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THEORY

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THEORY

OF

DIFFERENTIAL EQUATIONS.

PART IV.

PARTIAL DIFFERENTIAL EQUATIONS.

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CHAPTER XII.

General Integrals of Equations of Orders Higher than the First.

The present chapter is devoted to general explanations, connected with the existence-theorem and with the kinds of integrals that are possessed by equations of order higher than the first, particularly those equations having general integrals without partial quadratures. For the most part, though not entirely, the equations considered are of the second order in two independent variables. The discussion is based mainly upon the memoir of Ampère, quoted in § 179, and upon the first chapter of the memoir by Imschenetsky, quoted in § 180.

178. After the discussion of equations of the first order which involve only a single dependent variable, and a discussion of sets of equations of the first order which involve several dependent variables and are integrable by any generalisation of any process that is effective for equations involving only one dependent variable, the next subject for consideration is manifestly the theory of partial differential equations of the second order. We shall begin with the simplest aggregate of such equations; and, for that aggregate, we shall assume that there is only a single dependent variable z, and that there are only two independent variables x and y. Denoting the first derivatives of z by p and q, and the second derivatives by r, s and t, as usual, we may take an equation of the second order in the form

$$f(x, y, z, p, q, r, s, t) = 0.$$

Such an equation certainly possesses integrals. We shall assume that f either is in a form or can be brought into a form which makes it a regular function of its arguments: in that case, we have seen that Cauchy's existence-theorem applies and that integrals, characterised by certain properties, do exist. Thus there is an integral z determined by the characteristic properties:—

(i) it is a regular function of x and y within fields of variation round a and b, given by

$$|x-a| \leqslant \rho, \quad |y-b| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) when $x=a$, the integral z reduces to $\phi_0(y)$ and the derivative $\frac{\partial z}{\partial x}$ reduces to $\phi_1(y)$, where $\phi_0(y)$ and $\phi_1(y)$ are regular functions of y within the domain of b and are otherwise arbitrary.

There is a single condition, of a formal type, which must be satisfied, or the existence of the foregoing integral cannot be established: it is that, if

$$\phi_0(b)=c, \quad \phi_1(b)=\lambda, \quad \phi_0'(b)=\mu, \quad \phi_1'(b)=\beta, \quad \phi_0''(b)=\gamma,$$

the equation

$$f(a, b, c, \lambda, \mu, \theta, \beta, \gamma)=0,$$

regarded as an equation in θ, should have at least one simple root. When this condition is satisfied, each simple root θ determines an integral z as above: and the integral thus associated with that simple root is unique.

When the condition is not satisfied so that $f=0$, regarded as an equation in θ, has no simple root, then the existence of such an integral is not established. But it may then happen that $f=0$, regarded as an equation in γ, has simple roots. In that case, the theorem establishes the existence of an integral z, regular as before in the domain of a and b, but now such that, when $y=b$, the integral z reduces to $\psi_0(x)$ and the derivative $\frac{\partial z}{\partial y}$ reduces to $\psi_1(x)$, where $\psi_0(x)$ and $\psi_1(x)$ are regular functions of x within the domain of a and are otherwise arbitrary.

If, however, $f=0$, regarded as an equation in γ, has no simple roots and, as before, has no simple roots when regarded as an equation in θ, it may have simple roots when regarded as an equation in β. In that case, there is an integral of a similar type, obtainable most easily through a transformation of the independent variables.

Thus the equation is proved to possess an integral characterised by definite properties except only in the case where $f=0$, regarded as an equation in θ, β, γ in turn, possesses no simple roots, so that we should have

$$\frac{\partial f}{\partial \theta}=0, \quad \frac{\partial f}{\partial \beta}=0, \quad \frac{\partial f}{\partial \gamma}=0,$$

concurrently with $f=0$. Returning now to the differential equation

$$f(x, y, z, p, q, r, s, t)=0,$$

the quantities

$$\frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial s}, \quad \frac{\partial f}{\partial t}$$

will usually be variable quantities; and then values can usually be given to their arguments such that, while f vanishes for those values, not all the three quantities $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ vanish. It may, however, happen that there are values of the arguments which satisfy the four equations

$$f=0, \quad \frac{\partial f}{\partial r}=0, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=0;$$

and then the Cauchy existence-theorem does not apply. In that case, there may be variable values of z (and of x and y) which satisfy all the four equations: such values of z, if any, will be called *singular integrals*. In all other cases, the existence-theorem establishes the existence of an integral with the specified properties: as its existence was first established by Cauchy, it frequently is called the *Cauchy integral*.

It will be noticed that two arbitrary functions enter into the expression of the specified properties.

The form thus stated is the simplest form of the Cauchy integral, in so far that the initial conditions are in their simplest form. As indicated (§ 24) in the discussion of the existence theorem, the initial conditions can be taken in an ampler form as follows:

> For all the values of x and y satisfying a given relation that is not critical with regard to the form of the differential equation, that is, for all points of a given analytical plane

curve, the variable z and its derivative in any direction, that is not tangential to the curve, acquire values represented by arbitrarily assigned functions of x and y.

This undoubtedly is more general. However, as it arises through a transformation of the variables from the simpler case, and does not otherwise add any element of generality to the solution, we shall usually be content to take the initial conditions in their simpler form.

It will be convenient to assume, for the purpose of immediate discussion as well as for simplicity of statement, that the equation can be resolved with regard to r, so that it takes the form

$$r = g(x, y, z, p, q, s, t):$$

the original equation can be regarded as the aggregate of all these resolved equations. The Cauchy integral is then a regular function of the variables in the domain of a and b: two arbitrary functions $\phi_0(y)$ and $\phi_1(y)$, subject solely to the condition of being regular in the domain of b, affect its form: and it is a unique integral as satisfying these conditions.

But while it thus possesses arbitrarily assigned elements, which frequently can be specialised so as to include integrals otherwise obtained, there is no certainty that specialisation or definition of these elements will secure that the integral shall include every integral; and therefore there is no certainty that the Cauchy integral is completely comprehensive. A question thus arises as to whether the equation possesses any integral that is more comprehensive; a further question is stirred as to the different kinds of integral that the equation may possess. Even so, limiting assumptions have been made: all singularities and other deviations from regularity in the form of the original equation have been avoided.

Two Definitions of the General Integral.

179. It is usual to assign, to the most comprehensive integral known, the name of the *general integral*, for partial equations of order higher than the first; but there are two definitions of the general integral. One of these definitions is due* to Ampère;

* *Journ. de l'Éc. Polytechnique, cah.* XVII (1815), p. 550.

the other, given* by Darboux, is based upon the researches of Cauchy.

According to Ampère, an integral (no matter how obtained) is general when the only relations, which are free from the arbitrary elements and to which the integral leads among the variables and the derivatives of the dependent variable up to any order whatever, are those expressed by the differential equation and by equations deduced from the equation by differentiation. Thus, in this sense,

$$z = \phi(x + y) + \psi(x - y)$$

is a general integral of the equation

$$r = t;$$

for the relations to which the integral equation gives rise are

$$\frac{\partial^{2+m+n} z}{\partial x^{2+m} \partial y^n} = \frac{\partial^{2+m+n} z}{\partial x^m \partial y^{2+n}},$$

for all positive integer values of m and n, and all of these relations are derivatives of the differential equation. But

$$z = a(x^2 + y^2) + 2hxy + bx + cy + d,$$

where a, b, c, d, h are arbitrary constants, is not a general integral of the same equation: for it satisfies relations

$$\frac{\partial^3 z}{\partial x^3} = 0, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 z}{\partial y^3} = 0,$$

no one of which can be derived from the differential equation, though derivatives of the differential equation are not inconsistent with the relations.

According to Darboux, an integral (no matter how obtained) is general if the arbitrary elements which it contains can be determined so as to give the Cauchy integral, involving assigned functional values to z and to one of its derivatives in specified circumstances. Thus, in this sense also,

$$z = \phi(x + y) + \psi(x - y)$$

is a general integral of the equation

$$r = t:$$

* *Théorie générale des surfaces*, t. II, pp. 97, 98.

for, if the initial conditions of the Cauchy integral are that $z=f(y)$ and $\frac{\partial z}{\partial x}=g(y)$, when $x=a$, then if

$$\sigma(u)=\tfrac{1}{2}f(u)+\tfrac{1}{2}\int_0^u g(u)\,du,$$

$$\rho(v)=\tfrac{1}{2}f(v)-\tfrac{1}{2}\int_0^v g(v)\,dv,$$

the required Cauchy integral is given by

$$z=\sigma(x+y-a)+\rho(a-x+y),$$

the functions ϕ and ψ thus having been appropriately determined.

The tenour of the Ampère definition of a general integral is different from that of the Cauchy general integral. Though the difference between the integrals is often of no account, yet for particular equations it can be significant: and therefore it is worth while to estimate which of the two integrals is the more comprehensive.

It seems clear that, in the matter of comprehensiveness, the Cauchy general integral has some advantage over the Ampère general integral.

The limitations, which are imposed in the course of establishing the Cauchy integral, are of a qualitative kind: they are restrictions as to the position and the extent of the domains within which the various functions that occur are regular, or they are hypotheses as to the resolubility of the differential equation: but no positive relations (other than derivatives of the differential equation) are used or are required in order to secure the convergence of the power-series obtained, or the continuity of the functions, or the freedom of the initial conditions. Consequently, an integral that is general in the sense of the Darboux-Cauchy definition is general also in the sense of Ampère's definition. As against this inference, it must be borne in mind that, however arbitrarily the initial conditions are chosen either as regards the position of the domain or the forms of the assigned functions, the Cauchy integral is always a regular function of the variables and that deviations from regularity have been excluded from consideration: there is no such restriction on the Ampère integral.

The restriction can often be partly removed by considering a part of a domain and by taking a regular expression for a branch of a non-regular integral in that region: but this modification is not always possible, and there are deviations from regularity such that the removal of the restriction cannot be made complete.

But on the other hand, classes of equations can be constructed which have integrals that are general in the Ampère sense and certainly are not general in the Darboux-Cauchy sense. It is true that such classes of equations are special in type, and that a similar difference need not exist for equations that are not of any special type: but the mere existence of such equations is a limitation upon the comparative comprehensiveness of the Ampère integral.

An instance is adduced* by Goursat in the example

$$s = yq:$$

simple quadratures lead to an integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

where θ and ϕ are arbitrary functions, and a is any constant. Now the quantity z', where

$$z' = \int_a^y e^{xu} \phi(u)\, du,$$

satisfies the differential equation: all the relations between the variables and the derivatives of z', which are free from arbitrary elements, are constituted by the differential equation and by derivatives of the differential equation. Thus z' is an integral of the differential equation which is general in the sense of Ampère's definition: it clearly is less comprehensive than the integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

which is easily seen to be general in the Darboux-Cauchy sense.

We shall return to a further discussion of the matter when dealing with linear equations of the second order. It is manifest that the foregoing explanations can be applied to equations of

* *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II, p. 212. This treatise, when quoted hereafter, will be referred to as Goursat, *S. O.*

order higher than the second: and meanwhile, we may regard the general integral as one that is largely (though not universally) comprehensive.

Classes of Integrals.

180. Before proceeding to the discussion of certain properties of general integrals, taken according to either of the definitions just indicated, we shall mention some other classes of integrals and briefly outline some of their relations with one another*.

Speaking broadly, we may define an integral of a partial differential equation of the second (or of higher) order as a relation between the variables such that, in virtue of the relation itself and of derivatives from it, the differential equation is satisfied. When the integral relation does not involve derivatives of the dependent variable, it is sometimes called a *primitive*: the more frequent practice is not to give any special title to such a relation. When the integral relation does involve derivatives of the dependent variable, these derivatives being of order lower than that of the equation, the integral relation is usually called an *intermediate integral*: it has not been proved, and it is not a fact, that intermediate integrals are always possessed by differential equations of order higher than the first.

We have already referred to *general integrals*: after the provisional explanations and for the sake of simplicity, we shall regard the Ampère definition as giving a necessary qualification (though not a complete qualification) for a general integral. A *particular integral* is a special case of the general integral: it is distinguished by the property that while, in conjunction with its derivatives, it leads to the differential equation and to derivatives of the differential equation, it leads also to other differential equations not obtainable as derivatives of the differential equation.

It has been customary with writers, following Lagrange, to refer to *complete integrals* or *complete primitives*: Ampère however considered, and gave† reasons for considering, that such

* In this connection, reference may be made to the first chapter of Imschenetsky's memoir on equations of the second order with two independent variables, *Grunert's Archiv*, t. LIV (1872), pp. 209—360.

† See the memoir (p. 554) cited in § 179.

integrals are always particular integrals. Their special occurrence is due to the fact that Lagrange proceeded to a differential equation from an integral relation, by eliminating from the latter and from its derivatives, as many arbitrary elements as possible. Thus, let an integral equation

$$g(x, y, z, a_1, a_2, a_3, a_4, a_5) = 0$$

be given, involving five arbitrary constants. When the two first derivatives and the three second derivatives are formed, there are six equations in all: from these, the five constants can be eliminated and the eliminant may be a single equation of the second order. Also, if the constants be independent of one another, not more than a single equation will result, unless some functional combinations occur: and similarly, subject to the same exceptional occurrence of functional combinations, not more than five independent constants could be eliminated. Thus five is the greatest number of arbitrary constants which could be expected in an integral, when its generality depends on arbitrary constants alone; hence the name assigned* to such an integral by Lagrange.

There are other integrals of various types. We have seen that the equations

$$f = 0, \quad \frac{\partial f}{\partial r} = 0, \quad \frac{\partial f}{\partial s} = 0, \quad \frac{\partial f}{\partial t} = 0,$$

could be satisfied simultaneously; if they lead to an integral, on the analogy of the corresponding case for equations of the first order, it is called *singular*.

It is sufficiently clear that, just as in the case of equations of the first order where the more obvious classes of integrals are not sufficiently comprehensive to include all types of special integrals, so in the case of equations of higher order there will be integrals (which may be called *special*, for convenience), possessed by particular equations and not included in the preceding classes.

* A similar argument, applied to an equation

$$f(x, y, z, p, q, r, s, t) = 0,$$

shews that, if initial values chosen for x and y be regarded as pure constants, the values of the six quantities z, p, q, r, s, t for those initial values are connected by a single relation, so that it might be considered that there are five independent arbitrary constants at our disposal.

Frequently they will be peculiarly associated with the form of the equation when it is quite regular: when the equation is not regular, special integrals will frequently occur, particularly associated with singularities of the form or with other deviations from regularity*.

It is not unusual to attempt some classification of intermediate integrals, though this is the less important because such integrals do not always exist. Still, when they do exist, two classes are selected as being of wider range than others. If an intermediate integral involves one arbitrary function in its expression, it is usually called an *intermediate general integral.* If it involves two arbitrary independent constants (this being usually the greatest number of constants that can be eliminated from an integral and its two derivatives leading to an equation of the second order), it is sometimes called an *intermediate complete integral.* Well-known instances of equations possessing an intermediate general integral are provided by the equations

$$U(rt-s^2)+Rr+2Ss+Tt=V,$$

when certain conditions are satisfied, the quantities R, S, T, U, V being functions of x, y, z, p, q. An instance of an equation possessing a complete intermediate integral is given by

$$(rt-s^2)^2+(pt-qs)(ps-qr)=0:$$

the intermediate integral is

$$c^2q-cz+p=a,$$

where a and c are arbitrary constants.

Instances hereafter will occur freely in which it appears that a differential equation of the second order does not possess any intermediate integral. Thus the equation†

$$s=z$$

cannot possess an intermediate integral. Such an integral, if possessed, would have one of the forms

$$p=f(x, y, z, q), \quad q=g(x, y, z), \quad p=h(x, y, z).$$

* See the Supplementary Note, at the end of Chapter XVI.

† The example is quoted by Imschenetsky, (*l.c.*) p. 222, from Raabe.

If the first exists, the equation $s=z$ should be obtainable from

$$\left.\begin{aligned} p &= f \\ s &= \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial q} \\ r &= \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial q} \end{aligned}\right\}.$$

Obviously, the last of these relations cannot be used to obtain the equation $s=z$. It is clear that t must not occur: hence the second relation gives $\frac{\partial f}{\partial q}=0$, so that f does not involve q. Unless $\frac{\partial f}{\partial z}=0$, the second relation will reintroduce q, which ought not to remain if the equation $s=z$ is to be obtained. Hence f must not involve z: and then it is obvious that, as f does not involve z, no combination of

$$p=f, \quad s=\frac{\partial f}{\partial y},$$

can lead to $s=z$. Thus no relation of the form $p=f$ is an intermediate integral of the original equation. Similarly, no relation of the form $q=g(x, y, z)$ can be an intermediate integral of the original equation.

It must not however be assumed that, when an equation possesses a complete primitive, it necessarily possesses an intermediate complete integral. Let the primitive be

$$f(x, y, z, a_1, a_2, a_3, a_4, a_5)=0;$$

it is not generally possible to eliminate more than two of the arbitrary constants between $f=0$ and its two first derivatives: the eliminant generally contains three arbitrary constants, and thus it is not an intermediate complete integral. Indeed, the differential equation of the second order could not generally be deduced from the eliminant: for the three included constants could not generally be eliminated between the eliminant-equation and its two first derivatives. While this is the case in general, it is not the case universally: for constants may coalesce at either of the stages in such a way as to make the elimination possible.

The real importance of the intermediate integral, when it occurs, lies in the fact that it enables us to construct the primitive of an equation of the second order by two operations of what may be called grade unity: ordinarily the construction of the primitive is an irresoluble operation of grade two.

Ex. The relations of the various integrals to one another can be illustrated simply by reference to the equation

$$r=t,$$

of which a general integral (as has already been seen) is given by

$$z=\phi(x+y)+\psi(x-y),$$

where ϕ and ψ are arbitrary functions.

A complete integral is given by

$$z=a+\beta x+\gamma y+\delta xy+\epsilon(x^2+y^2);$$

as it can be expressed in the form

$$z=a+\tfrac{1}{2}(\beta+\gamma)u+\tfrac{1}{4}(2\epsilon+\delta)u^2$$
$$+\tfrac{1}{2}(\beta-\gamma)v+\tfrac{1}{4}(2\epsilon-\delta)v^2,$$

where $u=x+y$, $v=x-y$, it is a particular form of the general integral. Also, for this complete integral,

$$p=\beta+\delta y+2\epsilon x,$$
$$q=\gamma+\delta x+2\epsilon y,$$

so that

$$p+q=\beta+\gamma+(2\epsilon+\delta)(x+y),$$
$$p-q=\beta-\gamma+(2\epsilon-\delta)(x-y),$$

both of which are intermediate integrals: for the former leads to

$$r+s=s+t,$$

and the latter leads to

$$r-s=-(s-t),$$

both of which are the original equation.

Another complete integral is given by

$$z=a+\beta x+\gamma y+\delta xy+\epsilon(x^3+3xy^2);$$

it also is a particular form of the general integral, because it can be expressed in the form

$$z=a+\tfrac{1}{2}(\beta+\gamma)u+\tfrac{1}{4}\delta u^2+\tfrac{1}{2}\epsilon u^3$$
$$+\tfrac{1}{2}(\beta-\gamma)v-\tfrac{1}{4}\delta v^2+\tfrac{1}{2}\epsilon v^3,$$

where $u=x+y$, $v=x-y$, as before. For this complete integral,

$$p=\beta+\delta y+3\epsilon(x^2+y^2),$$
$$q=\gamma+\delta x+6\epsilon xy,$$

so that

$$p+q=\beta+\gamma+\delta(x+y)+3\epsilon(x+y)^2,$$
$$p-q=\beta-\gamma-\delta(x-y)+3\epsilon(x-y)^2,$$

which are not in the form of intermediate integrals, as both of them involve effectively three arbitrary constants. Elimination, however, is possible from their derived equations: thus, from the first,

$$r+s=\delta+6\epsilon(x+y)=s+t,$$

there being a combination of quantities in the elimination.

Similarly,

$$p+q=\theta(x+y),\quad p-q=\chi(x-y),$$

are intermediate general integrals, θ and χ denoting arbitrary functions.

Character of the General Integral.

181. We have seen that the general integral of an equation of the second order involves two arbitrary functions in its expression: for when a position is selected, with only the limitation that it shall be on an assigned curve, the values of the dependent variable and one of its derivatives at any such position become assigned functions of the independent variables. Now the general integral, whether obtained in the Ampère sense or in the Darboux-Cauchy sense, must satisfy the tests implied in the Ampère definition: for we have seen that an integral, which is general in the Darboux-Cauchy sense, is general also in the Ampère sense.

Suppose, then, that the differential equation of the second order is of the customary form

$$f(x, y, z, p, q, r, s, t)=0:$$

and let derivatives of f, of all orders up to and including those of $n-2$ be formed, where $n \geqslant 2$: the total number of equations thus possessed is

$$\begin{aligned}&1+2+\ldots+(n-1)\\ &=\tfrac{1}{2}n(n-1).\end{aligned}$$

Each of these equations is free from all the arbitrary elements that occur in the integral: and the derivatives of the dependent variable that occur are of all orders, up to and including those of order n.

Next, consider the integral equation; and suppose that it is given in finite form, whether it furnishes the value of z explicitly or implicitly. When all the derivatives of the integral equation, of orders up to n inclusive, are formed, there are

$$\begin{aligned}&1+2+\ldots+(n+1),\\ &=\tfrac{1}{2}(n+1)(n+2),\end{aligned}$$

equations in all. When all the arbitrary elements are eliminated from this tale of $\frac{1}{2}(n+1)(n+2)$ equations, it is required (from the property that the integral is general, in Ampère's sense) that the resulting equations should be exactly equivalent to the preceding tale of $\frac{1}{2}n(n-1)$ equations derived from the differential equation. Hence the number of such elements to be eliminated is

$$\begin{aligned}&\tfrac{1}{2}(n+1)(n+2)-\tfrac{1}{2}n(n-1)\\ &=2n+1,\end{aligned}$$

being therefore the number of arbitrary elements that occur in the integral equation and in derivatives from the integral equation, when derivatives of the dependent variable up to order n are formed.

It therefore appears that *a general integral in finite form contains arbitrary elements in such a manner that their number increases with successive differentiations of the integral.*

As a matter of fact it can be verified that, in even the simplest instances such as in the integral of $s=0$, the increase in the number of arbitrary elements arises through the introduction of new derivatives of arbitrary functions. The arguments of the two arbitrary functions, which occur in the general integral of an equation of the second order, may be different or they may be the same: it is to be noted that, when we proceed from derivatives of the integral equation in successive orders, two arbitrary elements (being the next higher derivatives of the arbitrary functions of specific arguments) are introduced at each successive stage.

Thus the general integral of the equation

$$r=t$$

is

$$z=\phi(x+y)+\psi(x-y):$$

the general integral (as will be proved later) of the equation

$$r-t=\frac{2p}{x}$$

is

$$z=\phi(x+y)+\psi(y-x)-x\{\phi'(x+y)-\psi'(y-x)\}:$$

where, in each case, ϕ and ψ are arbitrary functions. It is easy to verify the foregoing theorem for each of these equations.

Among the methods of analysis applied to partial equations of order higher than the first, there are two modes of occurrence of arbitrary functions in an equation giving a general integral which arise more frequently than others.

In one of these modes, the arbitrary functions present themselves as possessed of determinate arguments which are expressed, explicitly or implicitly, in terms of the independent variables.

Thus, for the equation

$$r=t,$$

already quoted, the general integral is

$$z=\phi(x+y)+\psi(y-x),$$

where the arguments of the arbitrary functions ϕ and ψ are explicit functions of x and y.

The general integral of the equation

$$pqr=s(1+p^2)$$

is given by the elimination of u between the two equations

$$\left.\begin{array}{r} z-\phi(u)-ux-(1+u^2)^{\frac{1}{2}}\ f(y)=0 \\ \phi'(u)+x+u(1+u^2)^{-\frac{1}{2}}f(y)=0 \end{array}\right\};$$

the argument of the arbitrary function ϕ is an implicit function of x and y, affected also by the occurrence and the form of the arbitrary function f.

In the other of the modes referred to, the arbitrary functions present themselves as possessed of arguments involving parameters, which are subject to quadratures of a multiplicity dependent upon the number of independent variables. This mode of occurrence is frequent in the integrals of many of the partial differential equations of mathematical physics: and there are two distinct forms of this mode of occurrence, according as the parameter is independent, or is not independent, of the variables.

Thus the equation

$$r=q,$$

which effectively is an equation in Fourier's theory of the conduction of heat, is satisfied by

$$z=\int_{-\infty}^{\infty}\phi(x+2uy^{\frac{1}{2}})\,e^{-u^2}du:$$

the parameter u of integration is independent of x and y. Again, the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which is the equation obeyed by gravitational potential in free space, is satisfied* by

$$V=\int_0^{2\pi}f(z+ix\cos u+iy\sin u,\ u)\,du,$$

* Whittaker, *Math. Ann.*, t. LVII (1903), p. 337.

where f is an arbitrary function of its two arguments, and the parameter u of integration is independent of x and y. The equation

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0$$

is satisfied by*

$$V=\int_0^{2\pi}\int_0^{\pi} f(x_1 \sin u \cos v+x_2 \sin u \sin v+x_3 \cos u+ix_4,\ u,\ v)\ du\, dv,$$

and by†

$$V=\int_0^{2\pi} f(x_1 \cos t+x_2 \sin t+ix_3,\ x_1 \sin t-x_2 \cos t+ix_4,\ t)\, dt.$$

In all of these forms the parameters of integration are independent of the variables.

On the other hand, equations occur possessing integrals in which there are quadratures with regard to variables while all other magnitudes in the subject of quadrature are kept constant. Thus a primitive of the equation

$$r-t+\frac{4p}{x+y}=0$$

is

$$(x+y)\, z+e^{\frac{2y}{x+y}} F(x+y)+e^{\frac{2y}{x+y}}\int e^{-\frac{2y}{a}} f(2y-a)\, dy=0,$$

where, after the quadrature, a is to be replaced by $x+y$, and F, f are arbitrary functions: but it is to be noted that an equivalent form is

$$z+e^{\frac{2y}{x+y}}\, G(x+y)=g(y-x)+(x+y)\, g'(y-x)+(x+y)^2\, g''(y-x)+\ldots,$$

provided the series on the right-hand side converges, the functions G and g being arbitrary. In the latter expression, the integral is in finite form as regards the function G but not as regards the function g.

Partial Quadratures.

182. There is a fundamental difference between the two classes of equations thus constituted. In the latter class, there is a quadrature, either definite or indefinite, with regard to a parameter or a variable, while all other magnitudes occurring in the subject of quadrature remain constant: it is usual to describe these as *integrals with partial quadratures*. In the former class, there are no such quadratures; and it is usual to describe‡ such general integrals as *integrals without partial quadratures*.

* Whittaker, *Math. Ann.*, t. LVII (1903), p. 345.

† Bateman, *Proc. Lond. M. S.*, Ser. 2, Vol. I (1904), p. 457.

‡ Ampère called them the *first class* (*l.c.*, p. 558): but he did not develope the classification, and so a more definite description is preferable.

The difference between the two classes of equations is not solely formal, as regards the absence or the presence of partial quadratures: it affects the character of the dependent variable and its derivatives.

In the case of equations, the integrals of which do not involve partial quadratures, derivatives of the integral equations with regard to the independent variables introduce no new arguments but only direct derivatives of the functional forms that occur in the integral. But in the case of equations the integrals of which do involve partial quadratures, such derivation often leads to subjects of quadrature quite distinct from those that occur in the original integral; and so it can lead to new arguments for the functional forms.

Accordingly, the class of equations whose integrals do not involve partial quadratures is simpler than the class of equations whose integrals are affected by partial quadratures: and the properties of the simpler class have been more fully developed than those of the other.

As already remarked, the two types indicated are the two more usual modes of occurrence: but they are not completely comprehensive. For example, the arbitrary elements might be defined in connection with some other differential equation, either of lower order than the given equation and not involving more independent variables, or involving fewer independent variables than the given equation and not of higher order. It is at least conceivable that precise selection from such a mode of determining the arbitrary elements might lead to new classes of integrable equations. Even so, it is clear that the methods of occurrence of arbitrary functions are not completely exhausted.

183. As the class of equations, whose general integrals (in finite form as regards one or other of the arbitrary functions) do not involve partial quadratures, is thus marked off from the others and is the simplest of all, it is convenient to have a means of testing whether a given equation does or does not belong to the class. Such a test (not, however, absolute) was devised by Ampère, as follows: the explanation is associated with an equation of the second order in two independent variables, but the test is easily seen to be applicable to equations of higher orders also and to equations in a greater number of independent variables.

Let the equation be

$$f(x, y, z, p, q, r, s, t) = 0,$$

and suppose the equation so taken that f is a polynomial function of the arguments r, s, t. Let α be an argument of an arbitrary function, that occurs in the general integral in finite form; and transform the independent variables so that α becomes one of them, say to x and α, so that y is a function of x and α. Now

$$\begin{aligned} dq &= s\,dx + t\,dy \\ &= \left(s + t\frac{\partial y}{\partial x}\right) dx + t\frac{\partial y}{\partial \alpha} d\alpha, \end{aligned}$$

so that

$$\frac{dq}{dx} = s + t\frac{\partial y}{\partial x},$$

$$\frac{dq}{d\alpha} = t\frac{\partial y}{\partial \alpha}.$$

Similarly,

$$\frac{dp}{dx} = r + s\frac{\partial y}{\partial x};$$

and therefore

$$s = \frac{dq}{dx} - t\frac{\partial y}{\partial x},$$

$$r = \frac{dp}{dx} - \frac{dq}{dx}\frac{\partial y}{\partial x} + t\left(\frac{\partial y}{\partial x}\right)^2,$$

and the value of t is given by

$$t = \frac{\dfrac{dq}{d\alpha}}{\dfrac{\partial y}{\partial \alpha}}.$$

Let substitution for r and s in terms of t be made in $f = 0$: and let the result, arranged in powers of t, be

$$P + Qt + Rt^2 + \ldots + Xt^n = 0,$$

where $P, Q, \ldots, X$ involve derivatives of p and q with regard to x but not derivatives with regard to α. The differential equation is to be satisfied identically when the integral is substituted in it. Also, as t has the value

$$\frac{dq}{d\alpha} \div \frac{\partial y}{\partial \alpha},$$

it contains a derivative of the arbitrary function of α in the general integral which is of at least one order higher than derivatives of that function occurring in derivatives of p and q with regard to x: that is, t contains a derivative of an arbitrary function that does not occur in $P, Q, \ldots, X$. In order therefore that the equation may be satisfied identically, we must have

$$P=0, \quad Q=0, \quad R=0, \ldots, X=0,$$

in virtue of the given equation. When these relations are consistent with one another, in virtue of the given equation and of a value of $\frac{\partial y}{\partial x}$, all the necessary conditions are satisfied. Unless they are satisfied, the equation cannot possess a general integral, which can be expressed in a finite form without partial quadratures as regards at least one of the arbitrary functions. But, though the conditions are necessary, we are not in a position to declare them sufficient to secure an integral of the character indicated; they only provide a qualifying test.

Moreover, as $\frac{\partial y}{\partial x}$ is the derivative of y with regard to x when α is constant, the value of $\frac{\partial y}{\partial x}$ determines α. We may take

$$\alpha = y - \int \frac{\partial y}{\partial x}\, dx,$$

or we may take α any function of the right-hand side: or if $\theta(x, y) = \text{constant}$ be an integral of the relation

$$dy - \frac{\partial y}{\partial x}\, dx = 0,$$

we may take

$$\alpha = \theta(x, y).$$

In any case, the value of $\frac{\partial y}{\partial x}$ determines the argument α: but in order to obtain α explicitly, an explicit value of $\frac{\partial y}{\partial x}$ in terms of y and x would require to be known.

We shall recur to this analysis in a later chapter (Chapter XVII).

Ex. 1. Consider the equation (§ 181)

$$r - t + \frac{4p}{x+y} = 0,$$

the general integral of which is of finite form as regards one arbitrary function, and may be free from partial quadratures though not then of finite form as regards the other arbitrary function.

Making the same assumptions as in the text, and writing

$$\theta=\frac{\partial y}{\partial x},$$

we find the Ampère conditions to be

$$\frac{dp}{dx}-\frac{dq}{dx}\theta+\frac{4p}{x+y}=0,$$
$$\theta^2-1=0.$$

These are consistent with one another: and, as

$$\frac{dp}{dx}=r+s\theta,\quad \frac{dq}{dx}=s+t\theta,$$

they are consistent with the original equation. Hence the original equation may possess a general integral which is free from partial quadratures.

The two values of θ are 1, -1: the two values of a for this equation can be taken

$$a=y+x,\qquad a=y-x:$$

that is, the arbitrary functions which occur are functions of $y+x$ and $y-x$ respectively.

Ex. 2. Consider the equation *

$$st+x(rt-s^2)^2=0.$$

Making the assumptions in the text, and writing

$$\theta=\frac{\partial y}{\partial x},$$

we have

$$rt-s^2=t\left(\frac{dp}{dx}+\frac{dq}{dx}\theta\right)-\left(\frac{dq}{dx}\right)^2,$$

so that, when substitution in the differential equation is effected, the equation becomes

$$x\left(\frac{dq}{dx}\right)^4+t\left\{-2x\left(\frac{dq}{dx}\right)^2\left(\frac{dp}{dx}+\frac{dq}{dx}\theta\right)+\frac{dq}{dx}\right\}+t^2\left\{x\left(\frac{dp}{dx}+\frac{dq}{dx}\theta\right)^2-\theta\right\}=0.$$

If then the equation can have a general integral in finite form free from partial quadratures, we must have

$$x\left(\frac{dq}{dx}\right)^4=0,$$

$$\frac{dq}{dx}\left\{1-2x\frac{dq}{dx}\left(\frac{dp}{dx}+\frac{dq}{dx}\right)\theta\right\}=0,$$

$$x\left(\frac{dp}{dx}+\frac{dq}{dx}\theta\right)^2-\theta=0.$$

* Ampère, *l.c.*, p. 608.

In order that these equations may be satisfied, we must have

$$\frac{dq}{dx}=0,$$

$$x\left(\frac{dp}{dx}\right)^2-\theta=0;$$

the three equations are consistent with one another in virtue of these two equations. These two equations are also consistent with the original equation: the first of them implies

$$s+t\theta=0,$$

and the second of them implies

$$x(r+s\theta)^2-\theta=0;$$

the elimination of θ between the two gives

$$x(rt-s^2)^2+st=0,$$

which is the original equation. Thus the two equations are consistent with one another and with the original equation: so far there is nothing to prevent the general integral from being expressible in a finite form without partial quadratures.

The equation

$$-\frac{s}{t}=\theta=\frac{\partial y}{\partial x}$$

corresponds to the determination of the argument a. The construction of the argument is a matter for later investigation; the immediate purpose is to test whether there is any reason to prevent the equation from possessing an integral of the required type.

Ex. 3. Prove that the equation

$$rt-a^2x^2t^2+py-qs=0,$$

where a is a constant, cannot have a general integral expressible in finite form without partial quadratures unless the integral also satisfies the equation

$$(py-qs)^2=a^2x^2q^2t^2. \qquad \text{(Ampère.)}$$

The Arbitrary Elements in General Integrals.

184. There are two properties of general integrals without partial quadratures that can be established: one of them relates to the number of arbitrary functions which can occur in a general integral of this type*: the other of them relates to the number of arguments occurring in an arbitrary function. It will now be proved *that the number of independent arbitrary functions in the*

* Ampère, (*l.c.*), p. 583; Imschenetsky, (in the memoir quoted in § 180), p. 236. See also a memoir by the author, *Proc. L. M. S.*, t. xxix (1898), p. 5.

general integral of a differential equation of any order, when that general integral is of finite form and without partial quadratures, is equal to the order of the equation; also that, usually though not universally, the number of arguments in each such arbitrary function is less by unity than the number of independent variables.

Most of the preceding explanations have been concerned with equations of the second order in two independent variables. As the propositions just stated are of wider application, we shall assume that the differential equation is of order m in n independent variables.

Accordingly, we take an equation

$$F = 0$$

of order m in n independent variables $x_1, \ldots, x_n$; as usual, the dependent variable is denoted by z.

The integral system will, of course, contain these variables. Suppose that, in addition to them, it involves a number k of variable quantities $\alpha, \beta, \gamma, \ldots$, these quantities being not necessarily independent of one another. As the integral system is to be equivalent to a single relation, which shall express z explicitly or implicitly in terms of the independent variables, that integral system must contain $k + 1$ equations from which the k variable quantities $\alpha, \beta, \gamma, \ldots$ can be conceived as eliminable.

Further suppose, firstly, that g independent arbitrary functions $\phi, \psi, \ldots$ occur in the integral system and, secondly, that, as a rule, each such function has r arguments though in particular cases the number of arguments may be less than r. These arguments may be considered as connected with the k variable quantities $\alpha, \beta, \gamma, \ldots$. Also it may happen that derivatives (some or all up to a specified order) of the arbitrary functions with regard to their arguments occur in the integral system. Let the highest derivatives of ϕ, which thus occur, be of order p_ϕ; and similarly for the derivatives of the other arbitrary functions.

Using Ampère's test as to whether an integral of a differential equation is general, we construct all the derivatives of the integral system of all orders up to s inclusive, where s is any integer equal to or greater than m: from the aggregate of equations thus obtained, all the arbitrary elements are to be eliminated: the surviving equations are to be equivalent to the equation $F = 0$ and to the equations deduced from $F=0$ by forming its derivatives of all orders up to $s-m$ inclusive.

The quantities to be eliminated from the equations deduced from the integral system are of two groups: one group is constituted by the quantities $\alpha, \beta, \gamma, \ldots$ and their derivatives, the other by the arbitrary functions $\phi, \psi, \ldots$ and their derivatives.

As regards the total number in the former group, it is made up of $\alpha, \beta, \gamma, \ldots$ and their derivatives of all orders of all orders up to s inclusive: hence this number is

$$\frac{(s+n)!}{s!\ n!}k$$
$$=\frac{(s+1)(s+2)\ldots(s+n)}{n!}k.$$

Some of the derivatives may vanish identically, and then corresponding quantities to be eliminated would not occur: the offset, to be allowed on this account, will be considered later.

Next, we require the total number of quantities connected with the arbitrary functions that have to be eliminated. In the case of any function ϕ, the highest derivative which occurs in the integral system is p_ϕ: if differentiation is being effected with regard to a variable not involved in any of the r arguments of ϕ, no new derivative is then introduced: but when the variable of differentiation is involved in one (or in more than one) of the r arguments of ϕ, then a new derivative is introduced. Now all the r arguments are variable magnitudes; hence the first differentiations of the integral system will introduce the various derivatives of ϕ of order $p_\phi+1$, the second differentiations will introduce those of order $p_\phi+2$, and so on up to the sth differentiations which will introduce those of order $p_\phi+s$; and each of these is a derivative with regard to some combination of the r arguments. Hence the total number of derivatives of ϕ in all (including ϕ itself) is

$$1+r+\frac{r(r+1)}{2!}+\ldots+\frac{r(r+1)\ldots(r+s+p_\phi-1)}{(s+p_\phi)!}$$
$$=\frac{(r+1)(r+2)\ldots(r+s+p_\phi)}{(s+p_\phi)!}$$
$$=\frac{(s+p_\phi+1)(s+p_\phi+2)\ldots(s+p_\phi+r)}{r!}.$$

Similarly for the derivatives of the other arbitrary functions: and therefore the total number of the derivatives of the arbitrary functions (including the arbitrary functions themselves) is

$$\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\},$$

where the summation is to be taken for the g arbitrary functions, and the number p may vary from term to term in this sum. This number really is an upper limit for any value of s. It may happen that, owing to the form of the integral system, not all these derivatives actually occur; there then would not be the corresponding quantities to be eliminated.

Consequently, the total number of quantities in the two groups, which have to be eliminated from the integral system and the equations deduced from the integral system, can be as great as

$$\frac{1}{n!}k(s+1)(s+2)\dots(s+n)$$
$$+\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}.$$

We have seen, however, that there may be an offset, on account of possibly vanishing derivatives of arguments on the one hand, and of possibly non-occurrent derivatives of arbitrary functions on the other hand: let N denote the aggregate number of quantities of this kind within the range considered, which otherwise would be included in the preceding aggregate. Hence, if I is the total number of quantities to be eliminated from the integral system and the equations derived from it, then

$$I=\frac{1}{n!}k(s+1)(s+2)\dots(s+n)$$
$$+\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}-N.$$

Next, when the integral system of $k+1$ equations is differentiated with regard to all possible combinations of the independent variables so as to give derivatives of z of all orders up to s inclusive, the complete tale of equations (including the original integral system) is J, where

$$J=\frac{(s+1)(s+2)\dots(s+n)}{n!}(k+1).$$

It is from these J equations that the foregoing number I of quantities must be eliminated, in order to give partial differential equations satisfied by z. Usually, the various eliminable quantities disappear singly during the elimination: in that case, the number of eliminant equations is $J - I$. But it may happen that some of the quantities disappear in a combination of several together, and also that this simultaneous disappearance may occur for several combinations: in that case, the number of eliminant equations will be increased say by S. Accordingly, we may say that the number of eliminant equations is

$$J - I + S:$$

each of them is a differential equation satisfied in virtue of the integral system, and s is the highest order of derivative that occurs.

Now by Ampère's test of a general integral, this aggregate of $J - I + S$ equations is to be an exact algebraic equivalent of the partial differential equation $F = 0$ of order m and of the equations deduced from $F = 0$ by effecting upon it all differentiations with respect to the independent variables of all orders up to $s - m$ inclusive, so that the deduced equations combined will involve derivatives of z of all orders up to s. The total number of equations in this set (including $F = 0$) is

$$\frac{1}{n!}(s - m + 1)(s - m + 2) \ldots (s - m + n),$$

which number therefore must be equal to $J - I + S$. Thus we have the relation

$$\begin{aligned}&\frac{1}{n!}(s+1)(s+2) \ldots (s+n) + N + S\\ &-\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\ldots(s+p+r)\}\\ &=\frac{1}{n!}(s-m+1)(s-m+2)\ldots(s-m+n),\end{aligned}$$

which must hold for all integer values of s such that $s \geqslant m$.

Of the integers that occur in this equation, the various numbers $p_\phi, p_\psi, \ldots$ are given: no one of them depends upon s. Also m, the order of the original differential equation, and n, the number of independent variables, are known and do not depend upon s. The number r, being the number of arguments in the arbitrary function, and the number g, being the number of arbitrary functions and also the number of different products in the summation typified by Σ, are not yet known: they do not, however, depend upon s. On the other hand, N and S may depend upon s, and, if they are different from zero, they usually do depend upon s; but for comparatively large values of s, both N and S are integers that are small compared with the number of quantities and of equations respectively in question.

With these explanations, let the preceding numerical relation be transformed so as to be of the form

$$\frac{1}{r\,!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}-N-S$$

$$=\frac{1}{n\,!}\{(s+1)(s+2)\dots(s+n)-(s-m+1)(s-m+2)\dots(s-m+n)\},$$

and let both sides of this form of the relation be expanded in descending powers of s. On the left-hand side, the term containing the highest power of s is

$$\frac{1}{r\,!}gs^r,$$

on the assumption (which will be made, after the statement made concerning N and S) that neither N nor S contains so high a power of s. On the right-hand side, the term in s^n disappears, and the coefficient of s^{n-1} is

$$\frac{1}{n\,!}\{1+2+\dots+n+(m-1)+(m-2)+\dots+(m-n)\}$$

$$=\frac{m}{(n-1)\,!}.$$

Hence

$$\frac{1}{r\,!}gs^r=\frac{m}{(n-1)\,!}s^{n-1},$$

which is to hold for all values of s. Consequently,

$$g=m,$$

that is, the number of arbitrary functions in a general integral without partial quadratures is equal to the order of the equation: and

$$r=n-1,$$

that is, the number of arguments in an arbitrary function can be one less than the number of independent variables.

These are the two propositions which were to be established.

Note 1. These propositions are of wide range: some special cases are worthy of special mention.

Let $m=1$: we infer that the general integral of a partial differential equation of the first order in n independent variables contains one arbitrary function which can have $n-1$ arguments.

Let $m=2$, $n=2$: we infer that the general integral of a partial differential equation of the second order in two independent variables, when it is free from partial quadratures, contains two arbitrary functions each of a single argument. But there is nothing in the preceding discussion to shew whether the two arguments are different or are the same.

Let $m=2$, $n=3$: we infer that the general integral of a partial differential equation of the second order in three independent variables, when it is free from partial quadratures, contains two arbitrary functions, each of two arguments. But there is nothing in the preceding discussion to shew what relation, if any, exists between the arguments.

Note 2. The preceding discussion has taken no account of the precise form of the equation $F=0$; and therefore it may be found not to apply to equations of special types. In such cases, all that can be inferred is that, if arbitrary functions contain $n-1$ arguments, the number of them in the general integral is not greater than the order of the equation: while this last property is not necessarily maintained if the functions contain fewer than $n-1$ arguments.

Ex. 1. Prove that the equations

$$x=F_1\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$
$$y=F_2\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$
$$z=F_3\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$

cannot represent the general integral of an equation of the second order.

(Goursat.)

Ex. 2. Prove that each of the quantities

$$z=X_1+X_2+X_3+(x_2-x_3)X_1'+(x_3-x_1)X_2'+(x_1-x_2)X_3',$$
$$z=X_1+X_2+X_3-(2x_1+x_3)X_1'+(x_1+x_3)X_2'+(x_2-x_3)X_3',$$

where X_1, X_2, X_3 are arbitrary functions of x_1, x_2, x_3 respectively and X_1', X_2', X_3' are their derivatives, satisfies a partial differential equation of the second order: and apply Ampère's test to prove that, in neither case, is the integral a general integral.

Ex. 3. The equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

is satisfied by

$$\begin{aligned}u=f\,(ix+y\cos a+z\sin a)\\ +g\,(x\cos\beta+iy+z\sin\beta)\\ +h\,(x\cos\gamma+y\sin\gamma+iz),\end{aligned}$$

where a, β, γ are arbitrary constants, and where f, g, h are arbitrary functions: discuss the character of the integral thus given.

185. The two results, as regards the number of arbitrary functions and the number of arguments in an arbitrary function contained in a general integral without partial quadratures, can be brought into relation with Cauchy's existence-theorem for an equation of general order in any number of independent variables, as has already been done for an equation of the second order in two variables. It was proved (§ 25) that, for an equation

$$\frac{\partial^m z}{\partial x_1{}^m}=Z,$$

where Z is a regular function of all the variables and of all the derivatives (save only the derivative on the left-hand side) of all orders up to m inclusive, then an integral exists, which is a regular function of the variables in non-infinitesimal domains and is such that, when $x_1=a_1$, the quantities

$$z,\quad \frac{\partial z}{\partial x_1},\ \ldots,\ \frac{\partial^{m-1}z}{\partial x_1{}^{m-1}}$$

become functions of $x_2, \ldots, x_n$, which are regular in the specified domains and otherwise are quite arbitrary. Thus m arbitrary functions occur: and each of them involves $n-1$ arguments which, when $x_1=a_1$, are algebraically equivalent to $x_2, \ldots, x_n$.

A more general form of the theorem is obtained merely by transformation of the variables, as follows: an integral exists of the same regular character as before which is such that, when any relation

$$u\,(x_1, \ldots, x_n)=0$$

exists among the variables, the variable z and the $m-1$ derivatives of the successive orders $1, \ldots, m-1$ become arbitrarily assigned functions of $x_1, \ldots, x_n$. The number of arbitrary functions is the same as before, viz. it is m: the number of arguments is $n-1$, for the arbitrary functions in the initial conditions involve all the n variables subject to the single relation $u=0$.

It thus appears, whether from Cauchy's existence-theorem or from Ampère's investigation on general integrals without partial quadratures, that an equation of any order m contains m arbitrary functions in an integral. Two examples have been given (§ 184, Ex. 2) in which integrals occurred having three arbitrary functions in finite form without quadratures and yet satisfied equations of the second order in three variables: in those instances, the arbitrary functions each involved only a single argument: whereas Ampère's investigation and Cauchy's theorem alike insist on two arguments in the arbitrary functions which occur in the general integral of the equations in question.

It should however be remarked that the integral, as given in the establishment of the existence-theorem, is found as a converging series and usually cannot be changed so as to have a finite form: the Ampère investigation only deals with integrals that are in a finite form and are without partial quadratures.

No inference, however, can be deduced as to the number of arbitrary elements occurring in the explicit expression of an integral involving partial quadratures. As will be seen hereafter (§ 209), it is possible to express the general integral of a linear equation of the second order in terms of only one arbitrary function: the matter will be considered during the discussion of those linear equations.

Equation characteristic of the Argument of an Arbitrary Function in a General Integral.

186. There is still another result which can be obtained when the general integral of the equation is in finite form without partial quadratures.

