, 5 inches side, the vertex, E, of the triangle lying outside the square. Find the greatest and least moments of inertia of the figure, and hence find the moment of inertia about BD, the diagonal of the square.

(21) If I_0 is the moment of inertia of an area about a straight line in the same plane passing through its centre, and I is its moment of inertia about a parallel line in the plane, there is a rule which enables us to calculate I if we know I_0. Prove the rule: If for a circle I_0 is $\frac{\pi}{4}r^4$, what is I about a tangent to the circle ? (B. of E., 1913.)

CHAPTER XVII

132. The work of this chapter will be devoted to the consideration of areas, centroids, and moments of inertia of irregular figures.

The Trapezoidal Rule. Let the base line be divided into a certain number of equal parts and ordinates erected to the curve from the points of division. The area is thus divided into a number of strips of equal breadth, and for n strips there will be $n + 1$ ordinates (Fig. 74).

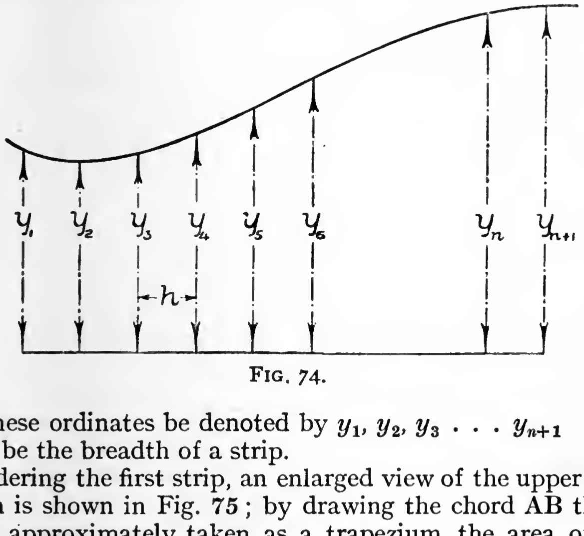


FIG. 74.

Let these ordinates be denoted by $y_1, y_2, y_3 \ldots y_{n+1}$

Let h be the breadth of a strip.

Considering the first strip, an enlarged view of the upper portion of which is shown in Fig. 75; by drawing the chord AB the strip may be approximately taken as a trapezium, the area of which is $\frac{h}{2}(y_1 + y_2)$.

If the other strips are taken in the same way, the whole area will be approximately equal to the sum of all these trapeziums.

$$\text{Area} = \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \frac{h}{2}(y_3 + y_4) + \ldots \frac{h}{2}(y_n + y_{n+1})$$

$$= \frac{h}{2}\{(y_1 + y_{n+1}) + 2(y_2 + y_3 + y_4 + \ldots y_n)\}$$

$$= \frac{h}{2}\{A + 2B\} \quad \ldots \ldots \ldots \ldots \quad (1)$$

where A = sum of the first and last ordinates
and B = sum of the remaining ordinates.

133. *The Mid-ordinate Rule.* If the mid-ordinate is drawn meeting the curve at C (see Fig. 75), and the tangent ECF is drawn to the curve at that point, the strip may then be taken approximately as a trapezium, the top side of which is the tangent ECF. In this case the lengths of the parallel sides of the trapezium are not known, but it is evident that the mid-ordinate is half the sum of the parallel sides, and therefore the area of the strip is hy'_1 where y'_1 is the mid-ordinate. It should be noticed that this is equivalent to taking the strip as being approximately a rectangle the height of which is the mid-ordinate.

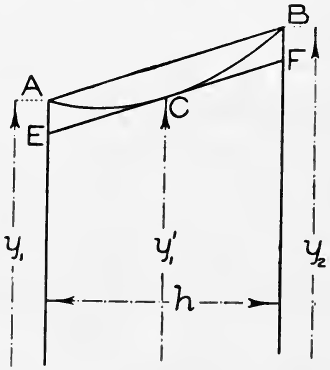


FIG. 75.

If the other strips are treated in the same way, the whole area will be approximately equal to the sum of all these equivalent rectangles.

$$\text{Area} = h(y'_1 + y'_2 + y'_3 + \ . \ . \ . \ y'_n)$$
$$= \text{breadth of strip} \times \text{sum of the mid-ordinates} \quad . \quad . \quad . \quad (2)$$

For good work it is not safe to use these rules separately, but it is better to take the mean of the results obtained by working with each. The reason for this may be seen from a study of Fig. 75. Using the trapezoidal rule for that strip will give a value for the area in excess of the true value by an amount equal to the area ACB. Using the mid-ordinate rule for the same strip will give a value for the area which is less than the true value by an amount equal to the sum of the areas AEC and BFC. The errors thus involved are opposite in nature, and by taking the mean of the two results there is the tendency for these errors to neutralise each other.

134. *Simpson's Rule for an Odd Number of Ordinates.*

Let the base line be divided into an even number of parts and the ordinates drawn to the curve from the points of division (Fig. 76). The figure is thus divided into an even number of strips of equal breadth.

Let h be the breadth of a strip, $2n$ the number of strips, and $y_1, y_2, y_3 \ldots y_{2n}, y_{2n+1}$ the ordinates.

Considering the first two strips, the ordinates for which are y_1, y_2, and y_3. Let OX and OY, the axes of reference, be so chosen that the ordinate y_2 coincides with the axis OY. Therefore, with reference to these axes the co-ordinates of the points A, B, and C will be $(-h, y_1)$, $(0, y_2)$, and (h, y_3) respectively. Let that part of the curve which passes through the points A, B, and C be represented by the equation $y = a + bx + cx^2$ where a, b, and c are constants.

These constants can be expressed in terms of the ordinates.

For when $x = -h$, $y = y_1$, and $y_1 = a - bh + ch^2$
when $x = 0$, $y = y_2$, and $y_2 = a$
when $x = h$, $y = y_3$, and $y_3 = a + bh + ch^2$

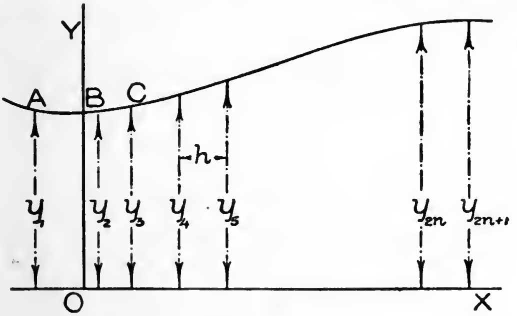


FIG. 76.

Then $$b = \frac{1}{2h}\{y_3 - y_1\}$$

and $$c = \frac{1}{2h^2}\{y_1 - 2y_2 + y_3\}$$

Denoting the area of the first two strips by A_1,

Then
$$\begin{aligned} A_1 &= \int_{-h}^{h} y\,dx \\ &= \int_{-h}^{h} (a + bx + cx^2)\,dx \\ &= \left[ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3\right]_{-h}^{h} \\ &= 2ah + \frac{2}{3}ch^3 \\ &= 2hy_2 + \frac{h}{3}\{y_1 - 2y_2 + y_3\} \\ &= \frac{h}{3}\{y_1 + 4y_2 + y_3\} \end{aligned}$$

This gives the area of the first two strips in terms of the ordinates y_1, y_2, and y_3. Taking the other strips in pairs and treating each pair in a similar manner,

$$\begin{aligned}
A_1 &= \frac{h}{3}\{y_1 + 4y_2 + y_3\} \\
A_2 &= \frac{h}{3}\{y_3 + 4y_4 + y_5\} \\
A_3 &= \frac{h}{3}\{y_5 + 4y_6 + y_7\} \\
&\vdots \\
A_n &= \frac{h}{3}\{y_{2n-1} + 4y_{2n} + y_{2n+1}\}
\end{aligned}$$

Hence total area

$$\begin{aligned}
&= A_1 + A_2 + A_3 \ldots + A_n \\
&= \frac{h}{3}\{(y_1 + y_{2n+1}) + 4(y_2 + y_4 + \ldots y_{2n}) + 2(y_3 + y_5 + \ldots y_{2n-1})\} \\
&= \frac{h}{3}\{A + 4B + 2C\}
\end{aligned}$$

where A = sum of the first and last ordinates
B = sum of the even ordinates
C = sum of the remaining odd ordinates.

It should be noticed that as this rule has been obtained by adding the strips in pairs, it can only be used when the figure is divided into an even number of strips and then there must be an odd number of ordinates.

135. *Simpson's Second Rule.* There is no such general rule that can be applied when the figure is divided into an odd number

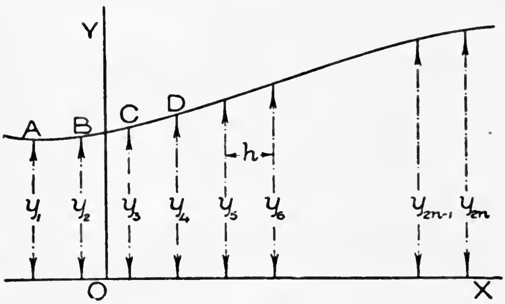


FIG. 77.

of strips, for the nature of the rule must depend upon the number of strips taken at a time.

Let three strips be taken together, and let the axes of reference be so chosen that the axis of y comes midway between the ordinates y_2 and y_3 (Fig. 77).

The curve must pass through the four points A, B, C, and D, and will be of the form $y = a + bx + cx^2 + dx^3$ where a, b, c, and d are constants.

When $x = -\frac{3h}{2}$ $y = y_1$ $\quad y_1 = a - \frac{3}{2}bh + \frac{9}{4}ch^2 - \frac{27}{8}dh^3$ . . (1)

$x = -\frac{h}{2}$ $y = y_2$ $\quad y_2 = a - \frac{1}{2}bh + \frac{1}{4}ch^2 - \frac{1}{8}dh^3$ . . (2)

$x = \frac{h}{2}$ $y = y_3$ $\quad y_3 = a + \frac{1}{2}bh + \frac{1}{4}ch^2 + \frac{1}{8}dh^3$ . . (3)

$x = \frac{3h}{2}$ $y = y_4$ $\quad y_4 = a + \frac{3}{2}bh + \frac{9}{4}ch^2 + \frac{27}{8}dh^3$ . . (4)

and these equations can be solved for the constants.

$$\text{Area of the three strips} = \int_{-\frac{3h}{2}}^{\frac{3h}{2}} y\,dx$$

$$= \int_{-\frac{3h}{2}}^{\frac{3h}{2}} (a + bx + cx^2 + dx^3)\,dx$$

$$= \left[ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 + \frac{1}{4}dx^4\right]_{-\frac{3h}{2}}^{\frac{3h}{2}}$$

$$A_1 = 3ah + \frac{9}{4}ch^3$$

Thus to express A_1 in terms of the ordinates, it is only necessary to find the constants a and c.

Adding (1) and (4) $\quad y_1 + y_4 = 2a + \frac{9}{2}ch^2$

Adding (2) and (3) $\quad y_2 + y_3 = 2a + \frac{1}{2}ch^2$

Then $\quad y_1 + y_4 - y_2 - y_3 = 4ch^2.$

or $\quad c = \frac{1}{4h^2}\{y_1 - y_2 - y_3 + y_4\}$

Also $\quad 9(y_2 + y_3) - (y_1 + y_4) = 16a$

or $\quad a = \frac{1}{16}\{-y_1 + 9y_2 + 9y_3 - y_4\}$

Hence $$A_1 = h\left\{\frac{3}{16}(-y_1 + 9y_2 + 9y_3 - y_4) + \frac{9}{16}(y_1 - y_2 - y_3 + y_4)\right\}$$

$$= \frac{h}{16}\{6y_1 + 18y_2 + 18y_3 + 6y_4\}$$

$$= \frac{3h}{8}\{y_1 + 3y_2 + 3y_3 + y_4\}$$

If A_2 be the area of the next three strips,

$$A_2 = \frac{3h}{8}\{y_4 + 3y_5 + 3y_6 + y_7\}$$

$$A_3 = \frac{3h}{8}\{y_7 + 3y_8 + 3y_9 + y_{10}\}$$

and $$A_n = \frac{3h}{8}\{y_{3n-2} + 3y_{3n-1} + 3y_{3n} + y_{3n+1}\}$$

where $3n$ is the number of strips into which the figure has been divided.

Hence $$\text{Total area} = A_1 + A_2 + A_3 + \ldots A_n$$

$$= \frac{3h}{8}\{(y_1 + y_{3n+1}) + 2(y_4 + y_7 + \ldots y_{3n+1}) + 3(y_2 + y_3 + y_5 + y_6 + \ldots y_{3n-1} + y_{3n})\}$$

$$= \frac{3h}{8}\{A + 3B + 2C\}$$

where A = sum of the first and last ordinates

$B = y_2 + y_3 + y_5 + y_6 + \ldots y_{3n-1} + y_{3n}$

$C = y_4 + y_7 + y_{10} + \ldots y_{3n+1}$

This rule cannot be used in the same general manner as the first rule, since from the nature of its formation, in taking three strips at a time, the number of strips into which the figure is divided must be a multiple of three.

In actual practice the result is too approximate if the figure is divided into a number of strips less than 10, and for a number greater than 10 Simpson's second rule only provides for the few cases when the number of strips is 15, 21, 27, etc.

136. *The Prismoidal Rule.* Simpson's first rule can be applied to find the volume of the frustum of a pyramid or a cone, a wedge, or to any solid in which the area of a section taken parallel to the base is proportional to the square of the perpendicular distance of that section from the base.

Taking the case of the frustum of a rectangular pyramid of height h, the sides of the base being a and b. Let H be the height of the imaginary vertex, and let k and l be the sides of the rectangular section at a distance x from the base.

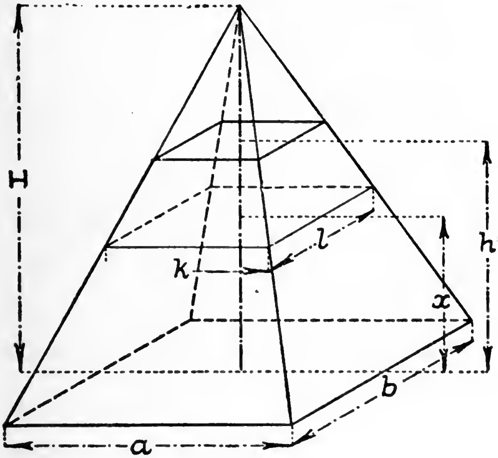


FIG. 78.

Then $$\frac{k}{a} = \frac{H - x}{H}, \quad \text{or } k = \frac{a}{H}(H - x)$$

also $$\frac{l}{b} = \frac{H - x}{H}, \quad \text{or } l = \frac{b}{H}(H - x)$$

Thus the area of section $= kl$.

$$A = \frac{ab}{H^2}(H - x)^2$$

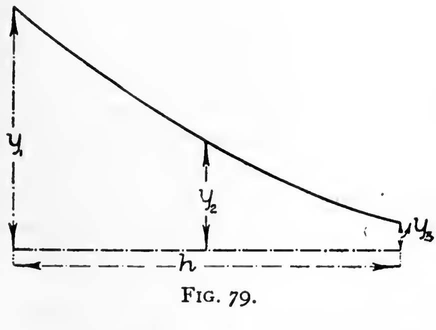


FIG. 79.

If the curve connecting x and A is drawn between the limits $x = 0$ and $x = h$ (Fig. 79), the area under this curve will give the volume of the frustum, but the curve is a parabola, and as

Simpson's rule is derived from such a curve, the rule can be applied by dividing the area into two strips of equal breadth.

The breadth of each strip is $\frac{h}{2}$.

Let the ordinates be y_1, y_2, and y_3.

Then The volume $= \frac{h}{6}(y_1 + 4y_2 + y_3)$

where

$$y_1 = \text{area of the base}$$
$$y_3 = \text{area of the top}$$
$$y_2 = \text{area of the mid-section.}$$

137. *The Centroid.* In dealing with a closed, irregular figure, the figure can be enclosed in a rectangle and two adjacent sides of this rectangle can be taken as the axes of reference. The position of the centroid of the figure can then be determined with respect to these axes.

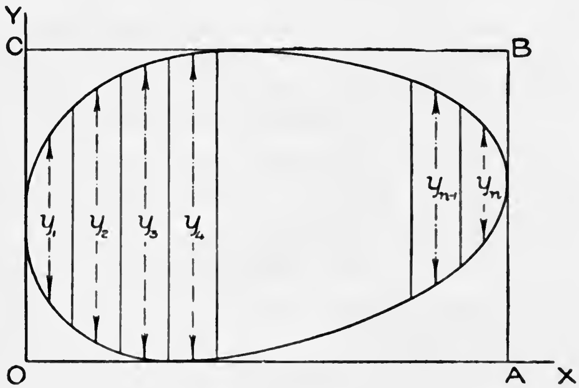


FIG. 80.

Let an irregular figure (Fig. 80) be enclosed in a rectangle **OABC**, and let the base OA be divided into n equal parts, and the figure divided into n strips of equal breadth by lines drawn through the points of division, perpendicular to OA. Let b be the breadth of each strip, and $y_1, y_2, y_3 \ldots y_n$ the mid-ordinates of the strips.

Taking the first strip and treating it as a rectangle,

Area of the strip $= by_1$

Distance of centroid of the strip from OY $= \frac{b}{2}$

Moment of the strip about OY $= \frac{1}{2}b^2y_1.$

Taking the second strip and treating it as a rectangle,

Area of the strip $= by_2$

Distance of the centroid of the strip from OY $= \dfrac{3b}{2}$

Moment of the strip about OY $= \dfrac{3}{2}b^2y_2.$

The whole area will be obtained by taking the sum of all these strips.

$$A = b(y_1 + y_2 + y_3 + \ . \ . \ . \ y_n)$$

The moment of the whole area will be obtained by taking the sum of the moments of all these strips.

$$M_{OY} = \frac{1}{2}b^2y_1 + \frac{3}{2}b^2y_2 + \frac{5}{2}b^2y_3 + \ . \ . \ . \ \frac{2n-1}{2}b^2y_n$$

$$= \frac{b^2}{2}\{y_1 + 3y_2 + 5y_3 + \ . \ . \ . \ (2n-1)y_n\}$$

If $\bar{x}$ is the perpendicular distance of the centroid from the axis OY,

$$\bar{x} = \frac{M_{OY}}{A}$$

$$= \frac{b}{2}\,\frac{y_1 + 3y_2 + 5y_3 + \ . \ . \ . \ (2n-1)y_n}{y_1 + y_2 + y_3 + \ . \ . \ . \ y_n}$$

If the area is made to rotate about the axis OY describing a surface of revolution,

$$V_{OY} = 2\pi A\bar{x}$$

$$= 2\pi\ M_{OY}$$

$$= \pi b^2\{y_1 + 3y_2 + 5y_3 + \ . \ . \ . \ (2n-1)y_n\}$$

Let the side OC be divided into n equal parts and the figure divided into n strips of equal breadth by lines drawn through the points of division, parallel to the axis OX. Let a be the breadth of each strip, and x_1, x_2, x_3 . . . x_n the mid-ordinates of the strips.

Then $\quad A = a(x_1 + x_2 + x_3 + \ . \ . \ . \ x_n)$

$$M_{OX} = \frac{a^2}{2}\{x_1 + 3x_2 + 5x_3 + \ . \ . \ . \ (2n-1)x_n\}$$

If $\bar{y}$ is the perpendicular distance of the centroid from the axis OX,

$$\bar{y} = \frac{M_{OX}}{A}$$

$$= \frac{a}{2}\,\frac{x_1 + 3x_2 + 5x_3 + \ . \ . \ . \ (2n-1)x_n}{x_1 + x_2 + x_3 + \ . \ . \ . \ x_n}$$

If the area is made to rotate about the axis OX describing a surface of revolution,

$$\begin{aligned} V_{OX} &= 2\pi A\bar{y} \\ &= 2\pi M_{OX} \\ &= \pi a^2\{x_1 + 3x_2 + 5x_3 + \ldots (2n-1)x_n\} \end{aligned}$$

138. *The Moment of Inertia.* Considering the figure (Fig. 80) to be divided into vertical strips, each of breadth b, and y_1, y_2, y_3 . . . y_n being the mid-ordinates of the strips.

Taking each strip as a rectangle,

Moment of inertia of the first strip about the axis OY $= I_1$,

and
$$\begin{aligned} I_1 &= \frac{1}{12}b^3y_1 + by_1 \times \left(\frac{b}{2}\right)^2 \\ &= \frac{1}{12}b^3y_1 + \frac{1}{4}b^3y_1 \end{aligned}$$

For the second strip
$$\begin{aligned} I_2 &= \frac{1}{12}b^3y_2 + by_2 \times \left(\frac{3b}{2}\right)^2 \\ &= \frac{1}{12}b^3y_2 + \frac{9}{4}b^3y_2 \end{aligned}$$

For the last strip
$$\begin{aligned} I_n &= \frac{1}{12}b^3y_n + by_n \times \left\{\frac{(2n-1)b}{2}\right\}^2 \\ &= \frac{1}{12}b^3y_n + \frac{(2n-1)^2}{4}b^3y_n \end{aligned}$$

The moment of inertia of the whole figure about the axis OY will be obtained by taking the sum of the moments of inertia of all these strips.

$$\begin{aligned} I_{OY} &= \frac{b^3}{12}\{y_1 + y_2 + y_3 + \ldots y_n\} + \frac{b^3}{4}\{y_1 + 9y_2 + 25y_3 \\ &\quad + \ldots (2n-1)^2y_n\} \\ &= \frac{b^2}{12}A + \frac{b^3}{4}\{y_1 + 9y_2 + 25y_3 + \ldots (2n-1)^2y_n\} \end{aligned}$$

If the figure is divided into horizontal strips each of breadth a, and x_1, x_2, x_3 . . . x_n are the mid-ordinates of the strips,

Then

$$\begin{aligned} I_{OX} &= \frac{a^3}{12}\{x_1 + x_2 + x_3 + \ldots x^n\} + \frac{a^3}{4}\{x_1 + 9x_2 + 25x_3 \\ &\quad + \ldots (2n-1)^2x_n\} \\ &= \frac{a^2}{12}A + \frac{a^3}{4}\{x_1 + 9x_2 + 25x_3 + \ldots (2n-1)^2x_n\} \end{aligned}$$

139. In dealing with irregular figures, the work is rendered much simpler by adopting a tabular method of working, and this will be seen by a consideration of the following example. The given irregular figure (Fig. 81) is contained in a rectangle whose base is 9 inches and height 6 inches. The figure is divided into 10 vertical strips each of breadth 0·9 inches, and 10 horizontal strips each of breadth 0·6 inches. The mid-ordinates of these strips have been measured, and are given in the tables below.

(1) Working with the vertical strips :

y_n	$(2n-1)y_n$	$(2n-1)^2y_n$
2·46	2·46	2·5
3·96	11·88	35·6
4·52	22·60	113·0
5·10	35·70	249·9
5·73	51·57	464·1
5·94	65·34	718·7
5·53	71·89	934·6
4·38	65·70	985·5
3·02	51·34	872·8
1·79	34·01	646·2
42·43	412·49	5022·9

The breadth of the strip $b = 0·9$

Then Area $= 42·43 \times 0·9$

$= 38·19$ sq. in.

$$\bar{x} = \frac{0·9}{2} \times \frac{412·49}{42·43}$$

$= 4·374$ in.

$$V_{OY} = \pi \times (0·9)^2 \times 412·49$$

$= 1059$ cub. in.

$$I_{OY} = \frac{(0·9)^3}{12} \times 42·43 + \frac{(0·9)^3}{4} \times 5022·9$$

$= 917·8$ inch units

(2) Working with horizontal strips:

x_n	$(2n-1)x_n$	$(2n-1)^2x_n$
2·00	2·00	2·0
5·22	15·66	47·0
6·62	33·10	165·5
8·44	59·08	413·6
8·95	80·55	724·9
8·88	97·68	1074·5
8·29	107·77	1401·0
7·12	106·80	1602·0
5·02	85·34	1450·8
2·98	56·62	1075·8
63·52	644·60	7957·1

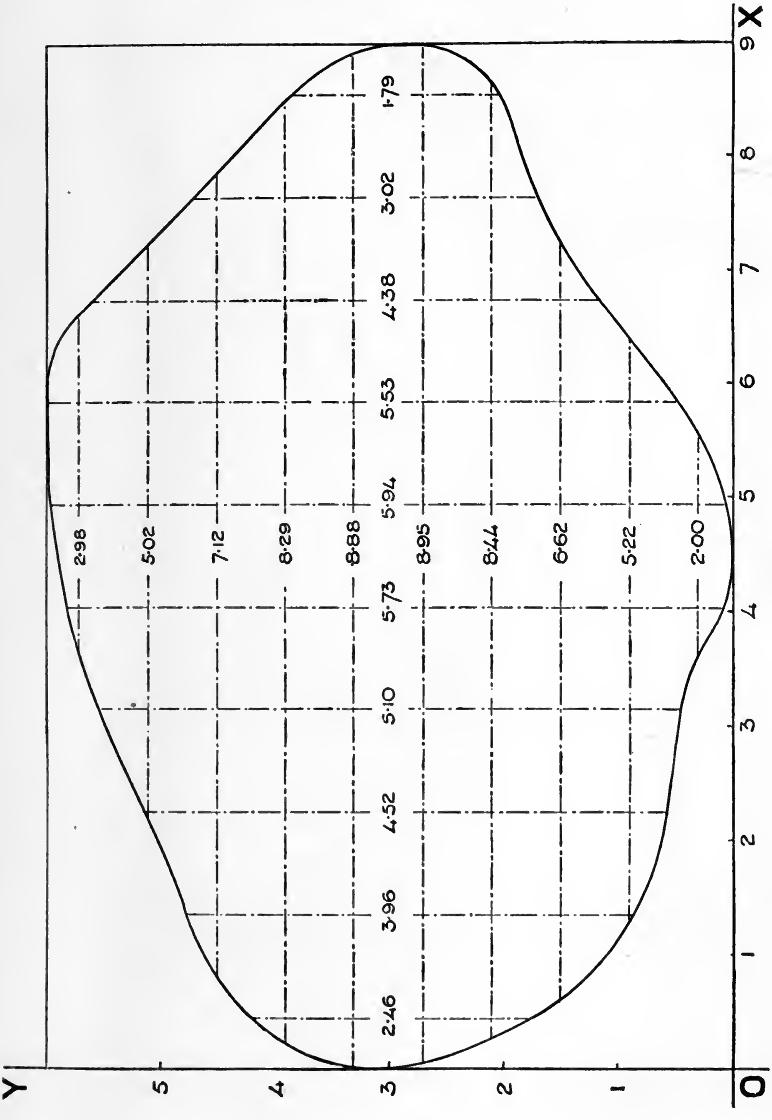
The breadth of the strip $a = 0{\cdot}6$

Then
$$\text{Area} = 63{\cdot}52 \times 0{\cdot}6$$
$$= 38{\cdot}12 \text{ sq. in.}$$
$$\bar{y} = \frac{0{\cdot}6}{2} \times \frac{644{\cdot}6}{63{\cdot}52}$$
$$= 3{\cdot}045 \text{ in.}$$
$$V_{OX} = \pi \times (0{\cdot}6)^2 \times 644{\cdot}6$$
$$= 729{\cdot}1 \text{ cub. in.}$$
$$I_{OX} = \frac{(0{\cdot}6)^3}{12} \times 63{\cdot}52 + \frac{(0{\cdot}6)^3}{4} \times 7957{\cdot}1$$
$$= 430{\cdot}8 \text{ inch units}$$

140. *The Derived Figures.* Let an irregular figure be enclosed in a rectangle, and two adjacent sides OM and OL be taken as the axes of reference (Fig. 82).

Let P be the point where the axis OX touches the boundary of the figure.

Let the irregular area be divided into a large number of horizontal strips each of breadth δy.



Y
X
O
1
2
3
4
5
6
7
8
9
2·46
3·96
4·52
5·10
5·73
5·94
5·53
4·38
3·02
1·79
2·98
5·02
7·12
8·29
8·88
8·95
8·44
6·62
5·22
2·00

Fig. 81.

Let the length of one of these strips be AB, the perpendicular distance of this strip from the axis OX being y.

$$\text{The area of the strip} = AB\ \delta y$$

$$\text{The moment of the strip about OX} = AB y\ \delta y$$

Let QT be the projection of AB on LN and the lines **PQ and PT** drawn cutting AB in A_1 and B_1 respectively.

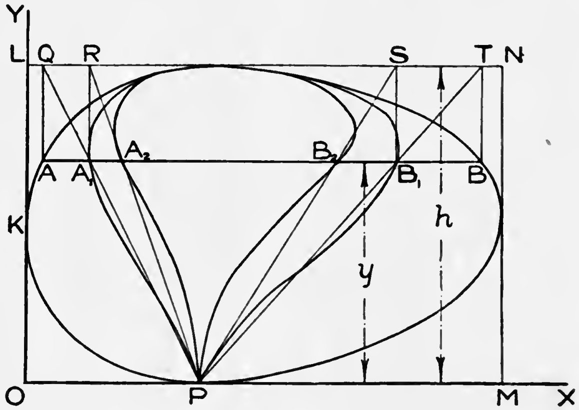


FIG. 82.

Then if h is the height of the rectangle,

$$\text{By similar triangles } \frac{QT}{h} = \frac{A_1B_1}{y}$$

or
$$y\ QT = h\ A_1B_1$$

$$y\ AB = h\ A_1B_1, \quad \text{since } AB = QT$$

and
$$y\ AB\ \delta y = h\ A_1B_1\ \delta y$$

Hence the moment of the strip about OX $= hA_1B_1\ \delta y$.

The moment of the irregular area about OX would be obtained by taking the sum of the moments of all these strips,

and
$$M_{OX} = \Sigma AB\ y\ \delta y$$
$$= h\Sigma A_1B_1\ \delta y$$

Now $\Sigma A_1B_1\ \delta y$ is the area of the figure obtained by joining all the points derived in the same manner as A_1 and B_1 for different positions of the horizontal line AB between the limits $y = 0$ and $y = h$. This figure is spoken of as the "first derived figure."

$$\text{Then } M_{OX} = h \times \text{area of the first derived figure}$$

If $\bar{y}$ is the perpendicular distance of the centroid from the axis OX,

Then $$\bar{y} = \frac{M_{OX}}{\text{area}}$$

$$= \frac{h \times \text{area of the first derived figure}}{\text{area of the figure}}$$

Let RS be the projection of A_1B_1 on LN and the lines PR and PS drawn cutting A_1B_1 in A_2 and B_2 respectively.

By similar triangles $\dfrac{RS}{h} = \dfrac{A_2B_2}{y}$

Then
$$h\,A_2B_2 = y\,RS$$
$$= y\,A_1B_1, \quad \text{since } RS = A_1B_1$$
$$= \frac{y^2}{h}\,AB, \quad \text{since } y\,AB = h\,A_1B_1$$

and $$h^2\,A_2B_2 = y^2\,AB$$

Hence $$h^2\,A_2B_2\,\delta y = AB\,y^2\,\delta y$$

But $AB\,y^2\,\delta y$ is the moment of inertia of the strip about the axis OX. The moment of inertia of the irregular area about OX would be obtained by taking the sum of the moments of inertia of all these strips,

and
$$I_{OX} = \Sigma AB\,y^2\,\delta y$$
$$= h^2\Sigma A_2B_2\,\delta y$$

Now $\Sigma A_2B_2\,\delta y$ is the area of the figure obtained by joining all the points derived in the same manner as A_2 and B_2 for different positions of the horizontal line AB between the limits $y = 0$ and $y = h$. This figure is spoken of as the "second derived figure."

Then $$I_{OX} = h^2 \times \text{area of the second derived figure}$$

If K is the point where the axis OY touches the boundary of the irregular figure and vertical lines are drawn across the figure, then by working with the projections of these lines on NM, the other pair of derived figures can be obtained,

and $$\bar{x} = \frac{k \times \text{area of the first derived figure}}{\text{area of the figure}}$$

$$I_{OY} = k^2 \times \text{area of the second derived figure}$$

when $k = OM$, the length of the rectangle.

141. Working with the irregular figure given in the previous example, and let this figure be divided into 10 horizontal strips of equal breadth and the mid-ordinate of each strip drawn.

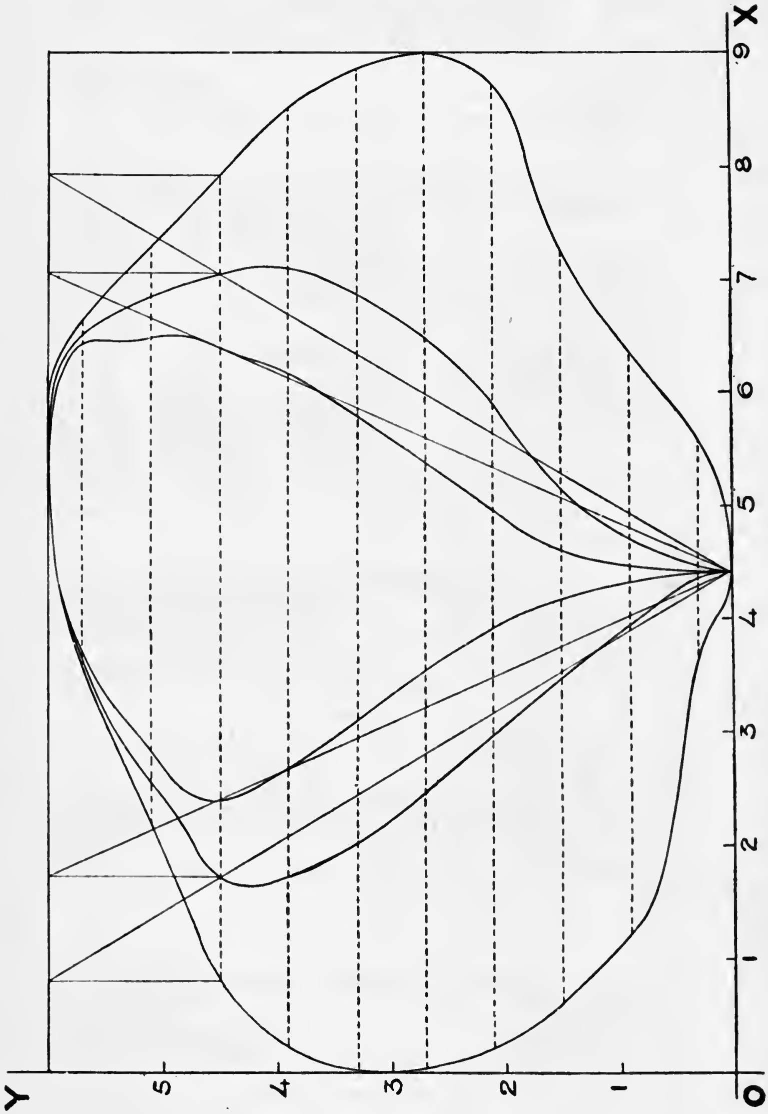


FIG. 83.

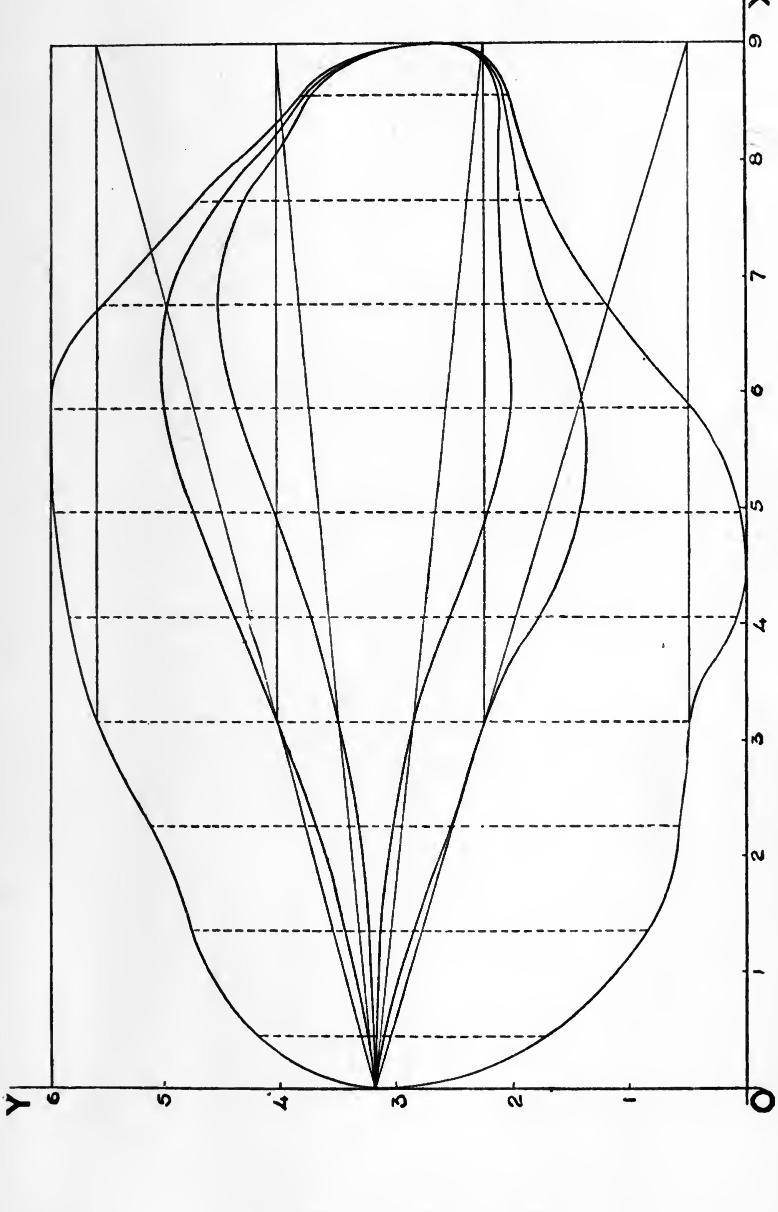


FIG. 84.

Find for each mid-ordinate the corresponding points on the derived figures (Fig. 83), and in this way the corresponding mid-ordinates of the derived figures can be found.

x_n	x_n'	x_n''
2·00	0·11	0·01
5·22	0·78	0·12
6·62	1·58	0·42
8·44	2·98	1·06
8·95	4·01	1·79
8·88	4·82	2·66
8·29	5·35	3·46
7·12	5·30	3·95
5·42	4·24	3·59
2·98	2·84	2·70
63·52	32·01	19·76

Breadth of strip = 0·6″
Area of figure = 63·52 × 0·6 = 38·12 sq. in.
Area of first derived figure = 32·01 × 0·6 = 19·21 sq. in.
Area of second derived figure = 19·76 × 0·6 = 11·86 sq. in.

$$\bar{y} = 6 \times \frac{19{\cdot}21}{38{\cdot}12}$$
$$= 3{\cdot}024 \text{ in.}$$

$$\mathrm{I}_{\mathrm{OX}} = 6^2 \times 11{\cdot}86$$
$$= 427{\cdot}2 \text{ inch units}$$

Dividing the irregular figure into 10 vertical strips (Fig. 84), and working in the same way with the mid-ordinates of these strips:

y_n	y_n'	y_n''
2·46	0·14	0·01
3·96	0·58	0·09
4·52	1·15	0·28
5·10	1·78	0·64
5·73	2·59	1·16
5·94	3·30	1·82
5·53	3·61	2·36
4·38	3·28	2·48
3·02	2·60	2·21
1·79	1·72	1·64
42·43	20·75	12·69

Breadth of strip = 0·9″
Area of figure = 42·43 × 0·9 = 38·19 sq. in.
Area of first derived figure = 20·75 × 0·9 = 18·67 sq. in.
Area of second derived figure = 12·69 × 0·9 = 11·42 sq. in.

$$\bar{x} = 9 \times \frac{18 \cdot 67}{38 \cdot 19}$$
$$= 4 \cdot 399 \text{ in.}$$
$$I_{OY} = 9^2 \times 11 \cdot 42$$
$$= 925 \cdot 1 \text{ inch units}$$

Examples XVII

(1) The following values of y and x being given, tabulate $\delta y/\delta x$ and $y\ \delta x$ in each interval. If $y\ \delta x$ be called δA, tabulate the values of A if A is 0 where $x = 0$. (B. of E., 1905.)

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
y	1·428	1·561	1·691	1·820	1·947	2·071	2·193	2·314	2·431	2·547

(2) By tabulation give approximately a table of values of $\frac{dy}{dx}$ and $\int y\ dx$ if the following values of x and y are given. (B. of E., 1908.)

x	0	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
y	1·2679	1·3640	1·4663	1·5774	1·7002	1·8391	2·0000	2·1918	2·4281	2·7321

(3) The following values of x and y being given, tabulate $\delta y/\delta x$ in each interval, also $\delta A = y\ \delta x$ and $A = \int y\ dx$. (B. of E., 1910.)

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
y	6·428	7·071	7·660	8·192	8·660	9·063	9·397	9·659	9·848

(4) The diameter of a circle of 5 inches radius is divided into 20 equal parts, and ordinates are drawn to the circle from each point of division. Calculate the lengths of these ordinates. Find the area of the circle—(1) using the Trapezoidal Rule, (2) using Simpson's Rule—and calculate the percentage error in each case.

(5) The following values of x and y give the co-ordinates of a number of points on a curve. Plot the points and draw the curve.

x inches	0	0·8	1·9	3·1	4·3	5·5	6·6	7·9	9·0
y inches	2·00	2·10	2·32	2·69	3·11	3·69	4·30	5·17	6·00

Take the area bounded by the curve, the ordinates at $x = 0$ and $x = 9$, and the axis of x. By dividing this area into 18 vertical strips of equal breadth, find (1) the area, (2) the distance of the centroid from the axis of y, (3) the volume of the surface of revolution generated as the area rotates about the axis of y, (4) the moment of inertia of the area about the axis of y.

(6) Divide the area in Question 5 into 12 horizontal strips of equal breadth, and find (1) the area, (2) the height of the centroid above the axis of x, (3) the volume of he surface of revolution generated as the area rotates about the axis of x, (4) the moment of inertia of the area about the axis of x.

(7) In the figure of Question 5 take P, a point on the axis of x, so that $x = 4·5$. Use this point to draw the first and second derived figures. Find the areas of these figures and use your results to find (1) the height of the centroid above the axis of x, (2) the moment of inertia of the area about the axis of x.

(8) In the figure of Question 5 take Q, a point on the axis of y, so that $y = 2$. Use this point to draw the first and second derived figures. Find the areas of these figures, and use your results to find (1) the distance of the centroid from the axis of y, (2) the moment of inertia of the area about the axis of y.

(9) The following values of x and y give the co-ordinates of a number of points on a closed curve. Plot the points and draw the figure.

x inches	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0	6·5	7·0	7·5	8·0
y inches	2·0	3·32	0·35	4·35	0·09	5·00	0	5·53	0·09	5·88	0·30	6·0	0·59	5·70	1·05	4·71	3·50

By dividing the figure into 16 vertical strips of equal breadth, find (1) the area, (2) the distance of the centroid from the axis of y, (3) the volume of the surface of revolution generated as the figure rotates about the axis of y, (4) the moment of inertia about the axis of y.

(10) By dividing the figure of Question 9 into 12 horizontal strips of equal breadth, find (1) the area, (2) the height of the centroid above the axis of x, (3) the volume of the surface of revolution generated as the figure rotates about the axis of x, (4) the moment of inertia about the axis of x.

(11) In the figure of Question 9, take P as the point of contact of the bounding line of the figure and the axis of x. Use this point to draw the first and second derived figures. Find the areas of these figures, and use your results to find (1) the height of the centroid above the axis of x, (2) the moment of inertia about the axis of x.

(12) In the figure of Question 9, take Q as the point of contact of the bounding line of the figure and the axis of y. Use this point to draw the first and second derived figures. Find the areas of these figures, and use your results to find (1) the distance of the centroid from the axis of y, (2) the moment of inertia about the axis of y.

(13) Draw a circle of 3 inches radius, and let PT be a tangent to the circle, P being the point of contact. Using this point P, draw the first and second derived figures. Find the areas of these figures, and use your results to find the height of the centroid above the tangent PT, and the moment of inertia about that tangent. Verify your results.

(14) The co-ordinates of five points A, B, C, D, and E are (1·5, 0), (3·5, 0), (6, 3·5), (2, 6), and (0, 2·5) respectively, and these are the five angular points of a polygon ABCDE. Plot the points and draw the polygon. Let P be the mid-point of the side AB. Using this point, draw the first and second derived figures, and find their areas. Use your results to find the height of the centroid of the polygon above the side AB and the moment of inertia about the side AB.

(15) The top of a reservoir is a rectangle of sides $2a$ and $2b$, the depth is h, and the sides are inclined to the horizontal at 45°. Prove that the volume contained by the reservoir is

$$\frac{2h}{3}\left[6ab - 3h(a+b) + 2h^2\right]$$

(Int. Sci., 1913.)

(16) A wedge has a rectangular base 24 inches by 16 inches and the height is 6 inches. The faces corresponding to the larger sides of the base are inclined to the horizontal at 45°, while those corresponding to the smaller sides are inclined to the horizontal at 60°. Find the volume of the wedge.

(17) In Question 16, what would be the height of the wedge so that the top surface becomes a straight line ? What is the length of this edge and what is the volume of the resulting wedge ?

(18) The basis of Simpson's Rule is that if three successive equidistant ordinates (distant h apart), y_1, y_2, y_3, are drawn to any curve, the three points may be taken as lying on the curve $y = a + bx + cx^2$. Imagine y_2 to be the axis of y, so that $(-h, y_1)$, $(0, y_2)$, and (h, y_3) are the three points. Substitute these values in the equation, and find a and c. Integrate $a + bx + cx^2$ between the limits h and $-h$ and divide by $2h$. This gives the average value of y. Express it in terms of y_1, y_2, and y_3. (B. of E., 1907.)

CHAPTER XVIII

142. *Lengths of Curves.* If P and Q are two points taken very close together on a curve, the length of that part of the curve between P and Q being δs. Then δs can be taken approximately as the hypotenuse of a right-angled triangle, the base of which is δx and the perpendicular δy. The smaller δx is made the more nearly true does this approximation become, and it becomes actually true when δx is made infinitely small.

Then
$$\delta s^2 = \delta x^2 + \delta y^2$$

or
$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

and also
$$\left(\frac{\delta s}{\delta y}\right)^2 = 1 + \left(\frac{\delta x}{\delta y}\right)^2$$

In the limit when δx is made infinitely small,
$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

or
$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Thus, to get s, the length of a certain portion of the curve, the first of these expressions must be integrated with respect to x between assigned limits, or the second expression must be integrated with respect to y between assigned limits.

Then
$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or
$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 1. Find the length of the arc of curve of $y^2 = 8x^3$ between the limits $x = 1$ and $x = 3$.

$$y = 2\sqrt{2}x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = 3\sqrt{2}x^{\frac{1}{2}}$$

Then $$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+18x}$$

and $$s = \int_1^3 \sqrt{1+18x}\,dx$$

Let $$z = 1 + 18x$$
$$dz = 18dx$$

and $$s = \frac{1}{18}\int_{19}^{55} z^{\frac{1}{2}}\,dz$$
$$= \frac{1}{27}\left[z^{\frac{3}{2}}\right]_{19}^{55}$$
$$= \frac{1}{27}\{55^{1\cdot5} - 19^{1\cdot5}\}$$
$$= 12{\cdot}05$$

Example 2. Find the length of the arc of the curve $y^2 = 8x$ between the limits $x = 2$ and $x = 4$.

The limits for y are $y = 4$ and $y = 4\sqrt{2}$.

Now $$x = \frac{y^2}{8}$$
$$\frac{dx}{dy} = \frac{y}{4}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{16}$$
$$= \frac{1}{16}(16 + y^2)$$

Then $$s = \frac{1}{4}\int_4^{4\sqrt{2}} \sqrt{16+y^2}\,dy$$

Put $$y^2 = 16\sinh^2\theta$$
and $$\sqrt{16+y^2} = 4\cosh\theta$$
$$dy = 4\cosh\theta\,d\theta$$

Then $$s = 4\int \cosh^2\theta\,d\theta$$
$$= 2\int (1 + \cosh 2\theta)\,d\theta$$
$$= 2\left[\theta + \frac{1}{2}\sinh 2\theta\right]$$
$$= 2\left[\sinh^{-1}\frac{y}{4} + \frac{y\sqrt{16+y^2}}{16}\right]$$

$$= 2\left[\log_e \frac{y + \sqrt{16 + y^2}}{4} + \frac{y\sqrt{16 + y^2}}{16}\right]_4^{4\sqrt{2}}$$

$$= 2\left[\log_e \frac{4\sqrt{2} + 4\sqrt{3}}{4} - \log \frac{4 + 4\sqrt{2}}{4} + \frac{16\sqrt{6}}{16} - \frac{16\sqrt{2}}{16}\right]$$

$$= 2\left\{\log_e \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} + 1} + \sqrt{6} - \sqrt{2}\right\}$$

$$= 2\{0{\cdot}2648 + 1{\cdot}0353\}$$

$$= 2{\cdot}6002$$

143. *The Sag in a Telegraph Wire.* Let l be the half span, d the sag, and s the whole length of the wire, and suppose that the resulting curve is a parabola whose equation is $x^2 = 4ay$.

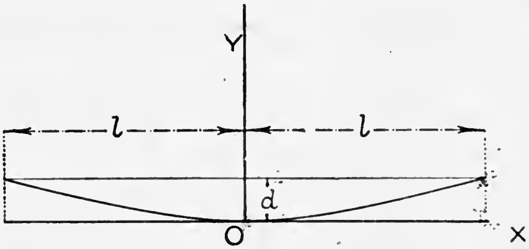


FIG. 85.

Now when $x = l$, $y = d$.

Hence $\quad l^2 = 4ad$

and $\quad a = \dfrac{l^2}{4d}$

Also $\quad y = \dfrac{x^2}{4a}$

$$\frac{dy}{dx} = \frac{x}{2a}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{4a^2}}$$

Then length of wire $= s = 2\displaystyle\int_0^l \sqrt{1 + \frac{x^2}{4a^2}}\, dx$

$$= \frac{1}{a}\int_0^l \sqrt{4a^2 + x^2}\, dx$$

Put $\quad x^2 = 4a^2 \sinh^2 \theta$

Then $\sqrt{4a^2 + x^2} = 2a \cosh \theta$

and $\quad dx = 2a \cosh \theta \, d\theta$

Hence
$$s = 4a\int \cosh^2\theta\, d\theta$$
$$= 2a\int (1+\cosh 2\theta)d\theta$$
$$= 2a\left[\theta + \frac{1}{2}\sinh 2\theta\right]$$
$$= 2a\left[\sinh^{-1}\frac{x}{2a} + \frac{x\sqrt{x^2+4a^2}}{4a^2}\right]$$
$$= 2a\left[\log_e \frac{x+\sqrt{x^2+4a^2}}{2a} + \frac{x\sqrt{x^2+4a^2}}{4a^2}\right]_0^l$$
$$= 2a\left[\log_e \frac{l+\sqrt{l^2+4a^2}}{2a} + \frac{l\sqrt{l^2+4a^2}}{4a^2}\right]$$
$$= 2a\left[\log_e\left\{\frac{l}{2a} + \sqrt{\left(\frac{l}{2a}\right)^2+1}\right\} + \frac{l}{2a}\sqrt{\left(\frac{l}{2a}\right)^2+1}\right]$$
$$= 2a\left[\log_e\{\alpha + \sqrt{\alpha^2+1}\} + \alpha\sqrt{\alpha^2+1}\right]$$

where
$$\alpha = \frac{l}{2a}$$

Now
$$\alpha = \frac{l}{2}\frac{4d}{l^2} = \frac{2d}{l}$$

and since d is very small compared with l, α is very small compared with 1, and $(1+\alpha^2)^{\frac{1}{2}} = 1 + \frac{1}{2}\alpha^2$ approximately.

Hence
$$s = 2a\left[\log_e\left(1+\alpha+\frac{\alpha^2}{2}\right) + \alpha\left(1+\frac{\alpha^2}{2}\right)\right]$$
$$= 2a\left[\left(\alpha+\frac{\alpha^2}{2}\right) - \frac{1}{2}\left(\alpha+\frac{\alpha^2}{2}\right)^2 + \frac{1}{3}\left(\alpha+\frac{\alpha^2}{2}\right)^3 - \dots + \alpha + \frac{\alpha^3}{2}\right]$$
$$= 2a\left[2\alpha + \frac{1}{3}\alpha^3\right] \text{ neglecting the higher powers of } \alpha$$
$$= 2a\left\{\frac{l}{a} + \frac{l^3}{24a^3}\right\}$$
$$= 2l + \frac{l^3}{12a^2}$$
$$= 2l + \frac{4}{3}\frac{d^2}{l}$$

Then
$$\frac{4}{3}\frac{d^2}{l} = s - 2l$$

and
$$d = \frac{1}{2}\sqrt{3ls - 6l^2}$$

This result can be obtained in quite a different manner. For since the sag d is very small compared with the half-span l, the curve is one of very small slope, and $\frac{dy}{dx}$ is therefore very small for any value of x between 0 and l.

Hence $$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 \text{ approximately}$$

$$= 1 + \frac{x^2}{8a^2}$$

Then $$s = 2\int_0^l \left(1 + \frac{x^2}{8a^2}\right) dx$$

$$= 2\left[x + \frac{x^3}{24a^2}\right]_0^l$$

$$= 2l + \frac{l^3}{12a^2}$$

$$= 2l + \frac{4}{3}\frac{d^2}{l}$$

and $$d = \frac{1}{2}\sqrt{3ls - 6l^2}$$

144. *Areas of Surfaces of Revolution.* Let PQ be a small arc of the curve of length δs, and this may be approximately taken as a small chord of length δs. (Fig. 86.)

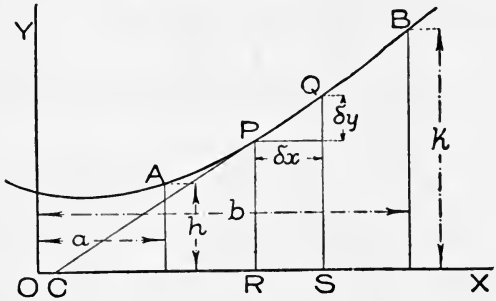


Fig. 86.

The area described as this chord rotates about the axis of x is the small part of a conical surface. This area can be taken as the difference between the curved surfaces of the two cones, one whose slant side is CQ and the radius of the base is

QS, and the other whose slant side is CP and the radius of the base is PR.

Area of elementary surface $= \pi(\mathrm{CQ}\ \mathrm{QS} - \mathrm{CP}\ \mathrm{PR})$

$$\begin{aligned}\delta \mathrm{S} &= \pi\{(l+\delta s)(y+\delta y) - ly\}\\ &= \pi\{l\,\delta y + y\,\delta s + \delta s\,\delta y\}\\ &= \pi\{l\,\delta y + y\,\delta s\}\end{aligned}$$

By similar triangles $\quad \dfrac{y}{l} = \dfrac{\delta y}{\delta s}$

and $\quad l\,\delta y = y\,\delta s$

Hence $\quad \delta \mathrm{S} = 2\pi y\,\delta s$

Area of the whole surface $= 2\pi \sum y\,\delta s$

$$\mathrm{S} = 2\pi \int y\,ds$$

Hence $\quad \mathrm{S} = 2\pi \displaystyle\int_a^b y\,\frac{ds}{dx}\,dx, \quad$ where $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

and also $\mathrm{S} = 2\pi \displaystyle\int_h^k y\,\frac{ds}{dy}\,dy, \quad$ where $\dfrac{ds}{dy} = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$

Example. Find the area of the surface described by the arc of the curve $y^2 = 8x$, between the limits $x = 2$ and $x = 4$, rotating about the axis of x.

The limits for y are $y = 4$ and $y = 4\sqrt{2}$.

$$x = \frac{y^2}{8}$$

$$\frac{dx}{dy} = \frac{y}{4}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{16}} = \frac{1}{4}\sqrt{16+y^2}$$

Hence $\quad \mathrm{S} = 2\pi \displaystyle\int y\,\frac{ds}{dy}\,dy$

$$= \frac{2\pi}{4}\int_4^{4\sqrt{2}} y\sqrt{16+y^2}\,dy$$

Put $\quad z = 16 + y^2$

Then $\quad dz = 2\,y\,dy$

and $$S = \frac{\pi}{4}\int z^{\frac{1}{2}}\,dz$$

$$= \frac{\pi}{4}\left[\frac{2}{3}z^{\frac{3}{2}}\right]$$

$$= \frac{\pi}{6}\left[(16 + y^2)^{\frac{3}{2}}\right]_4^{4\sqrt{2}}$$

$$= \frac{\pi}{6}\{48^{\frac{3}{2}} - 32^{\frac{3}{2}}\}$$

$$= \frac{32\pi}{3}\{3\sqrt{3} - 2\sqrt{2}\}$$

$$= 79 \cdot 36$$

145. Let the whole arc of a curve be divided into a very large number of small elementary arcs.

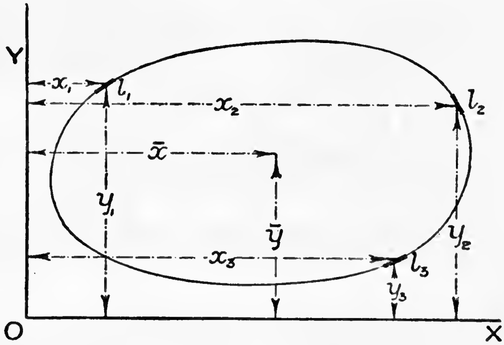


FIG. 87.

Let $l_1, l_2, l_3 \ldots$ be the lengths of these arcs,

$x_1, x_2, x_3 \ldots$ their distances from the axis OY,

and $y_1, y_2, y_3 \ldots$ their distances from the axis OX (Fig. 87).

Let the whole area rotate about the axis OX and S_{OX} be the area of the surface of the resulting surface of revolution.

The elementary length l_1 will describe a surface of area $2\pi l_1 y_1$.

The elementary length l_2 will describe a surface of area $2\pi l_2 y_2$.

The total surface will be the sum of all these elementary surfaces.

$$\text{Hence } S_{OX} = 2\pi\{l_1y_1 + l_2y_2 + l_3y_3 \ldots\}$$
$$= 2\pi s\bar{y}$$

where s is the whole length of the curve and $\bar{y}$ is the height of the centroid of that length of curve above the axis OX.

If the whole area rotates about the axis OY, then

$$S_{OY} = 2\pi s\bar{x}$$

Example. To find the area of the curved surface of a spherical cap of height h and radius of base r.

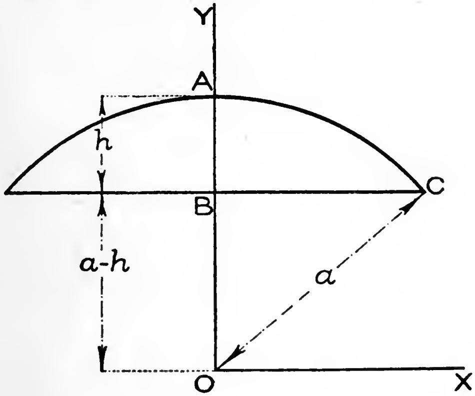


FIG. 88.

Let a be the radius of the sphere. The whole surface of the sphere can be produced by the semicircle rotating about the vertical diameter or the axis OY. The surface of the spherical cap is produced by the arc AC rotating about the axis OY.

Choosing the centre of the circle as origin, the equation to the circle is $x^2 + y^2 = a^2$, while the limits of y for the arc AC will be $y = a - h$ and $y = a$.

Since $$x^2 + y^2 = a^2$$

Then $$2x + 2y\frac{dy}{dx} = 0$$

and $$\frac{dx}{dy} = -\frac{y}{x}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{x^2}$$

$$= \frac{a^2}{x^2}$$

Now $$S_{OY} = 2\pi\int x\,ds$$

$$= 2\pi\int_{a-h}^{a} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$= 2\pi\int_{a-h}^{a} x\frac{a}{x}\,dy$$

$$= 2\pi a\int_{a-h}^{a} dy$$

$$= 2\pi a\,[a - (a - h)]$$

$$= 2\pi ah$$

But $$r^2 = a^2 - (a - h)^2$$

Hence $$a = \frac{1}{2h}\{r^2 + h^2\}$$

Then $$S_{OY} = \frac{2\pi h}{2h}\{r^2 + h^2\}$$
$$= \pi(r^2 + h^2)$$

146. *The Cycloid.*

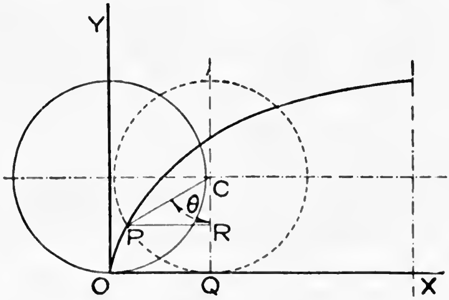


FIG. 89.

Let a be the radius of the rolling circle.

Then $$OQ = \text{arc } PQ = a\theta$$

If x, y are the co-ordinates of P,

Then $$\begin{aligned} x &= OQ - PR \\ &= a\theta - a\sin\theta \\ &= a(\theta - \sin\theta) \end{aligned}$$

Also $$\begin{aligned} y &= CQ - CR \\ &= a - a\cos\theta \\ &= a(1 - \cos\theta) \end{aligned}$$

The curve is evidently symmetrical about a vertical centre line and for half the curve, the limits of θ are 0 to π radians, the limits of y are 0 to $2a$, and the limits of x are 0 to πa.

Also $$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}$$

(*a*) To find the area.

$$\text{Area} = \int y\,dx$$
$$= \int_0^{2\pi} a(1-\cos\theta)\times a(1-\cos\theta)\,d\theta$$
$$= a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2 2\theta)\,d\theta$$
$$= a^2\int_0^{2\pi}\left\{1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right\}d\theta$$
$$= a^2\left[\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= 3\pi a^2$$

(*b*) To find the volume of the surface of revolution generated as the area rotates about the base.

$$V_{OX} = \pi\int y^2\,dx$$
$$= \pi\int_0^{2\pi} a^2(1-\cos\theta)^2\times a(1-\cos\theta)\,d\theta$$
$$= \pi a^3\int_0^{2\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta$$
$$= \pi a^3\int_0^{2\pi}\left\{1-3\cos\theta+\frac{3}{2}(1+\cos 2\theta)-\frac{1}{4}(\cos 3\theta+3\cos\theta)\right\}d\theta$$
$$= \pi a^3\int_0^{2\pi}\left\{\frac{5}{2}-\frac{15}{4}\cos\theta+\frac{3}{2}\cos 2\theta-\frac{1}{4}\cos 3\theta\right\}d\theta$$
$$= \pi a^3\left[\frac{5\theta}{2}-\frac{15}{4}\sin\theta+\frac{3}{4}\sin 2\theta-\frac{1}{12}\sin 3\theta\right]_0^{2\pi}$$
$$= 5\pi^2 a^3$$

(*c*) To find the height of the centroid of the cycloidal area above the base.

By symmetry the centroid is evidently situated in the vertical centre line. Let $\bar{y}$ be the height.

Then $$V_{OX} = 2\pi A\bar{y}$$
$$\bar{y} = \frac{V_{OX}}{2\pi A}$$
$$= \frac{5\pi^2 a^3}{2\pi\, 3\pi a^2}$$
$$= \frac{5a}{6}$$

(*d*) To find the length of the cycloidal curve,

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{2(1-\cos\theta)}{(1-\cos\theta)^2}$$

$$= \frac{2}{1-\cos\theta}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{2}}{(1-\cos\theta)^{\frac{1}{2}}}$$

Now

$$s = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{2}\int_0^{2\pi}\frac{a(1-\cos\theta)}{(1-\cos\theta)^{\frac{1}{2}}}\,d\theta$$

$$= \sqrt{2}a\int_0^{2\pi}(1-\cos\theta)^{\frac{1}{2}}\,d\theta$$

$$= 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= -4a\left[\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4a\{\cos\pi - \cos 0\}$$

$$= 8a$$

(*e*) To find the area of the surface of revolution generated as the curve rotates about the base.

$$\text{Now } \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{2}}{(1-\cos\theta)^{\frac{1}{2}}}$$

$$S_{OX} = 2\pi\int y\,\frac{ds}{dx}\,dx$$

$$= 2\pi\int_0^{2\pi} a(1-\cos\theta)\times\frac{\sqrt{2}}{(1-\cos\theta)^{\frac{1}{2}}}\times a(1-\cos\theta)d\theta$$

$$= 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}\,d\theta$$

$$= 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\left[\frac{1}{3}\cos^3\frac{\theta}{2} - \cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 16\pi a^2\left[\frac{1}{3}(-1-1) - (-1-1)\right]$$

$$= 16\pi a^2\left(2 - \frac{2}{3}\right)$$

$$= \frac{64\pi a^2}{3}$$

(*f*) To find the height of the centroid of the cycloidal curve above the base.