Let the equation be

$$f(x, y, z, p, q, r, s, t)=0,$$

and suppose that its general integral is of the type specified. Differentiating m times with respect to y, and writing

$$z_{\mu,\nu} = \frac{\partial^{\mu+\nu} z}{\partial x^{\mu} \partial y^{\nu}},$$

we have

$$\frac{\partial f}{\partial r} z_{2,m} + \frac{\partial f}{\partial s} z_{1,m+1} + \frac{\partial f}{\partial t} z_{0,m+2} + U = 0,$$

where the derivatives in U are of order not higher than $m+1$.

Let α be the argument of an arbitrary function in the general integral and, assuming that α involves y, change the independent variables from x and y to x and α: then y is a function of x and α. Also let θ denote the value of $\frac{\partial y}{\partial x}$ when α is constant. Thus

$$\frac{dz_{0,m+1}}{dx} = z_{1,m+1} + z_{0,m+2}\,\theta,$$

$$\frac{dz_{1,m}}{dx} = z_{2,m} + z_{1,m+1}\,\theta,$$

using the same notation as in § 183 for new derivations with regard to x: whence, substituting for $z_{1,m+1}$ and $z_{2,m}$, we have the above equation in the form

$$\left(\frac{\partial f}{\partial t} - \theta\frac{\partial f}{\partial s} + \theta^2\frac{\partial f}{\partial r}\right) z_{0,m+2} + U + \frac{\partial f}{\partial s}\frac{dz_{0,m+1}}{dx} + \frac{\partial f}{\partial r}\left(\frac{dz_{1,m}}{dx} - \theta\frac{dz_{0,m+1}}{dx}\right) = 0.$$

Now, for all values of n, we have

$$\frac{d^n z}{d\alpha^n} = z_{0,n}\left(\frac{\partial y}{\partial \alpha}\right)^n + \ldots,$$

the unspecified derivatives of z being of order less than n. Substituting for $z_{0,m+2}$ and for the other derivatives according to similar formulæ of transformation, we have a term in $\frac{d^{m+2}z}{d\alpha^{m+2}}$ of which

$$\frac{\partial f}{\partial t} - \theta\frac{\partial f}{\partial s} + \theta^2\frac{\partial f}{\partial r}$$

is the coefficient. This quantity $\frac{d^{m+2}z}{d\alpha^{m+2}}$ does not occur elsewhere in the equation, the other derivatives with regard to α being of order not higher than $m+1$; when the value of z is substituted, the quantity $\frac{d^{m+2}z}{d\alpha^{m+2}}$ introduces a derivative of the arbitrary function of α, which is of order higher than any other derivative that occurs in the equation. But the equation must be identically satisfied when this value of z, given by the general integral, is substituted; and therefore the term involving the highest derivative of the arbitrary function of α in the general integral must vanish, that is,

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0.$$

Also, θ is the value of $\frac{\partial y}{\partial x}$ when α is constant, so that

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\theta = 0:$$

hence

$$\frac{\partial f}{\partial r}\left(\frac{\partial \alpha}{\partial x}\right)^2 + \frac{\partial f}{\partial s}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y} + \frac{\partial f}{\partial t}\left(\frac{\partial \alpha}{\partial y}\right)^2 = 0,$$

being an equation satisfied by α.

The same equation would be satisfied by the argument, say β, of the other arbitrary function: and it may happen that α and β are the same.

We have said that the equation is satisfied by α and by β. In general, derivatives of z would occur in this equation; and it could not be used for the immediate determination of α and β. But if f is linear in r, s, t, and has functions of x and y only for the coefficients of r, s, t, then the determination of α and β is effected by integrating

$$\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial f}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial f}{\partial s} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial f}{\partial t} = 0,$$

a partial differential equation of the first order.

Short of this actuality, however, which belongs to only a restricted class of equations, we can make other inferences from the equation

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0,$$

which is found to recur continually in the investigations on $f=0$.

Let m and n be its roots, when it is regarded as a quadratic in θ: then we may take

$$\frac{\partial \alpha}{\partial x} + m \frac{\partial \alpha}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial x} + n \frac{\partial \beta}{\partial y} = 0.$$

Various cases occur.

(i) Let m and n be distinct from one another, both being finite. There are two distinct arguments: and the two arbitrary functions in the general integral have these for their respective arguments. The only condition is one of inequality, viz. that

$$\left(\frac{\partial f}{\partial s}\right)^2 - 4 \frac{\partial f}{\partial r} \frac{\partial f}{\partial t}$$

does not vanish.

(ii) Let m and n be the same, and be finite. There is only one quantity: it is the common argument of the two arbitrary functions in the general integral. The condition

$$\left(\frac{\partial f}{\partial s}\right)^2 - 4 \frac{\partial f}{\partial r} \frac{\partial f}{\partial t} = 0$$

must be satisfied.

(iii) If m is zero and n finite though not zero, then α is a function of y alone, and β is not thus restricted. In this case,

$$\frac{\partial f}{\partial t} = 0:$$

hence, when the equation does not involve t explicitly, one of the arbitrary functions in the general integral involves y alone.

(iv) If m is infinite and n is finite though not zero, then α is a function of x alone and β is not thus restricted. In this case,

$$\frac{\partial f}{\partial r} = 0:$$

so that, when the equation does not involve r explicitly, one of the arbitrary functions in the general integral involves x alone.

(v) If m is infinite and n is zero, then α is a function of x only, and β is a function of y only. In this case,

$$\frac{\partial f}{\partial r} = 0, \qquad \frac{\partial f}{\partial t} = 0,$$

so that r and t do not occur explicitly. Resolving the equation with regard to s, we have it in the form

$$s = g(x, y, z, p, q):$$

when it possesses a general integral without partial quadratures, the arguments of the two arbitrary functions in that general integral are respectively x and y. In particular, when the equation

$$s + ap + bq + cz = 0,$$

a, b, c being functions of x and y only, has a general integral without partial quadratures, the arbitrary functions in that general integral are X and Y, arbitrary functions of x and of y respectively.

(vi) If both m and n are zero, then α and β are functions of y alone. In this case,

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial s} = 0,$$

so that s and t do not occur explicitly. Moreover, q cannot then occur explicitly: for, as

$$z = F(x, y, \alpha, \alpha', \ldots, \beta, \beta', \ldots),$$

the occurrence of q in the differential equation would give rise to derivatives of α and of β which (on the assumption that F is in finite form) are of order higher than the derivatives of those quantities found in z, p, or r. Thus the equation can only be of the form

$$f(x, y, z, p, r) = 0:$$

which effectively is an ordinary equation of the second order, having z for its dependent variable, x for its independent variable, and having two arbitrary functions of the parametric variable y for the two arbitrary elements in its integral.

(vii) If both m and n are infinite, then α and β are functions of x alone. The case is similar to the last case with the interchange of x and y, with the corresponding interchanges: in particular, the original differential equation is

$$f(x, y, z, q, t) = 0.$$

Ex. 1. When the equation is

$$r=t,$$

the equation for the arguments is

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = 0:$$

that is, the arguments of the arbitrary functions in the general integral are

$$\alpha = x+y, \qquad \beta = x-y.$$

Ex. 2. The equation

$$(b+cq)^2 r - 2(b+cq)(a+cp)s + (a+cp)^2 t = 0$$

satisfies the Ampère tests (§ 183): it may therefore have a general integral in finite form without partial quadratures. The arguments of the arbitrary functions in this general integral are given by

$$(b+cq)^2\left(\frac{\partial u}{\partial x}\right)^2 - 2(b+cq)(a+cp)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + (a+cp)^2\left(\frac{\partial u}{\partial y}\right)^2 = 0:$$

the two arguments are one and the same, and may be taken as the simplest integral of

$$(b+cq)\frac{\partial u}{\partial x} - (a+cp)\frac{\partial u}{\partial y} = 0.$$

This simplest integral is an integral of the equation

$$\frac{dx}{b+cq} = \frac{dy}{-(a+cp)},$$

that is, of

$$a\,dx + b\,dy + c\,(p\,dx + q\,dy) = 0,$$

that is, of

$$a\,dx + b\,dy + c\,dz = 0:$$

hence the common argument of the two arbitrary functions in the general integral is $ax+by+cz$.

Ex. 3. Find the arguments of the arbitrary functions in the general integrals of the equations:

(i) $x^2r + 2xys + y^2t = 0$;

(ii) $x^2r - 2xys + y^2t = 0$;

(iii) $q^2r - 2pqs + p^2t = 0$;

(iv) $q^2r + 2pqs + p^2t = 0$;

(v) $x^2r - y^2t = 0$;

(vi) $q(1+q)r - (p+q+2pq)s + p(1+p)t = 0$;

proving that, in the case of each equation, the Ampère tests which allow it to possess a general integral expressible in finite form without partial quadratures are satisfied.

Ex. 4. Prove that an integral of the equation

$$x(r+t) + p = 0$$

is given by

$$z=\int_0^\pi f(x\cos\phi+iy)\,d\phi+\int_0^\pi g(x\cos\psi-iy)\,d\psi.$$

Does this integral involve two arbitrary functions that are independent of one another? Does the equation possess a general integral in finite form without partial quadratures?

187. A similar discussion, in connection with an equation of order n in two independent variables, leads to a similar result as regards the arguments of the arbitrary functions. If the equation be

$$F(z_{n,0},\ z_{n-1,1},\ \ldots,\ z_{0,n},\ \ldots)=0,$$

where

$$z_{p,q}=\frac{\partial^{p+q}z}{\partial x^p\,\partial y^q},$$

and if u be any one of those arguments, then the equation

$$\frac{\partial F}{\partial z_{n,0}}\left(\frac{\partial u}{\partial x}\right)^n+\frac{\partial F}{\partial z_{n-1,1}}\left(\frac{\partial u}{\partial x}\right)^{n-1}\frac{\partial u}{\partial y}+\ldots+\frac{\partial F}{\partial z_{0,n}}\left(\frac{\partial u}{\partial y}\right)^n=0$$

is satisfied: it being always remembered that the general integral of $F=0$ is assumed to be of finite form and without partial quadratures.

Usually this equation would be only one of a set of equations satisfied in connection with the general integral of $F=0$. For the very restricted class of equations, in which F is linear in the derivatives of order n and has functions of x and y only for the coefficients of these derivatives, the foregoing equation becomes a partial differential equation of the first order for the actual determination of the arguments u.

188. Corresponding results can similarly be obtained for equations of any order in any number of independent variables; it will be sufficient to state them for an equation of the second order in three independent variables. A change in the notation will be made: we denote the dependent variable by v, the independent variables by x, y, z, and we write

$$l,\ m,\ n=\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z},$$

$$a,\ b,\ c,\ f,\ g,\ h=\frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^2 v}{\partial y\partial z},\ \frac{\partial^2 v}{\partial z\partial x},\ \frac{\partial^2 v}{\partial x\partial y},$$

respectively. Then an equation of the second order is of the form

$$F(x, y, z, v, l, m, n, a, b, c, f, g, h) = 0:$$

we may suppose that F is a regular function of all its arguments, and we shall assume that F is polynomial in the derivatives a, b, c, f, g, h.

The Ampère test as to the possession of a general integral in finite form without partial quadratures can be applied as before. Let u be an argument of an arbitrary function in that integral, and let the variables be changed from x, y, z to x, y, u, so that z becomes a function of x, y, u: forming the derivatives of z, let

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$$

We substitute

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + cp^2,$$

$$b = \frac{dm}{dy} - q\frac{dn}{dy} + cq^2,$$

$$f = \frac{dn}{dy} - cq,$$

$$g = \frac{dn}{dx} - cp,$$

$$h = \frac{dl}{dy} - q\frac{dn}{dx} + cpq = \frac{dm}{dx} - p\frac{dn}{dy} + cpq,$$

in $F = 0$, and arrange it in powers of c in the form

$$F_0 + cF_1 + c^2F_2 + \dots + c^m F_m = 0:$$

in order that a general integral of the specified type may be possessed, the equations

$$F_0 = 0, \quad F_1 = 0, \dots, \quad F_m = 0,$$

must be consistent with one another, with

$$F = 0,$$

and with

$$\frac{dl}{dy} - q\frac{dn}{dx} = \frac{dm}{dx} - p\frac{dn}{dy}.$$

But, as before, these conditions are necessary, though not universally sufficient: they provide a qualifying test.

In particular, the equation $F_m = 0$ is

$$p^2 \frac{\partial F}{\partial a} + q^2 \frac{\partial F}{\partial b} + \frac{\partial F}{\partial c} - q \frac{\partial F}{\partial f} - p \frac{\partial F}{\partial g} + pq \frac{\partial F}{\partial h} = 0.$$

Now p and q are the derivatives of z with regard to x and y when u is constant: thus

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} = 0.$$

Hence u, the argument of an arbitrary function in a general integral of $F = 0$, supposed in finite form and without partial quadratures, satisfies the equation

$$\frac{\partial F}{\partial a}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial F}{\partial b}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial F}{\partial c}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial F}{\partial f}\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial g}\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial h}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0,$$

in connection with an integral of the partial differential equation.

We shall return to the subject during the discussion of the methods of integration of equations of the second order involving more than two independent variables (Chap. XXIV).

Ex. 1. One of the most important of these equations is

$$a + b + c = 0,$$

which recurs continually in mathematical physics. A general integral has already (§ 181) been given in a form which requires partial quadratures.

The qualifying conditions that it should possess a general integral without partial quadratures are easily found to be that the relations

$$\frac{dl}{dx} - p \frac{dn}{dx} + \frac{dm}{dy} - q \frac{dn}{dy} = 0,$$

$$\frac{dl}{dy} + p \frac{dn}{dy} - \frac{dm}{dx} - q \frac{dn}{dx} = 0,$$

$$p^2 + q^2 + 1 = 0,$$

shall be consistent with one another and with the original equation. It is easy to see that these conditions are satisfied: and thus the equation possesses a general integral without partial quadratures.

Also, the last relation leads to

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

as the equation satisfied by an argument of an arbitrary function in the integral. A general integral of this partial differential equation of the first order is given by the equations

$$\left.\begin{aligned} u &= \gamma\,(z + ix\cos a + iy\sin a) + \phi\,(\gamma, a) \\ 0 &= z + ix\cos a + iy\sin a + \frac{\partial\phi}{\partial\gamma} \\ 0 &= i\gamma\,(-x\sin a + y\cos a) + \frac{\partial\phi}{\partial a} \end{aligned}\right\}.$$

Ex. 2. In connection with the last result, verify that, if

$$\phi\,(\gamma, a) = \gamma\theta\,(a) + \psi\,(a),$$

where θ and ψ are arbitrary functions of a, then

$$v = F(u),$$

where F denotes an arbitrary function, satisfies the equation

$$a + b + c = 0.$$

CHAPTER XIII.

Linear Equations of the Second Order in Two Independent Variables: the Laplace-Transformations.

The greater part of the present chapter is devoted to what is usually called *Laplace's method* for the integration of an equation of the second order that is linear in the dependent variable and its derivatives. The method* given by Laplace was developed and extended by Darboux†: and it is upon his exposition that the present account of the method is based. Some detailed references are given in the course of the chapter: and general mention may here be made of Goursat's discussion of the method‡.

189. We proceed now to consider, in detail, equations of the second order; and we begin with those equations which involve two independent variables. Among them, two classes are marked out from the rest by their simplicity of form: one of these classes is constituted by the equations which are linear in the dependent variable and its derivatives: the other is constituted by the equations which possess an intermediate integral involving the first derivatives of the dependent variable. The two classes are not completely exclusive of one another; but the main methods of dealing with them are quite distinct. We discuss first the equations which are linear.

The most general form of equation, which is linear in the dependent variable and its derivatives, and which is of the second order, is (in the ordinary notation)

$$Rr + 2Ss + Tt + 2Pp + 2Qq + Zz = U,$$

* Originally given in his memoir *Mémoires de l'Acad. royale des sciences*, 1777, the memoir itself being dated 1773: see also *Œuvres complètes de Laplace*, t. IX, pp. 5—68.

† *Théorie générale des surfaces*, t. II, pp. 23 *et seq.*

‡ In chapter V of his treatise already (p. 7) quoted.

where R, S, T, P, Q, Z, U, are functions of x and y only. We may take U as zero: for, if U is not zero in any given instance, and if ζ is any particular value of z (no matter how special) which satisfies the equation, then writing

$$z = \zeta + z',$$

the equation for z' is of the same form save that U is zero. We shall therefore assume that U is zero.

Let the independent variables be changed from x and y to u and v: the equation is unaltered in form, so that it is

$$R'r' + 2S's' + T't' + 2P'p' + 2Q'q' + Zz = 0,$$

where p', q', r', s', t' are the derivatives of z with regard to the new variables, and

$$R' = R\left(\frac{\partial u}{\partial x}\right)^2 + 2S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2,$$

$$S' = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y},$$

$$T' = R\left(\frac{\partial v}{\partial x}\right)^2 + 2S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2.$$

The quantities u and v are at our disposal: the simplest form of changed equation depends upon the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0.$$

Firstly, let the roots of this quadratic be unequal, and denote them by $-m$, $-n$; and then let u and v be determined by the equations

$$\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + n\frac{\partial v}{\partial y} = 0.$$

Then we have

$$R' = 0, \quad T' = 0;$$

also

$$S' = 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\left(T - \frac{S^2}{R}\right),$$

which is not zero. The equation takes the form

$$\frac{\partial^2 z}{\partial u \partial v} + a\frac{\partial z}{\partial u} + b\frac{\partial z}{\partial v} + cz = 0,$$

where a, b, c are functions of the independent variables alone.

Secondly, let $R=0$, $T=0$: the changed equation will still be of the last form, provided

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0,$$

and

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\neq 0.$$

We thus may make

$u=$ any function of x, and $v=$ any function of y:

or

$u=$ any function of y, and $v=$ any function of x.

For either of these transformations, the deduced form of equation is invariantive.

Thirdly, let the roots of the quadratic be equal, having $-m$ for their common value. We then take

$$u=x,\quad \frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}=0,$$

and we have

$$R'=R,$$
$$S'=(Rm+S)\frac{\partial v}{\partial y}=0,$$
$$T'=0.$$

The equation takes the form

$$\frac{\partial^2 z}{\partial x^2}+a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial v}+cz=0,$$

where a, b, c are functions of the independent variables.

Fourthly, let $S=0$, $T=0$. The equation originally is of the form last obtained: and it remains unchanged by the transformation if

$$\frac{\partial u}{\partial x}\neq 0,\quad \frac{\partial v}{\partial x}=0,$$

that is, if v is any function of y, and if u is any function of x and y which certainly involves x.

Fifthly, let $R=0$, $S=0$: taking

$$u=y,\quad v=x,$$

we again obtain the form in the last case.

Similarly, we can obtain one or other of the forms if only one of the three quantities R, S, T should vanish. Hence in every case, *the linear equation can, by change of the independent variables, be transformed so as to become either*

$$s + ap + bq + cz = 0,$$

or

$$r + ap + bq + cz = 0,$$

where a, b, c are functions of the independent variables alone.

This reduction to one or other of two alternative forms may be compared with the determination of the arguments in the arbitrary functions in the general integral, there supposed to be of finite form and devoid of partial quadratures. From that determination, we infer that, if the equation

$$s + ap + bq + cz = 0$$

possesses a general integral of the specified type, the general integral will contain an arbitrary function of x and an arbitrary function of y in its expression. Also we infer that if, in the equation

$$r + ap + bq + cz = 0,$$

the coefficient b is not zero, a general integral devoid of partial quadratures cannot be of finite form, while, if the coefficient b is zero and if the equation possesses a general integral of finite form and free from partial quadratures, that general integral will involve two arbitrary functions of y in its expression.

Ex. 1. If R, S, T, P, Q, Z are constants, then, by the transformations $u = ax + \beta y$, $v = a'x + \beta' y$, $z = \zeta e^{a''x + \beta'' y}$, with appropriate determinations of a, β, a', β', a'', β'', the linear equation can be changed to one of the forms

$$r + q = 0, \quad s + z = 0, \quad s + p + q = 0. \qquad \text{(A. Schwartz.)}$$

Ex. 2. Obtain the condition that the system

$$\xi p + \eta q + \zeta z = z_1, \quad a p_1 + \beta q_1 + \gamma z_1 = 0,$$

where ξ, η, ζ, a, β, γ are functions of x and y, may be equivalent to a linear equation of the second order: and, assuming the condition satisfied, integrate the equation. (A. Schwartz.)

190. In the preceding reduction to one or other of two forms, the discrimination is made by the equality or the inequality of the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0.$$

In many investigations, particularly those concerned with the general theory of surfaces and with characteristics, the independent variables are real: and consequently it may be of importance to note the form of the equation according as the roots of the quadratic are real or are complex, the quantities R, S, T being real.

In the first place, let the roots be conjugate complex quantities, so that $RT - S^2 > 0$: write

$$RT - S^2 = \theta^2.$$

Then

$$R'T' - S'^2 = (RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2,$$

so that $R'T' - S'^2$ must be positive, a condition that will be satisfied by taking

$$R' = T', \quad S' = 0.$$

Then

$$R'R = \left(R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y}\right)^2 + \theta^2\left(\frac{\partial u}{\partial y}\right)^2,$$

$$T'T = \left(R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y}\right)^2 + \theta^2\left(\frac{\partial v}{\partial y}\right)^2,$$

$$0 = \left(R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y}\right)\left(R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y}\right) + \theta^2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y};$$

and therefore

$$R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y} = \mu\frac{\partial v}{\partial y},$$

$$R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y} = -\frac{\theta^2}{\mu}\frac{\partial u}{\partial y},$$

where

$$\frac{\mu^2}{\theta^2} = \frac{R}{T},$$

using the relation $R' = T'$. The equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{S}{R}\frac{\partial u}{\partial y} + \frac{\mu}{R}\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{T\mu}{R}\frac{\partial u}{\partial y} - \frac{S}{R}\frac{\partial v}{\partial y}\end{aligned}\right\}$$

are of the form considered in §§ 8—13: they give values of u and v. The transformed equation is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + l\frac{\partial z}{\partial u} + m\frac{\partial z}{\partial v} + nz = 0,$$

that is,

$$r + t + lp + mq + nz = 0,$$

where l, m, n are functions of the independent variables. This is sometimes called the *elliptic* case.

In the second place, let the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0$$

be real and different: the earlier analysis shews that the transformed equation is

$$s + ap + bq + cz = 0,$$

where now the variables are real. This is sometimes called the *hyperbolic* case.

In the third place, let the roots of the quadratic be equal: they are real. The transformed equation is

$$r + ap + bq + cz = 0,$$

where now the variables are real. This is sometimes called the *parabolic* case.

So far as concerns most of the processes of integration, the distinction between real and complex variables is insignificant: it becomes important in certain applications to physics, to the geometry of ordinary space, and particularly in regard to characteristics. So far as concerns the processes of integration discussed for the linear equation, there is no distinction between real and complex variables: by taking

$$x + iy = x', \quad x - iy = y',$$

we change the elliptic case into the hyperbolic case. Accordingly, as here we are concerned with processes of integration, it will be sufficient to discuss the two forms

$$s + ap + bq + cz = 0,$$
$$r + ap + bq + cz = 0.$$

THE EQUATION $s + ap + bq + cz = 0$: ITS INVARIANTS.

191. We proceed to consider the equation

$$s + ap + bq + cz = 0,$$

one of the two forms to which every linear equation can be reduced. The form of the equation is unaltered if we introduce a new dependent variable such that

$$z = \lambda z',$$

where λ is any function of x and y: it is unaltered if the independent variables are changed to x' and y', where

$$x = \phi(x'), \quad y = \psi(y'),$$

the functions ϕ and ψ being arbitrary: likewise for the transformations

$$x = \psi(y'), \quad y = \phi(x').$$

Consider the effect of these in turn.

Substituting $z = \lambda z'$, we have

$$s' + a'p' + b'q' + c'z' = 0,$$

where

$$a' = a + \frac{1}{\lambda}\frac{\partial \lambda}{\partial y},$$

$$b' = b + \frac{1}{\lambda}\frac{\partial \lambda}{\partial x},$$

$$c' = c + \frac{a}{\lambda}\frac{\partial \lambda}{\partial x} + \frac{b}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial x \partial y};$$

consequently,

$$c' - a'b' = \frac{\partial^2 (\log \lambda)}{\partial x \partial y} + c - ab,$$

and therefore

$$\frac{\partial a'}{\partial x} + a'b' - c' = \frac{\partial a}{\partial x} + ab - c = h,$$

$$\frac{\partial b'}{\partial y} + a'b' - c' = \frac{\partial b}{\partial y} + ab - c = k.$$

The quantities h and k are thus unaltered for the substitution

$$z = \lambda z'.$$

Making the transformation

$$x = \phi(x'), \quad y = \psi(y'),$$

we find

$$\frac{h'}{h} = \phi'(x')\,\psi'(y') = \frac{k'}{k};$$

and making the transformation

$$x = \psi(y'), \quad y = \phi(x'),$$

we find

$$\frac{h'}{k} = \phi'(x')\,\psi'(y') = \frac{k'}{h}.$$

Consequently, for all the transformations which leave the form of the differential equation unaltered, the combinations of the coefficients denoted by h and k reproduce themselves, either exactly or save as to a factor which does not depend upon the equation. Accordingly, these quantities h and k are called the *invariants of the equation.*

Ex. Obtain the expressions of these invariants in terms of the coefficients of the original equation

$$Rr+2Ss+Tt+2Pp+2Qq+Zz=0.$$

(Imschenetsky.)

One important property can be associated with the invariants: it is that, if either of the invariants should vanish, the equation can be immediately integrated. As will be seen, this property is made the basis of Laplace's method of integration.

If h vanishes, then

$$c=ab+\frac{\partial a}{\partial x},$$

and so the equation becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}+ax\right)+b\left(\frac{\partial z}{\partial y}+az\right)=0.$$

Hence

$$\frac{\partial z}{\partial y}+az=Ye^{-\int b\,dx},$$

and therefore

$$ze^{\int a\,dy}=X+\int Ye^{\int(a\,dy-b\,dx)}\,dy,$$

where X is an arbitrary function of x and Y of y.

If k vanishes, then

$$c=ab+\frac{\partial b}{\partial y}:$$

proceeding similarly, we find

$$ze^{\int b\,dx}=Y_1+\int X_1e^{\int(b\,dx-a\,dy)}\,dx,$$

where X_1 is an arbitrary function of x and Y_1 of y.

In either case, it is clear that the only inverse operations required are quadratures.

192. As it thus appears that different forms of the equation, all of the same type, are obtainable by the transformations, it is convenient to have some *canonical form* to which the equation can be uniquely reduced. The reduced equation or the canonical form, chosen by Darboux*, is that for which

$$c' = a'b'.$$

When this relation is used, we have

$$\frac{\partial a'}{\partial x} = h, \quad \frac{\partial b'}{\partial y} = k, \quad \frac{\partial^2 \log \lambda}{\partial x \partial y} = ab - c.$$

To obtain a' and b' explicitly, we determine the functions of the variables introduced in the quadratures by the conditions, (i) that a' shall vanish when x has an assigned value x_0 whatever y may be, and (ii) that b' shall vanish when y has an assigned value y_0 whatever x may be. Thus

$$a' = \int_{x_0}^{x} h\,dx, \quad b' = \int_{y_0}^{y} k\,dy;$$

and

$$c' = a'b',$$

so that, as h and k are known from any form of the equation, the coefficients in the reduced form are known. As regards the multiplier λ, we have

$$\frac{\partial \log \lambda}{\partial x} = b' - b = \int_{y_0}^{y} k\,dy - b,$$

$$\frac{\partial \log \lambda}{\partial y} = a' - a = \int_{x_0}^{x} h\,dx - a,$$

and λ is therefore known save as to a constant factor, which is trivial because the equation is linear and homogeneous in z and its derivatives.

Two other forms might be chosen. Thus we might assign

$$a' = 0$$

as characteristic of a reduced form: then

$$-c' = h, \quad \frac{\partial b'}{\partial y} - c' = k,$$

so that

$$c' = -h, \quad b' = \int_{y_0}^{y} (k - h)\,dy,$$

and the reduced form is

$$s' + q' \int_{y_0}^{y} (k - h)\,dy - hz' = 0.$$

* *Théorie générale des surfaces*, t. II, p. 26.

Similarly, if we were to assign

$$b' = 0$$

as characteristic of a reduced form, it would be

$$s' + p'\int_{x_0}^{x}(h-k)\,dx - kz' = 0.$$

Obviously the condition, necessary and sufficient to secure that an equation shall be reducible to a form

$$s = \mu z,$$

is that the invariants shall be equal: their common value is μ, and the multiplier λ needed to lead to this form is e^u, where

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}.$$

There is one other form to which the equation can be conditionally reduced. Suppose that some integral, no matter how particular, is known: let it be denoted by u. Then, when we substitute

$$z = u\zeta,$$

where ζ is a new dependent variable, the equation for ζ is

$$\frac{\partial^2 \zeta}{\partial x \partial y} + \left(a + \frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial \zeta}{\partial x} + \left(b + \frac{1}{u}\frac{\partial u}{\partial x}\right)\frac{\partial \zeta}{\partial y} = 0,$$

say

$$\frac{\partial^2 \zeta}{\partial x \partial y} + a'\frac{\partial \zeta}{\partial x} + b'\frac{\partial \zeta}{\partial y} = 0,$$

where a' and b' are functions of x and y. The term involving ζ alone is absent.

The invariants of this equation are the same as before, as is to be expected: for

$$h' = \frac{\partial a'}{\partial x} + a'b'$$

$$= \frac{\partial a}{\partial x} + ab - c,$$

on reduction: and similarly

$$k' = \frac{\partial b'}{\partial y} + a'b'$$

$$= \frac{\partial b}{\partial x} + ab - c.$$

This form of the equation occurs frequently in the general theory of surfaces. In particular, it is the equation satisfied by each of the Cartesian coordinates of a point on a surface, expressed in terms of the parameters of conjugate directions on the surface.

Relation of the two Laplace-Transformations.

193. Having discussed the simple cases when one or other of the invariants vanishes, we may now suppose that neither vanishes.

In the first place, we take

$$\frac{\partial z}{\partial y} + az = z_1;$$

then

$$\frac{\partial z_1}{\partial x} + bz_1 = \frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + \left(\frac{\partial a}{\partial x} + ab\right) z$$

$$= hz,$$

and therefore

$$\frac{\partial}{\partial y}\left\{\frac{1}{h}\left(\frac{\partial z_1}{\partial x} + bz_1\right)\right\} + \frac{a}{h}\left(\frac{\partial z_1}{\partial x} + bz_1\right) = z_1,$$

so that the equation for z_1 is

$$s_1 + a_1 p_1 + b_1 q_1 + c_1 z_1 = 0,$$

where

$$a_1 = a - \frac{1}{h}\frac{\partial h}{\partial y} = a - \frac{\partial \log h}{\partial y},$$

$$b_1 = b,$$

$$c_1 = c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{b}{h}\frac{\partial h}{\partial y}.$$

The equation for z_1 is of exactly the same type as the equation for z. Let h_1 and k_1 denote its invariants: then

$$\left.\begin{aligned} h_1 &= 2h - k - \frac{\partial^2 \log h}{\partial x \partial y} \\ k_1 &= h \end{aligned}\right\}.$$

In the next place, we take

$$\frac{\partial z}{\partial x} + bz = Z_1,$$

so that

$$\frac{\partial Z_1}{\partial y} + aZ_1 = kz;$$

the equation for Z_1 is similarly found to be

$$S_1 + A_1 P_1 + B_1 Q_1 + C_1 Z_1 = 0,$$

where

$$A_1 = a,$$

$$B_1 = b - \frac{1}{k}\frac{\partial k}{\partial x} = b - \frac{\partial \log k}{\partial x},$$

$$C_1 = c + \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} - \frac{a}{k}\frac{\partial k}{\partial x}.$$

The equation for Z_1 is of the same type as the equation for z. Denoting its invariants by H_1 and K_1, we have

$$\left.\begin{aligned} H_1 &= k \\ K_1 &= 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \end{aligned}\right\}.$$

It would appear as if a couple of distinct transformations could thus be obtained, independent of one another; as a matter of fact, they are in a sense the inverses of each other. Taking

$$z_1 = \frac{\partial z}{\partial y} + az = \sigma(z),$$

$$Z_1 = \frac{\partial z}{\partial x} + bz = \Sigma(z),$$

we had

$$hz = \frac{\partial z_1}{\partial x} + bz_1 = \Sigma(z_1) = \Sigma\sigma(z),$$

$$kz = \frac{\partial Z_1}{\partial y} + aZ_1 = \sigma(Z_1) = \sigma\Sigma(z).$$

Thus the two transformations, effected in succession upon z, give merely a multiple of z and so (§ 191) lead to an equation with the same invariants as the original equation. Effectively, we may write

$$\Sigma\sigma = h, \quad \sigma\Sigma = k,$$

as operators: or, as multiplication of z by a factor does not affect the invariants, we can regard the operations Σ and σ as inverses of one another.

Hence, when we take any number of these substitutions in turn and in any order, it is unnecessary to frame combinations of σ and Σ in immediate succession, so far as our quest is the form of the invariants of the successively transformed equations. All the independent sets of invariants will be obtained by taking

$$\sigma, \ \sigma^2, \ \sigma^3, \ \ldots,$$

$$\Sigma, \ \Sigma^2, \ \Sigma^3, \ \ldots,$$

where

$$\sigma^2(z) = \sigma\{\sigma(z)\},$$

and so on. Moreover, as the invariants of the equation satisfied by $\Sigma\sigma z$ are the invariants of the equation satisfied by z, we can write

$$\Sigma = \sigma^{-1}, \quad \Sigma^2 = \sigma^{-1}(\sigma^{-1}) = \sigma^{-2},$$

and so on: and, similarly,

$$\sigma = \Sigma^{-1}, \quad \sigma^2 = \Sigma^{-2},$$

and so on. Thus all the independent sets of invariants will be obtained by effecting upon z the set of operations

$$\ldots, \ \sigma^{-3}, \ \sigma^{-2}, \ \sigma^{-1}, \ 1, \ \sigma, \ \sigma^2, \ \sigma^3, \ \ldots,$$

or the set of operations

$$\ldots, \ \Sigma^{-3}, \ \Sigma^{-2}, \ \Sigma^{-1}, \ 1, \ \Sigma, \ \Sigma^2, \ \Sigma^3, \ \ldots,$$

the two sets being the same as one another in reversed order.

It should be noted, that the coefficient a in the equation is unaltered for the operation Σ or σ^{-1}, and that the coefficient b is unaltered for the operation σ or Σ^{-1}.

This inverse character of the two transformations relatively to each other can be illustrated by the two sets of equations giving the two sets of invariants. We have

$$H_1 = k,$$

$$K_1 = 2k - h - \frac{\partial^2 \log k}{\partial x \, \partial y}.$$

Expressing Σ in the form σ^{-1}, we should naturally express H_1 and K_1 in the forms h_{-1} and k_{-1}: and so

$$h_{-1} = k,$$

$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial x \, \partial y},$$

giving a relation between the invariants of an equation and those of the equation next after it in the ascending series of transformations σ. Repeating the relation, we have

$$h = k_1,$$
$$k = 2k_1 - h_1 - \frac{\partial^2 \log k_1}{\partial x \partial y},$$

that is,

$$k_1 = h,$$
$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y},$$

which are the equations expressing the invariants of the equation in σz in terms of the invariants of the equation in z.

Invariants of Successively Transformed Equations.

194. Expressions for the invariants of the equation, having $\sigma^n z$ for its dependent variable, can be found in terms of the invariants of the equations that occur earlier in the series. Since

$$h_n = 2h_{n-1} - k_{n-1} - \frac{\partial^2 \log h_{n-1}}{\partial x \partial y},$$
$$k_n = h_{n-1},$$

we have

$$h_n - 2h_{n-1} + h_{n-2} = -\frac{\partial^2 \log h_{n-1}}{\partial x \partial y},$$
$$h_{n-1} - 2h_{n-2} + h_{n-3} = -\frac{\partial^2 \log h_{n-2}}{\partial x \partial y},$$
$$h_{n-2} - 2h_{n-3} + h_{n-4} = -\frac{\partial^2 \log h_{n-3}}{\partial x \partial y},$$
$$\ldots\ldots\ldots\ldots = \ldots\ldots,$$
$$h_1 - 2h + k = -\frac{\partial^2 \log h}{\partial x \partial y};$$

adding, we find

$$h_n - h_{n-1} = h - k - \frac{\partial^2}{\partial x \partial y} \log (h h_1 \ldots h_{n-1}).$$

Taking this for $n, n-1, \ldots$ and adding, we have

$$h_n - h = n(h - k) - \frac{\partial^2}{\partial x \partial y} \log (h^n h_1^{n-1} \ldots h^2_{n-2} h_{n-1}),$$

that is,

$$h_n = (n+1) h - nk - \frac{\partial^2}{\partial x \partial y} \log (h^n h_1^{n-1} \ldots h^2_{n-2} h_{n-1}).$$

Also

$$k_n = h_{n-1}$$
$$= nh - (n-1)k - \frac{\partial^2}{\partial x \partial y} \log (h^{n-1} h_1^{n-2} \dots h^2_{n-3} h_{n-2}).$$

And then, knowing the invariants, we can write down a canonical form of the equation.

Similar expressions can be obtained for the invariants of the equation having $\Sigma^n z$ for its dependent variable.

It is important also to have, in explicit form, the relation between the dependent variables z and $\sigma^n z$. We have seen that the coefficient b is unaltered by the application of the σ-substitution: also that, if

$$z_{m+1} = \frac{\partial z_m}{\partial y} + a_m z_m = \sigma z_m = \sigma^{m+1} z,$$

then

$$h_m z_m = \frac{\partial z_{m+1}}{\partial x} + b z_{m+1}.$$

Consequently,

$$z_m e^{\int b\,dx} = \frac{1}{h_m} \frac{\partial}{\partial x} (z_{m+1} e^{\int b\,dx}),$$

and therefore

$$z e^{\int b\,dx} = \frac{\partial}{h \partial x} \cdot \frac{\partial}{h_1 \partial x} \cdot \dots \cdot \frac{\partial}{h_{n-1} \partial x} (z_n e^{\int b\,dx}),$$

where z_n, $= \sigma^n z$, is the dependent variable in the nth transformation. Thus z is expressible in terms of z_n by a series of direct operations.

Similarly,

$$z e^{\int a\,dy} = \frac{\partial}{k \partial y} \cdot \frac{\partial}{K_1 \partial y} \cdot \dots \cdot \frac{\partial}{K_{\mu-1} \partial y} (Z_\mu e^{\int a\,dy})$$

is the relation between z and Z_μ, $= \Sigma^\mu z$, the dependent variable in the μth equation in the series of successive transformations Σ.

These expressions are important for the practical integration of the equation because, in this method, it is the variable z_n or Z_μ which is first explicitly obtained.

Ex. 1. The invariants of the equation satisfied by $\sigma^n z$, when the equation satisfied by z is

$$s + \frac{\alpha}{x+y} p + \frac{\beta}{x+y} q + \frac{\gamma}{(x+y)^2} z = 0,$$

where α, β, γ are constants, can be derived from the preceding results.

We have

$$a=\frac{\alpha}{x+y},\quad b=\frac{\beta}{x+y},\quad c=\frac{\gamma}{(x+y)^2}:$$

hence

$$h=\frac{\alpha\beta-\alpha-\gamma}{(x+y)^2},\quad k=\frac{\alpha\beta-\beta-\gamma}{(x+y)^2}.$$

Also

$$\frac{\partial^2 \log h}{\partial x\,\partial y}=\frac{2}{(x+y)^2};$$

therefore

$$h_i=\frac{\mu_i}{(x+y)^2},\quad k_i=\frac{\mu_{i-1}}{(x+y)^2},$$

where μ_i is a constant partly dependent upon i. Now

$$h_i=(i+1)\,h-ik-\frac{\partial^2}{\partial x\,\partial y}\log\,(h^i h_1^{\,i-1}\ldots h^2_{i-2}h_{i-1})$$

$$=(i+1)\,h-ik-\frac{2}{(x+y)^2}\{i+(i-1)+\ldots+2+1\}$$

$$=(i+1)\,h-ik-\frac{i(i+1)}{(x+y)^2};$$

consequently,

$$\mu_i=(i+1)\,(\alpha\beta-\alpha-\gamma)-i\,(\alpha\beta-\beta-\gamma)-i\,(i+1)$$

$$=-\gamma+(\alpha+i)\,(\beta-i-1).$$

Hence

$$h_n=\frac{(\alpha+n)\,(\beta-n-1)-\gamma}{(x+y)^2},$$

$$k_n=\frac{(\alpha+n-1)\,(\beta-n)-\gamma}{(x+y)^2}.$$

In the special case when γ is a prime number, α and β also being integers, neither h_n nor k_n can vanish unless

$$\gamma=\alpha+\beta-2.$$

If both conditions, viz. that γ is a prime number and is equal to $\alpha+\beta-2$, be satisfied, then n is $\beta-2$ when $h_n=0$ or (what is the same thing in effect) n is $\beta-1$ when $k_n=0$.

In the special case, when γ is an integer that is not a prime, then particular forms of α and β may make h_n zero or k_n zero.

In the special case, when the constants α, β, γ (not being integers) are such that, for one (or more than one) value of n, either of the relations

$$\gamma=(\alpha+n)\,(\beta-n-1),$$

$$\gamma=(\alpha+n-1)\,(\beta-n),$$

is satisfied, then at the corresponding stage, we have $h_n=0$ or $k_n=0$.

In no other case will either of the invariants vanish at any stage in the successive transformations.

Ex. 2. Shew that, if the invariants of the equation satisfied by σz are the same as those of the equation satisfied by z, both equations are represented by

$$s = z,$$

on making the appropriate changes of variables. (Darboux.)

Ex. 3. Shew that, if the invariants of the equation satisfied by $\sigma^2 z$ are the same as those of the equation satisfied by z, then

$$\frac{\partial^2}{\partial x \, \partial y} \{\log (hk)\} = 0,$$

and that, by appropriate changes of x into a function of itself alone and of y into a function of itself alone, the values of h and k can be deduced from the integral of

$$\frac{\partial^2 \omega}{\partial x \, \partial y} = \sin \omega.$$

Obtain a reduced form of the original differential equation. (Darboux.)

Ex. 4. Shew that, if $h_2 = k$, then $k_2 = h$ subject to a transformation of x into a function of itself and of y also into a function of itself.

Ex. 5. The equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2l \frac{\partial z}{\partial x} + 2m \frac{\partial z}{\partial y} + nz = 0$$

is transformed by a substitution

$$z_1 = \lambda z,$$

where λ is a function of x and y only: prove that the quantities

$$J = \frac{\partial l}{\partial y} - \frac{\partial m}{\partial x},$$

$$K = \frac{\partial l}{\partial x} + \frac{\partial m}{\partial y} + l^2 + m^2 - n,$$

are invariants for all such transformations. (Burgatti.)

Ex. 6. Shew that, if both the invariants J and K of the preceding equation vanish, the equation is reducible to

$$\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} = 0:$$

if J vanishes but not K, the equation is reducible to

$$\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} + c z_1 = 0:$$

and that, if K vanishes but not J, the equation is reducible to

$$\beta \frac{\partial^2 (\alpha z)}{\partial x^2} + \alpha \frac{\partial^2 (\beta z)}{\partial y^2} = 0,$$

where

$$\alpha = e^{\int l \, dx}, \quad \beta = e^{\int m \, dy}.$$ (Burgatti.)

Ex. 7. Apply the results in Exx. 5, 6 to the equation

$$s + ap + bq + cz = 0.$$

195. From the relation between the dependent variable in the original equation and the dependent variables in the equations that arise by successive applications of one or other of the two transformations, it is clear that the general integral of the original equation can be obtained when once the general integral of one of the transformed equations is known.

The integration is certainly possible for an equation when either of its invariants vanishes. Suppose that the nth equation, in the succession of σ-transformations, is the first of the equations characterised by the possession of a vanishing invariant: then h_n must be the vanishing invariant, because the value of k_n is h_{n-1} which is a non-vanishing invariant of the next earlier equation. Thus

$$\frac{\partial}{\partial x}\left(\frac{\partial z_n}{\partial y}+a_n z_n\right)+b\left(\frac{\partial z_n}{\partial y}+a_n z_n\right)=\left(\frac{\partial a_n}{\partial x}+a_n b-c_n\right)z_n$$
$$=0,$$

because h_n is zero: the coefficient b is the coefficient b in the original equation which is unaffected by the σ-transformations; and therefore

$$\frac{\partial z_n}{\partial y}+a_n z_n=Ye^{-\int b\,dx},$$
$$z_n=e^{-\int a_n dy}\left\{X+\int Ye^{\int(a_n dy-b\,dx)}dy\right\},$$

where X and Y are arbitrary functions respectively. If

$$-\gamma=\int(a_n dy-b\,dx),\quad e^{-\gamma}=\beta,\quad \alpha=e^{-\int a_n dy},$$

we have

$$z_n=\alpha\left(X+\int Y\beta\,dy\right),$$

where α and β are determinate functions of x and y.

Now the relation between z and z_n is

$$ze^{\int b\,dx}=\frac{\partial}{h\partial x}\cdot\frac{\partial}{h_1\partial x}\cdot\ldots\cdot\frac{\partial}{h_{n-1}\partial x}\left(z_n e^{\int b\,dx}\right)$$
$$=\frac{\partial}{h\partial x}\cdot\frac{\partial}{h_1\partial x}\cdot\ldots\cdot\frac{\partial}{h_{n-1}\partial x}\left\{e^{\gamma}\left(X+\int Ye^{-\gamma}dy\right)\right\};$$

and therefore, effecting the differential operations, we have the value of z expressed in the form

$$z=A\left(X+\int Y\beta\,dy\right)+A_1\left(X'+\int Y\frac{\partial\beta}{\partial x}dy\right)+\ldots$$
$$\ldots+A_n\left(X^{(n)}+\int Y\frac{\partial^n\beta}{\partial x^n}dy\right),$$

where $A, A_1, \ldots, A_n$ are determinate functions of x and y, and where $X^{(m)}$ is the mth derivative of X with respect to its argument x.

The functions X and Y are arbitrary: consequently, the integral obtained is a general integral and, in the form obtained, it involves indefinite partial quadratures. As Y is arbitrary, we shall have a specialised integral on making Y zero: and this specialised integral is

$$z = AX + A_1X' + \ldots + A_nX^{(n)},$$

so that an integral exists, involving homogeneously and linearly an arbitrary function and its derivatives, when one of the invariants in the succession of equations, constructed by a repeated σ-transformation, vanishes: and the specialised integral is in finite form, without partial quadratures.

The converse of this result is also true: that is to say, if an equation

$$s + ap + bq + cz = 0$$

possesses an integral

$$z = AX + A_1X' + \ldots + A_nX^{(n)},$$

where $A_1, \ldots, A_n$ are determinate functions of x and y, where X is an arbitrary function of x and $X', \ldots, X^{(n)}$ are its first n derivatives, then the successive application of the σ-transformation will, after n operations at most, produce an equation for which the invariant h is zero. To prove this assertion, let the specified value of z be substituted in the differential equation: the result is

$$B_{n+1}X^{(n+1)} + B_nX^{(n)} + \ldots = 0,$$

where

$$B_{n+1} = \frac{\partial A_n}{\partial y} + aA_n,$$

$$B_n = \frac{\partial A_{n-1}}{\partial y} + aA_{n-1} + \frac{\partial^2 A_n}{\partial x \partial y} + a\frac{\partial A_n}{\partial x} + b\frac{\partial A_n}{\partial y} + cA_n,$$

and, for values of $m \leqslant n$,

$$B_m = \frac{\partial A_{m-1}}{\partial y} + aA_{m-1} + \frac{\partial^2 A_m}{\partial x \partial y} + a\frac{\partial A_m}{\partial x} + b\frac{\partial A_m}{\partial y} + cA_m.$$

As X is an arbitrary function of x, and as the differential equation must be satisfied identically by the postulated value of z, we must have

$$B_{n+1}=0,\quad B_n=0,\ \ldots,\ B_m=0,$$

for all values of $m \leqslant n$. Thus

$$\frac{\partial A_n}{\partial y}+aA_n=0,$$

from $B_{n+1}=0$; and therefore

$$\frac{\partial A_{n-1}}{\partial y}+aA_{n-1}=hA_n.$$

Now let the σ-transformation be applied to z: a new dependent variable z_1 is introduced, and we have

$$\begin{aligned} z_1 &= \frac{\partial z}{\partial y}+az \\ &= \left(\frac{\partial A_n}{\partial y}+aA_n\right)X^{(n)}+\left(\frac{\partial A_{n-1}}{\partial y}+aA_{n-1}\right)X^{(n-1)}+\ldots \\ &= hA_nX^{(n-1)}+\ldots, \end{aligned}$$

so that the order of the highest derivative of X in z_1 is certainly less by unity than it is in z and, if $h=0$, it is certainly less by two units than it is in z.

Similarly, when the corresponding σ-transformation is applied to the equation in z_1, a new variable z_2 is obtained such that the order of the highest derivative of X which it contains is certainly less by unity than the corresponding order in z_1 and, if $h_1=0$, the order is certainly less by two units than the highest order in z_1.

Hence taking these substitutions in succession, we reduce the order of the highest derivative of X in the successive dependent variables by one unit at least in each operation: and therefore, after n operations at most, either we obtain an equation such that the invariant h of the preceding equation vanishes, or we obtain a dependent variable ζ such that

$$z_\mu=\zeta=CX,$$

where C is a determinate function of x and y, and $\mu \leqslant n$. Now an equation of the type under consideration, which is satisfied by z_μ, is

$$s_\mu+a_\mu p_\mu+b_\mu q_\mu+c_\mu z_\mu=0,$$

where

$$a_\mu = -\frac{1}{C}\frac{\partial C}{\partial y}, \qquad b_\mu = \frac{1}{C}\frac{\partial C}{\partial x}, \qquad c_\mu = -\frac{1}{C}\frac{\partial^2 C}{\partial x \partial y};$$

and therefore

$$h_\mu = \frac{\partial a_\mu}{\partial x} + a_\mu b_\mu - c_\mu$$

$$= 0,$$

on substitution.

Ex. The equation

$$s + \frac{a}{x+y} p + \frac{\beta}{x+y} q + \frac{\gamma}{(x+y)^2} z = 0$$

possesses an integral of the preceding type in finite form involving an arbitrary function X and its first n derivatives, if

$$\gamma = (a+n)(\beta - n - 1),$$

where n is a positive integer.

Obtain the general integral for the conditional value of γ when $n=1$.

Integrals of Finite Rank.

196. When the series of σ-transformations is finite in the sense that, after a finite number n of operations, an equation is obtained having its invariant h equal to zero, we have seen that an integral of the original equation exists in the form

$$z = AX + A_1 X' + \ldots + A_n X^{(n)}.$$

Conversely, if an integral of the original equation of this form exists, then μ of the σ-transformations (where $\mu \leqslant n$) lead to an equation having its invariant h equal to zero.

If $\mu = n$, the two properties are the exact reciprocals of each other.

If $\mu < n$, then an integral of the original equation exists in the form

$$z = CX_1 + C_1 X_1' + \ldots + C_\mu X_1^{(\mu)},$$

where X_1 is another arbitrary function of x. Hence it is necessary to consider whether expressions of this type can, by change of the arbitrary function, be changed so that the new form involves, in a diminished order, the derivatives of the new arbitrary function.

It is clear that the highest order in such an expression can always be increased by taking

$$X_1 = \alpha X + \alpha_1 X' + \ldots + \alpha_{n-\mu} X^{(n-\mu)},$$

where $\alpha, \alpha_1, \ldots, \alpha_{n-\mu}$ are specific functions of x alone: it is not clear (and it is not, in fact, the case) that it is always possible to decrease the order of such an expression by taking

$$\xi = \beta X + \beta_1 X' + \ldots + \beta_\rho X^{(\rho)},$$

where $\beta, \beta_1, \ldots, \beta_\rho$ are functions of x, specifically at our disposal.

An expression

$$AX + A_1 X' + \ldots + A_n X^{(n)}$$

is declared* to be of *rank* $n+1$, (or to be *irreducible*), when it is not possible, by any transformation

$$\xi = \alpha X + \alpha_1 X' + \ldots + \alpha_\mu X^{(\mu)},$$

to make the order of the highest derivative of ξ in the transformed expression less than n. We may therefore say that, when n is the number of σ-transformations applied in succession and needed to produce the first equation having its invariant h equal to zero, the original equation possesses an integral of rank $n+1$.

All these properties are associated with the σ-transformations of which the first is

$$\frac{\partial z}{\partial y} + az = z_1.$$

Similar considerations occur in association with the Σ- (or σ^{-1}) transformations of which the first is

$$\frac{\partial z}{\partial x} + bz = Z_1.$$

The general result, which can be established in a precisely similar manner, is that, if the equation resulting after m applications of the Σ-transformation is the first in the succession for which an invariant (now k_m) vanishes, then the original equation possesses an integral of rank $m+1$, of the form

$$z = BY + B_1 Y' + \ldots + B_m Y^{(m)},$$

where Y is an arbitrary function of y and $Y', Y'', \ldots$ are its derivatives, and where $B, B_1, \ldots, B_m$ are definite functions of x

* Darboux uses the term *rang*.

and y. Conversely, if an integral of this form is possessed by the original equation, the rank of the integral may be equal to $m+1$ but, if not, it is less than $m+1$.

197. As regards the possible reducibility of a given expression, the rank of which is less than $n+1$ though it involves derivatives of the arbitrary function of order n, it is sufficient to reduce the highest order of derivative by one unit at a time. For if an expression is reducible by a transformation

$$X_1 = \beta X + \alpha X' + X'',$$

where α and β are specific functions of x alone, let a quantity λ be determined by the equation

$$\lambda' + \alpha\lambda - \lambda^2 = \beta,$$

and then take

$$\mu = \alpha - \lambda:$$

if

$$\xi = X' + \lambda X,$$

then

$$X_1 = \xi' + \mu\xi,$$

that is, the original expression is reducible by the transformations

$$\xi = X' + \lambda X,$$
$$X_1 = \xi' + \mu\xi,$$

in succession. Similarly, if the expression in question can have the order of the highest derivative reduced by more than two units, we can secure the reduction by successive reductions of a single unit at a time.

Ex. 1. The condition for reducibility (and, if the condition is satisfied, the reduced form) of an expression Θ, where

$$\Theta = aX + a_1 X' + a_2 X'',$$

where a, a_1, a_2 are functions of x and y, are easily obtained. Let a relation

$$\xi = X' + \lambda X,$$

where λ is a function of x only, represent the arbitrary function in a reduced form of Θ: then

$$\xi' = X'' + \lambda X' + \lambda' X,$$

so that

$$a_2\xi' + p\xi = a_2 X'' + (a_2\lambda + p) X' + (a_2\lambda' + p\lambda) X.$$

In order that this expression may be the same as Θ, we must have

$$a_2\lambda + p = a_1,$$
$$a_2\lambda' + p\lambda = a;$$

and therefore λ satisfies the equation

$$a_2\lambda' = a - a_1\lambda + a_2\lambda^2,$$

while, when λ is known, the value of p is given by

$$p = a_1 - a_2\lambda.$$

Now the equation satisfied by λ is

$$\lambda' - \lambda^2 = \frac{a}{a_2} - \lambda\frac{a_1}{a_2},$$

and λ is a function of x only: hence, taking

$$\lambda = -\frac{1}{u}\frac{du}{dx},$$

where u is a function of x only, we have

$$a_2\frac{d^2u}{dx^2} + a_1\frac{du}{dx} + au = 0.$$

Hence also

$$\frac{\partial a_2}{\partial y}\frac{d^2u}{dx^2} + \frac{\partial a_1}{\partial y}\frac{du}{dx} + \frac{\partial a}{\partial y}u = 0;$$

and therefore

$$\frac{\dfrac{d^2u}{dx^2}}{a_1\dfrac{\partial a}{\partial y} - a\dfrac{\partial a_1}{\partial y}} = \frac{\dfrac{du}{dx}}{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}} = \frac{u}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}.$$

Consequently, necessary and sufficient conditions are

$$\frac{\partial}{\partial y}\left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right) = 0,$$

$$\frac{a_1\dfrac{\partial a}{\partial y} - a\dfrac{\partial a_1}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}} = \frac{\partial}{\partial x}\left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right) + \left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right)^2.$$

The conditions can be simplified by taking

$$a = \beta a_2, \quad a_1 = \beta_1 a_2:$$

they then become

$$\frac{\partial}{\partial y}\left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right) = 0,$$

and

$$\frac{\beta_1\dfrac{\partial\beta}{\partial y} - \beta\dfrac{\partial\beta_1}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}} + \frac{\partial}{\partial x}\left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right) = \left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right)^2,$$

the latter of which can also be written

$$\beta\left(\frac{\partial\beta_1}{\partial y}\right)^2-\beta_1\frac{\partial\beta}{\partial y}\frac{\partial\beta_1}{\partial y}+\left(\frac{\partial\beta}{\partial y}\right)^2=\frac{\partial\beta_1}{\partial y}\frac{\partial^2\beta}{\partial x\,\partial y}-\frac{\partial\beta}{\partial y}\frac{\partial^2\beta_1}{\partial x\,\partial y}.$$

The value of λ is

$$\lambda=\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}},$$

and that of p is

$$p=a_2\left(\beta_1-\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right).$$

Ex. 2. Shew that the expression

$$X'''+a_1X''+a_2X'+a_3X=0$$

can be transformed into an expression involving only X_1, X_1', X_1'', where X_1 is an arbitrary function, provided the following conditions are satisfied, viz.:—Let quantities A, B, C be defined by the relations

$$A\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_2}{\partial y}\\ \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_2}{\partial y^2}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_3}{\partial y}\\ \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_3}{\partial y^2}\end{vmatrix},$$

$$(A^2-B)\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_2}{\partial y}\\ \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_2}{\partial y^2}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial a_2}{\partial y}, & \dfrac{\partial a_3}{\partial y}\\ \dfrac{\partial^2 a_2}{\partial y^2}, & \dfrac{\partial^2 a_3}{\partial y^2}\end{vmatrix},$$

$$C-3AB+A^3=a_3+a_1(A^2-B)-a_2A;$$

then the conditions are that the equations

$$\frac{\partial A}{\partial y}=0,\quad \frac{\partial A}{\partial x}=B,\quad \frac{\partial B}{\partial x}=C,$$

shall be satisfied.

Shew also that the reduced form is

$$X_1''+(a_1-A)X_1'+(a_2-a_1A+A^2-2B)X_1,$$

where

$$X_1=X'+AX.$$

Ex. 3. Obtain conditions necessary and sufficient to secure that the expression

$$X'''+a_1X''+a_2X'+a_3X=0$$

is of rank not greater than two.

Equations having an Integral of Finite Rank.

198. We can now construct the aggregate of equations which admit an integral of rank $n+1$ obtainable by means of the σ-transformation.

When an integral is of rank $n+1$, the invariant h_n of the equation, which results from n successive applications of the σ-transformation, is equal to zero. Let a_n and b_n be chosen arbitrarily: then, as $h_n = 0$, we have

$$c_n = a_n b_n + \frac{\partial a_n}{\partial x},$$

which determines c_n; and then

$$\begin{aligned} k_n &= \frac{\partial b_n}{\partial y} + a_n b_n - c_n \\ &= \frac{\partial b_n}{\partial y} - \frac{\partial a_n}{\partial x} \\ &= \theta, \end{aligned}$$

say. Now the relations

$$h_{m+1} = 2h_m - k_m - \frac{\partial^2 \log h_m}{\partial x \partial y},$$

$$k_{m+1} = h_m,$$

give

$$h_m = k_{m+1},$$

$$k_m = 2k_{m+1} - h_{m+1} - \frac{\partial^2 \log k_{m+1}}{\partial x \partial y}.$$

Consequently,

$$h_{n-1} = k_n = \theta,$$

$$k_{n-1} = 2k_n - \frac{\partial^2 \log k_n}{\partial x \partial y} = 2\theta - \frac{\partial^2 \log \theta}{\partial x \partial y};$$

all the invariants for all the equations in the series can thus be calculated in backward succession. Also we have

$$b_{j+1} = b_j,$$

so that the coefficient b in what is to be the original equation is given by

$$b = b_n.$$

Again,

$$a_{j+1} = a_j - \frac{\partial \log h_j}{\partial y},$$

and therefore the coefficient a in what is to be the original equation is given by

$$a = a_n + \frac{\partial}{\partial y} \log (hh_1 \ldots h_{n-1}).$$

Lastly,

$$c = \frac{\partial a}{\partial x} + ab - h = \frac{\partial b}{\partial y} + ab - k.$$

The coefficients in the equation are thus determined, so that the equation can be regarded as known: its actual expression involves the two arbitrary elements a_n and b_n.

The integral of the equation is of rank $n+1$. Let

$$\gamma = \int (b_n dx - a_n dy),$$

so that γ may be considered known and, in particular,

$$\frac{\partial^2 \gamma}{\partial x \partial y} = \theta;$$

then the actual expression of the full general integral of the equation is

$$ze^{\int b_n dx} = \frac{\partial}{h\partial x} \cdot \frac{\partial}{h_1 \partial x} \cdot \ldots \cdot \frac{\partial}{h_{n-1}\partial x} \left\{ e^{\gamma} \left(X + \int Y e^{-\gamma} \, dy \right) \right\},$$

where X and Y denote arbitrary functions of x and of y respectively.

A similar process leads to the aggregate of equations which admit an integral of finite rank $m+1$ obtainable by means of the Σ-transformation. We take arbitrary quantities a_m, b_m: as K_m vanishes, we have

$$c_m = a_m b_m + \frac{\partial b_m}{\partial y}.$$

The successive invariants K_i and H_i are given by the relations

$$H_{i+1} = K_i,$$
$$K_{i+1} = 2K_i - H_i - \frac{\partial^2 \log K_i}{\partial x \partial y};$$

the coefficients in what is to be the original equation are given by

$$a = a_m,$$
$$b = b_m + \frac{\partial}{\partial x} \log (kK_1 \ldots K_{m-1}),$$
$$c = \frac{\partial a}{\partial x} + ab - h = \frac{\partial b}{\partial y} + ab - k;$$

and if

$$\delta = \int (a_m dy - b_m dx),$$

the full general integral of the equation is given by

$$ze^{\int a_m dy} = \frac{\partial}{k\partial y} \cdot \frac{\partial}{K_1 \partial y} \cdot \ldots \cdot \frac{\partial}{K_{m-1}\partial y}\left\{e^{\delta}\left(Y + \int Xe^{-\delta} dx\right)\right\},$$

where X and Y denote arbitrary functions of x and of y respectively.

Ex. 1. The equations admitting an integral of rank unity are of the form

$$s + ap + bq + cz = 0,$$

where either

$$c = ab + \frac{\partial a}{\partial x},$$

when h vanishes, and the integral is

$$ze^{\int a\, dy} = X + \int Ye^{\int (a\, dy - b\, dx)}\, dy :$$

or

$$c = ab + \frac{\partial b}{\partial y},$$

when k vanishes, and the integral is

$$ze^{\int b\, dx} = Y + \int Xe^{\int (b\, dx - a\, dy)}\, dx.$$

The coefficients a and b are chosen arbitrarily.

Ex. 2. For equations admitting an integral of rank two, we have a couple of forms according as the vanishing invariant arises through the σ-transformation or through the Σ-transformation.

Taking the σ-transformation, the invariant h_1 is to vanish: we choose two arbitrary quantities $\boldsymbol{a}$ and $\boldsymbol{\beta}$, these being a_1 and b_1. Then

$$\frac{\partial \boldsymbol{\beta}}{\partial y} - \frac{\partial \boldsymbol{a}}{\partial x} = k_1$$

$$= h;$$

and then

$$b = \boldsymbol{\beta},$$

$$a = \boldsymbol{a} + \frac{\partial (\log h)}{\partial y},$$

$$c = -h + \boldsymbol{a\beta} + \boldsymbol{\beta} \frac{\partial \log h}{\partial y} + \frac{\partial \boldsymbol{a}}{\partial x} + \frac{\partial^2 \log h}{\partial x\, \partial y}.$$

These give the coefficients of the equation: it is easy to deduce from them the tests as to whether an equation with coefficients a', b', c' belongs to the type. We have

$$h = \frac{\partial a'}{\partial x} + a'b' - c',$$

and then

$$\beta = b',$$

$$a = a' - \frac{\partial \log h}{\partial y},$$

so that, as

$$h = \frac{\partial \beta}{\partial y} - \frac{\partial a}{\partial x},$$

we have

$$h = \frac{\partial b'}{\partial y} - \frac{\partial a'}{\partial x} + \frac{\partial^2 \log h}{\partial x \, \partial y},$$

as the necessary condition*.

For the general integral of the equation

$$s + ap + bq + cz = 0,$$

when it admits an integral of rank two associated with the σ-transformation, let

$$\begin{aligned} \gamma &= \int (\beta dx - a dy) \\ &= \int (b dx - a dy) + \log h \\ &= u + \log h, \end{aligned}$$

say: then, by the general result, the full integral is

$$\begin{aligned} ze^{\int b dx} &= \frac{1}{h} \frac{\partial}{\partial x} \left\{ h e^u \left(X + \int \frac{Y}{h} e^{-u} dy \right) \right\} \\ &= \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) e^u \left(X + \int \frac{Y}{h} e^{-u} dy \right) \\ &\qquad + e^u \left\{ X' + \int Y \frac{\partial}{\partial x} \left(\frac{e^{-u}}{h} \right) dy \right\} \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} ze^{\int a dy} &= X' + \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) X \\ &\quad + \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) \int Y \frac{e^{-u}}{h} dy + \int Y \frac{\partial}{\partial x} \left(\frac{e^{-u}}{h} \right) dy. \end{aligned}$$

The integral of rank two is, of course,

$$z = e^{-\int a dy} X' + e^{-\int a dy} \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) X.$$

Corresponding results hold when the integral of rank two is associated with the Σ-transformation. We take two arbitrary quantities a' and β': as K_1 is zero in this case, we have

$$\frac{\partial a'}{\partial x} - \frac{\partial \beta'}{\partial y} = H_1 = k,$$

* It is, in effect, $h_1 = 0$.

and then

$$a=a',$$
$$b=\beta'+\frac{\partial \log k}{\partial x},$$
$$c=-k+a'\beta'+a'\frac{\partial \log k}{\partial x}+\frac{\partial \beta'}{\partial y}+\frac{\partial^2 \log k}{\partial x\,\partial y}.$$

The condition that a given equation

$$s+a''p+b''q+c''z=0$$

should belong to this type is that

$$k=\frac{\partial a''}{\partial x}-\frac{\partial b''}{\partial y}+\frac{\partial^2 \log k}{\partial x\,\partial y},$$

where

$$k=-c''+a''b''+\frac{\partial b''}{\partial y}.$$

Also, the full integral of the equation

$$s+ap+bq+cz=0,$$

when it admits an integral of rank two associated with the Σ-transformation, is

$$ze^{\int a\,dy}=\frac{1}{k}\frac{\partial}{\partial y}\left\{ke^{v}\left(Y+\int\frac{X}{k}e^{-v}dx\right)\right\},$$

where

$$v=\int(a\,dy-b\,dx):$$

and the integral of rank two is

$$z=e^{-\int b\,dx}Y'+e^{-\int b\,dx}\left(\frac{1}{k}\frac{\partial k}{\partial y}+\frac{\partial v}{\partial y}\right)Y.$$

Ex. 3. Prove that, if an equation

$$s+ap+bq+cz=0$$

admits an integral of rank $n+1$, the term involving the highest derivative of the arbitrary function is

$$e^{-\int a\,dy}X^{(n)} \quad \text{or} \quad e^{-\int b\,dx}Y^{(n)}$$

in the respective cases.

Ex. 4. Integrate the equations

(i) $s+xp+yq+(1+xy)z=0$;

(ii) $s+mxp+nyq+(2m-n+mn\,xy)z=0$;

(iii) $s+myp+e^{cy}q+(2c+my)e^{cy}z=0$;

where m, n, c are constants in the last two equations.

Ex. 5. Prove that the equation

$$s+xyq+nxz=0,$$

where n is a finite integer, possesses an integral of finite rank; and obtain the integral in the two cases $n=2$, $n=-1$. (Imschenetsky.)

Ex. 6. Let the equation

$$s+ap+bq=0$$

be of rank $m+1$ in one of the variables; prove that the equation

$$s+p(a+u)+q(b+v)=0$$

is of rank $m+2$ in one of the variables, where

$$u=\frac{\partial}{\partial y}\log\left(\frac{z_3 J}{\frac{\partial\sigma}{\partial y}}\right), \quad v=\frac{\partial}{\partial x}\log\left(\frac{z_3 J}{\frac{\partial\sigma}{\partial x}}\right),$$

z_1, z_2, z_3 are three integrals of the original equation, σ is the ratio of either z_1 or z_2 to z_3, and

$$J=\frac{\partial\left(\frac{z_1}{z_3},\ \frac{z_2}{z_3}\right)}{\partial(x,\ y)}.$$

Apply this result to the equations

(i) $s=0$;

(ii) $xys+xp+yq=0$. (R. Liouville.)

Equations having Integrals of Doubly-finite Rank.

199. Hitherto, the equations considered have been such that they have admitted an integral which involves an arbitrary function of one of the variables so as to be of finite rank in that variable: but the general integral, in the actual expressions obtained, involved partial quadratures so far as concerns the occurrence of the other variable. It is manifest that one specially select class of equations will be constituted by those possessing general integrals which are of finite rank in both variables and for which, therefore, both sets of transformations lead to a vanishing invariant after only a finite number of operations in each series effected upon the given equation.

We have seen that, when the equation

$$s+ap+bq+cz=0$$

possesses an integral of rank $n+1$ in the form

$$z=AX+A_1X'+\ldots+A_nX^{(n)},$$

and when the σ-transformation

$$\frac{\partial z}{\partial y}+az=z_1$$

is effected, then the rank of z_1 is n. On the other hand, if the Σ-transformation

$$\frac{\partial z}{\partial x} + bz = Z_1$$

is effected, then the rank of Z_1 in the variable x is obviously $n+2$. Similarly, if the equation possesses an integral of rank $m+1$ in the form

$$z = BY + B_1 Y' + \ldots + B_m Y^{(m)},$$

and if the σ-transformation

$$\frac{\partial z}{\partial y} + az = z_1$$

is effected, the rank of z_1 is $m+2$, while if the Σ-transformation

$$\frac{\partial z}{\partial x} + bz = Z_1$$

is effected, the rank of Z_1 is m.

Now suppose that both sets of transformations are finite in the sense that, in each set, only a finite number of operations is needed to produce a vanishing invariant; then obviously the general integral of the equation is

$$z = AX + A_1 X' + \ldots + A_n X^{(n)} + BY + B_1 Y' + \ldots + B_m Y^{(m)}.$$

The effect of the σ-transformation on this quantity z is to increase the rank of the new variable z_1 in y by one unit and to decrease the rank in x by one unit: that is, the integer $m+n$ is the same for z as for z_1, and therefore it is invariantive for the σ-transformation. Similarly, this integer $m+n$ is invariantive for the Σ-transformation. Accordingly, Darboux* calls this invariantive integer $m+n$ the *characteristic number* of the equation.

In preceding investigations, when only a single series of transformations leading to a vanishing invariant was considered, the general integral contained terms of one of the two forms

$$\int Y \frac{\partial^\mu \alpha}{\partial x^\mu} dy, \quad \int X \frac{\partial^\mu \beta}{\partial y^\mu} dx,$$

respectively: it is not impossible that, on integration by parts or through some other process, such terms could be replaced by

* *L.c.*, t. II, p. 38.

quantities that are of finite rank in the respective variables. Hence it is desirable to consider equations which are of finite rank in each of the variables. Accordingly, we assume that the equation

$$s + ap + bq + cz = 0$$

possesses a general integral which is of finite rank $n+1$ in the variable x and of finite rank $m+1$ in the variable y.

Let the σ-transformation be applied to this equation so as to construct n equations in succession: the dependent variable z_n of the last of these equations is of rank 1 in the variable x and of rank $m+n+1$ in the variable y so that, writing

$$\mu = m + n,$$

we have

$$z_n = KX + CY + C_1 Y' + \ldots + C_\mu Y^{(\mu)}.$$

Moreover, the invariant h_n of the last equation in the series vanishes, and so the equation is

$$\frac{\partial}{\partial x}\left(\frac{\partial z_n}{\partial y} + a_n z_n\right) + b_n\left(\frac{\partial z_n}{\partial y} + a_n z_n\right) = 0:$$

also the values of K and C_μ, the coefficients of X and of $Y^{(\mu)}$ in z_n, are* given by

$$K = e^{-\int a_n dy}, \quad C_\mu = e^{-\int b_n dx}.$$

Taking

$$u = e^{\int (a_n dy - b_n dx)},$$

$$\phi = z_n e^{\int a_n dy},$$

the equation is

$$\frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial \phi}{\partial y}\right) = 0,$$

and therefore

$$\phi = X + \int u Y_1 dy,$$

where X and Y_1 are arbitrary functions of x and of y respectively. Having regard to the expression for z_n and to the value of K in that expression, we see that the X in z_n and the X in ϕ are the same; and then

$$\int u Y_1 dy = GY + G_1 Y' + \ldots + G_\mu Y^{(\mu)},$$

* Ex. 3, § 198.

where

$$G_r = \frac{C_r}{K}, \qquad (r = 0, 1, \ldots, \mu),$$

and, in particular,

$$G_\mu = \frac{C_\mu}{K} = e^{\int a_n dy - b_n dx} = u.$$

Hence Y_1, which is a function of y only, must be of the form

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)},$$

where $\lambda, \lambda_1, \ldots, \lambda_\mu$ are determinate functions of y, the function Y being arbitrary. Thus

$$\begin{aligned} &\frac{\partial}{\partial y}\{GY + G_1 Y' + \ldots + G_{\mu-1} Y^{(\mu-1)} + uY^{(\mu)}\} \\ &\quad = uY_1 \\ &\quad = u\{\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}\}; \end{aligned}$$

and therefore, as Y is an arbitrary function of y, we must have

$$\begin{aligned} u\lambda_\mu &= G_{\mu-1} + \frac{\partial u}{\partial y}, \\ u\lambda_{\mu-1} &= G_{\mu-2} + \frac{\partial G_{\mu-1}}{\partial y}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ u\lambda_r &= G_{r-1} + \frac{\partial G_r}{\partial y}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ u\lambda_1 &= G + \frac{\partial G_1}{\partial y}, \\ u\lambda &= \frac{\partial G}{\partial y}. \end{aligned}$$

The elimination of the quantities $G, G_1, \ldots, G_{\mu-1}$ among these equations leads to the relation

$$u\lambda - \frac{\partial}{\partial y}(u\lambda_1) + \frac{\partial^2}{\partial y^2}(u\lambda_2) - \ldots + (-1)^\mu \frac{\partial^\mu}{\partial y^\mu}(u\lambda_\mu) + (-1)^{\mu+1}\frac{\partial^{\mu+1} u}{\partial y^{\mu+1}} = 0,$$

which may be regarded as an equation for the determination of the quantity u, where

$$u = e^{\int (a_n dy - b_n dx)}.$$

Moreover, when u is known (on the assumption that $\lambda, \lambda_1, \ldots, \lambda_\mu$ are known), then $G_{\mu-1}, \ldots, G_1, G$ are immediately derivable from the foregoing equations. The equation, having z_n for its dependent variable, can be constructed: and thence, by the inverse substitutions repeated n times in succession, we construct the original equation.

A similar result follows if we proceed from the original equation, supposed to possess a general integral of what may be called doubly-finite rank, by using the Σ (or σ^{-1}) transformations. For the construction of the equation, it is necessary to solve an ordinary differential equation, still of order $\mu+1$ and having x for its independent variable: the problem is of the same order of difficulty as under the preceding process.

Accordingly, continuing the solution of the problem under the former analysis, we choose the quantities $\lambda, \lambda_1, \ldots, \lambda_\mu$ at will as functions of y: the variable x remains parametric in the determination of u as an integral of the ordinary equation of order $\mu+1$: and then, u being known, the quantities $G, G_1, \ldots, G_{\mu-1}$ are obtained (in reverse order) merely by differential operations. The knowledge of these quantities gives the value of ϕ: we return, by the inverted σ-transformations, to the original equation and to the value of z.

We have seen that an expression, involving an arbitrary function and derivatives of that function up to any order p, may have its rank less than p: if such be the case, one or other of the transformations considered in connection with the equations under discussion will lead to a vanishing invariant after a number of applications which is less than p. In the present case, the same question arises as to the rank of the quantity ϕ: the expression

$$GY + G_1 Y' + \ldots + G_{\mu-1} Y^{(\mu-1)} + uY^{(\mu)}$$

must not be reducible because otherwise the rank of ϕ in the variable y would be less than $\mu+1$. The expression can only be reducible if

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}$$

is reducible, that is, if the equation

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)} = 0$$

is reducible. The quantities $\lambda, \lambda_1, \ldots, \lambda_\mu$ are at our disposal: they therefore must be chosen so that the preceding equation is irreducible: and this choice can always be made*.

* A method for constructing irreducible equations has been given by Frobenius, *Crelle*, t. LXXX (1875), p. 332: see vol. IV of this work, § 80.

200. Now that the existence of the result is definitely established, its form can be materially simplified by the use of other properties. Let $y_1, \ldots, y_{\mu+1}$ be $\mu+1$ linearly independent integrals of the equation

$$\lambda v + \lambda_1 v' + \ldots + \lambda_\mu v^{(\mu)} + v^{(\mu+1)} = 0,$$

so that the determinant

$$\Delta = \begin{vmatrix} y_1, & y_2, & \ldots, & y_{\mu+1} \\ y_1', & y_2', & \ldots, & y'_{\mu+1} \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(\mu)}, & y_2^{(\mu)}, & \ldots, & y^{(\mu)}_{\mu+1} \end{vmatrix}$$

does not vanish, its actual value being $Ae^{-\int \lambda_\mu dy}$, where A is a non-vanishing constant. Then, for each of these integrals, the expression

$$\frac{\partial}{\partial y}(Gy_i + G_1 y_i' + \ldots + G_{\mu-1} y_i^{(\mu-1)} + u y_i^{(\mu)})$$

vanishes, so that

$$Gy_i + G_1 y_i' + \ldots + G_{\mu-1} y_i^{(\mu-1)} + u y_i^{(\mu)} + \xi_i = 0,$$

where ξ_i is a function of x alone, and i has the values $1, \ldots, \mu+1$. Solving these $\mu+1$ linear equations for $G, G_1, \ldots, G_{\mu-1}$, and for u (which is G_μ), and substituting their values in

$$\begin{aligned} \phi &= X + \int u Y_1 dy \\ &= X + GY + G_1 Y' + \ldots + G_\mu Y^{(\mu)}, \end{aligned}$$

we have

$$\phi\Delta = \begin{vmatrix} X, & Y, & Y', & \ldots, & Y^{(\mu)} \\ \xi_1, & y_1, & y_1', & \ldots, & y_1^{(\mu)} \\ \xi_2, & y_2, & y_2', & \ldots, & y_2^{(\mu)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_{\mu+1}, & y_{\mu+1}, & y'_{\mu+1}, & \ldots, & y^{(\mu)}_{\mu+1} \end{vmatrix} = \Theta,$$

say: and then

$$\begin{aligned} z_n &= e^{-\int a_n dy} \phi \\ &= \frac{\Theta}{\Delta} e^{-\int a_n dy}. \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= u Y_1 \\ &= u(\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}); \end{aligned}$$

substituting for ϕ and equating coefficients of $Y^{(\mu+1)}$, we have

$$u\Delta = (-1)^{\mu+1} \begin{vmatrix} \xi_1 , & y_1 , & y_1' , & \dots , & y_1^{(\mu-1)} \\ \xi_2 , & y_2 , & y_2' , & \dots , & y_2^{(\mu-1)} \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{\mu+1}, & y_{\mu+1}, & y'_{\mu+1}, & \dots , & y_{\mu+1}^{(\mu-1)} \end{vmatrix},$$

which gives the value of u. Also, since the expression

$$Y^{(\mu+1)} + \lambda Y + \lambda_1 Y' + \dots + \lambda_\mu Y^{(\mu)}$$

vanishes when $Y = y_1, y_2, \dots, y_{\mu+1}$, we have

$$\Delta\,(Y^{(\mu+1)} + \lambda Y + \lambda_1 Y' + \dots + \lambda_\mu Y^{(\mu)})$$
$$= (-1)^{\mu+1} \begin{vmatrix} Y , & Y' , & \dots , & Y^{(\mu)} , & Y^{(\mu+1)} \\ y_1 , & y_1' , & \dots , & y_1^{(\mu)} , & y_1^{(\mu+1)} \\ \dots & \dots & \dots & \dots & \dots \\ y_{\mu+1}, & y'_{\mu+1}, & \dots , & y_{\mu+1}^{(\mu)}, & y_{\mu+1}^{(\mu+1)} \end{vmatrix}$$
$$= (-1)^{\mu+1}\Phi,$$

say, so that

$$\frac{\partial\phi}{\partial y} = (-1)^{\mu+1}\frac{u}{\Delta}\Phi,$$

where u and Δ do not involve the arbitrary function Y.

It is to be noted that no linear relation can exist among the quantities $\xi_1, \xi_2, \dots, \xi_{\mu+1}$: if such a relation existed, there would be a linear relation among the quantities G of the form

$$G = \alpha_1 G_1 + \dots + \alpha_{\mu-1} G_{\mu-1} + \alpha_\mu G_\mu$$
$$= \alpha_1 G_1 + \dots + \alpha_{\mu-1} G_{\mu-1} + \alpha_\mu u,$$

which by the use of the relations between u and these quantities G, would lead to a linear equation of order μ satisfied by u, contrary to the property that the linear equation of order $\mu + 1$ satisfied by u is irreducible.

We can now proceed to the construction of the original equation as well as to the derivation of its general integral. We have

$$ze^{\int b\,dx} = \frac{\partial}{h\partial x} \cdot \frac{\partial}{h_1\partial x} \cdot \dots \cdot \frac{\partial}{h_{n-1}\partial x}(z_n e^{\int b\,dx}),$$

and therefore

$$z = Az_n + A_1\frac{\partial z_n}{\partial x} + \dots + A_n\frac{\partial^n z_n}{\partial x^n}.$$

When the value

$$z_n = \frac{\Theta}{\Delta} e^{-\int a_n dy}$$

is substituted, we see at once that terms involving $X, X', \ldots, X^{(n)}$ will occur linearly: moreover, there will be terms involving Y and its derivatives linearly, and it is known that the highest derivative of Y that should occur is $Y^{(m)}$. Hence we have

$$z = \alpha X + \alpha_1 X' + \ldots + \alpha_n X^{(n)} + \beta Y + \beta_1 Y' + \ldots + \beta_m Y^{(m)},$$

where the coefficients $\alpha, \ldots, \alpha_n, \beta, \ldots, \beta_m$ have yet to be determined.

Owing to the presence of the determinant Θ in the expression for z_n, it is clear that z_n vanishes when

$$X = \xi_i, \quad Y = y_i,$$

simultaneously; and this is true for all values of i. Also forming the derivatives of z_n with regard to x, it is clear that every one of these derivatives vanishes similarly when

$$X = \xi_i, \quad Y = y_i,$$

simultaneously: consequently, as z is a linear combination of z_n and these derivatives, z itself also vanishes in these circumstances. Hence

$$\alpha\xi_i + \alpha_1 \xi_i' + \ldots + \alpha_n \xi_i^{(n)} + \beta y_i + \beta_1 y_i' + \ldots + \beta_m y_i^{(m)} = 0,$$

for $i = 1, 2, \ldots, \mu + 1$, where $\mu = m + n$. These $m + n + 1$ relations, linear and homogeneous in the $m + n + 2$ coefficients α and β, determine the ratios of these quantities and can be regarded as determining all these $m + n + 2$ coefficients, save as to an unknown common factor. Consequently z is known, save as to this factor.

But the differential equation, so far as concerns its two invariants, is unaffected by the association of any factor with z: and therefore, within this range, we can neglect the factor or, what is the same thing, we can make it unity. Hence, writing

$$\begin{vmatrix} X, & X', & \ldots, & X^{(n)}, & Y, & Y', & \ldots, & Y^{(m)} \\ \xi_1, & \xi_1', & \ldots, & \xi_1^{(n)}, & y_1, & y_1', & \ldots, & y_1^{(m)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_{\mu+1}, & \xi'_{\mu+1}, & \ldots, & \xi^{(n)}_{\mu+1}, & y_{\mu+1}, & y'_{\mu+1}, & \ldots, & y^{(m)}_{\mu+1} \end{vmatrix} = Z,$$

we may take

$$z = Z;$$

and in this expression, $y_1, \ldots, y_{\mu+1}$ are $\mu + 1$ linearly independent functions of y, while $\xi_1, \ldots, \xi_{\mu+1}$ are $\mu + 1$ linearly independent functions of x.

Next, to obtain the differential equation of the second order satisfied by z, we suppose that Z is expanded and, when expanded, has the form

$$Z = \alpha X + \alpha_1 X' + \ldots + \alpha_n X^{(n)} + \beta Y + \beta_1 Y' + \ldots + \beta_m Y^{(m)},$$

where the coefficients $\alpha, \ldots, \alpha_n, \beta, \ldots, \beta_m$ are now known functions of x and y, and, in particular,

$$\alpha = \begin{vmatrix} \xi_1', & \ldots, & \xi_1^{(n)}, & y_1', & \ldots, & y_1^{(m)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi'_{\mu+1}, & \ldots, & \xi^{(n)}_{\mu+1}, & y'_{\mu+1}, & \ldots, & y^{(m)}_{\mu+1} \end{vmatrix},$$

with similar expressions for α_n, β, β_m. With the foregoing value Z of z, we have

$$\frac{\partial z}{\partial x} = \frac{\partial \alpha}{\partial x} X + \ldots + \alpha_n X^{(n+1)} + \frac{\partial \beta}{\partial x} Y + \ldots + \frac{\partial \beta_m}{\partial x} Y^{(m)},$$

$$\frac{\partial z}{\partial y} = \frac{\partial \alpha}{\partial y} X + \ldots + \frac{\partial \alpha_n}{\partial y} X^{(n)} + \frac{\partial \beta}{\partial y} Y + \ldots + \beta_m Y^{(m+1)},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 \alpha}{\partial x \partial y} X + \ldots + \frac{\partial \alpha_n}{\partial y} X^{(n+1)} + \frac{\partial^2 \beta}{\partial x \partial y} Y + \ldots + \frac{\partial \beta_m}{\partial x} Y^{(m+1)};$$

and therefore, taking the equation for z in the form

$$s + ap + bq + cz = 0,$$

and substituting these values of s, p, q, we have a relation

$$\frac{\partial \alpha_n}{\partial y} + a\alpha_n = 0,$$

from the coefficient of $X^{(n+1)}$, a relation

$$\frac{\partial \beta_m}{\partial x} + b\beta_m = 0,$$

from the coefficient of $Y^{(m+1)}$, and a relation

$$\frac{\partial^2 \alpha}{\partial x \partial y} + a\frac{\partial \alpha}{\partial x} + b\frac{\partial \alpha}{\partial y} + c\alpha = 0,$$

from the coefficient* of X. These three relations give the values of a, b, c: and so the differential equation is fully known.

It thus appears that equations having general integrals which are doubly finite in rank can be constructed. The integrals are formed by means of a number of functions $\xi_1, \ldots, \xi_{\mu+1}$ of x, having no linear relations with one another, and a number of functions $y_1, \ldots, y_{\mu+1}$ of y, likewise having no linear relations with one another. The number of different kinds of integrals is the same as the partition of the integer μ into two positive integers n and m: each integral, thus provided by a partition of the integer μ, determines a differential equation

$$s + ap + bq + cz = 0$$

uniquely. The integer μ is the characteristic number of the equations: and thus there are $\mu + 1$ different types of equations having one and the same characteristic number.

Ex. 1. From the values of α and β, expressed in the forms of determinants of $\xi_1, \ldots, \xi_{\mu+1}, y_1, \ldots, y_{\mu+1}$ and of their derivatives, verify that the relations

$$\frac{\partial^2 \alpha}{\partial x \partial y} + a\frac{\partial \alpha}{\partial x} + b\frac{\partial \alpha}{\partial y} + c\alpha = 0,$$

$$\frac{\partial^2 \beta}{\partial x \partial y} + a\frac{\partial \beta}{\partial x} + b\frac{\partial \beta}{\partial y} + c\beta = 0,$$

are equivalent to one another.

Ex. 2. Prove that the expression for Z given in the text can vanish identically only when X is a linear combination of $\xi_1, \ldots, \xi_{\mu+1}$ with constant coefficients and, at the same time, Y is the same linear combination of $y_1, \ldots, y_{\mu+1}$. (Darboux.)

Ex. 3. There is only one equation with the characteristic number zero, for there is only one partition of 0.

To construct the equation and its integral, we require a single function of x and a single function of y, say ξ and η respectively. Then

$$Z = \begin{vmatrix} X, & Y \\ \xi, & \eta \end{vmatrix} = \xi\eta\left(\frac{X}{\xi} - \frac{Y}{\eta}\right):$$

* There is also a relation

$$\frac{\partial^2 \beta}{\partial x \partial y} + a\frac{\partial \beta}{\partial x} + b\frac{\partial \beta}{\partial y} + c\beta = 0$$

from the coefficient of Y. Both of these relations involving c are expressions of the condition that α and β are solutions of the equation obtained by taking $X=1$, $Y=0$, and $X=0$, $Y=1$, respectively.

we know that any factor associated with z can be neglected without affecting the invariants of the equation, and so we neglect the factor $\xi\eta$: and then, taking

$$X_1=\frac{X}{\xi},\quad Y_1=\frac{Y}{\eta},$$

as new arbitrary functions of x and of y respectively, we have

$$z=X_1-Y_1.$$

The differential equation is

$$s=0\ ;$$

and we have

$$h=0,\quad k=0.$$

The equation $s=0$ is, in fact, the simplest reduced form of equation for which $h=0$, $k=0$: the less simple form is

$$s+p\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial x}+z\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x\,\partial y}\right)=0,$$

where u is any function of the variables.

Ex. 4. There are two types of equation with the characteristic number unity, corresponding to the partitions $1+0$ and $0+1$ of 1. We take two functions ξ_1 and ξ_2 of x, and two functions η_1 and η_2 of y.

For one type, corresponding to partition $1+0$, the value of Z is

$$Z=\begin{vmatrix} X, & X', & Y \\ \xi_1, & \xi_1', & \eta_1 \\ \xi_2, & \xi_2', & \eta_2 \end{vmatrix}=\alpha X+\alpha_1 X'+\beta Y.$$

No loss of generality as regards the invariants of the equation is caused by taking

$$\xi_2=x,\quad \eta_2=y\ ;$$

and then

$$\alpha=y\xi_1'-\eta_1,\quad \alpha_1=x\eta_1-y\xi_1,\quad \beta=\xi_1-x\xi_1',$$

so that

$$a=-\frac{1}{\alpha_1}\frac{\partial\alpha_1}{\partial y}=-\frac{x\eta_1'-\xi_1}{x\eta_1-y\xi_1},$$

$$b=-\frac{1}{\beta}\frac{\partial\beta}{\partial x}=\frac{x\xi_1''}{\xi_1-x\xi_1'},$$

$$c=ab,$$

the last being obtained either from the equation

$$\frac{\partial^2\beta}{\partial x\,\partial y}+a\frac{\partial\beta}{\partial x}+b\frac{\partial\beta}{\partial y}+c\beta=0,$$

on noticing that b does not involve y, or from the condition

$$k=0,$$

so that

$$c=ab+\frac{\partial b}{\partial y}=ab.$$

The value of h is given by

$$h=\frac{\partial a}{\partial x}+ab-c$$

$$=\frac{\partial a}{\partial x}$$

$$=-\frac{(x\xi_1'-\xi_1)(y\eta_1'-\eta_1)}{(x\eta_1-y\xi_1)^2};$$

and then

$$h_1=2h-k-\frac{\partial^2(\log h)}{\partial x\,\partial y}$$

$$=2\frac{\partial a}{\partial x}+2\frac{\partial^2}{\partial x\,\partial y}(\log a_1)$$

$$=2\frac{\partial a}{\partial x}-2\frac{\partial a}{\partial x}=0.$$

For the other type of equation, corresponding to the partition 0+1, the value of Z is

$$Z=\begin{vmatrix} X, & Y, & Y' \\ \xi_1, & \eta_1, & \eta_1' \\ \xi_2, & \eta_2, & \eta_2' \end{vmatrix}=aX+\beta Y+\beta_1 Y'.$$

As in the previous case, we may take

$$\xi_2=x,\quad \eta_2=y;$$

and then

$$a=\eta_1-y\eta_1',\quad \beta=x\eta_1'-\xi_1,\quad \beta_1=y\xi_1-x\eta_1,$$

so that

$$a=-\frac{1}{a}\frac{\partial a}{\partial y}=\frac{y\eta_1''}{\eta_1-y\eta_1'},$$

$$b=-\frac{1}{\beta}\frac{\partial\beta}{\partial x}=-\frac{y\xi_1'-\eta_1}{y\xi_1-x\eta_1},$$

$$c=ab.$$

And, for this type of equation, we have

$$h=0,\quad K_1=0.$$

Ex. 5. Integrate the equations

(i) $$s-\frac{1}{y}p+\frac{k}{x}q-\frac{k}{xy}z=0,$$

k being a constant;

(ii) $$s+\left(\frac{1}{y}-\frac{1}{x-y}\right)p-\frac{2}{x}q-\frac{2}{x}\left(\frac{1}{y}-\frac{1}{x-y}\right)z=0;$$

(iii) $$s+\frac{2}{x-y}p-\frac{2}{x-y}q-\frac{4}{(x-y)^2}z=0.$$

Ex. 6. Obtain the three types of equation

$$s+ap+bq+cz=0,$$

which have 2 for their characteristic number.

Is it possible to determine c so that 2 shall be the characteristic number, if

$$a=\frac{(x-y)\,\eta''}{\xi-\eta-(x-y)\,\eta'},\quad b=\frac{(x-y)\,\xi''}{\xi-\eta-(x-y)\,\eta'},$$

where ξ and η are functions of x and of y respectively?

Ex. 7. Prove that the equation

$$s+mxyp+nyz=0$$

is of finite rank in one of the variables if $m\div n$ is an integer, m and n being constants.

Obtain the successive invariants when this condition is satisfied; and integrate the equation so as to obtain the general integral.

Ex. 8. Transform the equation

$$r+2\lambda s+(\lambda^2-\mu^2)\,t+ap+\beta q=0,$$

where λ, μ, a, β are functions of the independent variables, so that it becomes

$$s+ap+bq=0;$$

and express the invariants of the latter in terms of λ, μ, a, β.

Apply this method to the equation

$$r+\tfrac{3}{2}s+\tfrac{1}{2}t-\frac{2}{x}(p+q)=0,$$

shewing that the transformed equation is of doubly-finite rank.

(Winckler.)

Ex. 9. Solve the equation

$$r-t=\frac{1}{x}(p-q),$$

by making a transformation similar to that in the last example.

Ex. 10. Prove that the equation

$$s+\frac{p+q}{x+y}-\tfrac{1}{4}\,\frac{n\,(n+1)}{(x+y)^2}\,z=0$$

is of doubly-finite rank when n is a positive integer (including zero).

Ex. 11. Shew that the integral of the equation

$$r-t=2n\frac{p}{x},$$

where n is a positive integer, is

$$z=\sum_{m=0}^{m=n}(-1)^m\,\frac{2^m}{m!}\,\frac{\binom{n}{m}}{\binom{2n}{m}}\,x^m\left[\phi^{(m)}(x-y)+\psi^{(m)}(x+y)\right],$$

where ϕ and ψ are arbitrary functions; and that the integral of the equation

$$r-t=-2n\frac{p}{x},$$

where n is a positive integer, is

$$z=\sum_{m=0}^{m=n-1}(-1)^m\frac{2^m}{m!}\frac{\binom{n-1}{m}}{\binom{2n-2}{m}}x^{-2n+m+1}\left[\phi^{(m)}(x-y)+\psi^{(m)}(x+y)\right].$$

Deduce these results also by transforming both equations into Laplace's linear form. (Sersawy.)

DARBOUX'S MODIFIED FORMS.

201. Darboux has given* another form for the succession of invariants and for the equation itself, when it is of finite rank in either of the variables.