By symmetry, the centroid is evidently situated in the vertical centre line. Let $\bar{y}_c$ be the height.

Then

$$S_{OX} = 2\pi s\bar{y}_c$$

$$\bar{y}_c = \frac{S_{OX}}{2\pi s}$$

$$= \frac{64\pi a^2}{48\pi a}$$

$$= \frac{4a}{3}$$

147. *Polar Co-ordinates.* If P is a point whose rectangular co-ordinates are x, y, its position with respect to the axes of reference can also be determined by means of a distance r measured from the origin along a radial line which is inclined to the axis OX at an angle θ. r and θ are spoken of as the polar co-ordinates of the point. (Fig. 90.)

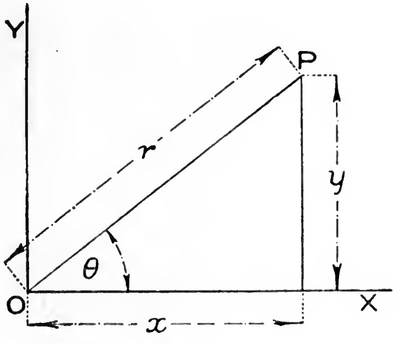


FIG. 90.

The relations between the rectangular and polar co-ordinates can be very easily determined.

For $\qquad x = r \cos \theta, \quad \text{and } y = r \sin \theta$

also $\qquad x^2 + y^2 = r^2$

These relations can be used to transform the equation of a curve from one system to the other.

For example, the equation of a parabola in rectangular co-ordinates is $y^2 = 4ax$.

Then
$$r^2 \sin^2 \theta = 4ar \cos \theta$$
$$r = 4a \frac{\cos \theta}{\sin^2 \theta}$$
$$= \frac{8a \cos \theta}{1 - \cos 2\theta}$$

148. *The Area of a Curve in Polar Co-ordinates.* Let P and Q be two points on a curve, taken very close together. (Fig. 91.)

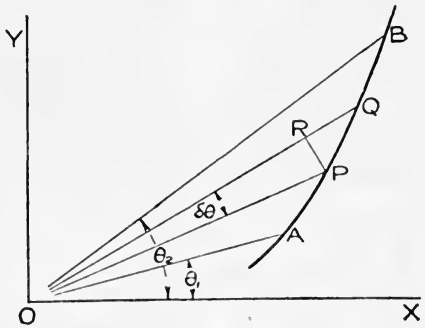


FIG. 91.

The polar co-ordinates of P being r, θ, and of Q $(r + \delta r), (\theta + \delta\theta)$.

Let PR be drawn perpendicular to OQ. Then if $\delta\theta$ is small the following relations are approximately true,

$$\text{OR} = \text{OP} = r$$
$$\text{RQ} = \delta r$$
and
$$\text{PR} = r\,\delta\theta$$

Area of the sector OPQ = area of triangle OPR + area of triangle RPQ

$$= \frac{1}{2} r^2 \sin \delta\theta + \frac{1}{2} r\,\delta\theta\,\delta r$$

When $\delta\theta$ is taken to be very small this area becomes $\frac{1}{2} r^2\,\delta\theta$.

The area of the sector OAB (working with θ_1 and θ_2 as the limits for θ) will be the sum of the areas of all these elementary sectors.

$$\text{Area of sector AOB} = \sum_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, \delta\theta$$

$$= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

when $\delta\theta$ is made infinitely small.

To find the length of the curve.

For a very small arc PQ

$$PQ^2 = QR^2 + PR^2$$

That is
$$\delta s^2 = \delta r^2 + r^2 \, \delta\theta^2$$

Then
$$\frac{\delta s}{\delta\theta} = \sqrt{\left(\frac{\delta r}{\delta\theta}\right)^2 + r^2}$$

or
$$\frac{\delta s}{\delta r} = \sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

When $\delta\theta$ becomes infinitely small,

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

or
$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$$

Hence to get s, the length of a certain part of the curve, the first expression must be integrated with respect to θ and the second with respect to r; in each case the integration being taken between given limits,

and
$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

or
$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \, dr$$

149. *Example* 1. To find the area and the length of the cardioid, the equation of which is $r = a(1 - \cos\theta)$.

To draw the curve, give θ some well-known values and calculate the corresponding values of r.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
r	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$	$\frac{3a}{2}$	a	$\frac{a}{2}$	0

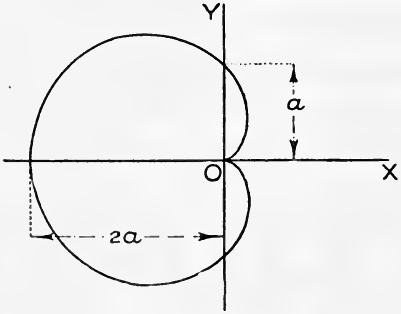


FIG. 92.

The curve is evidently symmetrical about the horizontal axis, and therefore the whole area will be twice the area of the top half.

$$\text{Area} = 2\int_0^\pi \frac{1}{2} r^2\, d\theta$$

$$= a^2\int_0^\pi (1 - \cos\theta)^2\, d\theta$$

$$= a^2\int_0^\pi (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= a^2\int_0^\pi \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi$$

$$= \frac{3\pi a^2}{2}$$

Since $\qquad r = a(1 - \cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= 2a^2(1 - \cos\theta)$$

$$\text{Length} = 2\int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\sqrt{2}a\int_0^\pi (1 - \cos\theta)^{\frac{1}{2}}\, d\theta$$

$$= 4a\int_0^\pi \sin\frac{\theta}{2}\, d\theta$$

$$= -8a\left[\cos\frac{\theta}{2}\right]_0^\pi$$

$$= -8a\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= 8a$$

150. *Example* 2. Transform the equation $x^3 + y^3 = 3xy$ into polar co-ordinates, and then find the area of the loop.

$$x^3 + y^3 = 3xy$$

$$r^3 \cos^3\theta + r^3 \sin^3\theta = 3r^2 \sin\theta \cos\theta$$

Then
$$r = \frac{3\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$$

Now $r = 0$ when θ has the values 0 and $\frac{\pi}{2}$, and so the loop evidently occurs between these values of θ.

To draw the loop, give θ some intermediate values and calculate the corresponding values of r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	1·677	2·121	1·677	0

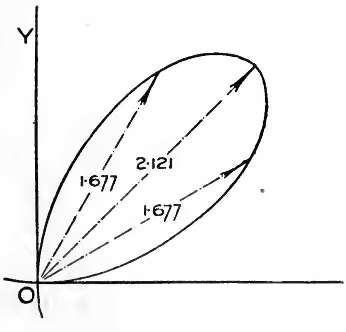


FIG. 93.

$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta\,d\theta}{(\sin^3\theta + \cos^3\theta)^2}$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\cos^6\theta}\;\frac{d\theta}{\left(\frac{\sin^3\theta}{\cos^3\theta} + 1\right)^2}$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta\sec^2\theta\,d\theta}{(\tan^3\theta + 1)^2}$$

Putting $x = \tan\theta$

Then $dx = \sec^2\theta\,d\theta$

and $$\int\frac{\tan^2\theta\sec^2\theta\,d\theta}{(\tan^3\theta+1)^2} = \int\frac{x^2\,dx}{(x^3+1)^2}$$

Putting $x^3 + 1 = y$

Then $dy = 3x^2\,dx$

and $$\int\frac{x^2\,dx}{(x^3+1)^2} = \frac{1}{3}\int\frac{dy}{y^2}$$

$$= -\frac{1}{3y}$$

$$= -\frac{1}{3(x^3+1)}$$

$$= -\frac{1}{3(\tan^3\theta+1)}$$

$$\text{Hence area} = \frac{9}{2}\left[-\frac{1}{3(\tan^3\theta+1)}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{3}{2}\Big[0 - 1\Big]$$

$$= \frac{3}{2}$$

Examples XVIII

(1) Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 0$ and $x = \frac{1}{2}$.

(2) Find the area of the surface of revolution produced by that part of the curve $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 0$ and $x = \frac{1}{2}$, rotating about the axis of x.

(3) Find the length of the curve $y^2 = 4x$ between $x = 0$ and $x = 1$.

(4) Find the area of the surface of revolution produced by that part of the curve $y^2 = 4x$ between $x = 0$ and $x = 1$, rotating about the axis of x.

(5) Use the results of Questions 3 and 4 to find the height of the centroid of the arc of the parabola $y^2 = 4x$ between $x = 0$ and $x = 1$ above the axis of x.

(6) Find the height of the centroid of a semicircular arc of radius a above the diameter.

(7) Find the length of the curve $y^2 = 9x^3$ between $x = 1$ and $x = 2$.

(8) Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ between $x = 0$ and $x = 8$.

(9) Find the area of the surface of revolution produced by that part of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ between $x = 0$ and $x = 8$, rotating about the axis of x. What is the height of the centroid of this part of the curve above the axis of x?

(10) Find the length of that part of the curve $y = 2x - x^2$ between $x = 0$ and $x = 1$.

(11) Find the length of that part of the curve $y = \log_e x$ between $x = 1$ and $x = 2$.

(12) Express the equations of the following curves in polar co-ordinates:

$$(1)\quad y^2 = \frac{x^3}{2a - x}$$

$$(2)\quad x^2y^2 = a^2(x^2 - y^2)$$

$$(3)\quad a^2y^4 = x^4(a^2 - x^2)$$

(13) Trace the curve $r^2 = 16 \sin^2 \theta + 25 \cos^2 \theta$, and find the area enclosed by it.

(14) Transform the equation of the curve $(x^2 + y^2)^2 = 9(x^2 - y^2)$ into polar co-ordinates. Trace the curve, and find the area of a loop.

(15) Trace each of the following curves between $\theta = 0$ and $\theta = 2\pi$:

$$(1)\quad r\theta = 4$$

$$(2)\quad r\theta^{\frac{1}{2}} = 4$$

$$(3)\quad r\theta^2 = 4$$

For each curve find the area of a sector between $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

CHAPTER XIX

151. When a beam is subjected to some system of loading, the beam is slightly bent out of its horizontal position. The bending action depends upon the extent, the character, and the position of the loads, and also this bending action varies at different sections of the beam. If A is the section of a beam (Fig. 94) situated at a distance x from the point of support, and R is the reaction of the support, then all the forces to the right of A help to produce the bending action at A.

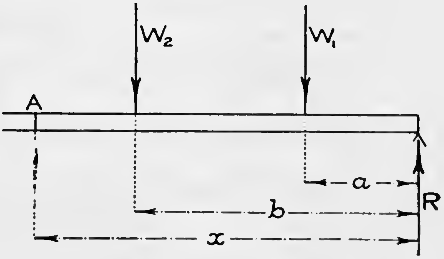


FIG. 94.

Let W_1 and W_2, situated at distances a and b respectively from the end of the beam, be the loads on that part of the beam to the right of A.

The bending action at A is measured by the algebraic sum of the moments of R, W_1, and W_2, and this is defined as the "Bending Moment" at A.

Thus the bending moment at A $= Rx - W_1(x-a) - W_2(x-b)$.

In general, the bending moment at any section of a beam may be defined as the algebraic sum of the moments of all the external forces acting on that part of the beam, to the right or to the left of that section.

Example 1. A beam 30 feet long is supported at the ends. It is divided into three equal spans, which carry uniformly distributed loads of $\frac{3}{4}$ ton per foot run, 1 ton per foot run, and $\frac{1}{2}$ ton per foot run respectively. Find expressions for the bending moment at any point in each span, and draw the bending moment diagram.

If R_1 and R_2 are the reactions at the supports

Then $$30R_2 = 5 \times 25 + 10 \times 15 + 7{\cdot}5 \times 5$$

$$R_2 = 10\frac{5}{12} \text{ tons}$$

Also $$30R_1 = 7{\cdot}5 \times 25 + 10 \times 15 + 5 \times 5$$

$$R_1 = 12\frac{1}{12} \text{ tons}$$

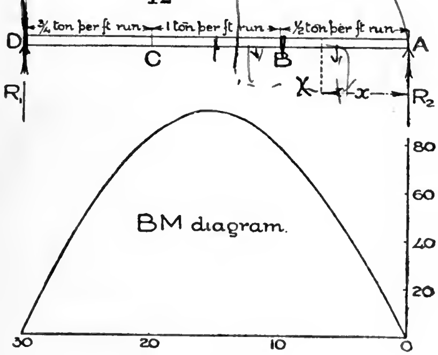


FIG. 95.

(1) Considering a section situated between A and B at a distance x feet from the end. The forces acting to the right of this section are:

(*a*) $R_2 = 10\frac{5}{12}$ tons vertically upwards at a distance x ft.

(*b*) $\frac{x}{2}$ tons acting vertically downwards at a distance $\frac{x}{2}$ ft. from the section.

Bending moment $= \frac{125}{12}x - \frac{x^2}{4}$ ft. tons, and this expression can only be used when x has values between 0 and 10 ft.

(2) Considering a section situated between B and C at a distance x feet from the end. The forces acting to the right of this section are:

(*a*) $R_2 = 10\frac{5}{12}$ tons acting vertically upwards at a distance x ft.

(*b*) 5 tons acting vertically downwards at a distance $(x - 5)$ ft.

(*c*) $(x - 10)$ tons acting vertically downwards at a distance $\frac{1}{2}(x - 10)$ ft.

Bending moment $= \frac{125}{12}x - 5(x - 5) - \frac{1}{2}(x - 10)^2$ ft. tons and this expression can only be used when x has values between 10 ft. and 20 ft.

(3) Considering a section between C and D situated at a distance x ft. from the end. The forces acting to the right of this section are:

(*a*) $R_2 = 10\frac{5}{12}$ tons acting vertically upwards at a distance x ft.
(*b*) 5 tons acting vertically downwards at a distance $(x - 5)$ ft.
(*c*) 10 tons acting vertically downwards at a distance $(x - 15)$ ft.
(*d*) $\frac{3}{4}(x - 20)$ tons acting vertically downwards at a distance $\frac{1}{2}(x - 20)$ ft.

Bending moment $= \frac{125}{12}x - 5(x - 5) - 10(x - 15) - \frac{3}{8}(x - 20)^2$ ft. tons, and this expression can only be used when x has values between 20 ft. and 30 ft.

Taking these three expressions for the bending moment and giving x values suitable to each, values of the bending moment can be calculated for different sections of the beam.

Between A and B.		Between B and C.		Between C and D.	
x ft.	M ft. tons.	x ft.	M ft. tons.	x ft.	M ft. tons.
0	0	10	79·17	20	83·33
2	19·83	12	88·00	22	72·67
4	37·67	14	92·83	24	59·00
6	53·50	16	93·67	26	42·33
8	67·33	18	90·50	28	22·67
10	79·17	20	83·33	30	0

The bending moment is evidently greatest at a section situated in the middle span and for that span

$$M = \frac{125}{12}x - 5(x - 5) - \frac{1}{2}(x - 10)^2$$

$$\frac{dM}{dx} = \frac{125}{12} - 5 - (x - 10)$$

For a maximum value $\frac{dM}{dx} = 0$

That is, when $x = 15\frac{5}{12}$ ft.

and $M_{max} = 93{\cdot}83$ ft. tons

Example 2. A beam 30 ft. long, supported at the ends, carries a load which increases uniformly from 0 at one end to 1 ton at the other end. Find the expression for the bending moment at any section and draw the bending moment diagram. What is the greatest bending moment and where does it occur ?

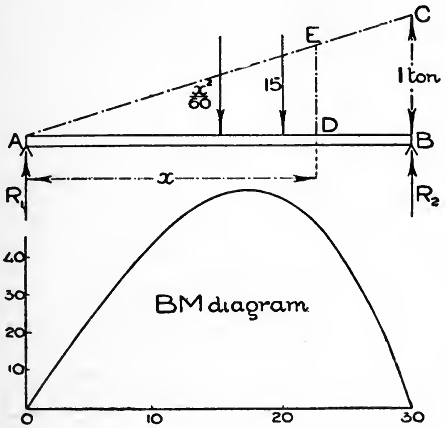


FIG. 96.

The total load on the beam is given by the area of the load diagram ABC and it acts at the centroid of that diagram. Hence the total load is 15 tons acting at a distance of 10 ft. from B.

The reactions of the supports are evidently

$$R_1 = 5 \text{ tons at A}$$

and

$$R_2 = 10 \text{ tons at B}$$

Considering a section D of the beam situated at a distance x ft. from A.

The forces acting on that part of the beam to the left of D are :

(*a*) $R_1 = 5$ tons acting vertically upwards at a distance x ft.

(*b*) That part of the load, the magnitude of which is given by the area of the load diagram ADE ; this acts vertically downwards through the centroid of the area ADE. Hence the magnitude of this force is $\frac{x^2}{60}$ tons and the distance from D is $\frac{x}{3}$ ft.

$$\text{Bending moment} = 5x - \frac{x^3}{180} \text{ ft. tons.}$$

If x be given any value between 0 and 30 ft. the value of the bending moment at any section can be calculated :

x ft.	0	3	6	9	12	15	18	21	24	27	30
M. ft. tons	0	14·85	28·80	40·95	50·40	56·25	57·60	53·55	43·20	25·60	0

Now $$M = 5x - \frac{x^3}{180}$$

$$\frac{dM}{dx} = 5 - \frac{x^2}{60}$$

M is a maximum when $\frac{dM}{dx} = 0$

That is, when $x^2 = 300$

or $x = 10\sqrt{3} = 17{\cdot}32$ ft.

$$M_{max} = 57{\cdot}73 \text{ ft. tons}$$

152. Let P be any section of a beam carrying some system of loads. The only force action at P is the action between the particles ; while at that section there is acting a vertical external force which is the equilibrant of all the forces acting on that part of the beam to the right, or to the left, of the section ; this force is spoken of as the " Shearing Force " at the section. Now the portion of the beam to the right of the section is in equilibrium against vertical translation, but it would have the tendency to rotate, and the magnitude and direction of this rotation is determined by the bending moment at the section. At the section there are elastic forces induced by the external loading, and these forces produce at that section a couple whose magnitude is equal to that of the bending moment, but the direction is opposite. These elastic forces consist of pulls decreasing uniformly to zero and thrusts increasing uniformly from zero. Hence the algebraic sum of the moments of all these pulls and thrusts, taken with reference to some fixed axis of the section, must balance the algebraic sum of the moments of the external forces acting on that portion of the beam to the right, or to the left of the section ; or,—

Moment of resistance of the section = bending moment at the section.

The axis about which the moment of resistance is taken is called the neutral axis of the section and is that axis, or that fibre of the beam which is in an unstrained condition. Thus, in brief, at a given section, the shearing force is the equilibrant of all the external forces acting on that portion of the beam to the right, or to the left of that section. The bending moment is the algebraic sum of the moments of all the external forces acting on that portion of the beam to the right, or to the left, of that section. The moment of resistance is the algebraic sum of the moments of all the induced tensile and compressive stresses, taken about the neutral axis.

153. Let Fig. 97 represent a portion of a bent beam, so taken that the form assumed is a circular arc of radius equal to the radius of curvature. **C** is the centre of curvature.

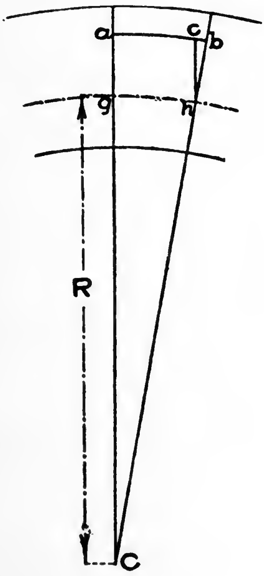


FIG. 97.

The top part of the beam is in tension and the lower part is in compression, while there is one surface in an unstrained condition, known as the Neutral Surface. The intersection of the neutral surface with a certain section of the beam gives the neutral axis of that section.

Let *ab* be a fibre situated at a distance y from the neutral surface. This fibre is in a strained condition, and

$$\text{The strain in } ab = \frac{\text{increase in length}}{\text{original length}}$$

$$= \frac{bc}{ac}$$

$$= \frac{bc}{gh}$$

But the figures *bch* and *ghc* are similar,

Hence $$\frac{bc}{gh} = \frac{hc}{gC} = \frac{y}{R}$$

Then the strain in $ab = \dfrac{y}{R}$ where R is the radius of curvature measured from the neutral surface.

But stress = E × strain, where E is Young's Modulus,

Then $$\text{Stress in } ab = E\frac{y}{R}$$

or $$p = E\frac{y}{R}$$

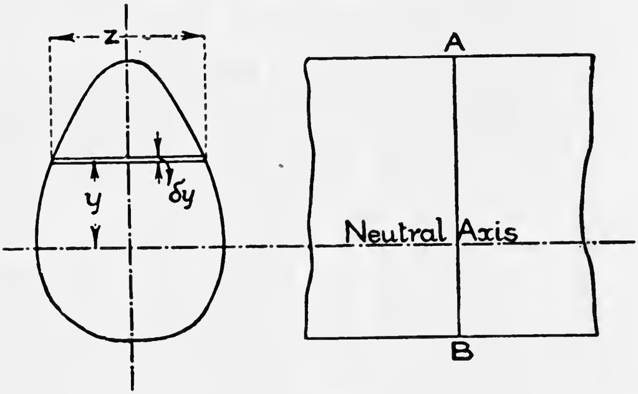


FIG. 98.

Let AB (Fig. 98) be any section of a beam, and considering a thin horizontal strip of that section, the breadth of the strip being δy, the length of the strip z, and y the height above the neutral axis,

Then $$\text{Stress at height } y = E\frac{y}{R}$$

$$\text{Total force on the strip} = \text{Stress} \times \text{Area}$$

$$= E\frac{y}{R} z\,\delta y$$

$$= \frac{E}{R} y\,\delta A$$

where $\delta A = z\,\delta y$ = Area of the strip.

$$\text{Total force acting on the whole section} = \frac{E}{R}\sum y\,\delta A$$

$$= \frac{E}{R} A\bar{y}$$

where $\bar{y}$ is the height of the centroid of the area above the neutral axis and A is the whole area.

But since a plane section always remains plane, the total force acting on that section must be zero, or the resultant pull must be equal to the resultant thrust.

Hence $$\frac{E}{R}A\bar{y} = 0$$

or $$\bar{y} = 0$$

Therefore the neutral axis must pass through the centroid of the section.

Referring again to the elementary strip,

$$\text{Total force on the strip} = E\frac{y}{R}z\,\delta y$$

$$= E\frac{y}{R}\,\delta A$$

$$\text{Moment of this force about the neutral axis} = \frac{E}{R}y^2\,\delta A$$

$$\text{The moment of resistance for the whole section} = \frac{E}{R}\sum y^2\,\delta A$$

$$= \frac{E}{R}I$$

where **I** is the moment of inertia of the section about the neutral axis.

Now the moment of resistance = Bending moment,

Hence $$M = \frac{E}{R}I$$

or $$\frac{M}{I} = \frac{E}{R}$$

Also, since **R** is the radius of curvature and the slope of the beam is very small, $$\frac{1}{R} = \frac{d^2y}{dx^2}$$

Then $$M = EI\frac{d^2y}{dx^2}$$

In general, for any section of a loaded beam, if

I = moment of inertia of the section about the neutral axis
M = bending moment at the section
E = Young's Modulus for the material
R = radius of curvature at the section
p = stress induced in the strained fibre
y = distance of that strained fibre from the neutral axis

Then $$\frac{M}{I} = \frac{E}{R} = \frac{p}{y}$$

and also $$M = EI\frac{d^2y}{dx^2}$$

154. *The Deflection of Beams.* (1) A cantilever of length l ft. carrying a load W tons at one end.

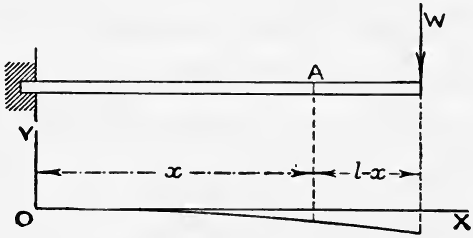


Fig. 99.

Let A be a section situated at a distance x ft. from the fixed end.

Bending moment at A = $W(l - x)$ ft. tons.

Hence $$EI\frac{d^2y}{dx^2} = W(l - x)$$

$$EI\frac{dy}{dx} = W\left(lx - \frac{1}{2}x^2\right) + \text{Const}$$

But at O, where $x = 0$, $\frac{dy}{dx} = 0$, since at that point the direction of the beam is horizontal. Therefore Const = 0.

$$EI\frac{dy}{dx} = W\left(lx - \frac{1}{2}x^2\right)$$

and $$EIy = W\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Hence Const = 0

Then $$y = \frac{W}{EI}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right)$$

$$= \frac{W}{6EI}(3lx^2 - x^3)$$

This gives the deflection at any point distant x ft. from the fixed end. The deflection is evidently greatest when $x = l$.

Then $$y_{max} = \frac{Wl^3}{3EI}$$

155. (2) A cantilever of length l carrying a uniformly distributed load of w tons per foot run.

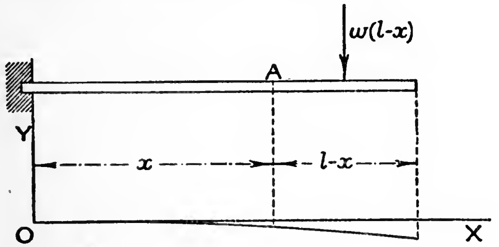


FIG. 100.

Let A be a section situated at a distance x ft. from the fixed end. The load on the portion of the cantilever to the right of A is $w(l - x)$ tons, and this acts at a distance of $\frac{1}{2}(l - x)$ ft. from A.

Bending moment at A $= \frac{w}{2}(l - x)^2$ ft. tons.

Hence $$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{w}{2}(l^2 - 2lx + x^2)$$

and $$\mathrm{EI}\frac{dy}{dx} = \frac{w}{2}\left(l^2x - lx^2 + \frac{1}{3}x^3\right) + \text{Const}$$

But at O, where $x = 0$, $\frac{dy}{dx} = 0$. Then Const $= 0$

$$\mathrm{EI}\frac{dy}{dx} = \frac{w}{2}\left(l^2x - lx^2 + \frac{1}{3}x^3\right)$$

Then $$\mathrm{EI}y = \frac{w}{2}\left(\frac{1}{2}\,l^2x^2 - \frac{1}{3}\,lx^3 + \frac{1}{12}\,x^4\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Then Const $= 0$

and $$y = \frac{w}{2\mathrm{EI}}\left\{\frac{1}{2}\,l^2x^2 - \frac{1}{3}\,lx^3 + \frac{1}{12}\,x^4\right\}$$

$$= \frac{w}{24\mathrm{EI}}\{6l^2x^2 - 4lx^3 + x^4\}$$

This gives the deflection at any point distant x feet from the fixed end. The deflection is evidently greatest when $x = l$.

Then $$y_{max} = \frac{wl^4}{8\mathrm{EI}}$$

$$= \frac{\mathrm{W}l^3}{8\mathrm{EI}}$$

where $\mathrm{W} = wl$, the total load on the cantilever.

156. (3) A beam of length l ft. supported at both ends and carrying a concentrated load W tons at the middle.

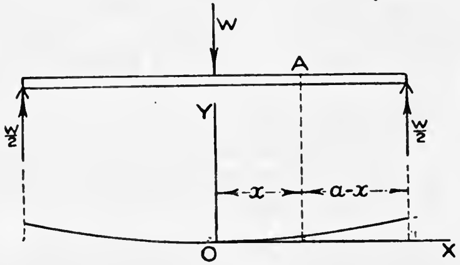


FIG. 101.

The reaction at each support is $\frac{1}{2}$W tons.

Let the centre of the beam be the origin, and let a be the length of the half span.

Let A be a section situated at a distance x ft. from O.

$$\text{Bending moment at A} = \frac{W}{2}(a - x) \text{ ft. tons}$$

Hence
$$EI\frac{d^2y}{dx^2} = \frac{W}{2}(a - x)$$

and
$$EI\frac{dy}{dx} = \frac{W}{2}\left(ax - \frac{1}{2}x^2\right) + \text{Const}$$

But at O, where $x = 0$, $\frac{dy}{dx} = 0$. Then Const $= 0$

$$EI\frac{dy}{dx} = \frac{W}{2}\left(ax - \frac{1}{2}x^2\right)$$

Then
$$EIy = \frac{W}{2}\left(\frac{1}{2}ax^2 - \frac{1}{6}x^3\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Then Const $= 0$

and
$$y = \frac{W}{2EI}\left(\frac{1}{2}ax^2 - \frac{1}{6}x^3\right)$$
$$= \frac{W}{12EI}(3ax^2 - x^3)$$

This gives the value of y for any point distant x ft. from the centre, and y is greatest when $x = a$.

$$y_{max} = \frac{Wa^3}{6EI}$$
$$= \frac{Wl^3}{48EI}, \quad \text{since } a = \frac{l}{2}$$

This also gives the maximum value of the deflection.

In order to obtain δ, the deflection at any point,

$$\delta = y_{max} - y$$

$$= \frac{Wa^3}{6EI} - \frac{W}{12EI}(3ax^2 - x^3)$$

$$= \frac{W}{12EI}\{2a^3 - 3ax^2 + x^3\}$$

157. (4) A beam of length l ft. supported at both ends and carrying a uniformly distributed load of w tons per foot run.

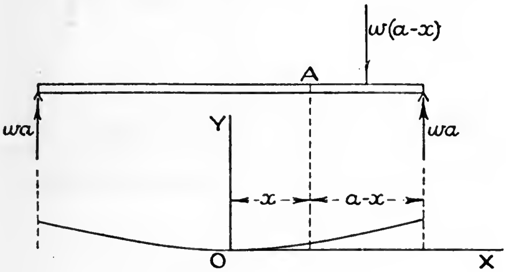


FIG. 102.

The reaction at each support is wa tons, where a is the length of the half span.

Let the centre of the beam be the origin, and let A be a section situated at a distance x ft. from O.

Bending moment at A $= wa(a-x) - \frac{w}{2}(a-x)^2$

$$= \frac{w}{2}(a^2 - x^2) \text{ ft. tons}$$

Then $$EI\frac{d^2y}{dx^2} = \frac{w}{2}(a^2 - x^2)$$

and $$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{1}{3}x^3\right) + \text{Const}$$

But at O, where $x = 0$, $\frac{dy}{dx} = 0$. Then Const $= 0$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{1}{3}x^3\right)$$

Then $$EIy = \frac{w}{2}\left(\frac{1}{2}a^2x^2 - \frac{1}{12}x^4\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Then Const $= 0$

and
$$y = \frac{w}{2\mathrm{EI}}\left(\frac{1}{2}a^2x^2 - \frac{1}{12}x^4\right)$$
$$= \frac{w}{24\mathrm{EI}}(6a^2x^2 - x^4)$$

This gives the value of y for any point distant x feet from the centre, and y is greatest when $x = a$.

$$y_{max} = \frac{5wa^4}{24\mathrm{EI}}$$
$$= \frac{5wl^4}{384\mathrm{EI}}, \text{ since } a = \frac{l}{2}$$
$$= \frac{5\mathrm{W}l^3}{384\mathrm{EI}}$$

where $\mathrm{W} = wl$ tons, the total load on the beam.

This also gives the maximum value of the deflection.

In order to obtain δ, the deflection at any point,

$$\delta = y_{max} - y$$
$$= \frac{5wa^4}{24\mathrm{EI}} - \frac{w}{24\mathrm{EI}}(6a^2x^2 - x^4)$$
$$= \frac{w}{24\mathrm{EI}}(5a^4 - 6a^2x^2 + x^4)$$

158. (5) A beam of length l ft, fixed at both ends, carrying a concentrated load W tons at the middle.

The effect of keeping each end horizontal is the same as applying at each end a couple of magnitude u, which acts in a clockwise direction. Also, since at the ends and at the centre the direction of the beam is horizontal at these points, $\frac{dy}{dx} = 0$.

The vertical reaction at each point of fixing is $\frac{\mathrm{W}}{2}$ tons.

Let $a =$ the half span, and let the centre of the beam be taken as the origin.

Let A be a section situated at a distance x ft. from O.

Bending moment at A $= \frac{\mathrm{W}}{2}(a - x) - u$ ft. tons

Hence
$$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{\mathrm{W}}{2}(a - x) - u$$

and
$$\mathrm{EI}\frac{dy}{dx} = \frac{\mathrm{W}}{2}\left(ax - \frac{1}{2}x^2\right) - ux + \text{Const}$$

But at O, where $x = 0$, $\frac{dy}{dx} = 0$. Then Const $= 0$

$$EI\frac{dy}{dx} = \frac{W}{2}\left(ax - \frac{1}{2}x^2\right) - ux$$

At the end, that is, where $x = a$, $\frac{dy}{dx} = 0$

and $$\frac{W}{2}\left(a^2 - \frac{1}{2}a^2\right) - ua = 0$$

or $$u = \frac{Wa}{4}$$

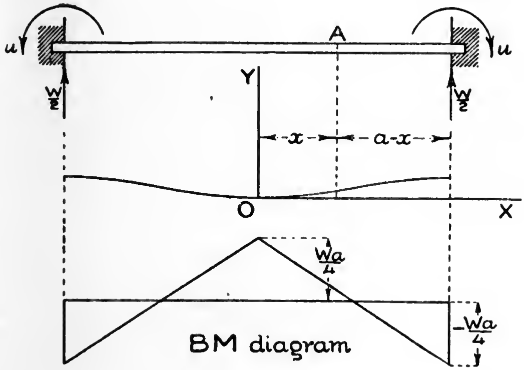


FIG. 103.

This gives the actual magnitude of the fixing couple

and $$EI\frac{dy}{dx} = \frac{W}{2}\left(ax - \frac{1}{2}x^2\right) - \frac{1}{4}Wax$$

$$= \frac{W}{4}(ax - x^2)$$

Then $$EIy = \frac{W}{4}\left(\frac{1}{2}ax^2 - \frac{1}{3}x^3\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Then Const $= 0$

and $$y = \frac{W}{4EI}\left(\frac{1}{2}ax^2 - \frac{1}{3}x^3\right)$$

$$= \frac{W}{24EI}(3ax^2 - 2x^3)$$

This gives the value of y for any point distant x feet from the centre, and y is greatest when $x = a$.

$$y_{max} = \frac{W}{24EI}(3a^3 - 2a^3)$$
$$= \frac{Wa^3}{24EI}$$
$$= \frac{Wl^3}{192EI}, \quad \text{since } a = \frac{l}{2}$$

This also gives the maximum value of the deflection.

In order to obtain δ, the deflection at any point,

$$\delta = y_{max} - y$$
$$= \frac{Wa^3}{24EI} - \frac{W}{24EI}(3ax^2 - 2x^3)$$
$$= \frac{W}{24EI}(a^3 - 3ax^2 + 2x^3)$$

Since the bending moment at any point is $\frac{W}{2}(a - x) - u$ and $u = \frac{Wa}{4}$

Then
$$M = \frac{W}{2}(a - x) - \frac{Wa}{4}$$
$$= \frac{W}{4}(a - 2x)$$

Thus when $x = a$, $M = -\frac{1}{4}Wa$; when $x = 0$, $M = \frac{1}{4}Wa$; and when $x = \frac{a}{2}$, $M = 0$.

Therefore the bending moment increases uniformly from $-\frac{1}{4}Wa$ at the point of fixing to $+\frac{1}{4}Wa$ at the centre of the beam, and at a point half-way it vanishes.

159. (6) A beam of length l feet fixed at both ends, carrying a uniformly distributed load of w tons per foot run.

The effect of keeping each end horizontal is the same as applying at each end a couple of magnitude u, which acts in a clockwise direction. Also, since at the ends and at the centre the direction of the beam is horizontal, at these points $\frac{dy}{dx} = 0$.

The vertical reaction at each point of fixing is wa tons, where a is the length of the half span.

Let the centre of the beam be taken as the origin, and let A be a section situated at a distance x ft. from O. The portion of the load on that part of the beam to the right of A is $w(a-x)$ tons.

$$\text{Bending moment at A} = wa(a-x) - \frac{1}{2}w(a-x)^2 - u$$

$$= \frac{w}{2}(a^2 - x^2) - u \text{ ft. tons}$$

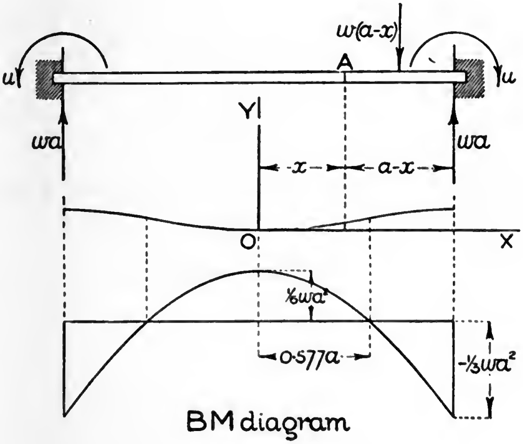


FIG. 104.

Hence $$EI\frac{d^2y}{dx^2} = \frac{w}{2}(a^2 - x^2) - u$$

and $$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{1}{3}x^3\right) - ux + \text{Const}$$

But at O, where $x = 0$, $\dfrac{dy}{dx} = 0$. Then Const $= 0$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{1}{3}x^3\right) - ux$$

At the end, that is, where $x = a$, $\dfrac{dy}{dx} = 0$

and $$\frac{w}{2}\left(a^3 - \frac{1}{3}a^3\right) - ua = 0$$

or $$u = \frac{wa^2}{3}$$

This gives the actual magnitude of the fixing couple,

and $$\mathrm{EI}\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{1}{3}x^3\right) - \frac{1}{3}wa^2x$$

$$= \frac{w}{6}(a^2x - x^3)$$

Then $$\mathrm{EI}y = \frac{w}{6}\left(\frac{1}{2}a^2x^2 - \frac{1}{4}x^4\right) + \text{Const}$$

But at O, where $x = 0$, $y = 0$. Then Const $= 0$

and $$y = \frac{w}{6\mathrm{EI}}\left(\frac{1}{2}a^2x^2 - \frac{1}{4}x^4\right)$$

$$= \frac{w}{24\mathrm{EI}}(2a^2x^2 - x^4)$$

This gives the value of y for any point distant x feet from the centre, and y is greatest when $x = a$.

$$y_{max} = \frac{w}{24\mathrm{EI}}(2a^4 - a^4)$$

$$= \frac{wa^4}{24\mathrm{EI}}$$

$$= \frac{wl^4}{384\mathrm{EI}}, \quad \text{since } a = \frac{l}{2}$$

$$= \frac{\mathrm{W}l^3}{384\mathrm{EI}}$$

where $\mathrm{W} = wl$, the total load on the beam.

This also gives the maximum value of the deflection.

In order to obtain δ, the deflection at any point,

$$\delta = y_{max} - y$$

$$= \frac{wa^4}{24\mathrm{EI}} - \frac{w}{24\mathrm{EI}}(2a^2x^2 - x^4)$$

$$= \frac{wa^4}{24\mathrm{EI}}(a^4 - 2a^2x^2 + x^4)$$

Since the bending moment at any point is $\frac{w}{2}(a^2 - x^2) - u$ and $u = \frac{wa^2}{3}$

Then $$\mathrm{M} = \frac{w}{2}(a^2 - x^2) - \frac{wa^2}{3}$$

$$= \frac{w}{6}(a^2 - 3x^2)$$

Thus when $x = a$, $M = -\frac{wa^2}{3}$; when $x = 0$, $M = \frac{wa^2}{6}$; and when $x = \frac{a}{\sqrt{3}} = 0{\cdot}577a$, $M = 0$.

Therefore the bending moment curve is a parabola, and the bending moment increases from $-\frac{wa^2}{3}$ at the point of fixing to $+\frac{wa^2}{6}$ at the centre of the beam, while at a point situated at a distance $0{\cdot}577a$ from the centre of the beam the bending moment vanishes.

160. Let A and B be two sections of a beam taken very close together (Fig. 105), δx being the distance between these sections and $w\,\delta x$ the load on this elementary length of the beam.

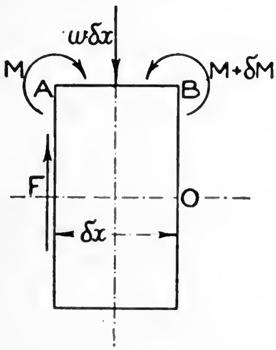


FIG. 105.

Let M be the bending moment at A and $M + \delta M$ the bending moment at B. Also F is the shearing force at A.

Taking moments about O,

$$M + \delta M = M + F\,\delta x - \frac{1}{2} w\,\delta x^2$$

or

$$\delta M = F\,\delta x$$

taking $w\,\delta x^2$ to be negligibly small in comparison with $F\,\delta x$.

When δx is made infinitely small,

$$F = \frac{dM}{dx}$$

that is, the shearing force at a section is the rate at which the bending moment is changing with respect to the length.

161. Let CD and C_1D_1 be two sections of a beam, δx apart (Fig. 106), the bending moment at CD being M, and at C_1D_1

$M + \delta M$. Considering a fibre situated at a distance y from the neutral axis, z being the breadth of this fibre.

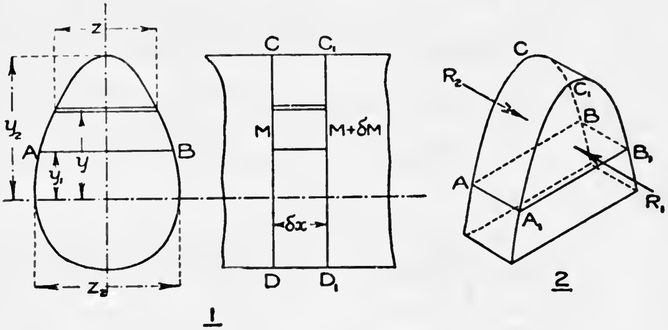


FIG. 106.

For the section CD, if p is the stress in the fibre,

Then $$p = \frac{M}{I} y$$

$$\text{Thrust on the fibre} = pz\,\delta y$$
$$= \frac{M}{I} yz\,\delta y$$

If AB is a fixed line in the section drawn parallel to the neutral axis, at a distance y_1 from it, and if R_2 be the resultant thrust on that part of the section above AB,

Then $$R_2 = \sum_{y_1}^{y_2} \frac{M}{I} yz\,\delta y$$

and this may be taken as acting from left to right.

For the section C_1D_1, if p is the stress in the fibre,

Then $$p = \frac{M + \delta M}{I} y$$

$$\text{Thrust on the fibre} = pz\,\delta y$$
$$= \frac{M + \delta M}{I} yz\,\delta y$$

and if R_1 is the resultant thrust on that part of the section above AB,

Then $$R_1 = \sum_{y_1}^{y_2} \frac{M + \delta M}{I} yz\,\delta y$$

and this may be taken as acting from right to left.

Hence the small portion of the beam contained by the two parallel sections C_1D_1 and CD, δx apart, and a horizontal plane situated at a distance y_1 from the neutral surface (Fig. 106 (2)) is acted upon by two horizontal forces.

$$R_1 = \frac{M + \delta M}{I}\sum_{y_1}^{y_2} yz\,\delta y \text{ acting from right to left, and}$$

$$R_2 = \frac{M}{I}\sum_{y_1}^{y_2} yz\,\delta y \text{ acting from left to right.}$$

The result is a horizontal force $R_1 - R_2$, tending to make this portion of the beam slide over the horizontal surface which is situated at a distance y_1 from the neutral axis. This tendency to slide is resisted by the shearing action at that surface, and if q is the intensity of the shearing stress there,

Then
$$qz_1\,\delta x = R_1 - R_2$$
$$= \frac{\delta M}{I}\sum_{y_1}^{y_2} yz\,\delta y$$
$$q = \frac{\delta M}{\delta x}\,\frac{1}{Iz_1}\sum_{y_1}^{y_2} yz\,\delta y$$

When δy is made infinitely small,

$$q = \frac{\delta M}{\delta x}\,\frac{1}{Iz_1}\int_{y_1}^{y_2} yz\,dy$$

When δx is made infinitely small,

$$q = \frac{F}{Iz_1}\int_{y_1}^{y_2} yz\,dy$$

where $F = \frac{dM}{dx}$, the shearing force at the section.

Example 1. A beam of circular section, to investigate the distribution of shear stress over a section at which the shearing force is F.

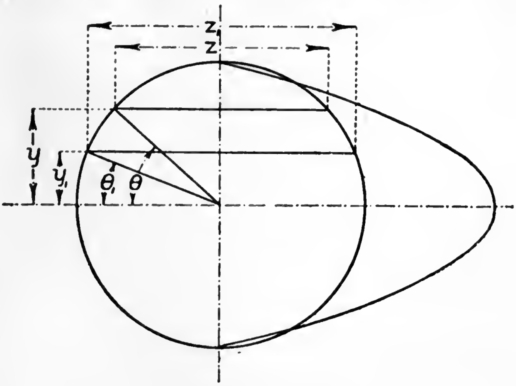


FIG. 107.

Now
$$z_1 = 2R \cos \theta_1$$
$$y_1 = R \sin \theta_1$$
$$z = 2R \cos \theta$$
$$y = R \sin \theta$$
$$dy = R \cos \theta \, d\theta$$

Then
$$q = \frac{F}{Iz_1}\int_{y_1}^{y_2} yz \, dy$$
$$= \frac{FR^2}{I \cos \theta_1}\int_{\theta_1}^{\frac{\pi}{2}} \cos^2 \theta \sin \theta \, d\theta$$
$$= \frac{FR^2}{I \cos \theta_1}\left[-\frac{\cos^3 \theta}{3}\right]_{\theta_1}^{\frac{\pi}{2}}$$
$$= \frac{FR^2}{I \cos \theta_1} \times \frac{\cos^3 \theta_1}{3}$$
$$= \frac{FR^2}{3I} \cos^2 \theta_1$$
$$= \frac{FR^2}{3I}\left(1 - \frac{y_1^2}{R^2}\right)$$
$$= \frac{4F}{3\pi R^2}\left(1 - \frac{y_1^2}{R^2}\right), \text{ since } I = \frac{\pi R^4}{4}$$

Taking a circle of 3 inches radius and calculating the values of q corresponding to horizontal sections situated at distances 0, $\frac{1}{2}''$, $1''$, etc., from the centre.

For any section $q = \dfrac{4F}{27\pi}\left(1 - \dfrac{y_1^2}{9}\right)$

$$= k\left(1 - \frac{y_1^2}{9}\right), \quad \text{where } k = \frac{4F}{27\pi}$$

y_1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
q	k	$\frac{35}{36}k$	$\frac{8}{9}k$	$\frac{3}{4}k$	$\frac{5}{9}k$	$\frac{11}{36}k$	0

Example 2. A beam of rectangular section, depth b, breadth a. To investigate the distribution of shear stress over a section at which the shearing force is F.

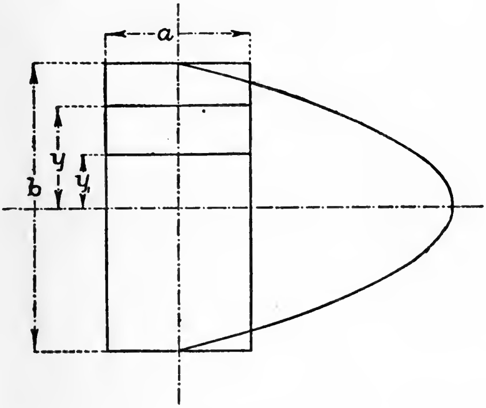


FIG. 108.

Now $$z = z_1 = a$$

and
$$q = \frac{F}{Iz_1}\int_{y_1}^{y_2} yz\,dy$$
$$= \frac{F}{Ia}\int_{y_1}^{\frac{b}{2}} ay\,dy$$
$$= \frac{F}{I}\int_{y_1}^{\frac{b}{2}} y\,dy$$
$$= \frac{F}{I}\left[\frac{y^2}{2}\right]_{y_1}^{\frac{b}{2}}$$
$$= \frac{F}{2I}\left(\frac{b^2}{4} - y_1^2\right)$$
$$= \frac{6F}{ab^3}\left(\frac{b^2}{4} - y_1^2\right), \quad \text{since } I = \frac{ab^3}{12}$$

Taking a rectangle 6 inches deep and 3 inches in width and calculating the values of q at horizontal sections situated at distances 0, ½″, 1″, etc., from the neutral axis.

For any section $q = \frac{6F}{ab^3}(9 - y_1^2)$

$$= k(9 - y_1^2), \quad \text{where } k = \frac{6F}{ab^3}$$

y_1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
q	$9k$	$8 \cdot 75k$	$8k$	$6 \cdot 75k$	$5k$	$2 \cdot 75k$	0

162. The following method for investigating the distribution of shear stress over a section will be found to give a very convenient relation to use providing the section is such that the height of centroid of any portion of it above the neutral axis can be easily determined. It can therefore be used with great advantage to any beam section which can be taken as being made up of rectangles.

Referring to Fig. 106 and considering the section CD.

If p_1 is the stress at AB and p is the stress at KL

Then $$\frac{p}{y} = \frac{p_1}{y_1}, \text{ and } p = \frac{p_1}{y_1} y$$

Thrust on the fibre KL $= pz\,\delta y$

$$= \frac{p_1}{y_1} yz\,\delta y$$

and if R_2 is the resultant thrust on that part of the section above AB,

Then $$R_2 = \frac{p_1}{y_1} \sum_{y_1}^{y_2} yz\,\delta y$$

$$= \frac{p_1}{y_1} \sum_{y_1}^{y_2} y\,\delta a$$

$= \frac{p_1}{y_1} A\bar{y}$ where A is the area of that part of the section above AB and $\bar{y}$ is the height of the centroid of that part of the section above the neutral axis.

Considering the section C_1D_1, if p_2 is the stress at A_1B_1 and p is the stress at K_1L_1

Then $$\frac{p}{y} = \frac{p_2}{y_1}, \text{ and } p = \frac{p_2}{y_1} y$$

and if R_1 is the resultant thrust on that part of the section above A_1B_1

Then
$$R_1 = \frac{p_2}{y_1}\sum_{y_1}^{y_2} yz\,\delta y$$
$$= \frac{p_2}{y_1}\sum_{y_1}^{y_2} y\,\delta a$$
$$= \frac{p_2}{y_1}A\bar{y}$$

These two horizontal forces R_1 and R_2 acting in opposite directions on the parts of the sections C_1D_1 and CD above A_1B_1 and AB respectively, are equivalent to a single horizontal force $R_1 - R_2$ acting in the direction of R_1. This will tend to make the portion of the beam under consideration (Fig. 106 (2)) slide over the surface ABB_1A_1. If q is the intensity of the shear stress at that surface

Then
$$qz_1\,\delta x = R_1 - R_2$$

But
$$R_1 - R_2 = \frac{p_2}{y_1}A\bar{y} - \frac{p_1}{y_1}A\bar{y}$$
$$= \frac{p_2 - p_1}{y_1}A\bar{y}$$

Hence
$$q = \frac{R_1 - R_2}{z_1\,\delta x}$$
$$= \frac{p_2 - p_1}{y_1} \times \frac{A\bar{y}}{z_1\,\delta x}$$

If δM is the increase in the bending moment occurring between C and C_1,

Then at C
$$\frac{M}{I} = \frac{p_1}{y_1}$$

and at C_1
$$\frac{M + \delta M}{I} = \frac{p_2}{y_1}$$

Hence
$$\frac{\delta M}{I} = \frac{p_2 - p_1}{y_1}$$

and
$$q = \frac{\delta M}{I} \times \frac{A\bar{y}}{z_1\,\delta x}$$
$$= \frac{\delta M}{\delta x} \times \frac{A\bar{y}}{Iz_1}$$
$$= \frac{FA\bar{y}}{Iz_1}.$$

where **F** is the shearing force at the section.

Example 1. To investigate the distribution of shear stress over the given section.

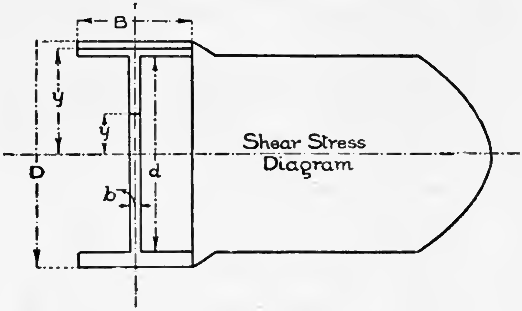


FIG. 109.

(1) For the flange, let q be the shear stress at a distance y from the neutral axis.

Then
$$A\bar{y} = B\left(\frac{D}{2} - y\right) \times \frac{1}{2}\left(\frac{D}{2} + y\right)$$
$$= \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)$$

and
$$q = \frac{FA\bar{y}}{Iz_1}$$
$$= \frac{F}{IB} \times \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)$$
$$= \frac{F}{2I}\left(\frac{D^2}{4} - y^2\right)$$

and this relation will only hold for values of y between $\frac{D}{2}$ and $\frac{d}{2}$, when $y = \frac{D}{2}$, $q = 0$, and when $y = \frac{d}{2}$, $q = \frac{F}{8I}\left(D^2 - d^2\right)$.

(2) For the web let q be the shear stress at a distance y from the neutral axis.

Then
$$A\bar{y} = \frac{B}{2}\left(D - d\right) \times \frac{1}{4}\left(D + d\right) + b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(\frac{d}{2} + y\right)$$
$$= \frac{B}{8}\left(D^2 - d^2\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

and
$$q = \frac{FA\bar{y}}{Iz_1}$$

$$= \frac{F}{Ib}\left\{\frac{B}{8}(D^2 - d^2) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right\}$$

$$= \frac{F}{8I} \times \frac{B}{b}(D^2 - d^2) + \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

and this relation will only hold for values of y between $\frac{d}{2}$ and 0.

$$\text{When } y = \frac{d}{2},\ q = \frac{F}{8I} \times \frac{B}{b}(D^2 - d^2)$$

$$\text{and when } y = 0,\ q = \frac{F}{8I} \times \frac{B}{b}(D^2 - d^2) + \frac{Fd^2}{8I}$$

$$= \frac{F}{8I}\left\{\frac{B}{b}(D^2 - d^2) + d^2\right\}$$

It should be noticed that, in passing from the flange to the web, the shear stress increases suddenly from $\frac{F}{8I}(D^2 - d^2)$ to $\frac{FB}{8Ib}(D^2 - d^2)$, and a consideration of the shear stress diagram for this section will show that the web practically takes all the shear.

Example 2. To investigate the distribution of shear stress over a square section in which the diagonal is horizontal.

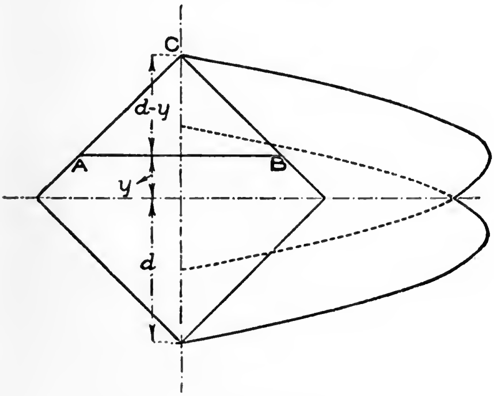


FIG. 110.

Let $2d$ be the length of the diagonal, and let q be the shear stress at AB, at distance y from the diagonal.

Then $$AB = 2(d - y)$$

Area of the triangle $ABC = (d - y)^2$

and
$$\bar{y} = y + \frac{1}{3}(d - y)$$
$$= \frac{1}{3}(d + 2y)$$

Hence
$$q = \frac{\mathrm{FA}\bar{y}}{\mathrm{I}b}$$
$$= \frac{\mathrm{F}}{3}\,\frac{(d-y)^2\,(d+2y)}{2\mathrm{I}\,(d-y)}$$
$$= \frac{\mathrm{F}}{6\mathrm{I}}(d-y)(d+2y)$$

Hence $q = 0$ when $y = d$ and when $y = -\frac{d}{2}$; and q is greatest when $d^2 + dy - 2y^2$ is greatest.

Differentiating this with respect to y and equating the result to zero,
$$d - 4y = 0$$
or
$$y = \frac{1}{4}d$$

The maximum value of $q = \frac{\mathrm{F}}{6\mathrm{I}} \times \frac{3d}{4} \times \frac{9d}{4}$
$$= \frac{9\mathrm{F}d^2}{32\mathrm{I}}$$

and at the centre
$$q = \frac{\mathrm{F}d^2}{6\mathrm{I}}$$

163. *Twisting Moment.* Let a cylinder of length l and radius r be fixed at one end and a twisting moment T applied at the other end. Let AB be the position of a generator of the cylinder before

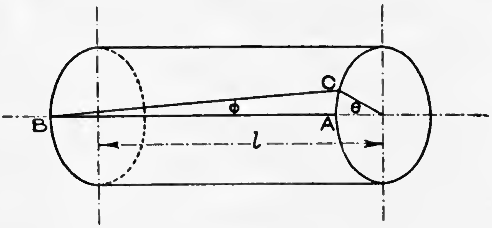


FIG. 111.

the twisting moment has been applied. If θ is the angle of twist, the point A moves to the position C, and the generator AB takes up the position CB on the surface of the cylinder (Fig. 111).

Now $$\text{Shear strain} = \frac{AC}{AB}$$

$$= \tan\varphi$$

$$= \varphi, \text{ if } \varphi \text{ is a small angle.}$$

Thus φ radians measures the shear strain at the surface of the cylinder.

Also $$AC = l\varphi = r\theta$$

and $$\varphi = \frac{r\theta}{l}$$

But shear stress = N × shear strain where N is the modulus of rigidity.

Then $$f_s = N\varphi$$

$$= \frac{N\theta}{l} r$$

where f_s is the shear stress at the surface of the cylinder.

Let q be the intensity of the shear stress at a point in the cylinder whose radial distance is x.

Then $$q = \frac{N\theta}{l} x$$

$$= \frac{x}{r} f_s$$

Let this shear stress take place over an elementary ring of width δx, situated at a radial distance x.

$$\text{Area of elementary ring} = 2\pi x\ \delta x$$

$$\text{Total shearing force} = q \times \text{area}$$

$$= \frac{2\pi}{r} f_s\, x^2\, \delta x$$

The moment of this force about the axis of the cylinder

$$= \frac{2\pi}{r} f_s\, x^3\, \delta x$$

Hence $$T = \text{twisting moment} = \frac{2\pi}{r} f_s \int_0^r x^3\, dx$$

$$= \frac{\pi}{2} r^3 f_s$$

$$= \frac{\pi}{16} d^3 f_s \text{ where } d \text{ is the diameter.}$$

If I is the moment of inertia of the circular section about the axis of the cylinder,

Then $$\mathrm{I} = \frac{\pi r^4}{2}$$

and $$\mathrm{T} = \frac{\pi r^3}{2} f_s$$
$$= \frac{\mathrm{I}}{r} f_s$$

Now $$\theta = \frac{l\varphi}{r}$$
$$= \frac{l}{\mathrm{N}r} f_s$$
$$= \frac{l}{\mathrm{N}r} \times \frac{2\mathrm{T}}{\pi r^3}$$
$$= \frac{2l\mathrm{T}}{\pi \mathrm{N} r^4} \text{ radians}$$
$$= \frac{l\mathrm{T}}{\mathrm{NI}} \text{ radians}$$

Let the cylinder be hollow, and let r_1 and r_2 be the internal and external radii respectively.

Then $$\mathrm{T} = \frac{2\pi}{r_2} f_s \int_{r_1}^{r_2} x^3\, dx$$
$$= \frac{2\pi}{r_2} f_s \left[\frac{x^4}{4}\right]_{r_1}^{r_2}$$
$$= \frac{\pi}{2}\left(\frac{r_2^4 - r_1^4}{r_2}\right) f_s$$
$$= \frac{\pi}{16}\left(\frac{d_2^4 - d_1^4}{d_2}\right) f_s$$

where d_2 and d_1 are the external and internal diameters respectively.

If I is the moment of inertia of the section about the axis of the cylinder,

Then $$\mathrm{I} = \frac{\pi}{2}(r_2^4 - r_1^4)$$

and $$\mathrm{T} = \frac{\pi}{2}\left(\frac{r_2^4 - r_1^4}{r_2}\right) f_s$$
$$= \frac{\mathrm{I}}{r_2} f_s$$

Now
$$\theta = \frac{l\phi}{r_2}$$
$$= \frac{l}{Nr_2} f_s$$
$$= \frac{l}{Nr_2} \times \frac{2r_2 T}{\pi(r_2^4 - r_1^4)}$$
$$= \frac{2lT}{\pi N(r_2^4 - r_1^4)} \text{ radians}$$
$$= \frac{lT}{NI} \text{ radians}$$

If a shaft is making n revolutions per minute, and H is the horse-power transmitted,

Then
$$H = \frac{2\pi n T}{12 \times 33000}$$
$$= \frac{\pi n T}{6 \times 33000}$$

where T is the twisting moment or torque in inch pounds.

Also
$$T = \frac{6 \times 33000 H}{\pi n} \text{ inch lb.}$$

and these two relations can be combined with those already obtained for solid and hollow cylindrical shafts.

Examples XIX

(1) A beam 30 ft. long is supported at the ends A and B. It carries loads of 5 tons, 10 tons, and 8 tons at points situated at distances of 5 ft., 16 ft., and 22 ft. respectively from A. Find the reactions at the supports and the values of the bending moment at points situated at distances of 10 ft., 18 ft., and 25 ft. from A.

(2) A beam 20 ft. long is supported at the ends and is divided into two equal lengths. The first length carries at its centre a load of 10 tons, while the second length carries a uniformly distributed load of 1 ton per foot run. Calculate the values of the bending moment at points situated at distances 5 ft., 10 ft., and 15 ft. from one end.

(3) A beam 20 ft. long is supported at the ends and is divided into two lengths of 8 ft. and 12 ft. The first length carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run, and the second length carries a uniformly distributed load of $\frac{3}{4}$ ton per foot run. Find expressions for the bending moment at any point for each

length and draw the bending moment diagram. What is the maximum bending moment and where does it occur ?

(4) A beam 30 ft. long is supported at the ends A and B ; C is a point 10 ft. from A. The part AC of the beam is unloaded, while the part BC carries a load which increases uniformly from 0 at C to 2 tons at B. Find expressions for the bending moment for the two parts of the beam and draw the bending moment diagram. What is the maximum bending moment and where does it occur ?

(5) A beam 30 ft. long is supported at the ends A and B ; C is a point 12 ft. from A. The part AC carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run, while the part CB carries a load which decreases uniformly from $1\frac{1}{2}$ tons at C to 0 at B. Find expressions for the bending moment for the two parts of the beam and draw the bending moment diagram. What is the maximum bending moment and where does it occur ?

(6) A cantilever of length l ft. is loaded at the free end with W tons; it also carries a uniformly distributed load of w tons per foot run. Find the bending moment, the slope and the deflection at a point P, which is situated at a distance x ft. from the point of fixing. What is the deflection at the free end ?

(7) A beam of length $2a$ ft. is supported at the ends and is loaded with W tons at the centre; it also carries a uniformly distributed load of w tons per foot run. P is a point situated at a distance x ft. from the centre of the beam. Find the bending moment, the slope, and the deflection at P.

(8) A beam of length $2a$ ft. is fixed at both ends and is loaded with W tons at the centre; it also carries a uniformly distributed load of w tons per foot run. Find the magnitude of the fixing couple. If P is a point situated at a distance x ft. from the centre of the beam, find the bending moment, the slope, and the deflection at P. For what value of x is the bending moment zero ?

(9) A cantilever of length l ft. carries a load which decreases uniformly from w tons at the fixed end to 0 at the free end. Find the bending moment, the slope, and the deflection at a point situated at a distance x ft. from the fixed end. What is the maximum deflection ?

(10) A beam of length $2a$ ft. is supported at the ends and carries a load which decreases uniformly from w tons at the centre to 0 at the ends. Find the bending moment, the slope, and the deflection at a point which is situated at a distance x ft. from the centre. What is the maximum deflection ?