Suppose that a linear equation is of finite rank in the variable x, and assume that the invariant h of the nth equation in the succession, obtained by the use of the σ-transformation, is zero. Thus, as $h_n=0$, we have

$$\begin{aligned} k_n &= \frac{\partial b_n}{\partial y}+a_n b_n-c_n \\ &= \frac{\partial b_n}{\partial y}-\frac{\partial a_n}{\partial x} \\ &= -\frac{\partial^2 \log \alpha}{\partial x \partial y}, \end{aligned}$$

where

$$\alpha=e^{\int(a_n dy-b_n dx)};$$

and the value of z_n is given by

$$z_n e^{\int a_n dy}=X+\int \alpha Y dy.$$

Now

$$\begin{aligned} h_{n-1} &= k_n \\ &= -\frac{\partial^2 \log \alpha}{\partial x \partial y}; \end{aligned}$$

and

$$h_{m+1}-2h_m+h_{m-1}=-\frac{\partial^2 \log h_m}{\partial x \partial y},$$

* *Théorie générale des surfaces*, t. II, pp. 123, et seq.

for all values of m, so that, taking $m = n - 1$, we have

$$\begin{aligned} h_{n-2} &= \quad 2h_{n-1} - \frac{\partial^2 \log h_{n-1}}{\partial x \partial y} \\ &= -2\frac{\partial^2 \log \alpha}{\partial x \partial y} - \frac{\partial^2}{\partial x \partial y}\left\{\log\left(\frac{1}{\alpha}\frac{\partial^2 \alpha}{\partial x \partial y} - \frac{1}{\alpha^2}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y}\right)\right\} \\ &= -\frac{\partial^2}{\partial x \partial y}\left\{\log\left(\alpha\frac{\partial^2 \alpha}{\partial x \partial y} - \frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y}\right)\right\}. \end{aligned}$$

This result suggests a new form: Darboux introduces quantities

$$H_0 = \quad \alpha \ ,$$

$$H_1 = \begin{vmatrix} \alpha\ , & \dfrac{\partial \alpha}{\partial x} \\ \dfrac{\partial \alpha}{\partial y}\ , & \dfrac{\partial^2 \alpha}{\partial x \partial y} \end{vmatrix},$$

$$H_2 = \begin{vmatrix} \alpha\ , & \dfrac{\partial \alpha}{\partial x}\ , & \dfrac{\partial^2 \alpha}{\partial x^2} \\ \dfrac{\partial \alpha}{\partial y}\ , & \dfrac{\partial^2 \alpha}{\partial x \partial y}\ , & \dfrac{\partial^3 \alpha}{\partial x^2 \partial y} \\ \dfrac{\partial^2 \alpha}{\partial y^2}\ , & \dfrac{\partial^3 \alpha}{\partial x \partial y^2}\ , & \dfrac{\partial^4 \alpha}{\partial x^2 \partial y^2} \end{vmatrix},$$

and so on. The expressions, by means of these quantities, of the two invariants already obtained are

$$\begin{aligned} h_n &= \quad 0, \\ h_{n-1} &= -\frac{\partial^2}{\partial x \partial y}(\log H_0), \\ h_{n-2} &= -\frac{\partial^2}{\partial x \partial y}(\log H_1); \end{aligned}$$

and it is natural to inquire whether the expression of h_{n-s} is

$$h_{n-s} = -\frac{\partial^2}{\partial x \partial y}(\log H_{s-1}).$$

This suggested expression for h_{n-s} is actually valid: to establish the validity, we proceed as follows.

Writing

$$\frac{\partial^{r+s} \alpha}{\partial x^r \partial y^s} = \alpha_{r,s},$$

consider the determinant

$$H_{p+1} = \begin{vmatrix} \alpha_{0,0}, & \alpha_{1,0}, & \dots, & \alpha_{p+1,0} \\ \alpha_{0,1}, & \alpha_{1,1}, & \dots, & \alpha_{p+1,1} \\ \dots & \dots & \dots & \dots \\ \alpha_{0,p+1}, & \alpha_{1,p+1}, & \dots, & \alpha_{p+1,p+1} \end{vmatrix}.$$

Let $A_{i,j}$ denote the minor of $\alpha_{i,j}$ in H_{p+1}: then, by a well-known property, we have

$$\begin{vmatrix} A_{p,p}, & A_{p+1,p} \\ A_{p,p+1}, & A_{p+1,p+1} \end{vmatrix} = H_{p+1} \begin{vmatrix} \alpha_{0,0}, & \dots, & \alpha_{p-1,0} \\ \dots & \dots & \dots \\ \alpha_{0,p-1}, & \dots, & \alpha_{p-1,p-1} \end{vmatrix}$$

$$= H_{p+1} H_{p-1}.$$

Now

$$A_{p+1,p+1} = H_p,$$

$$A_{p+1,p} = \frac{\partial H_{p+1}}{\partial \alpha_{p+1,p}} = -\frac{\partial H_p}{\partial y},$$

$$A_{p,p+1} = \frac{\partial H_{p+1}}{\partial \alpha_{p,p+1}} = -\frac{\partial H_p}{\partial x},$$

$$A_{p,p} = \frac{\partial H_{p+1}}{\partial \alpha_{p,p}} = \frac{\partial^2 H_p}{\partial x \partial y};$$

and therefore

$$H_{p+1} H_{p-1} = H_p \frac{\partial^2 H_p}{\partial x \partial y} - \frac{\partial H_p}{\partial x}\frac{\partial H_p}{\partial y}$$

$$= H_p^2 \frac{\partial^2}{\partial x \partial y} (\log H_p).$$

Now suppose that

$$h_{n-s-1} = -\frac{\partial^2}{\partial x \partial y} (\log H_s),$$

for $s = 0, 1, \dots, p$; then

$$H_{p+1} H_{p-1} = -H_p^2 h_{n-p-1},$$

so that

$$\frac{\partial^2}{\partial x \partial y} (\log H_{p+1})$$

$$= -\frac{\partial^2}{\partial x \partial y} (\log H_{p-1}) + 2\frac{\partial^2}{\partial x \partial y} (\log H_p) + \frac{\partial^2}{\partial x \partial y} (\log h_{n-p-1})$$

$$= -2h_{n-p-1} + h_{n-p} + \frac{\partial^2}{\partial x \partial y} (\log h_{n-p-1})$$

$$= -h_{n-p-2},$$

so that the form holds for $s=p+1$, if it holds for $s=0, 1, \ldots, p$. It is known to hold for $s=0, 1$, so that it holds generally; and therefore

$$h_{n-i}=-\frac{\partial^2}{\partial x\partial y}(\log H_{i-1}),$$

for all the values of i.

202. Similar analysis can be employed to construct the whole series of equations that occur through the successive applications of the Laplace σ-transformation. The integral of the nth equation is

$$z_n e^{\int a_n dy} = X + \int \alpha Y dy$$
$$= z_0,$$

say. Let z_m denote the determinant

$$z_m = \begin{vmatrix} z_0, & \dfrac{\partial z_0}{\partial x}, & \ldots, & \dfrac{\partial^m z_0}{\partial x^m} \\ \alpha, & \dfrac{\partial \alpha}{\partial x}, & \ldots, & \dfrac{\partial^m \alpha}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{m-1}\alpha}{\partial y^{m-1}}, & \dfrac{\partial^m \alpha}{\partial x\partial y^{m-1}}, & \ldots, & \dfrac{\partial^{2m-1}\alpha}{\partial x^m \partial y^{m-1}} \end{vmatrix};$$

let Z_μ denote the minor of $\dfrac{\partial^\mu z_0}{\partial x^\mu}$ in z_m, and let $C_{i,j}$ denote the minor of $\alpha_{i,j}$ in z_m. Then, using the same theorem as before, we have

$$\begin{vmatrix} Z_{m-1}, & Z_m \\ C_{m-1,m-1}, & C_{m,m-1} \end{vmatrix} = (-1)^{m-1} z_m H_{m-2};$$

also

$$Z_m = (-1)^m H_{m-1},$$
$$Z_{m-1} = (-1)^{m-1}\frac{\partial H_{m-1}}{\partial x},$$
$$C_{m-1,m-1} = -\frac{\partial z_{m-1}}{\partial x},$$
$$C_{m,m-1} = z_{m-1};$$

and therefore

$$z_m H_{m-2} = z_{m-1}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial z_{m-1}}{\partial x}.$$

Again,

$$\begin{vmatrix} Z_{m-1} \quad , & Z_m \\ C_{m-1,m-2}, & C_{m,m-2} \end{vmatrix} = (-1)^m z_m \begin{vmatrix} \alpha, & \dfrac{\partial\alpha}{\partial x}, & \dots, & \dfrac{\partial^{m-2}\alpha}{\partial x^{m-2}} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial^{m-3}\alpha}{\partial y^{m-3}}, & & \dots, & \dfrac{\partial^{2m-5}\alpha}{\partial x^{m-2}\partial y^{m-3}} \\ \dfrac{\partial^{m-1}\alpha}{\partial y^{m-1}}, & & \dots, & \dfrac{\partial^{2m-3}\alpha}{\partial x^{m-2}\partial y^{m-1}} \end{vmatrix}$$

$$= (-1)^m z_m \frac{\partial H_{m-2}}{\partial y};$$

and therefore, as

$$C_{m,m-2} = -\frac{\partial z_{m-1}}{\partial y}, \qquad C_{m-1,m-2} = \frac{\partial^2 z_{m-1}}{\partial x \partial y},$$

we have

$$z_m \frac{\partial H_{m-2}}{\partial y} = \frac{\partial z_{m-1}}{\partial y}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial^2 z_{m-1}}{\partial x \partial y}.$$

Hence

$$\frac{\partial^2 z_{m-1}}{\partial x \partial y} - \frac{\partial(\log H_{m-1})}{\partial x}\frac{\partial z_{m-1}}{\partial y} = -\frac{1}{H_{m-1}}\frac{\partial H_{m-2}}{\partial y} z_m$$

$$= -\frac{1}{H_{m-1}}\frac{\partial(\log H_{m-2})}{\partial y}\left\{z_{m-1}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial z_{m-1}}{\partial x}\right\},$$

by the former relation: consequently,

$$\frac{\partial^2 z_{m-1}}{\partial x \partial y} - \frac{\partial z_{m-1}}{\partial x}\frac{\partial(\log H_{m-2})}{\partial y} - \frac{\partial z_{m-1}}{\partial y}\frac{\partial(\log H_{m-1})}{\partial x}$$

$$+ z_{m-1}\frac{\partial(\log H_{m-1})}{\partial x}\frac{\partial(\log H_{m-2})}{\partial y} = 0.$$

This is a linear equation of the second order satisfied by z_{m-1}; and it has the canonical form constructed in § 192, for which $c = ab$. Moreover, its invariants are

$$-\frac{\partial^2}{\partial x \partial y}(\log H_{m-2}), \quad = h_{n-m+1},$$

and

$$-\frac{\partial^2}{\partial x \partial y}(\log H_{m-1}), \quad = h_{n-m}$$

$$= k_{n-m+1},$$

which are the invariants of the equation that is deduced from the original equation by $n-m+1$ successive applications of the

Laplace σ-transformation. Now an equation in canonical form is uniquely determined by its invariants; hence the foregoing equation, which is satisfied by z_{m-1}, is the $(n-m+1)$th equation in the series derived from the original equation by the successive application of the Laplace σ-transformations.

Moreover, an expression for z_{m-1} (which is the general integral of the equation) has been given which involves the two arbitrary functions X and Y of x and of y respectively, through the quantity z_0 and its derivatives. It also involves the quantity α, which is not known initially but belongs to the last equation in the series. If, however, α can be obtained (and α must be determined, if this process of integration is to be effective in practice), then the whole series of equations is known and the integral of every equation in the series is known.

When α is assumed at will, the preceding equation satisfied by z_{m-1} is the typical expression of equations which are of finite rank $m-1$ in the variable x. If, however, the equation is given, then the invariant h is known; and, writing

$$\log U = -\iint h\,dx\,dy,$$

we have

$$H_{m-2} = U,$$

which is an equation satisfied by α: but, in the absence of other information, the value of m is not known, and the determination of α is not practicable by this method.

Ex. Prove that

$$z_{m-1}\frac{\partial^2 z_{m-1}}{\partial x\,\partial y} - \frac{\partial z_{m-1}}{\partial x}\frac{\partial z_{m-1}}{\partial y} = z_m z_{m-2},$$

$$z_{m-1}\frac{\partial H_{m-2}}{\partial y} - H_{m-2}\frac{\partial z_{m-1}}{\partial y} = H_{m-1} z_{m-2}.$$

(Darboux.)

203. If however the equation is of finite rank in both variables, say of rank n in the variable x and of rank m in the variable y, then the complete series of equations can be represented by equating the successive expressions

$$\sigma^{-m}F,\ \sigma^{-(m-1)}F,\ \ldots,\ \sigma^{-1}F,\ F,\ \sigma F,\ \ldots,\ \sigma^n F$$

to zero. In that case, we can begin with the equation

$$\sigma^n F = 0,$$

as before, assuming a quantity α; and we can construct gradually the invariants of the equations in backward succession. Writing

$$m + n = \mu - 1,$$

we know that the invariants of $\sigma^{-m}F$ are

$$-\frac{\partial^2 \log H_{\mu-1}}{\partial x \partial y}, \qquad -\frac{\partial^2 \log H_{\mu-2}}{\partial x \partial y};$$

as the series terminates with the mth repetition of the transformation σ^{-1}, the former vanishes. Now

$$H_{p+1} H_{p-1} = H_p^2 \frac{\partial^2}{\partial x \partial y} (\log H_p),$$

for all values of p; and therefore

$$\begin{aligned} H_\mu H_{\mu-2} &= H^2_{\mu-1} \frac{\partial^2}{\partial x \partial y} (\log H_{\mu-1}) \\ &= 0, \end{aligned}$$

or, since $H_{\mu-2}$ does not vanish, we must have

$$H_\mu = 0,$$

that is,

$$\begin{vmatrix} \alpha\ , & \dfrac{\partial \alpha}{\partial x}\ , & \ldots, & \dfrac{\partial^\mu \alpha}{\partial x^\mu} \\ \dfrac{\partial \alpha}{\partial y}\ , & \dfrac{\partial^2 \alpha}{\partial x \partial y}\ , & \ldots, & \dfrac{\partial^{\mu+1} \alpha}{\partial x^\mu \partial y} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^\mu \alpha}{\partial y^\mu}\ , & \dfrac{\partial^{\mu+1} \alpha}{\partial x \partial y^\mu}\ , & \ldots, & \dfrac{\partial^{2\mu} \alpha}{\partial x^\mu \partial y^\mu} \end{vmatrix} = 0,$$

which is an equation for the determination of α.

The equation shews that a relation

$$x_0 \alpha + x_1 \frac{\partial \alpha}{\partial x} + \ldots + x_\mu \frac{\partial^\mu \alpha}{\partial x^\mu} = 0$$

exists, such that $x_0, \ldots, x_\mu$ are functions of x alone. Suppose that $\xi_1, \ldots, \xi_\mu$ are μ linearly independent integrals of such an equation; then the most general value of α is

$$\alpha = \eta_1 \xi_1 + \eta_2 \xi_2 + \ldots + \eta_\mu \xi_\mu,$$

where $\eta_1, \ldots, \eta_\mu$ are independent of x, that is, are functions of y. The equation also shews that a relation

$$y_0 \alpha + y_1 \frac{\partial \alpha}{\partial y} + \ldots + y_\mu \frac{\partial^\mu \alpha}{\partial y^\mu} = 0$$

exists, such that $y_0, \ldots, y_\mu$ are functions of y alone; the same argument as before shews that the most general value of α is

$$\alpha = \eta_1 \xi_1 + \eta_2 \xi_2 + \ldots + \eta_\mu \xi_\mu,$$

where $\xi_1, \ldots, \xi_\mu$ are μ linearly independent functions of x and $\eta_1, \ldots, \eta_\mu$ are μ linearly independent functions of y.

The value of z_m, the integral of the last equation of the series, depends upon z_0, which is

$$X + \int Y\alpha dy,$$

and which thus appears to involve partial quadratures. In the case just considered when the equations are of finite rank in each of the variables, z_m should be expressible without the use of such quadratures: the actual expression can be obtained as follows.

The quantity α, where

$$\alpha = \xi_1 \eta_1 + \ldots + \xi_\mu \eta_\mu,$$

is the most general integral of the equation

$$P(\alpha) = y_0 \alpha + y_1 \frac{\partial \alpha}{\partial y} + \ldots + y_\mu \frac{\partial^\mu \alpha}{\partial y^\mu} = 0,$$

where $\eta_1, \ldots, \eta_\mu$ are μ linearly independent particular integrals. Let

$$Q(\beta) = y_0' \beta + y_1' \frac{\partial \beta}{\partial y} + \ldots + y_\mu' \frac{\partial^\mu \beta}{\partial y^\mu} = 0$$

be the equation which is the adjoint of $P(\alpha) = 0$: then*

$$\beta P(\alpha) - \alpha Q(\beta) = \frac{dU}{dy},$$

where U is a quantity free from partial quadratures and of order $\mu - 1$ in the derivatives of β. Now replace Y in the integral $\int Y\alpha dy$ by $-Q(Y)$, which is permissible because Y is quite arbitrary: and let α be the foregoing integral of $P(\alpha) = 0$: then

$$\begin{aligned} \frac{dU}{dy} &= \beta P(\alpha) - \alpha Q(\beta) \\ &= -\alpha Q(Y), \end{aligned}$$

* See vol. IV of this work, § 82.

on taking $\beta = Y$; and now

$$z_0 = X - \int \alpha Q(Y)\,dy$$
$$= X + U,$$

where U now is of order $\mu - 1$ in the derivatives of y. This being the value of z_0, the value of z_m is explicitly free from partial quadratures: the construction of other forms of z_m can be made exactly as before, in §§ 201, 202.

Goursat's Theorem on the Rank of an Equation.

204. We have seen that the general integral of the equation

$$s + ap + bq + cz = 0$$

involves its arbitrary elements in linear fashion. It can be particularised in many ways: each particular form is an integral: and any linear combination of such particular integrals with constant coefficients is also an integral of the equation. If, in any aggregate of such integrals, no one of them can be expressed as a linear combination of the others with constant coefficients, they are said to be linearly distinct.

Now suppose that n successive applications of the σ-transformation lead to a vanishing invariant h_n, being the first vanishing invariant of the forward series of equations: the preceding theory shews that the original equation possesses an integral

$$z = AX + A_1X' + \ldots + A_nX^{(n)},$$

where X is an arbitrary function of x, and $A, A_1, \ldots, A_n$ are determinate functions of x and y. Let $n + 2$ linearly independent and arbitrary functions $X_1, \ldots, X_{n+2}$ be chosen in such a way that the $n + 2$ integrals, which they determine, are linearly distinct. Denoting these integrals by $z_1, \ldots, z_{n+2}$, and eliminating $A, A_1, \ldots, A_n$ by means of their expressions, we have

$$\begin{vmatrix} z_1, & X_1, & \ldots, & X_1^{(n)} \\ z_2, & X_2, & \ldots, & X_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ z_{n+2}, & X_{n+2}, & \ldots, & X_{n+2}^{(n)} \end{vmatrix} = 0,$$

a relation in which the coefficients of $z_1, \ldots, z_{n+2}$ are functions of x only. Hence, upon the supposition that n applications of the σ-transformation lead to a vanishing invariant h_n, we have found a linear relation, the coefficients in which are functions of the variables, among $n+2$ linearly distinct integrals.

The converse* is also true, viz.: *If $n+1$ linearly distinct integrals of the equation are connected by a homogeneous linear relation the coefficients of which are functions of one of the variables only, then $n-1$ applications at most of one of the Laplace transformations will lead to a vanishing invariant.* (The σ- or the Σ-transformation should be applied according as the coefficients in the homogeneous linear relation are functions of x or are functions of y).

First, let $n=1$, so that, if z_1 and z_2 are two linearly distinct integrals, we have

$$z_2 = z_1 u,$$

where u is a function of x only. Substituting z_2 in the equation

$$s + ap + bq + cz = 0,$$

and remembering that z_1 also is an integral, we have

$$\frac{du}{dx}\left(\frac{\partial z_1}{\partial y} + a z_1\right) = 0,$$

that is, since u is a function of x,

$$\frac{\partial z_1}{\partial y} + a z_1 = 0.$$

Consequently,

$$\frac{\partial^2 z_1}{\partial x \partial y} + a\frac{\partial z_1}{\partial x} + \frac{\partial a}{\partial x} z_1 = 0,$$

and therefore

$$-b\frac{\partial z_1}{\partial y} + \left(\frac{\partial a}{\partial x} - c\right) z_1 = 0,$$

or, inserting the earlier value of $\frac{\partial z_1}{\partial y}$ and removing the non-vanishing factor z_1, we have

$$h = \frac{\partial a}{\partial x} + ab - c = 0,$$

which verifies the theorem for $n=1$.

* The theorem is due to Goursat, *Amer. Jour. Math.*, t. XVIII (1896), p. 348.

Take now the general case of a homogeneous linear relation between $n+1$ integrals, and assume that the coefficients are functions of x only: dividing by the coefficient of z_{n+1}, we have the relation in the form

$$z_{n+1} = \xi_1 z_1 + \xi_2 z_2 + \ldots + \xi_n z_n.$$

It may be assumed that the coefficients $\xi_1, \ldots, \xi_n$ are not connected by a linear relation with constant coefficients: if any relation exists, such as

$$\xi_n = a_1 \xi_1 + \ldots + a_{n-1}\xi_{n-1} + a_n,$$

then

$$z_{n+1} - a_n z_n = \xi_1 (z_1 + a_1 z_n) + \xi_2 (z_2 + a_2 z_n) + \ldots + \xi_{n-1} (z_{n-1} + a_{n-1} z_n),$$

a homogeneous linear relation between n linearly distinct integrals $z_1 + a_1 z_n, \ldots, z_{n-1} + a_{n-1} z_n, z_{n+1} - a_n z_n$. Thus the assumption that the coefficients $\xi_1, \ldots, \xi_n$ are linearly independent of one another is really an assumption that $n+1$ integrals is the smallest number between which a homogeneous linear relation exists. Such an assumption is no limitation but only makes the problem more precise: it will therefore be made.

Let the Laplace σ-transformation be applied to the equation, and write

$$\zeta_r = \frac{\partial z_r}{\partial y} + a z_r,$$

for $r = 1, \ldots, n$: thus $\zeta_1, \ldots, \zeta_n$ are integrals of the transformed equation. Now

$$z_{n+1} = \sum_{r=1}^{n} \xi_r z_r,$$

so that

$$\frac{\partial z_{n+1}}{\partial y} = \sum_{r=1}^{n} \xi_r \frac{\partial z_r}{\partial y},$$

$$\frac{\partial z_{n+1}}{\partial x} = \sum_{r=1}^{n} \left(\xi_r \frac{\partial z_r}{\partial x} + z_r \frac{d\xi_r}{dx} \right),$$

$$\frac{\partial^2 z_{n+1}}{\partial x \partial y} = \sum_{r=1}^{n} \left(\xi_r \frac{\partial^2 z_r}{\partial x \partial y} + \frac{\partial z_r}{\partial y} \frac{d\xi_r}{dx} \right):$$

substituting z_{n+1} for z in

$$s + ap + bq + cz = 0,$$

and remembering that z_r is an integral for $r = 1, \ldots, n$, we have

$$\sum_{r=1}^{n} \left(\frac{\partial z_r}{\partial y} + a z_r \right) \frac{d\xi_r}{dx} = 0,$$

that is,

$$\zeta_1 \frac{d\xi_1}{dx} + \zeta_2 \frac{d\xi_2}{dx} + \ldots + \zeta_n \frac{d\xi_n}{dx} = 0.$$

Now no one of the quantities $\xi_1, \ldots, \xi_n$ is a constant: otherwise the original relation would effectively be a relation between only n integrals of the original equation: hence we can divide by $\frac{d\xi_n}{dx}$, and we have

$$\zeta_n = \xi_1'\zeta_1 + \xi_2'\zeta_2 + \ldots + \xi'_{n-1}\zeta_{n-1}.$$

We have assumed that there is no relation of the form

$$\xi_n = a_1\xi_1 + \ldots + a_{n-1}\xi_{n-1} + a_n,$$

and therefore there is no relation of the form

$$\frac{d\xi_n}{dx} = a_1 \frac{d\xi_1}{dx} + \ldots + a_{n-1} \frac{d\xi_{n-1}}{dx}:$$

consequently there is no relation of the form

$$a_1\xi_1' + \ldots + a_{n-1}\xi'_{n-1} + 1 = 0.$$

It therefore follows that a relation of the indicated type, among $n+1$ linearly distinct integrals of the original equation, leads to a relation of the same type among n integrals of the equation, which is the result of applying to the original equation the Laplace σ-transformation.

Applying the Laplace σ-transformation to the equation which has $\zeta_1, \zeta_2, \ldots, \zeta_n$ for its integrals, we obtain a new equation and are led to a relation, of the same type as before, existing among $n-1$ integrals of the new equation; and applying it to successive equations $n-1$ times in all, we are led at the end to a homogeneous linear relation between two integrals of the last equation.

From what has already been proved, we know that the h-invariant of this last equation is zero: that is,

$$h_n = 0.$$

A similar argument holds when the coefficients in the homogeneous linear relation between $n+1$ integrals are functions of y alone: the successive application of the Laplace Σ-transformation, $n-1$ times in all, leads to a vanishing invariant for the last equation: that is,

$$K_n = 0,$$

in this case.

Thus Goursat's theorem is established.

Ex. Shew that three integrals of the equation

$$s+\frac{2z}{(x-y)^2}=0$$

are connected by an equation

$$\begin{vmatrix} z_1, & a_1x^2+2b_1x+c_1, & a_1x+b_1 \\ z_2, & a_2x^2+2b_2x+c_2, & a_2x+b_2 \\ z_3, & a_3x^2+2b_3x+c_3, & a_3x+b_3 \end{vmatrix}=0;$$

and obtain them. Shew also that three integrals ζ_1, ζ_2, ζ_3 are connected by a relation

$$\begin{vmatrix} \zeta_1, & a_1y^2+2b_1y+c_1, & a_1y+b_1 \\ \zeta_2, & a_2y^2+2b_2y+c_2, & a_2y+b_2 \\ \zeta_3, & a_3y^2+2b_3y+c_3, & a_3y+b_3 \end{vmatrix}=0;$$

and obtain them also. (Goursat.)

Lévy's Transformation.

205. In the two Laplace transformations

$$z_1=\frac{\partial z}{\partial y}+az, \quad Z_1=\frac{\partial z}{\partial x}+bz,$$

the only quantities that occur are those which appear in the differential equation: and it might be deemed possible to secure a more general transformation by taking new variables of the form

$$z_1'=\frac{\partial z}{\partial y}+uz, \quad Z_1'=\frac{\partial z}{\partial x}+vz,$$

where u and v are quantities initially at our disposal. As a matter of fact, such a transformation is not so effective as the Laplace transformation: its main importance lies, partly in the analytical forms obtained, partly in some geometrical applications. As it does not lead to any new process for the integration of the equation, only a brief outline will be given.

The suggested transformation is adopted by Lévy* in the form

$$z'=\frac{\partial z}{\partial y}+(a+\alpha)z,$$

* *Journ. de l'Éc. Polytechnique*, t. XXXVII, Cah. LVI (1886), p. 67.

where α is the disposable quantity, the original equation still being

$$s + ap + bq + cz = 0.$$

Proceeding as before, we easily find that the equation for z' is

$$s' + a'p' + b'q' + c'z' + Az = 0,$$

where

$$a' = a - \frac{1}{\alpha}\frac{\partial \alpha}{\partial y},$$

$$b' = b,$$

$$c' = c + k - h - \frac{\partial \alpha}{\partial x} - \frac{b}{\alpha}\frac{\partial \alpha}{\partial y},$$

$$\frac{A}{\alpha} = \frac{\partial^2 \log \alpha}{\partial x \partial y} - \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right) + k - h.$$

In order that the transformed equation may have the same character as the original equation, we take

$$A = 0,$$

and then we have

$$s' + a'p' + b'q' + c'z' = 0.$$

Denoting the invariants of this equation by h' and k', we have

$$h' = \frac{\partial a'}{\partial x} + a'b' - c'$$

$$= 2h - k + \frac{\partial \alpha}{\partial x} - \frac{\partial^2 \log \alpha}{\partial x \partial y}$$

$$= h + \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right),$$

on account of $A = 0$: and

$$k' = \frac{\partial b'}{\partial y} + a'b' - c'$$

$$= h + \frac{\partial \alpha}{\partial x}.$$

Also

$$\frac{\partial}{\partial y}\left(\frac{k'}{\alpha}\right) = \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right) + \frac{\partial^2 \log \alpha}{\partial x \partial y}$$

$$= \frac{\partial \alpha}{\partial x} + h - k$$

$$= k' - k,$$

so that

$$k = k' - \frac{\partial}{\partial y}\left(\frac{k'}{\alpha}\right).$$

It is clear that k' cannot vanish unless $k = 0$; hence, in so far as Lévy's transformation aims at securing an integrable form of equation, it is unnecessary to take further account of the invariant k'.

Let h_1 be the h-invariant belonging to the equation obtained by applying the Laplace σ-transformation to the original equation, so that

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y}:$$

then

$$\begin{aligned} h' - h_1 &= \frac{\partial \alpha}{\partial x} - \frac{\partial^2 \log \alpha}{\partial x \partial y} + \frac{\partial^2 \log h}{\partial x \partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\alpha}{h}h\right) + \frac{\partial}{\partial x}\left\{\frac{\alpha}{h}\frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right)\right\} \\ &= \frac{\partial}{\partial x}\left(\frac{\alpha h'}{h}\right). \end{aligned}$$

It is clear that h' can vanish only if $h_1 = 0$: and therefore the Lévy transformation does not provide a vanishing invariant unless a vanishing invariant is provided by the Laplace σ-transformation. Hence from the point of view of integrating the equation, no particular advantage accrues from the Lévy transformation.

Ex. 1. Shew that, if the Laplace σ-transformation be applied any number of times in succession to the equation

$$s' + a'p' + b'q' + c'z' = 0,$$

as transformed by Lévy, and if h_1', h_2', h_3', ... are the h-invariants of the successive equations, then

$$h_n = h'_{n-1} - \frac{\partial}{\partial x}\left(\frac{\alpha h' h_1' \dots h'_{n-1}}{h h_1 \dots h_{n-1}}\right),$$

$$h_n' = h_n \quad + \frac{\partial}{\partial y}\left(\frac{h h_1 \dots h_n}{\alpha h' h_1' \dots h'_{n-1}}\right),$$

where $h_1, \dots, h_n$ denote, as usual, the h-invariants of the equations obtained by the successive application of the σ-transformation to the original equation. (Lévy.)

Ex. 2. Shew that the Laplace σ-transformation and a Lévy transformation are permutable with each other in the sense that the invariants of the doubly transformed equation are independent of the order of application of the two transformations. (Lévy.)

Ex. 3. Shew that, if ζ is any particular solution of the original equation, then the Lévy transformation can be expressed in the form

$$\frac{z'}{\zeta} = \frac{\partial}{\partial y}\left(\frac{z}{\zeta}\right). \qquad \text{(Lévy.)}$$

The Equation $r + 2\alpha p + 2\beta q + \gamma z = 0$.

206. We now come to consider, more briefly, the equation

$$r + a'p + b'q + c'z = 0;$$

this was found (§ 189) to be the alternative of

$$s + ap + bq + cz = 0$$

as a form to which every linear equation can be changed. It will be convenient to take it in a form

$$r + 2\alpha p + 2\beta q + \gamma z = 0,$$

where α, β, γ are functions of the independent variables x and y.

When we take

$$z = \lambda\zeta,$$

the new equation having ζ for its dependent variable is

$$\frac{\partial^2\zeta}{\partial x^2} + 2\alpha'\frac{\partial\zeta}{\partial x} + 2\beta'\frac{\partial\zeta}{\partial y} + \gamma'\zeta = 0,$$

where

$$\beta' = \beta,$$

$$\alpha' = \alpha + \frac{1}{\lambda}\frac{\partial\lambda}{\partial x},$$

$$\gamma' = \gamma + \frac{2\alpha}{\lambda}\frac{\partial\lambda}{\partial x} + \frac{2\beta}{\lambda}\frac{\partial\lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2\lambda}{\partial x^2}.$$

Obviously β is an invariant for the transformation in question. Again,

$$\gamma' - \frac{\partial\alpha'}{\partial x} - \alpha'^2 = \gamma - \frac{\partial\alpha}{\partial x} - \alpha^2 + \frac{2\beta}{\lambda}\frac{\partial\lambda}{\partial y},$$

and therefore

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta'}\left(\gamma' - \frac{\partial\alpha'}{\partial x} - \alpha'^2\right)\right\} - 2\frac{\partial\alpha'}{\partial y}$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial\alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial\alpha}{\partial y},$$

on the assumption that β does not vanish. Hence, if

$$I = \beta,$$

$$J = \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial \alpha}{\partial y},$$

then I *and* J *are invariants for transformations of the type*

$$z' = \lambda z,$$

which leave the form of the equation unaltered.

Further, if β should vanish or if λ should involve x only, then

$$\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2$$

is an invariant for all transformations of the type $z' = \lambda z$.

Again, the equation conserves its form when the independent variables are changed according to a law

$$x' = \text{a function of } x, \quad y' = \text{a function of } y,$$

and the form is conserved only for such a law. If we take

$$x' = \xi, \quad y' = \eta,$$

ξ being a function of x only, and η a function of y only, and if the new equation be

$$r' + 2\alpha' p' + 2\beta' q' + \gamma' z = 0,$$

we have

$$\beta' = \frac{\beta \eta'}{\xi'^2},$$

$$\alpha' = \frac{\alpha}{\xi'} + \tfrac{1}{2}\frac{\xi''}{\xi'^2},$$

$$\gamma' = \frac{\gamma}{\xi'};$$

and therefore

$$I' = \beta' = \frac{\eta'}{\xi'^2} I,$$

$$J' = \frac{\partial}{\partial \xi}\left\{\frac{1}{\beta'}\left(\gamma' - \frac{\partial \alpha'}{\partial \xi} - \alpha'^2\right)\right\} - 2\frac{\partial \alpha'}{\partial \eta}$$

$$= \frac{1}{\xi' \eta'} J - \frac{1}{2\xi' \eta'}\frac{\partial}{\partial x}\left[\frac{\{\xi, x\}}{\beta}\right],$$

where $\{\xi, x\}$ is the Schwarzian derivative of ξ.

Thus I is invariantive for all changes of the independent variables that conserve the form of the equation. In order that J may be invariantive, we must have

$$\frac{\partial}{\partial x}\left[\frac{\{\xi, x\}}{\beta}\right]=0,$$

of which obviously there are four cases, viz.

(i) if β is a function of x alone, the transformations of x are limited by the relation

$$\{\xi, x\}=k\beta,$$

where k is any constant:

(ii) if β is a function of y alone, the transformations of x are limited by the relation

$$\{\xi, x\}=-\tfrac{1}{2}k^2,$$

where k is an arbitrary constant, that is,

$$\xi=\frac{ae^{kx}+b}{ce^{kx}+d},$$

where a, b, c, d, k are arbitrary constants:

(iii) if β involves x and y and is expressible in a form

$$\beta=XY,$$

where X is a function of x alone and Y is a function of y alone, then the transformations of x are limited by the relation

$$\{\xi, x\}=kX,$$

where k is any constant:

(iv) if β involves x and y and is not expressible in a form

$$\beta=XY,$$

or if β is zero, the transformations of x are limited by the relation

$$\{\xi, x\}=0,$$

that is,

$$\xi=\frac{ax+b}{cx+d},$$

where a, b, c, d are arbitrary constants.

Thus the quantity J, which is invariantive for all transformations of the form

$$z' = \lambda z,$$

where λ is any function of x and y, is invariantive for only certain transformations of the independent variables. The more important transformations, however, are those which change the dependent variable only.

207. If the invariant I vanishes, the equation is of the form

$$\frac{\partial^2 z}{\partial x^2} + 2\alpha \frac{\partial z}{\partial x} + \gamma z = 0,$$

effectively an ordinary linear equation for which y is parametric. The one invariant is

$$K = \gamma - \frac{\partial \alpha}{\partial x} - \alpha^2,$$

being the usual invariant of the ordinary equation of the second order for transformations of the kind considered: writing

$$z' = z e^{\int \alpha \, dx},$$

we have the equation in the form

$$\frac{\partial^2 z'}{\partial x^2} + K z' = 0.$$

The case needs no special consideration in the present connection.

If the invariant J vanishes, that is, if

$$\frac{\partial}{\partial x} \left\{ \frac{1}{\beta} \left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2 \right) \right\} - 2 \frac{\partial \alpha}{\partial y} = 0,$$

then a function θ of x and y exists such that

$$\alpha = \frac{\partial \theta}{\partial x};$$

$$\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2 = 2\beta \frac{\partial \theta}{\partial y}.$$

When these values of α and γ are introduced into the equation, it is easily transformed to

$$\frac{\partial^2}{\partial x^2} (z e^{\theta}) + 2\beta \frac{\partial}{\partial y} (z e^{\theta}) = 0.$$

Various cases arise according to the form of β.

If neither I nor J should vanish, it is always possible to transform the equation so that it shall not contain the first derivative of z with regard to x. To make this transformation, take two quantities θ and ϕ such that

$$\theta = \int_{x_0}^{x} \alpha\, dx,$$

$$\phi = \int_{x_0}^{x} J\, dx;$$

then

$$\alpha = \frac{\partial \theta}{\partial x},$$

$$J = \frac{\partial \phi}{\partial x},$$

and consequently

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial^2 \theta}{\partial x \partial y} = J = \frac{\partial \phi}{\partial x},$$

so that we may take

$$\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right) - 2\frac{\partial \theta}{\partial y} = \phi,$$

the arbitrary additive function of y being deemed included in ϕ. Then, writing

$$z' = ze^{\theta},$$

we easily find

$$\frac{\partial^2 z'}{\partial x^2} + 2\beta\frac{\partial z'}{\partial y} + 2\beta\phi z' = 0:$$

the invariant J' of this equation is

$$J' = \frac{\partial \phi}{\partial x},$$

which, in fact, is J.

Ex. 1. Suppose that the invariant J of the equation vanishes and that β does not involve x: then, if

$$\zeta = ze^{\theta}, \quad dt = -\frac{dy}{2\beta},$$

the equation takes the well-known form

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial \zeta}{\partial t}.$$

If an integral of this equation is required which is such as to give

$$\zeta = f(t), \quad \frac{\partial \zeta}{\partial x} = g(t),$$

when $x=a$, it is easily obtainable in the form

$$\zeta=f(t)+\frac{(x-a)^2}{2\,!}f'(t)+\frac{(x-a)^4}{4\,!}f''(t)+\dots$$

$$+(x-a)g(t)+\frac{(x-a)^3}{3\,!}g'(t)+\frac{(x-a)^5}{5\,!}g''(t)+\dots.$$

The construction of an integral of the original equation, which is such as to give

$$z=F(y),\quad \frac{\partial z}{\partial x}=G(y),$$

when $x=a$, is now only a matter of transformation of the variables.

Ex. 2. Obtain the integral of the equation

$$\frac{\partial^2 z}{\partial x^2}=x\frac{\partial z}{\partial y},$$

such that

$$z=f(y),\quad \frac{\partial z}{\partial x}=g(y),$$

when $x=0$.

Ex. 3. The equation

$$\frac{\partial^2 z}{\partial x^2}+2a\frac{\partial z}{\partial x}+2\beta\frac{\partial z}{\partial y}+\gamma z=0$$

is transformed by a substitution

$$z_1=\frac{\partial z}{\partial x}+(a+u)z$$

into the equation

$$\frac{\partial^2 z_1}{\partial x^2}+2a_1\frac{\partial z_1}{\partial x}+2\beta_1\frac{\partial z_1}{\partial y}+\gamma_1 z_1=0:$$

prove that u satisfies the equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\frac{\partial u}{\partial x}-u^2\right)\right\}+2\frac{\partial u}{\partial y}=J,$$

and that a_1, β_1, γ_1 are given by

$$a_1=a-\frac{1}{2\beta}\frac{\partial \beta}{\partial x},$$

$$\beta_1=\beta,$$

$$\gamma_1=\gamma-\frac{a}{\beta}\frac{\partial \beta}{\partial x}+\frac{u}{\beta}\frac{\partial \beta}{\partial x}-2\frac{\partial u}{\partial x}.$$

Prove also that J_1, the J-invariant of the new equation, is given by

$$J_1-J=\frac{\partial}{\partial y}\left(\frac{1}{\beta}\frac{\partial \beta}{\partial x}\right)+\tfrac{1}{2}\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\frac{\partial}{\partial x}\left(\frac{1}{\beta}\frac{\partial \beta}{\partial x}\right)\right\}-\frac{\partial}{\partial x}\left\{\frac{1}{u}\frac{\partial}{\partial x}\left(\frac{u^2}{\beta}\right)\right\}.$$

Verify independently this last result for the special case

$$u=0,\quad J=0.$$

208. The somewhat cumbrous forms, occurring in the last example given, are the forms which are necessary for the maintenance of the type of the equation: and they suggest that no series of successive transformations, similar to the Laplace σ- and Σ-transformations, can usefully be constructed for the equation or can lead to types of equations the integrals of which are expressible in finite form.

Moreover, taking an equation for which neither of the invariants vanishes, we have seen that it can be transformed so as to become

$$\frac{\partial^2 z'}{\partial x^2} + 2\beta\frac{\partial z'}{\partial y} + 2\beta\phi z' = 0,$$

where neither β nor ϕ vanishes. The conclusions of § 186 and the application of Cauchy's theorem alike shew that there is a general integral involving two arbitrary functions in its expression, both of them having y for their argument: moreover, owing to the particular linear form of the equation, each of the functions and its derivatives enters linearly into the expression of the integral, when the integral is given explicitly. It is easy to see that, *when these functions are quite arbitrary, the integral cannot be expressed in finite form which is completely explicit and free from partial quadratures.* If possible, let such an integral be

$$z' = \alpha Y + \alpha_1 Y' + \ldots + \alpha_n Y^{(n)},$$

where Y is an arbitrary function of y: the substitution of this value of z' introduces, through $2\beta\dfrac{\partial z'}{\partial y}$, a term

$$2\beta\alpha_n Y^{(n+1)}$$

which cannot be cancelled, for $Y^{(n+1)}$ is not introduced elsewhere in the substitution.

It is thus useless to seek for finite explicit forms for the most general integral provided by Cauchy's theorem for an unconditioned equation of the present type; but for particular equations finite forms may be obtainable when the arbitrary functions are specialised. In the latter instances, the earlier argument does not hold: for in the case of a specialised function Y, the term

$$2\beta\alpha_n Y^{(n+1)}$$

may be balanced by other terms from

$$\frac{\partial^2 \alpha_n}{\partial x^2} Y^{(n)} + 2\beta \frac{\partial \alpha_n}{\partial y} Y^{(n)} + 2\beta\phi\alpha_n Y^{(n)}.$$

The result is limited to integrals that occur in explicit form: it does not necessarily hold for integrals the expressions of which involve partial quadratures.

It therefore appears that, for the unconditioned equation

$$\frac{\partial^2 z}{\partial x^2} + 2\alpha \frac{\partial z}{\partial x} + 2\beta \frac{\partial z}{\partial y} + \gamma z = 0,$$

the only integrals, which are of a general type and which are expressed explicitly, are not finite in form. Such integrals are given by Cauchy's theorem; and they are such that, when $x = a$, the quantity z assumes an assigned value $f(y)$ and the quantity $\frac{\partial z}{\partial x}$ assumes another assigned value $g(y)$, subject to conditions as to regularity—connected with the forms of α, β, γ, $f(y)$, $g(y)$. By taking all the functions f and g that are regular within a selected domain, and by taking all the domains within which the coefficients α, β, γ are regular, we obtain all the regular integrals of the equation. But the aggregate of integrals thus obtained is restricted to those which are regular; and if the integrals be constructed by the analysis used to obtain Cauchy's theorem, no one of them is in finite form.

Borel's Expression for Regular Integrals not finite in form.

209. In connection with these integrals, and as illustrating the generality of some integrals with partial quadratures, it is worthy of notice that Borel has shewn* that the aggregate of these regular integrals can be represented by one form which requires partial quadratures. His investigation deals with equations in n variables and of order p, which are algebraically resoluble with regard to $\frac{\partial^p z}{\partial x_1^p}$: it will be sufficiently explained for the present purpose by taking $n = 2$, $p = 2$.

* *Bull. des sciences math.*, Sér. II, t. XIX (1895), p. 122: the idea of this mode of representation of an aggregate of integrals was suggested to him by some remarks of Delassus, *ib.*, pp. 51 et seq.

Take any positions which are ordinary positions for the equation in the planes of x and of y respectively, and make these the origins. The integral given by Cauchy's theorem is such that $z = f(y)$ and $\frac{\partial z}{\partial x} = g(y)$, when $x = 0$, the functions f and g being regular functions in the domain of $y = 0$ and otherwise arbitrary. The initial conditions therefore give the values of

$$\frac{\partial^n z}{\partial y^n}, \quad \frac{\partial^{n+1} z}{\partial x \partial y^n},$$

when $x = 0$ and $y = 0$, for all values of n: let these quantities be arranged in any sequence such that the total order of derivation does not decrease in the sequence and denote them, so arranged, by $\mu_1, \mu_2, \mu_3, \ldots$.

Now take a function u such that

$$u = \frac{1}{\left(1 - \frac{x}{r}\right)\left(1 - \frac{y}{r}\right)}:$$

and, forming the values of

$$\frac{\partial^n u}{\partial y^n}, \quad \frac{\partial^{n+1} u}{\partial x \partial y^n},$$

when $x = 0$ and $y = 0$, for all values of n, arrange them in the same sequence as the derivatives of z above; denote them, so arranged, by $\sigma_1, \sigma_2, \sigma_3, \ldots$. Owing to the limitations of regularity imposed on $f(y)$ and $g(y)$, it is always possible to choose a value r' of r such that, for values of m equal to or greater than some selected integer k,

$$\sigma_m > |\mu_m|:$$

we shall assume that r is greater than r'.

Construct the Cauchy integral of the given linear differential equation of the second order resoluble with regard to $\frac{\partial^2 z}{\partial x^2}$, assigning as the initial conditions that

$$z = u, \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x},$$

when $x = 0$. On account of the linear form, this integral is of the form

$$z = \sigma_1 \psi_1 + \sigma_2 \psi_2 + \sigma_3 \psi_3 + \ldots,$$

where the quantities $\psi_1, \psi_2, \psi_3, \ldots$ are determinate regular functions of x and y in the domains considered. This integral is absolutely and uniformly convergent.

By means of the quantity z, so obtained, construct a quantity θ defined by the equation

$$\theta(x, y, \alpha) = \sigma_1\psi_1 \cos\alpha + \sigma_2\psi_2 \cos 2\alpha + \sigma_3\psi_3 \cos 3\alpha + \ldots,$$

where α is a real parameter: thus θ is convergent, and it is determinate, containing no arbitrary function. Also take

$$F(\alpha) = \frac{\mu_1}{\sigma_1}\cos\alpha + \frac{\mu_2}{\sigma_2}\cos 2\alpha + \frac{\mu_3}{\sigma_3}\cos 3\alpha + \ldots,$$

which is absolutely and uniformly convergent: as $F(\alpha)$ contains the quantities $\mu_1, \mu_2, \mu_3, \ldots$ derived through the arbitrary functions $f(y)$ and $g(y)$ initially given, it clearly is an arbitrary function. It is possible to integrate the product $\theta F(\alpha)$ with regard to α, on account of the character of the convergence of θ and of $F(\alpha)$. Obviously

$$\frac{1}{\pi}\int_0^{2\pi} \theta(x, y, \alpha) F(\alpha)\, d\alpha = \mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots,$$

and, in the integral with partial quadratures, $\theta(x, y, \alpha)$ is determinate while $F(\alpha)$ is arbitrary. Now just as the integral, determined by the assignment of initial values $\sigma_1, \sigma_2, \sigma_3, \ldots$ to certain derivatives of the dependent variable, is given by

$$\sigma_1\psi_1 + \sigma_2\psi_2 + \sigma_3\psi_3 + \ldots,$$

so the integral, determined by the assignment of $\mu_1, \mu_2, \mu_3, \ldots$ to the same derivatives of the dependent variable, is given by

$$\mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots.$$

But the latter aggregate of assignments is the equivalent of the initial conditions in Cauchy's theorem which require that z and $\frac{\partial z}{\partial x}$ shall acquire the values $f(y)$ and $g(y)$ respectively, when $x = 0$, the functions f and g being regular within the domain of $y = 0$ and being otherwise arbitrary. Denoting this integral by Z, we have

$$\begin{aligned} Z &= \mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots \\ &= \frac{1}{\pi}\int_0^{2\pi} \theta(x, y, \alpha)\, F(\alpha)\, d\alpha, \end{aligned}$$

where $\theta(x, y, \alpha)$ is a determinate function, and $F(\alpha)$ is an arbitrary function because of the arbitrary quality in its coefficients.

Thus the Cauchy integral, associated with the two arbitrary functions in the initial conditions, can be represented by an expression, requiring partial quadratures and involving only a single arbitrary function*.

Ex. 1. A simple example will shew how Borel's construction of the integral with partial quadratures works out in practice. Consider the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}:$$

when the initial conditions are that $z = f(y)$ and $\frac{\partial z}{\partial x} = g(y)$ when $x = 0$, on the assumption $f(y)$ and $g(y)$ are regular functions in a domain (say) of $y = 0$: the explicit integral (Ex. 1, § 207) is

$$Z = f(y) + \frac{x^2}{2!} f'(y) + \frac{x^4}{4!} f''(y) + \dots$$

$$+ xg(y) + \frac{x^3}{3!} g'(y) + \frac{x^5}{5!} g''(y) + \dots$$

For the actual expressions of $f(y)$ and $g(y)$, let

$$f(y) = \sum_0 \frac{\nu_m}{m!} y^m,$$

$$g(y) = \sum_0 \frac{\lambda_m}{m!} y^m;$$

and in order to secure uniform convergence, let the domain of y for each of these functions be a circle of radius r, where $r > r' > 1$, r' itself being a quantity that is greater than unity. Then if M be the greatest value of $|f(y)|$ within the domain, and if M' be the greatest value of $|g(y)|$ within the domain, it is known† that

$$|\nu_m| \leqslant Mr^{-m}, \quad |\lambda_m| \leqslant M'r^{-m},$$

or if N denote a quantity greater than M and M', then

$$|\nu_m| < Nr^{-m}, \quad |\lambda_m| < Nr^{-m}.$$

When values of $f(y)$ and $g(y)$ are substituted so as to give a doubly-infinite series which obviously converges, we have

$$Z = \sum_{n=0} \sum_{m=0} \frac{y^n}{n!} \left\{ \frac{x^{2m}}{2m!} \nu_{n+m} + \frac{x^{2m+1}}{(2m+1)!} \lambda_{n+m} \right\},$$

which is merely another form of the integral satisfying the initial conditions.

* The explanation of the paradox in the present case, if it be regarded as a paradox, lies in the fact that the two sets of arbitrary coefficients arising from the two functions $f(y)$ and $g(y)$ respectively are included in the single function $F(\alpha)$.

† *T. F.*, § 22.

Now construct the integral of the equation which is such that

$$z = Ne^y, \quad \frac{\partial z}{\partial x} = Ne^y,$$

when $x=0$. Keeping for the moment coefficients ν_n', λ_n' so as to correspond with ν_n, λ_n respectively, where, in fact,

$$\nu_n' = N, \quad \lambda_n' = N,$$

we have the integral (say ζ) in the form

$$\zeta = \sum_{n=0}\sum_{m=0} \frac{y^n}{n\,!}\left\{\frac{x^{2m}}{2m\,!}\nu'_{n+m} + \frac{x^{2m+1}}{(2m+1)\,!}\lambda'_{n+m}\right\}.$$

In accordance with the notation of the text, we take

$$\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6, \ldots = \nu_0',\ \lambda_0',\ \nu_1',\ \lambda_1',\ \nu_2',\ \lambda_2', \ldots,$$

and

$$\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \mu_5,\ \mu_6, \ldots = \nu_0,\ \lambda_0,\ \nu_1,\ \lambda_1,\ \nu_2,\ \lambda_2, \ldots;$$

and we construct a function $\theta(x, y, a)$ from ζ by changing σ_μ in ζ into $\sigma_\mu \cos \mu a$, so that

$$\theta(x, y, a) = \sum_{n=0}\sum_{m=0} \frac{y^n}{n\,!}\left[\frac{x^{2m}}{2m\,!}\nu'_{n+m}\cos\{(2n+2m+1)a\} + \frac{x^{2m+1}}{(2m+1)\,!}\lambda'_{n+m}\cos\{(2n+2m+2)a\}\right].$$

Inserting the values of ν'_{n+m} and λ'_{n+m}, we have

$$\begin{aligned}\theta(x, y, a) &= N\sum_{n=0}\sum_{m=0} \frac{y^n}{n\,!}\left[\frac{x^{2m}}{2m\,!}\cos\{(2n+2m+1)a\} + \frac{x^{2m+1}}{(2m+1)\,!}\cos\{(2n+2m+2)a\}\right]\\ &= N\sum_{n=0}\sum_{p=0} \frac{y^n}{n\,!}\frac{x^p}{p\,!}\cos\{(2n+p+1)a\}\\ &= Ne^{x\cos a + y\cos 2a}\cos(a + x\sin a + y\sin 2a).\end{aligned}$$

It is evident, on simple substitution, that the relation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial y}$$

is satisfied.

The function $F(a)$ is now required: in accordance with the construction in the text, we take

$$\begin{aligned}F(a) &= \frac{\mu_1}{\sigma_1}\cos a + \frac{\mu_2}{\sigma_2}\cos 2a + \ldots\\ &= \frac{\nu_0}{\nu_0'}\cos a + \frac{\lambda_0}{\lambda_0'}\cos 2a + \frac{\nu_1}{\nu_1'}\cos 3a + \frac{\lambda_1}{\lambda_1'}\cos 4a + \ldots\\ &= \frac{1}{N}\sum_{m=0}\left[\nu_m\cos\{(2m+1)a\} + \lambda_m\cos\{(2m+2)a\}\right].\end{aligned}$$

Because

$$\frac{|\nu_m|}{N}<\frac{1}{r^m}, \quad \frac{|\lambda_m|}{N}<\frac{1}{r^m},$$

and r is greater than a quantity which itself is greater than unity, this series for $F(a)$ converges absolutely and uniformly. Hence, writing

$$G(a)=\sum_{m=0}[\nu_m \cos\{(2m+1)a\}+\lambda_m \cos\{(2m+2)a\}],$$

we have

$$Z=\frac{1}{\pi}\int_0^{2\pi} e^{x\cos a+y\cos 2a}\cos(a+x\sin a+y\sin 2a)\,G(a)\,da,$$

which is an expression, requiring partial quadratures and involving the one arbitrary function $G(a)$, for the integral of the equation determined by the initial conditions that

$$\left.\begin{aligned} Z=f(y)=\Sigma\frac{\nu_m}{m!}y^m \\ \frac{\partial Z}{\partial x}=g(y)=\Sigma\frac{\lambda_m}{m!}y^m \end{aligned}\right\},$$

when $x=0$, and when the suppositions as to the domains of $f(y)$ and $g(y)$ are satisfied.

If the domains, within which assigned functions $f(y)$ and $g(y)$ are regular, be circles of radius r, where r is less than unity, the case can be changed to the case already discussed by taking new variables such that

$$\frac{x''}{r''}=\frac{x}{r}, \quad \frac{y''}{r''}=\frac{y}{r},$$

where r'' is greater than a quantity which itself is greater than unity.

Evidently

$$\nu_m=\frac{1}{\pi}\int_0^{2\pi}\cos\{(2m+1)a\}\,G(a)\,da,$$

$$\lambda_m=\frac{1}{\pi}\int_0^{2\pi}\cos\{(2m+2)a\}\,G(a)\,da.$$

Ex. 2. Prove that an integral of the equation in the preceding example, viz. of

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial y},$$

is given by

$$z=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\phi(x+2ay^{\frac{1}{2}})\,e^{-a^2}\,da,$$

where ϕ denotes any function of its argument. (Fourier.)

Ex. 3. The functions

$$f(y)=\sum_{m=0}\frac{\nu_m}{m!}y^m, \quad g(y)=\sum_{m=0}\frac{\lambda_m}{m!}y^m,$$

are regular functions of y in a domain, round the origin, of radius greater than a quantity which itself is greater than unity ; and $G(a)$ is given by

$$G(a)=\sum_{m=0}\left[\nu_m \cos\{(2m+1)\,a\}+\lambda_m \cos\{(2m+2)\,a\}\right],$$

which is a uniformly converging series. Prove that an integral of the equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0,$$

such that z and $\dfrac{\partial z}{\partial x}$ are equal to $f(y)$ and $g(y)$ respectively, when $x=0$, is given by

$$z=\frac{1}{2\pi}\int_0^{2\pi}\theta(x, y, a)\,G(a)\,da,$$

where

$$\theta(x, y, a)=e^{y\cos 2a - x\sin 2a}\{\cos(a+y\sin 2a+x\cos 2a)+\sin(y\sin 2a+x\cos 2a)\}$$
$$+e^{y\cos 2a + x\cos 2a}\{\cos(a+y\sin 2a-x\cos 2a)-\sin(y\sin 2a-x\cos 2a)\}.$$

CHAPTER XIV.

Adjoint Equations: Linear Equations having Equal Invariants.

The present chapter is devoted, partly to the consideration of the adjoint equation and of Riemann's investigation whereby the adjoint equation is employed to further the integration of the original equation, partly to the consideration of equations which are self-adjoint and are of finite rank in both of the variables. In the account here given, much use has been made of the exposition given by Darboux*: and the method devised by Moutard, in the fragment of the memoir quoted in § 218, is explained and some illustrations are given. Reference may also be made to a memoir† by R. Liouville.

210. Not a few of the characteristic properties of a linear ordinary equation are expressible by means of the properties of the associated equation which is usually called Lagrange's adjoint equation. It proves similarly convenient to associate an adjoint equation with a linear partial equation which, for the present purpose, will be limited to the form

$$s + ap + bq + cz = 0.$$

Generalising the usual definition‡ of the expression adjoint to an ordinary linear expression, which is the linear condition to be satisfied by v in order that

$$vP(w), \quad = v\left(P_0 \frac{d^n w}{dz^n} + \ldots + P_n w\right),$$

may be an exact differential, we say that, if $F(z)$ is a quantity which is linear in z and its derivatives with regard to x and y, then

* *Théorie générale des surfaces*, t. II, pp. 71—163.

† *Journ. de l'Éc. Polytechnique*, t. XXXVII (1886), pp. 7—62.

‡ See vol. IV of this work, § 82.

the linear expression adjoint to $F(z)$ is the quantity which must vanish in order that the double integral

$$\iint uF(z)\,dx\,dy$$

can be expressed as a simple integral. The condition therefore is that u should be such as to secure that a relation

$$uF(z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

should hold, where M and N are free from all quadrature operations. For our purpose, we take

$$F(z) = s + ap + bq + cz;$$

and then, if

$$uF(z) = u(s + ap + bq + cz) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y},$$

M and N must, quà functions of z, be of the forms

$$M = A\frac{\partial z}{\partial y} + Bz, \quad N = C\frac{\partial z}{\partial x} + Dz,$$

respectively, where A, B, C, D are functions of x, y, u, and do not involve z. Substituting these values of M and N, and equating coefficients of the derivatives of z, we have

$$u = A + C,$$

$$au = B + \frac{\partial C}{\partial y},$$

$$bu = \frac{\partial A}{\partial x} + D,$$

$$cu = \frac{\partial B}{\partial x} + \frac{\partial D}{\partial y}.$$

Evidently,

$$\frac{\partial}{\partial x}(au) + \frac{\partial}{\partial y}(bu) - cu = \frac{\partial^2 C}{\partial x \partial y} + \frac{\partial^2 A}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial x \partial y},$$

so that, writing

$$G(u) = \frac{\partial^2 u}{\partial x \partial y} - a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)u,$$

the necessary condition is

$$G(u) = 0.$$

When this condition is satisfied, the above four equations become equivalent to three only that are independent of one another: consequently, they are inadequate for the precise determination of A, B, C, D, and therefore M and N cannot be precisely determined. The latter result is, however, only a fair expectation from the form adopted; for $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ will remain unaltered in value if M and N be increased by $\frac{\partial \phi}{\partial y}$ and $-\frac{\partial \phi}{\partial x}$ respectively, where ϕ is any function of x, y, z, u, which is linear and homogeneous in z and u.

We therefore may satisfy all the equations and still may make one assumption that is not inconsistent with them: the simplest appears to be the assumption that $A = C$. Then

$$A = C = \tfrac{1}{2}u;$$

and so

$$B = au - \tfrac{1}{2}\frac{\partial u}{\partial y},$$

$$D = bu - \tfrac{1}{2}\frac{\partial u}{\partial x},$$

so that

$$M = \tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right) + auz,$$

$$N = \tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right) + buz.$$

Then, with these values of M and N, we have

$$uF(z) - \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = zG(u),$$

that is,

$$\left.\begin{aligned} uF(z) - zG(u) &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \\ zG(u) - uF(z) &= -\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \end{aligned}\right\}.$$

The former shews that $G(u) = 0$ is the condition that the double integral $\iint uF(z)\,dxdy$ should be expressible as a simple integral: the latter shews that $F(z) = 0$ is the condition that the double integral $\iint zG(u)\,dxdy$ should be expressible as a simple integral. Hence *the two equations*

$$F(z)=0, \quad G(u)=0,$$

are adjoint to each other, a property which is the same as for ordinary linear equations.

Also, $G(u)=0$ is a linear equation, similar in type to $F(z)=0$ and of the same order. Denoting the invariants of $F(z)=0$ by h_f and k_f, and the invariants of $G(u)=0$ by h_g and k_g, we have

$$h_g=-\frac{\partial a}{\partial x}+(-a)(-b)-\left(c-\frac{\partial a}{\partial x}-\frac{\partial b}{\partial y}\right)=k_f,$$

$$k_g=-\frac{\partial b}{\partial y}+(-a)(-b)-\left(c-\frac{\partial a}{\partial x}-\frac{\partial b}{\partial y}\right)=h_f.$$

Thus the invariants of either of two equations, which are reciprocally adjoint, are the invariants of the other merely interchanged: and it will be noticed that the coefficients a and b in the two equations differ only in sign.

Further, if $F(z)=0$ and $G(u)=0$ are effectively the same, so that each is self-adjoint, then $a=0$, $b=0$: the equation is

$$s+cz=0,$$

that is, the equation has equal invariants. The converse of this property is obviously true.

Relations between Adjoint Equations under Laplace Transformations.

211. Consider now the two series of equations derived from $F=0$ by the successive applications of the two Laplace transformations. The two sets of quantities, which (by vanishing) give the equations of the series, are expressible in the form

$$\ldots,\ \sigma^{-3}F,\ \ \sigma^{-2}F,\ \ \sigma^{-1}F,\ \ F,\ \ \sigma F,\ \ \sigma^2 F,\ \ \sigma^3 F,\ \ldots.$$

The invariants h_i and k_i of $\sigma^i F$, where i is positive, are given according to the law

$$h_{\mu+1}=2h_\mu-h_{\mu-1}-\frac{\partial^2\log h_\mu}{\partial x\partial y},$$

$$k_{\mu+1}=h_\mu;$$

and the invariants H_i and K_i of $\sigma^{-i}F$, where i is positive, are given according to the law

$$H_{\mu+1} = K_\mu,$$
$$K_{\mu+1} = 2K_\mu - K_{\mu-1} - \frac{\partial^2 \log K_\mu}{\partial x \partial y}.$$

Let the h-invariant of F be denoted by h_0 and the k-invariant be denoted by h_{-1}. Then

$$H_1 = k = h_{-1},$$
$$K_1 = 2k - h - \frac{\partial^2 \log k}{\partial x \partial y}$$
$$= 2h_{-1} - h_0 - \frac{\partial^2 \log h_0}{\partial x \partial y},$$

so that we may write

$$K_1 = h_{-2},$$

in conformity with the general law: and then

$$H_2 = K_1 = h_{-2},$$

and so on. Thus the invariants of the equations may be arranged in the form

$$\ldots, \left.\begin{matrix} h_{-2} \\ h_{-3} \end{matrix}\right\}, \left.\begin{matrix} h_{-1} \\ h_{-2} \end{matrix}\right\}, \left.\begin{matrix} h \\ h_{-1} \end{matrix}\right\}, \left.\begin{matrix} h_1 \\ h \end{matrix}\right\}, \left.\begin{matrix} h_2 \\ h_1 \end{matrix}\right\}, \ldots;$$

that is, we have a succession of functions

$$\ldots, h_{-3}, \quad h_{-2}, \quad h_{-1}, \quad h, \quad h_1, \quad h_2, \quad h_3, \ldots.$$

Now let the Laplace transformations be applied to the equation $G(u) = 0$ any number of times in succession. The two sets of quantities, which (by vanishing) give the equations of the series, are expressible in the form

$$\ldots, \sigma^{-3}G, \quad \sigma^{-2}G, \quad \sigma^{-1}G, \quad G, \quad \sigma G, \quad \sigma^2 G, \quad \sigma^3 G, \ldots.$$

The invariants of G are known to be k and h, being those of F interchanged. For σG, the coefficients are

$$a_1' = -a - \frac{\partial \log k}{\partial y},$$
$$b_1' = -b,$$
$$c_1' = c - 2\frac{\partial b}{\partial y} + b\frac{\partial \log k}{\partial y};$$

hence, for σG,

$$\begin{aligned} h_1' &= \frac{\partial a_1'}{\partial x} + a_1' b_1' - c_1' \\ &= 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \\ &= K_1 \\ &= h_{-2}, \end{aligned}$$

and

$$\begin{aligned} k_1' &= \frac{\partial b_1'}{\partial y} + a_1' b_1' - c_1' \\ &= k \\ &= h_{-1}; \end{aligned}$$

and so on. Thus the invariants of the equations may be arranged in the form

$$\ldots, \left.\begin{matrix} h_2 \\ h_3 \end{matrix}\right\}, \left.\begin{matrix} h_1 \\ h_2 \end{matrix}\right\}, \left.\begin{matrix} h \\ h_1 \end{matrix}\right\}, \left.\begin{matrix} h_{-1} \\ h \end{matrix}\right\}, \left.\begin{matrix} h_{-2} \\ h_{-1} \end{matrix}\right\}, \ldots;$$

that is, we have a succession of functions

$$\ldots, h_3, \quad h_2, \quad h_1, \quad h, \quad h_{-1}, \quad h_{-2}, \ldots,$$

being the same succession of functions as before, but in reversed order.

Thus the invariants of $\sigma^n G$ are those of $\sigma^{-n} F$ merely interchanged; and likewise for $\sigma^{-n} G$ and $\sigma^n F$.

Again, if one of the Laplace transformations is of finite rank for F because it leads to a vanishing invariant, the other of the Laplace transformations is of the same finite rank for G.

Again, if both of the Laplace transformations are of finite rank for F because each of them leads to a vanishing invariant, both of them are of finite rank for G, the orders of the finite ranks being interchanged: the characteristic number is the same for the two equations.

Hence, if an equation is integrable by Laplace's method, in the sense that a finite number of either of the transformations leads to an equation which admits of direct quadrature, the adjoint equation is also integrable by the method through the use of the complementary transformation: the number of operations is the same for the two equations.

212. In §§ 199—203 it was shewn how, when an equation is of finite rank in both variables, it was possible to construct all the equations derived from the original equation by both the Laplace transformations and also to give the integral of each of the equations. The method there given can be applied also to the adjoint equation, because then it also is of finite rank in both variables: and, as the tale of invariants for the two sets of equations is the same save only for reversal of order, practically the whole of the former calculations can be used.

The invariant h_n of $\sigma^{-n}G(u)$ vanishes; and so, as this is the vanishing invariant of $\sigma^n F(z)$, we have

$$h_n = -\frac{\partial^2 \log \alpha}{\partial x \partial y}.$$

In order to use the earlier results, we interchange x and y; and we begin as before. Let

$$\zeta_0 = Y + \int \alpha X \, dx,$$

and

$$\zeta_m = \begin{vmatrix} \zeta_0, & \dfrac{\partial \zeta_0}{\partial y}, & \ldots, & \dfrac{\partial^m \zeta_0}{\partial y^m} \\ \alpha, & \dfrac{\partial \alpha}{\partial y}, & \ldots, & \dfrac{\partial^m \alpha}{\partial y^m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{m-1} \alpha}{\partial x^{m-1}}, & \dfrac{\partial^m \alpha}{\partial x^{m-1} \partial y}, & \ldots, & \dfrac{\partial^{2m-1} \alpha}{\partial x^{m-1} \partial y^m} \end{vmatrix};$$

then ζ_m satisfies the equation

$$\frac{\partial^2 \zeta_m}{\partial x \partial y} - \frac{\partial \zeta_m}{\partial x} \frac{\partial (\log H_m)}{\partial x} - \frac{\partial \zeta_m}{\partial y} \frac{\partial (\log H_{m-1})}{\partial x} + \zeta_m \frac{\partial (\log H_m)}{\partial x} \frac{\partial (\log H_{m-1})}{\partial y} = 0,$$

on making the changes indicated. It is the linear equation of the second order satisfied by ζ_m; and it is, in fact, the equation

$$\sigma^{-(n-m)} G(u) = 0,$$

having the integral $u = \zeta_m$.

Now this equation has the same invariants as the equation for z_m, except that their order is reversed: hence it is equivalent to the adjoint of that equation for z_m, which is

$$\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}(\log H_{m-1})+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}(\log H_m)$$

$$+\left[\frac{\partial(\log H_m)}{\partial x}\frac{\partial(\log H_{m-1})}{\partial y}+\frac{\partial^2}{\partial x\partial y}\{\log(H_m H_{m-1})\}\right]\phi=0.$$

Thus

$$\zeta_m=\lambda\phi,$$

so that

$$-\frac{\partial}{\partial y}(\log H_m)+\frac{1}{\lambda}\frac{\partial\lambda}{\partial y}=\frac{\partial}{\partial y}(\log H_{m-1}),$$

$$-\frac{\partial}{\partial x}(\log H_{m-1})+\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}=\frac{\partial}{\partial x}(\log H_m),$$

and therefore

$$\lambda=H_m H_{m-1}.$$

Hence the equation, which is satisfied by

$$\frac{\zeta_m}{H_m H_{m-1}},$$

is the adjoint of the equation satisfied by z_m.

Ex. 1. Integrate the equations:—

(i) $s-xp-yq-(1-xy)z=0$;

(ii) $s-mxp-nyq+(m-2n+mnxy)z=0$,
where m and n are constants;

(iii) $s-myp-e^{cy}q+(c+my)e^{cy}z=0$,
where m and c are constants;

(iv) $s-xyq+mxz=0$,
where m is a finite integer;

(v) $s+\frac{1}{y}p-\frac{c}{x}q-\frac{c}{xy}z=0$,
where c is a constant;

(vi) $s-\frac{2}{x-y}p+\frac{2}{x-y}q=0$;

(vii) $s+\left(\frac{1}{x-y}-\frac{1}{y}\right)p+\frac{2}{x}q-z\left\{\frac{2}{xy}-\frac{2}{x(x-y)}-\frac{1}{(x-y)^2}\right\}=0$;

all of these being of finite rank.

Ex. 2. Form the equation which is adjoint to

$$s+\frac{m}{x+y}p+\frac{n}{x+y}q+\frac{l}{(x+y)^2}z=0:$$

construct its invariants and thence verify that, if the original equation is of finite rank in either variable, the adjoint equation is of finite rank in the other variable.

Ex. 3. Prove that, by using the adjoint equation, any linear equation

$$s+ap+bq+cz=0$$

can be expressed in the form

$$(a+\beta)\,s+\frac{\partial a}{\partial y}p+\frac{\partial\beta}{\partial x}q=0.$$

If z' and z'' be any two integrals of this equation, supposed of rank n in one of the variables, shew that the equation

$$\frac{\partial}{\partial y}\left(p\,\frac{q''}{q'}\right)-\frac{\partial}{\partial x}\left(q\,\frac{p''}{p'}\right)=0$$

is of rank $n+1$ in that variable; shew also that its integral can be obtained, merely by quadratures, from the integral of the original equation.

(R. Liouville.)

Ex. 4. With the notation and assumptions of the preceding example, shew that, if ζ be an integral of the equation

$$(a+\beta)\,s+\frac{\partial\beta}{\partial y}p+\frac{\partial a}{\partial x}q=0,$$

and if

$$\lambda=\int\left(\beta\,\frac{\partial\zeta}{\partial x}\,dx-a\,\frac{\partial\zeta}{\partial y}\,dy\right),$$

then the equation

$$(a+\beta)\,s+\frac{\lambda-\beta\zeta}{\lambda+a\zeta}\,\frac{\partial a}{\partial y}p+\frac{\lambda+a\zeta}{\lambda-\beta\zeta}\,\frac{\partial\beta}{\partial x}q=0$$

is of rank $n+1$. (R. Liouville.)

Riemann's use of Adjoint Equations.

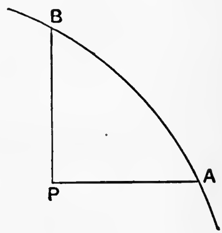
213. If, of two given equations which are reciprocally adjoint, the integral of either in finite form has been obtained by any process, the integral of the other can be deduced. When the obtained integral is free from quadratures, the earlier investigations have shewn that it can be constructed by Laplace's method through the use of a finite number of one of the transformations; the result established in § 210 shews that the adjoint equation can also be integrated by means of the same number of applications of the other of the transformations. When the obtained integral is not free from quadratures, the integral of the adjoint equation can be obtained by a method due to Riemann* who, indeed, was the first to use the equation that is adjoint to a given partial equation.

The object of the investigation is to determine an integral of the equation

$$F(z) = s + ap + bq + cz = 0,$$

which shall be the integral determined by the initial conditions in Cauchy's theorem in its most general form, that is, the variable z and one of its derivatives are to assume assigned functions of the variables as their values along a given curve in the plane.

Let AB represent the curve and, taking any point P in the plane, draw PA and PB parallel to the axes and in their positive directions. Let z denote an integral of the equation, and let u denote an integral of the adjoint equation



$$G(u) = 0;$$

then, with the notation of § 210, we have

$$uF(z) - zG(u) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

We can consider $PABP$ as an area in the plane, having PA, AB, BP for its boundary: hence

$$\iint \{uF(z) - zG(u)\}\, dx dy = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy,$$

the integrals being taken over the area. Hence, when z and u are integrals of their respective equations, we have

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = 0,$$

and therefore†, as

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = \int (M dy - N dx),$$

* It occurs in a memoir, published in 1860 and apparently written the year before: see *Ges. Werke* (1876), pp. 158 et seq. The exposition of Riemann's method as given by Darboux, *Théorie générale des surfaces*, t. II, pp. 75 et seq., is adopted in the account which follows.

† *T. F.*, § 16.

the single integral being taken positively round the boundary of the area over which the double integral is taken, we have

$$\int_A^B (M\,dy - N\,dx) + \int_B^P M\,dy + \int_P^A (-N)\,dx = 0,$$

that is,

$$\int_A^B (M\,dy - N\,dx) - \int_P^B M\,dy - \int_P^A N\,dx = 0.$$

Now the variable z and one of its derivatives, say $\frac{\partial z}{\partial x}$, possess assigned values along the curve AB: moreover, the relation

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

is always satisfied: and, along the curve, dy is known in terms of dx; hence $\frac{\partial z}{\partial y}$ is known* along the curve. If then, u is completely known, that is, if we know an integral of the adjoint equation subject to any conditions we choose to assign, then

$$\int_A^B (M\,dy - N\,dx)$$

can be regarded as a known quantity in connection with that integral and with the assigned initial conditions for z. Again,

$$\int_P^A N\,dx = \int_P^A \left\{\tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right) + buz\right\} dx$$

$$= \int_P^A \left\{\tfrac{1}{2}\frac{\partial}{\partial x}(uz) - z\left(\frac{\partial u}{\partial x} - bu\right)\right\} dx$$

$$= \tfrac{1}{2}(uz)_A - \tfrac{1}{2}(uz)_P - \int_P^A z\left(\frac{\partial u}{\partial x} - bu\right) dx;$$

and, similarly,

$$\int_P^B M\,dy = \tfrac{1}{2}(uz)_B - \tfrac{1}{2}(uz)_P - \int_P^B z\left(\frac{\partial u}{\partial y} - au\right) dy;$$

* If the curve be $\theta(x, y) = 0$, and if the values acquired by z and $\frac{\partial z}{\partial x}$ along the curve be $\zeta(x, y)$ and $\pi(x, y)$ respectively, then the value of $\frac{\partial z}{\partial y}$ along the curve is given by the equation

$$\left(\frac{\partial z}{\partial y} - \frac{\partial \zeta}{\partial y}\right)\frac{\partial \theta}{\partial x} = \left(\pi - \frac{\partial \zeta}{\partial x}\right)\frac{\partial \theta}{\partial y}.$$

whence, substituting these values, we have

$$(uz)_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx)$$
$$- \int_P^A z\left(\frac{\partial u}{\partial x} - bu\right) dx - \int_P^B z\left(\frac{\partial u}{\partial y} - au\right) dy.$$

In the last two quadratures, the quantity z occurs and it is as yet unknown; accordingly, we choose

$$\frac{\partial u}{\partial x} - bu = 0, \text{ along } PA,$$

and

$$\frac{\partial u}{\partial y} - au = 0, \text{ along } PB.$$

Hence, assuming that u is determined as an integral of the equation $G(u) = 0$ satisfying these conditions, we have the value of z at any point P of the plane given by

$$(uz)_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx).$$

Denote the coordinates of P by x, y; and let ξ, η denote current coordinates for u. Then since

$$\frac{\partial u}{\partial x} - bu = 0$$

along PA, that is, along the line $\eta = y$, we have

$$u = u_P\, e^{\int_{x, y}^{\xi, y} b\,dX},$$

for any point along that line. Similarly

$$u = u_P\, e^{\int_{x, y}^{x, \eta} a\,dY},$$

for any point along PB, which is the line $\xi = x$. It is clear that, without affecting the differential conditions imposed upon u, or without affecting the equation $G(u) = 0$, or without affecting the equation which gives z_P, we can remove the factor u_P: this removal can be effected by taking it as equal to unity. Thus the aggregate of conditions imposed upon u is:

(i) it satisfies the equation $G(u) = 0$;

(ii) it acquires the value unity at P;

(iii) when $\eta = y$, then $u = e^{\int_x^\xi b\,dX}$;

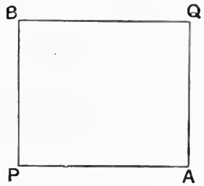
(iv) when $\xi = x$, then $u = e^{\int_y^\eta a\,dY}$.

When u is thus known, the value of z, in connection with the assigned initial conditions, is given by

$$z_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx):$$

that is, the integration of $F(z) = 0$ under arbitrarily assigned conditions is made to depend upon the integration of the adjoint equation $G(u) = 0$ under specified particular conditions. This is Riemann's result*.

214. Further significance can be given to the result by developing the reciprocal properties of the equations $F(z) = 0$ and $G(u) = 0$. For this purpose, choose the curve AB so that it shall consist of two straight lines AQ and QB, parallel to the axes; and let x', y' denote the coordinates of Q. Then



$$\int_A^B M\,dy = \int_A^Q M\,dy$$

$$= \int_A^Q \left\{\tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right) + auz\right\} dy$$

$$= \int_A^Q \left\{-\tfrac{1}{2}\frac{\partial}{\partial y}(uz) + u\left(\frac{\partial z}{\partial y} + az\right)\right\} dy$$

$$= \tfrac{1}{2}(uz)_A - \tfrac{1}{2}(uz)_Q + \int_A^Q u\left(\frac{\partial z}{\partial y} + az\right) dy;$$

and, similarly,

$$\int_A^B N\,dx = \int_Q^B N\,dx$$

$$= -\int_B^Q N\,dx$$

$$= \tfrac{1}{2}(uz)_Q - \tfrac{1}{2}(uz)_B - \int_B^Q u\left(\frac{\partial z}{\partial x} + bz\right) dx.$$

Substituting, we have

$$z_P = (uz)_Q - \int_A^Q u\left(\frac{\partial z}{\partial y} + az\right) dy - \int_B^Q u\left(\frac{\partial z}{\partial x} + bz\right) dx.$$

Now a definite curve AQ, QB has been chosen: and though initial (or boundary) conditions are imposed on z, nothing has

* *L.c.*, p. 161.

been said as to their explicit form in the present connection. Accordingly, we shall assume that z is given along AQ and QB by the conditions, (i), that

$$z_Q = 1;$$

(ii), along AQ, we are to have

$$\frac{\partial z}{\partial y} + az = 0,$$

that is,

$$z = e^{-\int_\eta^{y'} a\,dY},$$

along AQ; and, (iii), along BQ, we are to have

$$\frac{\partial z}{\partial x} + bz = 0,$$

that is,

$$z = e^{-\int_\xi^{x'} b\,dX}.$$

(It may be noted that, in passing from an equation $F(z) = 0$ to its adjoint equation $G(u) = 0$, the signs of a and b change though their values are unaltered otherwise: thus these conditions for z are the exact reciprocal of the conditions for u). Substituting the values thus obtained in the preceding relation, we find

$$z_P = u_Q;$$

or, as x and y are variables for z while x' and y' are parametric, and as x' and y' are variables for u while x and y are parametric, we can write the result in the form

$$z(x, y;\ x', y') = u(x', y';\ x, y).$$

It therefore appears that, if we can obtain an integral of either of two adjoint equations subject to selected associate conditions, we can obtain the integral of the other of the two equations subject to reciprocally selected associate conditions. In fact, *in order to effect the integration of both of the adjoint equations, it is sufficient to determine the function which has been denoted by*

$$u(x', y';\ x, y).$$

Ex. 1. Consider the equation

$$F(z) = s - \frac{1}{x+y}p - \frac{1}{x+y}q = 0.$$

The equation $G(u) = 0$, when the independent variables are x' and y', is

$$\frac{\partial^2 u}{\partial x' \partial y'} + \frac{1}{x'+y'}\frac{\partial u}{\partial x'} + \frac{1}{x'+y'}\frac{\partial u}{\partial y'} - \frac{2}{(x'+y')^2}u = 0.$$

There are three conditions imposed upon u by the preceding process :—

(i) when $x'=x$, $y'=y$, we must have

$$u=1\ ;$$

(ii) when $y'=y$, we must have

$$u=e^{\int_x^{x'}\frac{-dX}{X+y}}=\frac{x+y}{x'+y}$$

(iii) when $x'=x$, we must have

$$u=e^{\int_y^{y'}\frac{-dY}{x+Y}}=\frac{x+y}{x+y'},$$

the first of which is in accord with the second and the third.

Any form of u, which satisfies the equation $G(u)=0$ and is in accordance with these conditions, will be sufficient for our purpose; and the form, suggested by the conditions, is that of a rational fraction. Now the equation $G(u)=0$ is satisfied by

$$u=(x'+y')^m,$$

provided

$$m(m-1)+2m-2=0,$$

that is,

$$m=1, \quad \text{or} \quad m=-2.$$

Accordingly, we choose a form

$$u=\frac{\alpha+\beta x'+\gamma y'+\delta x'y'}{(x'+y')^2},$$

where α, β, γ, δ do not involve x' or y': they can be functions of x and y and, having regard to the values of u when $x'=x$ and when $y'=y$ respectively, we may expect that δ will be a pure constant, β and γ will be of the first degree in x and y, and α will be of the second degree in x and y. But the first requisite is that u should satisfy the equation $G(u)=0$: substituting, we find that the one necessary and sufficient condition is

$$\beta+\gamma=0,$$

so that

$$u=\frac{\alpha+\gamma(y'-x')+\delta x'y'}{(x'+y')^2}.$$

Having regard to the condition that $u=\dfrac{x+y}{x'+y}$ when $y=y'$, we see that

$$(x+y)(x'+y)=\delta x'y+\gamma(y-x')+\alpha:$$

so that, as α, γ, δ are independent of x', we must have

$$\left.\begin{aligned}\delta y-\gamma&=x+y\\ \gamma y+\alpha&=(x+y)\,y\end{aligned}\right\}.$$

Similarly, from the condition that $u=\dfrac{x+y}{x+y'}$ when $x=x'$, we find that

$$\left.\begin{aligned}\delta x+\gamma&=x+y\\ -\gamma x+\alpha&=(x+y)\,x\end{aligned}\right\}.$$

The four equations involving α, γ, δ are satisfied by

$$\delta=2, \quad \gamma=y-x, \quad \alpha=2xy\,;$$

and therefore

$$u=\frac{2\,(x'y'+xy)+(y-x)\,(y'-x')}{(y'+x')^2},$$

where x' and y' are current coordinates for u, and x and y are the coordinates of the point where the value of z is desired. We thus have found a value of u which, while completely known, satisfies the assigned conditions along the lines PA and PB which are given by

$$y'=y, \quad x'=x,$$

respectively.

By way of application, let it be required to find an integral z such that

$$z=\frac{(x'+y)^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial x}=3\,\frac{(x'+y)^2}{(x'+y')^3},$$

when $x=x'$, and

$$z=\frac{(x+y')^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial y}=3\,\frac{(x+y')^2}{(x'+y')^3},$$

when $y=y'$. Along AQ, let Y be the variable of integration; and along BQ, let X be the variable of integration. Hence, along AQ, we have

$$z=\frac{(x'+Y)^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial y}=3\,\frac{(x'+Y)^2}{(x'+y')^3},$$

$$u=\frac{2\,(x'Y+xy)+(Y-x')\,(y-x)}{(x'+Y)^2}\,;$$

and therefore

$$\int_A^Q u\left(\frac{\partial z}{\partial y}+az\right)dY$$

$$=\int_y^{y'} u\left(\frac{\partial z}{\partial y}-\frac{1}{x'+Y}\,z\right)dY$$

$$=\frac{2}{(x'+y')^3}\int_y^{y'}\{Y\,(2x'+y-x)+2xy-x'y+x'x\}\,dY$$

$$=\frac{y'-y}{(x'+y')^3}\{(2x'+y-x)\,(y'+y)+4xy-2x'y+2x'x\}$$

$$=\frac{1}{(x'+y')^3}\{-y^3-3xy^2+xy\,(4y'-2x')+y\,(y'^2-2x'y')+x\,(2x'y'-y'^2)+2x'y'^2\}.$$

Again, along BQ, we have

$$z=\frac{(X+y')^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial x}=3\,\frac{(X+y')^2}{(x'+y')^3},$$

$$u=\frac{2\,(Xy'+xy)+(y'-X)\,(y-x)}{(X+y')^2}\,;$$

and therefore

$$\int_B^Q u\left(\frac{\partial z}{\partial x}+bz\right)dX$$

$$=\int_x^{x'} u\left(\frac{\partial z}{\partial x}-\frac{1}{X+y'}z\right)dX$$

$$=\frac{2}{(x'+y')^3}\int_x^{x'}\{X(2y'+x-y)+2xy+yy'-xy'\}\,dX$$

$$=\frac{x'-x}{(x'+y')^3}\{(2y'+x-y)(x'+x)+4xy+2yy'-2xy'\}$$

$$=\frac{1}{(x'+y')^3}\{-x^3-3x^2y+xy(4x'-2y')+x(x'^2-2x'y')-y(x'^2-2x'y')+2x'^2y'\}.$$

Now

$$u_Q=\frac{2(x'y'+xy)+(y'-x')(y-x)}{(x'+y')^2},$$

$$z_Q=1;$$

consequently

$$z_P=(uz)_Q-\int_A^Q u\left(\frac{\partial z}{\partial y}+az\right)dY-\int_B^Q u\left(\frac{\partial z}{\partial x}+bz\right)dX$$

$$=\frac{(x+y)^3}{(x'+y')^3},$$

on reduction.

Ex. 2. Shew that the equation

$$\frac{\partial^2\theta}{\partial x\partial y}-\frac{m}{x-y}\frac{\partial\theta}{\partial x}-\frac{n}{x-y}\frac{\partial\theta}{\partial y}+\frac{l}{(x-y)^2}\theta=0$$

can, by a change of the dependent variable represented by

$$\theta=(x-y)^a z,$$

where a is an appropriately determined constant, be transformed into the equation

$$\frac{\partial^2 z}{\partial x\partial y}-\frac{\beta'}{x-y}\frac{\partial z}{\partial x}+\frac{\beta}{x-y}\frac{\partial z}{\partial y}=0.$$

Obtain integrals of this last equation in the form

$$z=x^{\lambda}F\left(-\lambda,\ \beta',\ 1-\beta-\lambda,\ \frac{y}{x}\right),$$

$$z=x^{-\beta}y^{\beta+\lambda}F\left(\beta,\ \beta'+\beta+\lambda,\ 1+\beta+\lambda,\ \frac{y}{x}\right),$$

where F is the ordinary symbol for the hypergeometric series, and λ is any arbitrary constant.

Hence shew that, unless β is a positive integer greater than unity, the transformed equation possesses an infinitude of polynomial integrals.

(Euler, Darboux.)

Ex. 3. Shew that, if $\phi(x, y)$ is an integral of the equation

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial z}{\partial x} + \frac{\beta}{x-y}\frac{\partial z}{\partial y} = 0,$$

then

$$z = (ax+b)^{-\beta}(ay+b)^{-\beta'}\phi\left(\frac{cx+d}{ax+b}, \frac{cy+d}{ay+b}\right)$$

is also an integral of the equation, where a, b, c, d are arbitrary constants. (Darboux.)

Ex. 4. Denoting by u the variable of the equation which is adjoint to

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial z}{\partial x} + \frac{\beta}{x-y}\frac{\partial z}{\partial y} = 0,$$

and writing

$$u = (x-y)^{\beta+\beta'} v,$$

prove that v satisfies the equation

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial v}{\partial x} + \frac{\beta'}{x-y}\frac{\partial v}{\partial y} = 0.$$

Shew that, if

$$\sigma = \frac{(x-x')(y-y')}{(x-y')(y-x')},$$

then an integral u of the adjoint equation is given by

$$u = (y'-x)^{-\beta'}(y-x')^{-\beta}(y-x)^{\beta+\beta'} F(\beta, \beta', 1, \sigma);$$

and verify that this integral satisfies the conditions imposed upon the function u in the Riemann method of constructing an integral of the original equation in z. (Darboux.)

Note. Further properties of these equations are given by Darboux*.

Ex. 5. In the equation

$$s + ap + bq + cz = 0,$$

let a, b, c be functions of x and y which are regular within a domain $|x| \leqslant 1$, $|y| \leqslant 1$; and let $\phi(x)$, $\psi(y)$ be two functions of x and y which also are regular within those domains. Also, let M, N, P, H, K be not less than the greatest values of $|a|$, $|b|$, $|c|$, $|\phi(x)|$, $|\psi(y)|$ respectively, within those domains.

Prove that the original equation possesses an integral, which is a regular function of x and y within the assigned domains, which reduces to $\phi(x)$ when $y=0$, and which reduces to $\psi(y)$ when $x=0$, provided the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{M}{1-x-y}\frac{\partial Z}{\partial x} + \frac{N}{1-x-y}\frac{\partial Z}{\partial y} + \frac{P}{(1-x-y)^2} Z$$

* *Théorie générale des surfaces*, t. II, pp. 54—70, 81—91.

possesses* a regular integral, which reduces to $\frac{H}{1-x}$, when $y=0$, and which reduces to $\frac{K}{1-y}$, when $x=0$.

Apply the results in the preceding examples to shew that the latter equation does possess the regular integral indicated. (Darboux.)

Ex. 6. Shew that the equation adjoint to

$$f(z)=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+2l\frac{\partial z}{\partial x}+2m\frac{\partial z}{\partial y}+nz=0$$

is

$$g(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2L\frac{\partial u}{\partial x}+2M\frac{\partial u}{\partial y}+Nu=0,$$

where

$$L=-l, \quad M=-m, \quad N=n-2\frac{\partial l}{\partial x}-2\frac{\partial m}{\partial y}.$$

Let J' and K' denote the invariants of $g(u)=0$, which correspond to the invariants J and K of $f(z)=0$ (see § 194, Ex. 5): verify that

$$J'=-J, \quad K'=K.$$

Hence shew that, if J vanishes, $f(z)=0$ and $g(u)=0$ are effectively one and the same equation; and that, if K' vanishes but not J, the equation $g(u)=0$ can be expressed in the form

$$a\frac{\partial^2}{\partial x^2}\left(\frac{u}{\beta}\right)+\beta\frac{\partial^2}{\partial y^2}\left(\frac{u}{a}\right)=0,$$

where

$$a=e^{\int l dx}, \quad \beta=e^{\int m dy}.$$

Also prove, in general, that

$$uf(z)-zg(u)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y},$$

where

$$\left.\begin{aligned} P&=u\frac{\partial z}{\partial x}-z\frac{\partial u}{\partial x}+2luz \\ Q&=u\frac{\partial z}{\partial y}-z\frac{\partial u}{\partial y}+2muz \end{aligned}\right\}.$$

(Burgatti.)

Ex. 7. Shew that the equation adjoint to

$$f(z)=\frac{\partial^2 z}{\partial x^2}+2a\frac{\partial z}{\partial x}+2\beta\frac{\partial z}{\partial y}+\gamma z=0$$

is

$$g(u)=\frac{\partial^2 u}{\partial x^2}+2A\frac{\partial u}{\partial x}+2B\frac{\partial u}{\partial y}+\Gamma u=0,$$

* In connection with this existence-theorem, a memoir by Goursat, *Annales de Toulouse*, 2[me] Sér., t. v (1903), pp. 405—436, may be consulted. It also contains a number of references to other memoirs on the subject.

where

$$A = -\alpha, \quad B = -\beta, \quad \Gamma = \gamma - 2\frac{\partial \alpha}{\partial x} - 2\frac{\partial \beta}{\partial y}.$$

Denoting the invariants of $g(u)=0$ by I' and J', which correspond (§ 206) to the invariants I and J of $f(z)=0$, prove that

$$I' = -I,$$

$$J' = -J + \frac{\partial^2 \log I}{\partial x \, \partial y};$$

and verify that

$$uf(z) - zg(u) = \frac{\partial T}{\partial x} + \frac{\partial U}{\partial y},$$

where

$$T = u\frac{\partial z}{\partial x} - 2\frac{\partial u}{\partial x} + 2\alpha uz,$$

$$U = 2\beta uz.$$

Ex. 8. Indicate how the results in the two preceding examples can be used to obtain integrals of the respective original differential equations satisfying the initial conditions imposed in Cauchy's existence-theorem.

Ex. 9. Shew that, if ζ is any integral of the equation

$$s = z \sin \phi,$$

where ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial x \, \partial y} + \cos \phi = 0,$$

then

$$-\frac{\partial \zeta}{\partial x}\frac{\partial \phi}{\partial x} dx + \zeta \cos \phi \, dy$$

and

$$\zeta \cos \phi \, dx - \frac{\partial \zeta}{\partial y}\frac{\partial \phi}{\partial y} dy$$

are perfect differentials. Denoting these perfect differentials by $d\rho$ and $d\sigma$ respectively, prove that

$$\rho\frac{\partial \phi}{\partial x} - \frac{\partial^2 \zeta}{\partial x^2}, \quad \sigma\frac{\partial \phi}{\partial y} - \frac{\partial^2 \zeta}{\partial y^2},$$

are integrals of the original equation to be satisfied by z. (Guichard.)

Ex. 10. Four linearly distinct integrals of the equation

$$F(z) = s + ap + bq + cz = 0$$

are denoted by z_1, z_2, z_3, z_4; and quantities u_1, u_2, u_3, u_4 are determined by the equations

$$z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4 = 0,$$

$$p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 = 0,$$

$$q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4 = 0.$$

Prove that these quantities u satisfy an equation

$$G(u) = \frac{\partial^2 u}{\partial x \partial y} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u = 0.$$

Prove, further, that, if v be the integral of the equation adjoint to $G(u)=0$, then

$$v\left(\frac{\partial u_\mu}{\partial x}+\beta u_\mu\right)dx+u_\mu\left(\frac{\partial v}{\partial y}-av\right)dy$$

is an exact differential dw_μ; and that, if

$$Z=z_1w_1+z_2w_2+z_3w_3+z_4w_4,$$

then Z is an integral of $F(z)=0$. (Darboux.)

Ex. 11. Two integrals, independent of one another and belonging to the equation

$$s+ap+bq=0,$$

are denoted by z_1 and z_2; and z_1z_2 is also an integral of the equation. Prove that

$$\frac{\partial}{\partial x}\left(a+\frac{b}{\rho}\right)=\frac{\partial}{\partial y}(a\rho+b),$$

where

$$\rho=ie^{\int(b\,dx-a\,dy)}.$$

Apply this result to obtain a relation among the coefficients of the equation

$$s+ap+bq+cz=0,$$

when four linearly independent integrals z_1, z_2, z_3, z_4 satisfy a relation

$$z_1^2+z_2^2+z_3^2+z_4^2=0.$$

Equations with Equal Invariants.

215. One of the more important classes of linear equations of the second order, in that their properties are more fully developed than those of the other classes, is composed of those equations which have their *invariants equal* to one another. As has already (§ 192) been proved, the equation can be changed so as to acquire the form

$$s=\lambda z,$$

where λ is the common value of the equal invariants: and this form is canonical for equations with equal invariants.

The equation, which is adjoint to

$$\frac{\partial^2 z}{\partial x\partial y}=\lambda z,$$

is

$$\frac{\partial^2 u}{\partial x\partial y}=\lambda u,$$

as is seen at once by taking the general form, and making a and b zero: thus *the equation with equal invariants is self-adjoint.* Moreover, *it is the only equation which is self-adjoint*: for if $F(z)=0$,

$G(u)=0$, be two reciprocally adjoint equations, their invariants are given by

$$h_g = k_f, \quad k_g = h_f,$$

and therefore, if $F(z)=0$ and $G(u)=0$ are effectively the same equation so that they have the same invariants,

$$h_f = h_g = k_f, \quad k_f = k_g = h_f,$$

that is, the invariants are equal.

Suppose that a self-adjoint equation is of finite rank in one of the variables, say in x; then the set of equations obtained by the Laplace σ-transformations is finite, in the sense that after a finite number of the transformations in succession an equation is obtained having a vanishing invariant. Let

$$F = \frac{\partial^2 z}{\partial x \partial y} - \lambda z = 0$$

be the original equation: let the last equation of the set derived by the σ-transformation be

$$\sigma^n F = 0.$$

We know that, in the case of any equation of finite rank in the variable x, the adjoint equation is of equal finite rank in the variable y. In the present case, F is self-adjoint; and therefore it is of finite rank in the variable y, and the first equation of the set given by means of the Laplace σ^{-1}-transformation, which has a vanishing invariant, is

$$\sigma^{-n} F = 0.$$

Consequently, when a self-adjoint equation is of finite rank in one of the variables, it is of equal finite rank in the other variable.