(11) A beam of length $2a$ ft. is fixed at the ends and carries a load which decreases uniformly from w tons at the centre to 0 at the ends. Find the magnitude of the fixing couple. If P is a point situated at a distance x ft. from the centre, find the bending moment, the slope, and the deflection at P. What is the bending moment at the end ?

(12) A cantilever AB, of length l ft., A being the fixed end, carries a load of W tons at a point C distant b ft. from A. Find expressions for the bending moment, the slope, and the deflection for the two parts AC and CB of the cantilever. What is the deflection at C, and the deflection at B ?

(13) A beam AB, of length $2a$ ft., is supported at the ends and carries equal loads, each W tons, at points D and E on either side of the centre C and at distances b ft. from it. Find expressions for the bending moment, the slope, and the deflection for the two parts CE and EB of the beam. What is the greatest deflection, and what is the deflection under one of the loads ?

(14) ACB is a cantilever of length l ft. A is the fixed end and C is a point situated at a distance b ft. from A. The part AC carries a uniformly distributed load of w tons per foot run, while the part CB is unloaded. Find expressions for the bending moment, the slope, and the deflection for the two parts AC and CB of the cantilever. What are the deflections at C and B ?

(15) ADCEB is a beam, of length $2a$ ft., supported at the ends A and B ; C is the centre and D and E are points situated at equal distances b ft. on either side of C. The part DE of the beam carries a uniformly distributed load of w tons per foot run, while the remainder is unloaded. Find expressions for the bending moment, the slope, and the deflection for the two parts CE and EB of the beam. What is the maximum deflection and what is the deflection at E ?

(16) If $\frac{\mathrm{M}}{c} = \frac{d^2y}{dx^2}$, $\frac{d\mathrm{M}}{dx} = \mathrm{S}$ and $\frac{d\mathrm{S}}{dx} = w$. Let w be a constant, find S. Let $\mathrm{S} = \mathrm{W}$, a constant, when $x = l$. Find M and let $\mathrm{M} = 0$ when $x = l$, c is a given constant, find $\frac{dy}{dx}$ and let its value be 0 when $x = 0$. Find y and let its value be 0 when $x = 0$. (B. of E., 1908.)

(17) There are two cantilevers of equal length and of the same cross section. The first carries a load which decreases uniformly from 2 tons at the fixed end to 0 at the free end, and the second carries a uniformly distributed load of w tons per foot run. What must be the value of w so that the free ends of the cantilevers will be deflected to the same amount, and what is the ratio of the deflections at the mid points ?

(18) If the section Fig. 63 is subjected to a shearing force of 1000 lb., what is the intensity of the shear stress at the neutral axis ? What are the intensities of the shear stresses at the under edge of the flange and at the top of the web ?

(19) If the section Fig. 73, No. 5, is subjected to a shearing force of 1000 lb., what is the intensity of the shear stress at the neutral axis (1) when the web is vertical (2) when the web is horizontal ?

(20) Find the intensity of the shear stress at the neutral axis, if the section Fig. 73, No. 8, is subjected to a shearing force of 1000 lb.

(21) If the section Fig. 73, No. 9, is subjected to a shearing force of 5000 lb., what is the intensity of the shear stress at the neutral axis, and what fraction is it of the average shear stress for the whole section ?

(22) A hollow circular section, external radius 5 inches, internal radius 3 inches, is subjected to a shearing force of 2000 lb. What is the intensity of the shear stress at the diameter ?

(23) What is the diameter of a steel shaft which will stand a twisting moment of 16,000 inch pounds, the maximum shear stress being 10,000 lb. per sq. inch ? If a hollow steel shaft of 4 inches external diameter will stand the same twisting moment, what is its internal diameter ? If the length of each shaft is 9 ft., what is the angle of twist in each case ? $N = 13 \times 10^6$ lb. per sq. inch.

(24) A hollow steel shaft is subjected to pure twisting and transmits 5000 H.P. at a speed of 95 revolutions per minute. If the shear stress must not exceed 10,000 lb. per sq. inch and the internal diameter is to be 75 per cent. of the external diameter, find the external diameter.

CHAPTER XX

164. A differential equation is an equation connecting x, y and a differential coefficient, or differential coefficients of y with respect to x. The order of a differential equation is the order of the highest differential coefficient occurring in it. Thus an equation of the first order is one containing $\frac{dy}{dx}$, one of the second order would contain $\frac{d^2y}{dx^2}$, while an equation of the nth order would contain $\frac{d^ny}{dx^n}$.

A differential equation can be obtained by the elimination of the constants in a law connecting x and y, and the following examples will show how differential equations can be obtained in this way.

(*a*) If
$$xy = a$$
Then
$$x\frac{dy}{dx} + y = 0$$
(*b*) If
$$x^2 + y^2 = a^2$$
Then
$$2x + 2y\frac{dy}{dx} = 0$$
or
$$x + y\frac{dy}{dx} = 0$$
(*c*) If
$$y = ax + bx^2$$
$$\frac{dy}{dx} = a + 2bx$$
$$\frac{d^2y}{dx^2} = 2b$$
Now
$$x\frac{dy}{dx} = ax + 2bx^2$$
Hence
$$x\frac{dy}{dx} - y = bx^2$$
$$= \frac{1}{2}x^2\frac{d^2y}{dx^2}$$
and
$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

(*d*) If
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Then
$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

or
$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0$$

$$\frac{x^2}{a^2}+\frac{xy}{b^2}\frac{dy}{dx}=0$$

Hence
$$\frac{y^2}{b^2}-\frac{xy}{b^2}\frac{dy}{dx}=1$$

and
$$y^2-xy\frac{dy}{dx}=b^2$$

Differentiating
$$2y\frac{dy}{dx}-\left\{xy\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(y+x\frac{dy}{dx}\right)\right\}=0$$

or
$$2y\frac{dy}{dx}-xy\frac{d^2y}{dx^2}-y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2=0$$

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

(*e*) If
$$x^2+y^2=2ay+b$$

$$2x+2y\frac{dy}{dx}=2a\frac{dy}{dx}$$

and
$$\frac{x}{\frac{dy}{dx}}+y=a$$

Differentiating
$$\frac{\frac{dy}{dx}-x\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^2}+\frac{dy}{dx}=0$$

or
$$\frac{dy}{dx}-x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$$

(*f*) If
$$x^2+y^2=2ax+b$$

$$2x+2y\frac{dy}{dx}=2a$$

or
$$x+y\frac{dy}{dx}=a$$

Differentiating
$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

It will be seen from these examples that if one constant is eliminated a differential equation of the first order is formed, while the elimination of two constants produces a differential equation of the second order. Therefore the solution of a differential equation of the first order may contain one arbitrary constant, while the solution of a differential equation of the second order may contain two arbitrary constants. Confining our work to differential equations of the first order, the two types which occur most frequently in actual practice are those equations in which the variables can be separated, and those equations which can be solved by the use of an integrating factor.

165. *When the Variables can be Separated.* Equations of this type are such that all the terms involving x can be placed with dx on one side, and all the terms involving y can be placed with dy on the other side. Then one side can be integrated with respect to x and the other side with respect to y.

These equations have the form, or can be reduced to the form,

$$X + Y\frac{dy}{dx} = 0 \quad \ldots\ldots \quad (1)$$

or

$$Y + X\frac{dy}{dx} = 0 \quad \ldots\ldots \quad (2)$$

where X is a function of x and Y is a function of y,

and

$$\int X\,dx = -\int Y\,dy + \text{Const} \quad \ldots\ldots \quad (1)$$

or

$$\int\frac{dx}{X} = -\int\frac{dy}{Y} + \text{Const} \quad \ldots\ldots \quad (2)$$

Example 1. Solve the equation $x\dfrac{dy}{dx} = y + xy$.

Now

$$x\frac{dy}{dx} = y(x+1)$$

and

$$\frac{dy}{dx}\,\frac{x}{x+1} = y$$

$$\int\frac{x+1}{x}\,dx + \text{Const} = \int\frac{dy}{y}$$

$$x + \log_e x + c = \log_e y$$

$$x + c = \log_e\frac{y}{x}$$

$$e^{x+c} = \frac{y}{x}$$

and

$$y = xe^x \times e^c$$

$$= Axe^x \quad \text{where } A = e^c$$

X

Example 2. Solve the equation $x^2y^2 - 4x^2 = (x^2y^2 - 9y^2)\dfrac{dy}{dx}$

Now $$x^2(y^2 - 4) = y^2(x^2 - 9)\frac{dy}{dx}$$

and $$\frac{x^2}{x^2 - 9} = \frac{y^2}{y^2 - 4}\,\frac{dy}{dx}$$

$$\int\frac{x^2dx}{x^2 - 9} = \int\frac{y^2dy}{y^2 - 4} + \text{Const}$$

$$\int\left(1 + \frac{9}{x^2 - 9}\right)dx = \int\left(1 + \frac{4}{y^2 - 4}\right)dy + \text{Const}$$

$$\int\left\{1 + \frac{3}{2}\left(\frac{1}{x - 3} - \frac{1}{x + 3}\right)\right\}dx = \int\left\{1 + \frac{1}{y - 2} - \frac{1}{y + 2}\right\}dy + \text{Const}$$

$$x + \frac{3}{2}\log_e\frac{x - 3}{x + 3} = y + \log_e\frac{y - 2}{y + 2} + c$$

Example 3. Solve the equation

$$\sin^2 x \sin^2 y - \cos^2 x \cos^2 y\frac{dy}{dx} = 0$$

Now $$\sin^2 x \sin^2 y = \cos^2 x \cos^2 y\frac{dy}{dx}$$

and $$\tan^2 x = \cot^2 y\frac{dy}{dx}$$

$$\int\tan^2 x\, dx = \int\cot^2 y\, dy + c$$

$$\int(\sec^2 x - 1)\,dx = \int(\text{cosec}^2\, y - 1)\,dy + c$$

$$\tan x - x = -\cot y - y + c$$

$$\tan x + \cot y - x + y = c$$

166. *The Use of the Integrating Factor.* The integrating factor is used to solve differential equations of the form, or differential equations which can be reduced to the form,

$$\frac{dy}{dx} + \text{P}y = \text{Q}$$

where **P** and **Q** can either be constants or functions of x, but they must be independent of y.

Now $e^{\int \text{P}\,dx}$ when differentiated with respect to x gives $\text{P}e^{\int \text{P}\,dx}$

For $$z = e^w \text{ where } w = \int \text{P}\,dx$$

and $$\frac{dz}{dw} = e^w \text{ and } \frac{dw}{dx} = \mathrm{P}$$

but $$\frac{dz}{dx} = \frac{dz}{dw}\frac{dw}{dx}$$

$$= \mathrm{P}e^{\int \mathrm{P}\,dx}$$

Then $$\frac{d}{dx}\left(ye^{\int \mathrm{P}\,dx}\right) = y\mathrm{P}e^{\int \mathrm{P}\,dx} + \frac{dy}{dx} \times e^{\int \mathrm{P}\,dx}$$

Thus, if the differential equation is multiplied throughout by $e^{\int \mathrm{P}\,dx}$, the left-hand side becomes the result which would be obtained by differentiating $ye^{\int \mathrm{P}\,dx}$,

and $$\frac{dy}{dx} \times e^{\int \mathrm{P}\,dx} + \mathrm{P}ye^{\int \mathrm{P}\,dx} = \mathrm{Q}e^{\int \mathrm{P}\,dx}$$

or $$\frac{d}{dx}\left(ye^{\int \mathrm{P}\,dx}\right) = \mathrm{Q}e^{\int \mathrm{P}\,dx}$$

Integrating, $$ye^{\int \mathrm{P}\,dx} = \int \mathrm{Q}e^{\int \mathrm{P}\,dx}\,dx + \text{Const}$$

$e^{\int \mathrm{P}\,dx}$ is known as the integrating factor.

Example 1. Solve the equation $\dfrac{dy}{dx} + 2xy = x$.

$$\text{Now } \int \mathrm{P}\,dx = 2\int x\,dx$$

$$= x^2$$

The integrating factor is e^{x^2}

$$\text{Hence } ye^{x^2} = \int xe^{x^2}dx + c$$

To find $\int xe^{x2}\,dx$, put $x^2 = z$

Then $$dz = 2x\,dx$$

and the integral becomes $\dfrac{1}{2}\displaystyle\int e^z\,dz = \dfrac{1}{2}e^z$

Therefore $$ye^{x^2} = \frac{1}{2}e^{x^2} + c$$

and $$y = \frac{1}{2} + ce^{-x^2}$$

Example 2. Solve the equation $\frac{dy}{dx} + y \cos x = \cos^3 x$.

Now
$$\int P\, dx = \int \cos x\, dx$$
$$= \sin x$$

The integrating factor is $e^{\sin x}$

Hence
$$ye^{\sin x} = \int e^{\sin x} \cos^3 x\, dx + c$$

To find $\int e^{\sin x} \cos^3 x\, dx$, put $z = \sin x$

Then
$$dz = \cos x\, dx, \quad \text{and } \cos^2 x = 1 - z^2$$

and
$$\int e^{\sin x} \cos^3 x\, dx = \int e^z (1 - z^2)\, dz$$
$$= \int e^z\, dz - \int z^2 e^z\, dz$$
$$= e^z - \{z^2 e^z - 2\int z e^z\, dz\}$$
$$= e^z - \{z^2 e^z - 2(z e^z - \int e^z\, dz)\}$$
$$= e^z - z^2 e^z + 2z e^z - 2e^z$$
$$= -e^z (z^2 - 2z + 1)$$
$$= -e^{\sin x} (1 - \sin x)^2$$

Therefore
$$ye^{\sin x} = c - e^{\sin x} (1 - \sin x)^2$$

and
$$y = ce^{-\sin x} - (1 - \sin x)^2$$

Example 3. Solve the equation $x\frac{dy}{dx} + y = x^2 \sin x$.

Then
$$\frac{dy}{dx} + \frac{y}{x} = x \sin x$$

and
$$\int P\, dx = \int \frac{dx}{x}$$
$$= \log_e x$$

The integrating factor $= e^{\log_e x}$
$$= x$$

Hence
$$xy = \int x^2 \sin x\, dx + c$$
$$= -x^2 \cos x + 2\int x \cos x\, dx + c$$

$$= -x^2 \cos x + 2\{x \sin x - \int \sin x \, dx\} + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

$$= (2 - x^2) \cos x + 2x \sin x + c$$

167. *The Motion of a Projectile.* One law of air resistance is that, if R lb. is the resistance, d ft. is the diameter of the projectile, and v ft. per second is the horizontal component of the velocity at any instant.

Then $$R = 2d^2(v - 850)$$

If m lb. is the mass of the projectile,

$$R = -\frac{m}{g} \times \text{acceleration}$$

$$2d^2(v - 850) = -\frac{m}{g}\frac{dv}{dt}$$

$$\frac{dv}{dt} = -\frac{2gd^2}{m}(v - 850)$$

$$\int \frac{dv}{v - 850} = -\frac{2gd^2}{m}\int dt + C$$

$$\log_e(v - 850) = -\frac{2gd^2}{m}t + C$$

Let v_0 ft. per second be the initial horizontal muzzle velocity Then, when $t = 0$, $v = v_0$, and $\log_e(v_0 - 850) = C$.

Hence $$\log_e(v - 850) - \log_e(v_0 - 850) = -\frac{2gd^2}{m}t$$

$$\log_e \frac{v - 850}{v_0 - 850} = -\frac{2gd^2}{m}t$$

$$\frac{v - 850}{v_0 - 850} = e^{-\frac{2gd^2}{m}t}$$

and $$v = 850 + (v_0 - 850)e^{-\frac{2gd^2}{m}t}$$

This gives the velocity at any instant, but $v = \dfrac{ds}{dt}$

$$\frac{ds}{dt} = 850 + (v_0 - 850)e^{-\frac{2gd^2}{m}t}$$

Integrating, $$s = 850t - \frac{m(v_0 - 850)}{2gd^2}e^{-\frac{2gd^2}{m}t} + C_1$$

But when $t = 0$, $s = 0$, and $0 = -\dfrac{m(v_0 - 850)}{2gd^2} + C_1$

Hence $$s = 850t + \frac{m(v_0 - 850)}{2gd^2}\left\{1 - e^{-\frac{2gd^2}{m}t}\right\}$$

This gives the horizontal distance in terms of the time of flight.

As an exercise, let the diameter of the shot be 12 inches and the mass 850 lb. The shot starts with a horizontal muzzle velocity of 2700 ft. per second.

$$2gd^2 = 2 \times 32{\cdot}2 = 64{\cdot}4$$

$$\frac{2gd^2}{m} = 0{\cdot}07577$$

$$\frac{m}{2gd^2} = 13{\cdot}20$$

$$s = 850t + 13{\cdot}2(v_0 - 850)(1 - e^{-0{\cdot}07577t})$$
$$= 850t + 24420(1 - \alpha), \text{ where } \alpha = e^{-0{\cdot}07577t}$$

t	α	$1 - \alpha$	$24420(1-\alpha)$	$850t$	s
0	1	0	0	0	0
2	0·8594	0·1406	3433	1700	5133
3·106	0·7903	0·2097	5122	2640	7762
4	0·7386	0·2614	6383	3400	9783
6	0·6348	0·3652	8917	5100	14017
6·212	0·6246	0·3754	9168	5299	14467
8	0·5455	0·4545	11100	6800	17900
9·315	0·4937	0·5063	12370	7916	20286
10	0·4687	0·5313	12970	8500	21470
12	0·4027	0·5973	14580	10200	24780
12·42	0·3901	0·6099	14890	10550	25440
14	0·3461	0·6539	15970	11900	27870
15·52	0·3083	0·6917	16900	13190	30090
16	0·2975	0·7025	17150	13600	30750
18	0·2557	0·7443	18180	15300	33480
18·63	0·2437	0·7563	18470	15830	34300
20	0·2197	0·7803	19050	17000	36050
22	0·1888	0·8112	19810	18700	38510
24	0·1623	0·8377	20460	20400	40860
24·86	0·1519	0·8481	20710	21130	41840
26	0·1394	0·8606	21010	22100	43110
28	0·1191	0·8801	21490	23800	45290
30	0·1030	0·8970	21910	25500	47410
31·04	0·0951	0·9049	22110	26380	48490

(1) Taking an initial vertical velocity of 100 ft. per sec.

Then $$y = 100t - \frac{1}{2}gt^2$$
$$= t(100 - 16{\cdot}1t)$$

The highest point is reached after 3·106 secs. and the full time of flight is 6·212 secs.

t	$16{\cdot}1t$	$100 - 16{\cdot}1t$	y	s
0	0	100·0	0	0
2	32·2	67·8	135·6	5133
3·106	50·0	50·0	155·3	7762
4	64·4	35·6	142·4	9783
6	96·6	3·4	20·4	14017
6·212	100·0	0	0	14467

(2) Taking an initial vertical velocity of 200 ft. per sec.

Then $$y = 200t - \frac{1}{2}gt^2$$
$$= t(200 - 16{\cdot}1t)$$

The highest point is reached after 6·212 secs. and the full time of flight is 12·42 secs.

t	$16{\cdot}1t$	$200 - 16{\cdot}1t$	y	s
0	0	200·0	0	0
2	32·2	167·8	335·6	5133
4	64·4	135·6	542·4	9783
6	96·6	103·4	620·4	14017
6·212	100·0	100·0	621·2	14467
8	128·8	71·2	569·6	17900
10	161·0	39·0	390·0	21470
12	193·2	6·8	81·6	24780
12·42	200·0	0	0	25440

(3) Taking an initial vertical velocity of 300 ft. per sec.

Then $$y = 300t - \frac{1}{2}gt^2$$
$$= t(300 - 16{\cdot}1t)$$

The highest point is reached after 9·315 secs. and the full time of flight is 18·63 secs.

t	$16{\cdot}1t$	$300 - 16{\cdot}1t$	y	s
0	0	300·0	0	0
2	32·2	267·8	535·6	5133
4	64·4	235·6	942·4	9783
6	96·6	203·4	1220·4	14017
8	128·8	171·2	1369·6	17900
9·315	150·0	150·0	1396·2	20286
10	161·0	139·0	1390·0	21470
12	193·2	106·8	1281·6	24780
14	225·4	74·6	1044·4	27870
16	257·6	42·4	678·4	30750
18	289·8	10·2	183·6	33480
18·63	300·0	0	0	34300

(4) Taking an initial vertical velocity of 400 ft. per sec.

Then
$$y = 400t - \frac{1}{2}gt^2$$
$$= t(400 - 16{\cdot}1t)$$

The highest point is reached after 12·42 secs. and the full time of flight is 24·84 secs.

t	$16{\cdot}1t$	$400 - 16{\cdot}1t$	y	s
0	0	400·0	0	0
2	32·2	367·8	735·6	5133
4	64·4	335·6	1342·4	9783
6	96·6	303·4	1820·4	14017
8	128·8	271·2	2169·6	17900
10	161·0	239·0	2390·0	21470
12	193·2	206·8	2481·6	24780
12·42	200·0	200·0	2484·0	25440
14	225·4	174·6	2444·4	27870
16	257·6	142·4	2278·4	30750
18	289·8	110·2	1983·6	33480
20	322·0	78·0	1560·0	36050
22	354·2	45·8	1007·6	38510
24	386·4	13·6	326·4	40860
24·84	400·0	0	0	41840

(5) Taking an initial vertical velocity of 500 ft. per sec.

$$\text{Then } y = 500t - \frac{1}{2}gt^2$$
$$= t(500 - 16{\cdot}1t)$$

The highest point is reached after 15·52 secs. and the full time of flight is 31·04 secs.

t	$16{\cdot}1t$	$500 - 16{\cdot}1t$	y	s
0	0	500·0	0	0
2	32·2	467·8	935·6	5133
4	64·4	435·6	1742·4	9783
6	96·6	403·4	2420·4	14017
8	128·8	371·2	2969·6	17900
10	161·0	339·0	3390·0	21470
12	193·2	306·8	3681·6	24780
14	225·4	274·6	3844·4	27870
15·52	250·0	250·0	3880·0	30090
16	257·6	242·4	3874·4	30750
18	289·8	210·2	3783·6	33480
20	322·0	178·0	3560·0	36050
22	354·2	145·8	3207·6	38150
24	386·4	113·6	2726·4	40860
26	418·6	81·4	2116·4	43110
28	450·8	49·2	1377·6	45290
30	483·0	17·0	510·0	47410
31·04	500·0	0	0	48490

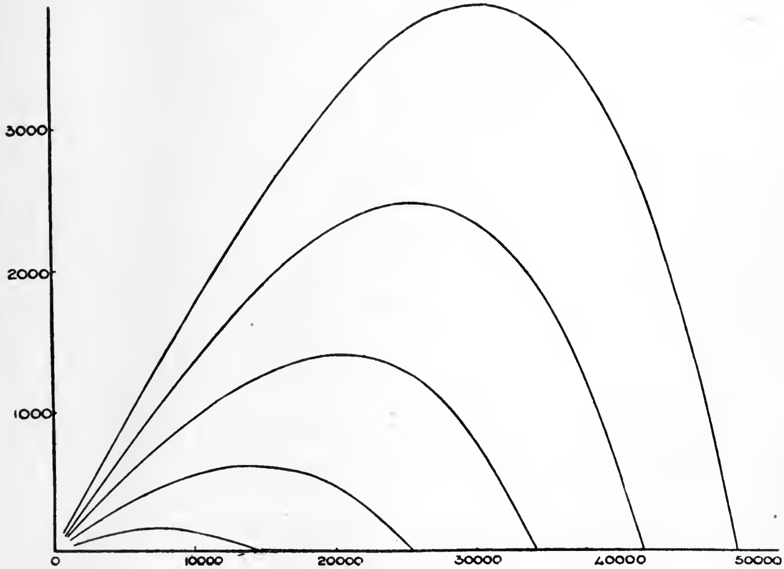


FIG. 112.

Fig. 112 shows the curves obtained by plotting the values of s horizontally and the values of y vertically.

168. If the voltage in an electric circuit is v volts, the current is C ampères, the resistance R ohms, the self-inductance L henries, then $v = \mathrm{RC} + \mathrm{L}\dfrac{d\mathrm{C}}{dt}$ where t is time in seconds.

(1) To express C in terms of t when v is constant.

Now
$$v = \mathrm{RC} + \mathrm{L}\frac{d\mathrm{C}}{dt}$$

$$v - \mathrm{RC} = \mathrm{L}\frac{d\mathrm{C}}{dt}$$

$$\frac{v}{\mathrm{R}} - \mathrm{C} = \frac{\mathrm{L}}{\mathrm{R}}\frac{d\mathrm{C}}{dt}$$

and
$$\int\frac{d\mathrm{C}}{\dfrac{v}{\mathrm{R}} - \mathrm{C}} = \frac{\mathrm{R}}{\mathrm{L}}\int dt + \text{Const}$$

Integrating,
$$-\log_e\left(\frac{v}{\mathrm{R}} - \mathrm{C}\right) = \frac{\mathrm{R}}{\mathrm{L}}t + \text{Const.}$$

(a) Let the initial condition be $\mathrm{C} = 0$ when $t = 0$.

Then
$$-\log_e\frac{v}{\mathrm{R}} = \text{Const}$$

Hence
$$\log_e\left(\frac{v}{\mathrm{R}} - \mathrm{C}\right) - \log_e\frac{v}{\mathrm{R}} = -\frac{\mathrm{R}t}{\mathrm{L}}$$

$$\frac{\dfrac{v}{\mathrm{R}} - \mathrm{C}}{\dfrac{v}{\mathrm{R}}} = e^{-\frac{\mathrm{R}t}{\mathrm{L}}}$$

$$\frac{v}{\mathrm{R}} - \mathrm{C} = \frac{v}{\mathrm{R}}e^{-\frac{\mathrm{R}t}{\mathrm{L}}}$$

and
$$\mathrm{C} = \frac{v}{\mathrm{R}}\left(1 - e^{-\frac{\mathrm{R}t}{\mathrm{L}}}\right).$$

It should be noticed that as t becomes large $e^{-\frac{\mathrm{R}t}{\mathrm{L}}}$ becomes small, and therefore C tends to the value $\dfrac{v}{\mathrm{R}}$.

(b) Let the initial condition be $\mathrm{C} = \mathrm{C}_0$ when $t = 0$

Then
$$-\log_e\left(\frac{v}{\mathrm{R}} - \mathrm{C}\right) = \frac{\mathrm{R}t}{\mathrm{L}} + \text{Const}$$

and $$-\log_e\left(\frac{v}{R} - C_0\right) = \text{Const}$$

$$\log_e\left(\frac{v}{R} - C\right) - \log_e\left(\frac{v}{R} - C_0\right) = -\frac{Rt}{L}$$

$$\log_e \frac{\frac{v}{R} - C}{\frac{v}{R} - C_0} = -\frac{Rt}{L}$$

$$\frac{\frac{v}{R} - C}{\frac{v}{R} - C_0} = e^{-\frac{Rt}{L}}$$

$$\frac{v}{R} - C = \left(\frac{v}{R} - C_0\right)e^{-\frac{Rt}{L}}$$

$$C = \frac{v}{R} - \left(\frac{v}{R} - C_0\right)e^{-\frac{Rt}{L}}$$

$$= \frac{v}{R}\left(1 - e^{-\frac{Rt}{L}}\right) + C_0 e^{-\frac{Rt}{L}}$$

(2) To express C in terms of t when $v = v_0 \sin pt$.

Then $$RC + L\frac{dC}{dt} = v_0 \sin pt$$

and $$\frac{dC}{dt} + \frac{R}{L}C = \frac{v_0}{L}\sin pt$$

Now $$\frac{R}{L}\int dt = \frac{Rt}{L}$$

The integrating factor is therefore $e^{\frac{Rt}{L}}$

and $$Ce^{\frac{Rt}{L}} = \frac{v_0}{L}\int e^{\frac{Rt}{L}} \sin pt\, dt + \text{Const}$$

$$= \frac{v_0}{L}\,\frac{e^{\frac{Rt}{L}}}{\frac{R^2}{L^2} + p^2}\left(\frac{R}{L}\sin pt - p\cos pt\right) + \text{Const}$$

(*a*) Let the initial condition be C = 0 when $t = 0$.

Then $$0 = -\frac{pv_0}{L\left(\frac{R^2}{L^2} + p^2\right)} + \text{Const}$$

and $$Ce^{\frac{Rt}{L}} = \frac{v_0 Le^{\frac{Rt}{L}}}{R^2 + L^2p^2}\left(\frac{R}{L} \sin pt - p \cos pt\right) + \frac{v_0 Lp}{R^2 + L^2p^2}$$

$$= \frac{v_0\, e^{\frac{Rt}{L}}}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \frac{v_0 Lp}{R^2 + L^2p^2}$$

Hence $$C = \frac{v_0}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \frac{v_0 Lp}{R^2 + L^2p^2} e^{-\frac{Rt}{L}}$$

$$= \frac{v_0}{\sqrt{R^2 + L^2p^2}} \sin (pt - \theta) + \frac{v_0 Lp}{R^2 + L^2p^2} e^{-\frac{Rt}{L}}$$

where $\theta = \tan^{-1} \frac{pL}{R}$

It should be noticed that as t becomes large $e^{-\frac{Rt}{L}}$ becomes small, and therefore C tends to the form $\frac{v_0}{\sqrt{R^2 + L^2p^2}} \sin (pt - \theta)$, a periodic function of the same frequency as v, but of amplitude $\frac{v_0}{\sqrt{R^2 + L^2p^2}}$

(*b*) Let the initial condition be $C = C_0$ when $t = 0$.

Then $$Ce^{\frac{Rt}{L}} = \frac{v_0 e^{\frac{Rt}{L}}}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \text{Const}$$

and $$C_0 = -\frac{pLv_0}{R^2 + L^2p^2} + \text{Const}$$

$$Ce^{\frac{Rt}{L}} - C_0 = \frac{v_0 e^{\frac{Rt}{L}}}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \frac{pLv_0}{R^2 + L^2p^2}$$

$$Ce^{\frac{Rt}{L}} = \frac{v_0 e^{\frac{Rt}{L}}}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \left(C_0 + \frac{pLv_0}{R^2 + L^2p^2}\right)$$

and $$C = \frac{v_0}{R^2 + L^2p^2} (R \sin pt - pL \cos pt) + \left(C_0 + \frac{pLv_0}{R^2 + L^2p^2}\right)e^{-\frac{Rt}{L}}$$

$$= \frac{v_0}{\sqrt{R^2 + L^2p^2}} \sin (pt - \theta) + \left(C_0 + \frac{pLv_0}{R^2 + L^2p^2}\right)e^{-\frac{Rt}{L}}$$

169. To investigate the motion of a body falling from rest under the action of gravity, the resistance of the air being taken into account.

(1) When the air resistance is proportional to the velocity of the body.

If R lb. is the resistance of the air, then $R = kv$ where k is a constant and if m lb. is the mass of the body.

$$mg - Rg = m \times \text{acceleration}$$

$$m\frac{dv}{dt} = mg - kgv$$

and $$\frac{dv}{dt} + \frac{kg}{m}v = g$$

Now $$\frac{kg}{m}\int dt = \frac{kg}{m}t$$

and the integrating factor is therefore $e^{\frac{kgt}{m}}$

Hence $$ve^{\frac{gkt}{m}} = g\int e^{\frac{gkt}{m}}\,dt + \text{Const}$$

$$= \frac{m}{k}e^{\frac{gkt}{m}} + \text{Const}$$

but when $t = 0$, $v = 0$, since the body falls from rest.

Then $$0 = \frac{m}{k} + \text{Const}$$

and $$ve^{\frac{gk}{m}} = \frac{m}{k}e^{\frac{gkt}{m}} - \frac{m}{k}$$

$$= \frac{m}{k}\left(e^{\frac{gkt}{m}} - 1\right)$$

$$v = \frac{m}{k}\left(1 - e^{-\frac{gkt}{m}}\right)$$

It should be noticed that as t becomes large, $e^{-\frac{gkt}{m}}$ becomes small, and therefore the velocity of the body tends to the limiting value $\frac{m}{k}$.

Now $$v = \frac{ds}{dt} = \frac{m}{k}\left(1 - e^{-\frac{gkt}{m}}\right)$$

Integrating, $$s = \frac{m}{k}\left(t + \frac{m}{gk}e^{-\frac{gkt}{m}}\right) + \text{Const}$$

but when $t = 0$, $s = 0$, then $0 = \frac{m^2}{gk^2} + \text{Const}$

and $$s = \frac{m}{k}\left(t + \frac{m}{gk}e^{-\frac{gkt}{m}}\right) - \frac{m^2}{gk^2}$$

$$= \frac{m}{k}t - \frac{m^2}{gk^2}\left(1 - e^{-\frac{gkt}{m}}\right)$$

(2) When the air resistance is proportional to the square of the velocity of the body.

If R lb. is the resistance of the air, then $R = kv^2$ where k is a constant; and if m lb. is the mass of the body,

$$mg - Rg = m \times \text{acceleration}$$

and
$$mg - gkv^2 = m\frac{dv}{dt}$$

$$\frac{dv}{dt} = g\left\{1 - \frac{kv^2}{m}\right\}$$

$$= g\{1 - \alpha^2 v^2\} \text{ where } \alpha^2 = \frac{k}{m}$$

Hence
$$g\int dt + \text{Const} = \int\frac{dv}{1 - \alpha^2 v^2}$$

$$gt + \text{Const} = \frac{1}{2}\int\frac{dv}{1 + \alpha v} + \frac{1}{2}\int\frac{dv}{1 - \alpha v}$$

$$= \frac{1}{2\alpha}\{\log_e(1 + \alpha v) - \log_e(1 - \alpha v)\}$$

$$= \frac{1}{2\alpha}\log_e\frac{1 + \alpha v}{1 - \alpha v}$$

But when $t = 0$, $v = 0$. Hence Const $= 0$

and
$$\frac{1}{2\alpha}\log_e\frac{1 + \alpha v}{1 - \alpha v} = gt$$

$$\log_e\frac{1 + \alpha v}{1 - \alpha v} = 2\alpha gt$$

$$\frac{1 + \alpha v}{1 - \alpha v} = e^{2\alpha gt}$$

$$\alpha v(e^{2\alpha gt} + 1) = e^{2\alpha gt} - 1$$

$$v = \frac{1}{\alpha}\frac{e^{2\alpha gt} - 1}{e^{2\alpha gt} + 1}$$

$$= \frac{1}{\alpha}\tanh \alpha gt$$

$$= \sqrt{\frac{m}{k}}\tanh g\sqrt{\frac{k}{m}}\,t,$$

giving the velocity of the body in terms of the time.

Now
$$v = \frac{ds}{dt} = \frac{1}{\alpha}\tanh \alpha gt$$

Integrating
$$s = \frac{1}{\alpha^2 g}\log_e\cosh \alpha gt + \text{Const}$$

But when $t = 0$, $s = 0$. Hence Const $= 0$

and
$$s = \frac{1}{\alpha^2 g} \log_e \cosh \alpha g t$$
$$= \frac{m}{kg} \log_e \cosh g\sqrt{\frac{k}{m}}\, t$$

Note.—To integrate $\tanh \alpha g t$, put $\alpha g t = x$.

Then
$$\int \tanh \alpha g t \, dt = \frac{1}{\alpha g}\int \tanh x \, dx$$
$$= \frac{1}{\alpha g}\int \frac{\sinh x}{\cosh x}\, dx$$
$$= \frac{1}{\alpha g} \log_e \cosh x$$

Since the numerator is the differential coefficient of the denominator,

hence
$$\int \tanh \alpha g t \; dt = \frac{1}{\alpha g} \log_e \cosh \alpha g t$$

170. To investigate the motion of a body projected vertically upwards with an initial velocity v_0, the resistance of the air being taken into account.

(1) When the air resistance is proportional to the velocity of the body.

If R lb. is the resistance of the air, then $R = kv$ where k is a constant and if m lb. is the mass of the body,

then
$$-mg - Rg = \text{mass} \times \text{acceleration}$$
$$m\frac{dv}{dt} = -mg - kgv$$

and
$$\frac{dv}{dt} + \frac{kg}{m} v = -g$$

Now
$$\frac{kg}{m}\int dt = \frac{kg}{m} t$$

and the integrating factor is therefore $e^{\frac{kgt}{m}}$

Hence
$$ve^{\frac{kgt}{m}} = -g\int e^{\frac{kgt}{m}}\, dt + \text{Const}$$
$$= -\frac{m}{k} e^{\frac{kgt}{m}} + \text{Const}$$

but when $t = 0$, $v = v_0$, since v_0 is the velocity of projection

then
$$v_0 = -\frac{m}{k} + \text{Const}$$

and $$ve^{\frac{kgt}{m}} - v_0 = \frac{m}{k} - \frac{m}{k}e^{\frac{kgt}{m}}$$

$$ve^{\frac{kgt}{m}} = v_0 - \frac{m}{k}\left(e^{\frac{kgt}{m}} - 1\right)$$

and $$v = v_0 e^{-\frac{kgt}{m}} - \frac{m}{k}\left(1 - e^{-\frac{kgt}{m}}\right)$$

Now $$v = \frac{ds}{dt} = v_0 e^{-\frac{kgt}{m}} - \frac{m}{k}\left(1 - e^{-\frac{kgt}{m}}\right)$$

Integrating $$s = -\frac{mv_0}{kg}e^{-\frac{kgt}{m}} - \frac{m}{k}\left(t + \frac{m}{kg}e^{-\frac{kgt}{m}}\right) + \text{Const}$$

but when $t = 0$, $s = 0$, $$0 = -\frac{mv_0}{kg} - \frac{m^2}{k^2g} + \text{Const}$$

and $$s = \frac{mv_0}{kg}\left(1 - e^{-\frac{kgt}{m}}\right) + \frac{m^2}{k^2g}\left(1 - e^{-\frac{kgt}{m}}\right) - \frac{m}{k}t$$

$$= \left(\frac{mv_0}{kg} + \frac{m^2}{k^2g}\right)\left(1 - e^{-\frac{kgt}{m}}\right) - \frac{m}{k}t$$

$$= \frac{m}{kg}\left(v_0 + \frac{m}{k}\right)\left(1 - e^{-\frac{kgt}{m}}\right) - \frac{m}{k}t$$

At the highest point the velocity is evidently **0**,

and $$v_0 e^{-\frac{kgt}{m}} - \frac{m}{k}\left(1 - e^{-\frac{kgt}{m}}\right) = 0$$

or $$v_0 e^{-\frac{kgt}{m}} = \frac{m}{k}\left(1 - e^{-\frac{kgt}{m}}\right)$$

$$\left(v_0 + \frac{m}{k}\right)e^{-\frac{kgt}{m}} = \frac{m}{k}$$

$$e^{-\frac{kgt}{m}} = \frac{m}{kv_0 + m}$$

$$-\frac{kg}{m}t = \log_e \frac{m}{kv_0 + m}$$

and $$t = \frac{m}{kg}\log_e \frac{kv_0 + m}{m}$$

and this gives the time taken to reach the highest point. By giving t this value in the expression for s, the vertical distance of the highest point above the point of projection can be determined.

Since $$s = \frac{m}{kg}\left\{v_0 + \frac{m}{k}\right\}\left\{1 - e^{-\frac{kgt}{m}}\right\} - \frac{m}{k}t$$

Then $$h = \frac{m}{kg}\left\{\frac{kv_0 + m}{k}\right\}\left\{1 - \frac{m}{kv_0 + m}\right\} - \frac{m^2}{k^2g}\log_e \frac{kv_0 + m}{m}$$

$$= \frac{m}{kg}\left\{\frac{kv_0 + m}{k}\right\}\left\{\frac{kv_0}{kv_0 + m}\right\} - \frac{m^2}{k^2g}\log_e \frac{kv_0 + m}{m}$$

$$= \frac{mv_0}{kg} - \frac{m^2}{k^2g}\log_e \frac{kv_0 + m}{m}$$

$$= \frac{m}{kg}\left\{v_0 - \frac{m}{k}\log_e \frac{kv_0 + m}{m}\right\}$$

(2) When the air resistance is proportional to the square of the velocity of the body.

If R lb. is the resistance of the air, then $R = kv^2$ where k is a constant, and if m lb. is the mass of the body,

then $$-mg - Rg = m \times \text{acceleration}$$

and $$-mg - gkv^2 = m\frac{dv}{dt}$$

$$\frac{dv}{dt} = -g\left(1 + \frac{kv^2}{m}\right)$$

$$= -g(1 + \alpha^2v^2), \quad \text{where } \alpha^2 = \frac{k}{m}$$

Hence $$\int\frac{dv}{1 + \alpha^2v^2} = -g\int dt + \text{Const}$$

Integrating $$\frac{1}{\alpha}\tan^{-1}\alpha v = -gt + \text{Const}$$

Since the initial velocity of projection is v_0, then when $t = 0$, $v = v_0$, and $\frac{1}{\alpha}\tan^{-1}\alpha v_0 = \text{Const}$.

Hence $$gt = \frac{1}{\alpha}\{\tan^{-1}\alpha v_0 - \tan^{-1}\alpha v\}$$

$$\alpha gt = \tan^{-1}\frac{\alpha v_0 - \alpha v}{1 + \alpha^2 v_0 v}$$

and $$\tan \alpha gt = \frac{\alpha v_0 - \alpha v}{1 + \alpha^2 v_0 v}$$

$$(1 + \alpha^2 v_0 v)\tan \alpha gt = \alpha v_0 - \alpha v$$

$$v(\alpha + \alpha^2 v_0 \tan \alpha gt) = \alpha v_0 - \tan \alpha gt$$

$$v = \frac{\alpha v_0 - \tan \alpha gt}{\alpha(1 + \alpha v_0 \tan \alpha gt)}$$

$$= \frac{1}{\alpha}\tan(\theta - \alpha gt), \quad \text{where } \tan\theta = \alpha v_0$$

At the highest point the velocity is evidently 0,

and $$\tan \alpha g t = \alpha v_0$$
$$\alpha g t = \tan^{-1} \alpha v_0$$
$$t = \frac{1}{\alpha g} \tan^{-1} \alpha v_0$$

Also $$v = \frac{ds}{dt} = \frac{1}{\alpha} \tan (\theta - \alpha g t)$$

and $$s = \frac{1}{\alpha}\int \tan (\theta - \alpha g t)\, dt + \text{Const}$$
$$= -\frac{1}{\alpha^2 g} \log_e \sec (\theta - \alpha g t) + \text{Const}$$

but when $t = 0$, $s = 0$, and $0 = -\dfrac{1}{\alpha^2 g} \log_e \sec \theta + \text{Const}$

Hence $$s = \frac{1}{\alpha^2 g}\{\log_e \sec \theta - \log_e \sec (\theta - \alpha g t)\}$$
$$= \frac{1}{\alpha^2 g} \log_e \frac{\sec \theta}{\sec (\theta - \alpha g t)}$$
$$= \frac{1}{\alpha^2 g} \log_e \frac{\cos (\theta - \alpha g t)}{\cos \theta}$$

The vertical distance of the highest point above the point of projection will be obtained when $t = \dfrac{1}{\alpha g} \tan^{-1} \alpha v_0$; that is, when $\theta = \alpha g t$.

Then $$h = \frac{1}{\alpha^2 g} \log_e \sec \theta$$

but $$\tan \theta = \alpha v_0 \quad \text{and } \sec \theta = \sqrt{1 + \alpha^2 v_0^2}$$

Hence $$h = \frac{1}{2\alpha^2 g} \log_e (1 + \alpha^2 v_0^2)$$

171. *The Compound Interest Law.*

If $$\frac{dy}{dx} = ny$$

Then $$\int \frac{dy}{y} = n \int dx + \text{Const}$$

and $$\log_e y = nx + c$$
$$y = e^{nx+c}$$
$$= \mathrm{A}e^{nx}, \quad \text{where } \mathrm{A} = e^c$$

If £P is the principal at any time t years and compound interest at r per cent. per annum is payable at every instant, then £$(P + \delta P)$ would be the principal at $(t + \delta t)$ years and δP is the interest on £P for δt years at r per cent.

Hence
$$\delta P = \frac{Pr\,\delta t}{100}$$

or
$$\frac{\delta P}{\delta t} = \frac{Pr}{100}$$

and when δt is made infinitely small,

$$\frac{dP}{dt} = \frac{Pr}{100}$$

Then
$$\int\frac{dP}{P} = \frac{r}{100}\int dt + \text{Const}$$

$$\log_e P = \frac{rt}{100} + \text{Const}$$

Let P_0 be the principal at the beginning, then when $t=0$, $P = P_0$, and $\log_e P_0 = \text{Const}$.

$$\log_e P - \log_e P_0 = \frac{rt}{100}$$

$$\log_e \frac{P}{P_0} = \frac{rt}{100}$$

$$\frac{P}{P_0} = e^{\frac{rt}{100}}$$

and
$$P = P_0 e^{\frac{rt}{100}}$$

Example. In what time would a certain sum of money invested at $2\frac{1}{2}$ per cent. per annum double itself if the interest is payable at every instant.

In this example $\quad P = 2P_0$

and
$$e^{\frac{rt}{100}} = 2$$

$$\frac{rt}{100} \times 0{\cdot}4343 = 0{\cdot}3010$$

$$t = \frac{30{\cdot}10}{0{\cdot}4343r}$$

$$= \frac{69{\cdot}33}{r}$$

$$= 27{\cdot}73 \text{ years,} \quad \text{when } r = 2\tfrac{1}{2} \text{ per cent.}$$

172. *The Slipping of a Belt on a Pulley.* Let α be the angle of lap and T_1 and T_2 the tensions in the belt at P and Q respectively (Fig. 113).

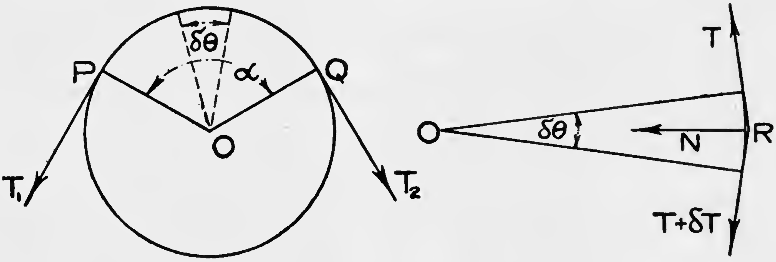


FIG. 113.

Considering an elementary length of belt subtending an angle $\delta\theta$ at the centre, and let the tensions on either side of this length of belt be T and $T + \delta T$.

The normal pressure N of this length of belt on the rim of the pulley will be found by resolving T and $T + \delta T$ in the direction OR.

$$\text{Thus } N = (T + \delta T) \cos\left(90° - \frac{\delta\theta}{2}\right) + T \cos\left(90° - \frac{\delta\theta}{2}\right)$$

$$= (2T + \delta T) \sin \frac{\delta\theta}{2}$$

$$= (2T + \delta T) \frac{\delta\theta}{2}, \text{ taking } \delta\theta \text{ as being small}$$

$$= T\,\delta\theta, \text{ taking } \delta T \text{ as being small}$$

Force of friction = normal pressure × coefficient of friction

$$= uT\,\delta\theta$$

When the force of friction is just equal to δT, slipping begins.

Then $\qquad \delta T = uT\,\delta\theta$

or $\qquad \dfrac{\delta T}{\delta\theta} = uT$

and $\qquad \dfrac{dT}{d\theta} = uT$, when $\delta\theta$ is infinitely small

Hence $\qquad \displaystyle\int\frac{dT}{T} = u\int d\theta + \text{Const}$

$$\log_e T = u\theta + \text{Const}$$

At P, $\theta = 0$ and $T = T_1$ $\qquad \log_e T_1 = \text{Const}$

At Q, $\theta = \alpha$ and $T = T_2$ $\qquad \log_e T_2 = u\alpha + \text{Const}$

$$\log_e T_2 - \log_e T_1 = u\alpha$$

$$\log_e \frac{T_2}{T_1} = u\alpha$$

$$\frac{T_2}{T_1} = e^{ua}$$

and $$T_2 = T_1 e^{ua}$$

If v ft. per sec. is the speed of the rim of the pulley and H is the horse-power transmitted,

Then $$H = \frac{(T_2 - T_1)v}{550}$$

Also $$T_2 - T_1 = T_1(e^{ua} - 1)$$

Hence $$H = \frac{vT_1(e^{ua} - 1)}{550}$$

and $$T_1 = \frac{550H}{v(e^{ua} - 1)}$$

Also $$T_2 = \frac{550H\ e^{ua}}{v(e^{ua} - 1)}$$

$$= \frac{550H}{v(1 - e^{-ua})}$$

173. *The Variation of Atmospheric Pressure with the Altitude.* Let p lb. per sq. ft. be the pressure of the air at a place h ft. above some datum level, and let $(p + \delta p)$ lb. per sq. ft. be the pressure of the air at a place $(h + \delta h)$ ft. above the same datum level.

The pressure of the air at distance h ft. = pressure of the air at distance $(h + \delta h)$ ft. + the weight of δh cubic ft. of air.

If w is the weight in lb. of a cubic foot of air at a distance h ft. above the datum level,

Then $$p = p + \delta p + w\,\delta h$$

or $$-\delta p = w\,\delta h$$

and $$\frac{dp}{dh} = -w, \text{ when } \delta h \text{ is made infinitely small.}$$

(1) If the temperature remains constant,

Then $$pv = \text{Const}, \quad \text{and } w = cp$$

$$\frac{dp}{dh} = -cp$$

$$\int \frac{dp}{p} = -c\int dh + \text{Const}$$

$$\log_e p = -ch + \text{Const}$$

but when $h = 0$, $p = p_0$, and $\log_e p_0 = \text{Const}$

Hence $$\log_e p - \log_e p_0 = -ch$$

$$\log_e \frac{p}{p_0} = -ch$$

$$p = p_0 e^{-ch}$$

Let p_0 lb. per sq. ft. be the pressure and w_0 lb. the weight of a cubic foot of air at datum level.

Then $$w_0 = cp_0, \quad \text{or } c = \frac{w_0}{p_0}$$

and $$p = p_0 e^{-\frac{w_0 h}{p_0}}$$

(2) When the temperature does not remain constant,

Then $$pv^n = \text{Const}, \quad \text{and } w = cp^{\frac{1}{n}}$$

$$\frac{dp}{dh} = -cp^{\frac{1}{n}}$$

$$\int p^{-\frac{1}{n}}\,dp = -c\int dh + \text{Const}$$

$$\frac{p^{1-\frac{1}{n}}}{1-\frac{1}{n}} = -ch + \text{Const}$$

but when $h = 0$, $p = p_0$, $$\frac{p_0^{1-\frac{1}{n}}}{1-\frac{1}{n}} = \text{Const}$$

$$\frac{n}{n-1}\left\{p_0^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right\} = ch$$

$$\frac{np_0^{1-\frac{1}{n}}}{n-1}\left\{1 - \left(\frac{p}{p_0}\right)^{1-\frac{1}{n}}\right\} = ch$$

but when $h = 0$, $p = p_0$ and $w = w_0$

Then $$w_0 = cp_0^{\frac{1}{n}} \text{ and } c = \frac{w_0}{p_0^{\frac{1}{n}}}$$

Hence $$h = \frac{n}{n-1}\,\frac{p_0}{w_0}\left\{1 - \left(\frac{p}{p_0}\right)^{1-\frac{1}{n}}\right\}$$

Also $pv = \text{RT}$, where T is the absolute temperature

and $$\frac{p}{p_0}\,\frac{v}{v_0} = \frac{\text{T}}{\text{T}_0}$$

but $$\frac{p}{p_0}\left(\frac{v}{v_0}\right)^n = 1, \quad \text{or } \left(\frac{p}{p_0}\right)^{\frac{1}{n}} = \frac{v_0}{v}$$

Then $$\frac{T}{T_0} = \frac{p}{p_0}\left(\frac{p}{p_0}\right)^{-\frac{1}{n}}$$

$$= \left(\frac{p}{p_0}\right)^{1-\frac{1}{n}}$$

Hence $$h = \frac{n}{n-1}\frac{p_0}{w_0}\left\{1 - \frac{T}{T_0}\right\}$$

174. *The Work done by an Expanding Gas.* If W is the work done in foot-pounds, p is the pressure in lb. per sq. ft. and v is the volume in cub. ft.

Then $$p = \frac{dW}{dv}$$

or $$W = \int p\, dv$$

Let the gas expand from volume v_1 to volume v_2, the pressure falling in consequence from p_1 to p_2.

If the law of expansion is $pv^n = \text{Const}$

Then $$p = cv^{-n}$$

and $$W = c\int_{v_1}^{v_2} v^{-n}\, dv$$

$$= \frac{c}{1-n}\Big[v^{1-n}\Big]_{v_1}^{v_2}$$

$$= \frac{c}{1-n}\Big[v_2^{1-n} - v_1^{1-n}\Big]$$

$$= \frac{p_1 v_1^{\,n}}{1-n}\Big[v_2^{1-n} - v_1^{1-n}\Big]$$

$$= \frac{p_1 v_1}{1-n}\left[\left(\frac{v_2}{v_1}\right)^{1-n} - 1\right]$$

$$= \frac{p_1 v_1}{n-1}\left[1 - \left(\frac{v_1}{v_2}\right)^{n-1}\right]$$

If the law of expansion is $pv = \text{Const}$

Then $$p = \frac{c}{v}$$

and $$W = c\int_{v_1}^{v_2}\frac{dv}{v}$$

$$= c\Big[\log_e v\Big]_{v_1}^{v_2}$$

$$= p_1 v_1\Big[\log_e v_2 - \log_e v_1\Big]$$

$$= p_1 v_1 \log_e \frac{v_2}{v_1}$$

175. *The Hypothetical Steam Engine Diagram.* Steam is admitted into the cylinder at constant pressure p_1, the volume increasing from 0 to v_1.

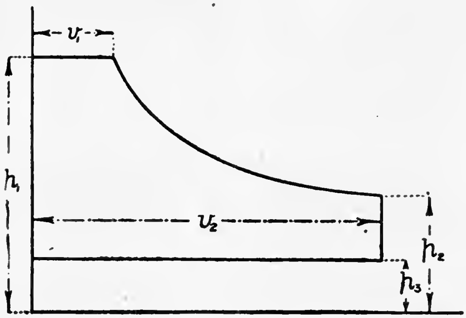


FIG. 114.

$$\text{Work done} = p_1 v$$

Let the steam then expand to volume v_2 according to the law $pv^n = \text{Const.}$

$$\text{Work done} = \frac{p_1 v_1}{n-1}\left[1 - \left(\frac{v_1}{v_2}\right)^{n-1}\right]$$

If p_3 is the back pressure, Work done during exhaust $= p_3 v_2$

$$\text{Total work done} = p_1 v_1 + \frac{p_1 v_1}{n-1}\{1 - r^{1-n}\} - p_3 v_2, \text{ where } r = \frac{v_2}{v_1}$$

If p_e = mean effective pressure,

$$\text{Total work done} = p_e v_2$$

and

$$p_e v_2 = p_1 v_1 + \frac{p_1 v_1}{n-1}\{1 - r^{1-n}\} - p_3 v_2$$

$$p_e = p_1 \frac{v_1}{v_2} + \frac{p_1}{n-1}\frac{v_1}{v_2}\{1 - r^{1-n}\} - p_3$$

$$= \frac{p_1}{r}\left\{1 + \frac{1 - r^{1-n}}{n-1}\right\} - p_3$$

$$= \frac{p_1}{n-1}\left\{\frac{n - r^{1-n}}{r}\right\} - p_3$$

$$= \frac{p_1}{n-1}\{nr^{-1} - r^{-n}\} - p_3$$

If the law of expansion is $pv = \text{Const}$

Then $$p_e v_2 = p_1 v_1 + p_1 v_1 \log_e \frac{v_2}{v_1} - p_3 v_2$$

$$p_e = p_1 \frac{v_1}{v_2}\left(1 + \log_e \frac{v_2}{v_1}\right) - p_3$$

$$= p_1 \left\{\frac{1 + \log_e r}{r}\right\} - p_3$$

Examples XX

Solve the following equations, evaluating the constants by using the special condition given in each case.

(1) $\dfrac{dy}{dx} = \dfrac{x^3 - 1}{y^3 - 1}$, given $y = 1$, when $x = 0$

(2) $(x^3y + y)\dfrac{dy}{dx} = x^2y^2 + x^2$, given $y = 0$, when $x = 0$

(3) $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 + 1}$, given $y = 2$, when $x = 0$

(4) $\dfrac{dy}{dx} = \dfrac{y^2 + 1}{x^2 - 1}$, given $y = 0$, when $x = 2$

(5) $\dfrac{dy}{dx} = \sin(x + y) - \sin(x - y)$, given $y = \dfrac{\pi}{2}$, when $x = 0$

(6) $\dfrac{dy}{dx} = \sin^2(x + y) - \sin^2(x - y)$, given $y = \dfrac{\pi}{4}$, when $x = 0$

(7) $\operatorname{Cos}^2 x \dfrac{dy}{dx} + y = 1$, given $y = 0$, when $x = 0$

(8) $\dfrac{dy}{dx} + 2xy = x$, given $y = \dfrac{1}{4}$, when $x = 1$

(9) $(x^2 - y^2)\dfrac{dy}{dx} = xy$, given $y = 1$, when $x = 1$

(10) $\dfrac{dy}{dx} = \dfrac{y(x - y)}{x(x + y)}$, given $y = 1$, when $x = 1$

(11) $x^2 \dfrac{dy}{dx} = x^2 + y^2$, given $y = \dfrac{1}{2}$, when $x = 1$

(In Questions 9, 10, and 11 put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Express the equation in terms of v and x, and solve by separating the variables.)

(12) $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x$, given $y = 0$, when $x = \dfrac{\pi}{2}$

(13) $\dfrac{dy}{dx} + y = \sin 2x$, given $y = 1$, when $x = 0$

(14) $\sqrt{1-x^2}\dfrac{dy}{dx}+y=1$, given $y=2$, when $x=0$

(15) $\dfrac{dy}{dx}+y\tan x=\sec x$, given $y=0$, when $x=0$

(16) $\dfrac{dy}{dx}+y\tan x=\sin^2 2x$, given $y=0$, when $x=0$

(17) $\dfrac{dy}{dx}+\dfrac{e^x}{1+e^x}y=x^2$, given $y=0$, when $x=0$

(18) Show that the differential equation $\dfrac{dy}{dx}+yx^m=y^nx^p$ reduces to the form $\dfrac{dz}{dx}+(1-n)z\,x^m=(1-n)x^p$ if $z=y^{1-n}$

(19) Apply the result of Question 18 to solve the differential equation $x^3\dfrac{dy}{dx}+x^2y=y^3$ subject to the condition that $y=1$ when $x=1$.

(20) Plot the values of s and t given in paragraph 167 on squared paper between $t=20$ and $t=28$. Use the graph to find the time of flight necessary for a horizontal range of 40,000 ft. What is the angular elevation at which the projectile must be fired to give this range ?

(21) The projectile in paragraph 167 is given an elevation of $\tan^{-1}\dfrac{2}{9}$; that is, for the horizontal muzzle velocity of 2700 ft. per sec., the vertical velocity must be 600 ft. per sec. Find the horizontal and vertical components of its velocity, 36 seconds after projection. What is the magnitude of the velocity at this instant and in what direction is it travelling ?

(22) A body of mass 5 lb. is projected upwards in a resisting medium with an initial velocity of 200 ft. per sec. ; the resistance of the medium being kv lb., where v is the velocity of the body and k is a constant. If the body takes 3·5 seconds to reach its highest point find the value of k (k lies between 0·04 and 0·05). What is the greatest height to which the body will rise and what will be the velocity of the body 2 seconds after projection ? Take $g=32{\cdot}2$ ft. per sec.^{2}.

(23) A body of mass 5 lb. is projected upwards in a resisting medium with an initial velocity of 200 ft. per sec. ; the resistance of the medium being kv^2 lb., where v is the velocity of the body and k is a constant. If the body takes 3·5 seconds to reach its highest point, find the value of k (k lies between 0·0004 and 0·0005). What is the greatest height to which the body will rise, and what will be the velocity of the body 2 seconds after projection ? Take $g=32{\cdot}2$ ft. per sec.^{2}.

(24) A body of mass m lb. falls from rest in a resisting medium, the resistance of which is kv lb., where v is the velocity of the body and k is a constant. If after the 1st second the velocity is 20 ft. per sec. and after the 2nd second it is 35 ft. per sec., show that the body after falling for a great length of time will tend to have a velocity whose value is 111·9 ft. per sec. ($g = 32·2$ *f.s.s.*).

(25) In dealing with the strength of thick cylinders, if p is the radial compressive stress and f the hoop tensile stress at a point whose distance from the axis is r,

Then $p + f = 2a$ and $p + r\dfrac{dp}{dr} = f$ where a is a constant. If at the internal surface $p = p_0$ when $r = r_0$ and at the external surface $p = p_1$ when $r = r_1$, express p in terms of r.

(26) In a hollow cylinder of nickel steel subjected to internal pressure p, and no pressure outside, when the material is all yielding, if p is the radial compressive stress and f the hoop tensile stress at a point whose distance from the axis is r, and if $f + ap = b$ where a and b are constants for a particular kind of steel, and if we also have the usual relation $r\dfrac{dp}{dr} + p + f = 0$, find p as a function of r. If the inside radius r_1 is 3 inches and the inside p_1 is 30 tons per sq. inch, what is r_0, the outer radius. (Take for nickel steel $a = \frac{3}{4}$, $b = 30$.) (B. of E., 1911.)

(27) A quantity of gas expands from 2·5 cubic ft. to 9 cubic ft., the law of expansion being $pv^n = \text{const.}$ If the pressure at the beginning of the expansion is 80 lb. per sq. inch, find the work done during expansion : (1) when $n = 1·0646$, (2) when $n = 1·131$.

(28) In the previous example, if the law of expansion is $pv = \text{const.}$ Find the work done during expansion.

(29) Let p denote the population of England and Wales in millions and t the time in years that has elapsed since 1801. If the increase of population per year is proportional to the population—that is, it follows the compound interest law—express p in terms of t. If p was 8·9 in 1801, and it was 36·1 in 1911, what is p likely to be in 1921 ? (B. of E., 1914.)

(30) If $y = \text{A}e^{ax}$ what is $\dfrac{dy}{dx}$? An electric condenser, of capacity k farads and leakage resistance R ohms, has been charged and the voltage v is diminishing according to the law $\dfrac{dv}{dt} = -\dfrac{v}{k\text{R}}$. Express v in terms of the time, t seconds. If k is $0·8 \times 10^{-6}$ farads ; if v is noted to be 30 and 15 seconds afterwards it is noted to be 26·43, find R. (B. of E., 1912.)

(31) In the atmosphere, if p is pressure and h height above datum level, if $w = cp^{1/\gamma}$ where c and γ are constants, and if $\frac{dp}{dh} = -w$, find an equation connecting p and h. What is the above c if $p = tw\mathrm{R}$? Assume $p = p_0$ and $t = t_0$ where $h = 0$. R is a known constant for air. Find an equation connecting h and t. (B. of E., 1904.)

(32) Water leaves a circular basin very slowly by a hole at the bottom, every particle describing a spiral which is very nearly circular. Let v be the speed at a point whose distance from the axis is r and height above some datum level h. Assume no "rotation" or "spin"—that is, $\frac{1}{2}\left(\frac{v}{r} + \frac{dp}{dr}\right) = 0$—and show that this means $v = \frac{c}{r}$ where c is some constant. Now at the atmospheric surface $\frac{v^2}{2g} + h = \mathrm{C}$ where C is a constant. Find from this the shape of the surface—that is, the law connecting r and h. (B. of E., 1905.)

(33) If v volts is the voltage in an electric circuit, C ampères the current, R ohms the resistance, L henries the self-inductance, and t seconds the time, then $v = \mathrm{RC} + \mathrm{L}\frac{d\mathrm{C}}{dt}$.

If v is constant and equal to 8 volts, $\mathrm{R} = 0{\cdot}75$ ohms, and $\mathrm{L} = 0{\cdot}08$ henry, express C in terms of t, knowing that when $t = 0$, $\mathrm{C} = 0$. What would be the value of C when $t = 0{\cdot}1$ sec.?

(34) If v volts is the voltage in an electric circuit, C ampères the current, R ohms the resistance, L henries the self-inductance, and t seconds the time, then $v = \mathrm{RC} + \mathrm{L}\frac{d\mathrm{C}}{dt}$.

If $v = v_0 \sin pt$ where v_0 and p are constants, and if $\mathrm{R} = 50$ ohms, $\mathrm{L} = 0{\cdot}1$ henry, $v_0 = 100$ volts, and $p = 500$, express C in terms of t, knowing that when $t = 0$, $\mathrm{C} = 0$. To what value does C ultimately tend if t is taken sufficiently great?

CHAPTER XXI

176. If $y = \mathrm{A} \sin nx + \mathrm{B} \cos nx$,

$$\frac{dy}{dx} = n\mathrm{A} \cos nx - n\mathrm{B} \sin nx$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -n^2\mathrm{A} \sin nx - n^2\mathrm{B} \cos nx \\ &= -n^2\{\mathrm{A} \sin nx + \mathrm{B} \cos nx\} \\ &= -n^2y \end{aligned}$$

Thus the solution of the differential equation $\frac{d^2y}{dx^2} + n^2y = 0$ can be assumed to be $y = \mathrm{A} \sin nx + \mathrm{B} \cos nx$ where A and B are constants.

If $y = \mathrm{A}e^{nx} + \mathrm{B}e^{-nx}$,

$$\frac{dy}{dx} = n\mathrm{A}e^{nx} - n\mathrm{B}e^{-nx}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= n^2\mathrm{A}e^{nx} + n^2\mathrm{B}e^{-nx} \\ &= n^2(\mathrm{A}e^{nx} + \mathrm{B}e^{-nx}) \\ &= n^2y \end{aligned}$$

Thus the solution of the differential equation $\frac{d^2y}{dx^2} - n^2y = 0$ can be assumed to be $y = \mathrm{A}e^{nx} + \mathrm{B}e^{-nx}$ where A and B are constants.

Also for the differential equation $\frac{d^2y}{dx^2} = \pm n^2y$ a general solution $y = ke^{\alpha x}$ can be assumed, k being a constant.

For if $y = ke^{\alpha x}$,

$$\frac{dy}{dx} = \alpha\, ke^{\alpha x}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \alpha^2\, ke^{\alpha x} \\ &= \alpha^2 y \end{aligned}$$

Hence $y = ke^{\alpha x}$ is a solution of the equation $\frac{d^2y}{dx^2} = n^2y$ if $\alpha^2 = n^2$ or $\alpha = \pm n$,

and $\quad y = \mathrm{A}e^{nx} + \mathrm{B}e^{-nx}$ is the complete solution

Also $y = ke^{\alpha x}$ is a solution of the equation $\frac{d^2y}{dx^2} = -n^2x$ if $\alpha^2 = -n^2$ or $\alpha = \pm ni$,

and $y = Ae^{inx} + Be^{-inx}$ is the complete solution

but $y = A(\cos nx + i \sin nx) + B(\cos nx - i \sin nx)$

$= (A + B) \cos nx + i(A - B) \sin nx$

$= C \cos nx + D \sin nx$ where C and D are constants

177. The general solution $y = ke^{\alpha x}$ can also be assumed for the differential equation $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + b^2y = 0$.

For if $y = ke^{\alpha x}$,

$$\frac{dy}{dx} = \alpha ke^{\alpha x}$$

$$\frac{d^2y}{dx^2} = \alpha^2 ke^{\alpha x}$$

Then $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + b^2y = \alpha^2ke^{\alpha x} + 2a\,\alpha ke^{\alpha x} + b^2 ke^{\alpha x}$

$= ke^{\alpha x}(\alpha^2 + 2a\alpha + b^2)$

$= 0$, if $\alpha^2 + 2a\alpha + b^2 = 0$

Hence $y = ke^{\alpha x}$ will be a solution of the equation $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + b^2y = 0$, providing α has the values which will satisfy the quadratic equation $\alpha^2 + 2a\alpha + b^2 = 0$.

Let α_1 and α_2 be the two roots of this equation.

Then $y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$ will be the complete solution of the differential equation.

For since α_1 and α_2 are the roots of the equation $\alpha^2 + 2a\alpha + b^2 = 0$.

Then $\alpha_1^2 + 2a\alpha_1 + b^2 = 0$

and $\alpha_2^2 + 2a\alpha_2 + b^2 = 0$

If $y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$,

$$\frac{dy}{dx} = \alpha_1Ae^{\alpha_1 x} + B\alpha_2e^{\alpha_2 x}$$

$$\frac{d^2y}{dx^2} = \alpha_1^2Ae^{\alpha_1 x} + \alpha_2^2Be^{\alpha_2 x}$$

Then $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + b^2y$

$= \alpha_1^2Ae^{\alpha_1 x} + \alpha_2^2Be^{\alpha_2 x} + 2a(\alpha_1Ae^{\alpha_1 x} + \alpha_2Be^{\alpha_2 x})$
$+ b^2(Ae^{\alpha_1 x} + Be^{\alpha_2 x})$

$= Ae^{\alpha_1 x}(\alpha_1^2 + 2a\alpha_1 + b^2) + Be^{\alpha_2 x}(\alpha_2^2 + 2a\alpha_2 + b^2)$

$= 0$

Solving the quadratic equation for α,

$$\alpha^2 + 2a\alpha + a^2 = -b^2 + a^2$$

$$\alpha + a = \pm\sqrt{a^2 - b^2}$$

and
$$\alpha_1 = -a + \sqrt{a^2 - b^2}$$
$$\alpha_2 = -a - \sqrt{a^2 - b^2}$$

The form of the solution of the differential equation depends entirely upon the nature of the values of α_1 and α_2—that is, upon the relation between the quantities a and b.

(1) If $a = 0$,

Then
$$\alpha_1 = bi, \quad \text{and } \alpha_2 = -bi$$
and
$$y = \mathrm{A}e^{ibx} + \mathrm{B}e^{-ibx}$$
$$= \mathrm{C}\cos bx + \mathrm{D}\sin bx$$

(2) If $a = b$.

Then α_1 and α_2 are each equal to $-a$, and $y = e^{-ax}(\mathrm{C} + \mathrm{D}x)$ is the complete solution. For let $-a$ and $-a+h$ be the values of α_1 and α_2 respectively, h being small,

Then
$$y = \mathrm{A}e^{-ax} + \mathrm{B}e^{(-a+h)x}$$
$$= e^{-ax}(\mathrm{A} + \mathrm{B}e^{hx})$$
$$= e^{-ax}\left\{\mathrm{A} + \mathrm{B}\left(1 + hx + \frac{h^2x^2}{\underline{|2}} + \ . \ . \ .\right)\right\}$$

and if h is taken to be very small,

$$y = e^{-ax}\{(\mathrm{A} + \mathrm{B}) + \mathrm{B}h\,x\}$$
$$= e^{-ax}(\mathrm{C} + \mathrm{D}x)$$

This result can be proved by direct differentiation.

For
$$y = e^{-ax}(\mathrm{C} + \mathrm{D}x)$$
$$\frac{dy}{dx} = \mathrm{D}e^{-ax} - ae^{-ax}\cdot(\mathrm{C} + \mathrm{D}x)$$
$$\frac{d^2y}{dx^2} = -a\mathrm{D}e^{-ax} - a\{\mathrm{D}e^{-ax} - ae^{-ax}(\mathrm{C} + \mathrm{D}x)\}$$
$$= -2a\mathrm{D}e^{-ax} + a^2e^{-ax}(\mathrm{C} + \mathrm{D}x)$$

Then
$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = -2a\mathrm{D}e^{-ax} + a^2e^{-ax}(\mathrm{C} + \mathrm{D}x) + 2a\mathrm{D}e^{-ax}$$
$$- 2a^2e^{-ax}(\mathrm{C} + \mathrm{D}x) + a^2e^{-ax}(\mathrm{C} + \mathrm{D}x)$$
$$= 0$$

(3) When $a < b$, $\sqrt{a^2 - b^2}$ becomes imaginary, and α_1 and α_2 become complex quantities.

$$\alpha_1 = -a + i\sqrt{b^2 - a^2}, \quad \text{or } -a + di$$
and
$$\alpha_2 = -a - i\sqrt{b^2 - a^2}, \quad \text{or } -a - di$$

Then $$y = Ae^{(-a+di)x} + Be^{(-a-di)x}$$
$$= e^{-ax}(Ae^{idx} + Be^{-idx})$$
$$= e^{-ax}(C \cos dx + D \sin dx)$$
or $$y = e^{-ax}(C \cos \sqrt{b^2 - a^2}\, x + D \sin \sqrt{b^2 - a^2}\, x)$$

178. *Simple Harmonic Motion.* This is defined as the motion of a body in which the retardation is proportional to the distance the body is from some given position; and if x is this distance and t is the time, the equation of motion will be $\frac{d^2x}{dt^2} = -n^2x$, where n^2 is the constant of proportion.

The solution will be $x = A \sin nt + B \cos nt$, where A and B are constants which can be evaluated if the initial conditions are known.

(1) Let the initial conditions be $x = 0$ and $v = v_0$ when $t = 0$.

Since $x = A \sin nt + B \cos nt$

and $x = 0$ when $t = 0$. Then $B = 0$.

Hence $x = A \sin nt$

also $v = \frac{dx}{dt} = nA \cos nt$

but $v = v_0$ when $t = 0$. Then $nA = v_0$

The final solution is $x = \frac{v_0}{n} \sin nt$.

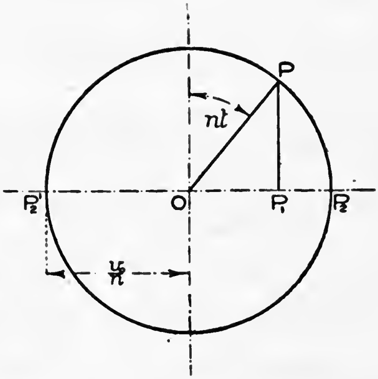


FIG. 115.

If a circle be drawn of radius $\frac{v_0}{n}$ and OP is a radius inclined at an angle nt to the vertical diameter, then P_1, the projection of P on the horizontal diameter, is such that $OP_1 = \frac{v_0}{n} \sin nt$ (Fig. 115).

If a particle P describes a circular path of radius $\frac{v_0}{n}$ ft. with uniform angular velocity n radians per second, the projection of this circular motion on the horizontal diameter will satisfy the condition that $x = \frac{v_0}{n} \sin nt$, and consequently $\frac{d^2x}{dt^2} = -n^2x$.

The body describes a complete oscillation, after passing from O to P_2, from P_2 through O to P_2', and from thence backwards again to O. This would take the same time as a complete revolution in the corresponding circular motion.

The periodic time = time of one complete revolution

$$T = \frac{2\pi}{n}$$

The frequency = number of complete oscillations per second

$$f = \frac{n}{2\pi}$$

(2) Let the initial conditions be $x = a$ and $v = 0$ when $t = 0$.

Then $\qquad x = A \sin nt + B \cos nt$

and $\qquad x = a$ when $t = 0$, then $B = a$

Also $\qquad v = \frac{dx}{dt} = nA \cos nt - nB \sin nt$

and $\qquad v = 0$ when $t = 0$, then $A = 0$

The final solution is $x = a \cos nt$.

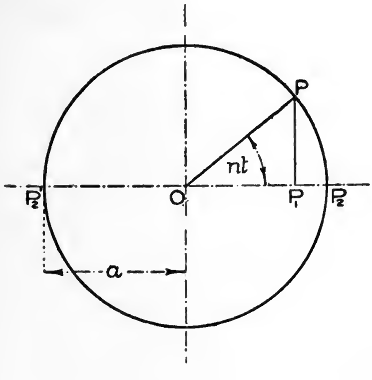


FIG. 116.

If a circle be drawn of radius a and OP is a radius inclined at an angle nt to the horizontal diameter (Fig. 116), then P_1, the projection of P on the horizontal diameter, is such that OP_1

$= a \cos nt$, and this satisfies the condition $x = a \cos nt$, and consequently $\frac{d^2x}{dt^2} = -n^2x$.

Here again the motion of the body is the horizontal projection of the motion of a particle describing a circular path of radius a with uniform angular velocity n radians per second.

$$\text{The periodic time} = \frac{2\pi}{n}$$

$$\text{Frequency} = \frac{n}{2\pi}$$

$$\text{Amplitude} = a$$

(3) Let the initial conditions be $x = a$ and $v = v_0$ when $t = 0$.

Then $\quad x = A \sin nt + B \cos nt$

and $\quad x = a$ when $t = 0$, then $B = a$

Also $\quad v = \frac{dx}{dt} = nA \cos nt - nB \sin nt$

and $\quad v = v_0$ when $t = 0$, then $A = \frac{v_0}{n}$

The final solution is $x = \frac{v_0}{n} \sin nt + a \cos nt$

$$= \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2} \sin (nt + \epsilon)$$

$$\text{where } \tan \epsilon = \frac{na}{v_0}$$

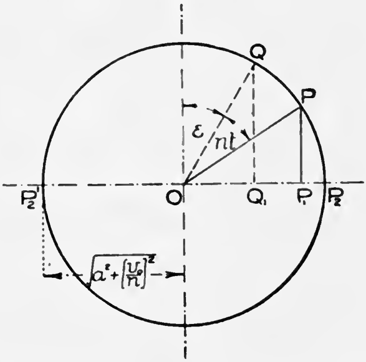


FIG. 117.

If a circle be drawn of radius $\sqrt{a^2 + \left(\frac{v_0}{n}\right)^2}$ and OQ is a radius inclined at an angle ϵ to the vertical diameter, while OP is another radius making an angle nt with OQ (Fig. 117), then Q_1 the projection of Q on the horizontal diameter, is such that

$$OQ_1 = \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2} \sin \epsilon$$

$$= \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2} \frac{a}{\sqrt{a^2 + \left(\frac{v_0}{n}\right)^2}}$$

$$= a$$

Q_1 is evidently the initial point.

P_1, the projection of P on the horizontal diameter, is such that $OP_1 = \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2} \sin (nt + \epsilon)$, which satisfies the condition $x = \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2} \sin (nt + \epsilon)$, and consequently $\frac{d^2x}{dt^2} = -n^2x$.

Here again the motion of the body is the horizontal projection of the motion of a particle describing a circular path of radius $\sqrt{a^2 + \left(\frac{v_0}{n}\right)^2}$ with uniform angular velocity n radians per second.

$$\text{Periodic time} = \frac{2\pi}{n}$$

$$\text{Frequency} = \frac{n}{2\pi}$$

$$\text{Amplitude} = \sqrt{a^2 + \left(\frac{v_0}{n}\right)^2}$$

The angle ϵ is spoken of as the Epoch of the simple harmonic motion.

179. The resultant of two simple harmonic motions of the same period and in the same straight line is a simple harmonic motion.

Let $\qquad x_1 = A_1 \sin (nt + \epsilon_1)$

and $\qquad x_2 = A_2 \sin (nt + \epsilon_2)$

be the two simple harmonic motions.

Then $x_1 + x_2 = A_1 \sin (nt + \epsilon_1) + A_2 \sin (nt + \epsilon_2)$ is the resultant.

Let OR be drawn making an angle nt with the vertical line OY (Fig. 118). Let OP_1 and OP_2 make angles ϵ_1 and ϵ_2 respectively with OR; also $OP_1 = A_1$ and $OP_2 = A_2$. Complete the parallelogram, of which OP_1 and OP_2 are adjacent sides, OP being the diagonal of this parallelogram and Q the horizontal projection of P.

$$\text{Then } OQ = OQ_2 + Q_2Q$$

$$= A_2 \sin (nt + \epsilon_2) + A_1 \sin (nt + \epsilon_1)$$

But $OQ = OP \sin(nt + E)$, which is a simple harmonic motion, of amplitude equal to the length of OP and the epoch E = angle ROP.

Then
$$OP = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\epsilon_2 - \epsilon_1)}$$

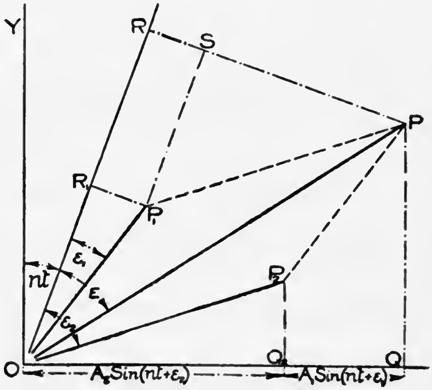


FIG. 118.

Let P_1R_1 and PR be drawn perpendicular to OR and P_1S parallel to OR

Then
$$\begin{aligned} OR &= OR_1 + R_1R \\ &= A_1 \cos \epsilon_1 + A_2 \cos \epsilon_2 \end{aligned}$$

and
$$\begin{aligned} PR &= PS + SR \\ &= PS + P_1R_1 \\ &= A_2 \sin \epsilon_2 + A_1 \sin \epsilon_1 \end{aligned}$$

Now
$$\begin{aligned} \tan E &= \frac{PR}{OR} \\ &= \frac{A_1 \sin \epsilon_1 + A_2 \sin \epsilon_2}{A_1 \cos \epsilon_1 + A_2 \cos \epsilon_2} \end{aligned}$$

180. *The Vibration of a Spring.* Let h be the stiffness of the spring—that is, a force of 1 lb., will elongate the spring h ft. If the spring is elongated x ft. the force required will be $\frac{x}{h}$ lb.

Let a body of mass m lb. be hung from this spring and then displaced from its equilibrium position and then let go.

If there are no frictional resistances to the motion of the body, the only force acting on the body will be the resistance of the spring, and when the body is at a distance x ft. from its equilibrium position this resistance is $\frac{x}{h}$ lb.

Then $$\frac{x}{h} = -\frac{m}{g}\frac{d^2x}{dt^2}$$

and $$\frac{d^2x}{dt^2} = -\frac{g}{mh}x$$

$$= -n^2x, \text{ where } n^2 = \frac{g}{mh}$$

Then $$x = \mathrm{A}\sin nt + \mathrm{B}\cos nt$$

Let the initial displacement of the body from its equilibrium position be d ft.

The initial conditions are $x = d$ and $v = 0$ when $t = 0$.

Then $x = d$ when $t = 0$, hence $\mathrm{B} = d$,

also $$v = \frac{dx}{dt} = n\mathrm{A}\cos nt - n\mathrm{B}\sin nt$$

but $v = 0$ when $t = 0$, hence $\mathrm{A} = 0$.

Finally, $x = d\cos nt$.

The body vibrates with simple harmonic motion the amplitude of which is d ft.

$$\text{The periodic time} = \frac{2\pi}{n}$$

$$= 2\pi\sqrt{\frac{mh}{g}} \text{ secs.}$$

$$\text{The frequency or the number of vibrations per sec.} = \frac{n}{2\pi}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{mh}}$$

181. *Theory of Struts. Euler's Formulæ.* In this work we have to find the buckling load—that is, the least load which can be applied at the ends of the strut to just cause the strut to bend. There are three distinct cases to consider :

(1) When the strut is free at both ends.
(2) When the strut is fixed at both ends.
(3) When the strut is free at one end and fixed at the other.

Case I. When the strut is free at both ends.

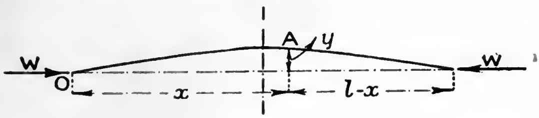


FIG. 119.

Let W be the load on the strut and y the deflection at a point A situated at a distance x from the end O.

$$\text{Bending moment at A} = -\mathrm{W}y$$

$$\text{but bending moment} = \mathrm{EI}\frac{1}{\mathrm{R}}$$

$$= \mathrm{EI}\frac{d^2y}{dx^2}$$

Hence
$$\mathrm{EI}\frac{d^2y}{dx^2} = -\mathrm{W}y$$

$$\frac{d^2y}{dx^2} = -\frac{\mathrm{W}}{\mathrm{EI}}y$$

$$= -n^2y, \text{ where } n^2 = \frac{\mathrm{W}}{\mathrm{EI}}$$

The solution of this is :

$$y = \mathrm{A}\sin nx + \mathrm{B}\cos nx$$

But when $x = 0,\ y = 0,\quad$ then $\mathrm{B} = 0$

and $y = \mathrm{A}\sin nx$

but when $x = l,\ y = 0,\quad$ then $\mathrm{A}\sin nl = 0$

Now A cannot be zero, for if so bending would not occur.

Hence
$$\sin nl = 0$$

and nl can have the values $0,\ \pi,\ 2\pi,\ 3\pi\ \ldots\ s\pi$. Confining our attention to the least value π, for the zero value is clearly inadmissible.

$$nl = \pi$$

$$n^2 = \frac{\pi^2}{l^2}$$

$$\frac{\mathrm{W}}{\mathrm{EI}} = \frac{\pi^2}{l^2}$$

$$\text{and } \mathrm{W} = \frac{\pi^2\mathrm{EI}}{l^2}$$

Case II. When the ends are fixed.

In this case there must be acting at each end a fixing couple

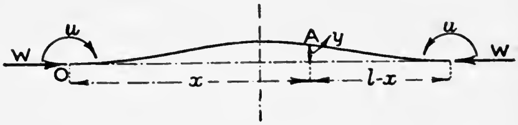


FIG. 120.

of magnitude u and this keeps the direction of an end of the strut in such a condition that the slope there is zero.

Bending moment at A $= u - Wy$

Then
$$EI\frac{d^2y}{dx^2} = u - Wy$$
$$\frac{d^2y}{dx^2} = -\frac{1}{EI}(Wy - u)$$
$$= -\frac{W}{EI}(y - a), \text{ where } a = \frac{u}{W}$$
$$= -n^2(y - a), \text{ where } n^2 = \frac{W}{EI}$$

If $z = y - a$ then $\dfrac{dz}{dx} = \dfrac{dy}{dx}$ and $\dfrac{d^2z}{dx^2} = \dfrac{d^2y}{dx^2}$

Hence
$$\frac{d^2z}{dx^2} = -n^2z$$

The solution of this is:
$$z = A\sin nx + B\cos nx$$
and
$$y - a = A\sin nx + B\cos nx$$
then
$$\frac{dy}{dx} = nA\cos nx - nB\sin nx$$
but when
$$x = 0, \frac{dy}{dx} = 0, \quad \text{then } A = 0$$
and
$$y - a = B\cos nx$$
Also, when
$$x = 0, y = 0, \text{ then } B = -a$$
Then
$$y - a = -a\cos nx$$
$$y = a(1 - \cos nx)$$
when
$$x = l, y = 0, \text{ then } a(1 - \cos nl) = 0$$
That is
$$1 - \cos nl = 0, \quad \text{or } \cos nl = 1$$
Hence nl must have the values $0, 2\pi, 4\pi \ldots$

Considering the least value, 2π, for the zero value is clearly inadmissible.
$$nl = 2\pi$$
$$n^2 = \frac{4\pi^2}{l^2}$$
$$\frac{W}{EI} = \frac{4\pi^2}{l^2}$$
and
$$W = \frac{4\pi^2 EI}{l^2}$$

Case III. When one end is fixed and the other end free.

The free end must be so constrained that it will always be in the same vertical line as the fixed end, while it should be free to take up its proper direction when bending occurs. Let the free end be considered to slide in a frictionless guide, the normal reaction at the guide being Q.

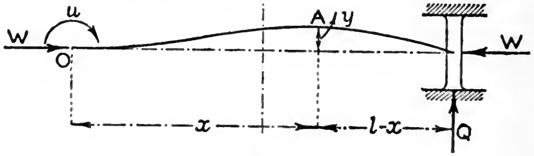


FIG. 121.

Considering the fixed end as the origin.

Bending moment at A $= Q(l - x) - Wy$

$$EI\frac{d^2y}{dx^2} = Q(l - x) - Wy$$

$$\frac{d^2y}{dx^2} = -\frac{W}{EI}\left\{y - \frac{Q}{W}(l - x)\right\}$$

$$= -n^2\left\{y - \frac{Q}{W}(l - x)\right\}, \text{ where } n^2 = \frac{W}{EI}.$$

Let $$z = y - \frac{Q}{W}(l - x)$$

$$\frac{dz}{dx} = \frac{dy}{dx} + \frac{Q}{W}$$

$$\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$$

Then $$\frac{d^2z}{dx^2} = -n^2z$$

The solution of this is:

$$z = A \sin nx + B \cos nx$$

$$y - \frac{Q}{W}(l - x) = A \sin nx + B \cos nx$$

when $x = 0$, $y = 0$, then $B = -\dfrac{Ql}{W}$

also $$\frac{dy}{dx} + \frac{Q}{W} = nA \cos nx - nB \sin nx$$

when $x = 0$, $\dfrac{dy}{dx} = 0$, then $nA = \dfrac{Q}{W}$

Hence $$y - \frac{Q}{W}(l - x) = \frac{Q}{nW} \sin nx - \frac{Ql}{W} \cos nx$$

$$= \frac{Q}{W}\left\{\frac{1}{n} \sin nx - l \cos nx\right\}$$

But when $x = l,\ y = 0$, then $\frac{1}{n} \sin nl - l \cos nl = 0$

or $\tan nl = nl$

This equation must be solved by means of a graph and neglecting the zero value, the least value of nl which satisfies the equation is found to be 4·493 radians or $257\frac{1}{2}°$.

Then
$$n^2l^2 = (4{\cdot}493)^2$$
$$= 2{\cdot}047\pi^2$$

Then
$$\frac{Wl^2}{EI} = 2{\cdot}047\pi^2$$

and
$$W = \frac{2{\cdot}047\pi^2 EI}{l^2}$$

182. *The Rankine-Gordon Formula for Struts.* For a very short strut where buckling plays no part, the breaking load should be Af, where f is the crushing strength of the material and A the cross sectional area. For a very long strut where crushing plays no part, the buckling load should be $\frac{c\pi^2 EI}{l^2}$ where c is a constant depending upon the nature of the ends.

Then if W is the load under which a strut of any length gives way, and if
$$W = \frac{Af}{1 + \frac{Afl^2}{c\pi^2 EI}}$$

When l is small, W becomes Af approximately.

When l is great, W becomes $\frac{c\pi^2 EI}{l^2}$ approximately.

This formula therefore makes W approximate to Af for very short struts, and to $\frac{c\pi^2 EI}{l^2}$ for very long struts.

Thus
$$W = \frac{Af}{1 + C\frac{Al^2}{I}}$$

where $C = \frac{f}{c\pi^2 E}$, a constant depending upon the material and the nature of the ends of the strut.

Now $I = Ak^2$ where k is the radius of gyration of the section with respect to that axis about which bending is most likely to take place—that is, the axis about which I is least.

Then
$$W = \frac{Af}{1 + C\left(\frac{l}{k}\right)^2}$$

If p is the intensity of the breaking load per square inch of section,

Then $$p = \frac{W}{A}$$

and $$p = \frac{f}{1 + C\left(\frac{l}{k}\right)^2}$$

When the ends are free $c = 1$ and $C = \dfrac{f}{\pi^2 E}$

When the ends are fixed $c = 4$ and $C = \dfrac{f}{4\pi^2 E}$

When one end is fixed and the other end is free $c = 2{\cdot}047$ and $C = \dfrac{f}{2{\cdot}047\pi^2 E}$

Example. A hollow cast-iron column 24 feet long has to carry safely a load of 50 tons. The external radius is 5 inches, find the internal radius, (1) when the ends are free, (2) when the ends are fixed, (3) when one end is free and the other fixed. $f = 36$ tons per sq. in., $E = 6000$ tons per sq. in., factor of safety $= 6$.

Let x be the internal radius.

$$\text{Area} = \pi(25 - x^2) \text{ sq. in.}$$

$$\text{Moment of inertia} = \frac{\pi}{4}(625 - x^4) \text{ inch units}$$

$$k^2 = \frac{25 + x^2}{4}$$

Allowing for the factor of safety, the column must be designed to carry a load of 300 tons.

Then $$p = \frac{300}{\pi(25 - x^2)} \text{ tons per sq. in}$$

(1) When the ends are free,

$$C = \frac{36}{\pi^2 \times 6000} = 6{\cdot}08 \times 10^{-4}$$

and $$\frac{300}{\pi(25 - x^2)} = \frac{36}{1 + \dfrac{6{\cdot}08 \times 10^{-4} \times (288)^2 \times 4}{25 + x^2}}$$

$$= \frac{36(25 + x^2)}{226{\cdot}8 + x^2}$$

Then $$2{\cdot}653(226{\cdot}8 + x^2) = 625 - x^4$$
$$x^4 + 2{\cdot}653x^2 = 23{\cdot}4$$
$$x^2 + 1{\cdot}327 = \pm\ 5{\cdot}016$$
$$x^2 = 3{\cdot}689$$
$$x = 1{\cdot}921$$

(2) When the ends are fixed,

$$C = \frac{36}{4\pi^2 \times 6000} = 1{\cdot}52 \times 10^{-4}$$

and $$\frac{300}{\pi(25 - x^2)} = \frac{36}{1 + \dfrac{50{\cdot}45}{25 + x^2}}$$
$$= \frac{36(25 + x^2)}{75{\cdot}45 + x^2}$$

Then $$2{\cdot}653(75{\cdot}45 + x^2) = 625 - x^4$$
$$x^4 + 2{\cdot}653x^2 = 424{\cdot}8$$
$$x^2 + 1{\cdot}327 = \pm\ 20{\cdot}65$$
$$x^2 = 19{\cdot}32$$
$$x = 4{\cdot}395$$

(3) When one end is free and the other fixed.

$$C = \frac{36}{2{\cdot}047\pi^2 \times 6000} = 2{\cdot}97 \times 10^{-4}$$

and $$\frac{300}{\pi(25 - x^2)} = \frac{36}{1 + \dfrac{98{\cdot}56}{25 + x^2}}$$
$$= \frac{36(25 + x^2)}{123{\cdot}6 + x^2}$$

Then $$2{\cdot}653(123{\cdot}6 + x^2) = 625 - x^4$$
$$x^4 + 2{\cdot}653x^2 = 297{\cdot}2$$
$$x^2 + 1{\cdot}327 = \pm\ 17{\cdot}30$$
$$x^2 = 15{\cdot}97$$
$$x = 3{\cdot}996$$

183. *Damped Vibrations.* A body of mass m lb. is suspended from a spring of stiffness h—that is, a force of 1 lb. will extend the spring by h feet. The body is displaced from its position of equilibrium and then allowed to oscillate in a resisting medium, the resistance of which is proportional to the velocity of the body.

When the body is at a distance x feet from its position of equilibrium the resistance of the spring is $\frac{x}{h}$ lb., while the resistance of the medium is kv lb., where k is a constant depending upon the nature of the medium; both of these forces tend to urge the body back to its equilibrium position.

Hence the total resistance to the motion is $\left(\frac{x}{h} + kv\right)$ lb.

and $$\frac{x}{h} + kv = -\frac{m}{g} \times \text{acceleration}$$

or $$\frac{d^2x}{dt^2} + \frac{gk}{m}\frac{dx}{dt} + \frac{g}{hm}x = 0$$

This can be expressed as

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2x = 0$$

where $$2a = \frac{gk}{m} \text{ and } b^2 = \frac{g}{hm}$$

Let $x = Ae^{\alpha t}$ be the solution of this equation.

Then $$\alpha^2 + 2a\alpha + b^2 = 0$$

$$\alpha + a = \pm\sqrt{a^2 - b^2}$$

and $$\alpha_1 = -a + \sqrt{a^2 - b^2}$$

$$\alpha_2 = -a - \sqrt{a^2 - b^2}$$

The complete solution is $x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$

Case I. When $b > a$. Let $a = 1$, and $b^2 = 10$.

Then $$\alpha = -1 \pm \sqrt{-9}$$
$$= -1 \pm 3i$$

Then $$x = Ae^{(-1+3i)t} + Be^{(-1-3i)t}$$
$$= e^{-t}(Ae^{i3t} + Be^{-i3t})$$
$$= e^{-t}(C\cos 3t + D\sin 3t)$$

where C and D are constants depending upon the initial conditions.

(*a*) Let the initial conditions be $x = 0$ and $v = 9$ when $t = 0$.

$$x = e^{-t}(C\cos 3t + D\sin 3t)$$

$$v = \frac{dx}{dt} = -e^{-t}(C\cos 3t + D\sin 3t) + e^{-t}(-3C\sin 3t + 3D\cos 3t)$$

But when $t = 0$, $x = 0$. Then $C = 0$

Also, when $t = 0$, $v = 9$. Then $D = 3$

$x = 3e^{-t}\sin 3t$ is the complete solution (*b*) Let the initial conditions be $x = 3$ and $v = 0$ when $t = 0$.

$$x = e^{-t}(\mathrm{C} \cos 3t + \mathrm{D} \sin 3t)$$

$$v = \frac{dx}{dt} = -e^{-t}(\mathrm{C} \cos 3t + \mathrm{D} \sin 3t) + e^{-t}(-3\,\mathrm{C} \sin 3t + 3\mathrm{D} \cos 3t)$$

But when $t = 0$, $x = 3$. Then $\mathrm{C} = 3$

Also, when $t = 0$, $v = 0$. Then $3\mathrm{D} - \mathrm{C} = 0$, or $\mathrm{D} = 1$

$$\begin{aligned} x &= e^{-t}(3 \cos 3t + \sin 3t) \\ &= \sqrt{10}e^{-t} \sin (3t + \alpha), \text{ where } \alpha = \tan^{-1} 3 \\ &= 3{\cdot}162\, e^{-t} \sin (3t + 1{\cdot}248) \end{aligned}$$

(*c*) Let the initial conditions be $x = 3$ and $v = 9$ when $t = 0$.

$$x = e^{-t}(\cos 3t + \mathrm{D} \sin 3t)$$

$$v = \frac{dx}{dt} = -e^{-t}(\cos 3t + \mathrm{D} \sin 3t) + e^{-t}(-3\mathrm{C} \sin 3t + 3\mathrm{D} \cos 3t)$$

But when $t = 0$, $x = 3$. Then $\mathrm{C} = 3$

Also, when $t = 0$, $v = 9$. Then $3\mathrm{D} - \mathrm{C} = 9$, or $\mathrm{D} = 4$

$$\begin{aligned} x &= e^{-t}(3 \cos 3t + 4 \sin 3t) \\ &= 5e^{-t} \sin (3t + \beta), \quad \text{where } \beta = \tan^{-1}\frac{3}{4} \\ &= 5e^{-t} \sin (3t + 0{\cdot}6435) \end{aligned}$$

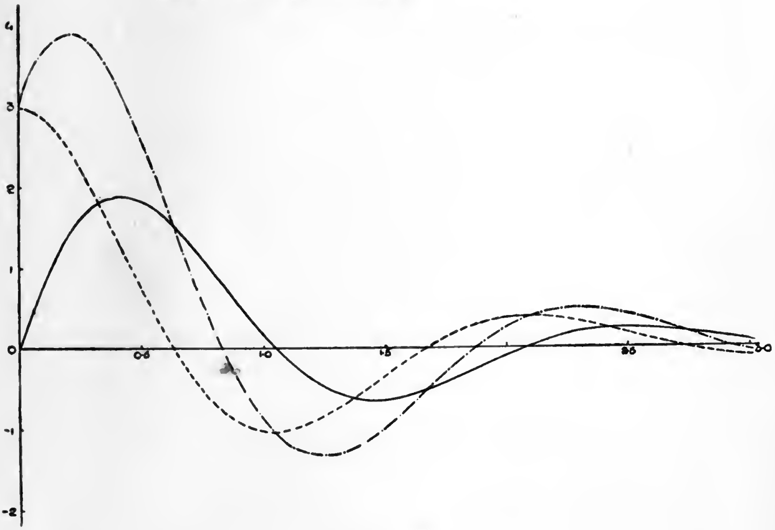


FIG. 122.

Fig. 122 shows the three relations plotted for values of t between 0 and 3. They each represent periodic functions of continuously diminishing amplitude, but in each case the periodic time is the same.

$$\text{Periodic time} = \frac{2\pi}{3} = 2{\cdot}0944 \text{ secs.}$$

$$\text{Frequency} = \frac{3}{2\pi} = 0{\cdot}4775$$

Case II. When $a = b$. Let $a = b = 2$.

Then $\alpha = -2$

$$x = e^{-2t}(\mathrm{A} + \mathrm{B}t)$$

where A and B are constants depending upon the initial conditions.

(*a*) Let the initial conditions be $x = 0$ and $v = 10$ when $t = 0$.

$$x = e^{-2t}(\mathrm{A} + \mathrm{B}t)$$

$$v = \frac{dx}{dt} = -2e^{-2t}(\mathrm{A} + \mathrm{B}t) + \mathrm{B}e^{-2t}$$

But when $t = 0$, $x = 0$. Then $\mathrm{A} = 0$

Also, when $t = 0$, $v = 10$. Then $\mathrm{B} = 10$

Hence $x = 10te^{-2t}$

(*b*) Let the initial conditions be $x = 5$ and $v = 0$ when $t = 0$.

$$x = e^{-2t}(\mathrm{A} + \mathrm{B}t)$$

$$v = \frac{dx}{dt} = -2e^{-2t}(\mathrm{A} + \mathrm{B}t) + \mathrm{B}e^{-2t}$$

But when $t = 0$, $x = 5$. Then $\mathrm{A} = 5$

Also, when $t = 0$, $v = 0$. Then $\mathrm{B} - 2\mathrm{A} = 0$, or $\mathrm{B} = 10$

Hence
$$x = e^{-2t}(5 + 10t)$$
$$= 5e^{-2t}(1 + 2t)$$

(*c*) Let the initial conditions be $x = 5$ and $v = 10$ when $t = 0$.

$$x = e^{-2t}(\mathrm{A} + \mathrm{B}t)$$

$$v = \frac{dx}{dt} = -2e^{-2t}(\mathrm{A} + \mathrm{B}t) + \mathrm{B}e^{-2t}$$

But when $t = 0$, $x = 5$. Then $\mathrm{A} = 5$

Also, when $t = 0$, $v = 10$. Then $\mathrm{B} - 2\mathrm{A} = 10$, or $\mathrm{B} = 20$

Hence
$$x = e^{-2t}(5 + 20t)$$
$$= 5e^{-2t}(1 + 4t)$$

Fig. 123 shows the three relations plotted for values of t between 0 and 2. In each case the maximum displacement of the body from the equilibrium position is quickly attained, and then the body goes slowly back to its equilibrium position although it never reaches it.

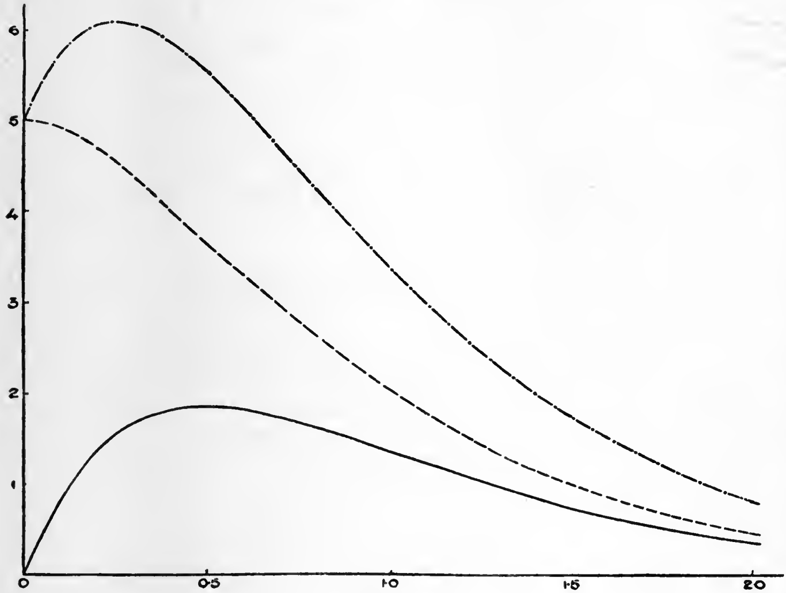


FIG. 123.

Case III. When $a > b$. Let $a = 3$, and $b^2 = 8$.

Then
$$\alpha = -3 \pm 1$$
$$= -4 \text{ or } -2$$
and
$$x = Ae^{-2t} + Be^{-4t}$$
where A and B are constants depending upon the initial conditions.

(*a*) Let the initial conditions be $x = 0$ and $v = 8$ when $t = 0$.

$$x = Ae^{-2t} + Be^{-4t}$$

$$v = \frac{dx}{dt} = -2Ae^{-2t} - 4Be^{-4t}$$

But when $t = 0$, $x = 0$, and $A + B = 0$

Also, when $t = 0$, $v = 8$, and $-2A - 4B = 8$

Hence
$$A = 4 \text{ and } B = -4$$
$$x = 4(e^{-2t} - e^{-4t})$$

(*b*) Let the initial conditions be $x = 4$ and $v = 0$ when $t = 0$.

$$x = Ae^{-2t} + Be^{-4t}$$

$$v = \frac{dx}{dt} = -2Ae^{-2t} - 4Be^{-4t}$$

But when $t = 0$, $x = 4$, and $A + B = 4$

Also, when $t = 0$, $v = 0$, and $-2A - 4B = 0$

Hence $A = 8$ and $B = -4$

$$x = 8e^{-2t} - 4e^{-4t}$$

$$= 4(2e^{-2t} - e^{-4t})$$

(*c*) Let the initial conditions be $x = 4$ and $v = 8$ when $t = 0$.

$$x = Ae^{-2t} + Be^{-4t}$$

$$v = \frac{dx}{dt} = -2Ae^{-2t} - 4Be^{-4t}$$

But when $t = 0$, $x = 4$, and $A + B = 4$

Also, when $t = 0$, $v = 8$, and $-2A - 4B = 8$

Hence $A = 12$ and $B = -8$

$$x = 12e^{-2t} - 8e^{-4t}$$

$$= 4(3e^{-2t} - 2e^{-4t})$$

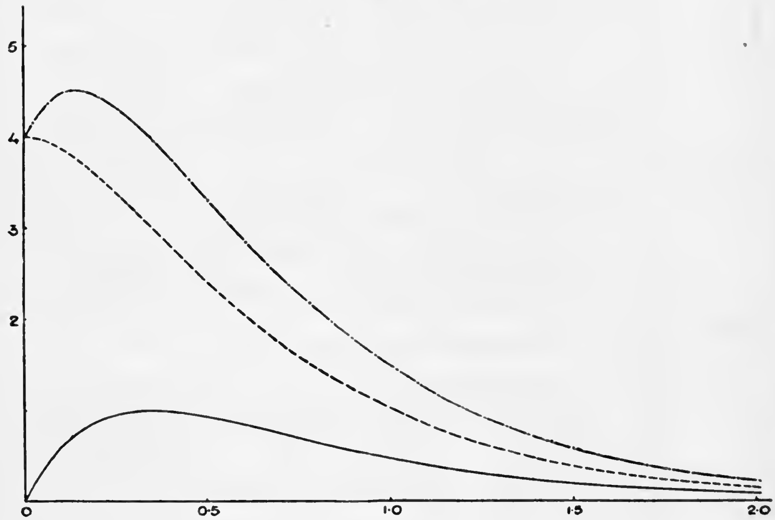


FIG. 124.

Fig. 124 shows the three relations plotted for values of t between 0 and 2. In each case the maximum displacement of the body from the equilibrium position is quickly attained and then the body goes slowly back to its equilibrium position although it never reaches it.

184. *Forced Vibrations.* A study of the curves obtained from the examples given in the previous paragraph indicates that the equation $\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2x = 0$ represents a motion which is damped—that is, it gradually dies away. In some cases this motion is oscillatory, while in other cases it is not. If the whole system has a vibration of known frequency forced upon it and it is necessary to make a study of this forced vibration, it is as well to remember that when the natural or damped vibration dies away, this forced vibration will remain. Let the equation $\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2x = c \sin pt$ represent the motion of a body which has a forced vibration of periodic time $\frac{2\pi}{p}$ impressed upon it. When the natural vibration dies away, the motion of the body will be one solely due to the forced vibration, and as the periodic time remains unchanged this motion will be given by $x = \mathrm{A} \sin pt + \mathrm{B} \cos pt$, and this solution must satisfy the complete equation of motion and the values of the constants A and B must be determined so that this will be so.

$$\frac{dx}{dt} = p\mathrm{A} \cos pt - p\mathrm{B} \sin pt$$

$$\frac{d^2x}{dt^2} = -p^2\mathrm{A} \sin pt - p^2\mathrm{B} \cos pt$$

$$\begin{aligned} \text{But } c \sin pt &= \frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2x \\ &= -p^2\mathrm{A} \sin pt - p^2\mathrm{B} \cos pt + 2a(p\mathrm{A} \cos pt - p\mathrm{B} \sin pt) \\ &\quad + b^2(\mathrm{A} \sin pt + \mathrm{B} \cos pt) \\ &= (-p^2\mathrm{A} - 2ap\mathrm{B} + b^2\mathrm{A}) \sin pt + (-p^2\mathrm{B} + 2ap\mathrm{A} \\ &\quad + b^2\mathrm{B}) \cos pt \end{aligned}$$

Equating coefficients of sin pt,

$$(b^2 - p^2)\mathrm{A} - 2ap\mathrm{B} = c \quad \dots\dots \quad (1)$$

Equating coefficients of cos pt,

$$2ap\mathrm{A} + (b^2 - p^2)\mathrm{B} = 0 \quad \dots\dots \quad (2)$$

and relations (1) and (2) can be solved as a pair of simultaneous equations for A and B.

$$\mathrm{A}(b^2 - p^2)^2 - 2ap(b^2 - p^2)\mathrm{B} = c(b^2 - p^2)$$

$$4a^2p^2\mathrm{A} + 2ap(b^2 - p^2)\mathrm{B} = 0$$

and $$\{(b^2 - p^2)^2 + 4a^2p^2\}\mathrm{A} = c(b^2 - p^2)$$

Hence
$$A = \frac{c(b^2 - p^2)}{(b^2 - p^2)^2 + 4a^2p^2}$$

Also
$$B = -\frac{2apA}{b^2 - p^2}$$
$$= -\frac{2apc}{(b^2 - p^2)^2 + 4a^2p^2}$$

As an example, let $a = 2$, $b = 3$, $c = 3$, and $p = 2$.

Then
$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 3 \sin 2t$$

Let
$$x = A \sin 2t + B \cos 2t.$$

Then
$$\frac{dx}{dt} = 2A \cos 2t - 2B \sin 2t$$

and
$$\frac{d^2x}{dt^2} = -4A \sin 2t - 4B \cos 2t$$

Hence $3 \sin 2t = (-4A - 8B + 9A) \sin 2t + (-4B + 8A + 9B) \cos 2t$

and
$$5A - 8B = 3$$
$$8A + 5B = 0$$
$$A = \frac{15}{89}, \quad B = -\frac{24}{89}$$

Thus
$$x = \frac{15}{89} \sin 2t - \frac{24}{89} \cos 2t$$
$$= \frac{3}{89}\{5 \sin 2t - 8 \cos 2t\}$$
$$= \frac{3}{\sqrt{89}} \sin (2t - \alpha), \quad \text{where } \tan \alpha = \frac{8}{5}$$

185. A condenser of capacity k is charged so that the potential difference of the plates is v_0. The two plates are connected by a wire.

If Q is the quantity and C the current,

Then
$$Q = kv, \text{ and } C = -\frac{dQ}{dt}$$

If the connecting wire is of resistance R but of negligible inductance,
$$v = RC$$
$$= -R\frac{dQ}{dt}$$
$$= -Rk\frac{dv}{dt}$$

Then
$$\frac{dv}{dt} = -\frac{v}{Rk}$$

Let $v = Ae^{at}$ be the solution of this equation.

$$\frac{dv}{dt} = \alpha Ae^{at}$$
$$= \alpha v$$

Hence $$\alpha = -\frac{1}{kR}$$

$$\text{and } v = Ae^{-\frac{t}{kR}}$$

But when $t = 0$, $v = v_0$, and $v_0 = A$

Thus $v = v_0 e^{-\frac{t}{kR}}$ is the relation expressing v the voltage in terms of t the time.

If the connecting wire is of resistance R and inductance L,

Then $$Q = kv,\quad C = -\frac{dQ}{dt},\quad \text{and } v = RC + L\frac{dC}{dt}$$

Thus $$C = -\frac{dQ}{dt}$$
$$= -k\frac{dv}{dt}$$
$$\frac{dC}{dt} = -k\frac{d^2v}{dt^2}$$

Hence $$v = -Rk\frac{dv}{dt} - kL\frac{d^2v}{dt^2}$$

and $$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{kL} = 0$$

Let $v = Ae^{at}$ be the solution.

Then $$\frac{dv}{dt} = \alpha Ae^{at}$$
$$\frac{d^2v}{dt^2} = \alpha^2 Ae^{at}$$

Hence $$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{kL} = 0$$

$$\alpha^2 + \frac{R}{L}\alpha + \frac{R^2}{4L^2} = -\frac{1}{kL} + \frac{R^2}{4L^2}$$

$$\alpha + \frac{R}{2L} = \pm\frac{1}{2L}\sqrt{R^2 - \frac{4L}{k}}$$

$$\alpha = \frac{1}{2L}\left\{-R \pm \sqrt{R^2 - \frac{4L}{k}}\right\}$$

Thus giving the required values of α.

It should be noticed that the form of the final solution depends upon the nature of the values of α; that is, according as R^2 is greater than, equal to, or less than $\frac{4L}{k}$. The method of procedure is exactly the same as that adopted for the analogous differential equation $\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2x = 0$.

Examples XXI

Solve the following differential equations:

(1) $\frac{d^2y}{dx^2} = -12y$

(2) $\frac{d^2y}{dx^2} = 18y$

(3) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$

(4) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

(5) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$

(6) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

(7) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$

(8) $\frac{d^3y}{dx^3} = -8y$

(9) $\frac{d^3y}{dx^3} = 8y$

(10) $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} + 16y = 0$

(11) $\frac{d^2x}{dt^2} = -20x$, where x ft. is distance and t secs. is time, represents the equation of motion of a body. Find an expression giving x in terms of t, when the initial conditions are (1) $t = 0$, $x = 0{\cdot}5$, and $\frac{dx}{dt} = 0$; (2) $t = 0$, $x = 0$, and $\frac{dx}{dt} = 4$; (3) $t = 0$, $x = 0{\cdot}5$, and $\frac{dx}{dt} = 4$.

(12) $\dfrac{d^2x}{dt^2} = 20x$, where x ft. is distance and t secs. is time, represents the equation of motion of a body. Find an expression giving x in terms of t when the initial conditions are: (1) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 0$; (2) $t = 0$, $x = 0$, and $\dfrac{dx}{dt} = 4$; (3) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 4$.

(13) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 20x = 0$, where x ft. is distance and t secs. is time, represents the equation of motion of a body. Find an expression giving x in terms of t when the initial conditions are: (1) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 0$; (2) $t = 0$, $x = 0$, and $\dfrac{dx}{dt} = 4$; (3) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 4$. What is the periodic time of the motion?

(14) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = 0$, where x ft. is distance and t secs. is time, represents the equation of motion of a body. Find an expression giving x in terms of t when the initial conditions are: (1) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 0$; (2) $t = 0$, $x = 0$, and $\dfrac{dx}{dt} = 4$; (3) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 4$. In each case find the maximum value of x and the time at which it occurs.

(15) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = 0$, where x ft. is distance and t secs. is time, represents the equation of motion of a body. Find an expression giving x in terms of t when the initial conditions are: (1) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 0$; (2) $t = 0$, $x = 0$, and $\dfrac{dx}{dt} = 4$; (3) $t = 0$, $x = 0{\cdot}5$, and $\dfrac{dx}{dt} = 4$. In each case find the maximum value of x and the time at which it occurs.

(16) A force of 10 lb. extends a spring by 1·6 inches. A mass of 8 lb. is suspended from such a spring and is displaced 2 inches from its equilibrium position and then let go. If x ft. is the distance of the mass from the equilibrium position at any subsequent time, t secs. find the equation of motion for the body and then find an expression giving x in terms of t. What is the periodic time of the motion and how many complete oscillations does the body make per minute? ($g = 32{\cdot}2$ f.s.s.)

(17) A force of 10 lb. extends a spring by 4 inches. A mass of 8 lb. is suspended from such a spring in a medium whose resistance to the motion is $2v$ lb. where v ft. per sec. is the velocity of the body at any instant. The body is displaced 3 inches from its equilibrium position and is then allowed to oscillate in the medium. If x ft. is the distance of the body from the equilibrium position at any subsequent time t secs., find the equation of motion for the body, and then find an expression giving x in terms of t. What is the periodic time of the motion ?

(18) The two simple harmonic motions $x_1 = 3{\cdot}2 \sin (nt + 0{\cdot}732)$ and $x_2 = 5{\cdot}6 \sin (nt + 1{\cdot}346)$ can be expressed as one simple harmonic motion $x = \text{A} \sin (nt + \text{E})$. Find the values of A and E.

(19) Solve $\dfrac{d^2x}{dt^2} + 2f\dfrac{dx}{dt} + n^2x = 0$. Take $n^2 = 200$, $f = 7{\cdot}485$.

Let $x = 0$ and also $\dfrac{dx}{dt} = 10$ when $t = 0$. (B. of E., 1912.)

(20) A body capable of damped vibration is acted upon by simply varying force which has a frequency f. If x is the displacement of the body at any instant t, and if the motion is defined by

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + n^2x = a \sin 2\pi ft$$

we wish to study the forced vibration.

Take $a = 1$, $b = 1{\cdot}5$, $n^2 = 4$ find x, first when $f = 0{\cdot}2547$ and second when $f = 0{\cdot}3820$. (B. of E., 1910.)

(21) $\dfrac{d^2x}{dt^2} + 2f\dfrac{dx}{dt} + n^2x = a \sin qt$ expresses the forced vibration of a system. Imagine the natural vibrations to have been damped out. Take $n^2 = 49$, $f = 3$, $q = 5$; find x. (B. of E., 1913.)

(22) A weight W lb. hangs from a spiral spring whose stiffness is such that a force of 1 lb. weight elongates it h feet. A downward force F lb., in addition to the force of gravity, acts upon the weight. At any instant the weight is x feet below the mean position it would have if F were zero. Neglecting friction and the mass of the spring, prove that $\dfrac{d^2x}{dt^2} + n^2x = n^2h\text{F}$ where $n^2 = g/\text{W}h$

If the natural frequency, f or $\dfrac{n}{2\pi}$, is 10 and if $h\text{F} = a \sin qt$, neglecting the natural vibration, find the forced vibration; first when the forced frequency f_1 or $q/2\pi$ is 2; second when it is 5; third when it is 15. (B. of E., 1914.)

(23) A condenser of capacity k farads is charged so that the potential difference of the plates is v_0 volts. The plates are con-

nected by a wire of resistance R ohms and inductance L henries. If v is the voltage at any subsequent time t seconds then

$$L\frac{d^2v}{dt^2} + R\frac{dv}{dt} + \frac{v}{k} = 0$$

If R = 100 ohms, $k = 5 \times 10^{-7}$ farad and $L = 10^{-3}$ henry. Express v in terms of t, having given that $v_0 = 500$, $\frac{dv}{dt} = 0$ when $t = 0$.

(24) In Question 23, if $L = 1{\cdot}25 \times 10^{-3}$ henry, all the other conditions remaining the same, express v in terms of t.

(25) In Question 23, if $L = 8 \times 10^{-3}$ henry, all the other conditions remaining the same, express v in terms of t.

(26) A straight steel rod of uniform circular section and 5 feet long is found to deflect 1 inch under a central load of 20 lb. when tested as a beam simply supported at the ends. Determine the critical load when the same rod is used as a vertical strut with free ends. (U.L., 1910.)

(27) A hollow cylindrical steel strut has to be designed for the following conditions: length 6 feet, axial load 12 tons, ratio of internal diameter to external diameter 0·8, factor of safety 10. Determine the external diameter of the strut and the thickness of the metal, if the ends of the strut are firmly built in. Use the Rankine-Gordon formula, taking $f = 21$ tons per sq. inch, and the constant for free ends as $\frac{1}{7500}$. (U.L., 1908.)

(28) A mild steel I joist, 9 inches deep over all, flanges 4·5 inches wide by 0·5 inch thick, web 0·5 inch thick, is used as a column, 20 feet high, loaded centrally. If the ends are firmly fixed, find the allowable load. Use the Rankine-Gordon formula; take the safe compressive stress as 12,000 lb. per sq. inch and the coefficient for free ends as $\frac{1}{9000}$. (U.L., 1909.)

CHAPTER XXII

186.

(a) $$\int_0^{2\pi} \sin n\theta \, d\theta = -\frac{1}{n}\Big[\cos n\theta\Big]_0^{2\pi} = -\frac{1}{n}(\cos 2n\pi - 1) = 0$$

$$\int_{-\pi}^{\pi} \sin n\theta \, d\theta = -\frac{1}{n}\Big[\cos n\theta\Big]_{-\pi}^{\pi} = -\frac{1}{n}\{\cos n\pi - \cos(-n\pi)\} = 0$$

$$\int_0^{\pi} \sin n\theta \, d\theta = -\frac{1}{n}\Big[\cos n\theta\Big]_0^{\pi} = -\frac{1}{n}(\cos n\pi - 1)$$

$= 0$ when n is even

$= \frac{2}{n}$ when n is odd

(b) $$\int_0^{2\pi} \cos n\theta \, d\theta = \frac{1}{n}\Big[\sin n\theta\Big]_0^{2\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos n\theta \, d\theta = \frac{1}{n}\Big[\sin n\theta\Big]_{-\pi}^{\pi} = 0$$

$$\int_0^{\pi} \cos n\theta \, d\theta = \frac{1}{n}\Big[\sin n\theta\Big]_0^{\pi} = 0$$

since $\sin n\pi = 0$ if n is a positive or negative integer.

(c) $$\int \sin n\theta \sin m\theta \, d\theta = \frac{1}{2}\int \{\cos(n-m)\theta - \cos(m+n)\theta\} \, d\theta$$

and it follows at once from (b) that this integral vanishes between the limits 0 and 2π; $-\pi$ and π; and 0 and π.

(d) $$\int \cos n\theta \cos m\theta \, d\theta = \frac{1}{2}\int \{\cos(n+m)\theta + \cos(n-m)\theta\} \, d\theta$$

and it follows at once from (b) that this integral vanishes between the limits 0 and 2π; $-\pi$ and π; and 0 and π.

(e) $$\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int \{\sin(n+m)\theta + \sin(n-m)\theta\} \, d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\theta}{n+m} - \frac{\cos(n-m)\theta}{n-m}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos 2(n+m)\pi - 1}{n+m} - \frac{\cos 2(n-m)\pi - 1}{n-m}\right]$$

$$= 0$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\theta}{n+m} - \frac{\cos(n-m)\theta}{n-m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\pi - \cos\{-(n+m)\}\pi}{n+m} - \frac{\cos(n-m)\pi - \cos\{-(n-m)\pi\}}{n-m}\right]$$

$$= 0$$

$$\int_0^{\pi} \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\theta}{n+m} - \frac{\cos(n-m)\theta}{n-m}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\pi - 1}{n+m} - \frac{\cos(n-m)\pi - 1}{n-m}\right]$$

$$= 0, \quad \text{when } n + m \text{ is even,}$$

$$= \frac{1}{n+m} + \frac{1}{n-m} \quad \text{when } n + m \text{ is odd}$$

Hence the integrals $\int \sin n\theta \, d\theta$, $\int \cos n\theta \, d\theta$, $\int \sin n\theta \sin m\theta \, d\theta$, $\int \cos n\theta \cos m\theta \, d\theta$, and $\int \sin n\theta \cos m\theta \, d\theta$ all vanish when taken between the limits 0 and 2π, and they all vanish when taken between the limits $-\pi$ and π. If 0 and π are taken as the limits of the integrals, then the integrals $\int \cos n\theta \, d\theta$, $\int \sin n\theta \sin m\theta \, d\theta$, and $\int \cos n\theta \cos m\theta \, d\theta$ vanish, while the integrals $\int \sin n\theta \, d\theta$ and $\int \sin n\theta \cos m\theta \, d\theta$ do not vanish.

187. These results do not apply to the case when $n = m$, except for the integral $\int \sin n\theta \cos m\theta \, d\theta$

$$\text{for} \quad \int_0^{2\pi} \sin n\theta \cos n\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} \sin 2n\theta \, d\theta$$

$$= -\frac{1}{4n}\Big[\cos 2n\theta\Big]_0^{2\pi}$$

$$= -\frac{1}{4n}\{\cos 4n\pi - 1\} = 0$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos n\theta \, d\theta = -\frac{1}{4n}\Big[\cos 2n\theta\Big]_{-\pi}^{\pi}$$

$$= -\frac{1}{4n}\{\cos 2n\pi - \cos(-2n\pi)\}$$

$$= 0$$

$$\int_0^{\pi} \sin n\theta \cos n\theta \, d\theta = -\frac{1}{4n}\Big[\cos 2n\theta\Big]_0^{\pi}$$

$$= -\frac{1}{4n}\{\cos 2n\pi - 1\}$$

$$= 0$$

The integral $\int \sin n\theta \sin m\theta \, d\theta$ becomes $\int \sin^2 n\theta \, d\theta$

$$\int_0^{2\pi} \sin^2 n\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \cos 2n\theta \, d\theta$$

$$= \pi$$

$$\int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} d\theta - \frac{1}{2}\int_{-\pi}^{\pi} \cos 2n\theta \, d\theta$$

$$= \frac{1}{2}\Big[\pi - (-\pi)\Big] = \pi$$

$$\int_0^{\pi} \sin^2 n\theta \, d\theta = \frac{1}{2}\int_0^{\pi} d\theta - \frac{1}{2}\int_0^{\pi} \cos 2n\theta \, d\theta$$

$$= \frac{\pi}{2}$$

The integral $\int \cos n\theta \cos m\theta \, d\theta$ becomes $\int \cos^2 n\theta \, d\theta$

$$\int_0^{2\pi} \cos^2 n\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2n\theta \, d\theta$$

$$= \pi$$

$$\int_{-\pi}^{\pi} \cos^2 n\theta \; d\theta = \frac{1}{2}\int_{-\pi}^{\pi} d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos 2n\theta \; d\theta$$

$$= \frac{1}{2}[\pi - (-\pi)] = \pi$$

$$\int_{0}^{\pi} \cos^2 n\theta \; d\theta = \frac{1}{2}\int_{0}^{\pi} d\theta + \frac{1}{2}\int_{0}^{\pi} \cos 2n\theta \; d\theta$$

$$= \frac{\pi}{2}$$

These results may be used as a means by which a periodic function can be expressed as a series of sines or cosines of multiple angles.

188. *The Sine Series.* The sine series is expressed in the form $y = f(x) = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \; . \; . \; . \; A_n \sin nx + \ldots$ where A_1, A_2, A_3, etc., are constant coefficients, and any integration which might be necessary must be taken between the limits $x = 0$ and $x = \pi$.

If we multiply throughout by $\sin nx$ and integrate each term with respect to x between the limits 0 and π,

Then $\int_0^\pi y \sin nx \; dx$

$$= A_1\int_0^\pi \sin x \sin nx \; dx + A_2\int_0^\pi \sin 2x \sin nx \; dx + \ldots A_n\int_0^\pi \sin^2 nx \; dx + \ldots$$

and all the integrals on the right-hand side vanish except

$$A_n\int_0^\pi \sin^2 nx \; dx, \quad \text{which becomes } \frac{\pi}{2}A_n$$

Hence $$\frac{1}{2}\pi A_n = \int_0^\pi y \sin nx \; dx$$

and $$A_n = \frac{2}{\pi}\int_0^\pi y \sin nx \; dx$$

If y is known in terms of x, the integral can be determined, and by giving n the values, 1, 2, 3, etc., the values of A_1, A_2, A_3 can be found.

Example. Expand the function $y = mx$ as a sine series, knowing that when $x = \pi$, $y = c$.