It therefore follows that, if a self-adjoint equation can be integrated by Laplace's method, the integral can be expressed in such a way that the two arbitrary functions are free from partial quadratures. Moreover, the integral is of the same finite rank in y as it is in x, and thus it must be of the form

$$z = AX + A_1 X' + \ldots + A_m X^{(m)} + BY + B_1 Y' + \ldots + B_m Y^{(m)};$$

we shall return later to the consideration of this form of the integral. The condition that a given self-adjoint equation

$$\frac{\partial^2 z}{\partial x \partial y} - \lambda z = 0$$

should be of finite rank $m+1$ in each of the variables is derivable at once from the earlier investigations: we have

$$h = \lambda = k,$$

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y} = \lambda - \frac{\partial^2 \log \lambda}{\partial x \partial y},$$

$$h_2 = 2h_1 - h - \frac{\partial^2 \log h_1}{\partial x \partial y} = \lambda - \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^2 h_1)\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$h_m = \lambda - \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^m h_1{}^{m-1} h_2{}^{m-2} \dots h^2_{m-2} h_{m-1})\},$$

and therefore, as the necessary and sufficient condition is that $h_m = 0$, this condition is

$$\lambda = \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^m h_1{}^{m-1} h_2{}^{m-2} \dots h^2_{m-2} h_{m-1})\},$$

being an equation satisfied by λ. The equation is of order $2m$.

216. The construction of all the linear equations, which have equal invariants and can be integrated in finite terms by Laplace's method, can be effected if this differential equation of order $2m$ for the determination of λ can be integrated in general. This integration has been obtained through a process, devised first by Moutard, of passing from one equation to another of contiguous rank; an exposition of the process will be given almost immediately. Meanwhile, the results (§§ 201—203) obtained by Darboux for a linear equation, which is of finite rank in both variables, can be applied when the linear equation is of the self-adjoint type.

The equations of the double set, derived from a self-adjoint equation $F=0$, can be represented by equating the expressions

$$\sigma^{-n}F, \quad \sigma^{-n+1}F, \dots, \sigma^{-1}F, \quad F, \quad \sigma F, \dots, \sigma^{n-1}F, \quad \sigma^n F$$

to zero; hence the quantity μ of § 203 is given by

$$2n = \mu - 1.$$

The invariant k_n of $\sigma^n F$ is the invariant h_{n-1} of $\sigma^{n-1}F$, that is,

$$k_n = -\frac{\partial^2 \log \alpha}{\partial x \partial y},$$

where α is of the form

$$\alpha = \xi_1 \eta_1 + \dots + \xi_{2n+1} \eta_{2n+1}:$$

and in this expression, $\eta_1, \ldots, \eta_{2n+1}$ are $2n+1$ linearly independent functions of y, while $\xi_1, \ldots, \xi_{2n+1}$ are $2n+1$ linearly independent functions of x. Again, the invariant k_{-n} of $\sigma^{-n}F$ is zero; and its invariant h_{-n} is the invariant k_{-n+1} of $\sigma^{-n+1}F$, that is,

$$h_{-n} = -\frac{\partial^2 \log \beta}{\partial x \partial y},$$

where β is of the form

$$\beta = x_1 y_1 + \ldots + x_{2n+1} y_{2n+1}:$$

and in this expression, $y_1, \ldots, y_{2n+1}$ are $2n+1$ linearly independent functions of y, while $x_1, \ldots, x_{2n+1}$ are $2n+1$ linearly independent functions of x. Now the invariants of $\sigma^m F$ are the same as those of $\sigma^{-m} F$ except as to order; and therefore

$$k_n = h_{-n},$$

that is,

$$\frac{\partial^2 \log \alpha}{\partial x \partial y} = \frac{\partial^2 \log \beta}{\partial x \partial y},$$

so that

$$\alpha = \beta \xi \eta,$$

where ξ and η are functions of x and of y, unrestricted so far as concerns this relation. As $x_1, \ldots, x_{2n+1}$ are $2n+1$ linearly independent functions of x, being $2n+1$ linearly independent integrals of an ordinary equation in x of order $2n+1$, the function ξ can be absorbed into each of them without affecting their linear independence; and similarly, the function η can be absorbed into each of the quantities $y_1, \ldots, y_{2n+1}$ without affecting their linear independence. Let this absorption take place in both cases: then we have

$$\alpha = \beta.$$

Adopting the same notation for the functional determinants in derivatives of α as before, we note that the invariant h_n of $\sigma^n F$ is zero, and that the equation $F=0$ is removed from $\sigma^n F = 0$ by n of the σ-transformations; hence $F=0$ can be expressed in the form

$$s - p\frac{\partial}{\partial y}(\log H_{n-1}) - q\frac{\partial}{\partial x}(\log H_n) + z\frac{\partial}{\partial y}(\log H_{n-1})\frac{\partial}{\partial x}(\log H_n) = 0.$$

Again, the invariant k_{-n} of $\sigma^{-n}F$ is zero, and the equation $F=0$ is removed from $\sigma^{-n}F=0$ by n of the σ^{-1}-transformations; hence, constructing quantities K from the magnitude β in the same way as the quantities H are constructed from the magnitude α, the equation $F=0$ can be expressed in the form

$$s - q\frac{\partial}{\partial x}(\log K_{n-1}) - p\frac{\partial}{\partial y}(\log K_n) + z\frac{\partial}{\partial x}(\log K_{n-1})\frac{\partial}{\partial y}(\log K_n) = 0.$$

Now we have

$$h_{n-s-1} = -\frac{\partial^2}{\partial x \partial y}(\log H_s),$$

$$k_{-(n-s'-1)} = -\frac{\partial^2}{\partial x \partial y}(\log K_{s'});$$

hence

$$h_{-m} = -\frac{\partial^2}{\partial x \partial y}(\log H_{m+n-1}),$$

$$\begin{aligned} h_{-\mu-1} &= k_{-\mu} \\ &= -\frac{\partial^2}{\partial x \partial y}(\log K_{n-\mu-1}). \end{aligned}$$

We therefore have

$$\frac{\partial^2}{\partial x \partial y}(\log K_n) = h_0 = -\frac{\partial^2}{\partial x \partial y}(\log H_{n-1}),$$

$$\frac{\partial^2}{\partial x \partial y}(\log K_{n-1}) = h_{-1} = -\frac{\partial^2}{\partial x \partial y}(\log H_n);$$

and therefore a common form of the equation, whether derived from $\sigma^n F = 0$ or from $\sigma^{-n} F = 0$, is

$$\begin{aligned} s - p\frac{\partial}{\partial y}(\log H_{n-1}) - q\frac{\partial}{\partial x}(\log K_{n-1}) & \\ + z\frac{\partial}{\partial y}(\log H_{n-1})\frac{\partial}{\partial x}(\log K_{n-1}) &= 0. \end{aligned}$$

The equation is in one of the canonical forms adopted in § 192. Its two invariants are

$$-\frac{\partial^2}{\partial x \partial y}(\log H_{n-1}), \quad -\frac{\partial^2}{\partial x \partial y}(\log K_{n-1});$$

also K_{n-1} is the same function of β as H_{n-1} is of α, and $\alpha = \beta$: hence the invariants are equal. When the equation is expressed in the binomial form, it is

$$\frac{\partial^2 Z}{\partial x \partial y} = -Z\frac{\partial^2 \log H_{n-1}}{\partial x \partial y},$$

where

$$z = ZH_{n-1}.$$

217. In order to find the integral of this equation, we use the two methods of derivation of the equation and of its integral. When we construct it from $\sigma^n F = 0$ as the initial equation, we have

$$z = \begin{vmatrix} z_0 , & \frac{\partial z_0}{\partial x} & , \ldots, & \frac{\partial^n z_0}{\partial x^n} \\ \alpha , & \frac{\partial \alpha}{\partial x} & , \ldots, & \frac{\partial^n \alpha}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{n-1} \alpha}{\partial y^{n-1}} , & \frac{\partial^n \alpha}{\partial x \partial y^{n-1}} & , \ldots, & \frac{\partial^{2n-1} \alpha}{\partial x^n \partial y^{n-1}} \end{vmatrix},$$

where

$$z_0 = X + U,$$

and U is homogeneous and linear in the derivatives of Y; hence the part of z dependent upon the derivatives of X only is

$$\theta(X) = \begin{vmatrix} X , & X' & , \ldots, & X^{(n)} \\ \alpha , & \frac{\partial \alpha}{\partial x} & , \ldots, & \frac{\partial^n \alpha}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{n-1} \alpha}{\partial y^{n-1}} , & \frac{\partial^n \alpha}{\partial x \partial y^{n-1}} & , \ldots, & \frac{\partial^{2n-1} \alpha}{\partial x^n \partial y^{n-1}} \end{vmatrix},$$

and the coefficient of $X^{(n)}$ in this expression is $(-1)^n H_{n-1}$. As

$$z = Z H_{n-1},$$

the coefficient of $X^{(n)}$ in Z is unity save as to sign.

Similarly, when we construct the equation from $\sigma^{-n} F = 0$ as the initial equation, we have

$$z' = \begin{vmatrix} \zeta_0 , & \frac{\partial \zeta_0}{\partial y} & , \ldots, & \frac{\partial^n \zeta_0}{\partial y^n} \\ \alpha , & \frac{\partial \alpha}{\partial y} & , \ldots, & \frac{\partial^n \alpha}{\partial y^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{n-1} \alpha}{\partial x^{n-1}} , & \frac{\partial^n \alpha}{\partial x^{n-1} \partial y} & , \ldots, & \frac{\partial^{2n-1} \alpha}{\partial x^{n-1} \partial y^n} \end{vmatrix},$$

where

$$\zeta_0 = Y + V,$$

and V is homogeneous and linear in the derivatives of X; hence the part of z' dependent upon the derivatives of Y only is

$$\vartheta(Y) = \begin{vmatrix} Y, & Y', & \dots, & Y^{(n)} \\ \alpha, & \dfrac{\partial\alpha}{\partial y}, & \dots, & \dfrac{\partial^n\alpha}{\partial y^n} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial^{n-1}\alpha}{\partial x^{n-1}}, & \dfrac{\partial^n\alpha}{\partial x^{n-1}\partial y}, & \dots, & \dfrac{\partial^{2n-1}\alpha}{\partial x^{n-1}\partial y^n} \end{vmatrix},$$

and the coefficient of $Y^{(n)}$ in this expression is $(-1)^n H_{n-1}$. As

$$z' = ZH_{n-1},$$

the coefficient of $Y^{(n)}$ in Z is unity save as to sign.

We therefore take

$$\begin{aligned} Z &= \frac{1}{H_{n-1}} \{\theta(X) + \vartheta(Y)\} \\ &= X^{(n)} - \frac{\partial \log H_{n-1}}{\partial x} X^{(n-1)} + \dots \\ &\quad + Y^{(n)} - \frac{\partial \log H_{n-1}}{\partial y} Y^{(n-1)} + \dots, \end{aligned}$$

absorbing the doubtful signs into X and Y respectively: and this is the integral of the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = -Z \frac{\partial^2 \log H_{n-1}}{\partial x \partial y}.$$

It should however be borne in mind that, in establishing this result, the relation

$$\alpha = \beta\xi\eta$$

was changed to

$$\alpha = \beta,$$

by absorbing the quantities ξ and η into the sets of quantities used for the composition of α and β. When this change is not made, then we no longer have $K_{n-1} = H_{n-1}$: but as K_{n-1} is the same function of β as H_{n-1} is of α, we have

$$H_{n-1} = K_{n-1}\xi_1\eta_1,$$

where ξ_1 and η_1 are functions of x and of y respectively. The equation

$$\frac{\partial^2 \log H_{n-1}}{\partial x \partial y} = \frac{\partial^2 \log K_{n-1}}{\partial x \partial y}$$

still holds; and the invariants of the equation as obtained are equal.

In that case, the value of Z satisfying the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = -Z \frac{\partial^2 \log H_{n-1}}{\partial x \partial y}$$

is found to have the form

$$Z = \frac{\theta(X)}{H_{n-1}\xi_2} + \frac{\vartheta(Y)}{K_{n-1}\eta_2},$$

where ξ_2 and η_2 are functions of x and of y only*.

The quantity α, which is subsidiary to the construction of the equation, occurs in the form

$$\alpha = \xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_{2n+1}\eta_{2n+1},$$

where $\xi_1, \xi_2, \dots, \xi_{2n+1}$ are $2n+1$ linearly independent functions of x, and $\eta_1, \eta_2, \dots, \eta_{2n+1}$ are $2n+1$ linearly independent functions of y; and it possesses this form as being an integral of the equation

$$H_{2n+1} = 0.$$

The appropriate values of α are obtained by Darboux† through a consideration of the properties of the ordinary linear equation satisfied by $\xi_1, \dots, \xi_{2n+1}$, regard being paid to the source of that equation. He proves that this equation is self-adjoint, and shews how the equations can be constructed for successive values of n.

Ex. 1. The simplest case occurs when $n=0$: then H_{n-1}, $=H_{-1}$ in this case, is zero; the equation is

$$\frac{\partial^2 Z}{\partial x\, \partial y} = 0,$$

and the value of Z is

$$Z = X + Y.$$

Ex. 2. The next simplest case occurs when $n=1$. Then

$$a = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$

where ξ_1, ξ_2, ξ_3 are three linearly independent functions of x, and likewise η_1, η_2, η_3 are three linearly independent functions of y.

Thus ξ_1, ξ_2, ξ_3 are a set of independent integrals of an ordinary linear equation of the third order which, as is known‡, can always be expressed in a form

$$\frac{d^3\xi}{dx^3} + R\xi = 0.$$

* See, on this matter, § 219 hereafter.

† *L.c.*, t. II, pp. 140 et seq.

‡ See a paper by the author, *Phil. Trans.*, A, 1888, p. 441.

As the equation is to be self-adjoint, R must be zero: and so we may take

$$\xi_1 = \tfrac{1}{2}x^2, \quad \xi_2 = x, \quad \xi_3 = 1.$$

Similarly

$$\eta_1 = 1, \quad \eta_2 = y, \quad \eta_3 = \tfrac{1}{2}y^2:$$

and

$$a = \tfrac{1}{2}(x+y)^2.$$

Thus

$$\frac{\partial^2}{\partial x\,\partial y}(\log H_0) = \frac{\partial^2}{\partial x\,\partial y}(\log a) = -\frac{2}{(x+y)^2};$$

the equation is

$$\frac{\partial^2 Z}{\partial x\,\partial y} = \frac{2}{(x+y)^2}Z,$$

and its integral is

$$Z = X' - \frac{2}{x+y}X + Y' - \frac{2}{x+y}Y.$$

MOUTARD'S THEOREM ON EQUATIONS WITH EQUAL INVARIANTS.

218. The process devised by Moutard* depends upon a theorem which facilitates the construction of the equations of successively increasing rank and, at the same time, puts the increase of rank in evidence.

Let ω denote any integral of the equation

$$\frac{\partial^2 \omega}{\partial x \partial y} = \lambda\omega,$$

where λ is a function of x and y; then

$$\omega\left(\frac{\partial^2 z}{\partial x\partial y} - \lambda z\right) = \omega\frac{\partial^2 z}{\partial x\partial y} - z\frac{\partial^2 \omega}{\partial x\partial y}$$

$$= \tfrac{1}{2}\frac{\partial}{\partial x}\left(\omega\frac{\partial z}{\partial y} - z\frac{\partial \omega}{\partial y}\right) + \tfrac{1}{2}\frac{\partial}{\partial y}\left(\omega\frac{\partial z}{\partial x} - z\frac{\partial \omega}{\partial x}\right),$$

* It is contained in a memoir, *Journ. de l'École Polyt.*, t. XXVIII (1878), pp. 1—11, which originally was merely the third or last section of a memoir presented to the Académie des Sciences in 1870; see *Comptes Rendus*, t. LXX (1870), pp. 834, 1068. The latter was never published: it seems to have disappeared in 1871 (Darboux, *Théorie générale des surfaces*, t. II, p. 53) during the fires of the Commune, which also caused the destruction of all the materials prepared by Bertrand for his work on differential equations that was to be the third volume of his *Traité de Calcul différentiel et de Calcul-intégral*. (Darboux, "Éloge de Bertrand," being the preface to Bertrand's *Éloges académiques, Nouvelle Série*, Hachette, 1902.)

so that, when z also is an integral of the same equation, a function ϕ must exist such that

$$\frac{\partial \phi}{\partial x} = \omega \frac{\partial z}{\partial x} - z \frac{\partial \omega}{\partial x} = \omega^2 \frac{\partial}{\partial x}\left(\frac{z}{\omega}\right),$$

$$-\frac{\partial \phi}{\partial y} = \omega \frac{\partial z}{\partial y} - z \frac{\partial \omega}{\partial y} = \omega^2 \frac{\partial}{\partial y}\left(\frac{z}{\omega}\right).$$

Hence

$$\frac{\partial}{\partial y}\left(\frac{1}{\omega^2}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\omega^2}\frac{\partial \phi}{\partial y}\right) = 0,$$

and therefore

$$\omega \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \omega}{\partial x}\frac{\partial \phi}{\partial y} = 0.$$

To express this equation in a canonical form, we take

$$\phi = \omega\theta,$$

and we easily find

$$\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x \partial y} = -\frac{1}{\omega}\frac{\partial^2 \omega}{\partial x \partial y} + \frac{2}{\omega^2}\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}$$

$$= \omega \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right)$$

$$= \frac{1}{\frac{1}{\omega}}\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right).$$

The relations between z and θ are simple: on the one hand, we have

$$\theta = \frac{1}{\omega}\phi$$

$$= \frac{1}{\omega}\int \omega^2 \left\{\frac{\partial}{\partial x}\left(\frac{z}{\omega}\right) dx - \frac{\partial}{\partial y}\left(\frac{z}{\omega}\right) dy\right\};$$

and, on the other hand, we have

$$z = \omega \int \frac{1}{\omega^2}\left(\frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial y} dy\right)$$

$$= \omega \int \frac{1}{\omega^2}\left\{\frac{\partial (\omega\theta)}{\partial x} dx - \frac{\partial (\omega\theta)}{\partial y} dy\right\}.$$

Thus we have Moutard's theorem :—

Writing

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z),$$

where Δ is regarded as a symbol of operation, then any integral of either of the equations

$$\Delta(z) = \Delta(\omega), \quad \Delta(z) = \Delta\left(\frac{1}{\omega}\right),$$

leads, by a definite simple quadrature, to an integral of the other equation.

The importance of the theorem lies in its application to the construction of the equations of successive rank with equal invariants; it obviously leads to the following method of proceeding:—

Let the general integral ζ of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z) = \lambda$$

be supposed known: and let ω denote a value obtained from ζ by assigning any general values to the arbitrary functions which occur in ζ: then the general integral of the equation*

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z) = \Delta\left(\frac{1}{\omega}\right) = \lambda_1,$$

is given by

$$z\omega = \int \left\{ \left(\omega \frac{\partial \zeta}{\partial x} - \zeta \frac{\partial \omega}{\partial x}\right) dx - \left(\omega \frac{\partial \zeta}{\partial y} - \zeta \frac{\partial \omega}{\partial y}\right) dy \right\};$$

and the rank of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_1$$

in each of the variables is greater by unity than the rank of the original equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda.$$

The first part of this statement is merely a repetition of Moutard's theorem in a slightly modified form. As regards the second part relating to the rank of the equation, the quantity $\frac{\partial \zeta}{\partial x}$ under the quadrature is of higher rank in x than ζ by one unit; and it is there multiplied by ω, so that this increase of

* The significance of this limitation will be illustrated later (Ex. 7): at present, it can be regarded merely as a direction not to take exceedingly special values of the arbitrary functions that occur in ζ.

rank is maintained after the quadrature; and similarly for $\frac{\partial \zeta}{\partial y}$. Thus the rank of the new equation in each of the variables is greater by one unit than the rank of the original equation.

Before proceeding to a general proposition dealing with the quadrature, some examples of the process will indicate its working.

Ex. 1. The general integral of

$$\Delta(z)=0$$

is

$$\zeta=X+Y.$$

Let ω denote the particular value of ζ represented by X_1+Y_1: then

$$\Delta\left(\frac{1}{\omega}\right)=\omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right)$$
$$=\frac{2X_1'Y_1'}{(X_1+Y_1)^2},$$

where X_1' and Y_1' as usual are the derivatives of X_1 and Y_1. It follows, from Moutard's theorem, that the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2X_1'Y_1'}{(X_1+Y_1)^2}$$

is given by

$$\begin{aligned}(X_1+Y_1)\,z=z\omega\\ &=\int\left\{\left(\omega\frac{\partial\zeta}{\partial x}-\zeta\frac{\partial\omega}{\partial x}\right)dx-\left(\omega\frac{\partial\zeta}{\partial y}-\zeta\frac{\partial\omega}{\partial y}\right)dy\right\}\\ &=\int[\{(X_1+Y_1)X'-(X+Y)X_1'\}\,dx\\ &\qquad-\{(X_1+Y_1)\,Y'-(X+Y)\,Y_1'\}\,dy].\end{aligned}$$

Now

$$(X_1+Y_1)\,X'-(X+Y)\,X_1'=\frac{\partial}{\partial x}\{(X_1+Y_1)(X-Y)\}-2XX_1',$$

$$-(X_1+Y_1)\,Y'+(X+Y)\,Y_1'=\frac{\partial}{\partial y}\{(X_1+Y_1)(X-Y)\}+2YY_1',$$

and therefore

$$\begin{aligned}(X_1+Y_1)\,z&=-2\int XX_1'\,dx+2\int YY_1'\,dy\\ &\quad+\int\left[\frac{\partial}{\partial x}\{(X_1+Y_1)(X-Y)\}\,dx+\frac{\partial}{\partial y}\{(X_1+Y_1)(X-Y)\}\,dy\right]\\ &=-2\int XX_1'\,dx+2\int YY_1'\,dy+(X_1+Y_1)(X-Y).\end{aligned}$$

Let two new arbitrary functions $f(x)$ and $g(y)$ be introduced by the defining relations

$$XX_1'=\frac{df}{dx},\quad YY_1'=-\frac{dg}{dy};$$

then

$$X=\frac{1}{X_1'}\frac{df}{dx}, \quad Y=-\frac{1}{Y_1'}\frac{dg}{dy},$$

and so

$$z=-2\frac{f+g}{X_1+Y_1}+\frac{1}{X_1'}\frac{df}{dx}+\frac{1}{Y_1'}\frac{dg}{dy};$$

or now writing X and Y for f and g respectively, we have the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2X_1'Y_1'}{(X_1+Y_1)^2}$$

given by

$$z=\frac{X'}{X_1'}+\frac{Y'}{Y_1'}-2\frac{X+Y}{X_1+Y_1}.$$

Note. The invariant h_1 connected with this equation vanishes: but

$$h_1=h-\frac{\partial^2 \log h}{\partial x\,\partial y},$$

and

$$h=\lambda=\frac{2X_1'Y_1'}{(X_1+Y_1)^2}.$$

Hence, if we take $h=e^{\zeta}$, we have

$$\frac{\partial^2\zeta}{\partial x\,\partial y}=e^{\zeta},$$

and

$$e^{\zeta}=\frac{2\phi'(x)\,\psi'(y)}{\{\phi(x)+\psi(y)\}^2}.$$

The latter is the general integral of the equation for ζ, a result first given by J. Liouville.

Ex. 2. Without any loss of generality, and merely by changing the independent variables, the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=2\frac{X_1'Y_1'}{(X_1+Y_1)^2}$$

can be changed into the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2};$$

and the general integral of the latter is (after the preceding example) given by

$$\zeta=X'+Y'-2\frac{X+Y}{x+y}.$$

Let ω denote the particular value that arises by taking

$$X=x^3, \quad Y=y^3,$$

in ζ: thus

$$\omega=3x^2+3y^2-2(x^2-xy+y^2)$$
$$=(x+y)^2.$$

Also

$$\Delta\left(\frac{1}{\omega}\right)=\frac{6}{(x+y)^2};$$

and therefore the integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$z(x+y)^2=\int\left[\left\{(x+y)^2\frac{\partial\zeta}{\partial x}-2(x+y)\zeta\right\}dx\right.$$
$$\left.-\left\{(x+y)^2\frac{\partial\zeta}{\partial y}-2(x+y)\zeta\right\}dy\right]=\Omega.$$

The quantity upon which a quadrature is to be effected is

$$d\Omega=(x+y)^2(X''dx-Y''dy)-4(x+y)(X'dx-Y'dy)$$
$$-2(x+y)(Y'dx-X'dy)+6(Ydx-Xdy)+6Xdx-6Ydy.$$

Now

$$d\{(x+y)^2(X'-Y')\}=(x+y)^2(X''dx-Y''dy)+2(x+y)(X'-Y')(dx+dy),$$

and therefore

$$d\{\Omega-(x+y)^2(X'-Y')\}$$
$$=-6(x+y)(X'dx-Y'dy)+6(Ydx-Xdy)+6Xdx-6Ydy$$
$$=-6d\{(x+y)(X-Y)\}+12(Xdx-Ydy):$$

hence

$$\Omega=(x+y)^2(X'-Y')-6(x+y)(X-Y)+12\int(Xdx-Ydy).$$

Writing

$$\xi=\int Xdx,\qquad \eta=-\int Ydy,$$

we have

$$\Omega=(x+y)^2(\xi''+\eta'')-6(x+y)(\xi'+\eta')+12(\xi+\eta);$$

and therefore the integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$z=\xi''+\eta''-6\frac{\xi'+\eta'}{x+y}+12\frac{\xi+\eta}{(x+y)^2},$$

where ξ and η are arbitrary functions of x and of y respectively.

Ex. 3. Deduce the integral of

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6X_1'Y_1'}{(X_1+Y_1)^2}.$$

Ex. 4. Comparing the integral of

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2},$$

which is

$$\zeta = X' + Y' - 2\,\frac{X+Y}{x+y},$$

with the integral of

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2X_1'Y_1'}{(X_1+Y_1)^2},$$

which is

$$\zeta = \frac{X'}{X_1'} + \frac{Y'}{Y_1'} - 2\,\frac{X+Y}{X_1+Y_1},$$

though (owing to a transformation of the independent variables) the two equations are essentially the same, we have an illustration of the remark (§ 217) that, according to the form of the equation, the derivative of highest order of the arbitrary function X has either unity or a function of x for its coefficient, and likewise for the derivative of highest order of the arbitrary function Y.

The same holds good of the two equations

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6}{(x+y)^2}, \quad \frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6X_1'Y_1'}{(X_1+Y_1)^2},$$

which are essentially the same: and so in other cases.

Ex. 5. Shew that the equation

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\mu}{(x+y)^2},$$

where μ is constant, is of finite rank in each of the variables when μ is of the form

$$\mu = n\,(n+1),$$

where n is an integer (which manifestly can be taken to be positive).

Assuming that μ has this value, prove that the general integral of the equation is

$$z = \sum_{r=0}^{n}\left\{(-1)^r\,\frac{(n+r)\,!}{(n-r)\,!\;r\,!}\,(x+y)^{-r}\left(\frac{d^{n-r}X}{dx^{n-r}} + \frac{d^{n-r}Y}{dy^{n-r}}\right)\right\},$$

where X and Y are arbitrary functions of x and of y respectively.

Ex. 6. The equation

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\mu}{(x+y)^2}$$

remains unaltered when x and y are changed into $\frac{1}{x}$ and $\frac{1}{y}$: discuss the effect of these changes upon the general integral in the cases

$$\mu = 2, \quad \mu = 6.$$

Ex. 7. As an illustration of the remark (in § 218, p. 141) that an equation, of rank next greater than a given equation, may not necessarily arise when any special form ω of the general integral ζ of the given equation is chosen, consider once more the equation

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2}{(x+y)^2}\,z,$$

the general integral of which is

$$\zeta = X' + Y' - 2\frac{X+Y}{x+y}.$$

If we take ω as the form of ζ defined by

$$X = \tfrac{1}{4}x^2, \quad Y = \tfrac{1}{4}y^2,$$

we have

$$\omega = \tfrac{1}{2}(x+y) - \tfrac{1}{2}\frac{x^2+y^2}{x+y}$$

$$= \frac{xy}{x+y};$$

and then

$$\omega \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right) = 0.$$

The resulting equation for the quantity θ, being

$$\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x \partial y} = \omega \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right) = 0,$$

is actually of lower rank than the equation from which it is derived.

On the other hand, the assumption

$$X = -\frac{1}{x}, \quad Y = -\frac{1}{y},$$

does lead to an equation of higher rank.

219. The law by which equations, of finite rank in both variables and having equal invariants, can be constructed in succession, may be expressed in a different form. Let the equation of rank n be

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_n,$$

and let its general integral be

$$z = \zeta_n,$$

where ζ_n contains two arbitrary functions; and let ω_n be a form of ζ_n. Then, by Moutard's theorem, we form the expression

$$\lambda_{n+1} = \omega_n \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega_n}\right);$$

and then the equation of rank $n+1$ is

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_{n+1}.$$

Moreover,

$$\lambda_n = \frac{1}{\omega_n}\frac{\partial^2 \omega_n}{\partial x \partial y},$$

because ω_n is an integral of the equation of rank n; hence

$$\lambda_{n+1} - \lambda_n = -\frac{1}{\omega_n}\frac{\partial^2 \omega_n}{\partial x \partial y} + \frac{2}{\omega_n^2}\frac{\partial \omega_n}{\partial x}\frac{\partial \omega_n}{\partial y} - \frac{1}{\omega_n}\frac{\partial^2 \omega_n}{\partial x \partial y}$$

$$= -2\frac{\partial^2}{\partial x \partial y}(\log \omega_n).$$

Consequently

$$\lambda_n = \lambda_1 - 2\frac{\partial^2}{\partial x \partial y}\{\log(\omega_1 \omega_2 \dots \omega_{n-1})\}$$

$$= -2\frac{\partial^2}{\partial x \partial y}\{\log(\omega_1 \omega_2 \dots \omega_{n-1})\},$$

because λ_1 is zero, the equation of rank unity being

$$\frac{\partial^2 z}{\partial x \partial y} = 0.$$

Each of the quantities $\omega_1, \omega_2, \dots, \omega_{n-1}$ contains two functions that can be regarded as arbitrary: hence this expression contains $2n-2$ arbitrary functions, $n-1$ of them being functions of x, and $n-1$ of them functions of y. Now it was seen (in § 217) that the relation, satisfied by a quantity λ which belongs to an equation of rank $m+1$ in each of the variables with equal invariants, is a partial differential equation of order $2m$; hence our quantity λ_n satisfies a partial differential equation of order $2n-2$. The expression obtained for λ_n contains $2n-2$ arbitrary functions: hence so long as they are kept arbitrary, the expression provides the general integral of that partial differential equation. What, however, is of greater importance for the present purpose is that the value of λ_n is given explicitly, when we know the general integral of each equation of lower rank in the series.

Integrals of Equations having Equal Invariants.

220. The result just established renders it possible to form the equation of any finite rank: and, as is known from Moutard's theorem, its integral can be obtained from the general integral of the equation of next lower rank by a process of quadrature. The process of quadrature can be effected in general through the following simplifications.

Let the equation, supposed to be of finite rank $n+1$ in each of the variables, be

$$\frac{\partial^2 z}{\partial x \partial y} = \lambda z;$$

and let the part of the general integral ζ, which involves the arbitrary function X and its derivatives linearly, be denoted by

$$\zeta = AX^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X.$$

This part of ζ must of course satisfy the differential equation identically: hence substituting and taking account of the highest derivative of X after the substitution, we have

$$\frac{\partial A}{\partial y} = 0,$$

and therefore

$$\begin{aligned} A &= \text{function of } x \text{ alone} \\ &= \xi, \end{aligned}$$

say. Let another arbitrary function X_1 be introduced by the relation

$$X_1 = X\xi;$$

then the new form of ζ is

$$\zeta = X_1{}^{(n)} + B_1 X_1{}^{(n-1)} + \ldots + B_n X_1;$$

in other words, we can take A as equal to unity without loss of generality. Accordingly, we take

$$\zeta = X^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X.$$

Thus

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial x \partial y} &= \frac{\partial A_1}{\partial y} X^{(n)} + \sum_{r=1}^{n-1} \left\{ \left(\frac{\partial^2 A_r}{\partial x \partial y} + \frac{\partial A_{r+1}}{\partial y} \right) X^{(n-r)} \right\} + \frac{\partial^2 A_n}{\partial x \partial y} X \\ &= \lambda \{ X^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X \}; \end{aligned}$$

and denoting by ω the particular value of ζ, to be used in constructing the integral of the equation next higher in rank, we have

$$\frac{\partial^2 \omega}{\partial x \partial y} = \lambda \omega.$$

Hence

$$\begin{aligned} \frac{\partial A_1}{\partial y} = \frac{1}{A_r} \left\{ \frac{\partial^2 A_r}{\partial x \partial y} + \frac{\partial A_{r+1}}{\partial y} \right\} &= \frac{1}{A_n} \frac{\partial^2 A_n}{\partial x \partial y} \\ &= \frac{1}{\omega} \frac{\partial^2 \omega}{\partial x \partial y}, \end{aligned}$$

for $r = 1, \ldots, n-1$.

Now we had

$$\frac{\partial(\omega\theta)}{\partial x} = \omega\frac{\partial\zeta}{\partial x} - \zeta\frac{\partial\omega}{\partial x}, \qquad -\frac{\partial(\omega\theta)}{\partial y} = \omega\frac{\partial\zeta}{\partial y} - \zeta\frac{\partial\omega}{\partial y};$$

hence

$$\begin{aligned}\frac{\partial(\omega\theta)}{\partial x} &= \omega\left\{X^{(n+1)} + A_1X^{(n)} + \sum_{r=1}^{n-1}\left(\frac{\partial A_r}{\partial x} + A_{r+1}\right)X^{(n-r)} + \frac{\partial A_n}{\partial x}X\right\}\\ &\quad - \frac{\partial\omega}{\partial x}\left\{X^{(n)} + \sum_{r=1}^{n-1} A_rX^{(n-r)} + A_nX\right\}\\ &= \omega X^{(n+1)} + \rho_1X^{(n)} + \sum_{r=1}^{n-1}\rho_{r+1}X^{(n-r)} + \rho_{n+1}X,\end{aligned}$$

where

$$\rho_1 = \omega A_1 - \frac{\partial\omega}{\partial x},$$

$$\rho_{n+1} = \omega\frac{\partial A_n}{\partial x} - A_n\frac{\partial\omega}{\partial x},$$

and

$$\rho_{r+1} = \omega A_{r+1} + \omega\frac{\partial A_r}{\partial x} - A_r\frac{\partial\omega}{\partial x},$$

the last holding for $r = 1, \ldots, n-1$. Similarly,

$$\frac{\partial(\omega\theta)}{\partial y} = \sigma_1X^{(n)} + \sum_{r=1}^{n-1}\sigma_{r+1}X^{(n-r)} + \sigma_{n+1}X,$$

where

$$\sigma_1 = \frac{\partial\omega}{\partial y},$$

$$\sigma_{r+1} = A_r\frac{\partial\omega}{\partial y} - \omega\frac{\partial A_r}{\partial y},$$

the last holding for $r = 1, \ldots, n$.

As regards these quantities $\rho_1, \ldots, \rho_{n+1}, \sigma_1, \ldots, \sigma_{n+1}$, we have

$$\begin{aligned}\frac{\partial\rho_r}{\partial y} - \frac{\partial\sigma_r}{\partial x} &= \frac{\partial}{\partial y}\left(\omega A_r + \omega\frac{\partial A_{r-1}}{\partial x} - A_{r-1}\frac{\partial\omega}{\partial x}\right) - \frac{\partial}{\partial x}\left(A_{r-1}\frac{\partial\omega}{\partial y} - \omega\frac{\partial A_{r-1}}{\partial y}\right)\\ &= A_r\frac{\partial\omega}{\partial y} + \omega\frac{\partial A_r}{\partial y} + 2\left(\omega\frac{\partial^2A_{r-1}}{\partial x\partial y} - A_{r-1}\frac{\partial^2\omega}{\partial x\partial y}\right)\\ &= A_r\frac{\partial\omega}{\partial y} + \omega\frac{\partial A_r}{\partial y} - 2\omega\frac{\partial A_r}{\partial y}\\ &= \sigma_{r+1},\end{aligned}$$

on using the relations between the coefficients A given by the condition that ζ satisfies the original equation. This equation holds for $r=1, \ldots, n$; also

$$\frac{\partial \rho_{n+1}}{\partial y} - \frac{\partial \sigma_{n+1}}{\partial x} = 2\left(\omega \frac{\partial^2 A_n}{\partial x \partial y} - A_n \frac{\partial^2 \omega}{\partial x \partial y}\right)$$
$$= 0.$$

Hence

$$\sigma_{r+1} = -\frac{\partial \sigma_r}{\partial x} + \frac{\partial \rho_r}{\partial y},$$

$$-\frac{\partial \sigma_r}{\partial x} = \frac{\partial^2 \sigma_{r-1}}{\partial x^2} - \frac{\partial^2 \rho_{r-1}}{\partial x \partial y},$$

....

$$(-1)^{r-1}\frac{\partial^{r-1}\sigma_2}{\partial x^{r-1}} = (-1)^r \frac{\partial^r \sigma_1}{\partial x^r} - (-1)^r \frac{\partial^r \rho_1}{\partial x^{r-1}\partial y},$$

$$= (-1)^r \frac{\partial^{r+1}\omega}{\partial x^r \partial y} + (-1)^{r-1}\frac{\partial^r \rho_1}{\partial x^{r-1}\partial y};$$

and therefore, on adding, we have

$$\sigma_{r+1} = \frac{\partial}{\partial y}\left\{\rho_r - \frac{\partial \rho_{r-1}}{\partial x} + \ldots + (-1)^{r-1}\frac{\partial^{r-1}\rho_1}{\partial x^{r-1}} + (-1)^r \frac{\partial^r \omega}{\partial x^r}\right\},$$

for the values $r=1, \ldots, n$. Similarly, if

$$\Omega = \rho_{n+1} - \frac{\partial \rho_n}{\partial x} + \ldots + (-1)^n \frac{\partial^n \rho_1}{\partial x^n} + (-1)^{n+1}\frac{\partial^{n+1}\omega}{\partial x^{n+1}},$$

then

$$\frac{\partial \Omega}{\partial y} = 0,$$

so that Ω is a function of x alone. Let

$$\pi_r = \rho_r - \frac{\partial \rho_{r-1}}{\partial x} + \ldots + (-1)^{r-1}\frac{\partial^{r-1}\rho_1}{\partial x^{r-1}} + (-1)^r \frac{\partial^r \omega}{\partial x^r},$$

for $r=1, \ldots, n$, so that

$$\frac{\partial \pi_r}{\partial y} = \sigma_{r+1}, \qquad \pi_{r+1} + \frac{\partial \pi_r}{\partial x} = \rho_{r+1},$$

with the convention $\pi_{n+1} = \Omega$; and let ψ denote the function

$$\psi = \omega X^{(n)} + \pi_1 X^{(n-1)} + \pi_2 X^{(n-2)} + \ldots + \pi_n X.$$

Then

$$\frac{\partial(\omega\theta - \psi)}{\partial y} = 0,$$

$$\frac{\partial(\omega\theta - \psi)}{\partial x} = \Omega X.$$

Now Ω is a function of x only; hence we may take

$$\omega\theta - \psi = \int^x \Omega X \, dx,$$

that is,

$$\omega\theta = \omega X^{(n)} + \pi_1 X^{(n-1)} + \ldots + \pi_n X + \int^x \Omega X \, dx.$$

Introduce a new arbitrary function X_1 such that

$$\int^x \Omega X \, dx = \Omega X_1,$$

so that

$$X = X_1' + X_1 \frac{\Omega'}{\Omega} = X_1' + X_1 \frac{d \log \Omega}{dx};$$

then

$$X^{(r)} = X_1^{(r+1)} + X_1^{(r)} \frac{d \log \Omega}{dx} + r X_1^{(r-1)} \frac{d^2 \log \Omega}{dx^2} + \ldots,$$

for $r = 1, \ldots, n+1$; substituting these in the expression for $\omega\theta$, and dividing by ω, we have

$$\theta = X_1^{(n+1)} + B_1 X_1^{(n)} + \ldots + B_{n+1} X_1,$$

so that θ is of grade $n+2$ in x.

Similar calculations, applied to the part of ζ which involves Y and its derivatives, lead to a quantity

$$\vartheta = Y_1^{(n+1)} + C_1 Y_1^{(n)} + \ldots + C_{n+1} Y_1,$$

where Y_1 is an arbitrary function of y: and then the quantity Θ, where

$$\Theta = \theta + \vartheta,$$

is the general integral of the equation

$$\frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial x \partial y} = \omega \frac{\partial^2}{\partial x \partial y} \left(\frac{1}{\omega} \right),$$

which thus is seen to be of finite rank $n+2$ in both variables.

Note. In stating this conclusion, two assumptions have been made. One of them is that the equation

$$\Omega = 0$$

is not satisfied: if it were satisfied, then θ would only be of rank $n+1$ in the variable x instead of being of rank $n+2$. The other is that the corresponding equation, say $\Upsilon = 0$, arising in connection with the part of $\omega\theta$ which involves Y and its derivatives, also is not satisfied.

Now Ω, which has been proved to be independent of y, is really a linear combination of the derivatives of ω with regard to x; and Υ, which is independent of x, is really a linear combination of the derivatives of ω with regard to y. If ω preserves some of the generality of ζ, then neither Ω nor Υ will vanish; and the equation for θ will then be of rank $n+2$. Even if ω is made exceedingly special, *the new equation for θ will be of increased rank, provided neither Ω nor Υ vanishes: and these conditions are sufficient as well as necessary.*

Ex. 1. The last proposition can be illustrated by one more reference to the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \frac{2}{(x+y)^2},$$

having

$$\zeta = X' + Y' - 2\frac{X+Y}{x+y},$$

for its general integral.

For the calculation of Ω in this case, we have

$$n=1,$$

$$A_1 = -\frac{2}{x+y},$$

$$\rho_1 = \omega A_1 - \frac{\partial \omega}{\partial x},$$

$$\rho_2 = \omega\frac{\partial A_1}{\partial x} - A_1\frac{\partial \omega}{\partial x},$$

$$\Omega = \rho_2 - \frac{\partial \rho_1}{\partial x} + \frac{\partial^2 \omega}{\partial x^2}$$

$$= 2\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{2}{x+y}\frac{\partial \omega}{\partial x}\right);$$

and similarly

$$\Upsilon = 2\left(\frac{\partial^2 \omega}{\partial y^2} + \frac{2}{x+y}\frac{\partial \omega}{\partial y}\right).$$

Hence the particular form ω of ζ, obtained by specialising X and Y, must be such that neither of the quantities

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{2}{x+y}\frac{\partial \omega}{\partial x}, \quad \frac{\partial^2 \omega}{\partial y^2} + \frac{2}{x+y}\frac{\partial \omega}{\partial y}$$

shall vanish.

It is easy to verify that both the quantities vanish if $X = \frac{1}{4}x^2$, $Y = \frac{1}{4}y^2$: for then

$$\omega = \frac{xy}{x+y}.$$

(See Ex. 7, § 218).

Ex. 2. As a particular example of the general method of proceeding, consider the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2},$$

having the general integral

$$\zeta=X'+Y'-2\frac{X+Y}{x+y}.$$

To obtain ω, let

$$X=x^3,\ \ Y=y^3,$$

and then

$$\omega=(x+y)^2,$$

so that (see the last example)

$$\Omega=2\left(\frac{\partial^2\omega}{\partial x^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial x}\right)=12,$$

and similarly

$$\Upsilon=12;$$

thus ω will lead to an equation of higher rank, and the equation is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right)=\frac{6}{(x+y)^2}.$$

Now, in the present case,

$$n=1,\ \ A_1=\frac{-2}{x+y},$$

so that

$$\rho_1=\omega A_1-\frac{\partial\omega}{\partial x}=-4(x+y),$$

$$\rho_2=\omega\frac{\partial A_1}{\partial x}-A\frac{\partial\omega}{\partial x}=6.$$

Hence

$$\pi_1=\rho_1-\frac{\partial\omega}{\partial x}=-6(x+y),$$

and therefore

$$\begin{aligned}\psi&=\omega X'+\pi_1 X\\&=(x+y)^2X'-6(x+y)X.\end{aligned}$$

The required part θ is given by

$$\begin{aligned}(x+y)^2\theta-\psi&=\omega\theta-\psi\\&=12\int Xdx:\end{aligned}$$

or if

$$X_1=\int Xdx,$$

we have

$$\theta=X_1''-\frac{6}{x+y}X_1'+\frac{12}{(x+y)^2}X_1.$$

Similarly, the other part ϑ is given by

$$\vartheta=Y_1''-\frac{6}{x+y}Y_1'+\frac{12}{(x+y)^2}Y_1.$$

The general integral of the equation

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$\Theta=\theta+\vartheta.$$

Ex. 3. We proceed to construct the equation of rank 3 having equal invariants which, subject to change of the independent variables, is the most general.

The equation of rank 1 is

$$s=0:$$

its most general integral is (say)

$$\zeta_1=X+Y,$$

and (in the notation of § 219) we can take

$$\omega_1=x+y,$$

this assumption effectively fixing the independent variables.

The equation of rank 2 is

$$s=\frac{2}{(x+y)^2}z:$$

its most general integral is

$$\zeta_2=X'+Y'-2\frac{X+Y}{x+y},$$

and (in the notation of § 219) we can take

$$\omega_2=\xi'+\eta'-2\frac{\xi+\eta}{x+y},$$

where ξ and η are functions of x and of y respectively.

Let the equation of rank 3 be

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\lambda;$$

then, by § 219,

$$\begin{aligned}\lambda&=-2\frac{\partial^2}{\partial x\,\partial y}\{\log(\omega_1\omega_2)\}\\&=-2\frac{\partial^2}{\partial x\,\partial y}[\log\{(x+y)(\xi'+\eta')-2\xi-2\eta\}].\end{aligned}$$

We proceed to construct Θ, by the use of the preceding analysis. We have

$$X'-\frac{2}{x+y}X$$

as the part of ζ_2 involving the arbitrary function of X: thus

$$n=1,\quad A_1=\frac{-2}{x+y},$$

$$\rho_1=-\frac{2\omega_2}{x+y}-\frac{\partial\omega_2}{\partial x},\quad \rho_2=\frac{2}{(x+y)^2}\omega_2+\frac{2}{x+y}\frac{\partial\omega_2}{\partial x},$$

$$\sigma_1=\frac{\partial\omega_2}{\partial y},\quad \sigma_2=\frac{-2}{x+y}\frac{\partial\omega_2}{\partial y}-\frac{2}{(x+y)^2}\omega_2.$$

Consequently,

$$\pi_1 = \rho_1 - \frac{\partial \omega_2}{\partial x}$$

$$= -2\frac{\omega_2}{x+y} - 2\frac{\partial \omega_2}{\partial x}$$

$$= 2\left(\frac{\xi' - \eta'}{x+y} - \xi''\right),$$

on reduction: also

$$\Omega = \rho_2 - \frac{\partial \rho_1}{\partial x} + \frac{\partial^2 \omega_2}{\partial x^2}$$

$$= 2\xi''',$$

on reduction. Hence

$$\omega_2\theta = \omega_2 X' + \pi_1 X + 2\int X\xi''' dx,$$

which gives the part of θ depending upon the arbitrary function of x. To exhibit the rank more clearly, we take

$$\int X\xi''' dx = X_1\xi''',$$

and we have

$$\theta = X_1'' + a_1 X_1' + a_2 X_1,$$

where

$$a_1 = \frac{\pi_1}{\omega_2} + \frac{d\log\xi'''}{dx}$$

$$= \frac{2}{\omega_2}\left(\frac{\xi' - \eta'}{x+y} - \xi''\right) + \frac{d\log\xi'''}{dx},$$

$$a_2 = \frac{2\xi'''}{\omega_2} + \frac{\pi_1}{\omega_2}\frac{d\log\xi'''}{dx} + \frac{d^2\log\xi'''}{dx^2}.$$

Similarly, the part of Θ depending upon the arbitrary function of Y is

$$\vartheta = Y_1'' + \beta_1 Y_1' + \beta_2 Y_1,$$

where

$$\beta_1 = \frac{\tau_1}{\omega_2} + \frac{d\log\eta'''}{dy}$$

$$= \frac{2}{\omega_2}\left(\frac{\eta' - \xi'}{x+y} - \eta''\right) + \frac{d\log\eta'''}{dy},$$

$$\beta_2 = \frac{2\eta'''}{\omega_2} + \frac{\tau_1}{\omega_2}\frac{d\log\eta'''}{dy} + \frac{d^2\log\eta'''}{dy^2}.$$

The value of Θ is

$$\Theta = \theta + \vartheta$$

$$= X_1'' + a_1 X_1' + a_2 X_1 + Y_1'' + \beta_1 Y_1' + \beta_2 Y_1;$$

and it is actually of rank three in each of the variables, provided neither ξ''' nor η''' vanishes, that is, provided neither ξ nor η is a quadratic polynomial in the respective variables.