Then $$m = \frac{c}{\pi} \text{ and } y = \frac{c}{\pi}x.$$

Then $y = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots A_n \sin nx + \ldots$

Then $$\int_0^\pi y \sin nx\,dx = A_n \int_0^\pi \sin^2 nx\,dx = \frac{1}{2}\pi A_n$$

Now $$\int_0^\pi y \sin nx\,dx = \frac{c}{\pi}\int_0^\pi x \sin nx\,dx$$

$$= \frac{c}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi$$

$$= \frac{c}{\pi n}(-\pi\cos n\pi)$$

$$= \frac{c}{n} \text{ when } n \text{ is odd}$$

$$= -\frac{c}{n} \text{ when } n \text{ is even}$$

Hence $$A_n = \frac{2c}{\pi n} \text{ when } n \text{ is odd}$$

$$= -\frac{2c}{\pi n} \text{ when } n \text{ is even}$$

Thus $$y = \frac{2c}{\pi}\sin x - \frac{2c}{2\pi}\sin 2x + \frac{2c}{3\pi}\sin 3x - \frac{2c}{4\pi}\sin 4x + \ldots$$

$$= \frac{2c}{\pi}\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right\}$$

working in terms of c.

or $$y = 2m\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right\}$$

working in terms of m.

Example. Expand the function $y = mx$ as a sine series, knowing that when $x = 0$, $y = 0$; when $x = \frac{\pi}{2}$, $y = c$, and when $x = \pi$, $y = 0$.

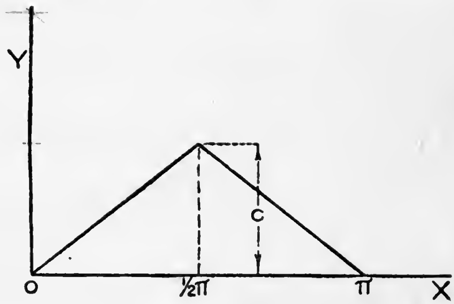


FIG. 125

Then $$m = \frac{2c}{\pi}$$

and between 0 and $\frac{\pi}{2}$ $$y = \frac{2c}{\pi}x$$

but between $\frac{\pi}{2}$ and π $$y = m(\pi - x) = \frac{2c}{\pi}(\pi - x)$$

Then $$y = A_1 \sin x + A_2 \sin 2x + \ldots A_n \sin nx + \ldots$$

and $$\int_0^\pi y \sin nx\, dx = A_n \int_0^\pi \sin^2 nx\, dx = \frac{1}{2}\pi A_n$$

But in taking the integral $\int_0^\pi y \sin nx \; dx$ it must be remembered that between 0 and $\frac{\pi}{2}$ $y = \frac{2c}{\pi}x$, but between $\frac{\pi}{2}$ and π $y = \frac{2c}{\pi}(\pi - x)$.

Therefore $\int_0^\pi y \sin nx\, dx$

$$= \frac{2c}{\pi}\int_0^{\frac{\pi}{2}} x \sin nx\, dx + \frac{2c}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx\, dx$$

$$= \frac{2c}{\pi}\left\{\int_0^{\frac{\pi}{2}} x \sin nx\, dx + \pi\int_{\frac{\pi}{2}}^{\pi} \sin nx\, dx - \int_{\frac{\pi}{2}}^{\pi} x \sin nx\, dx\right\}$$

$$= \frac{2c}{\pi}\left\{\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}} + \pi\left[-\frac{\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi} - \left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}\right\}$$

$$= \frac{2c}{\pi}\left\{\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi\cos n\pi}{n} - \left(-\frac{\pi\cos n\pi}{n} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right\}$$

$$= \frac{4c}{\pi n^2}\sin\frac{n\pi}{2}$$

But $$\frac{\pi}{2}A_n = \frac{4c}{\pi n^2}\sin\frac{n\pi}{2}$$

$$A_n = \frac{8c}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

when $n = 1$ $\sin\frac{n\pi}{2} = 1$ and $A_1 = \frac{8c}{\pi^2}\frac{1}{1^2}$

when $n = 2$ $\sin\frac{n\pi}{2} = 0$ and $A_2 = 0$

$$\text{when } n = 3 \quad \sin\frac{n\pi}{2} = -1 \quad \text{and } A_3 = -\frac{8c}{\pi^2}\frac{1}{3^2}$$

$$\text{when } n = 4 \quad \sin\frac{n\pi}{2} = 0 \quad \text{and } A_4 = 0$$

$$\text{when } n = 5 \quad \sin\frac{n\pi}{2} = 1 \quad \text{and } A_5 = \frac{8c}{\pi^2}\frac{1}{5^2}$$

$$\text{Then } y = \frac{8c}{\pi^2}\left\{\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\right\}$$

working in terms of c.

$$\text{or } y = \frac{4m}{\pi}\left\{\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\right\}$$

189. *The Cosine Series.* The cosine series is expressed in the form $y = f(x) = B_0 + B_1 \cos x + B_2 \cos 2x + \ \ldots\ B_n \cos nx + \ldots$ where B_0, B_1, B_2, B_3, etc., are constant coefficients and any integration which might be necessary must be taken between the limits $x = 0$ and $x = \pi$. It should be noticed that the cosine series differs from the sine series in having an initial constant term B_0.

In working with this series two operations are necessary, one operation to find the initial term B_0 and the other to find the general coefficient B_n.

If we integrate throughout with respect to x between the limits 0 and π.

Then $\int_0^\pi y\,dx$

$$= B_0\int_0^\pi dx + B_1\int_0^\pi \cos x\,dx + B_2\int_0^\pi \cos 2x\,dx + \ldots B_n\int_0^\pi \cos nx\,dx + \ldots$$

and all the integrals on the right-hand side vanish except $B_0\int_0^\pi dx$, which becomes πB_0.

Hence $$\pi B_0 = \int_0^\pi y\,dx$$

and $$B_0 = \frac{1}{\pi}\int_0^\pi y\,dx$$

If we multiply throughout by $\cos nx$ and integrate each term with respect to x between the limits 0 and π,

Then $\int_0^\pi y \cos nx\,dx$

$$= B_0\int_0^\pi \cos nx\,dx + B_1\int_0^\pi \cos x \cos nx\,dx + \ldots B_n\int_0^\pi \cos^2 nx\,dx + \ldots$$

and all the integrals on the right-hand side vanish except $B_n\int_0^\pi \cos^2 nx\,dx$, which becomes $\frac{1}{2}\pi B_n$.

Hence
$$\frac{1}{2}\pi B_n = \int_0^\pi y \cos nx\,dx$$

and
$$B_n = \frac{2}{\pi}\int_0^\pi y \cos nx\,dx$$

If y is known in terms of x, the integral can be determined and by giving n the values 1, 2, 3, etc., the values of B_1, B_2, B_3, etc., can be found.

Example. Expand the function $y = mx$ as a cosine series, knowing that $y = c$ when $x = \pi$.

Then
$$m = \frac{c}{\pi}, \text{ and } y = \frac{c}{\pi}x$$

$$y = B_0 + B_1 \cos x + B_2 \cos 2x + \ \ldots\ B_n \cos nx + \ldots$$

and
$$\int_0^\pi y\,dx = B_0\int_0^\pi dx = \pi B_0$$

Then
$$\pi B_0 = \frac{c}{\pi}\int_0^\pi x\,dx$$
$$= \frac{c}{\pi}\,\frac{\pi^2}{2}$$

and
$$B_0 = \frac{c}{2}$$

Also
$$\int_0^\pi y \cos nx\,dx = B_n\int_0^\pi \cos^2 nx\,dx = \frac{1}{2}\pi B_n$$

Now
$$\int_0^\pi y \cos nx\,dx = \frac{c}{\pi}\int_0^\pi x \cos nx\,dx$$
$$= \frac{c}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi$$
$$= \frac{c}{\pi n^2}(\cos n\pi - 1)$$

and
$$B_n = \frac{2c}{\pi^2 n^2}(\cos n\pi - 1)$$

When n is even, $\cos n\pi = 1$, and $B_n = 0$.
Hence B_2, B_4, B_6, etc., all become zero.

When n is odd, $\cos n\pi = -1$, and $B_n = -\frac{4c}{\pi^2 n^2}$

and $B_1 = -\frac{4c}{\pi^2}\frac{1}{1^2}$, $B_3 = -\frac{4c}{\pi^2}\frac{1}{3^2}$, $B_5 = -\frac{4c}{\pi^2}\frac{1}{5^2}$, etc.

Then $y = \frac{c}{2} - \frac{4c}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \frac{1}{49}\cos 7x + \ldots\right\}$

working in terms of c.

or $y = \frac{1}{2}m\pi - \frac{4m}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\}$

working in terms of m.

Example. Expand the function $y = mx^2$ as a cosine series, knowing that $y = c$ when $x = \pi$.

Then $$m = \frac{c}{\pi^2}, \quad \text{and } y = \frac{c}{\pi^2}x^2$$

and $y = B_0 + B_1 \cos x + B_2 \cos 2x + \ \ldots\ B_n \cos nx + \ldots$

Then $$B_0\int_0^\pi dx = \int_0^\pi y\,dx$$

$$\pi B_0 = \frac{c}{\pi^2}\int_0^\pi x^2\,dx$$

$$= \frac{c}{\pi^2}\frac{\pi^3}{3}, \quad \text{and } B_0 = \frac{c}{\pi^2}\frac{\pi^2}{3}$$

Also $$\int_0^\pi y\cos nx\,dx = B_n\int_0^\pi \cos^2 nx\,dx = \frac{1}{2}\pi B_n$$

But $$\int_0^\pi y\cos nx\,dx = \frac{c}{\pi^2}\int_0^\pi x^2\cos nx\,dx$$

$$= \frac{c}{\pi^2}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{c}{\pi^2}\frac{2\pi}{n^2}\cos n\pi$$

Then $$B_n = \frac{c}{\pi^2}\frac{4}{n^2}\cos n\pi$$

When n is even, $\cos n\pi = 1$, and $B_n = \frac{c}{\pi^2}\frac{4}{n^2}$

and $B_2 = \frac{c}{\pi^2}\frac{4}{2^2}$, $B_4 = \frac{c}{\pi^2}\frac{4}{4^2}$, $B_6 = \frac{c}{\pi^2}\frac{4}{6^2}$, etc.

When n is odd, $\cos n\pi = -1$, and $B_n = -\frac{c}{\pi^2}\frac{4}{n^2}$

and $B_1 = -\frac{c}{\pi^2}\frac{4}{1^2}$, $B_3 = -\frac{c}{\pi^2}\frac{4}{3^2}$, $B_5 = -\frac{c}{\pi^2}\frac{4}{5^2}$, etc.

Therefore

$$y = \frac{c}{\pi^2}\frac{\pi^2}{3} - \frac{4c}{\pi^2}\left\{\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + \ldots\right\}$$

$$= \frac{c}{3} - \frac{4c}{\pi^2}\left\{\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + \ldots\right\}$$

working in terms of c

or $$y = \frac{m\pi^2}{3} - 4m\left\{\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x \ldots\right\}$$

working in terms of m.

190. ***Fourier's Series.*** The general form of Fourier's Series is

$$y = f(x) = B_0 + B_1 \cos x + B_2 \cos 2x + \ldots B_n \cos nx + \ldots$$
$$+ A_1 \sin x + A_2 \sin 2x + \ldots A_n \sin nx + \ldots$$

Since the integrals $\int \sin nx\, dx$ and $\int \sin nx \cos mx\, dx$ do not vanish when taken between the limits 0 and π, but they do vanish when taken between the limits 0 and 2π, or between the limits $-\pi$ and π. Fourier's Series can only be worked when the necessary integration is performed between the limits 0 and 2π, or between the limits $-\pi$ and π.

In working with this series, three operations are necessary, the first to find the initial term B_0, the second to find the general coefficient B_n, and the third to find the general coefficient A_n.

(1) If we integrate throughout with respect to x between the limits 0 and 2π.

Then $\int_0^{2\pi} y\, dx$

$$= B_0\int_0^{2\pi} dx + B_1\int_0^{2\pi}\cos x\, dx + B_2\int_0^{2\pi}\cos 2x\, dx + \ldots B_n\int_0^{2\pi}\cos nx\, dx + \ldots$$
$$+ A_1\int_0^{2\pi}\sin x\, dx + A_2\int_0^{2\pi}\sin 2x\, dx + \ldots A_n\int_0^{2\pi}\sin nx\, dx + \ldots$$

and all the integrals on the right-hand side vanish except $B_0\int_0^{2\pi} dx$, which becomes $2\pi B_0$.

Hence $$2\pi B_0 = \int_0^{2\pi} y\, dx$$

and $$B_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, dx$$

(2) If we multiply throughout by $\cos nx$ and integrate each term with respect to x between the limits 0 and 2π,

Then $\int_0^{2\pi} y \cos nx\, dx$

$$= B_0\int_0^{2\pi}\cos nx\, dx + B_1\int_0^{2\pi}\cos x \cos nx\, dx + \ldots B_n\int_0^{2\pi}\cos^2 nx\, dx + \ldots$$

$$+ A_1\int_0^{2\pi}\sin x \cos nx\, dx + \ldots A_n\int_0^{2\pi}\sin nx \cos nx\, dx + \ldots$$

and all the integrals on the right-hand side vanish except $B_n\int_0^{2\pi}\cos^2 nx\, dx$, which becomes πB_n.

Hence
$$\pi B_n = \int_0^{2\pi} y \cos nx\, dx$$

or
$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\, dx$$

(3) If we multiply throughout by $\sin nx$ and integrate each term with respect to x between the limits 0 and 2π,

Then $\int_0^{2\pi} y \sin nx\, dx$

$$= B_0\int_0^{2\pi}\sin nx\, dx + B_1\int_0^{2\pi}\cos x \sin nx\, dx + \ldots B_n\int_0^{2\pi}\cos nx \sin nx\, dx + \ldots$$

$$+ A_1\int_0^{2\pi}\sin x \sin nx\, dx + \ldots A_n\int_0^{2\pi}\sin^2 nx\, dx + \ldots$$

and all the integrals on the right-hand side vanish except $A_n\int_0^{2\pi}\sin^2 nx\, dx$, which becomes πA_n.

Hence
$$\pi A_n = \int_0^{2\pi} y \sin nx\, dx$$

or
$$A_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\, dx$$

Working with the same series in a similar manner between the limits $-\pi$ and π, it can be shown that

$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y\, dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y \cos nx\, dx$$

and
$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y \sin nx\, dx$$

Example. Expand the function $y=e^x$ as a Fourier's Series, working between the limits $x = -\pi$ and $x = \pi$.

Then $y = B_0 + B_1 \cos x + B_2 \cos 2x + \ldots B_n \cos nx + \ldots$
$A_1 \sin x + A_2 \sin 2x + \ldots A_n \sin nx + \ldots$

and
$$2\pi B_0 = \int_{-\pi}^{\pi} y\, dx$$
$$= \int_{-\pi}^{\pi} e^x\, dx$$
$$= e^{\pi} - e^{-\pi}$$
$$B_0 = \frac{1}{2\pi}(e^{\pi} - e^{-\pi})$$

also
$$\pi B_n = \int_{-\pi}^{\pi} y \cos nx\, dx$$
$$= \int_{-\pi}^{\pi} e^x \cos nx\, dx$$
$$= \frac{1}{n^2+1}\Big[e^x \cos nx + ne^x \sin nx\Big]_{-\pi}^{\pi}$$
$$= \frac{1}{n^2+1}\{e^{\pi} \cos n\pi - e^{-\pi} \cos(-n\pi)\}$$
$$= \frac{\cos n\pi}{n^2+1}(e^{\pi} - e^{-\pi})$$

and
$$B_n = \frac{e^{\pi} - e^{-\pi}}{\pi(n^2+1)} \cos n\pi$$

When n is odd, $\cos n\pi = -1$ and $B_n = -\dfrac{e^{\pi} - e^{-\pi}}{\pi(n^2+1)}$

and $B_1 = -\dfrac{e^{\pi}-e^{-\pi}}{2\pi}$, $B_3 = -\dfrac{e^{\pi}-e^{-\pi}}{10\pi}$, $B_5 = -\dfrac{e^{\pi}-e^{-\pi}}{26\pi}$

When n is even, $\cos n\pi = 1$ and $B_n = \dfrac{e^{\pi} - e^{-\pi}}{\pi(n^2+1)}$

and $B_2 = \dfrac{e^{\pi} - e^{-\pi}}{5\pi}$, $B_4 = \dfrac{e^{\pi} - e^{-\pi}}{17\pi}$, $B_6 = \dfrac{e^{\pi} - e^{-\pi}}{37\pi}$

also
$$\pi A_n = \int_{-\pi}^{\pi} y \sin nx\, dx$$
$$= \int_{-\pi}^{\pi} e^x \sin nx\, dx$$
$$= \frac{1}{n^2+1}\Big[e^x \sin nx - ne^x \cos nx\Big]_{-\pi}^{\pi}$$

$$= -\frac{n}{n^2+1}\{e^{\pi}\cos n\pi - e^{-\pi}\cos(-n\pi)\}$$

$$= -\frac{n\cos n\pi}{n^2+1}(e^{\pi} - e^{-\pi})$$

and $$A_n = -\frac{n(e^{\pi} - e^{-\pi})}{\pi(n^2+1)}\cos n\pi$$

When n is odd, $\cos n\pi = -1$ and $A_n = \dfrac{n(e^{\pi} - e^{-\pi})}{\pi(n^2+1)}$

and $A_1 = \dfrac{e^{\pi} - e^{-\pi}}{2\pi}$, $A_3 = \dfrac{3(e^{\pi} - e^{-\pi})}{10\pi}$, $A_5 = \dfrac{5(e^{\pi} - e^{-\pi})}{26\pi}$

When n is even, $\cos n\pi = 1$ and $A_n = -\dfrac{n(e^{\pi} - e^{-\pi})}{\pi(n^2+1)}$

and $A_2 = -\dfrac{2(e^{\pi} - e^{-\pi})}{5\pi}$, $A_4 = -\dfrac{4(e^{\pi} - e^{-\pi})}{17\pi}$, $A_6 = -\dfrac{6(e^{\pi} - e^{-\pi})}{37\pi}$

Then $$y = \frac{e^{\pi} - e^{-\pi}}{\pi}\left\{\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x - \frac{1}{17}\cos 4x \ldots\right) + \left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \frac{4}{17}\sin 4x \ldots\right)\right\}$$

191. Hitherto we have been working with 0 and π, 0 and 2π, and $-\pi$ and π as the limits for x, but the work is not restricted to these limits; it is possible to work with any limits. Taking the function $y = f(\theta)$ and expressing it as a sine series between 0 and π.

Then $y = f(\theta) = A_1 \sin\theta + A_2 \sin 2\theta + \ldots A_n \sin n\theta + \ldots$

If θ is replaced by x in such a way that when $\theta = 0$, $x = 0$, and when $\theta = \pi$, $x = c$, then we have the relation $\theta = \dfrac{\pi x}{c}$, which renders it possible to work in terms of θ between the limits 0 and π.

Example. Expand the function $y = x^2$ as a sine series working between the limits $x = 0$ and $x = c$.

Then $y = A_1 \sin\theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \ldots A_n \sin n\theta + \ldots$ taken between the limits 0 and π.

Since $\theta = \dfrac{\pi x}{c}$, then $y = x^2 = \dfrac{c^2\theta^2}{\pi^2}$

and $$y = \frac{c^2\theta^2}{\pi^2} = A_1 \sin\theta + A_2 \sin 2\theta + \ldots A_n \sin n\theta + \ldots$$

working between the limits 0 and π,

$$\text{Then} \quad \frac{1}{2}\pi A_n = \int_0^\pi y \sin n\theta \, d\theta$$

$$= \frac{c^2}{\pi^2}\int_0^\pi \theta^2 \sin n\theta \, d\theta$$

$$= \frac{c^2}{\pi^2}\left[-\frac{\theta^2 \cos n\theta}{n} + \frac{2\theta \sin n\theta}{n^2} + \frac{2 \cos n\theta}{n^3}\right]_0^\pi$$

$$= \frac{c^2}{\pi^2}\left\{-\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right\}$$

$$\text{and} \quad A_n = \frac{2c^2}{\pi^3}\left\{-\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right\}$$

When n is odd, $\cos n\pi = -1$ and $A_n = \dfrac{2c^2}{\pi^3}\left(\dfrac{\pi^2}{n} - \dfrac{4}{n^3}\right)$

and $A_1 = \dfrac{2c^2}{\pi^3}(\pi^2 - 4)$, $A_3 = \dfrac{2c^2}{\pi^3}\left(\dfrac{\pi^2}{3} - \dfrac{4}{27}\right)$, $A_5 = \dfrac{2c^2}{\pi^3}\left(\dfrac{\pi^2}{5} - \dfrac{4}{125}\right)$, etc.

When n is even, $\cos n\pi = 1$ and $A_n = -\dfrac{2c^2}{\pi^3}\dfrac{\pi^2}{n}$

and $A_2 = -\dfrac{2c^2}{\pi^3}\dfrac{\pi^2}{2}$, $A_4 = -\dfrac{2c^2}{\pi^3}\dfrac{\pi^2}{4}$, $A_6 = -\dfrac{2c^2}{\pi^3}\dfrac{\pi^2}{6}$, etc.

$$\text{and } y = \frac{2c^2}{\pi^3}\left\{(\pi^2 - 4)\sin\theta - \frac{\pi^2}{2}\sin 2\theta + \left(\frac{\pi^2}{3} - \frac{4}{27}\right)\sin 3\theta - \frac{\pi^2}{4}\sin 4\theta + \ldots\right\}$$

Since $\theta = \dfrac{\pi x}{c}$

$$y = \frac{2c^2}{\pi^3}\left\{(\pi^2 - 4)\sin\frac{\pi x}{c} - \frac{\pi^2}{2}\sin\frac{2\pi x}{c} + \left(\frac{\pi^2}{3} - \frac{4}{27}\right)\sin\frac{3\pi x}{c} - \frac{\pi^2}{4}\sin\frac{4\pi x}{c} + \ldots\right\}$$

192. *Harmonic Analysis.* If the graph of a periodic function is given, it is possible to analyse the curve and express the result as a Fourier's Series. For if y is any ordinate of the curve and the base line of the curve is made to extend from 0 to 2π for a complete period

Then

$$y = f(\theta) = B_0 + B_1 \cos\theta + B_2 \cos 2\theta + \ldots B_n \cos n\theta + \ldots + A_1 \sin\theta + A_2 \sin 2\theta + \ldots A_n \sin n\theta + \ldots$$

Let the base line of such a curve be divided into m equal parts and the ordinates $y_0, y_1, y_2, \ldots y_m$ drawn to the curve at each point of division. The lengths of these ordinates will give the values of y when θ has the values $0, \frac{2\pi}{m}, \frac{4\pi}{m}, \frac{6\pi}{m} \ldots 2\pi$, respectively.

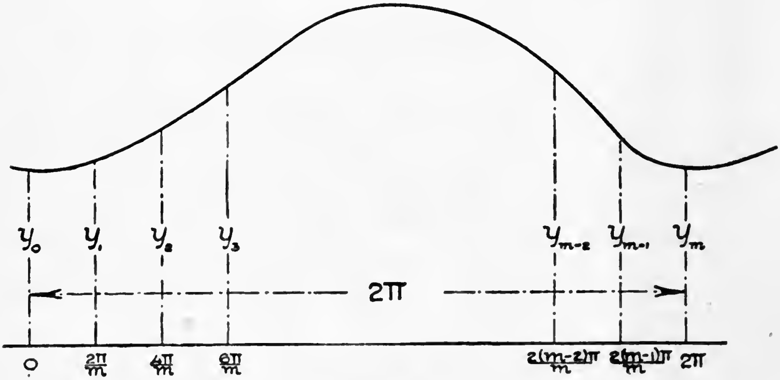


FIG. 126.

(1) $$B_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, d\theta$$

B_0 is evidently the average ordinate of the curve obtained by plotting θ horizontally and y vertically, and therefore

$$B_0 = \frac{1}{m}(\text{sum of the ordinates})$$

$$= \frac{1}{m}(y_0 + y_1 + y_2 + \ldots y_{m-1})$$

The ordinate y_m is not to be included, since it forms the initial ordinate of the next period.

(2) Now $$B_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta\, d\theta$$

B_n is evidently twice the average ordinate of the curve obtained by plotting θ horizontally and $y \cos n\theta$ vertically.

Then $B_n = \frac{2}{m}$ {sum of the ordinates of the θ, $y \cos n\theta$ curve}

$$= \frac{2}{m}\left\{y_0 \cos 0° + y_1 \cos \frac{2n\pi}{m} + y_2 \cos \frac{4n\pi}{m} + \ldots y_{m-1} \cos \frac{2n(m-1)\pi}{m}\right\}$$

This will give the coefficient of any cosine term in the resulting series by giving n the required value.

For when $n = 1$,

$$B_1 = \frac{2}{m}\left\{ y_0 + y_1 \cos\frac{2\pi}{m} + y_2 \cos\frac{4\pi}{m} + \ldots y_{m-1} \cos\frac{2(m-1)\pi}{m}\right\}$$

when $n = 2$,

$$B_2 = \frac{2}{m}\left\{ y_0 + y_1 \cos\frac{4\pi}{m} + y_2 \cos\frac{8\pi}{m} + \ldots y_{m-1} \cos\frac{4(m-1)\pi}{m}\right\}$$

(3) Now $A_n = \frac{1}{\pi}\int_0^{2\pi} y \sin n\theta \, d\theta$

A_n is evidently twice the average ordinate of the curve obtained by plotting θ horizontally and $y \sin n\theta$ vertically.

Then $A_n = \frac{2}{m}$ {sum of the ordinates of the θ, $y \sin n\theta$ curve}

$$= \frac{2}{m}\left\{ y_0 \sin 0° + y_1 \sin\frac{2n\pi}{m} + y_2 \sin\frac{4n\pi}{m} + \ldots y_{m-1} \sin\frac{2n(m-1)\pi}{m}\right\}$$

and this will give the coefficient of any sine term in the resulting series.

It follows, therefore, that in the operation necessary to obtain any coefficient of the form A_n, the ordinates y_0, y_1, y_2 . . . must be multiplied by the sines of the corresponding angles 0, $\frac{2n\pi}{m}$, $\frac{4n\pi}{m}$, $\frac{6n\pi}{m}$, etc., and twice the average of the sum of these products taken. While to get B_n the ordinates must be multiplied by the cosines of the corresponding angles and twice the average of the sum of the products taken.

If $\alpha = \frac{2n\pi}{m}$ and from a fixed point radial lines are drawn making angles 0, α, 2α, 3α, etc., to the horizontal (Fig. 127).

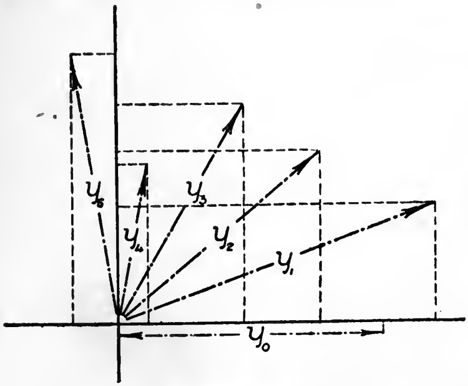


FIG. 127.

Then if the ordinates y_0, y_1, y_2, etc., are measured from the fixed point along these radial lines respectively,

$$A_n = \frac{2}{m} \{\text{sum of the vertical projections}\}$$

$$B_n = \frac{2}{m} \{\text{sum of the horizontal projections}\}$$

It should be noticed that the work will be made considerably simpler if the number of ordinates is so chosen as to make the angle $\frac{2\pi}{m}$ a simple fraction of 2π.

193. Considering a harmonic curve, the base line of which extends from 0 to 2π for a complete period. Let the base line be divided into 12 equal parts and the ordinates $y_0, y_1, y_2, \ldots y_{11}$ drawn to the curve at the points of division.

Then each division of the base will correspond to an angle $\frac{\pi}{6}$ or 30°.

Then $$B_0 = \frac{1}{12} \{y_0 + y_1 + y_2 + \ldots y_{11}\}$$

(1) Now $A_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin\theta \, d\theta$, and $B_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos\theta \, d\theta$

Hence $A_1 = \frac{1}{6}$ {sum of the ordinates of the θ, $y \sin\theta$ curve}

and $B_1 = \frac{1}{6}$ {sum of the ordinates of the θ, $y \cos\theta$ curve}

or $$A_1 = \frac{1}{6} \{y_0 \sin 0° + y_1 \sin 30° + y_2 \sin 60° + \ldots y_{11} \sin 330°\}$$

and $$B_1 = \frac{1}{6} \{y_0 \cos 0° + y_1 \cos 30° + y_2 \cos 60° + \ldots y_{11} \cos 330°\}$$

If from the point O radial lines are drawn at intervals of 30° and the ordinates y_0, y_1, etc., are measured along these lines. (Fig. 128.)

Then, resolving vertically,

$$A_1 = \frac{1}{6}\{(y_1 - y_7) \sin 30° + (y_2 - y_8) \sin 60° + (y_3 - y_9) + (y_4 - y_{10}) \sin 60° + (y_5 - y_{11}) \sin 30°\}$$

$$= \frac{1}{6}\{(y_1 + y_5 - y_7 - y_{11}) \sin 30° + (y_2 + y_4 - y_8 - y_{10}) \sin 60° + y_3 - y_9\}$$

and resolving horizontally.

$$B_1 = \frac{1}{6}\{(y_0 - y_6) + (y_1 - y_7) \cos 30° + (y_2 - y_8) \cos 60° - (y_4 - y_{10}) \cos 60° - (y_5 - y_{11}) \cos 30°\}$$

$$= \frac{1}{6}\{y_0 - y_6 + (y_1 + y_{11} - y_5 - y_7) \cos 30° + (y_2 + y_{10} - y_4 - y_8) \cos 60°\}$$

(2) Now $A_2 = \frac{1}{\pi}\int_0^{2\pi} y \sin 2\theta \, d\theta$, and $B_2 = \frac{1}{\pi}\int_0^{2\pi} y \cos 2\theta \, d\theta$

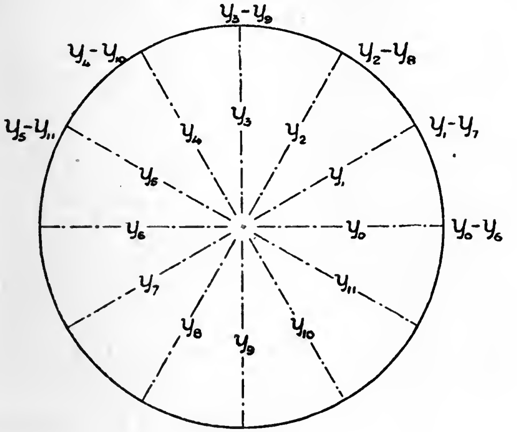


FIG. 128.

Hence $A_2 = \frac{1}{6}\{$sum of the ordinates of the θ, $y \sin 2\theta$ curve$\}$

and $B_2 = \frac{1}{6}\{$sum of the ordinates of the θ, $y \cos 2\theta$ curve$\}$

or $\quad A_2 = \frac{1}{6}\{y_0 \sin 0° + y_1 \sin 60° + y_2 \sin 120° + \ldots y_{11} \sin 660°\}$

and $B_2 = \frac{1}{6}\{y_0 \cos 0° + y_1 \cos 60° + y_2 \cos 120° + \ldots y_{11} \cos 660°\}$

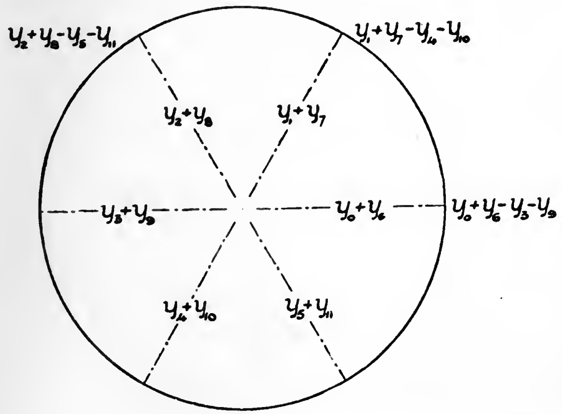


FIG. 129.

If from the point O radial lines are drawn at intervals of 60° and the ordinates y_0, y_1, etc., are measured along these lines.

Then, resolving vertically,

$$A_2 = \frac{1}{6}\{(y_1 + y_7 - y_4 - y_{10} + y_2 + y_8 - y_5 - y_{11}) \sin 60°\}$$

or $$A_2 = \frac{1}{6}\{(y_1 + y_2 + y_7 + y_8 - y_4 - y_5 - y_{10} - y_{11}) \sin 60°\}$$

and resolving horizontally,

$$B_2 = \frac{1}{6}\{(y_0 + y_6 - y_3 - y_9) + (y_1 + y_7 - y_4 - y_{10}) \cos 60° - (y_2 + y_8 - y_5 - y_{11}) \cos 60°\}$$

$$= \frac{1}{6}\{(y_0 + y_6 - y_3 - y_9) + (y_1 + y_5 + y_7 + y_{11} - y_2 - y_4 - y_8 - y_{10}) \cos 60°\}$$

(3) Now $A_3 = \frac{1}{\pi}\int_0^{2\pi} y \sin 3\theta \, d\theta$, and $B_3 = \frac{1}{\pi}\int_0^{2\pi} y \cos 3\theta \, d\theta$

Hence $A_3 = \frac{1}{6}$ {sum of the ordinates of the θ, $y \sin 3\theta$ curve}

and $B_3 = \frac{1}{6}$ {sum of the ordinates of the θ, $y \cos 3\theta$ curve}

or $$A_3 = \frac{1}{6}\{y_0 \sin 0° + y_1 \sin 90° + y_2 \sin 180° + \ldots y_{11} \sin 990°\}$$

and $$B_3 = \frac{1}{6}\{y_0 \cos 0° + y_1 \cos 90° + y_3 \cos 180° + \ldots y_{11} \cos 990°\}$$

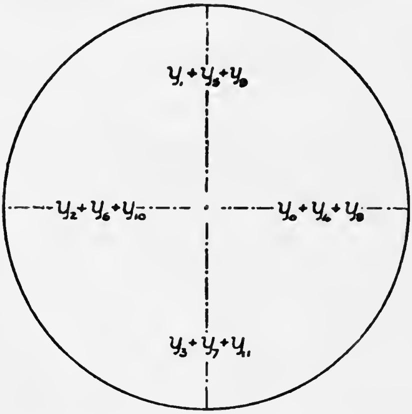


FIG. 130.

If from the point O radial lines are drawn at intervals of 90° and the ordinates y_0, y_1, etc., are measured along these lines.

Then, resolving vertically,

$$A_3 = \frac{1}{6}\{y_1 + y_5 + y_9 - y_3 - y_7 - y_{11}\}$$

and resolving horizontally,

$$B_3 = \frac{1}{6}\{y_0 + y_4 + y_8 - y_2 - y_6 - y_{10}\}$$

The other coefficients can be obtained in a similar manner.

194. *Example* 1. The following values are 12 equidistant ordinates of a periodic function taken for a complete period. Express the function as a Fourier's Series neglecting terms higher than 3θ.

3·40	5·41	6·42	5·40	3·58	2·29
1·60	1·43	1·92	2·80	2·78	2·47

Then $$B_0 = \frac{1}{12}\text{(sum of the ordinates)}$$

$$= \frac{39{\cdot}50}{12}$$

$$= 3{\cdot}29$$

To get A_1 and B_1 (Fig. 131).

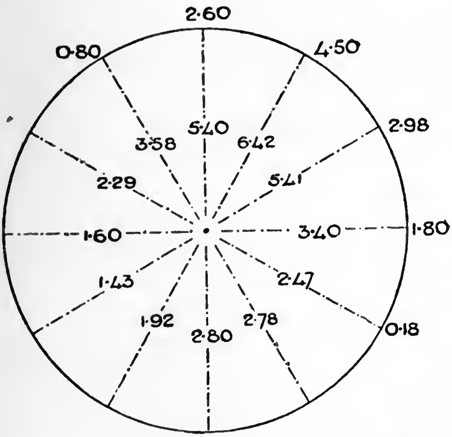


FIG. 131.

$$A_1 = \frac{1}{6}\{3{\cdot}98 \sin 30° + 4{\cdot}5 \sin 60° + 2{\cdot}6 + 0{\cdot}80 \sin 60° - 0{\cdot}18 \sin 30°\}$$

$$= \frac{1}{6}\{2{\cdot}60 + 3{\cdot}80 \sin 30° + 5{\cdot}3 \sin 60°\}$$

$$A_1 = 1{\cdot}51$$

$$B_1 = \frac{1}{6}\{1{\cdot}80 + 3{\cdot}98\cos 30° + 4{\cdot}5\cos 60° - 0{\cdot}80\cos 60° + 0{\cdot}18\cos 30°\}$$

$$= \frac{1}{6}\{1{\cdot}80 + 4{\cdot}16 \cos 30° + 3{\cdot}7 \cos 60°\}$$

$$B_1 = 1{\cdot}21$$

To get A_2 and B_2 (Fig. 132).

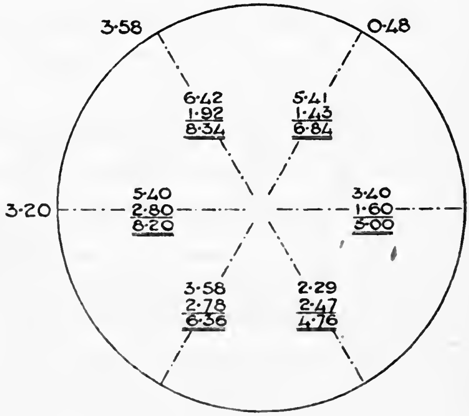


FIG. 132.

$$A_2 = \frac{1}{6}\{0{\cdot}48 \sin 60° + 3{\cdot}58 \sin 60°\}$$

$$= \frac{1}{6}\{4{\cdot}06 \sin 60°\}$$

$$= 0{\cdot}59$$

$$B_2 = \frac{1}{6}\{0{\cdot}48 \cos 60° - 3{\cdot}58 \cos 60° - 3{\cdot}20\}$$

$$= \frac{1}{6}\{-3{\cdot}10 \cos 60° - 3{\cdot}20\}$$

$$= -0{\cdot}79$$

To get A_3 and B_3 (Fig. 133).

$$A_3 = \frac{1{\cdot}2}{6} = 0{\cdot}2$$

$$B_3 = -\frac{1{\cdot}9}{6} = -0{\cdot}32$$

Hence $y = 3{\cdot}29 + 1{\cdot}21 \cos \theta - 0{\cdot}79 \cos 2\theta - 0{\cdot}32 \cos 3\theta$
$\qquad + 1{\cdot}51 \sin \theta + 0{\cdot}59 \sin 2\theta + 0{\cdot}20 \sin 3\theta$

also $1{\cdot}51 \sin \theta + 1{\cdot}21 \cos \theta = 1{\cdot}93 \sin (\theta + 38^\circ{\cdot}7)$

$$0{\cdot}59 \sin 2\theta - 0{\cdot}79 \cos 2\theta = 0{\cdot}97 \sin (2\theta - 53^\circ{\cdot}2)$$

$$0{\cdot}20 \sin 3\theta - 0{\cdot}32 \cos 3\theta = 0{\cdot}38 \sin (3\theta - 58^\circ)$$

Then $y = 3{\cdot}29 + 1{\cdot}93 \sin (\theta + 38^\circ{\cdot}7) + 0{\cdot}97 \sin (2\theta - 53^\circ{\cdot}2) + 0{\cdot}38 \sin (3\theta - 58^\circ)$

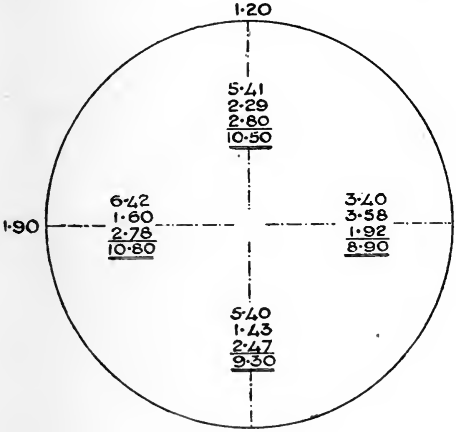


FIG. 133.

195. *Example* 2. The value of a periodic function of t is here given for twelve equidistant values of t covering the whole period. Express it in a Fourier's Series. Terms of the fourth and higher orders are negligible. (B. of E., 1911.)

2·340	3·012	3·685	4·149	3·685	2·203
0·825	0·513	0·875	1·085	1·189	1·637

$$\text{Then } B_0 = \frac{1}{12} \text{(sum of the ordinates)}$$

$$= \frac{25{\cdot}198}{12}$$

$$= 2{\cdot}100$$

To get A_1 and B_1 (Fig. 134).

$$\text{Then } A_1 = \frac{1}{6} \{2{\cdot}499 \sin 30^\circ + 2{\cdot}810 \sin 60^\circ + 3{\cdot}064 + 2{\cdot}496 \sin 60^\circ + 0{\cdot}566 \sin 30^\circ\}$$

$$= \frac{1}{6} \{3{\cdot}064 + 3{\cdot}065 \sin 30^\circ + 5{\cdot}306 \sin 60^\circ\}$$

$$= 1{\cdot}532$$

$$\text{and} \quad B_1 = \frac{1}{6}\{1{\cdot}515 + 2{\cdot}499 \cos 30° + 2{\cdot}810 \cos 60° - 2{\cdot}496 \cos 60° - 0{\cdot}566 \cos 30°\}$$

$$= \frac{1}{6}\{1{\cdot}515 + 1{\cdot}933 \cos 30° + 0{\cdot}314 \cos 60°\}$$

$$= 0{\cdot}558$$

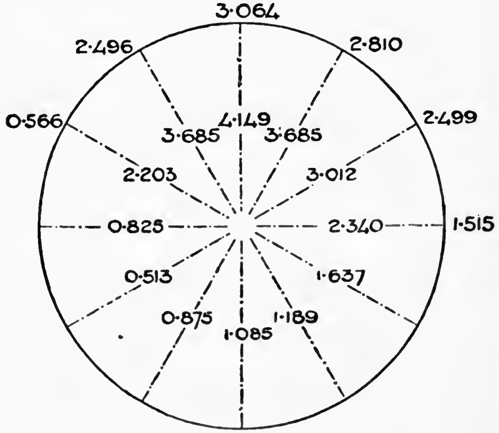


FIG. 134.

To get A_2 and B_2 (Fig. 135).

$$\text{Then } A_2 = \frac{1}{6}\{0{\cdot}720 \sin 60° - 1{\cdot}349 \sin 60°\}$$

$$= -\frac{0{\cdot}629 \sin 60°}{6}$$

$$= -0{\cdot}091$$

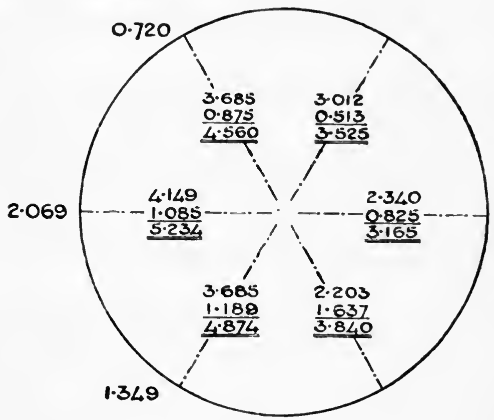


FIG. 135.

$$B_2 = \frac{1}{6}\{-2{\cdot}069 - 0{\cdot}720 \cos 60° - 1{\cdot}349 \cos 60°\}$$

$$= \frac{1}{6}\{-2{\cdot}069 - 2{\cdot}069 \cos 60°\}$$

$$= -0{\cdot}517$$

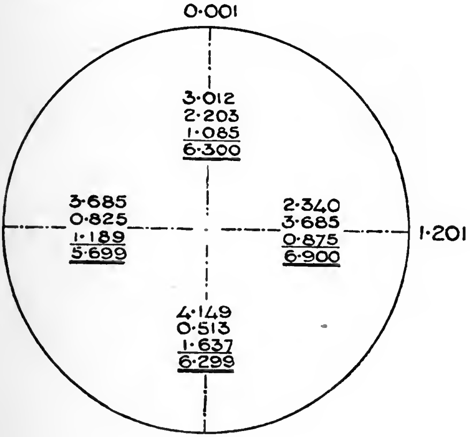


FIG. 136.

To get A_3 and B_3 (Fig. 136).

Then $$A_3 = 0$$

and $$B_3 = \frac{1{\cdot}201}{6} = 0{\cdot}2$$

$$x = 2{\cdot}1 + 0{\cdot}558 \cos \theta - 0{\cdot}517 \cos 2\theta + 0{\cdot}2 \cos 3\theta$$
$$+ 1{\cdot}532 \sin \theta - 0{\cdot}091 \sin 2\theta$$

where $\theta = \dfrac{2\pi t}{T}$ if T is the periodic time.

196. It has already been shown that between $x = 0$ and $x = \pi$

$$y = mx = 2m\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right\}$$

when $m = 1$, $x = 2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right\}$

and it is interesting to watch the development of the function $y = x$ from its component sine functions. Fig. 137 shows such a development.

Curve (1) shows $y_1 = 2 \sin x$

(2) shows $y_2 = 2 \sin x - \sin 2x$

(3) shows $y_3 = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x$

(4) shows $y_4 = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x$

(5) shows $y_5 = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x + \frac{2}{5} \sin 5x$

and it should be noticed how the addition or subtraction of a sine function brings the resultant curve nearer to the straight line $y = x$.

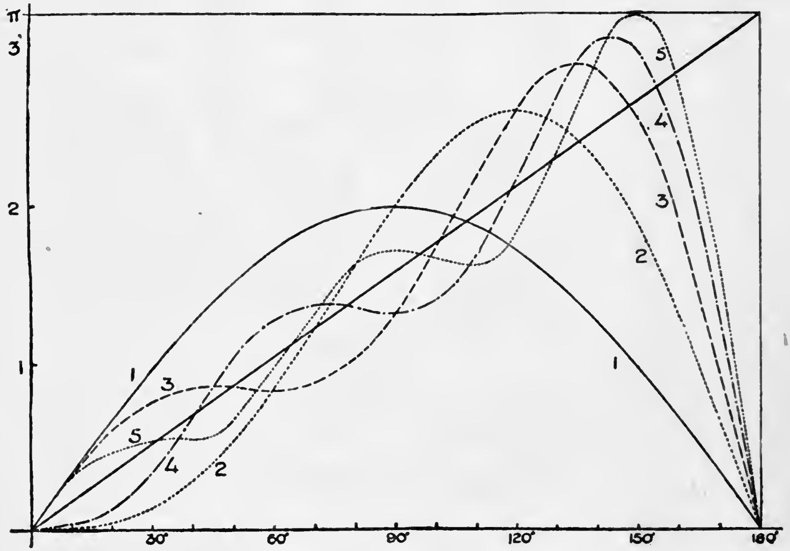


FIG. 137.

EXAMPLES XXII

(1) If $y = a \sin qt$ and $x = b \sin (qt - c)$ where t is time and a, q, b, c are constants; if $q = \frac{2\pi}{T}$ where T is the periodic time. Find the average value of xy during the time T. (B. of E., 1908.)

(2) Express $\sin at \cos bt$ as the sum of two terms and integrate with regard to t. If a is $\frac{2\pi}{T}$ and b is $3a$, what is the value of the integral between the limits 0 and T? (B. of E., 1913.)

(3) If $y = a \sin qt$ and $x = b \cos (qt - c)$ where t is time and a, q, b, c are constants. Find the average value of xy during the

periodic time $T = \frac{2\pi}{q}$. What is the average value of y^2 during the periodic time?

(4) If $y = a\cos qt$ and $x = b\cos(qt - c)$ where t is time and a, q, b, c are constants. Find the average value of xy during the periodic time $T = \frac{2\pi}{q}$. What is the average value of x^2 during the periodic time?

(5) If $y = a\cos qt$ and $x = b\sin(qt - c)$ where t is time and a, q, b, c are constants. Find the average value of xy during the periodic time $T = \frac{2\pi}{q}$. What is the average value of x^2 during the periodic time?

(6) Find a sine series for $y = mx^2$ between $x = 0$ and $x = \pi$.

(7) Find a sine series for $y = mx^3$ between $x = 0$ and $x = \pi$.

(8) Find a cosine series for $y = mx^3$ between $x = 0$ and $x = \pi$.

(9) Find a cosine series for $y = e^x$ between $x = 0$ and $x = \pi$.

(10) Find a cosine series for $y = e^{-x}$ between $x = 0$ and $x = \pi$.

(11) Find a Fourier's series for $y = x^2$ between $x = -\pi$ and $x = \pi$.

(12) Find a Fourier's series for $y = x^3$ between $x = -\pi$ and $x = \pi$.

(13) Find a Fourier's series for $y = e^{-x}$ between $x = -\pi$ and $x = \pi$.

(14) Find a sine series for $y = x$ between $x = 0$ and $x = c$.

(15) Find a cosine series for $y = x^2$ between $x = 0$ and $x = c$.

The value of a periodic function of t is given below for twelve equidistant values of t covering the whole period.

Neglecting terms of the fourth and higher orders, express it in a Fourier's series for the six different examples.

(16)

22·81	20·30	15·17	9·38	5·22	3·06
2·87	4·34	7·42	12·06	17·34	21·61

(17)

22·59	18·36	11·39	4·74	1·46	1·07
2·65	5·60	9·68	14·76	19·61	22·80

(18)

22·80	19·85	14·26	8·36	4·31	2·60
2·87	4·76	8·08	12·86	17·97	21·97

(19)

22·00	21·31	16·91	10·57	4·95	1·94
1·35	2·56	5·21	9·32	14·29	19·31

(20)

21·37	22·00	18·74	12·62	6·43	2·33
1·12	1·89	4·14	7·80	12·52	17·81

(21)

24·49	26·01	23·44	17·10	9·28	3·91
2·74	4·02	6·66	10·41	15·12	20·34

(22) The value of y, a periodic function of t, is here given for twelve equidistant values of t covering the whole period. Express y in a Fourier's series: **13·602, 18·468, 20·671, 20·182, 17·820, 14·346, 10·130, 5·612, 1·877, 0·486, 2·500, 7·506.** (B. of E., 1910.)

(23) A sliding piece has a periodic motion. Its distance x from a point in its path is measured at twenty-four equal intervals into which the whole periodic time is divided. Express x in a Fourier's series: **16·04, 16·74, 16·66, 15·86, 14·68, 13·42, 12·26, 11·16, 9·98, 8·76, 7·60, 6·68, 5·96, 5·34, 4·68, 4·14, 3·98, 4·50, 5·74, 7·46, 9·36, 11·24, 13·06, 14·70.** (B. of E., 1906.)

CHAPTER XXIII

197. If we consider the series

$$1,\ 4,\ 10,\ 20,\ 35,\ 56\ .\ .\ .$$

there does not seem to be any apparent connection between one term and the term immediately following it, neither is it obvious that the series is derived from some definite law of formation, but if successive differences are taken, a study of these differences will enable us to find the law of formation, providing such a law does exist.

	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
$u_0 = 1$				
	3			
$u_1 = 4$		3		
	6		1	
$u_2 = 10$		4		0
	10		1	
$u_3 = 20$		5		0
	15		1	
$u_4 = 35$		6		
	21			
$u_5 = 56$				

The values in the columns show the successive differences. The first set being the result of subtracting each number from its successor, the differences which are so obtained being changes in the value of u, may be denoted by Δu. The second set of differences are obtained by subtracting each value of Δu from its successor; these values may be denoted by $\Delta^2 u$, which may be taken to represent " Δ operating on u twice " ; in the example the values of $\Delta^2 u$ are in arithmetical progression ; therefore the third differences $\Delta^3 u$ are constant, each being equal to 1, and so all of the values of $\Delta^4 u$ will be zero. This indicates that the series has been derived from some definite law of formation; it will be shown later that this law is

$$u_n = \frac{1}{6}(n^3 + 6n^2 + 11n + 6)$$

198. Let a certain series be denoted by u_0, u_1, u_2, u_3, etc.

Then

$$
\begin{array}{ccccccc}
u & \Delta u & \Delta^2 u & \Delta^3 u & \Delta^4 u & \Delta^5 u & \Delta^6 u \\
u_0 & & & & & & \\
 & \Delta u_0 & & & & & \\
u_1 & & \Delta^2 u_0 & & & & \\
 & \Delta u_1 & & \Delta^3 u_0 & & & \\
u_2 & & \Delta^2 u_1 & & \Delta^4 u_0 & & \\
 & \Delta u_2 & & \Delta^3 u_1 & & \Delta^5 u_0 & \\
u_3 & & \Delta^2 u_2 & & \Delta^4 u_1 & & \Delta^6 u_0 \\
 & \Delta u_3 & & \Delta^3 u_2 & & \Delta^5 u_1 & \\
u_4 & & \Delta^2 u_3 & & \Delta^4 u_2 & & \Delta^6 u_1 \\
 & \Delta u_4 & & \Delta^3 u_3 & & \Delta^5 u_2 & \\
u_5 & & \Delta^2 u_4 & & \Delta^4 u_2 & & \\
 & \Delta u_5 & & \Delta^3 u_4 & & & \\
u_6 & & \Delta^2 u_5 & & & & \\
 & \Delta u_6 & & & & & \\
u_7 & & & & & &
\end{array}
$$

Then $u_1 = u_0 + \Delta u_0 \quad \ldots \ldots \quad (1)$

$$
\begin{aligned}
u_2 &= u_1 + \Delta u_1 \\
&= u_0 + \Delta u_0 + \Delta u_0 + \Delta^2 u_0 \\
&= u_0 + 2\Delta u_0 + \Delta^2 u_0 \quad \ldots \ldots \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
u_3 &= u_2 + \Delta u_2 \\
&= (u_0 + 2\Delta u_0 + \Delta^2 u_0) + (\Delta u_0 + 2\Delta^2 u_0 + \Delta^3 u_0) \\
&= u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0 \quad \ldots \ldots \quad (3)
\end{aligned}
$$

$$
\begin{aligned}
u_4 &= u_3 + \Delta u_3 \\
&= (u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0) + (\Delta u_0 + 3\Delta^2 u_0 + 3\Delta^3 u_0 \\
&\qquad + \Delta^4 u_0) \\
&= u_0 + 4\Delta u_0 + 6\Delta^2 u_0 + 4\Delta^3 u_0 + \Delta^4 u_0 \quad \ldots \ldots \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
u_5 &= u_4 + \Delta u_4 \\
&= (u_0 + 4\Delta u_0 + 6\Delta^2 u_0 + 4\Delta^3 u_0 + \Delta^4 u_0) + (\Delta u_0 + 4\Delta^2 u_0 \\
&\qquad + 6\Delta^3 u_0 + 4\Delta^4 u_0 + \Delta^5 u_0) \\
&= u_0 + 5\Delta u_0 + 10\Delta^2 u_0 + 10\Delta^3 u_0 + 5\Delta^4 u_0 + \Delta^5 u_0 \quad \ldots \quad (5)
\end{aligned}
$$

The multipliers of the differences are evidently the same as the Binomial coefficients in the expansions for which the powers are 1, 2, 3, 4, and 5 respectively.

Hence

$$u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\,\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0 + \ldots \text{ etc.} \quad (6)$$

This result enables us to find the value of any term, providing the first term u_0 is known and also the successive differences Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$ . . . are known. These differences evidently lie on a diagonal line running downwards from u_0.

Symbolically, then,

$$\begin{aligned}
u_1 &= (1+\Delta)u_0 &&= (1+\Delta)u_0\\
u_2 &= (1+2\Delta+\Delta^2)u_0 &&= (1+\Delta)^2 u_0\\
u_3 &= (1+3\Delta+3\Delta^2+\Delta^3)u_0 &&= (1+\Delta)^3 u_0\\
u_4 &= (1+4\Delta+6\Delta^2+4\Delta^3+\Delta^4)u_0 &&= (1+\Delta)^4 u_0\\
u_5 &= (1+5\Delta+10\Delta^2+10\Delta^3+5\Delta^4+\Delta^5)u_0 &&= (1+\Delta)^5 u_0
\end{aligned}$$

and $u_n = \left(1 + n\Delta + \dfrac{n(n-1)}{\underline{\lvert 2}}\Delta^2 + \dfrac{n(n-1)(n-2)}{\underline{\lvert 3}}\Delta^3 \ldots\right) u_0 = (1+\Delta)^n u_0$

In which $(1+\Delta)^2 u_0$ means that $(1+\Delta)$ operates on u_0 twice, or more generally $(1+\Delta)^n u_0$ means that $(1+\Delta)$ operates n times on u_0.

Example 1. Find the 9th term and the general term of the series

1, 4, 10, 20, 35, 56 . . .

$$\begin{array}{lccccc}
 & & \Delta u & \Delta^2 u & \Delta^3 u & \Delta^4 u\\
u_0 = & 1 & & & & \\
 & & 3 & & & \\
u_1 = & 4 & & 3 & & \\
 & & 6 & & 1 & \\
u_2 = & 10 & & 4 & & 0\\
 & & 10 & & 1 & \\
u_3 = & 20 & & 5 & & 0\\
 & & 15 & & 1 & \\
u_4 = & 35 & & 6 & & \\
 & & 21 & & & \\
u_5 = & 56 & & & &
\end{array}$$

The quantities in the diagonal line running downwards from u_0 will give the values of u_0, Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, etc., for this particular series, and $u_0 = 1$, $\Delta u_0 = 3$, $\Delta^2 u_0 = 3$, $\Delta^3 u_0 = 1$, and $\Delta^4 u_0 = 0$.

The 9th term is evidently u_8

and $$\begin{aligned} u_8 &= (1+\Delta)^8 u_0\\ &= u_0 + 8\Delta u_0 + 28\Delta^2 u_0 + 56\Delta^3 u_0\end{aligned}$$

The relation ends at the fourth term since $\Delta^4 u_0 = 0$.

Then $$\begin{aligned} u_8 &= 1 + 8 \times 3 + 28 \times 3 + 56 \times 1\\ &= 165\end{aligned}$$

Also $u_n = (1 + \Delta)^n u_0$

$$= u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0$$

$$= 1 + 3n + \frac{3}{2}n(n-1) + \frac{1}{6}n(n-1)(n-2)$$

$$= \frac{1}{6}\{6 + 18n + 9n^2 - 9n + n^3 - 3n^2 + 2n\}$$

$$= \frac{1}{6}\{n^3 + 6n^2 + 11n + 6\}$$

This result enables us to find the value of any term by giving n its necessary value.

The 9th term is obtained by putting $n = 8$

and $$u_8 = \frac{990}{6} = 165$$

Example 2. Find the 10th term and the general term of the series

$$1, \quad 5, \quad 15, \quad 35, \quad 69, \quad 121 \ldots$$

	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
$u_0 = 1$				
	4			
$u_1 = 5$		6		
	10		4	
$u_2 = 15$		10		0
	20		4	
$u_3 = 35$		14		0
	34		4	
$u_4 = 69$		18		
	52			
$u_5 = 121$				

Working along the diagonal line running downwards from u_0,

$$u_0 = 1,\ \Delta u_0 = 4,\ \Delta^2 u_0 = 6,\ \Delta^3 u_0 = 4, \text{ and } \Delta^4 u_0 = 0$$

The 10th term is evidently u_9,

and $u_9 = (1 + \Delta)^9 u_0$

$$= u_0 + 9\Delta u_0 + 36\Delta^2 u_0 + 84\Delta^3 u_0$$
$$= 1 + 9 \times 4 + 36 \times 6 + 84 \times 4$$
$$= 589$$

also $u_n = (1 + \Delta)^n u_0$

$$= u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0$$

$$= 1 + 4n + 3n(n-1) + \frac{2}{3}n(n-1)(n-2)$$

$$=\frac{1}{3}\{3+12n+9n^2-9n+2n^3-6n^2+4n\}$$

$$=\frac{1}{3}\{2n^3+3n^2+7n+3\}$$

Using this relation for u_n as a means of verifying the result obtained for the 10th term.

Then

$$u_9=\frac{1}{3}\{2\times 9^3+3\times 9^2+7\times 9+3\}$$

$$=\frac{1767}{3}$$

$$=589$$

199. In the preceding work it should be noticed that u_0 represents the first of the given values, and therefore the work can only be done by using the values of the successive differences which occur at the top of the table. It is possible to work with the differences lying on a diagonal line in any part of the table by altering the position of u_0.

Thus if u_0 is taken to be situated somewhere in the middle of the table, then we have to distinguish between the values of u, which occur on either side of u_0. Those values going downwards from u_0 are denoted, as before, by u_1, u_2, u_3, etc., while those working upwards from u_0 are denoted by u_{-1}, u_{-2}, u_{-3}, etc.

u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$	$\Delta^6 u$
u_{-5}						
	δu_{-4}					
u_{-4}		$\delta^2 u_{-3}$				
	δu_{-3}		$\delta^3 u_{-2}$			
u_{-3}		$\delta^2 u_{-2}$		$\delta^4 u_{-1}$		
	δu_{-2}		$\delta^3 u_{-1}$		$\delta^5 u_0$	
u_{-2}		$\delta^2 u_{-1}$		$\delta^4 u_0$		
	δu_{-1}		$\delta^3 u_0$			
u_{-1}		$\delta^2 u_0$				
	δu_0					
u_0						
	Δu_0					
u_1		$\Delta^2 u_0$				
	Δu_1		$\Delta^3 u_0$			
u_2		$\Delta^2 u_1$		$\Delta^4 u_0$		
	Δu_2		$\Delta^3 u_1$		$\Delta^5 u_0$	
u_3		$\Delta^2 u_2$		$\Delta^4 u_1$		
	Δu_3		$\Delta^3 u_2$			
u_4		$\Delta^2 u_3$				
	Δu_4					
u_5						

Hence from the term u_0 there can now be taken two different sets of successive differences. There is the first set Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, etc., running diagonally downwards, and from these values we are enabled to find the value of any term below u_0, for it has been already shown that $u_n = (1 + \Delta)^n u_0$.

There is the second set of differences δu_0, $\delta^2 u_0$, $\delta^3 u_0$, etc., running diagonally upwards, and from these values it should be possible to find the value of any term above u_0.

$$\text{Now } u_{-1} = u_0 - \delta u_0 \qquad (1)$$

$$\begin{aligned} u_{-2} &= u_{-1} - \delta u_{-1} \\ &= u_0 - \delta u_0 - (\delta u_0 - \delta^2 u_0) \\ &= u_0 - 2\delta u_0 + \delta^2 u_0 \qquad (2) \\ u_{-3} &= u_{-2} - \delta u_{-2} \\ &= u_0 - 2\delta u_0 + \delta^2 u_0 - (\delta u_0 - 2\delta^2 u_0 + \delta^3 u_0) \\ &= u_0 - 3\delta u_0 + 3\delta^2 u_0 - \delta^3 u_0 \qquad (3) \\ u_{-4} &= u_{-3} - \delta u_{-3} \\ &= u_0 - 3\delta u_0 + 3\delta^2 u_0 - \delta^3 u_0 - (\delta u_0 - 3\delta^2 u_0 + 3\delta^3 u_0 \\ &\qquad - \delta^4 u_0) \\ &= u_0 - 4\delta u_0 + 6\delta^2 u_0 - 4\delta^3 u_0 + \delta^4 u_0 \qquad (4) \\ u_{-5} &= u_{-4} - \delta u_{-4} \\ &= u_0 - 4\delta u_0 + 6\delta^2 u_0 - 4\delta^3 u_0 + \delta^4 u_0 - (\delta u_0 - 4\delta^2 u_0 \\ &\qquad + 6\delta^3 u_0 - 4\delta^4 u_0 + \delta^5 u_0) \\ &= u_0 - 5\delta u_0 + 10\delta^2 u_0 - 10\delta^3 u_0 + 5\delta^4 u_0 - \delta^5 u_0 \qquad (5) \end{aligned}$$

The multipliers of the differences are evidently the same as the binomial coefficients in the expansions for which the powers are 1, 2, 3, 4, and 5 respectively, but the signs are alternately positive and negative.

$$\text{Hence } u_{-n} = u_0 - n\,\delta u_0 + \frac{n(n-1)}{\lfloor 2}\delta^2 u_0 - \frac{n(n-1)(n-2)}{\lfloor 3}\delta^3 u_0 + \dots$$

Symbolically, then,

$$\begin{aligned} u_{-1} &= (1-\delta)u_0 &&= (1-\delta)u_0 \\ u_{-2} &= (1-2\delta+\delta^2)u_0 &&= (1-\delta)^2 u_0 \\ u_{-3} &= (1-3\delta+3\delta^2-\delta^3)u_0 &&= (1-\delta)^3 u_0 \\ u_{-4} &= (1-4\delta+6\delta^2-4\delta^3+\delta^4)u_0 &&= (1-\delta)^4 u_0 \\ u_{-5} &= (1-5\delta+10\delta^2-10\delta^3+5\delta^4-\delta^5)u_0 &&= (1-\delta)^5 u_0 \end{aligned}$$

$$\text{and } u_{-n} = \left(1 - n\delta + \frac{n(n-1)}{\lfloor 2}\delta^2 - \frac{n(n-1)(n-2)}{\lfloor 3}\delta^3 + \dots\right)u_0 = (1-\delta)^n u_0$$

In which $(1 - \delta)^2 u_0$ means that $(1 - \delta)$ operates on u_0 twice, or more generally $(1 - \delta)^n u_0$ means that $(1 - \delta)$ operates n times on u_0.