Ex. 4. Prove that, on taking

$$\xi = x^3, \quad \eta = y^3$$

in the preceding example, the equation for Θ is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y} = \frac{6}{(x+y)^2};$$

and that, on taking

$$\xi = x^4, \quad \eta = y^4,$$

in the preceding example, the equation for Θ is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y} = \frac{2}{(x^3+y^3)^2}(x^4 - 2x^3y + 12x^2y^2 - 2xy^3 + y^4).$$

Deduce the integrals of each of these equations.

Ex. 5. Verify that

$$\omega = (x-y)^m (x+y)^n$$

satisfies the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{m(m-1)}{(x-y)^2} + \frac{n(n-1)}{(x+y)^2};$$

and apply Moutard's theorem to prove that the equation can be integrated in finite terms when m and n are integers. (The integers evidently can be taken positive.)

Obtain the integral in the form

$$z = \frac{\partial^{m+n}}{\partial u^m \partial v^n}\{\phi(u^{\frac{1}{2}} + v^{\frac{1}{2}}) + \psi(u^{\frac{1}{2}} - v^{\frac{1}{2}})\}$$

where $u = (x-y)^2$, $v = (x+y)^2$, and ϕ, ψ are arbitrary functions. (Darboux.)

Ex. 6. Prove that, if an integral

$$\omega = \xi'' + \eta'' - 6\frac{\xi' + \eta'}{x+y} + 12\frac{\xi + \eta}{(x+y)^2}$$

of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6}{(x+y)^2}$$

is used, by Moutard's theorem, to construct another equation, the equation so constructed will not be of rank 4 in both variables unless the quantities

$$\frac{d^5\xi}{dx^5}, \quad \frac{d^5\eta}{dy^5}$$

are different from zero.

Assuming that ξ and η are not quartic polynomials in their respective variables, form the equation of rank 4 and obtain its general integral.

Ex. 7. Integrate the equations:—

(i) $\quad r - t = 2n\dfrac{p}{x}$;

(ii) $\quad c^2 x^{\frac{4}{3}} r - t = 0$;

(iii) $\quad (2n+1)^2 x^{\frac{4n}{2n+1}} r - t = 0$;

where n is an integer in (i) and in (iii). (Sersawy, Winckler.)

Ex. 8. Shew that the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y)$$

possesses an infinitude of integrals of the form*

$$\omega=f(x+y)\,g(x-y),$$

where the forms of the functions f and g are determined by the ordinary linear equations

$$\frac{f''(t)}{f(t)}=\phi(t)+a,\quad \frac{g''(t)}{g(t)}=\psi(t)+a,$$

the quantity a being a constant.

Denoting any other integral of the equation by

$$z=F(x+y)\,G(x-y),$$

prove that the integral of the equation

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial x\,\partial y}=\omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right),$$

which is derived from z by Moutard's theorem, is

$$Z=\left(F'-F\frac{f'}{f}\right)\left(G'-G\frac{g'}{g}\right).$$

(Darboux.)

Ex. 9. With the same notation as in the last example, shew that the general integral Z of the equation

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial x\,\partial y}=\frac{2f'^2-ff''}{f^2}-\psi(x-y)-a$$

is given by

$$Z=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}-2z\frac{f'}{f},$$

where z is the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y),$$

the quantity a being a constant, and the form f of $f(x+y)$ being determined by

$$\frac{f''(t)}{f(t)}=\phi(t)+a.$$

(Darboux.)

Ex. 10. Prove that, if ω be any integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y),$$

then

$$\Omega=\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}-2\omega\{\phi(x+y)+\psi(x-y)\}$$

is another integral of the equation. (Darboux.)

* The equation is called a *harmonic* equation: an integral, such as ω, is called a *harmonic* integral.

Ex. 11. Shew that, if the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\lambda(x, y)=\lambda$$

can be a harmonic equation, so that it is expressible in the form

$$\frac{1}{z}\frac{\partial^2 z}{\partial x'\,\partial y'}=\phi(x'+y')-\psi(x'-y'),$$

then the independent variables are given by

$$x'=\int X^{-\frac{1}{2}}dx,\quad y'=\int Y^{-\frac{1}{2}}dy,$$

where the equation

$$2X\frac{\partial^2\lambda}{\partial x^2}+3X'\frac{\partial\lambda}{\partial x}+\lambda X''=2Y\frac{\partial^2\lambda}{\partial y^2}+3Y'\frac{\partial\lambda}{\partial y}+\lambda Y''$$

must be satisfied.

When the equation in question is Euler's equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{m(1-m)}{(x-y)^2},$$

where m is a constant, prove that X is a quartic polynomial in x and that Y is the same quartic polynomial in y; and obtain the various forms of equation according to the equalities of the roots of X.

(Darboux.)

Ex. 12. Shew that the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{m(m-1)}{(x+y)^2}-\frac{n(n-1)}{(x-y)^2}+\frac{m'(m'-1)}{(1-xy)^2}-\frac{n'(n'-1)}{(1+xy)^2},$$

where m, n, m', n' are constants, can be made harmonic. (Darboux.)

CHAPTER XV.

FORMS OF EQUATIONS OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES HAVING THEIR GENERAL INTEGRALS IN EXPLICIT FINITE FORM.

THE substance of this chapter is based entirely on Cosserat's proof of the theorem enunciated by Moutard: the proof is contained in Note III at the end (pp. 405—422) of the fourth volume of Darboux's *Théorie générale des surfaces*, published in 1896.

It should be added that these investigations are the matter of a paper* by Tanner whose results, expressed in a slightly different form, are a clear anticipation of many of Cosserat's results. There is a difference in notation, but it is unessential to the main properties: Tanner uses the successive integrals of an arbitrary function of x or of y, instead of the successive derivatives. When the corresponding changes are made in the notation, a comparison of the two sets of results is easy and immediate.

Reference may also be made to a memoir by Goursat †.

221. In the two preceding chapters, we have been occupied mainly with the discussion of linear equations of the second order the integrals of which, whether half general or completely general, are expressible in finite form; and in each instance, it is the equation that is given while it is the integral that has to be determined.

But a different point of view may be adopted: as was the case with early investigations of classical analysts such as Euler and Lagrange, we may consider equations as determined by their integrals. In particular, we shall consider those equations of the second order the integral of which is composed of a single relation

* *Proc. L. M. S.*, t. VIII (1876), pp. 159—174.

† *Ann. de Toulouse*, 2me Sér., t. I (1899), pp. 31—78.

between x, y, z; this relation is to involve a couple of arbitrary functions, as well as derivatives of these arbitrary functions up to specified orders which are finite for each of them; the arguments of the arbitrary functions are to be given explicitly in terms of the variables, and are to be distinct from one another; and all expressions in the relation are to be free from partial quadratures. What is required is the aggregate of equations possessing integrals of this type: they are subject to the following theorem, first enunciated* by Moutard:—

Equations of the second order, which have an integral of the type indicated and which, by a transformation of the variables, cannot be expressed in Laplace's linear form

$$s + ap + bq + cz = 0,$$

or in Liouville's form

$$s = e^z,$$

are, with two simple exceptions, reducible to the form

$$s = \frac{\partial}{\partial x}(Ae^z) - \frac{\partial}{\partial y}(Be^{-z}),$$

where A and B are functions of the independent variables alone, satisfying certain conditions; and the integration of this equation can be made to depend uniquely upon the integration of

$$\frac{\partial^2 \alpha}{\partial x \partial y} = \frac{1}{A}\frac{\partial A}{\partial x}\frac{\partial \alpha}{\partial y} + AB\alpha,$$

which is of the linear type considered by Laplace.

In the preceding statement of the type of integral relation which is to lead to a partial differential equation of the second order, it has been laid down as a condition that the arguments of the two arbitrary functions are distinct from one another. This condition is effectively bound up with the finiteness of the form of the integral relation. For if the two arguments be the same, let a change in the independent variables be made whereby this common argument becomes the variable y; then the only deriva-

* The proof of the theorem formed the first part of the memoir, mentioned on p. 139, note, which was destroyed in 1871. The author did not rewrite this part; the statement of the theorem is taken from the abstract, as given in the *Comptes Rendus*, t. LXX (1870), pp. 834—838, and as reproduced in the *Journ. de l'Éc. Poly.*, t. XXXVII (1886), pp. 1—5.

tive of the second order that can occur is r. Suppose the differential equation resolved for r, and let the resolved form be

$$r = f(x, y, z, p, q).$$

If z is finite and explicit in form, so that derivatives of η and Y (the two arbitrary functions of y that occur in z) of only finite order occur in z, then (as in § 186) q contains derivatives of one order higher than those that occur in r, z, p: thus, if the equation is to be satisfied, q cannot occur in f. An equation

$$r = f(x, y, z, p)$$

is effectively an ordinary equation, not a partial equation*.

Accordingly, we adhere to the condition that, for the present purpose, the arguments of the two arbitrary functions are to be different from one another.

Cosserat's Proof of Moutard's Theorem.

222. The following process of establishing the theorem is due† to Cosserat.

Let x' and y' denote the arguments of the arbitrary functions in the integral relation; they are to be definite and explicit quantities involving x, y and (it may be) z; and they will be assumed different from one another. Let $\phi(x')$ and $\psi(y')$ be the arbitrary functions; and suppose that $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n$ are all the derivatives of ϕ and ψ that occur, the suffix in each case being the order of derivation with regard to the argument. The integral relation can be expressed in a form

$$F(x, y, z, \phi, \phi_1, \ldots, \phi_m, \psi, \psi_1, \ldots, \psi_n) = 0.$$

Change the independent variables so that they become x' and y'; denote ϕ, a function of one of the independent variables alone, by ξ and denote ψ, a function of the other independent variable alone, by η; and suppose that the equation, after the transforma-

* See also, on this argument, § 208.

† In Note III, pp. 405—422, added at the end of the fourth volume of Darboux's *Théorie générale des surfaces*.

tions have been effected, is resolved with regard to z. Then it can be taken in a form

$$z = f(x', y', \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n)$$
$$= f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n),$$

on dropping the dashes; and this relation is to be the general integral of an equation of the second order. As the arguments of the arbitrary functions in the integral are x and y respectively, the only derivative of the second order that can be expected (§ 186) to occur in the equation is s.

We have

$$p = \frac{\partial f}{\partial x} + \sum_{i=0}^{m} \frac{\partial f}{\partial \xi_i} \xi_{i+1}, \qquad q = \frac{\partial f}{\partial y} + \sum_{j=0}^{n} \frac{\partial f}{\partial \eta_j} \eta_{j+1},$$

$$s = \frac{\partial^2 f}{\partial x \partial y} + \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x \partial \eta_j} \eta_{j+1} + \sum_{i=0}^{m} \frac{\partial^2 f}{\partial y \partial \xi_i} \xi_{i+1} + \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial \xi_i \partial \eta_j} \xi_{i+1} \eta_{j+1}:$$

as already explained, r and t will not occur, all the more obviously because r alone contains ξ_{m+2} and t alone contains η_{n+2}, neither of which quantities occurs in z, p, q, s, and neither of which could be eliminated among these derivatives. To construct the differential equation, the two arbitrary functions and their derivatives have to be eliminated: eliminating ξ_{m+1} and η_{n+1}, which do not occur in z, we have an equation of the form

$$s + \zeta pq + ap + bq + c = 0,$$

where ζ, a, b, c are functions of x and y. When the other derivatives of ξ and η are eliminated by means of z, this equation being the required differential equation of the second order, ζ, a, b, c are functions of x, y, and z. Evidently,

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} + \zeta \frac{\partial f}{\partial \xi_m} \frac{\partial f}{\partial \eta_n} = 0,$$

so that, in its first form, ζ is a function of x and y only and involves no quantities that do not occur in z, and in its final form ζ is a function of x, y, z.

Let the value of $\int \zeta dz$ be obtained, on the hypothesis that x and y are constant through the quadrature: and writing

$$z_0 = e^{\int \zeta dz},$$

let a new variable Z be introduced, defined by the relation

$$Z = \int z_0 dz,$$

with the same hypothesis for the quadrature as before, so that Z is a function of x, y, z, which clearly will be of the same general type as the original quantity z. Taking Z as a new dependent variable, we have

$$P = pz_0 + \int \frac{\partial z_0}{\partial x} dz,$$

$$Q = qz_0 + \int \frac{\partial z_0}{\partial y} dz,$$

$$S = sz_0 + q\left(p\frac{\partial z_0}{\partial z} + \frac{\partial z_0}{\partial x}\right) + p\frac{\partial z_0}{\partial y} + \int \frac{\partial^2 z_0}{\partial x \partial y} dz,$$

$$= z_0 (s + pq\zeta) + q\frac{\partial z_0}{\partial x} + p\frac{\partial z_0}{\partial y} + \int \frac{\partial^2 z_0}{\partial x \partial y} dz;$$

and the quantities, for which quadrature with regard to z is required, are functions of x, y, z. Thus the differential equation for Z is of the form

$$S + AP + BQ + C = 0,$$

when z is eliminated from the coefficients by the relation

$$Z = \int z_0 dz.$$

The integral Z of the equation is of the same type as before. We may therefore make $\zeta = 0$; and thus we have to determine a, b, c as functions of x, y, z alone, such that the equation

$$s + ap + bq + c = 0$$

and the relation

$$z = f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n)$$

are equivalent to one another. But, as ζ now is zero, there is the limitation

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$$

upon the form of the integral relation.

THREE TYPES OF COEFFICIENTS a AND b.

223. Let the deduced expressions for s, p, q be substituted in the equation

$$s + ap + bq + c = 0,$$

which is to be satisfied identically. The term, which involves $\xi_{m+1}\eta_{n+1}$, arises through s alone: it disappears, owing to the foregoing limitation upon the form of f. The term, which involves ξ_{m+1}, must disappear: thus

$$\frac{\partial^2 f}{\partial y \partial \xi_m} + \sum_{\beta=0}^{n-1} \frac{\partial^2 f}{\partial \xi_m \partial \eta_\beta} \eta_{\beta+1} + a \frac{\partial f}{\partial \xi_m} = 0,$$

and this must be satisfied identically when the value of z is substituted in a. The term, which involves η_{n+1}, must disappear: thus

$$\frac{\partial^2 f}{\partial x \partial \eta_n} + \sum_{\alpha=0}^{m-1} \frac{\partial^2 f}{\partial \xi_\alpha \partial \eta_n} \xi_{\alpha+1} + b \frac{\partial f}{\partial \eta_n} = 0,$$

and this must be satisfied identically when the value of z is substituted in b.

It follows, from the condition $\dfrac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$, that $\dfrac{\partial f}{\partial \xi_m}$ does not contain η_n; hence* $\dfrac{\partial^2 f}{\partial y \partial \xi_m}$ does not contain η_n. The first of the two equations, which become identities when the value of z is substituted in a and in b, contains a term

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} \eta_n;$$

consequently a involves η_n only in the first power after the value of z has been substituted. The equation is then satisfied identically; hence, differentiating with regard to η_n, we have

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} \frac{\partial a}{\partial z} \frac{\partial f}{\partial \eta_n} = 0,$$

also satisfied identically. Now $\dfrac{\partial f}{\partial \xi_m}$ does not contain η_n, and

* The quantity $\dfrac{\partial^2 f}{\partial y \partial \xi_m}$ is only the partial, not the complete, derivative of $\dfrac{\partial f}{\partial \xi_m}$ with regard to y.

therefore $\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}}$ does not contain it: hence, differentiating the last identity with regard to η_n, we have

$$\frac{\partial}{\partial \eta_n}\left(\frac{\partial a}{\partial z}\frac{\partial f}{\partial \eta_n}\right) = 0,$$

that is,

$$\frac{\partial a}{\partial z}\frac{\partial^2 f}{\partial \eta_n{}^2} + \frac{\partial^2 a}{\partial z^2}\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0,$$

which is to be satisfied identically when the value of z is substituted in $\frac{\partial a}{\partial z}$ and $\frac{\partial^2 a}{\partial z^2}$.

This relation clearly will be satisfied identically, if a does not involve z: we thus have one possible case. For other cases, we have

$$\frac{\dfrac{\partial^2 a}{\partial z^2}}{\dfrac{\partial a}{\partial z}} = -\frac{\dfrac{\partial^2 f}{\partial \eta_n{}^2}}{\left(\dfrac{\partial f}{\partial \eta_n}\right)^2}.$$

Now, $\frac{\partial f}{\partial \eta_n}$ does not involve ξ_m because of the relation $\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$; hence the right-hand fraction, when expressed in terms of x, y, z, cannot involve z. If the fraction is zero, then

$$a = \mu + \lambda z,$$

where μ and λ can be functions of x and y; and then

$$\frac{\partial^2 f}{\partial \eta_n{}^2} = 0.$$

If the fraction is not zero, then

$$a = \mu + \lambda e^{z\rho},$$

where μ, λ, ρ can be functions of x and y; and then

$$\frac{\partial^2 f}{\partial \eta_n{}^2} + \rho\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0.$$

Consequently, *there are three possible forms for* a, viz.

$$\mu, \quad \mu + \lambda z, \quad \mu + \lambda e^{z\rho},$$

with corresponding limitations upon the form of f for the second and the third; and the quantities λ, μ, ρ can be functions of the independent variables.

Proceeding in the same way from the other of the equations which become identities when the value of z is substituted, we find that *there are three possible forms for b,* of the same type as those for a, and accompanied by the corresponding limitations on the form of f for the second and the third.

Ex. 1. Given an integral relation

$$F(u_1, \ldots, u_r, x, y, z)=0,$$

where $u_1, \ldots, u_r$ are r arbitrary functions, each having one definite argument, and where F is a definite function, shew that, if differential relations equivalent to $F=0$ are formed, the lowest aggregate will generally consist of r equations of order $2r-1$. (Falk.)

Ex. 2. Shew that, if

$$z=F(u, v)$$

be an integral equation in which F is a determinate function, and u, v are arbitrary functions of definite arguments, then z satisfies a differential equation of the second order if (and only if)

$$\frac{\dfrac{\partial^2 F}{\partial u\,\partial v}}{\dfrac{\partial F}{\partial u}\,\dfrac{\partial F}{\partial v}}$$

is independent of u and v or can be made independent of u and v by means of the primitive relation. (Falk.)

224. We proceed to consider the possible combinations. In the first place, let

$$a=\mu+\lambda e^{z\rho};$$

we then have

$$\frac{\partial^2 f}{\partial \eta_n{}^2}+\rho\left(\frac{\partial f}{\partial \eta_n}\right)^2=0,$$

so that, as ρ is a function of x and y only and as $\dfrac{\partial f}{\partial \eta_n}$ does not involve ξ_m, we have

$$\frac{\partial f}{\partial \eta_n}=\frac{1}{\rho\,(\eta_n+f_2)},$$

where f_2 is a quantity that does not involve ξ_m or η_n; and therefore

$$e^{z\rho}=e^{f\rho}=(\eta_n+f_2)\,F,$$

where F is a quantity that does not involve η_n. Now the value of b is to satisfy the equation

$$\frac{\partial^2 f}{\partial x\partial \eta_n}+\sum_{a=0}^{m-1}\frac{\partial^2 f}{\partial \xi_a\,\partial \eta_n}\,\xi_{a+1}+b\,\frac{\partial f}{\partial \eta_n}=0,$$

so that, on substituting the obtained value of $\dfrac{\partial f}{\partial \eta_n}$, we have

$$\left(b - \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)(\eta_n + f_2) = \frac{\partial f_2}{\partial x} + \sum_{\alpha=0}^{m-1} \frac{\partial f_2}{\partial \xi_\alpha}\xi_{\alpha+1},$$

and therefore

$$\left(b - \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)e^{z\rho} = \left(\frac{\partial f_2}{\partial x} + \sum_{\alpha=0}^{m-1} \frac{\partial f_2}{\partial \xi_\alpha}\xi_{\alpha+1}\right)F.$$

The right-hand side does not involve η_n: hence the left-hand side, when expressed in terms of the variables, must be explicitly free from z. It is therefore either zero or, if not zero, a function of x and y at the utmost.

If the right-hand side is not zero, let its value be denoted by σ, where σ can be a function of x and y; then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho}.$$

If the right-hand side is zero, then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x}.$$

These are the two forms for b which can be associated with $\mu + \lambda e^{z\rho}$ as the value for a: the combination

$$a = \mu + \lambda e^{z\rho}, \quad b = \mu' + \lambda' z,$$

is not possible. We take the two possible forms in turn.

First Combination of Coefficients.

225. Suppose that

$$a = \mu + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho},$$

the form of b being derived from the assumed form of a. A precisely similar argument can be used to deduce the form of a from an assumed form of b: we merely need to change the sign of ρ and to interchange the independent variables, and we have

$$\mu = \frac{1}{-\rho}\frac{\partial(-\rho)}{\partial y} = \frac{1}{\rho}\frac{\partial \rho}{\partial y}.$$

Moreover, as regards z, on taking account of these interchanges and changes, we have

$$e^{-z\rho} = e^{-f\rho} = (\xi_m + f_1)\, G,$$

where f_1 is a quantity that does not involve ξ_m or η_n, and G is a quantity that does not involve ξ_m; and also

$$\left(a - \frac{1}{\rho}\frac{\partial \rho}{\partial y}\right)(\xi_m + f_1) = \frac{\partial f_1}{\partial y} + \sum_{\beta=0}^{n-1} \frac{\partial f_1}{\partial \eta_\beta}\eta_{\beta+1}.$$

The two expressions for z, given by

$$e^{z\rho} = (\eta_n + f_2)\, F, \quad e^{-z\rho} = (\xi_m + f_1)\, G,$$

can be combined into the single expression

$$e^{z\rho} = \frac{\eta_n + f_2}{\xi_m + f_1} e^{f_3},$$

where now the quantities f_1, f_2, f_3 do not involve either ξ_m or η_n. Also, from the equation

$$\left(b - \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)(\eta_n + f_2) = \frac{\partial f_2}{\partial x} + \sum_{\alpha=0}^{m-1} \frac{\partial f_2}{\partial \xi_\alpha}\xi_{\alpha+1}$$

$$= \frac{d}{dx}(\eta_n + f_2),$$

we have

$$\sigma e^{-z\rho} = \frac{d}{dx}\{\log(\eta_n + f_2)\};$$

and, similarly,

$$\lambda e^{z\rho} = \frac{d}{dy}\{\log(\xi_m + f_1)\}.$$

The differential equation becomes, on substitution for a and b,

$$\frac{\partial^2 z}{\partial x \partial y} + \left(\lambda e^{z\rho} + \frac{1}{\rho}\frac{\partial \rho}{\partial y}\right)\frac{\partial z}{\partial x} + \left(\sigma e^{-z\rho} + \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)\frac{\partial z}{\partial y} + c = 0;$$

and so, introducing a new dependent variable Z such that

$$Z = z\rho,$$

we have

$$\frac{\partial^2 Z}{\partial x \partial y} + \frac{\partial}{\partial x}(\lambda e^Z) - \frac{\partial}{\partial y}(\sigma e^{-Z}) + C = 0,$$

where C is a function of x, y and (possibly) Z; and the integral of this equation is

$$Z = \log\left(\frac{\eta_n + f_2}{\xi_m + f_1}\right) + f_3.$$

Substituting this value of Z in the deduced equation, and taking account of the earlier values of $\lambda e^{z\rho}$ and $\sigma e^{-z\rho}$, we have

$$C = -\frac{\partial^2 f_3}{\partial x \partial y}.$$

Now, as f_3 does not explicitly involve either ξ_m or η_n, the right-hand side can involve both of them in a term

$$-\frac{\partial^2 f_3}{\partial \xi_{m-1} \partial \eta_{n-1}} \xi_m \eta_n,$$

and into no other term do both ξ_m and η_n enter. If C should involve Z, the quantities ξ_m and η_n enter into its expression (when substitution takes place) only through the fraction

$$\frac{\eta_n + f_2}{\xi_m + f_1}.$$

The two forms are incompatible; hence C is a function of x and y only. Consequently $\frac{\partial^2 f_3}{\partial x \partial y}$ also is a function of x and y only; and therefore we may take

$$f_3 = g_3 + X + Y,$$

where g_3 is a specific function of x and y only, free from the arbitrary functions ξ and η, while X and Y are any functions we please of ξ and of η respectively, subject to the limitation that f_3 does not involve ξ_m nor η_n. Let z' be a new variable, where

$$z' = Z - g_3 - X - Y,$$

and let the factors $e^{g_3 + X + Y}$ and $e^{-g_3 - X - Y}$ be absorbed into λ and σ respectively, making them A and B respectively. Then the equation takes the form

$$\frac{\partial^2 z'}{\partial x \partial y} + \frac{\partial}{\partial x}(A e^{z'}) - \frac{\partial}{\partial y}(B e^{-z'}) = 0;$$

and its integral is

$$z' = \log\left(\frac{\eta_n + f_2}{\xi_m + f_1}\right),$$

that is,

$$e^{z'} = \frac{\eta_n + f_2}{\xi_m + f_1},$$

where f_1 and f_2 do not involve either ξ_m or η_n. The quantities A and B are functions of x and y only; and

$$A e^{z'} = \frac{d}{dy}\{\log(\xi_m + f_1)\},$$

that is,

$$A\,(\eta_n + f_2) = \frac{d}{dy}\,(\xi_m + f_1);$$

and, similarly,

$$B\,(\xi_m + f_1) = \frac{d}{dx}\,(\eta_n + f_2).$$

The differential equation is to be free from any expression of the arbitrary functions, so that A and B are functions of x and y only; these two equations limit the forms of f_1 and f_2, and they may impose conditions upon A and B. If with given coefficients A and B, satisfying the conditions (if any), the values of f_1 and f_2 can be obtained, then the integral of the original equation can be regarded as known.

Assuming that the conditions affecting A and B are satisfied, we evidently have

$$\frac{d}{dx}\left\{\frac{1}{A}\frac{d}{dy}\,(\xi_m + f_1)\right\} = B\,(\xi_m + f_1);$$

so that, if

$$u = \xi_m + f_1,$$

u is an integral of the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{A}\frac{\partial A}{\partial x}\frac{\partial u}{\partial y} = ABu,$$

which is of Laplace's linear form. Similarly, if

$$v = \eta_n + f_2,$$

v is an integral of the equation

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial v}{\partial x} = ABv,$$

also of Laplace's linear form. Moreover,

$$Av = \frac{\partial u}{\partial y}, \qquad Bu = \frac{\partial v}{\partial x},$$

so that, if the integral of either equation can be obtained, the integral of the other will be known; and therefore the integration of the equation

$$\frac{\partial^2 z'}{\partial x \partial y} + \frac{\partial}{\partial x}\,(Ae^{z'}) - \frac{\partial}{\partial y}\,(Be^{-z'}) = 0,$$

where A and B satisfy the appropriate conditions, can be made to depend upon the integration of an equation of Laplace's linear form.

Thus the combination

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y} + \lambda e^{\rho z}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-\rho z},$$

leads to the establishment of part of the theorem enunciated.

Note. The relation between the differential equation and the two linear relations can be exhibited in a different form. The differential equation is

$$\frac{\partial}{\partial x}\left(\frac{\partial z'}{\partial y} + Ae^{z'}\right) = \frac{\partial}{\partial y}(Be^{-z'})$$

$$= \frac{d}{dy}\left\{\frac{d}{dx}\log(\eta_n + f_2)\right\}$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{v}\frac{\partial v}{\partial x}\right),$$

so that we may take

$$\frac{\partial z'}{\partial y} + Ae^{z'} = \frac{1}{v}\frac{\partial v}{\partial y},$$

that is,

$$\frac{\partial z'}{\partial y} + \frac{1}{u}\frac{\partial u}{\partial y} = \frac{1}{v}\frac{\partial v}{\partial y}.$$

Similarly, we have

$$\frac{\partial z'}{\partial x} - Be^{-z'} = -\frac{1}{u}\frac{\partial u}{\partial x},$$

that is,

$$\frac{\partial z'}{\partial x} - \frac{1}{v}\frac{\partial v}{\partial x} = -\frac{1}{u}\frac{\partial u}{\partial x};$$

both of these are immediate inferences from the integral of the differential equation, which is

$$z' = \log\left(\frac{v}{u}\right).$$

Ex. 1. The simplest case of all is provided by $m=0$, $n=0$: but it is trivial, for we easily see that

$$A=0, \quad B=0,$$

and the equation is merely

$$s=0.$$

Ex. 2. The case, next in simplicity, is provided by taking

$$m=0, \quad n=1:$$

the case $m=1$, $n=0$, can be derived from it, by an interchange of the independent variables and a change in the sign of z. Then

$$e^z = \frac{\eta' + f_2}{\xi + f_1},$$

where f_1 and f_2 may involve η but not ξ. Writing

$$u=\xi+f_1, \quad v=\eta'+f_2,$$

we have

$$\beta(\xi+f_1)=\frac{\partial v}{\partial x}=\frac{\partial f_2}{\partial x},$$

and f_2 does not involve ξ: hence β is zero, and f_2 does not involve x. Also

$$a(\eta'+f_2)=\frac{\partial u}{\partial y}=\frac{\partial f_1}{\partial \eta}\eta'+\frac{\partial f_1}{\partial y},$$

so that

$$f_1=a\eta+\theta,$$

and then

$$af_2=\eta\frac{\partial a}{\partial y}+\frac{\partial \theta}{\partial y},$$

that is,

$$f_2=\eta\frac{1}{a}\frac{\partial a}{\partial y}+\frac{1}{a}\frac{\partial \theta}{\partial y}.$$

Now f_2 is to be independent of x: thus we must have

$$a=g(x)\,h'(y),$$
$$\theta=g(x)\,k(y),$$

and then

$$f_2=\eta\frac{h''(y)}{h'(y)}+\frac{k'(y)}{h'(y)},$$
$$f_1=\{\eta h'(y)+k(y)\}\,g(x);$$

the equation is

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\{e^z g(x)\,h'(y)\}=0.$$

Without any loss of generality, we may take

$$h'(y)=1,$$

and $k(y)$ can be absorbed into η: hence the type of equation is

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\{e^z g(x)\}=0,$$

and its general integral is

$$e^z=\frac{\eta'}{\xi+\eta g(x)},$$

where ξ and η are arbitrary functions of x and of y respectively.

All other equations of this type, for which $m=0$, $n=1$, are deducible from the foregoing by transformations of the variables

$$x'=\phi(x), \quad y'=\psi(y).$$

Ex. 3. Next, consider the case

$$m=0, \quad n=2;$$

the case $m=2$, $n=0$ can be deduced from it by changing the sign of z and interchanging the independent variables. For the present, we have

$$e^z=\frac{v}{u},$$

where

$$v=\eta''+f_2, \quad u=\xi+f_1\ ;$$

in this case, f_1 and f_2 may contain η and η' but not ξ. Also

$$\alpha v=\frac{\partial u}{\partial y}, \quad \beta u=\frac{\partial v}{\partial x}.$$

As f_2 does not contain ξ, the last equation shews that

$$\beta=0, \quad \frac{\partial f_2}{\partial x}=0,$$

so that f_2 is a function of y only. Again,

$$\begin{aligned} a(\eta''+f_2)=av&=\frac{\partial u}{\partial y} \\ &=\frac{\partial f_1}{\partial \eta'}\eta''+\frac{\partial f_1}{\partial \eta}\eta'+\frac{\partial f_1}{\partial y}\ ; \end{aligned}$$

and therefore, as f_1 does not contain η'', we have

$$f_1=a\eta'+g_1,$$

where now g_1 may contain η but does not contain η'. Also, on substituting this value of f_1, we have

$$af_2=\eta'\left(\frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial \eta}\right)+\frac{\partial g_1}{\partial y}.$$

Hence, as f_2 is a function of y only, we must have

$$\left.\begin{aligned} \frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial \eta}&=ay_1 \\ \frac{\partial g_1}{\partial y}&=ay_2 \end{aligned}\right\},$$

where y_1 and y_2 are functions of y only. Hence

$$\begin{aligned} \frac{\partial^2 a}{\partial y^2}&=\frac{\partial}{\partial y}(ay_1)-\frac{\partial}{\partial \eta}(ay_2) \\ &=\frac{\partial}{\partial y}(ay_1)-a\frac{\partial y_2}{\partial \eta}\ ; \end{aligned}$$

hence, as a is independent of η, we may take y_2 and g_1 as linear in η, and y_1 as independent of η. Consequently, if

$$\begin{aligned} v&=\eta''+y_1\eta'+P\eta+Q, \\ u&=\xi\ +a\eta'\ +R\eta+S, \end{aligned}$$

where P and Q involve y alone, the relation

$$av=\frac{\partial u}{\partial y}$$

leads to

$$ay_1=\frac{\partial a}{\partial y}+R, \quad aP=\frac{\partial R}{\partial y}, \quad aQ=\frac{\partial S}{\partial y}.$$

We may make Q and S zero; for let

$$\eta=H-\bar{\eta},$$

where H is a new arbitrary function of y, and $\bar{\eta}$ is a function of y at our disposal. The new values of v and u are

$$v = H'' + y_1 H' + PH + (Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta}),$$
$$u = \xi \quad + aH' + RH + (S - a\bar{\eta}' - R\bar{\eta}).$$

Now

$$a(Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta}) = \frac{\partial S}{\partial y} - a\bar{\eta}'' - \frac{\partial a}{\partial y} a\bar{\eta}' - R\eta' - \bar{\eta}\frac{\partial R}{\partial y}$$
$$= \frac{\partial}{\partial y}(S - a\bar{\eta}' - R\bar{\eta}).$$

Choose $\bar{\eta}$, so that

$$Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta} = 0\,;$$

then

$$S - a\bar{\eta}' - R\bar{\eta}$$

is a function of x only, and it can be absorbed into ξ. Thus we may take

$$\left.\begin{aligned} v &= \eta'' + y_1\eta' + P\eta \\ u &= \xi \ + a\eta' + R\eta \end{aligned}\right\},$$

with the relations

$$ay_1 = \frac{\partial a}{\partial y} + R, \quad aP = \frac{\partial R}{\partial y}\,;$$

the equation for a is

$$\frac{\partial^2 a}{\partial y^2} = \frac{\partial}{\partial y}(ay_1) - aP,$$

so that, as y_1 and P are functions of x only, we have

$$a = g(x)\,\alpha(y) + h(x)\,b(y),$$

where g and h are functions of x only, α and b are functions of y only. Then

$$\alpha'' = y_1\alpha' + (y_1' - P)\,\alpha,$$
$$b'' = y_1 b' + (y_1' - P)\,b,$$

and therefore

$$y_1 = \frac{\alpha'' b - \alpha b''}{\alpha' b - \alpha b'} = \frac{\partial}{\partial y}\log(\alpha' b - \alpha b').$$

Hence

$$R = ay_1 - \frac{\partial a}{\partial y}$$
$$= a\frac{\partial}{\partial y}\left\{\log\left(\frac{\alpha' b - \alpha b'}{a}\right)\right\} = a\rho,$$

say ; and therefore

$$u = \xi + a\eta' + R\eta$$
$$= \xi + a(\eta' + \rho\eta),$$
$$v = \eta'' + y_1\eta' + P\eta$$
$$= \frac{1}{a}\frac{\partial u}{\partial y}$$
$$= \frac{1}{a}\frac{\partial}{\partial y}\{a(\eta' + \rho\eta)\}.$$

Hence the differential equation

$$\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial}{\partial x}(ae^z)=0,$$

where

$$a=g(x)\,a(y)+h(x)\,b(y),$$

has its general integral given by

$$e^z=\frac{\dfrac{\partial}{\partial y}\{a(\eta'+\rho\eta)\}}{a\xi+a^2(\eta'+\rho\eta)},$$

where

$$\rho=\frac{\partial}{\partial y}\left\{\log\left(\frac{a'b-ab'}{a}\right)\right\}.$$

Note. In the particular case, when

$$b(y)=Aa(y),$$

where A is a pure constant, we have

$$y_1=0, \quad \rho=0;$$

we can take $a(y)=1$ without loss of generality, and then

$$e^z=\frac{a\eta''}{a\xi+a^2\eta'}=\frac{\eta''}{\xi+a\eta'},$$

in effect, the result in Ex. 2, above.

Hence, *for the present case, where* $m=0$ *and* $n=2$, *we must have* $ab'-a'b$ *different from zero.*

Ex. 4. Consider the case provided by taking

$$m=1, \quad n=1;$$

the differential equation

$$\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial}{\partial x}(ae^z)-\frac{\partial}{\partial y}(\beta e^{-z})=0$$

has

$$e^z=\frac{\eta'+f_2}{\xi'+f_1}$$

for its integral, where f_1 and f_2 involve only ξ, η, x, y. Also, if

$$u=\xi'+f_1, \quad v=\eta'+f_2,$$

we have

$$av=\frac{\partial u}{\partial y}=\frac{\partial f_1}{\partial \eta}\eta'+\frac{\partial f_1}{\partial y},$$

$$\beta u=\frac{\partial v}{\partial x}=\frac{\partial f_2}{\partial \xi}\xi'+\frac{\partial f_2}{\partial x};$$

and therefore

$$a=\frac{\partial f_1}{\partial \eta}, \quad \beta=\frac{\partial f_2}{\partial \xi}.$$

Consequently

$$f_1=a\eta+g_1, \quad f_2=\beta\xi+g_2,$$

where g_1 may involve ξ but not η, and g_2 may involve η but not ξ: also

$$\frac{\partial f_1}{\partial y}=af_2=a\beta\xi+ag_2,$$
$$\frac{\partial f_2}{\partial x}=\beta f_1=a\beta\eta+\beta g_1,$$

that is,

$$\eta\frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial y}=a\beta\xi+ag_2,$$

$$\xi\frac{\partial \beta}{\partial x}+\frac{\partial g_2}{\partial x}=a\beta\eta+\beta g_1.$$

We therefore take

$$g_1=\theta_1\xi+\phi_1,\quad g_2=\theta_2\eta+\phi_2,$$

where ϕ_1 and ϕ_2 do not involve ξ or η. Substituting these values and equating coefficients, we find

$$\frac{\partial a}{\partial y}=a\theta_2,\quad \frac{\partial \theta_1}{\partial y}=a\beta,\quad \frac{\partial \phi_1}{\partial y}=a\phi_2,$$
$$\frac{\partial \beta}{\partial x}=\beta\theta_1,\quad \frac{\partial \theta_2}{\partial x}=a\beta,\quad \frac{\partial \phi_2}{\partial x}=\beta\phi_1;$$

and therefore

$$u=\xi'+f_1=\xi'+\frac{1}{\beta}\frac{\partial \beta}{\partial x}\xi+a\eta+\phi_1,$$

$$v=\eta'+f_2=\eta'+\beta\xi+\frac{1}{a}\frac{\partial a}{\partial y}\eta+\phi_2.$$

The two equations determining ϕ_1 and ϕ_2 are the same as those determining u and v: also, ϕ_1 and ϕ_2 do not involve ξ or η, so that, taking particular functions X of x and Y of y, we have

$$\phi_1=X'+\frac{1}{\beta}\frac{\partial \beta}{\partial x}X+aY,$$

$$\phi_2=Y'+\beta X+\frac{1}{a}\frac{\partial a}{\partial y}Y.$$

Obviously ϕ_1 and ϕ_2 can be absorbed into the other parts of u and v respectively by taking new arbitrary functions $\xi+X$, $\eta+Y$: hence, keeping ξ and η perfectly general, we have

$$\left.\begin{aligned} u&=\xi'+\frac{1}{\beta}\frac{\partial \beta}{\partial x}\xi+a\eta \\ v&=\eta'+\beta\xi+\frac{1}{a}\frac{\partial a}{\partial y}\eta \end{aligned}\right\}.$$

We still have to satisfy implicit limitations on a and β. Now

$$\theta_2=\frac{\partial \log a}{\partial y},\quad \theta_1=\frac{\partial \log \beta}{\partial x}:$$

and therefore

$$\frac{\partial^2 \log a}{\partial x\,\partial y}=\frac{\partial \theta_2}{\partial x}=a\beta=\frac{\partial \theta_1}{\partial y}=\frac{\partial^2 \log \beta}{\partial x\,\partial y}.$$

From the equality of the first and the last of these expressions, we have

$$\frac{\alpha}{X_1} = \frac{\beta}{Y_1} = \theta,$$

say; and then

$$\frac{\partial^2 \log \theta}{\partial x \, \partial y} = \theta^2 X_1 Y_1.$$

When we take

$$\theta^2 = e^\mu, \quad \sqrt{2} X_1 dx = dx', \quad \sqrt{2} Y_1 dy = dy',$$

this equation becomes

$$\frac{\partial^2 \mu}{\partial x' \, \partial y'} = e^\mu,$$

and therefore (§ 218, Ex. 1, *Note*)

$$e^\mu = 2 \frac{\phi'(x') \psi'(y')}{\{\phi(x') + \psi(y')\}^2}.$$

Let

$$\phi(x') = \rho(x), \quad \psi(y') = \sigma(y);$$

then

$$\phi'(x') X_1 \sqrt{2} = \rho'(x), \quad \psi'(y') Y_1 \sqrt{2} = \sigma'(y),$$

so that

$$\theta^2 = e^\mu = \frac{1}{X_1 Y_1} \frac{\rho'(x) \sigma'(y)}{\{\rho(x) + \sigma(y)\}^2},$$

and therefore

$$\alpha = \left(\frac{X_1}{Y_1}\right)^{\frac{1}{2}} \frac{\{\rho'(x) \sigma'(y)\}^{\frac{1}{2}}}{\rho(x) + \sigma(y)},$$

$$\beta = \left(\frac{Y_1}{X_1}\right)^{\frac{1}{2}} \frac{\{\rho'(x) \sigma'(y)\}^{\frac{1}{2}}}{\rho(x) + \sigma(y)}.$$

The quantities X_1 and Y_1 are at our disposal; we introduce new functions $h(x)$ and $k(y)$, such that

$$X_1 = \rho'(x) h^2(x), \quad Y_1 = \sigma'(y) k^2(y),$$

and then

$$\alpha = \frac{\rho'(x)}{\rho(x) + \sigma(y)} \frac{h(x)}{k(y)},$$

$$\beta = \frac{\sigma'(y)}{\rho(x) + \sigma(y)} \frac{k(y)}{h(x)}.$$

With these values of α and β, the general integral of the differential equation

$$\frac{\partial^2 z}{\partial x \, \partial y} + \frac{\partial}{\partial x}(\alpha e^z) - \frac{\partial}{\partial y}(\beta e^{-z}) = 0$$

is given by

$$e^z = \frac{\eta' + \beta \xi + \frac{1}{\alpha} \frac{\partial \alpha}{\partial y} \eta}{\xi' + \frac{1}{\beta} \frac{\partial \beta}{\partial x} \xi + \alpha \eta}.$$

The functions h, k, ρ, σ are at our disposal: special forms assumed for them will lead to special equations with corresponding integrals

Ex. 5. Integrate the equations :—

(i) $$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(\frac{e^z}{x+y}\right)-\frac{\partial}{\partial y}\left(\frac{e^{-z}}{x+y}\right)=0;$$

(ii) $$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left\{e^z\left(\frac{1}{x+y}-\frac{1}{x}\right)\right\}-\frac{\partial}{\partial y}\left\{e^{-z}\left(\frac{1}{x+y}-\frac{1}{y}\right)\right\}=0;$$

verifying in each case that α and β conform to the general conditions.

Ex. 6. Shew that, if the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}(e^{\alpha+z})-\frac{\partial}{\partial y}(e^{\beta-z})=0$$

possesses a general integral of the form

$$e^z=\frac{\eta''+f_2}{\xi'+f_1},$$

where f_1 and f_2 do not involve η'' or ξ', then α and β must satisfy the equations

$$\frac{\partial^2\beta}{\partial x\,\partial y}=e^{\alpha+\beta},$$

$$\frac{\partial^3\alpha}{\partial x\,\partial y^2}-\frac{\partial^2\alpha}{\partial x\,\partial y}\left(2\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial y}\right)=3e^{\alpha+\beta}\frac{\partial\alpha}{\partial y};$$

and prove that

$$f_2=\frac{\partial\beta}{\partial y}\eta'+\left\{\frac{\partial^2\beta}{\partial y^2}+\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}-\frac{\partial^2\alpha}{\partial y^2}-\left(\frac{\partial\alpha}{\partial y}\right)^2\right\}\eta+\xi e^\beta,$$

$$f_1=\eta' e^\alpha+e^\alpha\left(\frac{\partial\beta}{\partial y}-\frac{\partial\alpha}{\partial y}\right)\eta+\xi\frac{\partial\beta}{\partial y}.$$

Ex. 7. Obtain the equations which must be satisfied by α and β, in order that the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}(\alpha e^z)-\frac{\partial}{\partial y}(\beta e^{-z})=0$$

may possess a general integral of the form

$$e^z=\frac{\eta''+f_2}{\xi''+f_1},$$

where f_1 and f_2 do not involve ξ'' or η''.

Second Combination of Coefficients a and b.

226. In the next place, we consider the alternative form of b that can be associated with the value of a given by

$$a=\mu+\lambda e^{z\rho}:$$

we have

$$b=\frac{1}{\rho}\frac{\partial\rho}{\partial x}.$$

On reference to the analysis in § 224, it appears that

$$\frac{\partial f_2}{\partial x} + \sum_{a=0}^{m-1} \frac{\partial f_2}{\partial \xi_a} \xi_{a+1} = 0,$$

that is,

$$\frac{df_2}{dx} = 0.$$

Hence f_2 is independent of x; and therefore, as it does not involve η_n, it involves only $\eta_{n-1}, \ldots, \eta_1, \eta, y$. The differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + (\mu + \lambda e^{\rho z}) \frac{\partial z}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial z}{\partial y} + c = 0.$$

Let a new dependent variable Z be taken such that

$$Z = \rho z + \log \lambda,$$

so that

$$e^Z = \lambda e^{\rho z};$$

the differential equation becomes

$$\frac{\partial Z}{\partial x \partial y} + (e^Z + \phi) \frac{\partial Z}{\partial x} + \psi = 0,$$

where ϕ is a function involving x and y but not Z, and ψ is a function involving x, y, and Z, in general.

Repeating the application of the conditions, at present under consideration, to the equation in this form, we see that their effect will be guaranteed by taking $\rho = 1$, $\lambda = 1$, in the value of a. Thus

$$\frac{\partial f}{\partial \eta_n} = \frac{1}{\eta_n + f_2},$$

while we still have

$$\frac{df_2}{dx} = 0,$$

so that f_2 involves only $\eta_{n-1}, \ldots, \eta_1, \eta, y$; and therefore

$$f = \log(\eta_n + f_2) + f_1,$$

where f_1 does not involve η_n but may involve all the lower derivatives of η, and does involve all the derivatives of ξ of all orders up to ξ_m. Now, with the value $\lambda = 1$, we have

$$\begin{aligned} e^Z = e^z &= e^f \\ &= (\eta_n + f_2) e^{f_1}; \end{aligned}$$

and we had, in general,

$$\frac{\partial^2 f}{\partial y \partial \xi_m} + \sum_{\beta=0}^{n-1} \frac{\partial^2 f}{\partial \xi_m \partial \eta_\beta} \eta_{\beta+1} + a \frac{\partial f}{\partial \xi_m} = 0,$$

so that, as $a = e^Z + \phi$, we have

$$\frac{d}{dy}\left(\log \frac{\partial f_1}{\partial \xi_m}\right) + (\eta_n + f_2) e^{f_1} + \phi = 0:$$

an equation which also results from direct substitution, of the relation giving Z, in the differential equation satisfied by Z.

Again, we have (as before)

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} \frac{\partial f}{\partial \eta_n} \frac{\partial}{\partial Z} (e^Z + \phi) = 0,$$

that is,

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} e^Z \frac{\partial f}{\partial \eta_n} = 0:$$

also

$$e^Z \frac{\partial f}{\partial \eta_n} = e^f \frac{\partial f}{\partial \eta_n} = e^{f_1},$$

$$\frac{\partial f}{\partial \xi_m} = \frac{\partial f_1}{\partial \xi_m};$$

and therefore

$$\frac{\partial^2 f_1}{\partial \xi_m \partial \eta_{n-1}} + e^{f_1} \frac{\partial f_1}{\partial \xi_m} = 0.$$

Consequently

$$\frac{\partial f_1}{\partial \eta_{n-1}} + e^{f_1} = \text{quantity independent of } \xi_m$$

$$= \frac{\dfrac{\partial^2 f_4}{\partial \eta^2_{n-1}}}{\dfrac{\partial f_4}{\partial \eta_{n-1}}},$$

say, where f_4 is independent of ξ_m; and therefore, after a single integration,

$$e^{-f_1} \frac{\partial f_4}{\partial \eta_{n-1}} - f_4 = \text{quantity independent of } \eta_{n-1}$$

$$= f_3,$$

say, where f_3 involves no derivative of η of order higher than η_{n-2} and does involve ξ_m, while f_4 does involve η_{n-1}. Thus

$$e^{f_1} = \frac{1}{f_3 + f_4} \frac{\partial f_4}{\partial \eta_{n-1}}.$$

Further, the equation

$$\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right) + (\eta_n + f_2)\, e^{f_1} + \phi = 0$$

is satisfied identically, f_2 not involving η_n and being entirely independent of x, and ϕ being a function of x and y only; hence

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} + (\eta_n + f_2)\, e^{f_1}\frac{\partial f_1}{\partial \xi_m} = 0,$$

and therefore

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right) + \phi\right\}\frac{\partial f_1}{\partial \xi_m} = 0,$$

that is,

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right) = \phi\,\frac{\partial f_1}{\partial \xi_m}.$$

Now

$$e^{f_1} = \frac{1}{f_3 + f_4}\frac{\partial f_4}{\partial \eta_{n-1}},$$

and f_4 is independent of ξ_m: hence

$$\frac{\partial f_1}{\partial \xi_m} = -\frac{1}{f_3 + f_4}\frac{\partial f_3}{\partial \xi_m}.$$

This equation does not involve η_n: but it can involve η_{n-1}, which (though not occurring in f_3) can be introduced by f_4.