Thus if u_0 represents a term, in the middle of a set of given values, $u_{-n} = (1 - \delta)^n u_0$ will give the value of any term above u_0, and the differences to be used must lie on the diagonal line running upwards from u_0; while $u_n = (1 + \Delta\ {}^n u_0$ will give the value of any term below u_0, and the differences to be used must lie on the diagonal line running downwards from u_0.

Example. The values 12, 12, 6, 0, 0, 12, 42 are seven consecutive terms of a series of which the number 6 is the 5th term. Find the 1st term and the 11th term.

$$
\begin{array}{lccccc}
u_{-3} = 12 & & & & \\
 & 0 & & & \\
u_{-2} = 12 & & -6 & & \\
 & -6 & & 6 & \\
u_{-1} = \ \ 6 & & 0 & & 0 \\
 & -6 & & 6 & \\
u_0 \ \ = \ \ 0 & & 6 & & 0 \\
 & 0 & & 6 & \\
u_1 \ \ = \ \ 0 & & 12 & & 0 \\
 & 12 & & 6 & \\
u_2 \ \ = 12 & & 18 & & \\
 & 30 & & & \\
u_3 \ \ = 42 & & & &
\end{array}
$$

If the middle value is denoted by u_0, the 1st term will be u_{-5},

and $\qquad u_{-5} = (1 - \delta)^5 u_0 = u_0 - 5\delta u_0 + 10\delta^2 u_0 - 10\delta^3 u_0$

this relation ends at the 4th term, since $\delta^4 u_0 = 0$, and taking the differences on the diagonal line running upwards from u_0 $\delta u_0 = -6$, $\delta^2 u_0 = 0$, $\delta^3 u_0 = 6$.

Then
$$
\begin{aligned}
u_{-5} &= 0 - 5 \times (-6) + 10 \times 0 - 10 \times 6 \\
&= -30
\end{aligned}
$$

Also, the 11th term is u_5

and $\quad u_5 = (1 + \Delta)^5 u_0 = u_0 + 5\Delta u_0 + 10\Delta^2 u_0 + 10\Delta^3 u_0$

and taking the differences on the diagonal line running downwards from u_0, $\Delta u_0 = 0$, $\Delta^2 u_0 = 12$, and $\Delta^3 u_0 = 6$

Then
$$
\begin{aligned}
u_5 &= 0 + 5 \times 0 + 10 \times 12 + 10 \times 6 \\
&= 180
\end{aligned}
$$

200. The results $u_n = (1 + \Delta)^n u_0$ and $u_{-n} = (1 - \delta)^n u_0$ will hold for all values of n besides positive integers, and if the differences in a certain column, say $\Delta^r u$, are zero; exact results can be obtained by simply taking the binomial expansion for $(1 + \Delta)^n$ or $(1 - \delta)^n$ as far as the term involving $\Delta^{r-1} u_0$ or $\delta^{r-1} u_0$. Thus if the value of $u_{2\cdot 4}$ was needed, it would be better to alter the notation and call u_2 u_0, then $u_{2\cdot 4}$ could be taken as $u_{0\cdot 4}$ and the diagonal line of differences to be used would be the one going downwards from u_0. Or taking it another way, u_3 could be denoted by u_0, then $u_{2\cdot 4}$ could be taken as $u_{-0\cdot 6}$ and the diagonal line of differences to be used would be the one going upwards from u_0.

As an example of the application of the method of finite differences to interpolation, let us work with the following values of θ and p where θ is temperature in °C and p pressure in pounds per square inch.

θ	p	Δp	$\Delta^2 p$	$\Delta^3 p$
70	4·51			
		1·07		
75	5·58		0·21	
		1·28		0·03
80	6·86		0·24	
		1·52		0·02
85	8·38		0·26	
		1·78		0·06
90	10·16		0·32	
		2·10		0·02
95	12·26		0·34	
		2·44		0·05
100	14·70		0·39	
		2·83		0·05
105	17·53		0·44	
		3·27		0·03
110	20·80		0·47	
		3·74		0·08
115	24·54		0·55	
		4·29		0·04
120	28·83		0·59	
		4·88		0·07
125	33·71		0·66	
		5·54		0·04
130	39·25		0·70	
		6·24		0·09
135	45·49		0·79	
		7·03		
140	52·52			

The values of $\Delta^3 p$ indicate some slight irregularity, and this might be due to the fact that θ and p are experimental results ;

allowing a certain amount of latitude for this, let the values of $\Delta^4 p$ be neglected.

(1) To find the value of p when $\theta = 106$.

If $\theta = 105$ be denoted by u_0, then $\theta = 106$ will be denoted by $u_{0\cdot 2}$

$$\text{and} \quad u_{0\cdot 2} = (1+\Delta)^{0\cdot 2} u_0$$
$$= u_0 + 0\cdot 2\Delta u_0 - 0\cdot 08\Delta^2 u_0 + 0\cdot 048\Delta^3 u_0$$

where Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$ are the successive differences running diagonally downwards from $u_0 = 105$.

$$\text{Then } u_{0\cdot 2} = 17\cdot 53 + 0\cdot 2 \times 3\cdot 27 - 0\cdot 08 \times 0\cdot 47 + 0\cdot 048 \times 0\cdot 08$$
$$= 18\cdot 15$$

Or if $\theta = 110$ be denoted by u_0, then $\theta = 106$ will be denoted by $u_{-0\cdot 8}$

$$\text{and} \quad u_{-0\cdot 8} = (1-\delta)^{0\cdot 8} u_0$$
$$= u_0 - 0\cdot 8\delta u_0 - 0\cdot 08\delta^2 u_0 - 0\cdot 032\delta^3 u_0$$

where δu_0, $\delta^2 u_0$, and $\delta^3 u_0$ are the successive differences running diagonally upwards from $u_0 = 110$.

$$\text{Then } u_{-0\cdot 8} = 20\cdot 80 - 0\cdot 8 \times 3\cdot 27 - 0\cdot 08 \times 0\cdot 44 - 0\cdot 032 \times 0\cdot 05$$
$$= 18\cdot 15$$

Hence when $\theta = 106$, $p = 18\cdot 15$.

(2) To find the value of p when $\theta = 92$.

If $\theta = 90$ be denoted by u_0, then $\theta = 92$ will be denoted by $u_{0\cdot 4}$

$$\text{and} \quad u_{0\cdot 4} = (1+\Delta)^{0\cdot 4} u_0$$
$$= u_0 + 0\cdot 4\Delta u_0 - 0\cdot 12\Delta^2 u_0 + 0\cdot 064\Delta^3 u_0$$

where Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$ are the successive differences running diagonally downwards from $u_0 = 90$.

$$\text{Then } u_{0\cdot 4} = 10\cdot 16 + 0\cdot 4 \times 2\cdot 10 - 0\cdot 12 \times 0\cdot 34 + 0\cdot 064 \times 0\cdot 05$$
$$= 10\cdot 96$$

Or if $\theta = 95$ be denoted by u_0, then $\theta = 92$ will be denoted by $u_{-0\cdot 6}$

$$\text{and} \quad u_{-0\cdot 6} = (1-\delta)^{0\cdot 6} u_0$$
$$= u_0 - 0\cdot 6\delta u_0 - 0\cdot 12\delta^2 u_0 - 0\cdot 056\delta^3 u_0$$

where δu_0, $\delta^2 u_0$, and $\delta^3 u_0$ are the successive differences running diagonally upwards from $u_0 = 95$.

$$\text{Then } u_{-0\cdot 6} = 12\cdot 26 - 0\cdot 6 \times 2\cdot 10 - 0\cdot 12 \times 0\cdot 32 - 0\cdot 056 \times 0\cdot 06$$
$$= 10\cdot 96$$

Hence when $\theta = 92$, $p = 10\cdot 96$.

Example 1. The following values are the cubes of numbers from 8·0 to 8·5. Find the cube of 8·23.

8·0	512·000				
		19·441			
8·1	531·441		0·486		
		19·927		0·006	
8·2	551·368		0·492		0
		20·419		0·006	
8·3	571·787		0·498		0
		20·917		0·006	
8·4	592·704		0·504		
		21·421			
8·5	614·125				

Let $u_0 = (8{\cdot}2)^3$, then $(8{\cdot}23)^3$ will be denoted by $u_{0\cdot 3}$

and $$u_{0\cdot 3} = (1 + \Delta)^{0\cdot 3} u_0$$
$$= u_0 + 0{\cdot}3\Delta u_0 - 0{\cdot}105\Delta^2 u_0 + 0{\cdot}0595\Delta^3 u_0$$

where Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$ are the successive differences running diagonally downwards from u_0.

Then $(8{\cdot}23)^3 = 551{\cdot}368 + 0{\cdot}3 \times 20{\cdot}419 - 0{\cdot}105 \times 0{\cdot}498 + 0{\cdot}0595 \times 0{\cdot}006$
$= 557{\cdot}442$

Or taking $u_0 = (8{\cdot}3)^3$, then $(8{\cdot}23)^3$ will be denoted by $u_{-0\cdot 7}$

and $$u_{-0\cdot 7} = (1 - \delta)^{0\cdot 7} u_0$$
$$= u_0 - 0{\cdot}7\delta u_0 - 0{\cdot}105\delta^2 u_0 - 0{\cdot}0455\delta^3 u_0$$

where δu_0, $\delta^2 u_0$, and $\delta^3 u_0$ are the successive differences running diagonally upwards from u_0.

Then $(8{\cdot}23)^3 = 571{\cdot}787 - 0{\cdot}7 \times 20{\cdot}419 - 0{\cdot}105 \times 0{\cdot}492 - 0{\cdot}0455 \times 0{\cdot}006$
$= 557{\cdot}442$

Example 2. The following values are the tangents of angles from 70° to 77°. Find tan 70° 36′ and tan 76° 36′.

70°	2·7475				
		0·1567			
71°	2·9042		0·0168		
		0·1735		0·0029	
72°	3·0777		0·0197		0·0007
		0·1932		0·0036	
73°	3·2709		0·0233		0·0013
		0·2165		0·0049	
74°	3·4874		0·0282		0·0013
		0·2447		0·0062	
75°	3·7321		0·0340		0·0018
		0·2787		0·0080	
76°	4·0108		0·0420		
		0·3207			
77°	4·3315				

(1) If tan 70° be denoted by u_0, then tan 70° 36′ will be denoted by $u_{0\cdot6}$

$$\begin{aligned}\text{and}\quad u_{0\cdot6} &= (1+\Delta)^{0\cdot6}u_0\\ &= u_0 + 0\cdot6\Delta u_0 - 0\cdot12\Delta^2u_0 + 0\cdot056\Delta^3u_0 - 0\cdot0336\Delta^4u_0\end{aligned}$$

where Δu_0, Δ^2u_0, Δ^3u_0, and Δ^4u_0 are the successive differences running diagonally downwards from u_0.

$$\begin{aligned}\text{Then } \tan 70^\circ\ 36' &= 2\cdot7475 + 0\cdot6 \times 0\cdot1567 - 0\cdot12 \times 0\cdot0168\\ &\quad + 0\cdot056 \times 0\cdot0029 - 0\cdot0336 \times 0\cdot0007\\ &= 2\cdot8396\end{aligned}$$

(2) If tan 77° be denoted by u_0, then tan 76° 36′ will be denoted by $u_{-0\cdot4}$

$$\begin{aligned}\text{and}\quad u_{-0\cdot4} &= (1-\delta)^{0\cdot4}u_0\\ &= u_0 - 0\cdot4\delta u_0 - 0\cdot12\delta^2u_0 - 0\cdot064\delta^3u_0 - 0\cdot0416\delta^4u_0\end{aligned}$$

where δu_0, δ^2u_0, δ^3u_0, and δ^4u_0 are the successive differences running diagonally upwards from u_0.

$$\begin{aligned}\text{Then } \tan 76^\circ\ 36' &= 4\cdot3315 - 0\cdot4 \times 0\cdot3207 - 0\cdot12 \times 0\cdot0420\\ &\quad - 0\cdot064 \times 0\cdot008 - 0\cdot0416 \times 0\cdot0018\\ &= 4\cdot1976\end{aligned}$$

201. *The Method of finding the Best Value of* $\frac{\mathrm{dy}}{\mathrm{dx}}$ *from Tabular Values of* x *and* y.

Taylor's theorem states that if $A = f(x)$

$$\text{Then}\quad f(x+h) = A + h\frac{dA}{dx} + \frac{h^2}{\underline{|2}}\frac{d^2A}{dx^2} + \frac{h^3}{\underline{|3}}\frac{d^3A}{dx^3} + \ \ldots$$

$$\begin{aligned}\text{Now}\qquad \frac{dA}{dx} &= \frac{d}{dx}\{f(x)\} = f'(x)\\ \frac{d^2A}{dx^2} &= \frac{d}{dx}\{f'(x)\} = f''(x)\\ \frac{d^3A}{dx^2} &= \frac{d}{dx}\{f''(x)\} = f'''(x), \text{ and so on.}\end{aligned}$$

$$\text{Hence}\quad f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \ \ldots$$

If in a given set of tabular values of x and y, x and $x+h$ represent two consecutive values of x, and u_x and u_{x+h} represent the two corresponding values of y.

$$\begin{aligned}\text{Then}\qquad u_{x+h} &= u_x + \Delta u_x = (1+\Delta)u_x\\ \text{Also}\qquad u_x &= f(x)\\ \text{Then}\qquad u_{x+h} &= f(x+h)\\ \text{But}\qquad \frac{dy}{dx} &= \frac{f(x+h)-f(x)}{h}\end{aligned}$$

when h is made infinitely small.

Now h is a given increment in the value of x, and in consequence cannot be taken as becoming infinitely small. We have therefore to find the best value of $h\dfrac{\delta y}{\delta x}$ or $h\Delta$ for a definite value of h.

$$\begin{aligned}
\text{Now } u_{x+h} &= f(x+h) \\
&= f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \dots \\
&= \left(1 + h\Delta + \frac{h^2}{\lfloor 2}\Delta^2 + \frac{h^3}{\lfloor 3}\Delta^3 + \dots\right)f(x) \\
\text{and } (1+\Delta)u_x &= \left(1 + h\Delta + \frac{h^2\Delta^2}{\lfloor 2} + \frac{h^3\Delta^3}{\lfloor 3} + \dots\right)u_x \\
&= e^{h\Delta}\, u_x \\
\text{or } 1+\Delta &= e^{h\Delta}
\end{aligned}$$

$$\begin{aligned}
\text{Hence}\qquad h\Delta &= \log_e(1+\Delta) \\
&= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \\
\text{or}\qquad h\frac{\delta y}{\delta x} &= \Delta u_x - \frac{1}{2}\Delta^2 u_x + \frac{1}{3}\Delta^3 u_x - \frac{1}{4}\Delta^4 u_x + \dots
\end{aligned}$$

where Δu_x, $\Delta^2 u_x$, $\Delta^3 u_x$, $\Delta^4 u_x$ represent the successive differences running downwards in a diagonal direction from the term u_x. As u_x represents any term, the value of $h\dfrac{dy}{dx}$ can be obtained for any value of x given in the table, provided that the successive differences corresponding to that value of x are accessible.

If u_{x-h} is the value of y preceding u_x,

$$\begin{aligned}
\text{Then}\qquad u_{x-h} &= f(x-h) \\
\text{and also}\qquad u_{x-h} &= u_x - \delta u_x = (1-\delta)u_x \\
\text{Now}\qquad f(x-h) &= f(x) - hf'(x) + \frac{h^2}{\lfloor 2}f''(x) - \frac{h^3}{\lfloor 3}f'''(x) + \dots \\
&= \left(1 - h\Delta + \frac{h^2\Delta^2}{\lfloor 2} - \frac{h^3\Delta^3}{\lfloor 3} + \dots\right)f(x) \\
\text{and}\quad (1-\delta)u_x &= \left(1 - h\Delta + \frac{h^2\Delta^2}{\lfloor 2} - \frac{h^3\Delta^3}{\lfloor 3} + \dots\right\}u_x \\
&= e^{-h\Delta}\, u_x \\
\text{or}\qquad 1-\delta &= e^{-h\Delta} \\
\text{Hence}\qquad -h\Delta &= \log_e(1-\delta) \\
&= -\delta - \frac{\delta^2}{2} - \frac{\delta^3}{3} - \frac{\delta^4}{4} - \dots
\end{aligned}$$

or $$h\frac{\delta y}{\delta x} = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{3} + \frac{\delta^4}{4} - \dots$$

$$= \delta u_x + \frac{1}{2}\delta^2 u_x + \frac{1}{3}\delta^3 u_x + \frac{1}{4}\delta^4 u_x + \dots$$

where δu_x, $\delta^2 u_x$, $\delta^3 u_x$, $\delta^4 u_x$ represent the successive differences running upwards in a diagonal direction from the term u_x.

Thus the best value of $\frac{dy}{dx}$ for a given value of x is given by

$$h\frac{dy}{dx} = \Delta u_x - \frac{1}{2}\Delta^2 u_x + \frac{1}{3}\Delta^3 u_x - \frac{1}{4}\Delta^4 u_x + \dots$$

or $$h\frac{dy}{dx} = \delta u_x + \frac{1}{2}\delta^2 u_x + \frac{1}{3}\delta^3 u_x + \frac{1}{4}\delta^4 u_x + \dots$$

where h is the increment in the value of x.

As an example, from the given values of x and y it is required to find the value of $\frac{dy}{dx}$ when $x = 6$.

x	y				
4·5	9·69				
		3·21			
5·0	12·90		0·60		
		3·81		0·06	
5·5	16·71		0·66		0
		4·47		0·06	
6·0	21·18		0·72		0
		5·19		0·06	
6·5	26·37		0·78		0
		5·97		0·06	
7·0	32·34		0·84		
		6·81			
7·5	39·15				

Here the increment in the value of x is 0·5 and $h = 0{\cdot}5$.

Working upwards from $x = 6$

$$h\frac{dy}{dx} = \delta u_x + \frac{1}{2}\delta^2 u_x + \frac{1}{3}\delta^3 u_x + \frac{1}{4}\delta^4 u_x + \dots$$

$$= 4{\cdot}47 + \frac{1}{2} \times 0{\cdot}66 + \frac{1}{3} \times 0{\cdot}06$$

$$= 4{\cdot}82$$

Or, working downwards from $x = 6$,

$$h\frac{dy}{dx} = \Delta u_x - \frac{1}{2}\Delta^2 u_x + \frac{1}{3}\Delta^3 u_x - \frac{1}{4}\Delta^4 u_x \dots$$

$$= 5{\cdot}19 - \frac{1}{2} \times 0{\cdot}78 + \frac{1}{3} \times 0{\cdot}06$$

$$= 4{\cdot}82$$

Thus when $x = 6$, $\dfrac{dy}{dx} = \dfrac{4\cdot82}{h}$

$$= 9\cdot64$$

The above values have been calculated from the law $y = 0\cdot08x^3 + x - 2\cdot1$, and therefore this value of $\dfrac{dy}{dx}$ can easily be verified.

For if
$$y = 0\cdot08x^3 + x - 2\cdot1$$
$$\frac{dy}{dx} = 0\cdot24x^2 + 1$$
$$= 9\cdot64 \text{ when } x = 6.$$

202. If the set of tabular values for x and y is such that ultimately some difference column contains equal terms throughout and in the next column all of the terms are zero the two values of $h\dfrac{dy}{dx}$, one obtained by working diagonally downwards, and the other by working diagonally upwards, would be equal. But when dealing with experimental values or tabular values calculated correct to a certain number of significant figures, this is not the case, the two values of $h\dfrac{dy}{dx}$ differ slightly.

Hence the best value is taken as the mean of the two and

$$h\frac{dy}{dx} = \frac{1}{2}(\delta u_x + \Delta u_x) + \frac{1}{4}(\delta^2 u_x - \Delta^2 u_x) + \frac{1}{6}(\delta^3 u_x + \Delta^3 u_x) + \ \ldots$$

Example. The following table of values of x and y is given. Find $\dfrac{dy}{dx}$ when $x = 3$ with as great accuracy as possible. (B. of E., 1913.)

x	y				
0	6·98970				
		0·41393			
1	7·40363		− 0·03605		δ
		0·37788		0·00579	
2	7·78151		− 0·03026		− 0·00130
		0·34762		0·00449	
3	8·12913		− 0·02577		− 0·00094
		0·32185		0·00355	
4	8·45098		− 0·02222		− 0·00067
		0·29963		0·00288	
5	8·75061		− 0·01934		Δ
		0·28029			
6	9·03090				

Working along the diagonal lines indicated, it is evident that account can only be taken of the first three successive differences, the values of the following successive differences are clearly inaccessible.

h, the increment in the value of x, is 1.

$$\begin{aligned} \text{Now } \delta u_x + \Delta u_x &= 0{\cdot}34762 + 0{\cdot}32185 \\ &= 0{\cdot}66947 \\ \delta^2 u_x - \Delta^2 u_x &= -0{\cdot}03026 + 0{\cdot}02222 \\ &= -0{\cdot}00804 \\ \delta^3 u_x + \Delta^3 u_x &= 0{\cdot}00579 + 0{\cdot}00288 \\ &= 0{\cdot}000867 \end{aligned}$$

Then $h\frac{dy}{dx} = \frac{1}{2}(\delta u_x + \Delta u_x) + \frac{1}{4}(\delta^2 u_x - \Delta^2 u_x) + \frac{1}{6}(\delta^3 u_x + \Delta^3 u_x)$

$$= \frac{1}{2} \times 0{\cdot}66947 - \frac{1}{4} \times 0{\cdot}00804 + \frac{1}{6} \times 0{\cdot}00867$$

$$= 0{\cdot}33417$$

Hence when $x = 3$, $\frac{dy}{dx} = 0{\cdot}33417$

203. *The Best Value of* $\frac{d^2y}{dx^2}$ *that can be obtained from Tabular Values of* x *and* y.

It has already been shown that

$$h\Delta = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \ldots$$

Then $h^2\Delta^2 = \Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \ldots$

or $h\frac{d^2y}{dx^2} = \Delta^2 u_x - \Delta^3 u_x + \frac{11}{12}\Delta^4 u_x - \frac{5}{6}\Delta^5 u_x + \frac{137}{180}\Delta^6 u_x - \ldots$

where $\Delta^2 u_x$, $\Delta^3 u_x$, $\Delta^4 u_x$, etc., represent the successive differences running downwards in a diagonal direction from the term u_x.

Also $h\Delta = \delta + \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 + \frac{1}{4}\delta^4 + \frac{1}{5}\delta^5 + \ldots$

and $h^2\Delta^2 = \delta^2 + \delta^3 + \frac{11}{12}\delta^4 + \frac{5}{6}\delta^5 + \frac{137}{180}\delta^6 + \ldots$

or $h^2\Delta^2 = \delta^2 u_x + \delta^3 u_x + \frac{11}{12}\delta^4 u_x + \frac{5}{6}\delta^5 u_x + \frac{137}{180}\delta^6 u_x + \ldots$

where $\delta\ u_x$, $\delta^3 u_x$, $\delta^4 u_x$, etc., represent the successive differences running upwards in a diagonal direction from the term u_x.

As an example, take the set of values of x and y given in paragraph 201, and find the value of $\frac{d^2y}{dx^2}$ when $x = 6$. Thus working diagonally downwards, $\Delta^2u_x = 0{\cdot}78$, $\Delta^3u_x = 0{\cdot}06$, and $\Delta^4u_x = 0$.

Then
$$h^2\frac{d^2y}{dx^2} = \Delta^2u_x - \Delta^3u_x + \frac{11}{12}\Delta^4u_x - \ldots$$
$$= 0{\cdot}78 - 0{\cdot}06$$
$$= 0{\cdot}72$$
$$\frac{d^2y}{dx^2} = \frac{0{\cdot}72}{h^2}$$
$$= 2{\cdot}88$$

Working diagonally upwards, $\delta^2u_x = 0{\cdot}66$, $\delta^3u_x = 0{\cdot}06$, and $\delta^4u_x = 0$.

Then
$$h^2\frac{d^2y}{dx^2} = \delta^2u_x + \delta^3u_x + \frac{11}{12}\delta^4u_x + \ldots$$
$$= 0{\cdot}66 + 0{\cdot}06$$
$$= 0{\cdot}72$$
$$\frac{d^2y}{dx^2} = 2{\cdot}88$$

Now the law from which the values of x and y have been calculated is:
$$y = 0{\cdot}08x^3 + x - 2{\cdot}1$$
$$\frac{dy}{dx} = 0{\cdot}24x^2 + 1$$
$$\frac{d^2y}{dx^2} = 0{\cdot}48x$$
$$= 2{\cdot}88 \text{ when } x = 6$$

The expressions
$$h^2\frac{d^2y}{dx^2} = \Delta^2u_x - \Delta^3u_x + \frac{11}{12}\Delta^4u_x - \frac{5}{6}\Delta^5u_x + \frac{137}{180}\Delta^6u_x - \ldots$$
and
$$h^2\frac{d^2y}{dx^2} = \delta^2u_x + \delta^3u_x + \frac{11}{12}\delta^4u_x + \frac{5}{6}\delta^5u_x + \frac{137}{180}\delta^6u_x \ \ldots$$
will give exact results for $h^2\frac{d^2y}{dx^2}$ if ultimately some difference column contains equal terms throughout or in the next column all of the terms are zero. But when dealing with experimental values, or tabular values calculated correct to a certain number of significant figures, this is not the case, the two values of $h^2\frac{d^2y}{dx^2}$

differ slightly. Then the best value is taken as the mean of the two,

$$\text{and}\quad h^2\frac{d^2y}{dx^2} = \frac{1}{2}(\Delta^2 u_x + \delta^2 u_x) - \frac{1}{2}(\Delta^3 u_x - \delta^3 u_x) + \frac{11}{24}(\Delta^4 u_x + \delta^4 u_x)$$
$$- \frac{5}{12}(\Delta^5 u_x - \delta^5 u_x) + \ . \ . \ .$$

204. *The Case when* $\Delta^7 u_x = 0$. Very often in dealing with tabular values slight errors in the values themselves render the values of the higher differences worthless for interpolation; probably the first three difference columns can be relied upon. Also in many cases the actual number of tabular values given is not sufficient to enable us to find all of the successive differences. For example, if seven tabular values are given, the successive differences corresponding to the middle of these values can be found up to $\Delta^3 u_x$, but the successive differences higher than this are inaccessible. In order to provide for these cases, formulæ have been established which give the values of $h\dfrac{dy}{dx}$ and $h^2\dfrac{d^2y}{dx^2}$ in terms of the first three successive differences; these formulæ have been based on the assumption that $\Delta^7 u_x = 0$.

	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$	$\Delta^6 u_x$	$\Delta^7 u_x$
			$d-4c+8b-10a$		$b-4a$		0
				$c-\frac{7b}{2}+6a$		a	
			$d-3c+\frac{9b}{2}+4a$		$b-3a$		0
				$c-\frac{5b}{2}+3a$		a	
			$d-2c+2b-a$		$b-2a$		0
		$e-\frac{3d}{2}+c-\frac{b}{2}$		$c-\frac{3b}{2}+a$		a	
	$f-e+\frac{d}{2}$		$d-c+\frac{b}{2}$		$b-a$		0
		$e-\frac{d}{2}$		$c-\frac{b}{2}$		a	
	f		d		b		0
u_x		$e+\frac{d}{2}$		$c+\frac{b}{2}$		a	
	$f+e+\frac{d}{2}$		$d+c+\frac{b}{2}$		$b+a$		0
		$e+\frac{3d}{2}+c+\frac{b}{2}$		$c+\frac{3b}{2}+a$		a	
	$f+2e+2d+c+\frac{b}{2}$		$d+2c+2b+a$		$b+2a$		0
		$e+\frac{5d}{2}+3c+\frac{5b}{2}+a$		$c+\frac{5b}{2}+3a$		a	
			$d+3c+\frac{9b}{2}+4a$		$b+3a$		0
				$c+\frac{7b}{2}+6a$		a	
			$d+4c+8b+10a$		$b+4a$		0
				$c+\frac{9b}{2}+10a$		a	
			$d+5c+\frac{25b}{2}+20a$		$b+5a$		0

The above table gives some of the successive differences which have been obtained by working backwards from the seventh difference column, assuming that $\Delta^7 u_x = 0$.

(1) $h\dfrac{dy}{dx}$

$$= \frac{1}{2}(\Delta u_x + \delta u_x) - \frac{1}{4}(\Delta^2 u_x - \delta^2 u_x) + \frac{1}{6}(\Delta^3 u_x + \delta^3 u_x) - \frac{1}{8}(\Delta^4 u_x - \delta^4 u_x)$$
$$+ \frac{1}{10}(\Delta^5 u_x + \delta^5 u_x) - \frac{1}{12}(\Delta^6 u_x - \delta^6 u_x) + \frac{1}{14}(\Delta^7 u_x + \delta^7 u_x)$$

But
$$\Delta^7 u_x + \delta^7 u_x = 0$$
$$\Delta^6 u_x - \delta^6 u_x = 0$$
$$\Delta^5 u_x + \delta^5 u_x = 2b + a$$
$$\Delta^4 u_x - \delta^4 u_x = 4b + 2a = 2(2b + a)$$
$$\Delta^3 u_x + \delta^3 u_x = 2d + c + \frac{5b}{2} + a$$
$$\Delta^2 u_x - \delta^2 u_x = 2d + c + \frac{b}{2}$$

Hence $(\Delta^3 u_x + \delta^3 u_x) - (\Delta^2 u_x - \delta^2 u_x) = 2b + a$

and $\Delta^5 u_x + \delta^5 u_x = (\Delta^3 u_x + \delta^3 u_x) - (\Delta^2 u_x - \delta^2 u_x)$

also $\Delta^4 u_x - \delta^4 u_x = 2(\Delta^3 u_x + \delta^3 u_x) - 2(\Delta^2 u_x - \delta^2 u_x)$

Therefore $h\dfrac{dy}{dx}$

$$= \frac{1}{2}(\Delta u_x + \delta u_x) - \frac{1}{4}(\Delta^2 u_x - \delta^2 u_x) + \frac{1}{6}(\Delta^3 u_x + \delta^3 u_x)$$
$$- \frac{1}{4}\left\{(\Delta^3 u_x + \delta^3 u_x) - (\Delta^2 u_x - \delta^2 u_x)\right\}$$
$$+ \frac{1}{10}\left\{(\Delta^3 u_x + \delta^3 u_x) - (\Delta^2 u_x - \delta^2 u_x)\right\}$$
$$= \frac{1}{2}(\Delta u_x + \delta u_x) - (\Delta^2 u_x - \delta^2 u_x)\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{10}\right)$$
$$+ (\Delta^3 u_x + \delta^3 u_x)\left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right)$$
$$= \frac{1}{2}(\Delta u_x + \delta u_x) - \frac{1}{10}(\Delta^2 u_x - \delta^2 u_x) + \frac{1}{60}(\Delta^3 u_x + \delta^3 u_x)$$
$$= \frac{1}{60}\left\{30(\Delta u_x + \delta u_x) - 6(\Delta^2 u_x - \delta^2 u_x) + (\Delta^3 u_x + \delta^3 u_x)\right\} \quad \ldots (1)$$

(2) $$h^2\frac{d^2y}{dx^2} = \frac{1}{2}(\Delta^2 u_x + \delta^2 u_x) - \frac{1}{2}(\Delta^3 u_x - \delta^3 u_x) + \frac{11}{24}(\Delta^4 u_x + \delta^4 u_x)$$
$$- \frac{5}{12}(\Delta^5 u_x - \delta^5 u_x) + \frac{137}{360}(\Delta^6 u_x + \delta^6 u_x)$$

But $\quad \Delta^7 u_x - \delta^7 u_x = 0$

$$\mathrm{F} = \Delta^6 u_x + \delta^6 u_x = 2a$$
$$\mathrm{E} = \Delta^5 u_x - \delta^5 u_x = 5a$$
$$\mathrm{D} = \Delta^4 u_x + \delta^4 u_x = 2c + b + 4a$$
$$\mathrm{C} = \Delta^3 u_x - \delta^3 u_x = 3c + \frac{3b}{2} + a$$
$$\mathrm{B} = \Delta^2 u_x + \delta^2 u_x = 2e + d + c + \frac{b}{2}$$
$$\mathrm{A} = \Delta u_x - \delta u_x = e + \frac{d}{2}$$

Now $\quad \mathrm{B} - 2\mathrm{A} = c + \dfrac{b}{2}$

and
$$\begin{aligned} 4\mathrm{C} - \mathrm{D} &= 10c + 5b \\ &= 10(\mathrm{B} - 2\mathrm{A}) \end{aligned}$$

Then $\quad \mathrm{D} = 20\mathrm{A} - 10\mathrm{B} + 4\mathrm{C}$

Also
$$\begin{aligned} 3\mathrm{D} - 2\mathrm{C} &= 10a \\ &= 2\mathrm{E} \text{ or } 5\mathrm{F} \end{aligned}$$

Then
$$\begin{aligned} 5\mathrm{F} &= 3\mathrm{D} - 2\mathrm{C} \\ &= 60\mathrm{A} - 30\mathrm{B} + 12\mathrm{C} - 2\mathrm{C} \\ &= 60\mathrm{A} - 30\mathrm{B} + 10\mathrm{C} \\ \mathrm{F} &= 12\mathrm{A} - 6\mathrm{B} + 2\mathrm{C} \end{aligned}$$

Also
$$\begin{aligned} 2\mathrm{E} &= 3\mathrm{D} - 2\mathrm{C} \\ &= 60\mathrm{A} - 30\mathrm{B} + 10\mathrm{C} \\ \mathrm{E} &= 30\mathrm{A} - 15\mathrm{B} + 5\mathrm{C} \end{aligned}$$

But $h^2\dfrac{d^2y}{dx^2}$

$$= \frac{1}{2}\mathrm{B} - \frac{1}{2}\mathrm{C} + \frac{11}{24}\mathrm{D} - \frac{5}{12}\mathrm{E} + \frac{137}{360}\mathrm{F}$$

$$= \frac{1}{2}\mathrm{B} - \frac{1}{2}\mathrm{C} + \frac{11}{24}\{\,20\mathrm{A} - 10\mathrm{B} + 4\mathrm{C}\,\} - \frac{5}{12}\{\,30\mathrm{A} - 15\mathrm{B} + 5\mathrm{C}\,\}$$
$$+ \frac{137}{360}\{\,12\mathrm{A} - 6\mathrm{B} + 2\mathrm{C}\,\}$$

$$= \mathrm{A}\left\{\frac{55}{6} - \frac{75}{6} + \frac{137}{30}\right\} + \mathrm{B}\left\{\frac{1}{2} - \frac{55}{12} + \frac{25}{4} - \frac{137}{60}\right\}$$
$$+ \mathrm{C}\left\{-\frac{1}{2} + \frac{11}{6} - \frac{25}{12} + \frac{137}{180}\right\}$$

$$= \frac{37}{30}\mathrm{A} - \frac{7}{60}\mathrm{B} + \frac{1}{90}\mathrm{C}$$

$$= \frac{1}{180}\{222(\Delta u_x - \delta u_x) - 21(\Delta^2 u_x + \delta^2 u_x) + 2(\Delta^3 u_x - \delta^3 u_x)\} \ldots (2)$$

Example. Using the given tabular values of θ and p, find the best values of $\frac{dp}{d\theta}$ and $\frac{d^2p}{d\theta^2}$ when $\theta = 95$.

θ	80	85	90	95	100	105	110
p	6·86	8·38	10·16	12·26	14·70	17·53	20·80

In this case, as only seven tabular values are given, the first, second, and third successive differences can be worked with, but the higher differences are inaccessible.

θ	p	Δp	$\Delta^2 p$	$\Delta^3 p$	
80	6·86				
		1·52			δ
85	8·38		0·26		
		1·78		0·06	
90	10·16		0·32		
		2·10		0·02	
95	12·26		0·34		
		2·44		0·05	
100	14·70		0·39		
		2·83		0·05	
105	17·53		0·44		
		3·27			Δ
110	20·80				

Then
$$\Delta p + \delta p = 4{\cdot}54$$
$$\Delta^2 p - \delta^2 p = 0{\cdot}07$$
and
$$\Delta^3 p + \delta^3 p = 0{\cdot}11$$

But
$$h\frac{dp}{d\theta} = \frac{1}{60}\{30(\Delta p + \delta p) - 6(\Delta^2 p - \delta^2 p) + (\Delta^3 p + \delta^3 p)\}$$
$$= \frac{1}{60}\{30 \times 4{\cdot}54 - 6 \times 0{\cdot}07 + 0{\cdot}11\}$$
$$= 2{\cdot}2648$$
$$\frac{dp}{d\theta} = \frac{2{\cdot}2648}{5}$$
$$= 0{\cdot}4550$$

Also
$$\Delta p - \delta p = 0{\cdot}34$$
$$\Delta^2 p + \delta^2 p = 0{\cdot}71$$
$$\Delta^3 p - \delta^3 p = -\,0{\cdot}01$$

But $h^2\dfrac{d^2p}{d\theta^2} = \dfrac{1}{180}\{222(\Delta p - \delta p) - 21(\Delta^2 p + \delta^2 p) + 2(\Delta^3 p - \delta^3 p)\}$

$$= \frac{1}{180}\{222 \times 0{\cdot}34 - 21 \times 0{\cdot}71 - 2 \times 0{\cdot}01\}$$

$$= 0{\cdot}3364$$

$$\frac{d^2p}{d\theta^2} = \frac{0{\cdot}3364}{25}$$

$$= 0{\cdot}01346$$

Examples XXIII

(1) Find the 8th term and the general term of the series

20 30 42 56 72 . . .

(2) Find the 10th term and the general term of the series

10 4 0 − 2 − 2 . . .

(3) Find the 9th term and the general term of the series

18 9 6 15 42 . . .

(4) Find the 7th term and the general term of the series

3 9 20 38 65 . . .

(5) Find the 11th term and the general term of the series

36 27 12 27 132 411 . . .

(6) Find the 6th term and the general term of the series

32 38 44 56 80 . . .

(7) Working with the tabular values of p and θ given in paragraph 200, and denoting $\theta = 90$ by u_0, find p when $\theta = 92$. Also denoting $\theta = 95$ by u_0, find p when $\theta = 92$.

(8) Working with the tabular values of p and θ given in paragraph 200 and denoting $\theta = 120$ by u_0, find p when $\theta = 123$. Also denoting $\theta = 125$ by u_0, find p when $\theta = 123$.

(9) The following values are taken from the table of cubes:

6·1	6·2	6·3	6·4	6·5	6·6	6·7
226·981	238·328	250·047	262·144	274·625	287·496	300·763

Denoting $(6{\cdot}3)^3$ as u_0, find the value of $(6{\cdot}36)^3$, also denoting $(6{\cdot}4)^3$ as u_0, find the value of $(6{\cdot}36)^3$.

(10) The following values are taken from the table of cube roots:

6·1	6·2	6·3	6·4	6·5	6·6	6·7
1·827160	1·837091	1·846915	1·856636	1·866256	1·875777	1·885204

Denoting $\sqrt[3]{6 \cdot 3}$ as u_0, find the value of $\sqrt[3]{6 \cdot 36}$. Also, denoting $\sqrt[3]{6 \cdot 4}$ as u_0, find the value of $\sqrt[3]{6 \cdot 36}$.

(11) The following values are taken from the table of tangents:

60°	61°	62°	63°	64°	65°	66°	67°
1·7321	1·8040	1·8807	1·9626	2·0503	2·1445	2·2460	2·3559

Denoting tan 61° as u_0, find the value of tan 61° 24′. Also, denoting tan 66° as u_0, find the value of tan 65° 48′.

(12) From the given table of values of x and y, find the best value of $\frac{dy}{dx}$ when $x = 0 \cdot 95$.

x	0·8	0·85	0·9	0·95	1·0	1·05	1·1
y	0·7174	0·7513	0·7833	0·8134	0·8415	0·8674	0·8912

(13) The following values of p and θ being given, find $\frac{dp}{d\theta}$ when $\theta = 115$.

θ	100	105	110	115	120	125	130
p	14·70	17·53	20·80	24·54	28·83	33·71	39·25

(B. of E., 1905.)

(14) The following values of x and y being given, find the most probable value of $\frac{dy}{dx}$ when x is 3.

x	0	1	2	3	4	5	6
y	11·8	16·0	20·0	23·9	27·6	31·1	34·5

(B. of E., 1906.)

(15) If $L = ct\frac{dp}{dt}$ where L is latent heat (in foot-pounds), t is absolute temperature centigrade, p is pressure in pounds per square foot, c cubic feet is increase of volume if 1 lb. changes from lower to higher state. Calculate c at $t = 428$, if the following numbers are given for steam. When $t = 428$, L is $497 \cdot 2 \times 1393$.

t	413	418	423	428	433	438	443
p	7563	8698	9966	11380	12940	14680	16580

(B. of E., 1907.)

(16) The following table of values of x and y is given. Find $\frac{dy}{dx}$ when $x = 3$. (B. of E., 1913.)

x	0	1	2	3	4	5	6
y	6·98970	7·40363	7·78151	8·12913	8·45098	8·75061	9·03090

(17) From the given table of values of x and y, find the best values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 4{\cdot}5$

x	3·0	3·5	4·0	4·5	5·0	5·5	6·0
y	4·780	4·405	6·780	12·655	22·780	37·905	58·780

(18) Working with the tabular values of θ and p given in paragraph 200, find the best values of $\frac{dp}{d\theta}$ and $\frac{d^2p}{d\theta^2}$ when $\theta = 100$.

(19) Working with the tabular values of θ and p given in paragraph 200, find the best values of $\frac{dp}{d\theta}$ and $\frac{d^2p}{d\theta^2}$ when $\theta = 125$.

CHAPTER XXIV

205. *The Vector.* A vector is a quantity involving magnitude and direction, and in dealing with it as much importance must be placed upon its direction as is placed upon its magnitude. A force, a displacement, a velocity, an acceleration are examples of vectors, inasmuch as they have both magnitude and direction; while in the case of a velocity and an acceleration the magnitude and the direction can each be functions of the time. If, for example, a body is describing a circular path with uniform angular velocity, the linear velocity of the body at any instant is a vector

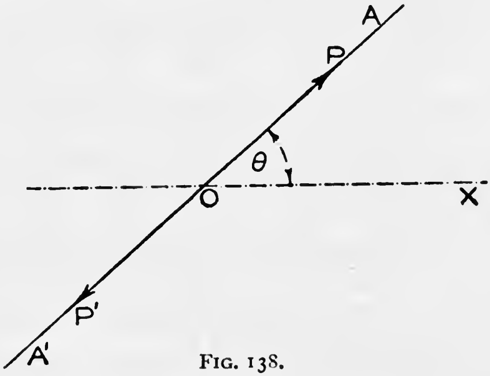


FIG. 138.

of constant magnitude and of varying direction; this direction is directly proportional to the time.

The four things which completely specify a vector are:

(1) The point of application.
(2) The magnitude.
(3) The line of action.
(4) The sense, or the direction along the line of action.

The "line of action" is determined by the angle it makes with some fixed direction, and usually this fixed direction is taken to be horizontal. The "sense" is fixed according as the vector acts away from or towards the point of application. Consequently a vector can be represented graphically by means of a straight line; for let ρ be the magnitude, θ the direction, and the sense be positive, that is, the vector acts away from the point of application.

Let O be the point of application, OX the horizontal direction, and OA the line of action of the vector. If $\widehat{XOA} = \theta$, $OP = \rho$ to some convenient scale, and the arrow indicates that the vector acts along the line of action from O to P, then the line OP represents the vector completely.

If the sense is negative, that is, the vector acts towards the point of application and the direction of the arrow must be reversed, then the length $OP' = \rho$ must be measured from O along OA in the opposite direction.

It is obvious that OP and OP′ represent two equal and opposite vectors, which, if taken together, would neutralise one another. Also if OP represents a positive vector, then OP′ will represent the corresponding negative vector, and therefore a positive vector can be made negative by simply changing its sense; this is very important, and has to be used in all cases of subtraction of vectors.

206. ***Resolution of Vectors.*** A vector can be resolved along any two assigned directions; that is, it can be replaced in effect by two vectors, the first of which acts along one of these directions, and the second along the other.

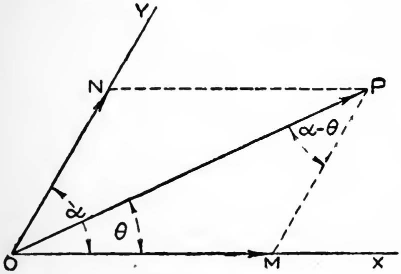


FIG. 139.

Thus if OX and OY (Fig. 139) are the two given directions, the angle $XOY = \alpha$, and $OP = \rho$ is the given vector whose line of action makes an angle θ with OX, then by making OP the diagonal of the parallelogram ONPM, ON represents the resolved part of the vector in the direction OY, and OM represents the resolved part in the direction OX. It is obvious that ON and OM should be measured to the same scale as OP.

Working with the triangle PMO and applying the sine rule.

$$\frac{OP}{\sin(180° - \alpha)} = \frac{PM}{\sin\theta} = \frac{OM}{\sin(\alpha - \theta)}$$

Hence
$$PM = ON = \frac{OP \sin\theta}{\sin(180° - \alpha)}$$
$$= \frac{\rho \sin\theta}{\sin\alpha}$$

also
$$OM = \frac{OP \sin(\alpha - \theta)}{\sin\alpha}$$
$$= \frac{\rho \sin(\alpha - \theta)}{\sin\alpha}$$

When $x = 90°$, ON and OM become the rectangular components of the given vector :

and $$ON = \rho \sin \theta$$
$$OM = \rho \sin (90° - \theta)$$
$$= \rho \cos \theta.$$

207. *Addition and Subtraction of Vectors.* In the previous paragraph, since the vector OP can be replaced in effect by the vectors OM and ON, it necessarily follows that the vector OP can replace in effect the two vectors OM and ON. Thus OP can be taken as the vector sum of OM and ON, and the sum of two vectors can be obtained by making two adjacent sides of a parallelogram represent in every respect the two vectors, and the diagonal of the parallelogram which passes through the point of intersection of their lines of action will represent in every respect the sum of the two vectors.

Let **A** be a vector of magnitude ρ and direction θ_1; let **B** be another vector of magnitude ρ_2 and direction θ_2.

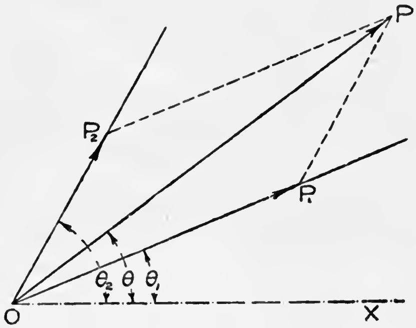


FIG. 140.

To find (A + B).

Let OX (Fig. 140) be the reference line, and let OP_1 and OP_2 make angles θ_1 and θ_2 respectively with OX. Let $OP_1 = \rho_1$ and $OP_2 = \rho_2$, and the parallelogram completed by drawing P_2P parallel to OP_1 and P_1P parallel to OP_2. Then OP will represent the sum of the two vectors, or (A + B), θ being the angle its line of action makes with OX, and OP measured to the same scale as OP_1 and OP_2, the magnitude. The parallelogram law can be used to find the difference of two vectors, since by altering the sense of the vector which has to be subtracted the question becomes one of addition of vectors.

To find (A − B).

Let OP_1 and OP_2 make angles θ_1 and θ_2 respectively with the reference line OX (Fig. 141), and let $OP_1 = \rho_1$ and $OP_2 = \rho_2$; the length OP_2 being now measured in the opposite direction.

The parallelogram is completed by drawing P_2P parallel to OP_1, and P_1P parallel to OP_2. Then OP will represent the difference of the two vectors, or (A – B), θ being the angle its line of action makes with OX, and OP the magnitude.

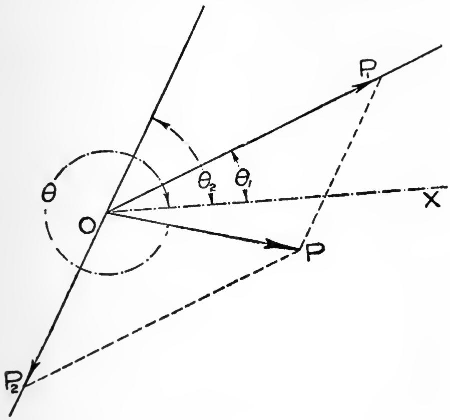


FIG. 141.

208. *The Vector Polygon.* In order to find the sum of a system of vectors the parallelogram law must be used time after time, and this continued application of the parallelogram law gives rise to the vector polygon.

Let A, B, C, D . . . be a system of vectors whose magnitudes are $\rho_1, \rho_2, \rho_3, \rho_4$ . . . and whose directions are $\theta_1, \theta_2, \theta_3, \theta_4$ . . .

Let OA, OB, OC, OD . . . make angles $\theta_1, \theta_2, \theta_3, \theta_4$ . . . with OX, and $OA = \rho_1$, $OB = \rho_2$, $OC = \rho_3$, $OD = \rho_4$. . . . (Fig. 142.)

By completing the parallelogram OAaB, the diagonal Oa will represent the sum of the vectors A and B.

By completing the parallelogram OabC, the diagonal Ob will represent the sum of the vectors Oa and C, that is the sum of the vectors A, B, and C.

Similarly Oc will represent the sum of the vectors A, B, C, and D. It is evident that by drawing OA_1 parallel to OA, and making $OA_1 = OA = \rho_1$, by drawing A_1a_1 parallel to Aa and making $Aa_1 = Aa = \rho_2$, Oa_1 will be exactly the same as Oa, and can therefore represent the sum of the vectors A and B.

By drawing a_1b_1 parallel to ab, and making $a_1b_1 = ab = \rho_3$, Ob_1 will be exactly the same as Ob, and can therefore represent the sum of the vectors A, B, and C. Similarly Oc_1, being exactly the same as Oc, will represent the sum of the vectors A, B, C, and D. Thus $OA_1a_1b_1c_1$ gives a polygon in which the arrows, denoting the sense of each vector, follow each other round in cyclic order; Oc_1 is the closing line of this polygon. If this polygon be drawn, the closing line Oc_1 will give the sum of the vectors A, B, C, and D, just as well as drawing the parallelogram of vectors time after time. It should be noticed that although the sense of each vector forming the sides of the polygon follow a cyclic order, the sense of the vector sum, or the closing side of the polygon, must be in the opposite cyclic direction.

If one of the vectors is to be subtracted its sense must be reversed, and the work can be proceeded with as in the case of addition.

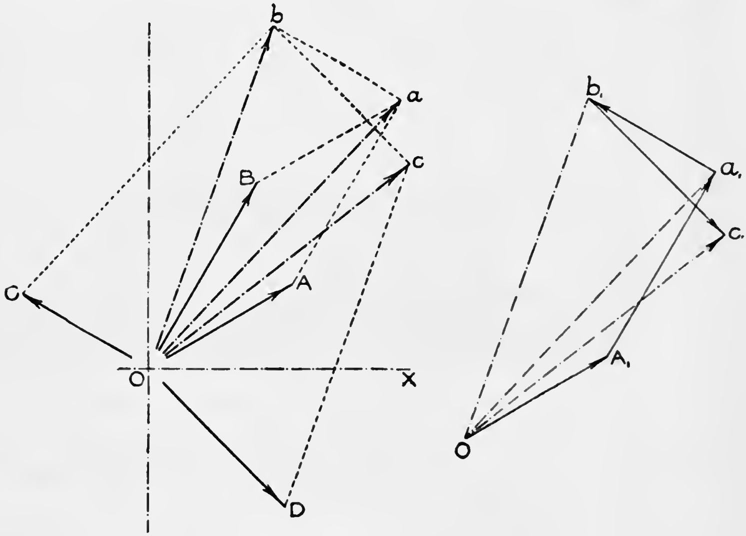


FIG. 142.

Example. A, B, C, and D are four vectors whose magnitudes and direction are given in the table below

A	5	30°
B	7	90°
C	6	150°
D	9	300°

Find (1) A + B + C + D, (2) A + B − C + D, (3) A − B + C − D.

Fig. 143 shows the vector polygons drawn to scale; the scale of magnitudes being shown at the bottom of the diagram. By actual measurement

(1) A + B + C + D
Magnitude = 9·1
Direction = 83°

(2) A + B − C + D
Magnitude = 11·9
Direction = 15°

(3) A − B + C − D
Magnitude = 3·5
Direction = 146°

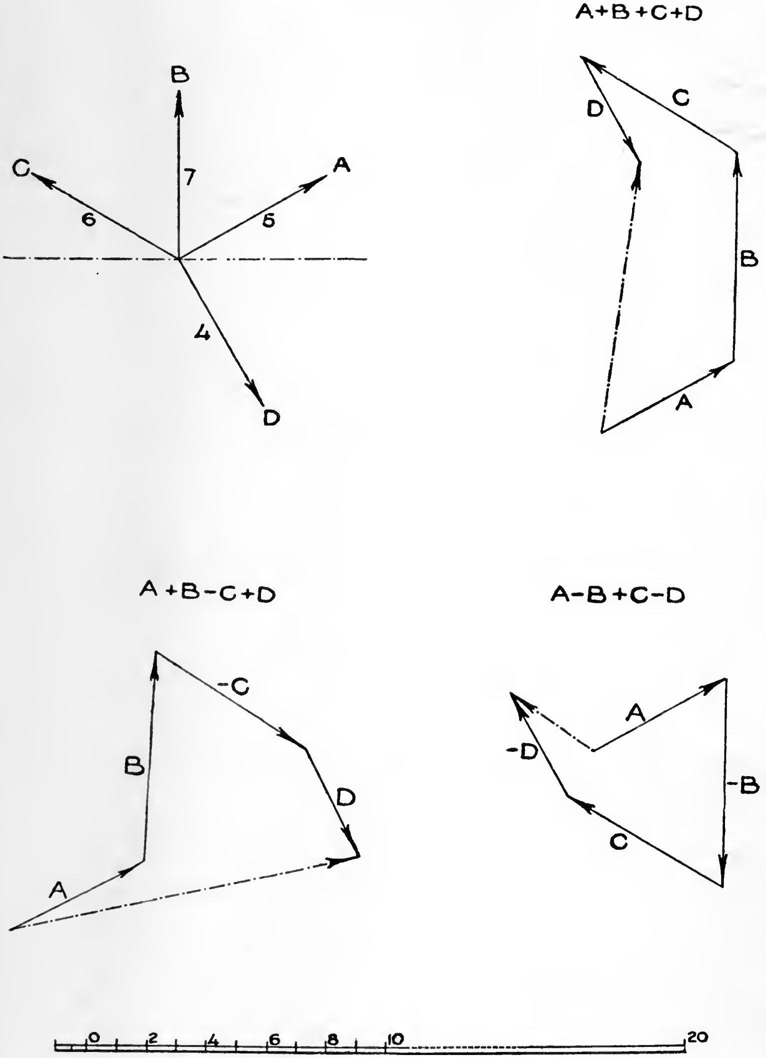


FIG. 143.

It should be noticed that in the second case the sense of C is changed, while in the third case the senses of B and D are changed, in their respective vector polygons.

209. *Addition and Subtraction of Vectors by Resolution.* If two vectors have the same line of action, their magnitudes can be added or subtracted according as their senses are like or unlike. This enables us to add and subtract vectors by resolving them in the horizontal and vertical directions, and then finding the algebraic sum of their horizontal and vertical components. Referring to the system of vectors in the previous paragraph, the horizontal components are $\rho_1 \cos \theta_1$, $\rho_2 \cos \theta_2$, $\rho_3 \cos \theta_3$, $\rho_4 \cos \theta_4$ . . . and since all these have the same line of action they can be reduced to one vector, of magnitude H, in the horizontal direction

$$\text{and} \quad H = \rho_1 \cos \theta_1 + \rho_2 \cos \theta_2 + \rho_3 \cos \theta_3 + \rho_4 \cos \theta_4 + \ldots$$

The vertical components are $\rho_1 \sin \theta_1$, $\rho_2 \sin \theta_2$, $\rho_3 \sin \theta_3$, $\rho_4 \sin \theta_4$ . . . and since all these have the same line of action, they can be reduced to one vector, of magnitude V, in the vertical direction

$$\text{and} \quad V = \rho_1 \sin \theta_1 + \rho_2 \sin \theta_2 + \rho_3 \sin \theta_3 + \rho_4 \sin \theta_4 + \ldots$$

Thus the whole system is reduced to the sum of two vectors: one, of magnitude H, acting horizontally; the other, of magnitude V, acting vertically. On applying the parallelogram law, the parallelogram becomes a rectangle.

Then $$\rho = \sqrt{H^2 + V^2}$$

and $$\theta = \tan^{-1} \frac{V}{H}$$

where ρ is the magnitude and θ the direction of the resulting sum of all the vectors.

If one of the vectors is to be subtracted, then the algebraic signs of its horizontal and vertical components must be changed. Taking the same example as in the previous paragraph and working it in this manner—

Vector.	Magnitude.	Direction.	H.		V.	
A	5	30°	5 cos 30°	4·330	5 sin 30°	2·500
B	7	90°	7 cos 90°	0	7 sin 90°	7·000
C	6	150°	6 cos 150°	−5·196	6 sin 150°	3·000
D	4	300°	4 cos 300°	2·000	4 sin 300°	−3·464

(1) To find A + B + C + D

$$H = 4{\cdot}330 - 5{\cdot}196 + 2{\cdot}000$$
$$= 1{\cdot}134$$
$$V = 2{\cdot}500 + 7{\cdot}000 + 3{\cdot}000 - 3{\cdot}464$$
$$= 9{\cdot}036$$
$$\rho = \sqrt{1{\cdot}134^2 + 9{\cdot}036^2}$$
$$= 9{\cdot}109$$
$$\theta = \tan^{-1}\frac{9{\cdot}036}{1{\cdot}134}$$
$$= 82^\circ\ 52'$$

(2) To find A + B − C + D

$$H = 4{\cdot}330 + 5{\cdot}196 + 2{\cdot}000$$
$$= 11{\cdot}526$$
$$V = 2{\cdot}500 + 7{\cdot}000 - 3{\cdot}000 - 3{\cdot}464$$
$$= 3{\cdot}036$$
$$\rho = \sqrt{11{\cdot}526^2 + 3{\cdot}036^2}$$
$$= 11{\cdot}92$$
$$\theta = \tan^{-1}\frac{3{\cdot}036}{11{\cdot}526}$$
$$= 14^\circ\ 46'$$

(3) To find A − B + C − D

$$H = 4{\cdot}330 - 5{\cdot}196 - 2{\cdot}000$$
$$= -2{\cdot}866$$
$$V = 2{\cdot}500 - 7{\cdot}000 + 3{\cdot}000 + 3{\cdot}464$$
$$= 1{\cdot}964$$
$$\rho = \sqrt{2{\cdot}866^2 + 1{\cdot}964^2}$$
$$= 3{\cdot}474$$
$$\theta = \tan^{-1}\frac{1{\cdot}964}{-2{\cdot}866}$$
$$= 145^\circ\ 35'$$

210. So far the work has been confined to vectors which are in the same plane. In dealing with vectors which are not in the same plane, the line of action of such a vector must be determined by means of the three angles it makes with the axes OX, OY, and OZ respectively.

Thus in space, a vector is specified completely by

(1) O, the point of application.

(2) The line of action which makes angles α, β, and γ with the axes OX, OY, and OZ respectively.

(3) A length OP measured from O along the line of action and equal to ρ, the magnitude of the vector.

(4) The sense ; this is positive if the vector acts away from O and negative if the vector acts towards O.

Only two angles need be given since the third can always be found from the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Thus in dealing with a system of vectors in space, having a common point of application,

If	ρ_1, ρ_2, ρ_3 . . .	are the magnitudes
	$\alpha_1, \alpha_2, \alpha_3$ . . .	the angles made with the axis of x
or	l_1, l_2, l_3 . . .	the corresponding direction cosines
	$\beta_1, \beta_2, \beta_3$ . . .	the angles made with the axis of y
or	m_1, m_2, m_3 . . .	the corresponding direction cosines
	$\gamma_1, \gamma_2, \gamma_3$ . . .	the angles made with the axis of z
or	n_1, n_2, n_3 . . .	the corresponding direction cosines

The components in the direction OX are $\rho_1 \cos \alpha_1$, $\rho_2 \cos \alpha_2$, $\rho_3 \cos \alpha_3$ . . . and since all these have the same line of action they can be reduced to one vector, of magnitude X, in the direction OX,

and
$$\begin{aligned} X &= \rho_1 \cos \alpha_1 + \rho_2 \cos \alpha_2 + \rho_3 \cos \alpha_3 \ldots \\ &= l_1\rho_1 + l_2\rho_2 + l_3\rho_3 + \ldots \end{aligned}$$

Similarly, resolving the vectors in the direction OY, and if Y is the algebraic sum of the components in that direction,

then
$$\begin{aligned} Y &= \rho_1 \cos \beta_1 + \rho_2 \cos \beta_2 + \rho_3 \cos \beta_3 \ldots \\ &= m_1\rho_1 + m_2\rho_2 + m_3\rho_3 + \ldots \end{aligned}$$

Also resolving the vectors in the direction OZ, and if Z is the algebraic sum of the components in that direction,

then
$$\begin{aligned} Z &= \rho_1 \cos \gamma_1 + \rho_2 \cos \gamma_2 + \rho_3 \cos \gamma_3 \ldots \\ &= n_1\rho_1 + n_2\rho_2 + n_3\rho_3 + \ldots \end{aligned}$$

Thus the whole system can be reduced to one of three vectors whose magnitudes are X, Y, Z, and whose lines of action are the three axes of reference, OX, OY, and OZ. If ρ is the magnitude of the vector sum, and α, β, and γ the angles it makes with the axes of reference,

then
$$\rho = \sqrt{X^2 + Y^2 + Z^2}$$
$$\alpha = \cos^{-1}\frac{X}{\rho}$$
$$\beta = \cos^{-1}\frac{Y}{\rho}$$
$$\gamma = \cos^{-1}\frac{Z}{\rho}$$

Example. A, B, and C are three vectors whose magnitudes and directions in space are given in the table below. Find A + B + C and A − B + C.

Vector	Magnitude	α	β
A	5	30°	70°
B	4	65°	32°
C	6	45°	63°

		l	m	n	$l\rho$	$m\rho$	$n\rho$
A	5	0·8660	0·3420	0·3648	4·3300	1·7100	1·8240
B	4	0·4226	0·8480	0·3200	1·6904	3·3920	1·2800
C	6	0·7071	0·4540	0·5421	4·2426	2·7240	3·2526

(1) To find A + B + C

$$\begin{aligned} X &= 4{\cdot}330 + 1{\cdot}6904 + 4{\cdot}2426 \\ &= 10{\cdot}263 \end{aligned}$$
$$\begin{aligned} Y &= 1{\cdot}7100 + 3{\cdot}3920 + 2{\cdot}7240 \\ &= 7{\cdot}8260 \end{aligned}$$
$$\begin{aligned} Z &= 1{\cdot}8240 + 1{\cdot}2800 + 3{\cdot}2526 \\ &= 6{\cdot}3566 \end{aligned}$$
$$\begin{aligned} \rho &= \sqrt{10{\cdot}263^2 + 7{\cdot}8260^2 + 6{\cdot}3566^2} \\ &= 14{\cdot}38 \end{aligned}$$

$$\alpha = \cos^{-1}\frac{10{\cdot}263}{14{\cdot}38}$$
$$= 44^\circ\ 30'$$
$$\beta = \cos^{-1}\frac{7{\cdot}826}{14{\cdot}38}$$
$$= 57^\circ\ 2'$$
$$\gamma = \cos^{-1}\frac{6{\cdot}357}{14{\cdot}38}$$
$$= 63^\circ\ 50'$$

(2) To find A − B + C

$$X = 4{\cdot}3300 - 1{\cdot}6904 + 4{\cdot}2426$$
$$= 6{\cdot}8822$$
$$Y = 1{\cdot}7100 - 3{\cdot}3920 + 2{\cdot}7240$$
$$= 1{\cdot}0420$$
$$Z = 1{\cdot}8240 - 1{\cdot}2800 + 3{\cdot}2526$$
$$= 3{\cdot}7966$$
$$\rho = \sqrt{6{\cdot}8822^2 + 1{\cdot}0420^2 + 3{\cdot}7966^2}$$
$$= 7{\cdot}929$$
$$\alpha = \cos^{-1}\frac{6{\cdot}882}{7{\cdot}929}$$
$$= 29^\circ\ 47'$$
$$\beta = \cos^{-1}\frac{1{\cdot}042}{7{\cdot}929}$$
$$= 82^\circ\ 27'$$
$$\gamma = \cos^{-1}\frac{3{\cdot}797}{7{\cdot}929}$$
$$= 60^\circ\ 53'$$

211. *The Scalar Product of two Vectors.* The scalar product of two vectors, taken in a given direction, is the product of the effective parts of the vectors in that direction; that is, the algebraic product of the components of the vectors in that direction.

Taking two vectors, one of magnitude ρ_1 and direction θ_1; the other of magnitude ρ_2 and direction θ_2.

Let OX (Fig. 144) be the line of reference, $P_1\widehat{O}X = \theta_1$, $P_2\widehat{O}X = \theta_2$ and $P\widehat{O}X = \theta$. Also let $OP_1 = \rho_1$ and $OP_2 = \rho_2$.

Then $$ON_1 = \rho_1 \cos(\theta_1 - \theta)$$
and $$ON_2 = \rho_2 \cos(\theta_2 - \theta)$$
are the components of the vectors taken in the direction OP.

The scalar product $= ON_1 \times ON_2$

$$= \rho_1\rho_2 \cos(\theta_1 - \theta)\cos(\theta_2 - \theta)$$

$$= \frac{1}{2}\rho_1\rho_2\{\cos(\theta_2 + \theta_1 - 2\theta) + \cos(\theta_2 - \theta_1)\}$$

This is evidently a maximum when

$$\cos(\theta_1 + \theta_2 - 2\theta) = 1$$

or $$\theta = \frac{1}{2}(\theta_1 + \theta_2)$$

That is when the line **OP** bisects the angle between the vectors.

The maximum value $= \frac{1}{2}\rho_1\rho_2\{1 + \cos(\theta_2 - \theta_1)\}$

$$= \rho_1\rho_2 \cos^2\frac{\theta_2 - \theta_1}{2}$$

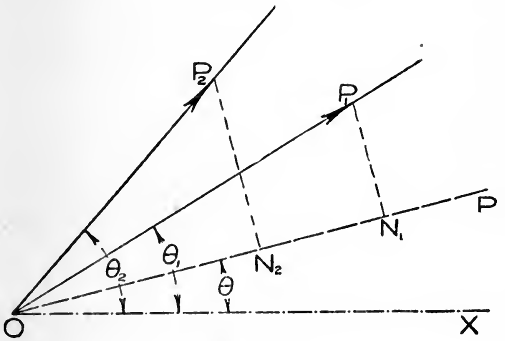


FIG. 144.

Also if $\theta = \theta_1$, that is, when **OP** coincides with the line of action of the first vector,

the scalar product $= \frac{1}{2}\rho_1\rho_2\{\cos(\theta_2 - \theta_1) + \cos(\theta_2 - \theta_1)\}$

$$= \rho_1\rho_2\cos(\theta_2 - \theta_1)$$

and if $\theta = \theta_2$, that is, when **OP** coincides with the line of action of the second vector,

the scalar product $= \frac{1}{2}\rho_1\rho_2\{\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_1)\}$

$$= \rho_1\rho_2\cos(\theta_2 - \theta_1)$$

Hence if the scalar product is taken in the direction of one or other of the two vectors, it becomes the " product of the magnitudes of the vectors and the cosine of the angle between them." This is the definition of the scalar product as applied to actual practice.

If, for example, a body is made to move with uniform velocity v feet per second, in a given direction, under the action of a force F lb., the line of action of this force making an angle α with the direction of motion,

The displacement of the body per second $= v$ ft.

The work done per second is the scalar product of the force and the displacement, taken in the direction of the displacement.

Hence the work done per sec. $= \mathrm{F}\, v \cos \alpha$ ft. lb.

212. *The Mathematical Representation of a Vector.* A vector can be considered to be the vector sum of its horizontal and vertical components, and the parallelogram of vectors used to find this sum becomes a rectangle.

It has already been shown in paragraph 31 that a complex quantity can be represented graphically by a magnitude measured in the direction of real quantities, that is, the horizontal direction, and a magnitude measured in the direction of imaginary quantities, that is, the vertical direction. The two directions, then, in which real and imaginary quantities are measured correspond to the two directions in which the horizontal and vertical components of a vector are taken. A vector can therefore be represented mathematically by a trigonometrical complex quantity, in which the real part represents the horizontal component of the vector, while the imaginary part represents the vertical component.

Thus $$\text{the vector} = \rho\,(\cos\theta + i\sin\theta)$$

where ρ is the magnitude and θ the direction.

It should be noticed that if this is recognised as a standard expression for a vector, then all vectors should ultimately reduce to this form, and this enables us to test whether this is a suitable expression for a vector. A velocity is a vector for which both magnitude and direction can be functions of the time; an acceleration is also a vector for which magnitude and direction can be functions of the time, but an acceleration is the direct result of differentiating a velocity with respect to the time.

Thus let v be a velocity, whose magnitude ρ and direction θ are both functions of t, the time.

Then $v = \rho\,(\cos\theta + i\sin\theta)$

$$\frac{dv}{dt} = \frac{d\rho}{dt}(\cos\theta + i\sin\theta) + \rho\frac{d\theta}{dt}(-\sin\theta + i\cos\theta)$$

$$= \left(\frac{d\rho}{dt}\cos\theta - \rho\frac{d\theta}{dt}\sin\theta\right) + i\left(\frac{d\rho}{dt}\sin\theta + \rho\frac{d\theta}{dt}\cos\theta\right)$$

$$= (\mathrm{A}\cos\theta - \mathrm{B}\sin\theta) + i\,(\mathrm{A}\sin\theta + \mathrm{B}\cos\theta)$$

where $$\mathrm{A} = \frac{d\rho}{dt} \text{ and } \mathrm{B} = \rho\frac{d\theta}{dt}$$

Now $$\frac{dv}{dt} = \sqrt{A^2+B^2}\left\{\left(\cos\theta\,\frac{A}{\sqrt{A^2+B^2}} - \sin\theta\,\frac{B}{\sqrt{A^2+B^2}}\right)\right.$$
$$\left. + i\left(\sin\theta\,\frac{A}{\sqrt{A^2+B^2}} + \cos\theta\,\frac{A}{\sqrt{A^2+B^2}}\right)\right\}$$
$$= \sqrt{A^2+B^2}\{(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$+ i(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\}$$
$$= \sqrt{A^2+B^2}\{\cos(\theta+\alpha) + i\sin(\theta+\alpha)\}\text{ where }\tan\alpha = \frac{B}{A}$$

and this gives the acceleration in the recognised form for a vector.

Hence if a velocity is a vector of magnitude ρ and direction θ, then the corresponding acceleration is also a vector of magnitude $\sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\theta}{dt}\right)^2}$ and direction $(\theta+\alpha)$ where $\tan\alpha = \rho\frac{d\theta}{dt}\Big/\frac{d\rho}{dt}$.

The angle α is evidently the angle between the direction of the velocity and the direction of the acceleration corresponding to that velocity.

Taking the case of a body describing a circular path, of radius r feet, with uniform velocity v feet per second. This velocity can be expressed as a vector of magnitude ρ and direction θ and $\rho = v$ a constant and $\theta = \frac{vt}{r}$.

Then $$\frac{d\rho}{dt} = 0 \text{ and } \frac{d\theta}{dt} = \frac{v}{r}$$

also $$\rho\frac{d\theta}{dt} = \frac{v^2}{r}$$

Now the velocity $= \rho(\cos\theta + i\sin\theta)$

and the acceleration $= \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\theta}{dt}\right)^2}\{\cos(\theta+\alpha) + i\sin(\theta+\alpha)\}$

Now $$\sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\theta}{dt}\right)^2} = \frac{v^2}{r}$$

and $$\tan\alpha = \rho\frac{d\theta}{dt}\Big/\frac{d\rho}{dt}$$
$$= \infty \quad\text{since } \frac{d\rho}{dt} = 0$$

Hence $$\alpha = 90°.$$

Thus the acceleration $= \frac{v^2}{r}\{\cos(\theta+90°) + i\sin(\theta+90°)\}$

That is, the acceleration is of magnitude $\frac{v^2}{r}$ and its direction is at right angles to the direction of the velocity. Now the direction of the velocity at any instant is along a tangent, hence the direction of the acceleration is along the corresponding radius. Thus if a body describes a circular path of radius r ft., with uniform velocity v ft. per sec., the acceleration is $\frac{v^2}{r}$ ft. per sec.^{2}, and is directed towards the centre.

213. *Example* 1. The value of a vector may be stated as a_θ where a is the amount and θ is the angle measured anti-clockwise from a found direction. The vector keeps in a plane. A point has the following velocities in feet per second at the following times (seconds). (B. of E., 1911.)

Velocity .	$100_{30°}$	$103{\cdot}3_{35°}$	$105{\cdot}7_{42°}$	$107{\cdot}2_{51°}$	$107{\cdot}8_{62°}$
Time .	10	10·01	10·02	10·03	10·04

Find approximately the value of the acceleration when $t = 10{\cdot}02$.

Now velocity $= a\,(\cos\theta + i\sin\theta)$

and acceleration $= \sqrt{\left(\frac{da}{dt}\right)^2 + \left(a\frac{d\theta}{dt}\right)^2}\{\cos(\theta + \alpha) + i\sin(\theta + \alpha)\}$

where $$\tan\alpha = a\frac{d\theta}{dt}\Big/\frac{da}{dt}$$

t	a	$\theta°$	δt	δa	$\delta\theta$	$\frac{\delta a}{\delta t}$	$\frac{\delta\theta}{\delta t}$
10·00	100	30					
			0·01	3·3	5	330	500
10·01	103·3	35					
			0·01	2·4	7	240	700
10·02	105·7	42					
			0·01	1·5	9	150	900
10·03	107·2	51					
			0·01	0·6	11	60	1100
10·04	107·8	62					

It is evident that when $t = 10{\cdot}02$ $\dfrac{da}{dt} = 195.$

Also $$\frac{d\theta}{dt} = \frac{800\pi}{180} = 13{\cdot}96$$

the angle being expressed in radians,

$$a\frac{d\theta}{dt} = 105{\cdot}7 \times 13{\cdot}96 = 1476$$

Then $$\sqrt{\left(\frac{da}{dt}\right)^2 + \left(a\frac{d\theta}{dt}\right)^2} = \sqrt{195^2 + 1476^2}$$

$$= 1489$$

$$\tan\alpha = \frac{1476}{195} = 7{\cdot}568$$

$$\alpha = 82^\circ\ 28'$$

Thus when $t = 10{\cdot}02$ the magnitude of the acceleration is 1489 ft. per sec. per sec., and its direction makes an angle of 82° 28′ with the direction of the corresponding velocity.

It also makes an angle of 124° 28′ with the fixed line.

Example 2. The velocity of a body is continually changing in direction and magnitude. The values given in the table below give ρ, the magnitude, and θ, the direction measured from a fixed line, at any time t seconds. Find the value of the acceleration when $t = 5{\cdot}3$.

t	5·0	5·1	5·2	5·3	5·4	5·5	5·6	5·7
ρ	10	13·7	18·7	25·2	33·4	43·5	55·7	70·2
θ	0	4·2°	10·8°	20·6°	34·0°	53·0°	77·2°	107·8°

t	ρ	$\delta\rho$	$\delta^2\rho$	$\delta^3\rho$	$\delta^4\rho$
5·0	10				
		3·7			
5·1	13·7		1·3		
		5·0		0·2	
5·2	18·7		1·5		0
		6·5		0·2	
5·3 —	25·2		1·7		0
		8·2		0·2	
5·4	33·4		1·9		0
		10·1		0·2	
5·5	43·5		2·1		0
		12·2		0·2	
5·6	55·7		2·3		
		14·5			
5·7	70·2				

When $$t = 5\cdot3 \quad 0\cdot1\,\frac{d\rho}{dt} = 6\cdot5 + \frac{1\cdot5}{2} + \frac{0\cdot2}{3}$$

or $$\frac{d\rho}{dt} = 73\cdot17$$

also $$0\cdot1\,\frac{d\rho}{dt} = 8\cdot2 - \frac{1\cdot9}{2} + \frac{0\cdot2}{3}$$

$$\frac{d\rho}{dt} = 73\cdot17$$

t	θ	$\delta\theta$	$\delta^2\theta$	$\delta^3\theta$	$\delta^4\theta$
5·0	0				
		4·2			
5·1	4·2		2·4		
		6·6		0·8	
5·2	10·8		3·2		0
		9·8		0·8	
5·3	20·6		4·0		0
		13·8		0·8	
5·4	34·4		4·8		0
		18·6		0·8	
5·5	53·0		5·6		0
		24·2		0·8	
5·6	77·2		6·4		
		30·6			
5·7	107·8				

When $$t = 5\cdot3 \quad 0\cdot1\frac{d\theta}{dt} = 9\cdot8 + \frac{3\cdot2}{2} + \frac{0\cdot8}{3}$$

$$\frac{d\theta}{dt} = 116\cdot7$$

also $$0\cdot1\,\frac{d\theta}{dt} = 13\cdot8 - \frac{4\cdot8}{2} + \frac{0\cdot8}{3}$$

$$= 116\cdot7$$

Expressing the angle in radians,

$$\frac{d\theta}{dt} = \frac{116\cdot7\pi}{180} = 2\cdot036$$

and $$\rho\,\frac{d\theta}{dt} = 25\cdot2 \times 2\cdot036$$

$$= 51\cdot33$$

Now velocity $= \rho\,(\cos\theta + i\sin\theta)$

and acceleration $= \sqrt{\left(\dfrac{d\rho}{dt}\right)^2 + \left(\rho\,\dfrac{d\theta}{dt}\right)^2}\,\{\cos(\theta+\alpha) + i\sin(\theta+\alpha)\}$

where $$\tan\alpha = \rho\,\frac{d\theta}{dt}\bigg/\frac{d\rho}{dt}$$

$$\sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\,\frac{d\theta}{dt}\right)^2} = \sqrt{73{\cdot}17^2 + 51{\cdot}33^2}$$

$$= 89{\cdot}37$$

$$\tan\alpha = \frac{51{\cdot}33}{73{\cdot}17}$$

$$\alpha = 35^\circ\ 3'$$

Hence when $t = 5{\cdot}3$ the magnitude of the acceleration $= 89{\cdot}37$ f.s.s., and the direction $= 20^\circ\ 36' + 35^\circ\ 3' = 55^\circ\ 39'$.

214. *The Multiplication of Vectors.* If A is a vector of magnitude ρ_1 and direction θ_1

then $\quad A = \rho_1\,(\cos\theta_1 + i\sin\theta_1)$.

Also if B is a vector of magnitude ρ_2 and direction θ_2

then $\quad B = \rho_2\,(\cos\theta_2 + i\sin\theta_2)$.

The product $AB = \rho_1\rho_2\,(\cos\theta_1 + i\sin\theta_1)\,(\cos\theta_2 + i\sin\theta_2)$

$$= \rho_1\rho_2\,\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$
$$= \rho_1\rho_2\,\{\cos(\alpha+2\theta_1) + i\sin(\alpha+2\theta_1)\}$$

where $\alpha = \theta_2 - \theta_1$, the angle between the vectors, and θ_1 is the inclination of the line of action of vector A with the initial line.

If $\theta_1 = 0$, that is, the initial line is so chosen that it coincides with the line of action of vector A,

then the product $\quad AB = \rho_1\rho_2\,(\cos\alpha + i\sin\alpha)$.

This is an expression in the form of a complex quantity, the real part being $\rho_1\rho_2\cos\alpha$ and the imaginary part $\rho_1\rho_2\sin\alpha$. Now the imaginary part represents a quantity which must be measured in a direction perpendicular to the direction in which the corresponding real part is measured. It has already been shown that the real part, the product $\rho_1\rho_2\cos\alpha$, is the scalar product of the two vectors taken in the direction of either of the vectors. Hence the product $\rho_1\rho_2\sin\alpha$ must be taken in a direction which is perpendicular to the lines of action of the two vectors. That is, this product must be taken in a direction which is perpendicular to the plane containing the lines of action of the two vectors.

Thus the effect of multiplying two vectors is to give rise to two distinct products:

(1) The product $\rho_1\rho_2 \cos\alpha$; this is the "scalar product," and must be taken in a direction corresponding to either of the lines of action of the vectors.

(2) The product $\rho_1\rho_2 \sin\alpha$; this is the "vector product," and must be taken in a direction perpendicular to the plane containing the lines of action of the vectors.

Examples XXIV

(1) A and B are two vectors; if $A = 8_{32^\circ}$ and $B = 5_{77^\circ}$, find (1) A + B and (2) A − B.

(2) A and B are two vectors; if $A = 13_{57^\circ}$ and $B = 22_{231^\circ}$, find (1) A + B and (2) A − B.

(3) Find the components of a vector of magnitude 12 along directions which make angles of 25° and 55° with the line of action of the vector.

(4) A and B are two vectors; if $A = 12_{35^\circ}$, find B so that $A + B = 17_{50^\circ}$.

(5) A and B are two vectors: if $A = 14_{40^\circ}$, find B so that $A - B = 6_{15^\circ}$.

A, B, C, and D are four vectors whose magnitudes and directions are given in the table below:

	ρ	θ
A	7	38°
B	3	84°
C	9	128°
D	5	294°

(6) Find A + B + C + D.

(7) Find A − B + C − D.

(8) Find A − B − C + D.

(9) Working with the three vectors A, B, and C given in the example of paragraph 210. Find B + C − A.

A, B, and C are three vectors whose magnitudes and positions in space are given in the table below:

	ρ	α	β
A	8	30°	75°
B	13	70°	45°
C	10	55°	55°

(10) Find A + B + C.

(11) Find B + C − A.

(12) Find A + B − C.

(13) Working with the values given in Example 1 of paragraph 213, find the acceleration when $t = 10{\cdot}03$.

(14) Working with the values given in Example 2 of paragraph 213, find the acceleration when $t = 5{\cdot}4$.

(15) A body is moving in such a way that the velocity at any instant is given by $v = a + bt$, and the direction is given by θ (radians) $= \alpha + \beta t$, where t is time in seconds and a, b, α and β are constants. Find the magnitude and direction of the acceleration at any instant.

(16) A vector α is changing in direction and magnitude; what is $\frac{d\alpha}{dt}$ if t is time? Illustrate this by one example, say, by centripetal acceleration of a point moving with constant speed in a circular path. (B. of E., 1909.)

(17) A body is moving in such a way that the velocity at any instant is given by $v = a + bt$, and the direction is given by θ (radians) $= \beta t^2$, where t is time in seconds and a, b, and β are constants. Find the magnitude and direction of the acceleration at any instant.

(18) The scalar product of two vectors is 12·74 and the vector product is 15·76. Find the angle between the vectors. If the magnitude of one of the vectors is 5·6, what is the magnitude of the other?

(19) Fifty pounds of shot per second moving horizontally with a velocity of 2,500 feet per second due north strike an armour-plate and leave the plate horizontally with a velocity of 800 feet per second due east. What force is exerted on the plate? Note that momentum and force are vectors.

Force is rate of change of momentum per second.

Momentum is mass multiplied by velocity.

The mass of 50 lb. of shot is $50 \div 32{\cdot}2$. (B. of E., 1906.)

(20) An aeroplane which is propelled at a speed of 50 miles per hour relatively to the air, is steered in a circular course during a steady wind of 15 miles per hour from the south. What are the actual speeds of the aeroplane when going north, south, east, and west?

(21) Rework Question 20. For a steady wind of 20 miles per hour from the north-east.

CHAPTER XXV

215. *The Straight Line Law.*

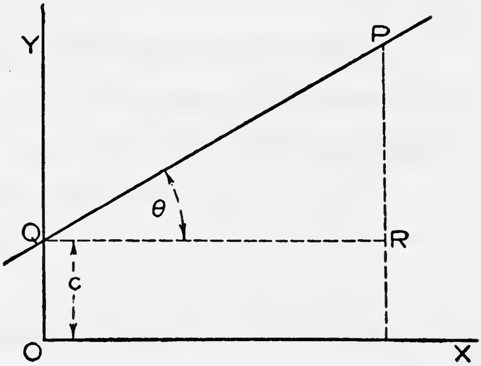


FIG. 145.

Let **PQ** (Fig. 145) be any straight line, **Q** being the point where this line cuts the axis of y; let $OQ = c$, when c is positive this point is above the origin and when c is negative the point **Q** is below the origin.

Let θ be the inclination of the line to the axis of x, the slope of the line is therefore $\tan\theta$; let this be denoted by m.

If **P** is any point on the line, its co-ordinates being (x, y); then by drawing **PR** parallel to the axis of y, and **QR** parallel to the axis of x, the right-angled triangle **PQR** is produced.

Then $$\frac{PR}{QR} = \tan\theta$$

or $$\frac{y - c}{x} = m$$

Hence $$y = mx + c.$$

This is the general equation of a straight line, and m and c are constants for any particular straight line. A straight line will be completely determined if the numerical values of m and c are found.

In dealing with questions on the straight line law, it is well to give to the quantities x and y their most general meaning. x represents the quantity plotted horizontally, and y the quantity plotted vertically. For greater generality the straight line law could be stated as

$$V = mH + c$$

where V is the quantity plotted vertically, and H the quantity plotted horizontally. It does not matter in what way the quantities V and H have been derived, or what form they take, but if the above relation connects them, and m and c are constants, then a straight line must be the result of plotting V vertically and H horizontally. Thus, for example, if a set of tabular values of x and y is given, the quantity V can be derived in some way from the values of both x and y, and this also can be the case for the quantity H; there can be a straight line law connecting V and H, although there need not necessarily be a straight line law connecting y and x.

216. *The Determination of the Constants.* If a straight line has been obtained by plotting some quantity V vertically and another quantity H horizontally, this line can be expressed in the form of an algebraic law connecting V and H if the numerical values of the constants m and c are found. The values of the constants can be found in two different ways.

(*a*) It has already been shown that m represents the slope of the line and c is the distance, above or below the origin, of the point of intersection of the line and the axis of y. Thus the values of these constants can be found from these statements. If *the origin is accessible* the point of intersection of the line and the axis of y can be found and the value of c can be read off along the axis of y. The slope m can be measured in the usual way. Take two points A and B on the line, make AB the hypotenuse of a right angled triangle, the base of which is parallel to the axis of x. Let the perpendicular of this triangle be measured by means of the vertical scale and the base measured by means of the horizontal scale,

then $$m = \frac{\text{perpendicular}}{\text{base}}$$

Generally speaking, this is not the best way of determining the values of the constants, because in actual practice it need not be necessary, except in a very few cases, to work from the origin, and if *the origin is inaccessible* the value of the constant c cannot be found directly. Then the constants can be found by solving a pair of simultaneous equations, and this gives rise to the second way.

(*b*) Take two points on the line as far removed as the range of values permits; it is unwise to work outside this range. If

$V = mH + c$ is the equation of the line, and the co-ordinates of the two selected points are (H_1, V_1) and (H_2, V_2) respectively

then for the first point $\quad V_1 = mH_1 + c$
and for the second point $\quad V_2 = mH_2 + c.$

Solving these two equations for m and c,

$$m = \frac{V_2 - V_1}{H_2 - H_1}$$

and

$$c = \frac{V_1H_2 - V_2H_1}{H_2 - H_1}$$

217. In actual practice the work with the straight line law can be considered in the following way. A set of tabular values of two varying quantities, x and y, is given ; a probable law connecting x and y is known or assumed, and it is necessary to prove that the given tabular values do actually satisfy the law. This can only be done by obtaining a straight line, and, in order to do this, the probable law must be so changed or adapted that it reduces to a form which can be compared with the general straight line law $V = mH + c$.

This adaptation is a very simple matter, for in the general straight line law the constant c stands alone. Hence, before the probable law can be compared with the straight line law, it must be so changed that it contains *an isolated constant.* This process of *isolation of constant* is very easily done by division.

Thus, for example, if a given set of tabular values of x and y are supposed to follow the law $y = ax^2 + b \log_{10}x$, the constant a can be isolated by dividing throughout by x^2, and the law becomes

$$\frac{y}{x^2} = b\,\frac{\log_{10}x}{x^2} + a$$

Comparing this with the straight line law $V = mH + c$, it follows that if $V = \dfrac{y}{x^2}$ be plotted vertically, and $H = \dfrac{\log_{10}x}{x^2}$ be plotted horizontally, and the result is a straight line, then the given tabular values of x and y will follow the law.

Again, the constant b can be isolated by dividing throughout by $\log_{10}x$, and the law becomes

$$\frac{y}{\log_{10}x} = a\,\frac{x^2}{\log_{10}x} + b$$

Comparing this with the straight line law $V = mH + c$, it follows that if $V = \dfrac{y}{\log_{10}x}$ be plotted vertically, and $H = \dfrac{x^2}{\log_{10}x}$ be plotted horizontally, and the result is a straight line, then the given tabular values of x and y will follow the law. The values of the constants a and b can be found from the straight line which proves the law.

218. *Example.* The following values of x and y are supposed to follow the law $y = ax^2 + b \log_{10} x$. Test if this is so, and if so, find the most probable values of the constants a and b.

x	2·86	3·88	4·66	5·69	6·65	7·77	8·67
y	16·7	26·4	35·1	47·5	60·6	77·5	93·4

(*a*) Dividing throughout by x^2 the law becomes

$$\frac{y}{x^2} = a + b\frac{\log_{10} x}{x^2}$$

Hence, plotting $\frac{y}{x^2}$ vertically and $\frac{\log_{10} x}{x^2}$ horizontally will give a straight line if the values follow the law.

$\frac{y}{x^2}$	2·041	1·754	1·616	1·468	1·371	1·283	1·243
$\frac{\log_{10} x}{x^2}$	0·0558	0·03912	0·03078	0·02332	0·01861	0·01475	0·01248

Fig. **146** shows the resulting straight line.

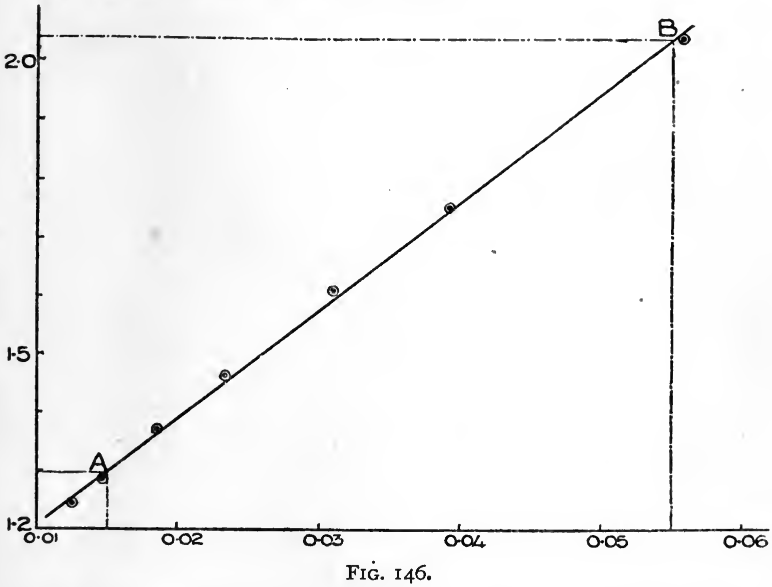


Fig. 146.

Let A and B be two points taken on this straight line.

For the point A when $\frac{y}{x^2} = 1{\cdot}297$, $\frac{\log_{10}x}{x^2} = 0{\cdot}015$

then $$a + 0{\cdot}015b = 1{\cdot}297 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the point B when $\frac{y}{x^2} = 2{\cdot}037$, $\frac{\log_{10}x}{x^2} = 0{\cdot}055$,

then $$a + 0{\cdot}055b = 2{\cdot}037 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2) $0{\cdot}04b = 0{\cdot}738$

$$b = 18{\cdot}45$$

and $$a = 1{\cdot}297 - 0{\cdot}015 \times 18{\cdot}45$$
$$= 1{\cdot}020$$

The law is $$y = 1{\cdot}02x^2 + 18{\cdot}45 \log_{10}x.$$

(*b*) Dividing throughout by $\log_{10}x$, the law becomes

$$\frac{y}{\log_{10}x} = b + a\frac{x^2}{\log_{10}x}$$

Hence, plotting $\frac{y}{\log_{10}x}$ vertically and $\frac{x^2}{\log_{10}x}$ horizontally will give a straight line if the values follow the law

$\frac{y}{\log_{10}x}$	36·59	44·83	52·50	62·91	73·65	87·04	99·56
$\frac{x^2}{\log_{10}x}$	17·93	25·57	32·48	42·87	53·74	67·79	80·13

Fig. 147 shows the resulting straight line.
Let A and B be two points taken on this straight line.

For the point A when $\frac{y}{\log_{10}x} = 38{\cdot}8$, $\frac{x^2}{\log_{10}x} = 20$,

then $$b + 20a = 38{\cdot}8 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the point B when $\frac{y}{\log_{10}x} = 100{\cdot}0$, $\frac{x^2}{\log_{10}x} = 80$,

then $$b + 80a = 100{\cdot}0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2) $60a = 61{\cdot}2$

$$a = 1{\cdot}02$$

and $$b = 38{\cdot}8 - 20 \times 1{\cdot}02$$
$$= 18{\cdot}40$$

The law is $$y = 1{\cdot}02x^2 + 18{\cdot}40 \log_{10}x.$$

219. As a further example, let us take a few laws and investigate, in each case, the treatment necessary to make them comparable with the general straight line law.

(1) $$y = a \log_{10} cx$$

Then
$$y = a\ \{\log_{10} c + \log_{10} x\}$$
$$= a \log_{10} c + a \log_{10} x$$

Hence, to obtain a straight line y must be plotted vertically and $\log_{10} x$ horizontally.

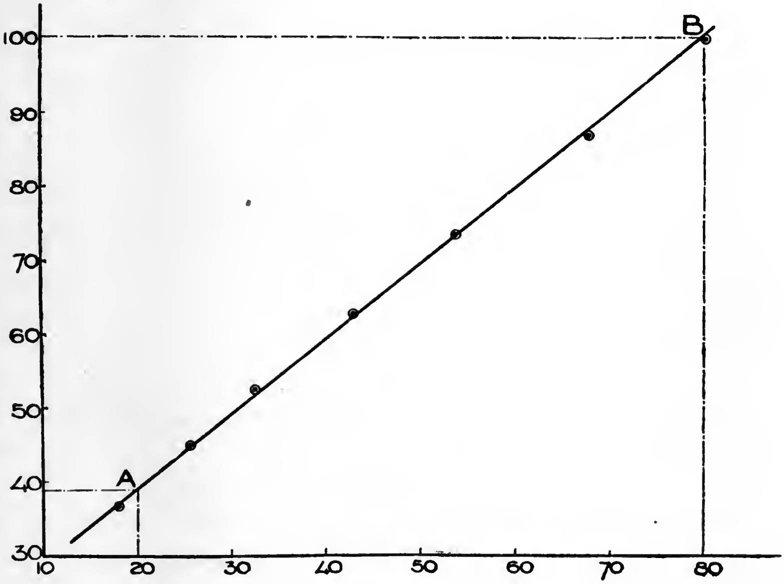


FIG. 147.

(2) $$y = ax + b \log_{10} x$$

Dividing throughout by x the law becomes

$$\frac{y}{x} = a + b\,\frac{\log_{10} x}{x}$$

Hence, to obtain a straight line $\frac{y}{x}$ must be plotted vertically, and $\frac{\log_{10} x}{x}$ horizontally.

Again dividing throughout by $\log_{10} x$ the law becomes

$$\frac{y}{\log_{10} x} = b + a\,\frac{x}{\log_{10} x}$$

Hence, to obtain a straight line $\frac{y}{\log_{10}x}$ must be plotted vertically and $\frac{x}{\log_{10}x}$ horizontally.

(3)
$$y = \frac{a}{b + x}$$

Then
$$by + xy = a$$

or
$$y = \frac{a}{b} - \frac{1}{b}\,xy$$

Hence, to obtain a straight line y must be plotted vertically and xy horizontally.

(4)
$$y = ae^{bx}$$

Taking common logarithms of both sides

$$\begin{aligned}\log_{10}y &= \log_{10}a + bx\log_{10}e\\ &= \log_{10}a + 0{\cdot}4343bx\end{aligned}$$

Hence, to obtain a straight line $\log_{10}y$ must be plotted vertically and x horizontally.

(5)
$$y = ab^{x}$$

Taking common logarithms of both sides

$$\log_{10}y = \log_{10}a + x\log_{10}b$$

Hence, to obtain a straight line $\log_{10}y$ must be plotted vertically and x horizontally.

(6)
$$y = ax^{bx}$$

Taking common logarithms of both sides

$$\log_{10}y = \log_{10}a + bx\log_{10}x$$

Hence, to obtain a straight line $\log_{10}y$ must be plotted vertically and $x\log_{10}x$ horizontally.

220. *General Determination of Laws.* In actual practice a question often arises of finding a law connecting a set of tabular values, and nothing further is known of these values except that they give a regular curve when plotted on squared paper.

It is highly probable that one of the three laws

$$y = a + bx^{n} \quad . \quad . \quad . \quad (1)$$
$$y = b\,(x + a)^{n} \quad . \quad . \quad . \quad (2)$$
$$y = a + be^{nx} \quad . \quad . \quad . \quad (3)$$

will suit the given values.

It should be noticed that these three laws are chosen as typical laws on account of the general nature of them. By adjusting the constants many simpler algebraic laws can be derived from them.

For example, the first law becomes $y = bx^n$ if $a = o$; or the second law becomes $y = \dfrac{b}{x+a}$ if $n = -1$.

The selection of the best of these three laws can easily be effected, because there is for each of them a simple relation connecting the variables, x and y, and the slope $\dfrac{dy}{dx}$.

(1) Taking the law $$y = a + bx^n$$

then $$y - a = bx^n$$

also $$\frac{dy}{dx} \text{ (or } s) = nbx^{n-1}$$

By division $$\frac{y-a}{s} = \frac{x}{n}$$

or $$y = \frac{1}{n}xs + a$$

Hence, if the tabular values of x and y suit the law, then a straight line must be the result of plotting y vertically and xs horizontally.

(2) Taking the law $$y = b(x+a)^n$$

$$\frac{dy}{dx} \text{ (or } s) = nb(x+a)^{n-1}$$

By division $$\frac{y}{s} = \frac{x+a}{n}$$

$$= \frac{1}{n}x + \frac{a}{n}$$

Hence, if the tabular values of x and y suit the law, then a straight line must be the result of plotting $\dfrac{y}{s}$ vertically and x horizontally.

(3) Taking the law $$y = a + be^{nx}$$

then $$y - a = be^{nx}$$

$$\frac{dy}{dx} \text{ (or } s) = nbe^{nx}$$

By division $$\frac{y-a}{s} = n$$

or $$y = ns + a$$

Hence, if the tabular values of x and y suit the law, then a straight line must be the result of plotting y vertically and s horizontally.

Thus the discrimination can be carried out in the following way:

Draw the curve connecting the tabular values of x and y on squared paper, and take a series of points on the curve. For each point it will be necessary to find the values of x, y, and s; the value of s can be found by drawing the tangent to the curve at the point and measuring its slope in the usual way. When a sufficient number of values has been obtained, plot the following three curves on the same sheet of squared paper, so adjusting the scales that the curves practically cover the same space.

Curve I. y vertically, xs horizontally; for the law $y = a + bx^n$.

Curve II. $\frac{y}{s}$ vertically, x horizontally; for the law $y = b(x + a)^n$.

Curve III. y vertically, s horizontally; for the law $y = a + be^{nx}$.

Then select the curve which most nearly approaches a straight line, and this will indicate the most suitable law.

This method is open to the objection that there is no way of drawing the correct tangent to a curve, but, if reasonable care is taken in drawing the tangent, the errors should not be great enough to make the discrimination impossible.

221. *Example.* The following values of x and y give a number of points on a curve obtained by bending a thin spline of wood. Find the law of the curve.

x	4	5	6	7	8	9	10	11	12	13
y	20·1	27·8	35·0	41·5	47·6	53·0	58·1	62·7	66·8	70·6

The curve was drawn to a large scale on a sheet of drawing paper, tangents drawn to the curve at the above points, and the values of the slope determined.

Tabulating the results

x	y	s	xs	$\frac{y}{s}$
4	20·1	7·98	31·92	2·56
5	27·8	7·43	37·15	3·74
6	35·0	6·75	40·50	5·19
7	41·5	6·27	43·89	6·63
8	47·6	5·74	45·92	8·31
9	53·0	5·24	47·16	10·11
10	58·1	4·83	48·30	12·02
11	62·7	4·41	48·51	14·22
12	66·8	3·87	46·44	17·24
13	70·6	3·47	43·81	20·36

Fig. 148 shows the three curves:

(1) y vertically, xs horizontally; for the law $y = a + bx^n$.

(2) $\frac{y}{s}$ vertically, x horizontally; for the law $y = b\,(x + a)^n$.

(3) y vertically, s horizontally; for the law $y = a + be^{nx}$.

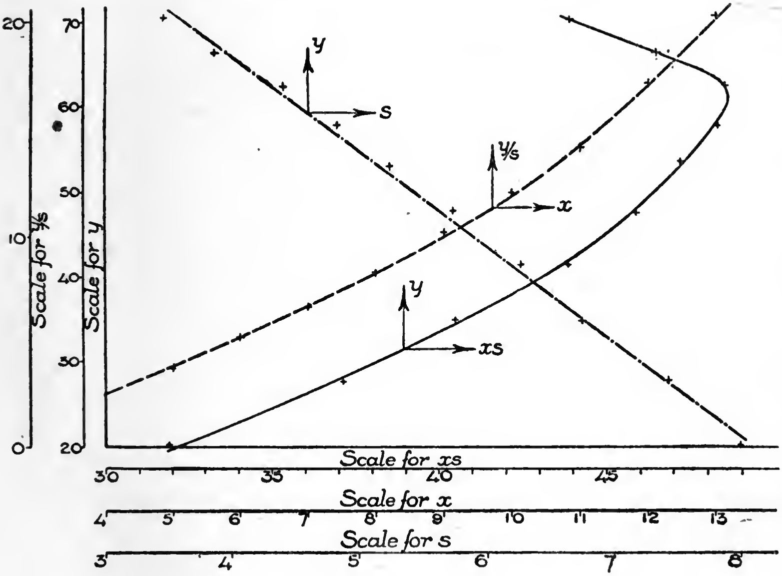


FIG. 148.

Of the three curves the third most nearly approaches a straight line, and, therefore, $y = a + be^{nx}$ is the law which suits the tabular values of x and y the best.

In order to find the constants of the law take three points on the original curve; for simplicity of calculation let these points be so chosen that the values of x are equidistant.

when $x = 4$, $y = 20{\cdot}1$ and $\quad a + be^{4n} = 20{\cdot}1 \quad . \quad . \quad . \quad (1)$

when $x = 8$, $y = 47{\cdot}6$ and $\quad a + be^{8n} = 47{\cdot}6 \quad . \quad . \quad . \quad (2)$

when $x = 12$, $y = 66{\cdot}8$ and $\quad a + be^{12n} = 66{\cdot}8 \quad . \quad . \quad . \quad (3)$

subtracting (1) from (2) $\quad be^{4n}(e^{4n} - 1) = 27{\cdot}5 \quad . \quad . \quad . \quad (4)$

subtracting (2) from (3) $\quad be^{8n}(e^{4n} - 1) = 19{\cdot}2 \quad . \quad . \quad . \quad (5)$

dividing (5) by (4) $\quad e^{4n} = \frac{19{\cdot}2}{27{\cdot}5} = 0{\cdot}6982$

and $\quad n = -0{\cdot}0898.$

from (4) $$b = \frac{27\cdot5}{e^{4n}(e^{4n}-1)}$$

$$= -\frac{27\cdot5}{0\cdot6982 \times 0\cdot3018}$$

$$= -130\cdot5$$

from (1) $$a = 20\cdot1 + 130\cdot5 \times 0\cdot6982$$

$$= 111\cdot2$$

The probable law is therefore

$$y = 111\cdot2 - 130\cdot5\, e^{-0\cdot0898x}$$

222. Another question arises out of the work of the previous paragraph; that is: knowing that a certain curve follows one or other of the laws, to find the constants of the law. In general, if a law contains three constants, then three points must be taken on the curve, and, substituting the values of the co-ordinates of these points in the law will give three equations to be solved for the three constants. The solution of these equations depends upon the way the values of x are chosen when taking the points on the curve.

Case I. For the law $y = a + bx^n$, let the values of x be so chosen that they are in Geometrical Progression; that is, they increase by a common ratio.

Let the co-ordinates of the three points be (h, y_1), (hr, y_2), and (hr^2, y_3).

Then for the first point	$a + bh^n = y_1$	. . . (1)
for the second point	$a + bh^nr^n = y_2$	. . . (2)
for the third point	$a + bh^nr^{2n} = y_3$	. . . (3)
subtracting (1) from (2)	$bh^n(r^n - 1) = y_2 - y_1$	. . . (4)
subtracting (2) from (3)	$bh^nr^n(r^n - 1) = y_3 - y_2$	. . . (5)

dividing (5) by (4) $$r^n = \frac{y_3 - y_2}{y_2 - y_1}$$

Thus giving a relation from which the value of n can be calculated, and knowing n, a and b can be found.

Example. The curve $y = a + bx^n$ passes through the three points (3, 21·47), (6, 37·09), and (12, 94·36). Find the values of the constants a, b, and n.

$$a + b\,3^n = 21\cdot47$$

$$a + b\,6^n = 37\cdot09$$

$$a + b\,12^n = 94\cdot36$$

Then $$b\,3^n(2^n - 1) = 15\cdot62$$

and $$b\,3^n\,2^n(2^n - 1) = 57\cdot27$$

Hence $$2^n = \frac{57 \cdot 27}{15 \cdot 62}$$

$$n = 1 \cdot 875$$

also $$b = \frac{15 \cdot 62}{3^n\,(2^n - 1)}$$

$$= 0 \cdot 747$$

and $$a = 21 \cdot 47 - b\,3^n$$

$$= 15 \cdot 65$$

223. *Case II.* For the law $y = b\,(x + a)^n$, the only way in which the value of a can be found is by the graphical solution of an equation, but it will simplify the calculation if very simple values of x are chosen.

Let the co-ordinates of the three points be (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Then for the first point $$b\,(x_1 + a)^n = y_1 \quad \ldots \quad (1)$$

for the second point $$b\,(x_2 + a)^n = y_2 \quad \ldots \quad (2)$$

for the third point $$b\,(x_3 + a)^n = y_3 \quad \ldots \quad (3)$$

dividing (2) by (1) $$\left(\frac{x_2 + a}{x_1 + a}\right)^n = \frac{y_2}{y_1} \quad \ldots \quad (4)$$

and $$n \log\left(\frac{x_2 + a}{x_1 + a}\right) = \log \frac{y_2}{y_1}$$

or $$n\{\log(x_2 + a) - \log(x_1 + a)\} = \log y_2 - \log y_1 \quad \ldots \quad (5)$$

dividing (3) by (2) $$\left(\frac{x_3 + a}{x_2 + a}\right)^n = \frac{y_3}{y_2}$$

or $$n\{\log(x_3 + a) - \log(x_2 + a)\} = \log y_3 - \log y_2 \quad \ldots \quad (6)$$

dividing (6) by (5) $$\frac{\log(x_3 + a) - \log(x_2 + a)}{\log(x_2 + a) - \log(x_1 + a)} = \frac{\log y_3 - \log y_2}{\log y_2 - \log y_1}$$

thus giving an equation which must be solved graphically for a.

Example. The curve $y = b\,(x + a)^n$ passes through the three points (2, 9·49), (6, 30·03), and (10, 59·70). Find the values of the constants a, b, and n.

$$b(2 + a)^n = 9 \cdot 49$$

$$b(6 + a)^n = 30 \cdot 03$$

$$b(10 + a)^n = 59 \cdot 70$$

Then $$\left(\frac{6 + a}{2 + a}\right)^n = 3 \cdot 163$$

and $$\left(\frac{10 + a}{6 + a}\right)^n = 1 \cdot 988$$

$$\text{Hence} \qquad n\{\log_{10}(6+a) - \log_{10}(2+a)\} = 0{\cdot}5002$$

$$\text{and} \qquad n\{\log_{10}(10+a) - \log_{10}(6+a)\} = 0{\cdot}2985$$

$$\frac{\log_{10}(10+a) - \log_{10}(6+a)}{\log_{10}(6+a) - \log_{10}(2+a)} = 0{\cdot}5967$$

Giving a the values 2, 2·1, 2·2, 2·3, 2·4, the values of the fraction are 0·5849, 0·5893, 0·5939, 0·5980, and 0·6023 respectively.

Fig. 149 shows the curve obtained by plotting these values with their corresponding values of a, and the point **P** is the point on the curve for which the value of the fraction is **0·5967**. This corresponds to a value for a of 2·26.

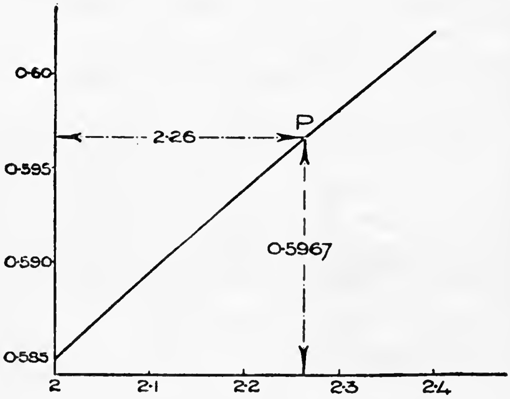


FIG. 149.

$$\text{Then } n = \frac{0{\cdot}5002}{\log_{10} 8{\cdot}26 - \log_{10} 4{\cdot}26}$$

$$= 1{\cdot}74$$

$$\text{and } b = \frac{9{\cdot}49 \times 1{\cdot}74}{4{\cdot}26}$$

$$= 0{\cdot}762.$$

224. *Case III.* For the curve $y = a + be^{nx}$ let the values of x be so chosen that they are in Arithmetical Progression; that is, they increase by the same amount.

Let the co-ordinates of the three points be (h, y_1), $(h+d, y_2)$, and $(h+2d, y_2)$.

Then for the first point	$a + be^{nh}$	$= y_1$	(1)
for the second point	$a + be^{n(h+d)}$	$= y_2$	(2)
for the third point	$a + be^{n(h+2d)}$	$= y_3$	(3)
subtracting (1) from (2)	$be^{nh}(e^{hd} - 1)$	$= y_2 - y_1$	(4)
subtracting (2) from (3)	$be^{n(h+d)}(e^{hd} - 1)$	$= y_3 - y_2$	(5)
dividing (5) by (4)	e^{nd}	$= \dfrac{y_3 - y_2}{y_2 - y_1}$	

Thus giving a relation from which the value of n can be calculated, and knowing n, a and b can be found.

A fully worked out example for this curve will be found in paragraph 221.

EXAMPLES XXV

(1) There are errors of observation in the following values of y and x:

x	4	5	6	7	8	9	10	11
y	6·29	5·72	5·22	4·78	4·39	4·06	3·75	3·48

It is found that the following two empirical formulæ seem to be nearly equally good:

$$y = \frac{a}{b + x} \text{ and } y = \alpha e^{-\beta x}.$$

Find the best values of a and b, α and β. (B. of E., 1906.)

(2) The following numbers are authentic; t seconds is the record time of a trotting (in harness) race of m miles:

m	1	2	3	4	5	10	20	30	50	100
t	119	257	416	598	751	1575	3505	6479	14141	32153

It is found that there is approximately a law $t = a\,m^b$, where a and b are constants. Test if this is so, and find the most probable values of a and b. The average speed in a race is $s = \frac{m}{t}$; express s in terms of m. (B. of E., 1907.)

(3) The following quantities measured in a laboratory are thought to follow the law $y = ab^{-x}$. Try if this is so, and if so, find the most probable values of a and b. (B. of E., 1908.)

x	0·1	0·2	0·4	0·6	1·0	1·5	2·0
y	350	316	120	63	12·86	2·57	0·425

(4) The equilibrium position for a certain governor is that a ball should be at a certain position r from an axis about which it revolves, when the centrifugal force is equal to $r\left(\frac{200 + 80\,h}{h}\right)$ where $h = \sqrt{2{\cdot}25 - r^2}$. Now a certain mathematical investigation becomes too complex if this law is used, whereas it is known that, if the centrifugal force were equal to $br - a$ where a and b are mere

numbers, the investigation would be easy. Find if there is approximately such a law within the limits $r = 0{\cdot}5$ and $r = 0{\cdot}7$, and what is the maximum error in making such an assumption. (B. of E., 1908.)

(5) When a shaft fails under the combined action of a bending moment M and a twisting moment T, according to what is called the internal friction hypothesis, $M + a\sqrt{M^2 + T^2}$ ought to be constant where a is a constant. Test if this is so, using the following numbers which have been published. Considerable errors in the observations must be expected. (B. of E., 1910.)

M	0	0	0	1,200	1,160	1,240	2,800	2,840
T	4,320	4,360	4,308	4,338	4,326	4,368	3,836	3,846
M	2,760	4,400	4,320	4,600	5,020	5,180	5,360	
T	3,804	2,416	2,438	2,060	0	0	0	

(6) In the following table C denotes the radio-activity of a substance, t hours after the observations were commenced. There is reason for believing that $\dfrac{dC}{dt} = aC$ where a is a constant. Try if this is so, and if so, find the most probable value of a. (B. of E., 1911.)

t	0	7·9	11·8	23·4	29·2	32·6	49·2	62·1	71·4
C	100	64	47·4	19·6	13·8	10·3	3·7	1·86	0·86

(7) The following values of x and y were observed in a laboratory and theory suggested that there might be a law $y = ax + b \log_{10} x$. There are errors of observation. Try if there is such a law, and if so, find the most probable values of a and b. (B. of E., 1912.)

x	10·2	31·0	52·0	75·0	104	132	181
y	3·75	6·26	7·99	9·54	11·39	12·94	15·67

(8) It is thought that the following observed values of x and y follow the law $y = Ae^{bx}$. There are errors of observation. Test if such a law is probably true, and if so, find the values of A and b. (B. of E., 1913.)

x	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5
y	13·28	15·04	17·53	19·80	23·11	26·00	30·50	34·40

(9) The following values of x and y were observed in a laboratory and theory suggested that there might be a law $y = ax^2 + b \times 10^x$. There are errors of observation. Try if there is such a law, and if so, find the probable values of a and b. (B. of E., 1913.)

x	0·1	0·2	0·3	0·4	0·5	0·6
y	1·61	2·26	3·20	4·47	6·21	8·07

(10) The curve $y = a + bx^n$ passes through the three points (2, 11·5), (4, 18·8), and (8, 39·7); find the values of the constants a, b, and n.

Find the value of y when $x = 5$.

(11) The curve $y = a + be^{nx}$ passes through the three points (1, 6·8), (3, 13·1), and (5, 25·7); find the values of the constants a, b, and n. Find the value of y when $x = 4$.

(12) The curve $y = a + bc^x$ passes through the three points (2, 5·3), (4, 12·8), and (6, 30·2); find the values of the constants a, b, and c. Find the value of y when $x = 3$.

(13) The curve $y = b\,(x + a)^n$ passes through the three points (1, 29·30), (3, 40·47), and (5, 52·28); find the values of the constants a, b, and n. Find the value of y when $x = 2$.

(14) The following values of x and y might follow the law $y = ax + bx \log_{10} x$. Try if this is so, and if so, find the most probable values of the constants a and b. (B. of E., 1914.)

x	20	30	40	60	80	100	120	140
y	40	66	96	154	221	284	356	420

(15) The following values of x and y are thought to follow the law $y = ax^{bx}$. Try if this is so, and if so, find the most probable values of the constants a and b.

x	1·34	2·52	3·41	4·13	4·86	5·25	5·71	6·00
y	1·06	1·56	2·34	3·26	4·68	5·84	7·43	8·59

(16) Find which of the three laws $y = a + bx^n$, $y = b\,(x + a)^n$, and $y = a + be^{nx}$ suit the following values of x and y.

x	2	3	4	5	6	7	8	9	10	11	12	13
y	12·83	13·48	14·28	15·28	16·52	18·05	19·95	22·31	25·24	28·87	33·37	38·44

Having decided upon the probable law, take three points on the curve and find the values of the constants a, b, and n.

(17) Find which of the three laws $y = a + bx^n$, $y = b(x + a)^n$, and $y = a + be^{nx}$ suit the following value of x and y.

x	2	3	4	5	6	7	8	9	10	11	12	13
y	9·49	13·70	18·54	23·98	30·03	36·64	43·78	51·47	59·70	68·42	77·66	87·36

Having decided upon the probable law, take three points on the curve and find the values of the constants a, b, and n.

(18) Find which of the three laws $y = a + bx^n$, $y = b(x + a)^n$, and $y = a + be^{nx}$ suit the following values of x and y.

x	2	3	4	5	6	7	8	9	10	11	12	13
y	18·36	21·47	25·65	30·87	37·09	44·28	52·43	61·52	71·56	82·50	94·36	107·1

Having decided upon the probable law, take three points on the curve and find the values of the constants a, b, and n.

ANSWERS

I (pages 16, 17, 18)

(1) (a) $(x-1\cdot112)(x+5\cdot052)$; (b) $(x-4\cdot707)(x-1\cdot013)$; (c) $(x-9\cdot435)(x+0\cdot515)$.

(2) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$ (3) $x+\dfrac{3}{2(x-2)}+\dfrac{5}{2(x+2)}$

(4) $\dfrac{19}{13(2x+3)}-\dfrac{22}{13(3x-2)}$ (5) $\dfrac{3}{4(3-x)}-\dfrac{5}{4(5x-3)}$

(6) $\dfrac{1}{6(x-1)}+\dfrac{11}{15(x+2)}+\dfrac{1}{10(x-3)}$

(7) $\dfrac{71}{63(3x-2)}+\dfrac{7}{9(3x+1)}-\dfrac{5}{7(2x+1)}$

(8) $x^2-2x+13+\dfrac{241}{30(x-3)}-\dfrac{239}{6(x+3)}+\dfrac{29}{5(x+2)}$

(9) $\dfrac{1}{19(3-2x)}+\dfrac{62}{551(x+8)}-\dfrac{7}{29(5-3x)}$

(10) $\dfrac{5}{x+4}-\dfrac{32}{(x+4)^2}+\dfrac{36}{(x+4)^3}$

(11) $\dfrac{8}{x-5}+\dfrac{120}{(x-5)^2}+\dfrac{600}{(x-5)^3}+\dfrac{993}{(x-5)^4}$

(12) $\dfrac{7}{9(3x-2)^2}+\dfrac{55}{9(3x-2)^3}+\dfrac{91}{9(3x-2)^4}$

(13) $\dfrac{1}{64}\left\{\dfrac{5}{4x+3}-\dfrac{45}{(4x+3)^2}+\dfrac{87}{(4x+3)^3}+\dfrac{73}{(4x+3)^4}\right\}$

(14) $\dfrac{1}{3}\left\{\dfrac{1}{x-1}+\dfrac{2x+1}{x^2+x+1}\right\}$ (15) $x+\dfrac{4}{3}\left\{\dfrac{1}{x-2}+\dfrac{2-x}{x^2+2x+4}\right\}$

(16) $x+\dfrac{1}{3}\left\{\dfrac{2}{x+1}+\dfrac{1-2x}{x^2-x+1}\right\}$

(17) $\dfrac{1}{6}\left\{\dfrac{2}{(x+2)^2}-\dfrac{1}{x+2}+\dfrac{x}{x^2-2x+4}\right\}$

(18) $\dfrac{1}{8}\left\{\dfrac{21}{x-3}+\dfrac{44}{(x-3)^2}+\dfrac{3}{x+1}\right\}$

(19) $\dfrac{14}{27(x+1)}-\dfrac{5}{9(x+1)^2}-\dfrac{14}{27(x-2)}+\dfrac{19}{9(x-2)^2}$

(20) $\dfrac{x}{2(x^2-x+1)} - \dfrac{x}{2(x^2+x+1)}$ (21) $\dfrac{0\cdot7034}{x-1\cdot112} + \dfrac{2\cdot296}{x+5\cdot052}$

(22) $\dfrac{1}{x-4\cdot707} + \dfrac{1}{x-1\cdot013}$ (23) $\dfrac{0\cdot0409}{x+3} + \dfrac{0\cdot266-0\cdot0409x}{x^2-3\cdot5x+4\cdot94}$

(24) (*a*) $1+14x+84x^2+280x^3+560x^4+\ldots$

(*b*) $1+5x+15x^2+35x^3+70x^4+\ldots$

(*c*) $1+x-\dfrac{x^2}{2}+\dfrac{x^3}{2}-\dfrac{5x^4}{8}+\ldots$

(*d*) $1-\dfrac{x}{3}+\dfrac{2x^2}{9}-\dfrac{14x^3}{81}+\dfrac{35x^4}{243}-\ldots$

(*e*) $1-x^2+x^4-x^6+x^8-\ldots$

(*f*) $1-\dfrac{x^2}{2}+\dfrac{3x^4}{8}-\dfrac{5x^6}{16}+\dfrac{35x^8}{128}-\ldots$

(25) (*a*) 11·75, 11·74740; (*b*) 25·04, 25·03997; (*c*) 9·02058, 9·02053; (*d*) 7·9375, 7·93701; (*e*) 7·95313, 7·95272; (*f*) 5·01, 5·00997.

(26) $1+\dfrac{x}{2}+\dfrac{3x^2}{8}+\dfrac{5x^3}{16}+\dfrac{35x^4}{128}+\ldots$, 1·0260.

(27) (*a*) 0·3429, 0·3429; (*b*) 0·3762, 0·3781, 0·503 per cent.

(28) 16·41 secs. lost, 12·31 secs. gained.

(29) 0·1823. (30) 1·6487, 1·3956, 0·60653.

(31) 0·69315, 1·09861, 1·38629, 1·60944.

(32) 0·4002, – 3·93, 5·787. (33) 1·176, 18·58, 0·1258.

(34) 0·1681, 0·3119, 0·4376, 0·5495, 0·6498, 0·7413.

(35) 0·01766, 0·01843, 0·01919, 0·01995.

II (pages 36, 37)

(1) (*a*) 29° 11′, 50° 3′, 100° 46′. (*b*) 31°, 50° 13′, 98° 47′.

(2) (*a*) 28·36, 43° 12′, 74° 48′. (*b*) 4·343, 48° 19′, 76° 41′

(3) (*a*) 7·92, 68° 51′, 55° 9′; 2·146, 111° 9′, 12° 51′.

(*b*) 4·239, 61° 10′, 86° 50′; 2·069, 118° 50′, 29° 10′.

(4) (*a*) 30·44, 31° 33′, 110° 27′. (*b*) 7·848, 28° 26′, 107° 34′.

(5) (*a*) 12·35, 8·618, 107°. (*b*) 2·599, 1·662, 85°.

(6) 6·071, 29° 31′, 50° 45′, 99° 44′, 2·168, 2·759, 4·336.

(7) 43° 1′, 61° 48′, 70° 20′.

(8) 5·762, 63°, 106° 17′, 148° 12′, 42° 31′, 4·651.

(9) – 0·5736, – 0·0523, – 0·8098, – 0·5592, 0·9272, 0·0699, 0·2250, 0·6157, 0·1584, 0·7431, – 0·9613, 0·3443.

(10) 0·9805, – 0·9481, – 6·535, 0·4441, – 0·9517, 2·2096, 0·6598, 0·0642, – 0·6954.

(11) 0, 0·159, 0·3429, 0·575, 1·1, 1·575, 1·891, 2.

(12) 7·389, 5·652, 2·718, 1, 0·3679, 0·1769, 0·1353, θ = 20° 16′.

(13) A = 15·81, α = 55° 18′, θ = 124° 42′.

(14) $A = 26{\cdot}40$, $\alpha = 37^\circ\ 18'$, max. when $\theta = 322^\circ\ 42'$, min. when $\theta = 142^\circ\ 42'$
(15) $13^\circ\ 12'$, $33^\circ\ 12'$
(16) $201^\circ\ 29'$, $225^\circ\ 53'$
(17) $339^\circ\ 6'$, $317^\circ\ 42'$
(18) $111^\circ\ 39'$, $140^\circ\ 51'$
(19) $161^\circ\ 4'$
(20) $22^\circ\ 38'$
(21) $126^\circ\ 2'$
(22) 0·2301, 0·9731, 0·2361, 0·8, 0·6, 1·3333, 0·9837, 0·1788, 5·5
(23) $16^\circ\ 16'$, $53^\circ\ 8'$
(24) 204·1 cub. in.
(25) 12·53, $61^\circ\ 21'$
(26) 13·93, $111^\circ\ 2'$
(27) 18·03, $236^\circ\ 19'$
(28) 13·90, $329^\circ\ 45'$

III (pages 46, 47)

(1) $115 + 133i$
(2) $295i - 211$
(3) $2{\cdot}52 + 0{\cdot}64i$
(4) $0{\cdot}2511 - 0{\cdot}2738i$
(5) $0{\cdot}4270i - 0{\cdot}4832$
(6) $-1{\cdot}670 - 0{\cdot}8738i$
(7) $3{\cdot}972 + 0{\cdot}8802i$
(8) $3{\cdot}522 + 1{\cdot}845i$
(9) $4{\cdot}152 - 2{\cdot}289i$
(10) $4 - i$
(11) $1{\cdot}905 + 3{\cdot}410i$
(12) $1{\cdot}246 + 4{\cdot}422i$
(13) $4{\cdot}336i - 2{\cdot}191$
(14) $4{\cdot}868i - 1{\cdot}644$
(15) $0{\cdot}7071(1 + i)$
(16) $0{\cdot}7071(1 - i)$
(17) 8·545, $20^\circ\ 33'$
(18) 21·10, $31^\circ\ 26'$
(19) 18·61, $306^\circ\ 15'$
(20) 12·10, $318^\circ\ 22'$
(21) 8·062, $144^\circ\ 28'$
(22) 19·72, $120^\circ\ 28'$
(23) 16·64, $214^\circ\ 44'$
(24) 20·81, $234^\circ\ 47'$
(25) $1,\ 45^\circ,\ -1,\ -\dfrac{1}{\sqrt{2}}(1 + i),\ \dfrac{1}{2}(\sqrt{3} + i),\ \dfrac{1}{2}(1 + \sqrt{3}i)$
(26) $0,\ -2i$
(27) 32
(28) $16\sqrt{3}i$
(29) $-0{\cdot}4618 + 4{\cdot}174i$
(30) $2{\cdot}523 + 1{\cdot}019i$
(31) $0{\cdot}2440 + 1{\cdot}736i$
(32) $0{\cdot}3485 + 2{\cdot}404i$
(33) $-(0{\cdot}7796 + 0{\cdot}5527i)$
(34) $-1{\cdot}072 + 0{\cdot}9674i$
(35) $-1{\cdot}096 + 0{\cdot}6805i$
(36) $-0{\cdot}6457 + 1{\cdot}805i$

IV (page 60)

(1) $5{\cdot}385 \sin (qt + 1{\cdot}191)$
(2) $12{\cdot}09 \sin (qt + 1{\cdot}148)$
(3) $9{\cdot}434 \sin (qt - 1{\cdot}012)$
(4) $13 \sin (qt - 0{\cdot}3948)$
(5) $0{\cdot}1280 \sin (qt - 0{\cdot}876)$
(6) $0{\cdot}0767 \sin (qt - 0{\cdot}5665)$
(7) $0{\cdot}1162 \sin (qt + 0{\cdot}620)$
(8) $0{\cdot}0718 \sin (qt + 1{\cdot}204)$
(9) $\sin (qt + 0{\cdot}7854)$
(10) $\sin (qt - 0{\cdot}7854)$
(11) $3{\cdot}072 \sin (qt + 0{\cdot}2793)$
(12) $3{\cdot}728 \sin (qt - 0{\cdot}2644)$
(13) $0{\cdot}6203 \sin (qt - 0{\cdot}1242)$
(14) $1{\cdot}092 \sin (qt + 2{\cdot}340)$
(15) 0·9153, 1·3557, 0·6751
(16) 0·6108
(17) 1·451
(18) 0·9731
(19) $0{\cdot}9895 + 0{\cdot}2498i$
(20) $0{\cdot}3210 + 0{\cdot}3455i$
(21) $\log_e\left\{\dfrac{x + 6 + \sqrt{x^2 + 12x + 48}}{2\sqrt{3}}\right\}$
(21) $\log_e\left\{\dfrac{x + 8 + \sqrt{x^2 + 16x + 36}}{2\sqrt{7}}\right\}$

(22) $C = 0{\cdot}0002596 \sin(5000t - 3{\cdot}798)$ (23) $\frac{1}{4}\{3 \sin\theta - \sin 3\theta\}$

(24) $\frac{1}{32}\{10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\}$

(25) $\frac{1}{16}\{\cos 5\theta + 5\cos 3\theta + 10\cos\theta\}$

(26) $\frac{1}{32}\{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\}$

(27) $\frac{1}{32}\{3\sin 2\theta - \sin 6\theta\}$ (28) $\frac{1}{128}\{\cos 8\theta - 4\cos 4\theta + 3\}$

(29) $\frac{1}{128}\{6\sin 2\theta + 2\sin 4\theta - 2\sin 6\theta - \sin 8\theta\}$

(30) $\frac{1}{32}\{\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\}$

V (pages 77, 78)

(1) 9·110, 109° 14′, 56° 42′, 140° 13′
(2) 6·170, 70° 6′, 56° 33′, 40° 23′
(3) 7·174, 116° 30′, 44° 42′, 57° 4′
(4) 15·59, 86° 19′, 134° 53′, 45° 7′
(5) 3·952, 61° 16′, 127° 24′, 50° 46′ (6) 7·668
(7) 4·123, 5·477, 5·916, 10·88, 74° 33′, 42° 13′, 63° 14′
(8) 59° 33′, 4·952
(9) (1) 2, – 1, 2, 0·8166, 65° 54′, 144° 45′, 65° 54′, 0·333, – 0·667, 0·333. (2) 5, 4, – 6, 2·771, 56° 20′, 46° 9′, 117° 30′, 1·535, 1·919, – 1·279. (3) 2, 3, 4, 1·536, 39° 49′, 59° 12′, 67° 25′, 1·1805, 0·787, 0·5902

(10) 9° 18′ (11) $\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}$; $\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}$; Yes (12) 0·7588

(13) 4·801, 62° 41′, 60° (14) 2·305, 4·335, 5·265 (15) 42° 33′

VI (pages 97, 98)

(1) $e^{\sin x}\cos x$ (2) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ (3) $\cot x$ (4) $x\cos x + \sin x$

(5) $\frac{\cos x}{x} - \frac{\sin x}{x^2}$ (6) $\operatorname{cosec} x(1 - x\cot x)$

(7) $x^2(3\sin x + x\cos x)$ (8) $x^2e^x(x+3)$

(9) $x^{n-1}(1 + n\log_e x)$ (10) $x^{n-1}\left(\frac{1}{\log_e a} + n\log_e x\right)$

(11) $x^3(x\tan^2 x + 4\tan x + x)$ (12) $x^{n-1}(x\cot x + n\log_e \sin x)$

(13) $nx^{n-1}\operatorname{cosec} nx(1 - x\cot nx)$ (14) $\dfrac{n}{x^{n+1}}(x\cos nx - \sin nx)$

(15) $ne^{-nx}(\cos nx - \sin nx)$ (16) $ne^{nx}\operatorname{cosec} nx(1 - \cot nx)$

(17) $ne^{nx}(\cos nx - \sin nx)$ (18) $nx^{n-1}e^{nx}\sin nx(1 + x + x\cot nx)$

(19) $ne^{nx}(\tan^2 nx + \tan nx + 1)$ (20) $ne^{nx}\sec nx(1 + \tan nx)$

(21) $ne^{nx}\sec nx(2\tan^2 nx + \tan nx + 1)$

(22) $\dfrac{x^2}{e^x \sin x}\{3 - x - x\cot x\}$ (23) $\dfrac{(a-c)\{bx^2 + 2(a+c)x + b\}}{(cx^2 + bx + a)^2}$

(24) $(x+a)^{n-1}(x+b)^{m-1}\{x(m+n) + am + bn\}$

(25) $\dfrac{2}{1-x^2}$ (26) $\dfrac{4x}{1-x^4}$ (27) $\dfrac{2(1-x^2)}{1+x^2+x^4}$ (28) $\dfrac{2}{\sqrt{1+x^2}}$

(29) $e^x\left\{\dfrac{1}{x+a} + \log_e(x+a)\right\}$ (30) $a^x\left\{\dfrac{1}{x+a} + \log_e(x+a)\times\log_e a\right\}$

(31) $\dfrac{1+e^x}{x+e^x}$ (32) $e^x + \dfrac{1}{x}$ (33) $\dfrac{e^x}{\log_e x}\left\{1 - \dfrac{1}{x\log_e x}\right\}$ (34) $\dfrac{x+1}{x}$

(35) $-\cos x\cos 2x\cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$

(36) $\dfrac{\sin x\sin 2x}{\sin 3x}(\cot x + 2\cot 2x - 3\cot 3x)$ (37) $\dfrac{e^{\sqrt{\sin x}}\cos x}{2\sqrt{\sin x}}$

(38) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (39) $\dfrac{2}{1+x^2}$ (40) $\dfrac{x}{1+x^2} + \tan^{-1}x$

(41) $\dfrac{\log_e x}{1+x^2} + \dfrac{\tan^{-1}x}{x}$ (42) $\dfrac{x}{(1+x^2)\tan^{-1}x} + \log_e(\tan^{-1}x)$

(43) $\dfrac{\tan x}{\sqrt{1-x^2}} + \sin^{-1}x\,(1 + \tan^2 x)$

(44) $\sec x\left(\tan x\tan^{-1}x + \dfrac{1}{1+x^2}\right)$

(45) $\dfrac{e^x}{\sqrt{1-x^2}}\{1 + \sqrt{1-x^2}\sin^{-1}x\}$ (46) $\dfrac{1}{(1-x)\sqrt{1-x^2}}$

(47) $-\dfrac{1}{(1+x)\sqrt{1-x^2}}$ (48) $-\dfrac{2x}{(1+x^2)\sqrt{1-x^4}}$

(49) $-\dfrac{3x^2}{(1+x^3)\sqrt{1-x^6}}$ (50) $-\dfrac{x(x^2+3)}{(1+x^2)\sqrt{1+x^2}}$

(51) $\dfrac{x^2-2x-2}{2(1+x+x^2)\sqrt{1-x^3}}$ (52) $\dfrac{2(1-x^2)}{(x^2-x+1)^2}$

(53) $-\dfrac{1+x}{(1+x^2)\sqrt{1+x^2}}$ (54) $-\dfrac{1}{1+x^2}$ (55) $\sqrt{\dfrac{2}{2-5x-2x^2}}$

VII (pages **112, 113, 114**)

(10) $e^x(x+1)$, $e^x(x+2)$, $e^x(x+3)$, $e^x(x+n)$

(11) $\sin nx = nx - \frac{n^3x^3}{\underline{|3}} + \frac{n^5x^5}{\underline{|5}} - \frac{n^7x^7}{\underline{|7}} + \ldots$

$\cos nx = 1 - \frac{n^2x^2}{\underline{|2}} + \frac{n^4x^4}{\underline{|4}} - \frac{n^6x^6}{\underline{|6}} + \ldots$

(13) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$

(14) $e^x(n \tan^{n-1} x + \tan^n x + n \tan^{n+1} x)$, $e^x(1 + \tan x + \tan^2 x)$
$e^x(2 + 3 \tan x + 2 \tan^2 x + 2 \tan^3 x)$
$e^x(5 + 7 \tan x + 11 \tan^2 x + 6 \tan^3 x + 6 \tan^4 x)$
$e^x(12 + 29 \tan x + 36 \tan^2 x + 52 \tan^3 x + 24 \tan^4 x + 24 \tan^5 x)$
$e^x(41 + 101 \tan x + 221 \tan^2 x + 220 \tan^3 x + 300 \tan^4 x + 120 \tan^5 x + 120 \tan^6 x)$

$e^x \tan x = x + x^2 + \frac{5x^3}{6} + x^4 + \frac{41x^5}{120} + \ldots$

(18) $-\frac{1}{a(1-\cos\theta)^2}$ (19) 4·4936 (20) 0·765 (21) 1·172

(22) 3·424 (23) $2x + 2x^2 - \frac{x^3}{3} - x^4 - \frac{19x^5}{60} + \frac{11x^6}{180} + \ldots$

(24) $x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots$

(25) $1 - \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} - \frac{x^3}{(\underline{|3})^2} + \frac{x^4}{(\underline{|4})^2} - \frac{x^5}{(\underline{|5})^2} + \ldots$

VIII (pages **127, 128, 129**)

(1) $y_{max} = 35$ when $x = 1$, $y_{min} = 26$ when $x = 2$
(2) $y_{max} = 124$ when $x = -5$, $y_{min} = -76$ when $x = 5$
(3) $y_{min} = 80$ when $x = -2$, $y_{max} = 87$ when $x = -1$, $y_{min} = 48$ when $x = 2$
(4) $y_{min} = 7{\cdot}213$ when $x = 0{\cdot}8236$
(5) $x = 1{\cdot}877$, $t = 0{\cdot}4163$ (6) $x = 3$, $t = 0$
(7) $x = 3{\cdot}877$, $t = 0{\cdot}2019$ (8) $x = 1{\cdot}840$, $t = 0{\cdot}5$
(9) $x = 5$, $t = 0$ (10) $x = 6{\cdot}065$, $t = 0{\cdot}25$
(11) $x = 1$, $t = 0{\cdot}3465$ (12) $x = 4$, $t = 0$
(13) $x = 4{\cdot}5$, $t = 0{\cdot}1439$ (14) $y_{max} = 1{\cdot}345$ when $x = 3$
(15) rad. = 8·166″, ht. = 11·54″ (16) rad. = 7·07″, ht. = 14·14″
(17) rad. = 8·507″, ht. = 10·51″
(18) rad. = 9·588 yds., ht. = 13·56 yds.
(19) $y_{max} = 0{\cdot}231$ when $x = 0{\cdot}5777$
(20) 17·01 knots, £2488, £2460, £2483 (21) 15·7 knots

(22) 10·6 knots (23) 1·81″ (24) $1\frac{2}{3}''$ (25) $h = 20{\cdot}78''$, $b = 12''$
(26) $y_{\max} = 1{\cdot}705$ when $t = 0{\cdot}002331$, $y_{\min} = -0{\cdot}265$ when $t = 0{\cdot}008414$ (27) rad. = $1\frac{1}{3}''$, ht. = $1\frac{2}{3}''$
(28) rad. = 1″, ht. = $2\frac{1}{2}''$ (29) rad. = $1\frac{2}{3}''$, ht. = $\frac{5}{6}''$
(30) $a = 900$, $b = 60{,}000$, $m = -406$, $n = 733{\cdot}6$, $V = 358{\cdot}3$
(31) $\theta = 79°\ 16'$, $x = 0{\cdot}9103$ ft.

IX (pages 141, 142)

(1) 3·75, 9, $4y = 15x - 7$, $y = 9x - 11$, 8° 36′
(2) 8, $y = 8x - 9$, $8y + x = 58$
(3) (3, 8), (− 2, 3), 35° 32′, 59° 2′
(4) $y = 48x - 64$, $x + 48y = 1538$
(5) $y = 1{\cdot}059x + 3{\cdot}539$, $y + 0{\cdot}9535x = 7{\cdot}564$
(6) (1·871, 1·225), 66° 26′ (7) (2·058, 0·4897), 90°
(8) (2·121, 1·885), (− 2·693, − 1·485), 103° 21′, 116° 8′
(9) $a = 4{\cdot}315$, $n = 0{\cdot}765$, 2·805, (0·2782, 1·621), (8·507, 22·20)
(10) (8, 8), 36° 52′
(11) $a = 3{\cdot}036$, $b = 0{\cdot}1423$, $y = 0{\cdot}88x + 1{\cdot}744$, $y = 11{\cdot}86 - 1{\cdot}136x$
(12) 14·27, 73·16, 196·3
(13) $a = 1{\cdot}32$, $b = 0{\cdot}5$, 11·68, (− 6·507, 10·75)
(14) $a = 21{\cdot}49$, $b = 7{\cdot}131$, $c = 1{\cdot}993$, $R = 141{\cdot}6$
(15) (34·27, 58·55), 22° 56′ (16) − 6·25, − 3·2
(17) 2·828, (4, 4) (18) (2, 4) (19) (0·7071, 0·6065)

X (pages 155, 156)

(1) $\frac{1}{2}\log_e(3x^2 + 4x + 7)$ (2) $\sqrt{3x^2 + 4x - 7}$
(3) $\log_e(\log_e x)$ (4) $\log_e \sin^{-1} x$
(5) $\log_e \tan^{-1} x$ (6) $\log_e(e^x + e^{-x})$
(7) $\log_e \cosh x$ (8) $\frac{1}{2}(\tan^{-1} x)^2$
(9) $\frac{1}{2}(\sin^{-1} x)^2$ (10) $2\sqrt{\tan x}$
(11) $-2\sqrt{\cos x}$ (12) $\sqrt{x^2 - 7}$
(13) $\frac{1}{3}\log_e(x^3 - 1)$ (14) $\frac{2}{3}(2x^2 - 3x + 1)^{\frac{3}{2}}$
(15) $\frac{9}{2}\log_e(x - 5) - \frac{5}{2}\log_e(x - 3)$ (16) $x + \log_e \frac{x-2}{x+2}$
(17) $-2\log_e(3 - x) - 3\log_e(3 + x)$ (18) $\log_e \frac{4x-3}{3x-2}$
(19) $\frac{1}{2}\log_e(x - 4) - \frac{1}{10}\log_e(5x + 2)$

(20) $-\frac{7}{44}\log_e(3-4x)-\frac{12}{11}\log_e(x+2)$

(21) $\frac{4}{15}\log_e(x+2)+\frac{9}{10}\log_e(x-3)-\frac{1}{6}\log_e(x-1)$

(22) $\frac{21}{22}\log_e(3-2x)-\log_e(x-2)-\frac{4}{33}\log_e(3x+1)$

(23) $5\log_e(x-1)-\frac{15}{x-1}-\frac{15}{2(x-1)^2}-\frac{2}{3(x-1)^3}$

(24) $\log_e(x+3)+\frac{6}{x+3}-\frac{9}{2(x+3)^2}$

(25) $\frac{1}{96}\left\{6\log_e(2x+3)+\frac{54}{2x+3}-\frac{45}{(2x+3)^2}-\frac{82}{(2x+3)^3}\right\}$

(26) $\frac{1}{54}\left\{10\log_e(3x-4)-\frac{38}{3x-4}-\frac{23}{(3x-4)^2}\right\}$

(27) $\frac{3}{4}\log_e\frac{x-1}{x+1}+\frac{1}{2(x+1)}$

(28) $\frac{3}{4}\log_e(x-2)+\frac{1}{4}\log_e(x+2)-\frac{1}{x-2}$

(29) $\frac{1}{4}\log_e\frac{x+1}{x-1}-\frac{x}{2(x^2-1)}$

(30) $\frac{1}{32}\log_e\frac{x-2}{x+2}+\frac{x-8}{8(x-2)^2}$ (31) $\tan^{-1}x$ (32) $\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}$

(33) $\tan^{-1}x+\frac{1}{2}\log_e(x^2+1)$ (34) $\sqrt{5}\tan^{-1}\frac{x}{\sqrt{5}}+\frac{1}{2}\log_e(x^2+5)$

(35) $\frac{1}{5}\tan^{-1}\frac{x+4}{5}$ (36) $\frac{1}{2}\log_e(x^2+6x+25)-\frac{3}{4}\tan^{-1}\frac{x+3}{4}$

(37) $\frac{1}{\sqrt{87}}\tan^{-1}\frac{x+5}{\sqrt{87}}$ (38) $\frac{1}{2}\log_e(x^2+4x+16)-\frac{5}{\sqrt{3}}\tan^{-1}\frac{x+2}{2\sqrt{3}}$

(39) $x-\log_e(x^2+2x+4)-\sqrt{3}\tan^{-1}\frac{x+1}{\sqrt{3}}$

(40) $\frac{1}{2\sqrt{7}}\log_e\frac{x-\sqrt{7}}{x+\sqrt{7}}$ (41) $\frac{1}{2}\log_e(x^2-8)+\frac{3}{4\sqrt{2}}\log_e\frac{x-2\sqrt{2}}{x+2\sqrt{2}}$

(42) $\frac{1}{2\sqrt{3}}\log_e\frac{\sqrt{3}+x}{\sqrt{3}-x}$ (43) $\frac{\sqrt{5}}{2}\log_e\left[\frac{\sqrt{5}+x}{\sqrt{5}-x}\right]-x$

(44) $\tan^{-1}(x+2)$ (45) $\frac{1}{2}\log_e(x^2-6x+6)-\frac{1}{2\sqrt{3}}\log_e\frac{x-3-\sqrt{3}}{x-3+\sqrt{3}}$

(46) $x-\frac{1}{2}\log_e(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

(47) $\dfrac{1}{2\sqrt{10}}\log_e\dfrac{\sqrt{10}+x+2}{\sqrt{10}-x-2}$

(48) $-\log_e(12-10x-x^2)-\dfrac{11}{2\sqrt{37}}\log_e\dfrac{\sqrt{37}+x+5}{\sqrt{37}-x-5}$

(49) $\dfrac{1}{6}\log_e\dfrac{x^2-2x+1}{x^2+x+1}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

(50) $\dfrac{1}{6}\log_e\dfrac{1+x+x^2}{1-2x+x^2}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

(51) $\dfrac{1}{3}\log_e\dfrac{x^2-x+1}{x^2+2x+1}$ (52) $\dfrac{1}{12}\log_e\dfrac{x^2+4x+4}{x^2-2x+4}+\dfrac{\sqrt{3}}{2}\tan^{-1}\dfrac{x-1}{\sqrt{3}}$

(53) $-x+\dfrac{1}{3}\log_e\dfrac{4+2x+x^2}{4-4x+x^2}+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x+1}{\sqrt{3}}$

(54) $\dfrac{1}{4}\log_e\dfrac{x-1}{x+1}-\dfrac{1}{2}\tan^{-1}x$ (55) $\dfrac{1}{4}\log_e\dfrac{x^2-1}{x^2+1}$

(56) $\dfrac{1}{4}\log_e\dfrac{x-1}{x+1}+\dfrac{1}{2}\tan^{-1}x$ (57) $\dfrac{1}{4}\log_e(x^4-1)$

(58) $\dfrac{1}{6}\log_e(3x^2-6x+14)+\dfrac{5}{\sqrt{33}}\tan^{-1}\dfrac{\sqrt{3}(x-1)}{\sqrt{11}}$

(59) $-\dfrac{1}{6}\log_e(10-4x-3x^2)-\dfrac{1}{3\sqrt{34}}\log_e\dfrac{\sqrt{34}+3x+2}{\sqrt{34}-3x-2}$

(60) $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x^2+1}{\sqrt{3}}$

XI (pages 171, 172)

(1) $\sin^{-1}\dfrac{x}{3}$ (2) $-3\sqrt{9-x^2}-2\sin^{-1}\dfrac{x}{3}$

(3) $\dfrac{9}{2}\left\{\sin^{-1}\dfrac{x}{3}+\dfrac{x\sqrt{9-x^2}}{9}\right\}$

(4) $\dfrac{1}{2}\left\{33\sin^{-1}\dfrac{x}{3}+(14-3x)\sqrt{9-x^2}\right\}$ (5) $\sin^{-1}\dfrac{x-3}{3}$

(6) $-4\sqrt{6x-x^2}+11\sin^{-1}\dfrac{x-3}{3}$

(7) $\dfrac{9}{2}\left\{\sin^{-1}\dfrac{x-3}{3}+\dfrac{(x-3)\sqrt{6x-x^2}}{9}\right\}$

(8) $\dfrac{1}{2}\left\{37\sin^{-1}\dfrac{x-3}{3}-(x+13)\sqrt{6x-x^2}\right\}$

(9) $\sin^{-1}\dfrac{x-4}{5}$ (10) $-2\sqrt{9+8x-x^2}+5\sin^{-1}\dfrac{x-4}{5}$

(11) $\frac{25}{2}\left\{\sin^{-1}\frac{x-4}{5}+\frac{(x-4)\sqrt{9+8x-x^2}}{25}\right\}$

(12) $52\sin^{-1}\frac{x-4}{5}-(x+12)\sqrt{9+8x-x^2}$

(13) $\frac{1}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}(x-1)}{\sqrt{13}}$

(14) $\frac{5}{3}\sqrt{10+6x-3x^2}-\frac{1}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}(x-1)}{\sqrt{13}}$

(15) $\frac{1}{2}\left\{\frac{13}{\sqrt{3}}\sin^{-1}\frac{\sqrt{3}(x-1)}{\sqrt{13}}+(x-1)\sqrt{10+6x-3x^2}\right\}$

(16) $\frac{1}{18}\left\{167\sqrt{3}\sin^{-1}\frac{\sqrt{3}(x-1)}{\sqrt{13}}-15(x+3)\sqrt{10+6x-3x^2}\right\}$

(17) $\sinh^{-1}\frac{x}{5}$ (18) $2\sqrt{x^2+25}-7\sinh^{-1}\frac{x}{5}$

(19) $\frac{1}{2}\left\{25\sinh^{-1}\frac{x}{5}+x\sqrt{x^2+25}\right\}$

(20) $\frac{1}{2}\left\{(3x+8)\sqrt{x^2+25}-71\sinh^{-1}\frac{x}{5}\right\}$ (21) $\sinh^{-1}\frac{x-6}{4}$

(22) $6\sqrt{x^2-12x+52}+31\sinh^{-1}\frac{x-6}{4}$

(23) $\frac{1}{2}\left\{16\sinh^{-1}\frac{x-6}{4}+(x-6)\sqrt{x^2-12x+52}\right\}$

(24) $\frac{1}{2}\left\{28\sinh^{-1}\frac{x-6}{4}+(x+12)\sqrt{x^2-12x+52}\right\}$

(25) $\frac{1}{\sqrt{2}}\sinh^{-1}\frac{2x+3}{\sqrt{5}}$

(26) $\frac{1}{4}\left\{6\sqrt{2x^2+6x+7}-23\sqrt{2}\sinh^{-1}\frac{2x+3}{\sqrt{5}}\right\}$

(27) $\frac{1}{8}\left\{5\sqrt{2}\sinh^{-1}\frac{2x+3}{\sqrt{5}}+2(2x+3)\sqrt{2x^2+6x+7}\right\}$

(28) $\frac{1}{4}\left\{7\sqrt{2}\sinh^{-1}\frac{2x+3}{\sqrt{5}}+2(2x-9)\sqrt{2x^2+6x+7}\right\}$

(29) $\cosh^{-1}\frac{x}{4}$ (30) $5\sqrt{x^2-16}-12\cosh^{-1}\frac{x}{4}$

(31) $\frac{1}{2}\left\{x\sqrt{x^2-16}-16\cosh^{-1}\frac{x}{4}\right\}$

(32) $\frac{1}{2}\left\{50\cosh^{-1}\frac{x}{4}+(3x-16)\sqrt{x^2-16}\right\}$ (33) $\cosh^{-1}\frac{x+5}{5}$

(34) $6\sqrt{x^2+10x} - 35\cosh^{-1}\dfrac{x+5}{5}$

(35) $\dfrac{1}{2}\left\{(x+5)\sqrt{x^2+10x} - 25\cosh^{-1}\dfrac{x+5}{5}\right\}$

(36) $\dfrac{1}{2}\left\{(x+9)\sqrt{x^2+10x} - 59\cosh^{-1}\dfrac{x+5}{5}\right\}$ (37) $\cosh^{-1}\dfrac{x-2}{5}$

(38) $3\sqrt{x^2-4x-21} - \cosh^{-1}\dfrac{x-2}{5}$

(39) $\dfrac{1}{2}\left\{(x-2)\sqrt{x^2-4x-21} - 25\cosh^{-1}\dfrac{x-2}{5}\right\}$

(40) $\dfrac{1}{2}\left\{(3x+4)\sqrt{x^2-4x-21} + 97\cosh^{-1}\dfrac{x-2}{5}\right\}$

(41) $\dfrac{1}{\sqrt{5}}\cosh^{-1}\dfrac{\sqrt{5}(x+1)}{\sqrt{21}}$

(42) $3\sqrt{5x^2+10x-16} - \dfrac{23}{\sqrt{5}}\cosh^{-1}\dfrac{\sqrt{5}(x+1)}{\sqrt{21}}$

(43) $\dfrac{1}{2}\left\{(x+1)\sqrt{5x^2+10x-16} - \dfrac{21\sqrt{5}}{5}\cosh^{-1}\dfrac{\sqrt{5}(x+1)}{\sqrt{21}}\right\}$

(44) $\dfrac{1}{25}\left\{30(x-3)\sqrt{5x^2+10x-16} + 221\sqrt{5}\cosh^{-1}\dfrac{\sqrt{5}(x+1)}{\sqrt{21}}\right\}$

XII (pages 181, 182)

(1) $\tan x$ (2) $-\cot x$ (3) $\dfrac{1}{3}\tan^3 x$ (4) $-\dfrac{1}{3\tan^3 x}$

(5) $\dfrac{1}{a}\log_e \sec(ax+b)$ (6) $\dfrac{1}{a}\log_e \sin(ax+b)$

(7) $\dfrac{1}{4}\tan^4 x - \dfrac{1}{2}\tan^2 x + \log_e \sec x$

(8) $\dfrac{1}{5}\tan^5 x - \dfrac{1}{3}\tan^3 x + \tan x - x$

(9) $-\dfrac{1}{2}\cot^2 x - \log_e \sin x$ (10) $-\dfrac{1}{3}\cot^3 x + \cot x + x$

(11) $\dfrac{1}{5}\log_e\left\{\tan\dfrac{1}{2}\left(x+\tan^{-1}\dfrac{4}{3}\right)\right\}$

(12) $\dfrac{1}{5}\log_e\left\{\tan\dfrac{1}{2}\left(x-\tan^{-1}\dfrac{3}{4}\right)\right\}$ (13) $\dfrac{1}{\sqrt{2}}\log_e\tan\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)$

(14) $\dfrac{1}{\sqrt{2}}\log_e\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)$ (15) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}\tan x}{2}$

(16) $\dfrac{1}{4\sqrt{3}} \log_e \dfrac{\sqrt{3} \tan x - 2}{\sqrt{3} \tan x + 2}$

(17) $\dfrac{1}{2} \log_e \dfrac{1 + \tan x}{1 - \tan x} - \tan x$

(18) $\dfrac{1}{\sqrt{7}} \log_e \dfrac{\sqrt{7} + \tan \dfrac{x}{2}}{\sqrt{7} - \tan \dfrac{x}{2}}$

(19) $\dfrac{1}{\sqrt{7}} \log_e \dfrac{3 \tan \dfrac{x}{2} + 4 - \sqrt{7}}{3 \tan \dfrac{x}{2} + 4 + \sqrt{7}}$

(20) $\dfrac{2}{\sqrt{7}} \tan^{-1} \dfrac{4 \tan \dfrac{x}{2} + 3}{\sqrt{7}}$

(21) $\dfrac{2}{\sqrt{7}} \tan^{-1} \dfrac{\tan \dfrac{x}{2}}{\sqrt{7}}$

(22) $\dfrac{1}{3} \cos^3 x - \cos x$

(23) $\dfrac{1}{32}(\sin 4x - 8 \sin 2x + 12x)$

(24) $\sin x - \dfrac{2}{3} \sin^3 x + \dfrac{1}{5} \sin^5 x$

(25) $\dfrac{1}{192}(\sin 6x + 9 \sin 4x + 45 \sin 2x + 60x)$

(26) $\dfrac{1}{5} \cos^5 x - \dfrac{1}{3} \cos^3 x$

(27) $\dfrac{1}{5} \sin^5 x - \dfrac{1}{7} \sin^7 x$

(28) $\dfrac{1}{6} \sin^6 x - \dfrac{1}{4} \sin^8 x + \dfrac{1}{10} \sin^{10} x$

(29) $\dfrac{1}{1024}(\sin 8x - 8 \sin 4x + 24x)$

(30) $\dfrac{1}{192}(\sin 6x - 3 \sin 4x - 3 \sin 2x + 12x)$

(31) $\dfrac{1}{3} \sin^3 x - \dfrac{2}{5} \sin^5 x + \dfrac{1}{7} \sin^7 x$

XIII (pages 190, 191)

(1) $\dfrac{x^8}{64}(8 \log_e x - 1)$

(2) $\dfrac{2x^{\frac{3}{2}}}{9}(3 \log_e x - 2)$

(3) $-\dfrac{1}{16x^4}(4 \log_e x + 1)$

(4) $2\sqrt{x}(\log_e x - 2)$

(5) $\dfrac{x^4}{32}\{8(\log_e x)^2 - 4 \log_e x + 1\}$

(6) $\dfrac{2x^{\frac{3}{2}}}{27}\{9(\log_e x)^3 - 18(\log_e x)^2 + 24 \log_e x - 16\}$

(7) $\dfrac{e^{2x}}{8}\{4x^3 - 6x^2 + 6x - 3\}$

(8) $-\dfrac{e^{-2x}}{4}\{2x^2 + 2x + 1\}$

(9) $-(1 - \sin x)^2 e^{\sin x}$

(10) $-2(1 - \sin x) e^{\sin x}$

(11) $\left(\frac{1}{4}-\frac{x^2}{2}\right)\cos 2x+\frac{x}{2}\sin 2x$

(12) $\left(\frac{x^3}{3}-\frac{2x}{9}\right)\sin 3x+\left(\frac{x^2}{3}-\frac{2}{27}\right)\cos 3x$

(13) $\left(-\frac{x^4}{2}+\frac{3x^2}{2}-\frac{3}{4}\right)\cos 2x+\left(x^3-\frac{3x}{2}\right)\sin 2x$

(14) $\frac{x^3}{6}-\frac{x}{4}\cos 2x-\left(\frac{x^2}{4}-\frac{1}{8}\right)\sin 2x$

(15) $\frac{x^4}{8}+\left(\frac{x^3}{4}-\frac{3x}{8}\right)\sin 2x+\left(\frac{3x^2}{8}-\frac{3}{16}\right)\cos 2x$

(16) $\frac{3x}{2}\sin x-\left(\frac{3x^2}{4}-\frac{3}{2}\right)\cos x-\frac{x}{18}\sin 3x+\left(\frac{x^2}{12}-\frac{1}{54}\right)\cos 3x$

(17) $\left(\frac{3x^2}{4}-\frac{3}{2}\right)\sin x+\frac{3x}{2}\cos x+\left(\frac{x^2}{12}-\frac{1}{54}\right)\sin 3x+\frac{x}{18}\cos 3x$

(18) $\frac{e^{2x}}{13}(2\sin 3x-3\cos 3x)$ (19) $\frac{e^{2x}}{13}(3\sin 3x+2\cos 3x)$

(20) $-\frac{e^{-3x}}{13}(3\sin 2x+2\cos 2x)$

(21) $-\frac{e^{-3x}}{13}(3\cos 2x-2\sin 2x)$

(22) $\frac{e^{2x}}{8}(2-\cos 2x-\sin 2x)$ (23) $\frac{e^{2x}}{8}(2+\cos 2x+\sin 2x)$

(24) $\frac{e^{3x}}{8}\left\{\frac{1}{5}\cos\left(4x-\tan^{-1}\frac{4}{3}\right)-\frac{4}{\sqrt{13}}\cos\left(2x-\tan^{-1}\frac{2}{3}\right)+1\right\}$

(25) $\frac{e^{3x}}{8}\left\{\frac{1}{5}\cos\left(4x-\tan^{-1}\frac{4}{3}\right)+\frac{4}{\sqrt{13}}\cos\left(2x-\tan^{-1}\frac{2}{3}\right)+1\right\}$

(26) $\frac{e^{-2x}}{8}(\cos 2x-\sin 2x-2)$ (27) $\frac{e^{-2x}}{8}(\sin 2x-\cos 2x-2)$

(28) $x\tan^{-1}x-\frac{1}{2}\log_e(1+x^2)$

(29) $\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{2}\log_e(1+x^2)$ (30) $x\sin^{-1}x+\sqrt{1-x^2}$

XIV (pages 205, 206, 207)

(1) $\frac{h}{6}(6a+3bh+2ch^2)$ (2) $\frac{2h}{3}(3a+ch^2)$ (3) 6·389 (4) 0·8647

(5) 1 (6) $\frac{\pi}{4}$ (7) $\frac{\pi}{4}$ (8) $\frac{1}{2}(e^{\pi}+1)$ (9) $-\frac{1}{2}(e^{\pi}+1)$ (10) π

(11) π^2-4 (12) $2a^2+\frac{16\sqrt{2}}{5}ab+4b^2$ (13) $2a+\frac{b}{\log_e c}(c^2-1)$

(14) $\frac{\pi}{2}$ (15) 4π (16) 0·8812 (17) 10·328 (18) 1·3168
(19) 26·84 (20) 0·2028 (21) 0·1419 radians (22) 3·957
(23) $a = 1·32$, $b = 0·5$, 7·909 (24) $a = 3·106$, $b = 0·1509$, 31·22
(25) 4·264 (26) $a = 10·19$, $b = 16·43$, $c = 1·553$, 30·79
(27) (3, 8), (– 2, 3), 20·83 (28) $\frac{32}{27}, \frac{32}{27}$ (29) $\frac{8}{3}$ (30) 12·07
(31) 0·7541 (32) 22·5, 4·5 (33) 17·07 (34) 5·94, 40·65
(35) 12566·4 (36) 2964 (37) 3141·6
(38) $a = 2·451$, $n = 1·606$; for area A, $V_{OX} = 453·1$ and $V_{OY} = 220·1$; for area B, $V_{OX} = 1457$ and $V_{OY} = 176·8$
(39) $a = 1·32$, $b = 0·5$, 118 (40) $a = 3·036$, $b = 0·1423$,1617
(41) $\frac{\pi^2}{2}$ (42) 14·14, 7·069 (43) 184·4 (44) 94·25, 157·1
(45) 520·4, 98·97

XV (pages 220, 221)

(1) $\bar{x} = 2·15''$, $\bar{y} = 2·28''$ (3) $\frac{2a}{3\pi}$ (6) 3·5355″ (7) 8·660″
(8) $\bar{x} = 0·869''$, $\bar{y} = 2·618''$ (9) 1·074″ (10) 1·009″
(11) $\bar{x} = 4·174''$, $\bar{y} = 3·423''$ (12) 2·411″ (13) 2·401″
(14) $\bar{x} = 3·438''$, $\bar{y} = 3·438''$ (15) $\bar{x} = 2·25$, $\bar{y} = 8·1$
(16) $\bar{x} = 0·9$, $\bar{y} = 5·58$ (17) $\bar{x} = 4·5$, $\bar{y} = 0·9$
(18) $\bar{x} = 3·6$, $\bar{y} = 3·6$ (19) $\bar{x} = 3·056$, $\bar{y} = 4·351$
(20) $\bar{x} = 2·122$, $\bar{y} = 1·697$
(21) $a = 1·515$, $n = 1·826$, $\bar{x} = 3·871$, $\bar{y} = 9·264$
(22) 2·77 (23) $\bar{x} = 2$, $\bar{y} = 11·4$

XVI (pages 244, 245, 246)

(1) 12·35 ft. (2) 7·3214 ft. (3) $\frac{h^3}{12}(3a + b)$, $\frac{h^3}{36}\left(a + b + \frac{2ab}{a + b}\right)$
(4) 7·639 ft., 12·953 ft.
(5) $I_{OX} = 86·16$, $I_{OY} = 11·594$, $I_{GX} = 36·03$, $I_{GY} = 6·077$
(6) 5·90, 5·245 (7) 111·71, 15·48, 0·269″, 0·722″
(8) 111·71, 10·83, 36·05 (9) 165·5, 13·11, 1·111″, 3·948″
(10) $I_{OX} = 110·55$, $I_{OY} = 132·32$, $I_{GX} = 34·40$, $I_{GY} = 19·18$
(11) 77·58, 36·91 (12) 2·384″, 5·899″
(13) 753·2, 498·7, 0·2246″, 0·2761″ (14) 29·15
(15) 79·79 ft.-lb. (16) 66·49 ft.-lb. (17) 0·7747a (18) 0·5478a
(19) 941·9 inch-lb. units (20) 180·8, 63·36, 148·4

XVII (pages 265, 266, 267)

(1) When $x = 0{\cdot}9$, $A = 1{\cdot}80155$

(2) When $x = 0{\cdot}09$, $A = 0{\cdot}16567$

(3) When $x = 0{\cdot}8$, $A = 6{\cdot}7840$

(4) True area 78·54 sq. in., area by Simpson's Rule 78·177 sq. in., error 0·458 per cent.; area by Trapezoidal Rule 77·614 sq. in., error 1·171 per cent.

(5) Area = 31 sq. in., $\bar{x} = 5{\cdot}392''$, $V_{OY} = 1051$ cub. in., $I_{OY} = 1099$ inch units

(6) Area = 31·14 sq. in., $\bar{y} = 1{\cdot}935''$, $V_{OX} = 378{\cdot}7$ cub. in., $I_{OX} = 171{\cdot}1$ inch units

(7) 1st derived fig. 10·11 sq. in., 2nd derived fig. 4·81 sq. in., $\bar{y} = 1{\cdot}947''$, $I_{OX} = 173{\cdot}2$ inch units

(8) 1st derived fig. 18·67 sq. in., 2nd derived fig. 13·59 sq. in., $\bar{x} = 5{\cdot}419''$, $I_{OY} = 1101$ inch units

(9) Area = 36·19 sq. in., $\bar{x} = 4{\cdot}127''$, $V_{OY} = 938{\cdot}5$ cub. in., $I_{OY} = 764{\cdot}3$ inch units

(10) Area = 36·36 sq. in., $\bar{y} = 2{\cdot}827''$, $V_{OX} = 645{\cdot}9$ cub. in., $I_{OX} = 369{\cdot}5$ inch units

(11) 1st derived fig. 17·025 sq. in., 2nd derived fig. 10·25 sq. in., $\bar{y} = 2{\cdot}809''$, $I_{OX} = 369$ inch units

(12) 1st derived fig. 18·73 sq. in., 2nd derived fig. 12·06 sq. in., $\bar{x} = 4{\cdot}141''$, $I_{OY} = 771{\cdot}8$ inch units

(13) 1st derived fig. 14·137 sq. in., 2nd derived fig. 8·836 sq. in., $\bar{y} = 3''$, $I = 318{\cdot}09$ inch units

(14) Area = 21·19 sq. in., 1st derived fig. 9·74 sq. in., 2nd derived fig. 5·63 sq. in., $\bar{y} = 2{\cdot}758''$, $I = 202{\cdot}7$ inch units

(16) 1268·3 cub. in. (17) $h = 8''$, volume = 1338·9 cub. in.

(18) $\frac{1}{6}(y_1 + 4y_2 + y_3)$

XVIII (pages 284, 285)

(1) 0·5211 (2) 3·418 (3) 2·295 (4) 15·31 (5) 1·062

(6) $\frac{2a}{\pi}$ (7) 5·576 (8) 12 (9) 241·3, 3·2 (10) 1·479

(11) 1·222

(12) $r = 2a \sin\theta \tan\theta$, $r^2 = 4a^2 \cot 2\theta \operatorname{cosec} 2\theta$, $r^2 = \frac{a^2 \cos 2\theta}{\cos^6 \theta}$

(13) 64·42 (14) $r^2 = 9 \cos 2\theta$, 4·5 (15) 2·546, 1·386, 1·215

XIX (pages 315, 316, 317, 318)

(1) 84·67 ft. tons, 112·4 ft. tons, 60·17 ft. tons

(2) 50 ft. tons, 50 ft. tons, 37·5 ft. tons

(3) $M = 12{\cdot}3\,x - 0{\cdot}75x^2$, $M = 12{\cdot}3x - 12(x-4) - 0{\cdot}375(x-8)^2$, $M_{max} = 50{\cdot}46$ ft. tons when $x = 8{\cdot}4'$

(4) For AC, $M = \dfrac{80x}{9}$; for CB, $M = \dfrac{80x}{9} - \dfrac{(x-10)^3}{30}$; $M_{max} = 144{\cdot}8$ ft. tons when $x = 19{\cdot}43'$

(5) For BC, $M = 11{\cdot}7x - \dfrac{x^3}{72}$; for CA, $M = 11{\cdot}7x - 13{\cdot}5(x-12) - \dfrac{3}{4}(x-18)^2$; $M_{max} = 130{\cdot}74$ ft. tons when $x = 16{\cdot}76'$

(6) $M = W(l-x) + \dfrac{w}{2}(l-x)^2$, $\dfrac{dy}{dx} = \dfrac{W}{EI}\left(lx - \dfrac{x^2}{2}\right) + \dfrac{w}{2EI}\left(l^2x - lx^2 + \dfrac{x^3}{3}\right)$,

$y = \dfrac{W}{EI}\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right) + \dfrac{w}{2EI}\left(\dfrac{l^2x^2}{2} - \dfrac{lx^3}{3} + \dfrac{x^4}{12}\right)$, at the end $\delta = \dfrac{Wl^3}{3EI} + \dfrac{wl^4}{8EI}$

(7) $M = \dfrac{W}{2}(a-x) + \dfrac{w}{2}(a^2 - x^2)$, $\dfrac{dy}{dx} = \dfrac{W}{2EI}\left(ax - \dfrac{x^2}{2}\right) + \dfrac{w}{2EI}\left(a^2x - \dfrac{x^3}{3}\right)$,

at the centre $\delta = \dfrac{Wa^3}{6EI} + \dfrac{5wa^4}{24EI}$

(8) Fixing couple $= \dfrac{Wa}{4} + \dfrac{wa^2}{3}$, $M = \dfrac{W}{2}\left(\dfrac{a}{2} - x\right) + \dfrac{w}{2}\left(\dfrac{a^2}{3} - x^2\right)$,

$\dfrac{dy}{dx} = \dfrac{W}{2EI}\left(\dfrac{ax}{2} - \dfrac{x^2}{2}\right) + \dfrac{w}{2EI}\left(\dfrac{a^2x}{3} - \dfrac{x^3}{3}\right)$, at the centre $\delta = \dfrac{Wa^3}{24EI} + \dfrac{wa^4}{24EI}$

(9) $M = \dfrac{w}{6l}(l-x)^3$, $\dfrac{dy}{dx} = \dfrac{w}{6lEI}\left\{l^3x - \dfrac{3l^2x^2}{2} + lx^3 - \dfrac{x^4}{4}\right\}$,

$y = \dfrac{w}{6lEI}\left\{\dfrac{l^3x^2}{2} - \dfrac{l^2x^3}{2} + \dfrac{lx^4}{4} - \dfrac{x^5}{20}\right\}$, $y_{max} = \dfrac{wl^4}{30EI}$

(10) $M = \dfrac{wa}{2}(a-x) - \dfrac{w}{6a}(a-x)^3$, $y_{max} = \dfrac{2wa^4}{15EI}$

$\dfrac{dy}{dx} = \dfrac{wa}{2EI}\left(ax - \dfrac{x^2}{2}\right) - \dfrac{w}{6aEI}\left(a^3x - \dfrac{3a^2x^2}{2} + ax^3 - \dfrac{x^4}{4}\right)$

(11) Fixing couple $= \dfrac{5wa^2}{24}$, $M = \dfrac{wa}{2}(a-x) - \dfrac{w}{6a}(a-x)^3 - \dfrac{5wa^2}{24}$,

$\dfrac{dy}{dx} = \dfrac{wa}{2EI}\left(ax - \dfrac{x^2}{2}\right) - \dfrac{w}{6aEI}\left(a^3x - \dfrac{3a^2x^2}{2} + ax^3 - \dfrac{x^4}{4}\right) - \dfrac{5wa^2x}{24}$

$y_{max} = \dfrac{7wa^4}{240EI}$

(12) For AC, $M = W(b-x)$, $\dfrac{dy}{dx} = \dfrac{W}{EI}\left(bx - \dfrac{x^2}{2}\right)$, $y = \dfrac{W}{EI}\left(\dfrac{bx^2}{2} - \dfrac{x^3}{6}\right)$;

for CB, $M = 0$, $\dfrac{dy}{dx} = \dfrac{Wb^2}{2EI}$, $y = \dfrac{W}{EI}\left(\dfrac{b^2x}{2} - \dfrac{b^3}{6}\right)$; at C, $\delta = \dfrac{Wb^3}{3EI}$,

at B, $\delta = \dfrac{Wb^2}{6EI}(3l-b)$

(13) At B, $\delta = \dfrac{W}{6EI}\{2a^3 - 3ab^2 + b^3\}$; at E, $\delta = \dfrac{W}{3EI}\{a^3 - 3ab^2 + 2b^3\}$

(14) At C, $\delta = \dfrac{wb^4}{8EI}$; at B, $\delta = \dfrac{wb^3}{24EI}(4a-b)$

(15) At B, $\delta = \dfrac{wb}{24EI}\{8a^3 - 4ab^2 + b^3\}$; at E, $\delta = \dfrac{wb}{3EI}\{a^3 - 2ab^2 + b^3\}$

(16) $y = \dfrac{w}{24c}(x^4 - 4lx^3 + 6l^2x^2) + \dfrac{W}{6c}(x^3 - 3lx^2)$ (17) $\dfrac{8}{15}, \dfrac{136}{147}$

(18) 569·2 lb. per sq. in., 472·6 lb. per sq. in., 29·53 lb. per sq. in.

(19) Web vertical 234·8 lb. per sq. in., web horizontal 31·88 lb. per sq. in.

(20) 163·5 lb. per sq. in. (21) 536 lb. per sq. in., 4·074

(22) 76·45 lb. per sq. in. (23) 2·012″, 3·867″ (24) 13·52″

XX (pages 345, 346, 347, 348)

(1) $(x^4 - y^4) - 4(x-y) - 3 = 0$ (2) $(y^2+1)^3 = (x^3+1)^2$

(3) $y = \dfrac{3 + e^{2\tan^{-1}x}}{3 - e^{2\tan^{-1}x}}$ (4) $y = \tan\left\{\log_e \sqrt{\dfrac{3(x-1)}{x+1}}\right\}$

(5) $y = 2\tan^{-1}(e^{2\sin x})$ (6) $y = \tan^{-1}(e^{-\cos 2x})$

(7) $y = 1 - e^{-\tan x}$ (8) $y = \dfrac{1}{2}\left\{1 - \dfrac{1}{2x^2}\right\}$

(9) $y = e^{\frac{1}{2}\left(1-\frac{x^2}{y^2}\right)}$ (10) $xy = e^{\frac{x}{y}-1}$

(11) $\log_e x = \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2y-x}{\sqrt{3}x}$ (12) $y = \dfrac{\sin x - 1}{x} - \cos x$

(13) $y = \dfrac{1}{\sqrt{5}}\sin(2x - \alpha) + 1{\cdot}4e^{-x}$ where $\tan\alpha = 2$

(14) $y = 1 + e^{-\sin^{-1}x}$ (15) $y = \sin x$ (16) $y = \dfrac{4}{3}\sin^3 x\cos x$

(17) $y(1+e^x) = \dfrac{x^3}{3} + e^x(x^2 - 2x + 2) - 2$ (19) $y^2 = \dfrac{2x^2}{x^4+1}$

(20) $t = 23{\cdot}26$ secs., 7° 54′

(21) 971 f.s., 559·3 f.s. downwards, 1120 f.s., 150° 4′

(22) $k = 0{\cdot}04682$, 289·2 ft., 61 f.s.

(23) $k = 0{\cdot}0004728$, $h = 256{\cdot}9$ ft., $90{\cdot}6$ f.s.

(25) $pr^2(r_1^2 - r_0^2) = r^2(p_1r_1^2 - p_0r_0^2) + r_0^2r_1^2(p_0 - p_1)$

(26) $p = \mathrm{A}r^{a-1} + \dfrac{b}{a-1}$, $\mathrm{A} = 197{\cdot}4$, $r = 7{\cdot}324''$

(27) 36130 ft. lb., 33950 ft. lb. (28) 36890 ft. lb.

(29) $\mathrm{P} = \mathrm{P}_0e^{ct}$, $c = 0{\cdot}01273$, $\mathrm{P} = 41{\cdot}01$

(30) $v = v_0e^{-\frac{t}{\mathrm{KR}}}$, $1{\cdot}48 \times 10^8$

(31) $ch = \dfrac{\gamma}{\gamma-1}\left\{p_0^{1-\frac{1}{\gamma}} - p^{1-\frac{1}{\gamma}}\right\}$, $c = \dfrac{p}{\mathrm{R}t}^{1-\frac{1}{\gamma}}$, $h = \dfrac{\gamma \mathrm{R}t_0}{\gamma-1}\left\{1 - \dfrac{t}{t_0}\right\}$

(32) $h = \mathrm{C} - \dfrac{c^2}{2gr^2}$ (33) $\mathrm{C} = 10{\cdot}67(1 - e^{-9{\cdot}375t})$, $6{\cdot}5$

(34) $\mathrm{C} = 0{\cdot}707\sin(500t - 0{\cdot}7854) + e^{-500t}$, $\mathrm{C} = 0{\cdot}707\sin(500t - 0{\cdot}7854)$

XXI (pages 372, 373, 374, 375)

(1) $\mathrm{A}\sin 2\sqrt{3}x + \mathrm{B}\cos 2\sqrt{3}x$ (2) $\mathrm{A}e^{3\sqrt{2}x} + \mathrm{B}e^{-3\sqrt{2}x}$

(3) $\mathrm{A}e^{-2x} + \mathrm{B}e^{-5x}$ (4) $e^{5x}(\mathrm{A} + \mathrm{B}x)$

(5) $e^{-x}(\mathrm{A}\sin 3x + \mathrm{B}\cos 3x)$ (6) $\mathrm{A}e^{x} + \mathrm{B}e^{2x} + \mathrm{C}e^{-2x}$

(7) $\mathrm{A}e^{x} + \mathrm{B}\sin 2x + \mathrm{C}\cos 2x$

(8) $\mathrm{A}e^{-2x} + e^{x}(\mathrm{B}\sin\sqrt{3}x + \mathrm{C}\cos\sqrt{3}x)$

(9) $\mathrm{A}e^{2x} + e^{-x}(\mathrm{B}\sin\sqrt{3}x + \mathrm{C}\cos\sqrt{3}x)$

(10) $(\mathrm{A}e^{x} + \mathrm{C}e^{-x})\sin\sqrt{3}x + (\mathrm{B}e^{x} + \mathrm{D}e^{-x})\cos\sqrt{3}x$

(11) $0{\cdot}5\cos 2\sqrt{5}t$, $0{\cdot}8944\sin 2\sqrt{5}t$, $1{\cdot}024\sin(2\sqrt{5}t + 0{\cdot}5096)$

(12) $0{\cdot}25(e^{2\sqrt{5}t} + e^{-2\sqrt{5}t})$, $0{\cdot}4472(e^{2\sqrt{5}t} - e^{-2\sqrt{5}t})$
$0{\cdot}6972e^{2\sqrt{5}t} - 0{\cdot}1972e^{-2\sqrt{5}t}$

(13) $0{\cdot}559e^{-2t}\sin(4t + 1{\cdot}108)$, $e^{-2t}\sin 4t$, $0{\cdot}9434\sin(4t + 0{\cdot}5585)$

(14) $e^{-2t}(0{\cdot}5 + t)$, $4te^{-2t}$, $e^{-2t}(0{\cdot}5 + 5t)$

(15) $0{\cdot}75e^{-t} - 0{\cdot}25e^{-3t}$, $2(e^{-t} - e^{-3t})$ $2{\cdot}75e^{-t} - 2{\cdot}25e^{-3t}$

(16) $x = \dfrac{1}{6}\cos 17{\cdot}38t$, $\mathrm{T} = 0{\cdot}3616$ sec., $n = 166$

(17) $x = 0{\cdot}1927e^{-4.025t}\sin(10{\cdot}22t + 1{\cdot}196)$, $\mathrm{T} = 0{\cdot}6148$ sec.

(18) $8{\cdot}422\sin(nt + 1{\cdot}125)$ (19) $0{\cdot}8333e^{-7.485t}\sin 12t$

(20) $0{\cdot}3574\sin(1{\cdot}6t - 1{\cdot}030)$, $-0{\cdot}2496\sin(2{\cdot}4t + 1{\cdot}116)$

(21) $0{\cdot}02625a\sin(5t - 0{\cdot}896)$

(22) $1{\cdot}0417a\sin 4\pi t$, $1{\cdot}333a\sin 10\pi t$, $-0{\cdot}8a\sin 30\pi t$

(23) $v = 809{\cdot}1e^{-27640t} - 309{\cdot}1e^{-72360t}$

(24) $v = 500e^{-40000t}(1 + 40000t)$

(25) $v = 354{\cdot}6e^{-6250t}\sin(7262t + 0{\cdot}859)$

(26) 246·7 lb. (27) 4·666″, 0·4666″ (28) 16·44 tons

XXII (pages 400, 401, 402)

(1) $\frac{1}{2}ab\cos c$ (2) 0 (3) $\frac{1}{2}ab\sin c,\quad \frac{1}{2}a^2$

(4) $\frac{1}{2}ab\cos c,\quad \frac{1}{2}b^2$ (5) $-\frac{1}{2}ab\sin c,\quad \frac{1}{2}b^2$

(6) $$\frac{2m}{\pi}\left[\left(\frac{\pi^2}{1}-\frac{4}{1^3}\right)\sin x+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin 3x+\left(\frac{\pi^2}{5}-\frac{4}{5^3}\right)\sin 5x+\right.$$
$$\left.-\left(\frac{\pi^2}{2}\sin 2x+\frac{\pi^2}{4}\sin 4x+\frac{\pi^2}{6}\sin 6x+\ \ldots\right)\right]$$

(7) $$2m\left[\left(\frac{\pi^2}{1}-\frac{6}{1^3}\right)\sin x-\left(\frac{\pi^2}{2}-\frac{6}{2^3}\right)\sin 2x+\left(\frac{\pi^2}{3}-\frac{6}{3^3}\right)\sin 3x-\ldots\right]$$

(8) $$\frac{m\pi^3}{4}-\frac{6m}{\pi}\left[\left\{\left(\frac{\pi}{1}\right)^2-\frac{4}{1^4}\right\}\cos x-\left(\frac{\pi}{2}\right)^2\cos 2x+\left\{\left(\frac{\pi}{3}\right)^2-\frac{4}{3^4}\right\}\cos 3x\right.$$
$$\left.-\left(\frac{\pi}{4}\right)^2\cos 4x+\ \ldots\right]$$

(9) $$\frac{2}{\pi}\left[\frac{1}{2}(e^\pi-1)-(e^\pi+1)\left\{\frac{1}{2}\cos x+\frac{1}{10}\cos 3x+\frac{1}{26}\cos 5x+\ \ldots\right\}\right.$$
$$\left.+(e^\pi-1)\left\{\frac{1}{5}\cos 2x+\frac{1}{17}\cos 4x+\frac{1}{37}\cos 6x\ \ldots\right\}\right]$$

(10) $$\frac{2}{\pi}\left[\frac{1}{2}(1-e^{-\pi})+(1+e^{-\pi})\left\{\frac{1}{2}\cos x+\frac{1}{10}\cos 3x+\frac{1}{26}\cos 5x+\ \ldots\right\}\right.$$
$$\left.+(1-e^{-\pi})\left\{\frac{1}{5}\cos 2x+\frac{1}{17}\cos 4x+\frac{1}{37}\cos 6x+\ \ldots\right\}\right]$$

(11) $$\frac{\pi^2}{3}-4\left\{\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-\frac{1}{16}\cos 4x+\frac{1}{25}\cos 5x-\ldots\right\}$$

(12) $$2\left[\left(\frac{\pi^2}{1}-\frac{6}{1^3}\right)\sin x-\left(\frac{\pi^2}{2}-\frac{6}{2^3}\right)\sin 2x+\left(\frac{\pi^2}{3}-\frac{6}{3^3}\right)\sin 3x-\ \ldots\right]$$

(13) $$\frac{e^\pi-e^{-\pi}}{\pi}\left[\frac{1}{2}-\left(\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x-\frac{4}{17}\sin 4x+\ \ldots\right)\right.$$
$$\left.-\left(\frac{1}{2}\cos x-\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x-\frac{1}{17}\cos 4x+\ \ldots\right)\right]$$

(14) $$\frac{2c}{\pi}\left[\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\frac{1}{4}\sin 4\theta+\ \ldots\right]\text{ where }\theta=\frac{\pi x}{c}$$

(15) $$\frac{c^2}{3}-\frac{4c^2}{\pi^2}\left[\cos\theta-\frac{1}{4}\cos 2\theta+\frac{1}{9}\cos 3\theta-\frac{1}{16}\cos 4\theta+\ \ldots\right]$$
where $\theta=\frac{\pi x}{c}$

(16) $11{\cdot}798-1{\cdot}292\sin\theta+9{\cdot}960\cos\theta+1{\cdot}050\cos 2\theta+0{\cdot}015\sin 3\theta$
$+\,0{\cdot}012\cos 3\theta$

(17) $11{\cdot}226 - 4{\cdot}790 \sin\theta + 9{\cdot}956 \cos\theta + 0{\cdot}013 \sin 2\theta + 1{\cdot}431 \cos 2\theta + 0{\cdot}175 \sin 3\theta + 0{\cdot}013 \cos 3\theta$

(18) $11{\cdot}724 - 2{\cdot}816 \sin\theta + 9{\cdot}949 \cos\theta - 0{\cdot}001 \sin 2\theta + 1{\cdot}122 \cos 2\theta + 0{\cdot}037 \sin 3\theta + 0{\cdot}015 \cos 3\theta$

(19) $10{\cdot}810 + 0{\cdot}664 \sin\theta + 10{\cdot}408 \cos\theta + 0{\cdot}794 \sin 2\theta + 0{\cdot}890 \cos 2\theta + 0{\cdot}022 \sin 3\theta - 0{\cdot}065 \cos 3\theta$

(20) $10{\cdot}731 + 2{\cdot}417 \sin\theta + 10{\cdot}236 \cos\theta + 1{\cdot}108 \sin 2\theta + 0{\cdot}528 \cos 2\theta - 0{\cdot}032 \sin 3\theta - 0{\cdot}073 \cos 3\theta$

(21) $13{\cdot}627 + 3{\cdot}157 \sin\theta + 11{\cdot}055 \cos\theta + 1{\cdot}657 \sin 2\theta - 0{\cdot}065 \cos 2\theta - 0{\cdot}188 \sin 3\theta - 0{\cdot}145 \cos 3\theta$

(22) $11{\cdot}100 + 9{\cdot}848 \sin\theta + 1{\cdot}736 \cos\theta + 0{\cdot}643 \sin 2\theta + 0{\cdot}766 \cos 2\theta$

(23) $10 + 3{\cdot}265 \sin\theta + 5{\cdot}030 \cos\theta + \cos 2\theta + 0{\cdot}2 \sin 4\theta$

XXIII (pages 423, 424, 425)

(1) $(n+4)(n+5)$, 132

(2) $(n-2)(n-5)$, 28

(3) $n^3 - 10n + 18$, 450

(4) $\frac{1}{6}\left\{2n^3 + 9n^2 + 25n + 18\right\}$, 154

(5) $n^4 - 10n^2 + 36$, 9036

(6) $n^3 - 3n^2 + 8n + 32$, 122

(7) 10·96, 10·96

(8) 31·68, 31·68

(9) 257·259456

(10) 1·8527598, 1·8527597

(11) 1·83409, 2·22510

(12) 0·5827

(13) 0·8013

(14) 3·813

(15) 5·456

(16) 3·340755

(17) 15·75, 17·0

(18) 0·5252, 0·01560

(19) 1·0402, 0·02678

XXIV (pages 444, 445)

(1) $A + B = 12{\cdot}07_{49^\circ\,3'}$, $A - B = 5{\cdot}694_{353^\circ\,37'}$

(2) $A + B = 9{\cdot}181_{222^\circ\,29'}$, $A - B = 34{\cdot}96_{53^\circ\,14'}$

(3) 5·148, 9·982

(4) $6{\cdot}233_{79^\circ\,56'}$

(5) $8{\cdot}933_{56^\circ\,28'}$

(6) $10{\cdot}09_{76^\circ\,41'}$

(7) $13{\cdot}21_{100^\circ\,22'}$

(8) $16{\cdot}44_{321^\circ\,2'}$

(9) 5·416, 72° 47′, 35° 32′, 59° 58′

(10) 29·68, 54° 48′, 55° 4′, 54° 20′

(11) 16·90, 78° 54′, 40° 21′, 51° 41′

(12) 9·69, 54° 23′, 55° 10′, 54° 32′

(13) $1874_{137^\circ\,48'}$

(14) $130{\cdot}7_{80^\circ\,10'}$

(15) Magnitude $= \sqrt{b^2 + a^2\beta^2 + 2ab\beta^2 t + b^2\beta^2 t^2}$,

direction $= \theta + \tan^{-1} \beta\left(t + \frac{a}{b}\right)$

(16) See paragraph 212.

(17) Magnitude $= \sqrt{b^2 + 4a^2\beta^2t^2 + 8ab\beta^2t^3 + 4b^2\beta^2t^4}$,

direction $= \theta + \tan^{-1} 2\beta t\left(t + \frac{a}{b}\right)$

(18) 51° 3′, 3·62 (19) 4076 lb.

(20) Going N., 35 m.p.h. due N.; going E., 52·2 m.p.h. 16° 42′ S. of E.; going S., 65 m.p.h. due S.; going W., 52·2 m.p.h. 16° 42′ S. of W.

(21) Going N., 65·69 m.p.h. 12° 26′ E. of N.; going E., 65·69 m.p.h. 12° 26′ N. of E.; going S., 38·56 m.p.h. 21° 31′ E. of S.; going W., 38·56 m.p.h. 21° 31′ N. of W.

XXV (pages 459, 460, 461, 462)

(1) $y = \frac{56 \cdot 81}{x + 4 \cdot 984}$, $y = 8 \cdot 718e^{-0 \cdot 08504x}$

(2) $t = 110 \cdot 7m^{1 \cdot 21}$, $sm^{0 \cdot 21} = 0 \cdot 009036$

(3) $y = 552 \cdot 7 \times 36 \cdot 52^{-x}$ (4) $F = 254 \cdot 5r - 17$

(5) $M = 6 \cdot 08\sqrt{M^2 + T^2} - 26190$ (6) $a = -0 \cdot 0646$

(7) $y = 0 \cdot 0443x + 3 \cdot 275 \log_{10} x$ (8) $y = 10 \cdot 14e^{0 \cdot 2717x}$

(9) $y = 9 \cdot 08x^2 + 1 \cdot 206 \times 10^x$

(10) $y = 7 \cdot 591 + 1 \cdot 369x^{1 \cdot 518}$, 23·34

(11) $y = 0 \cdot 5 + 4 \cdot 454e^{0 \cdot 3465x}$, 18·31

(12) $y = 2 \cdot 452 \times 1 \cdot 523^x - 0 \cdot 387$, 8·275

(13) $y = 2 \cdot 76(x + 5 \cdot 72)^{1 \cdot 24}$, 34·79

(14) $y = 0 \cdot 404x + 1 \cdot 225x \log_{10} x$

(15) $y = 0 \cdot 986x^{0 \cdot 202x}$ (16) $y = 10 \cdot 12 + 1 \cdot 76e^{0 \cdot 215x}$

(17) $y = 0 \cdot 762(x + 2 \cdot 26)^{1 \cdot 74}$ (18) $y = 15 \cdot 62 + 0 \cdot 746x^{1 \cdot 875}$

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