Again,

$$\begin{aligned}\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\log(f_3 + f_4)\right\} &= \frac{\partial}{\partial \xi_m}\left\{\frac{1}{f_3 + f_4}\left(\frac{df_3}{dy} + \frac{df_4}{dy}\right)\right\}\\ &= \frac{1}{f_3 + f_4}\frac{\partial}{\partial \xi_m}\left(\frac{df_3}{dy}\right) - \frac{1}{(f_3 + f_4)^2}\left(\frac{df_3}{dy} + \frac{df_4}{dy}\right)\frac{\partial f_3}{\partial \xi_m}\\ &= -\frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right);\end{aligned}$$

and consequently

$$\begin{aligned}\phi\,\frac{\partial f_1}{\partial \xi_m} &= \frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right)\\ &= \frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_3}{\partial \xi_m}\right)\right\}.\end{aligned}$$

Now f_3 involves no derivative of η higher than η_{n-2}, so that the right-hand side can, at the utmost, only be linear in η_{n-1}; also f_4

does involve η_{n-1} and $\dfrac{\partial f_3}{\partial \xi_m}$ is not zero, so that $\dfrac{\partial f_1}{\partial \xi_m}$ cannot be only linear in η_{n-1}. Hence ϕ, which is to involve none of the derivatives of the arbitrary quantities ξ and η, must be zero. The differential equation now becomes

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + \psi = 0,$$

where ψ can involve x, y, Z, and where Z is given by

$$Z = f = \log(\eta_n + f_2) + f_1,$$

$$e^{f_1} = \frac{1}{f_3 + f_4} \frac{\partial f_4}{\partial \eta_{n-1}};$$

the quantity f_2 does not involve η_n and is a function of $\eta_{n-1}, \ldots, \eta, y$ only; the quantity f_3 does not involve η_n or η_{n-1}, but it does involve ξ_m in such a way as to have

$$\frac{d}{dy}\left(\log \frac{\partial f_3}{\partial \xi_m}\right)$$

independent of ξ_m; and the quantity f_4 does not involve ξ_m or η_n, but does involve η_{n-1}.

Let the value of Z be substituted in the equation: then as

$$\begin{aligned}\frac{\partial Z}{\partial y} &= \frac{d}{dy}\{\log(\eta_n + f_2)\} + \frac{df_1}{dy} \\ &= \frac{d}{dy}\{\log(\eta_n + f_2)\} + \frac{\partial f_1}{\partial \eta_{n-1}} \eta_n + \ldots,\end{aligned}$$

and therefore

$$\frac{\partial^2 Z}{\partial x \partial y} = \eta_n \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial \eta_{n-1}}\right) + \ldots,$$

$$\frac{\partial Z}{\partial x} = \frac{df_1}{dx},$$

we have

$$\psi + \eta_n \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial \eta_{n-1}}\right) + \ldots + (\eta_n + f_2) e^{f_1} \frac{df_1}{dx} = 0.$$

Thus ψ is of the form

$$\alpha \eta_n + \beta,$$

where α and β are independent of η_n; and ψ is known to be a function of x, y, and Z alone. Hence

$$\psi = m e^Z + n,$$

where m and n are functions of x and y only; and therefore the equation in Z is

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + m e^Z + n = 0.$$

Let a new dependent variable u be introduced by the relation

$$\frac{\partial Z}{\partial x} + m = e^{-u},$$

so that the differential equation for Z then gives

$$e^Z = \frac{\partial u}{\partial y} - k e^u,$$

where

$$k = n - \frac{\partial m}{\partial y};$$

and therefore, on the elimination of Z, we have

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial}{\partial x}(k e^u) + (m - e^{-u}) \frac{\partial u}{\partial y} - m k e^u + k = 0.$$

When the equation in Z has an integral of the specified type, the preceding relations prescribe that type also for the value of u, and conversely.

If k is not zero, this equation is of the form

$$\frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + c = 0,$$

where

$$a = -k e^u, \quad b = m - e^{-u};$$

that is, the equation belongs to the first alternative of the former type. In this case, $\rho = 1$; and therefore

$$m = \frac{1}{\rho} \frac{\partial \rho}{\partial x} = 0.$$

If k is zero, the equation for u is

$$\frac{\partial^2 u}{\partial x \partial y} + (m - e^{-u}) \frac{\partial u}{\partial y} = 0,$$

or changing the variables from x and y to $-y$ and x, and changing the sign of u, we have

$$\frac{\partial^2 u}{\partial x \partial y} + (e^u - m) \frac{\partial u}{\partial x} = 0,$$

which is a particular case of

$$\frac{\partial^2 Z}{\partial x \partial y} + (e^Z + \phi)\frac{\partial Z}{\partial x} + \psi = 0.$$

For the latter, it was proved that ϕ could be made zero: hence

$$m = 0.$$

Thus, whether k is zero or is not zero, our equation for Z can be taken in the form

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + n = 0,$$

where n is a function of x and y only, not involving z. Let a quantity θ be taken such that

$$n = \frac{\partial^2 \theta}{\partial x \partial y},$$

and take a new variable ζ such that

$$\zeta = Z + \theta.$$

Then

$$\frac{\partial^2 \zeta}{\partial x \partial y} = -e^Z \frac{\partial Z}{\partial x} = -\frac{\partial}{\partial x}(e^{\zeta - \theta}),$$

that is,

$$\frac{\partial^2 \zeta}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^{\zeta}) = 0,$$

where

$$A = e^{-\theta}.$$

We thus have a particular case of the former equation in § 225, now given by making $B = 0$ in that equation.

The general integral can be at once obtained in the form

$$e^{\zeta} = \frac{Y}{X + \int AY dy},$$

which involves partial quadratures; there must be limitations upon the form of A which allow this expression to take a finite explicit form free from partial quadratures. If we write the integral

$$e^{\zeta} = \frac{v}{u},$$

where

$$v = \eta^{(n)} + y_1\eta^{(n-1)} + y_2\eta^{(n-2)} + \ldots + y_n\eta,$$
$$u = \xi_1 + P_1\eta^{(n-1)} + P_2\eta^{(n-2)} + \ldots + P_n\eta,$$

we have

$$P_1 = A,$$
$$Av = \frac{\partial u}{\partial y},$$

where $y_1, y_2, \ldots, y_n$ are functions of y only; and then

$$Ay_r = \frac{\partial P_r}{\partial y} + P_{r+1},$$

for $r = 1, \ldots, n-1$, with

$$A = P_1, \quad Ay_n = \frac{\partial P_n}{\partial y}.$$

Thus A is easily seen to satisfy the equation

$$\frac{\partial^n A}{\partial y^n} - \frac{\partial^{n-1}(Ay_1)}{\partial y^{n-1}} + \frac{\partial^{n-2}(Ay_2)}{\partial y^{n-2}} - \ldots + (-1)^n Ay_n = 0,$$

so that it is of the form

$$A = \xi_1 Y_1 + \ldots + \xi_n Y_n,$$

where $Y_1, \ldots, Y_n$ are linearly independent integrals of this ordinary equation of order n, and $\xi_1, \ldots, \xi_n$ are functions of x at our disposal.

On the whole, it appears that the equation determined by the combination

$$a = \mu + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x},$$

is only a particular form of the equation determined by the combination

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y} + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho}.$$

Examples of the particular form have already been given.

Ex. Work out the detailed form of A, and of the integral

$$e^z = \frac{\eta''' + y_1\eta'' + y_2\eta' + y_3\eta}{\xi + A\eta'' + P\eta' + Q\eta},$$

of the differential equation

$$\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial}{\partial x}(Ae^z) = 0.$$

Third Combination of Coefficients a and b.

227. In the next place, after having considered all the cases that arise when a has the form $\mu+\lambda e^{\rho z}$, we proceed to consider the cases that arise when

$$a=\mu+\lambda z.$$

As the equation is unaltered when x and y are interchanged as well as A and B, provided the sign of z is changed, it follows (from the knowledge of the forms of b which can be associated with $\mu+\lambda e^{\rho z}$ as the form of a) that b cannot have the form $\mu'+\lambda' e^{\rho' z}$; it can only be $\mu'+\lambda' z$ (which will be seen not to be possible) or σ, where μ', λ', σ are functions of x and y alone.

In the initial investigation, the equation

$$\frac{\partial a}{\partial z}\frac{\partial^2 f}{\partial \eta_n{}^2}+\frac{\partial^2 a}{\partial z^2}\left(\frac{\partial f}{\partial \eta_n}\right)^2=0$$

occurred, as holding in general: hence, for the present case, on the assumption that λ is different from zero, we have

$$\frac{\partial^2 f}{\partial \eta_n{}^2}=0.$$

But $\frac{\partial f}{\partial \eta_n}$ cannot be zero, for z would then not involve η_n. Also, the equation

$$\frac{\partial^2 f}{\partial x\partial \eta_n}+\sum_{a=0}^{m-1}\frac{\partial^2 f}{\partial \xi_a\partial \eta_n}\xi_{a+1}+b\frac{\partial f}{\partial \eta_n}=0$$

holds in general: and it is satisfied identically when substitution takes place for z. Taking derivatives with regard to η_n, and using the property that $\frac{\partial^2 f}{\partial \eta_n{}^2}$ vanishes for the present case, we have

$$\frac{\partial b}{\partial z}\left(\frac{\partial f}{\partial \eta_n}\right)^2=0.$$

Thus b does not explicitly involve z: consequently

$$b=\sigma,$$

where σ is a function of x and y only.

The differential equation is

$$\frac{\partial^2 z}{\partial x\partial y}+(\mu+\lambda z)\frac{\partial z}{\partial x}+\sigma\frac{\partial z}{\partial y}+c=0.$$

Let

$$z = \theta Z + \phi,$$

choosing θ and ϕ to be functions of x and y such that

$$\frac{\partial \theta}{\partial x} + \sigma\theta = 0,$$

$$\frac{\partial \theta}{\partial y} + \mu\theta = -\lambda\theta\phi;$$

then the equation for Z is

$$\frac{\partial^2 Z}{\partial x \partial y} + \lambda\theta Z \frac{\partial Z}{\partial x} + \gamma = 0,$$

where $\lambda\theta$ is a function of x and y only, and γ involves x, y, Z explicitly. Thus the equation

$$\frac{\partial^2 z}{\partial x \partial y} + Mz \frac{\partial z}{\partial x} + c = 0$$

has an integral of the specified type, when M involves only x and y, and c involves x, y, z.

For this form of equation $b = 0$, and therefore

$$\frac{\partial^2 f}{\partial x \partial \eta_n} + \sum_{a=0}^{m-1} \frac{\partial^2 f}{\partial \xi_a \partial \eta_n} \xi_{a+1} = 0;$$

also

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$$

in general: hence, for the equation in question,

$$\frac{\partial}{\partial \eta_n}\left(\frac{df}{dx}\right) = 0,$$

that is, $\frac{df}{dx}$ does not contain η_n. Now, as $\frac{\partial^2 f}{\partial \eta_n{}^2}$ is zero, f (that is, z) is only linear in η_n; and therefore $Mz\frac{\partial z}{\partial x}$ is only linear in η_n. Also, as $\frac{\partial z}{\partial x}$ does not contain η_n, it follows that $\frac{\partial^2 z}{\partial x \partial y}$ can contain η_n only linearly at the utmost; so that, as the differential equation is to be satisfied identically when substitution is made for z, c can only be linear in η_n, that is, quà function of the variables x, y, z alone, c can only be linear in z. Let

$$c = \alpha z + \beta,$$

where α and β are functions of x and y only; the differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + Mz\frac{\partial z}{\partial x} + \alpha z + \beta = 0.$$

Now take a new dependent variable u, such that

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}Mz^2;$$

forming the derivatives of u, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial^2 z}{\partial x \partial y} + Mz\frac{\partial z}{\partial x} + \tfrac{1}{2}z^2\frac{\partial M}{\partial x}\\ &= \tfrac{1}{2}z^2\frac{\partial M}{\partial x} - \alpha z - \beta,\end{aligned}$$

so that z and $\frac{\partial z}{\partial y}$ can be expressed in terms of u and $\frac{\partial u}{\partial x}$; and radicals involving $\frac{\partial u}{\partial x}$ occur unless $\frac{\partial M}{\partial x}$ vanishes. Again,

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial M}{\partial x}z - \alpha\right)\frac{\partial z}{\partial y} + \tfrac{1}{2}z^2\frac{\partial^2 M}{\partial x \partial y} - z\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial y},$$

the right-hand side of which, on substitution for z and $\frac{\partial z}{\partial y}$, becomes a function of u, $\frac{\partial u}{\partial x}$, and of x, y. Owing to the explicit value of u, which is

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}Mz^2,$$

the form of u is of the same type as that of z: and it has just been seen that u satisfies an equation of the second order. But, at the earliest stage of the investigation, it was seen that such an equation must be of the form

$$\frac{\partial^2 u}{\partial x \partial y} + \rho\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c = 0,$$

where ρ, a, b, c are functions of x, y, u. The preceding equation is certainly not of this form, being irrational in $\frac{\partial u}{\partial x}$ unless $\frac{\partial M}{\partial x}$ vanishes: consequently

$$\frac{\partial M}{\partial x} = 0,$$

that is, M is a function of y only. Taking

$$Mdy = dy',$$

and dividing the equation in z by M, we effectively make $M = 1$: and thus the equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} + \alpha z + \beta = 0,$$

and the new variable u is

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}z^2.$$

Proceeding again to the formation of the equation of the second order satisfied by u, we have

$$\frac{\partial u}{\partial x} = -\alpha z - \beta,$$

so that, unless α is zero, z and $\frac{\partial z}{\partial y}$ are expressible in terms of u and $\frac{\partial u}{\partial x}$. Also

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= -\alpha\frac{\partial z}{\partial y} - \frac{\partial \beta}{\partial y} \\ &= -\frac{\partial \beta}{\partial y} - \alpha\left(u - \tfrac{1}{2}z^2\right) \\ &= -\frac{\partial \beta}{\partial y} - \alpha u + \frac{1}{2\alpha}\left(\frac{\partial u}{\partial x} + \beta\right)^2,\end{aligned}$$

which, though a differential equation of the second order, is certainly not of the required form. Hence we must have $\alpha = 0$, which alone will prevent this forbidden form from occurring; and therefore the differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} + \beta = 0,$$

where β is a function of x and y only.

228. We now proceed to prove that β must be zero. Let

$$e^{-v} = \frac{\partial z}{\partial x},$$

so that

$$z = \frac{\partial v}{\partial y} - \beta e^v,$$

and therefore

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(\beta e^v) - e^{-v} = 0.$$

Obviously v is of the same type as z; and v is seen to satisfy a differential equation of the second order. This equation belongs to the form

$$s + ap + bq + c = 0,$$

where

$$a = -\beta e^v, \quad b = 0,$$

that is, a is of the form $\mu + \lambda e^{\rho z}$. We have seen that the type of equation is then

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(\alpha e^z) = 0,$$

to which the foregoing equation does not conform: hence, so long as β is different from zero, we have a form of a which leads to an equation that cannot arise in connection with such a form. It is therefore necessary that $\beta = 0$: and our equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} = 0.$$

We can obtain the general integral. Integrating with regard to x, we have

$$\frac{\partial z}{\partial y} + \tfrac{1}{2}z^2 = \text{function of } y \text{ only.}$$

The integral of this is obviously of the form

$$z = \frac{v}{\phi(x) + u} + w,$$

where u, v, w are functions of y only: we find, on substituting, that

$$v = 2\frac{du}{dy}, \quad w = -\frac{\dfrac{d^2u}{dy^2}}{\dfrac{du}{dy}},$$

and then

$$-\{u, y\} = \text{above function of } y,$$

where $\{u, y\}$ is the Schwarzian derivative of u. As this function of y is arbitrary, we may take u as arbitrary, say $u = \psi(y)$; and then

$$z = \frac{2\psi'(y)}{\phi(x) + \psi(y)} - \frac{\psi''(y)}{\psi'(y)},$$

where ϕ and ψ are arbitrary functions.

Fourth Combination of Coefficients a and b.

229. Lastly, suppose that a has the third of its forms, so that we take

$$a = \mu.$$

Owing to the investigation in § 226, we may dispense with the consideration of the value $\mu' + \sigma e^{\rho z}$ of b; and owing to that in § 227, we may dispense with the consideration of the value $\mu' + \sigma z$ of b. There is thus only one form left: we take

$$b = \sigma,$$

that is, we may regard a and b in the equation

$$s + ap + bq + c = 0$$

as functions of x and y alone which do not involve z explicitly.

The two equations (§ 223) connecting a and b with the value of z are expressible in the form

$$\frac{d}{dy}\left(\log \frac{\partial f}{\partial \xi_m}\right) + a = 0, \qquad \frac{d}{dx}\left(\log \frac{\partial f}{\partial \eta_n}\right) + b = 0.$$

From the former, we have

$$\log\left(\frac{\partial f}{\partial \xi_m}\right) = -\int a\,dy + \text{a function of } x \text{ only}$$

$$= \log u_1 + \text{a function of } x \text{ only},$$

say; and from the latter we have

$$\log\left(\frac{\partial f}{\partial \eta_n}\right) = -\int b\,dx + \text{a function of } y \text{ only}$$

$$= \log u_2 + \text{a function of } y \text{ only},$$

say: the quantities u_1 and u_2 being such that

$$\frac{\partial u_1}{\partial y} + a u_1 = 0, \qquad \frac{\partial u_2}{\partial x} + b u_2 = 0.$$

Hence we may take

$$z = u_1 f_1(x, \xi, \xi_1, \ldots, \xi_m) + u_2 f_2(y, \eta, \eta_1, \ldots, \eta_n) + f_3$$
$$= u_1 f_1 + u_2 f_2 + f_3,$$

where ξ_m occurs only in f_1, and η_n occurs only in f_2, while f_3 does not contain either ξ_m or η_n. Substituting this value of z in the differential equation, and taking account of the relations which define u_1 and u_2, we find

$$-u_1 f_1\left(\frac{\partial a}{\partial x}+ab\right)-u_2 f_2\left(\frac{\partial b}{\partial y}+ab\right)+\frac{\partial^2 f_3}{\partial x\partial y}+a\frac{\partial f_3}{\partial x}+b\frac{\partial f_3}{\partial y}+c=0:$$

and this relation must be satisfied identically. It is to be noted that u_1 is not zero, for otherwise z would not involve ξ_m; and similarly u_2 is not zero, for otherwise z would not involve η_n; also ξ_m does not occur in f_2 or in any of its derivatives, while η_n does not occur in f_1 or in any of its derivatives. But ξ_m and η_n do occur in c, after substitution of the value of z, unless c is free from z; and they occur, in lineo-linear fashion, in the combination of the derivatives of f_3, their form being

$$\frac{\partial^2 f_3}{\partial\xi_{m-1}\partial\eta_{n-1}}\xi_m\eta_n+a\frac{\partial f_3}{\partial\xi_{m-1}}\xi_m+b\frac{\partial f_3}{\partial\eta_{n-1}}\eta_n.$$

It therefore follows that, when the preceding relation is differentiated with regard to ξ_m twice, and with regard to η_n twice, derivatives of f_3 will not occur: the results, on dropping the non-vanishing factors u_1 and u_2 respectively, are

$$\left(\frac{\partial c}{\partial z}-\frac{\partial a}{\partial x}-ab\right)\frac{\partial^2 f_1}{\partial\xi_m^{\,2}}+u_1\frac{\partial^2 c}{\partial z^2}\left(\frac{\partial f_1}{\partial\xi_m}\right)^2=0,$$

$$\left(\frac{\partial c}{\partial z}-\frac{\partial b}{\partial y}-ab\right)\frac{\partial^2 f_2}{\partial\eta_n^{\,2}}+u_2\frac{\partial^2 c}{\partial z^2}\left(\frac{\partial f_2}{\partial\eta_n}\right)^2=0.$$

Assuming in the first place that $\frac{\partial^2 c}{\partial z^2}$ is not zero (the alternative assumption will come later), we have

$$\frac{1}{\dfrac{\partial c}{\partial z}-\dfrac{\partial a}{\partial x}-ab}\frac{\partial^2 c}{\partial z^2}=-\frac{1}{u_1}\left(\frac{\partial f_1}{\partial\xi_m}\right)^{-2}\frac{\partial^2 f_1}{\partial\xi_m^{\,2}},$$

$$\frac{1}{\dfrac{\partial c}{\partial z}-\dfrac{\partial b}{\partial y}-ab}\frac{\partial^2 c}{\partial z^2}=-\frac{1}{u_2}\left(\frac{\partial f_2}{\partial\eta_n}\right)^{-2}\frac{\partial^2 f_2}{\partial\eta_n^{\,2}}.$$

The quantities on the right-hand sides are independent of η_n and of ξ_m respectively: hence neither of them, when expressed in terms of the variables alone, can involve z; and therefore they are functions of x and y alone.

If the values of the two fractions on the left-hand sides, thus expressed in terms of x and y alone, are unequal, we take the quotients of the respective members of the two relations. The result is to give $\frac{\partial c}{\partial z}$ as a function of x and y alone: consequently c is a linear function of z, a contingency provisionally excluded. Accordingly, we may assume that the fractions are equal to one another: so that, as $\frac{\partial^2 c}{\partial z^2}$ is supposed not to vanish, we have

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}.$$

There thus exists a quantity θ, which is a function of x and y, such that

$$a = \frac{1}{\theta}\frac{\partial \theta}{\partial y}, \quad b = \frac{1}{\theta}\frac{\partial \theta}{\partial x};$$

and then, when we take a new dependent variable Z defined by the relation

$$Z = z\theta,$$

our differential equation becomes

$$\begin{aligned}\frac{\partial^2 Z}{\partial x \partial y} &= -c\theta + Z\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x \partial y}\\ &= -C,\end{aligned}$$

where C is a function of x, y, and Z. Hence, in the present case, we may take our equation in the form

$$\frac{\partial^2 z}{\partial x \partial y} + c = 0,$$

where c is a function of x, y, and z.

But now as a is zero, u_1 is a function of x only, so that the quantity

$$\frac{1}{u_1}\left(\frac{\partial f_1}{\partial \xi_m}\right)^{-2}\frac{\partial^2 f_1}{\partial {\xi_m}^2}$$

is a function of x alone; also, as b is zero, u_2 is a function of y only, the quantity

$$\frac{1}{u_2}\left(\frac{\partial f_2}{\partial \eta_n}\right)^{-2}\frac{\partial^2 f_2}{\partial {\eta_n}^2}$$

is a function of y alone. On the present assumption, the two quantities are equal; consequently, they are equal to one and the same constant, say $-k$, so that (as a and b now are zero) we have

$$\frac{\partial^2 c}{\partial z^2} = k \frac{\partial c}{\partial z},$$

and therefore

$$kc = e^{k(z+\mu)} + \rho,$$

where μ and ρ are functions of x and y only. Taking

$$v = k(z + \mu),$$

the differential equation for v is

$$\frac{\partial^2 v}{\partial x \partial y} + e^v + \rho = 0.$$

Let

$$\frac{\partial v}{\partial y} = - V:$$

then

$$\frac{\partial V}{\partial x} = e^v + \rho,$$

and therefore

$$\frac{\partial^2 V}{\partial x \partial y} + V \frac{\partial V}{\partial x} + \rho V - \frac{\partial \rho}{\partial y} = 0.$$

Now

$$V = -k \frac{\partial (z + \mu)}{\partial y},$$

and therefore V is of the same type as z: moreover, it satisfies an equation of the second order. Comparing the form of this equation with the admissible forms, we have

$$\rho = 0,$$

and so the differential equation is expressible in the form

$$\frac{\partial^2 v}{\partial x \partial y} + e^v = 0.$$

This is Liouville's equation: its general integral (§ 218, Ex. 1, *Note*) is

$$e^v = -2 \frac{\xi' \eta'}{(\xi + \eta)^2},$$

where ξ and η are arbitrary functions of x and y respectively.

Ex. Obtain the primitive of the equation

$$r + t = e^{az},$$

where a is a constant, in the form

$$e^{az} = \frac{2}{au^2} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right\},$$

the quantity u denoting

$$f(x+iy)+g(x-iy),$$

and f and g being arbitrary functions.

230. Lastly, we have to consider the omitted case set on one side in the preceding assumption, viz. when c is a linear function of z: let

$$c = \lambda z + \mu.$$

Then the differential equation is

$$s + ap + bq + \lambda z + \mu = 0.$$

If ζ is any particular integral, then, taking a new dependent variable Z such that

$$Z = z - \zeta,$$

we have

$$S + aP + bQ + \lambda Z = 0,$$

where a, b, λ are functions of x and y alone. This equation coincides, in form, with Laplace's linear equation which has already been discussed.

Summary of Results.

231. The results of the investigation can be summarised as follows.

I. When an integral relation is given in a form

$$z = f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n),$$

where ξ is an arbitrary function of x and η is an arbitrary function of y, and when it satisfies an equation of the second order, then, either directly or after a transformation of the dependent variable which does not affect the specific character of the integral relation, the equation of the second order can be made to acquire the form

$$s + ap + bq + c = 0,$$

where a, b, c are functions of x, y, z alone. The functions a and b can have any one of three possible forms, viz.

$$\mu + \lambda e^{\rho z}, \quad \mu + \lambda z, \quad \mu,$$

where μ, λ, ρ are functions of x and y only.

II. When both a and b are of the form $\mu + \lambda e^{\rho z}$, then

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y} + \lambda e^{\rho z}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-\rho z};$$

by transformation of the dependent variable which does not affect the character of the integral, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^z) - \frac{\partial}{\partial y}(Be^{-z}) = 0,$$

and its general integral is

$$e^z = \frac{\eta_n + f_2}{\xi_m + f_1} = \frac{v}{u},$$

where f_1 and f_2 do not involve ξ_m or η_n, and A and B are functions of x and y only such that

$$Av = \frac{\partial u}{\partial y}, \quad Bu = \frac{\partial v}{\partial x}.$$

III. If a is of the form $\mu + \lambda e^{\rho z}$ and if b is not of this form, then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x};$$

while, if b is of the form $\mu + \lambda e^{\rho z}$ and if a is not of this form, then

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y}.$$

In the former case, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^z) = 0,$$

having an integral

$$e^z = \frac{v}{u},$$

where

$$v = \eta^{(n)} + y_1 \eta^{(n-1)} + \ldots + y_n \eta,$$
$$u = \xi + A\eta^{(n-1)} + P_2 \eta^{(n-2)} + \ldots + P_n \eta,$$

and

$$\frac{\partial u}{\partial y} = Av;$$

in the latter case, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial}{\partial y}(Be^{-z}) = 0,$$

having an integral of corresponding form.

IV. If one of the two quantities a and b has the form $\mu + \lambda z$, the other has the form σ, where μ, λ, σ are functions of x and y alone. Then, by transformations of the variables which do not affect the character of the integral relation, the differential equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} = 0,$$

and the value of z is

$$z = \frac{2\eta'}{\xi + \eta} - \frac{\eta''}{\eta'}.$$

V. When a and b are functions of x and y alone, either the equation can be changed to Liouville's form

$$\frac{\partial^2 z}{\partial x \partial y} + e^z = 0,$$

and then

$$e^z = -\frac{2\xi'\eta'}{(\xi + \eta)^2};$$

or it can be changed so as to acquire the form of Laplace's linear equation

$$s + ap + bq + cz = 0,$$

where a, b, c are functions of x and y alone.

The two forms in (III) are obviously derivable from one another, by changing the sign of z and interchanging the variables x and y. All equations of the second order having their general integral of the specified type can, by transformations of the variables which do not affect the character of the integral, be expressed in one or other of the foregoing forms.

Ex. 1. Integrate the equations :—

(i) $s + \frac{e^z}{(x+y)^2} p + e^{-z} q = \frac{2e^z}{(x+y)^3} - \frac{1}{(x+y)^2}$;

(ii) $s + e^z p = 0$;

(iii) $s + e^z p + \frac{1}{(x+y)^2} = 0$;

(iv) $s + e^z p + \frac{n}{(x+y)^2} = 0$,

n being a positive integer. (Tanner.)

Ex. 2. Shew that the integral of the equation

$$s^2 = 4pq\lambda,$$

when λ is a function of x and y, is given by

$$z = \int \left\{ u^2\, dx + \frac{1}{\lambda} \left(\frac{\partial u}{\partial y} \right)^2 dy \right\},$$

where u is an integral of the equation

$$\frac{\partial^2 u}{\partial x\, \partial y} - \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x} \frac{\partial u}{\partial y} - \lambda u = 0.$$

Apply this property to integrate the equations

$$s^2 = \frac{4pq}{(x+y)^2},$$

$$s^2 = \frac{16pq}{(x+y)^2}.$$

(Goursat.)

Ex. 3. Obtain the general integral of the equation

$$sz = \{(1+p^2)(1+q^2)\}^{\frac{1}{2}}$$

in the form

$$z^2 = \left\{ \int X\, dx - \int \frac{dy}{Y} \right\} \left\{ \int Y\, dy - \int \frac{dx}{X} \right\},$$

where X and Y are arbitrary functions of x and y respectively.

(Goursat.)

Ex. 4. Integrate the equations:—

(i) $s \sin z = \{(1+p^2)(1+q^2)\}^{\frac{1}{2}}$;

(ii) $sz + \phi(x, p)\,\psi(y, q) = 0$,

where, in the latter equation, ϕ and ψ satisfy the conditions

$$\frac{\partial \phi}{\partial p} = \frac{p}{\phi} + a, \quad \frac{\partial \psi}{\partial q} = \frac{q}{\psi} + a,$$

and a is a constant. (Goursat.)

CHAPTER XVI.

Equations of the Second Order in Two Independent Variables, having an Intermediate Integral.

The present chapter is devoted mainly to the consideration of equations of the second order which are compatible with an equation of the first order or which (to use the customary phrase) possess an intermediate integral. A brief outline of the methods of Monge and of Boole is prefixed, those methods depending, for their proof, upon the assumption that an intermediate integral of a specified type does exist. Later, a method is given which does not depend upon that assumption and which leads to an intermediate integral, if it exists: in the process, the conditions under which the preceding integral exists are obtained.

In the preparation of the chapter, I have frequently used Boole's *Supplementary Volume*, quoted in § 236, and Imschenetsky's valuable memoir, quoted in § 180; other references are given in their appropriate connections. Some historical notes are given in Chapter III of Imschenetsky's memoir.

232. When we pass from strictly linear equations of the second order that are amenable to the Laplace process as developed by Darboux, and from the wider range of equations of the second order the general integral of which is expressible in finite and explicit form, the processes of integration that prove practicable are somewhat limited in range unless the results are allowed to be of a form that is not finite in expression. We always have the possibility of the application of Cauchy's theorem; the expression takes the form of a series which is at least singly infinite and may become doubly infinite, and there is no obvious mode of obtaining the integral in more compact form, even if there were certain knowledge that such compact form exists.

Other methods of proceeding to an integral have therefore to be devised. Among these, two are of prime importance—the methods devised by Ampère and by Darboux respectively: but even these are not of universal application, and they lead to results that are completely definite only when the equations to which they are applied are characterised by more or less well-defined properties. It is therefore worth while considering equations of classes which, though undoubtedly specialised, do possess something of a comprehensive character. Limiting ourselves still to equations of the second order in a single dependent variable and two independent variables, we shall here consider equations of the form

$$F(x, y, z, p, q, r, s, t) = 0,$$

which, in some form or other, possess integrals that are amenable to some finite processes of integration.

Among such equations, one of the most important classes (judged either from the historical development of the subject or from their occurrence in applications to subjects such as geometry or physics) is that class usually associated with the name of Monge*. The implicitly assumed property of these equations of the second order is that they possess an intermediate integral which involves derivatives of the first order and contains an arbitrary function in its expression: and the equation of the second order is assumed to be the unique equivalent, in that order, of the intermediate integral. When u and v denote two functions of x, y, z, p, q, which are distinct from one another, an intermediate integral of this type is represented by

$$f(u, v) = 0,$$

where f is any arbitrary function: then, as is well known†, the equation of the second order, which is the equivalent of this equation of the first order on the elimination of the arbitrary function, is

$$Rr + 2Ss + Tt + U(rt - s^2) = V,$$

where R, S, T, U, V are definite functions of x, y, z, p, q.

But the general converse is not valid: that is to say, if an equation of this form is propounded in which R, S, T, U, V are

* *Hist. de l'Acad. des Sciences*, 1784, pp. 118—192.

† In the following discussion, and for the sake of brevity, the customary method due to Monge will be assumed as belonging to the elements of the subject: it is expounded in the author's *Treatise on Differential Equations*, (third edition, 1903), §§ 229—241. Monge discussed only equations for which $U=0$: his method is applicable to equations without this restriction.

definite functions of x, y, z, p, q, there does not necessarily (and there certainly does not unconditionally) exist an intermediate integral of an equation of the first order equivalent to the propounded equation. In point of fact, the four quantities given by the ratios of R, S, T, U, V to one another are functions of the derivatives of u and v, when the equation is derived from the intermediate integral; and therefore, if the process is to be regarded as reversible, these four quantities must be expressible in terms of the derivatives of the two functions u and v. We should therefore expect that at least two conditions would be satisfied by the four quantities in question.

Assuming for the moment that the necessary conditions (whatever their number) are satisfied, so that the intermediate integral exists, there are various ways of proceeding to the construction of that intermediate integral.

One of these ways is Monge's method: it is actually comprised in Ampère's general method for the integration of partial equations which is applicable even when no intermediate integral exists. To give effect to the method, it is necessary to construct integrable combinations of certain ordinary equations which are homogeneous and linear in differential elements of the variables x, y, z, p, q.

Another method is that which customarily is associated with the name of Boole, though in effect it was given (at least partially) in earlier memoirs by De Morgan and Bour: it is actually comprised in Darboux's general method for the integration of partial equations which is applicable even when no intermediate integral exists. To give effect to the method, it is necessary to obtain the most general integral of a number of homogeneous linear partial differential equations of the first order which constitute a complete Jacobian system.

As is usual in such cases*, the conditions that the equations in the differential elements shall possess a number of integrable combinations are the same as the conditions that the simultaneous partial equations of the first order shall possess the same number of algebraically distinct integrals. The two methods, in so far as they are applicable to the equation in question, repose upon the

* See Part I of this work, §§ 26, 38.

same assumptions as to the fulfilment of implicit conditions: they are, in effect, equivalent to one another and are (so far as concerns the equation) merely different modes of arranging the analysis that contributes to the integration.

MONGE'S METHOD FOR THE EQUATION $Rr+2Ss+Tt+U(rt-s^2)=V$.

233. Monge's method is as follows, in outline*. Let the equation be

$$Rr+2Ss+Tt+U(rt-s^2)=V.$$

Two forms arise according as U is not zero, or is zero. We shall deal first with the form when U is not zero: we then divide the equation throughout by U, so that, without loss of generality in this case, we may take U as equal to unity, and the equation is

$$rt-s^2+Rr+2Ss+Tt=V.$$

The equations

$$dp=r\,dx+s\,dy,\quad dq=s\,dx+t\,dy,$$

are used to eliminate two of the three derivatives of the second order from the equation: when r and t are eliminated, the result is

$$\begin{aligned}dp\,dq+R\,dp\,dy+T\,dq\,dx-V\,dx\,dy&\\ =s\,(R\,dy\,dy-2S\,dx\,dy+T\,dx\,dx+dp\,dx+dq\,dy)&,\end{aligned}$$

and the equation will be satisfied if the equations

$$\begin{aligned}A&=\qquad dp\,dq+R\,dp\,dy+T\,dq\,dx-V\,dx\,dy=0,\\ B&=dp\,dx+dq\,dy+R\,(dy)^2-2S\,dx\,dy+T\,(dx)^2=0,\end{aligned}$$

are satisfied.

If r and s be eliminated, the result can be expressed in the form

$$tB+\frac{A\,dx-B\,dq}{dy}=0:$$

while, if s and t be eliminated, the result can be expressed in the form

$$rB+\frac{A\,dy-B\,dp}{dx}=0:$$

in each instance, it is sufficient to take $A=0$, $B=0$.

* The establishment of the various propositions, on the assumption that the necessary conditions are satisfied, is made in the various sections of the work quoted on p. 200, *note*.

Accordingly, the equations

$$\left.\begin{aligned} A &= dp\,dq + R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \\ B &= dp\,dx + dq\,dy + R\,(dy)^2 - 2S\,dx\,dy + T\,(dx)^2 = 0 \\ C &= dz - p\,dx - q\,dy = 0 \end{aligned}\right\}$$

are taken as a simultaneous set. Let

$$u = a, \quad v = b,$$

be two integrals of this set, the quantities a and b being constants: then it is proved* that the relation

$$u = f(v),$$

where f is an arbitrary function to be eliminated, leads to the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the proper relations between R, S, T, V being satisfied. As either u or v or both u and v will involve the derivatives of the first order, this equation

$$u = f(v)$$

is an intermediate integral. The method thus depends, for its effectiveness, upon the construction of the quantities u and v.

Now the equations $A = 0$ and $B = 0$ give

$$(dp + T\,dx)(dq + R\,dy) = (RT + V)\,dx\,dy,$$

$$(dp + T\,dx)\,dx + (dq + R\,dy)\,dy = 2S\,dx\,dy,$$

and therefore

$$dp + T\,dx + m\,dy = 0,$$

$$dq + R\,dy + n\,dx = 0,$$

where

$$mn = RT + V, \quad m + n = -2S.$$

Hence m and n are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0.$$

Two cases arise.

When the quadratic has equal roots, so that the condition

$$S^2 = RT + V.$$

* *L.c.*, § 232.

is satisfied, then $m = n = -S$; the set of equations in the differential elements can be uniquely represented by

$$\left.\begin{aligned} dp + Tdx - Sdy &= 0 \\ dq - Sdx + Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\}.$$

The conditions for the existence of an intermediate integral being supposed to be satisfied, it will be obtained in a form

$$u = f(v),$$

where $u = a$, $v = b$, are integral equations of the differential relations linear in the differential elements. It may be noticed that there are three differential relations and that therefore, if all the appropriate conditions are satisfied, there could be three integral relations

$$u = a, \quad v = b, \quad w = c,$$

equivalent to them, where a, b, c are constants. We shall return later to the consideration of this last possibility: meanwhile, an intermediate integral is obtainable on the supposition that the general conditions are satisfied.

When the quadratic has unequal roots, let

$$S^2 - RT - V = \theta^2,$$

where θ is not zero. Then m is not equal to n: we have

$$m, n = -S \pm \theta;$$

and therefore, taking

$$\rho = -S + \theta, \quad \sigma = -S - \theta,$$

the equations $A = 0$, $B = 0$ give either

$$dp + Tdx + \rho dy = 0,$$
$$dq + Rdy + \sigma dx = 0;$$

or

$$dp + Tdx + \sigma dy = 0,$$
$$dq + Rdy + \rho dx = 0;$$

and the equations $A = 0$, $B = 0$, may give both of these systems, though this is not necessarily a fact. Thus the set of equations in the differential elements can be replaced by one or other of the systems

$$\left.\begin{aligned} dp + Tdx + \rho dy &= 0 \\ dq + \sigma dx + Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + Tdx + \sigma dy &= 0 \\ dq + \rho dx + Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\},$$

where

$$\rho = -S + (S^2 - RT - V)^{\frac{1}{2}} = -S + \theta,$$

$$\sigma = -S - (S^2 - RT - V)^{\frac{1}{2}} = -S - \theta;$$

and so far, there is nothing to exclude the possibility of both systems (under proper conditions) being admissible. The original equations in the differential elements possessed integrals which because of the conditions that were satisfied, led to an intermediate integral; consequently, one or other of the two systems, linear in the differential elements and replacing the original equations, must possess these integrals. Let them belong to the first set in a form

$$u_1 = a, \quad v_1 = b,$$

so that the conditions are satisfied for the first set; the intermediate integral is

$$u_1 = f(v_1),$$

where f is an arbitrary function.

It may happen that the conditions are satisfied for both sets, and that the second set possess integrals in a form

$$u_2 = a', \quad v_2 = b',$$

where a' and b' are constants; an intermediate integral is

$$u_2 = g(v_2),$$

where g is an arbitrary function. In these circumstances, there are two distinct intermediate integrals; it is part of the theory, and it is proved*, that these two distinct intermediate integrals coexist, so that they can be used as simultaneous equations to express p and q in terms of x, y, z, the values of p and q given by them being such as to render

$$dz = pdx + qdy$$

* *L.c.*, § 236; also see hereafter, § 239.

an exact equation, quadrature of which gives a primitive of the original equation. More generally, however, if the conditions are satisfied, they are satisfied for only one of the two linear sets: and then there is only one intermediate integral.

234. The corresponding equations for the case when U vanishes, so that the equation has the form

$$Rr + 2Ss + Tt = V,$$

can be stated similarly: the assumption being made that the equation* possesses an intermediate integral. The equation $A = 0$ is now replaced by

$$A' = Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0,$$

and the equation $B = 0$ is now replaced by

$$B' = R\,(dy)^2 - 2S\,dx\,dy + T\,(dx)^2 = 0.$$

Let ρ_1 and σ_1 be the roots of

$$R\mu^2 - 2S\mu + T = 0,$$

so that

$$R\rho_1 = S + \alpha, \quad R\sigma_1 = S - \alpha,$$

where

$$\alpha^2 = S^2 - RT.$$

Then, if the quadratic has equal roots, so that $\alpha = 0$, the equations

$$A' = 0, \quad B' = 0, \quad dz - pdx - qdy = 0,$$

can uniquely be replaced by the system

$$\left.\begin{aligned} Rdy - Sdx &= 0 \\ Rdp + Sdq - Vdx &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\};$$

if $u = a$, $v = b$, where a and b are constants, be integrals of this linear system, then

$$u = f(v),$$

where f is an arbitrary function, is the single intermediate integral that can be obtained in this way.

* If R, S, T involve only x and y, and if V is homogeneous and linear in p, q, z, having functions of x and y for coefficients, the equation belongs to the linear form already discussed in Chapter XIII. In general, however, even for integrable equations of the type now under consideration, these limitations are not observed.

If the quadratic has unequal roots, it can be replaced by one or other of the two systems

$$\left.\begin{aligned} R\rho_1 dp + T dq - V\rho_1 dx = 0 \\ dy - \rho_1 dx = 0 \\ dz - (p + \rho_1 q)\, dx = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} R\sigma_1 dp + T dq - V\sigma_1 dx = 0 \\ dy - \sigma_1 dx = 0 \\ dz - (p + \sigma_1 q)\, dx = 0 \end{aligned}\right\}:$$

or what is the equivalent, by one or other of the systems

$$\left.\begin{aligned} dp + \sigma_1 dq - \frac{V}{R}\, dx = 0 \\ dy - \rho_1 dx = 0 \\ dz - (p + \rho_1 q)\, dx = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + \rho_1 dq - \frac{V}{R}\, dx = 0 \\ dy - \sigma_1 dx = 0 \\ dz - (p + \sigma_1 q)\, dx = 0 \end{aligned}\right\}.$$

The equation is supposed to possess an intermediate integral, so that the necessary conditions are satisfied; they are therefore satisfied in connection with one or other of the systems, say, with the first. If integrals of that first system are obtained in a form

$$u_1 = a, \quad v_1 = b,$$

where a and b are constants, an intermediate integral is given by the equation

$$u_1 = f(v_1),$$

where f is an arbitrary function.

It may happen that the conditions are satisfied also for the other linear system, so that it possesses integrals of the form

$$u_2 = a', \quad v_2 = b',$$

where a' and b' are constants: then an intermediate integral is given by the equation

$$u_2 = g(v_2),$$

where g is an arbitrary function. As before, these two distinct intermediate integrals coexist: when they are resolved, so as to express p and q in terms of x, y, z, the values of p and q so provided make

$$dz - p dx - q dy = 0$$

an exact equation, quadrature of which gives a primitive of the original equation. More generally, however, when the conditions are satisfied, they are satisfied in connection with only one of the two linear sets: and then only one intermediate integral can be obtained.

The form adopted for the linear system implies that R is not zero: the appropriate modifications when $R=0$ can easily be made.

235. To complete the integration, whether the original equation be

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

or be

$$Rr + 2Ss + Tt = V,$$

we integrate the intermediate integral, if only one has been obtainable, by the methods which apply to equations of the first order; according to the form of the intermediate integral, we may have one or more forms for the final primitive. We have seen that, when two intermediate integrals have been obtained, the final primitive is obtained by resolving the two equations for p and q and effecting a quadrature.

Such, in brief outline, is Monge's method of integrating the equations. It is effective only if the appropriate conditions are satisfied; and the explicit expression of these conditions must be obtained. We shall first, however, in similar brevity, give an outline of Boole's method of integrating the equations.

Boole's Method for the Equations.

236. Boole's method*, like Monge's, is based upon an assumption that an intermediate integral of the form

$$u = f(v),$$

where f is an arbitrary function, and u, v are definite functions of x, y, z, p, q, is possessed by the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

or by the equation

$$Rr + 2Ss + Tt = V,$$

* It is contained in chapters XXVIII and XXIX (in the latter more particularly) of the *Supplementary Volume* of his *Treatise on Differential Equations*: this volume was published in 1865, the year after his death. See also the memoir, *Crelle*, t. LXI (1863), pp. 309—333.

in the respective instances subjected to Monge's method. The difference from Monge's method lies in the fact that Boole obtains u and v as integrals of simultaneous partial equations of the first order, whereas Monge obtains them as integrals of equations in differential elements. Boole's procedure is as follows.

Denoting by $\frac{du}{dx}, \frac{du}{dy}, \frac{dv}{dx}, \frac{dv}{dy}$ the complete derivatives of u and v with regard to x and y, so that

$$\frac{d\theta}{dx} = \frac{\partial\theta}{\partial x} + p\frac{\partial\theta}{\partial z}, \quad \frac{d\theta}{dy} = \frac{\partial\theta}{\partial y} + q\frac{\partial\theta}{\partial z},$$

for any quantity θ, the assumed intermediate integral leads to the relations

$$\frac{du}{dx} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} = \left(\frac{dv}{dx} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q}\right) f'(v),$$

$$\frac{du}{dy} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} = \left(\frac{dv}{dy} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q}\right) f'(v).$$

When $f'(v)$ is eliminated between these two relations, and the resulting equation is arranged with reference to the combinations of r, s, t, we have

$$U_1(rt - s^2) + R_1 r + S_1 s + T_1 t = V_1,$$

where

$$R_1 = \frac{\partial u}{\partial p}\frac{dv}{dy} - \frac{\partial v}{\partial p}\frac{du}{dy},$$

$$S_1 = \frac{\partial u}{\partial q}\frac{dv}{dy} - \frac{\partial v}{\partial q}\frac{du}{dy} + \frac{\partial v}{\partial p}\frac{du}{dx} - \frac{\partial u}{\partial p}\frac{dv}{dx},$$

$$T_1 = \frac{\partial v}{\partial q}\frac{du}{dx} - \frac{\partial u}{\partial q}\frac{dv}{dx},$$

$$U_1 = \frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p},$$

$$V_1 = \frac{du}{dy}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dy}.$$

If this derived equation is the same as the original differential equation in either of the forms propounded for integration (with the appropriate conditions satisfied), then

$$\frac{R_1}{R} = \frac{S_1}{2S} = \frac{T_1}{T} = \frac{V_1}{V} = U_1,$$

for the form

$$rt - s^2 + Rr + 2Ss + Tt = V;$$

and

$$\frac{R_1}{R} = \frac{S_1}{2S} = \frac{T_1}{T} = \frac{V_1}{V}, \quad U_1 = 0,$$

for the form

$$Rr + 2Ss + Tt = V.$$

We take these in turn.

We have

$$R_1 \left(\frac{\partial u}{\partial q}\right)^2 - S_1 \frac{\partial u}{\partial q}\frac{\partial u}{\partial p} + T_1 \left(\frac{\partial u}{\partial p}\right)^2$$

$$= \left(\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q}\right)\left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)$$

$$= U_1 \left(\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q}\right);$$

hence

$$\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q} - R\left(\frac{\partial u}{\partial q}\right)^2 + 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} - T\left(\frac{\partial u}{\partial p}\right)^2 = 0$$

for the first form of the equation, and

$$R\left(\frac{\partial u}{\partial q}\right)^2 - 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} + T\left(\frac{\partial u}{\partial p}\right)^2 = 0$$

for the second form of the equation. Similarly,

$$R_1 \frac{du}{dx}\frac{\partial u}{\partial q} + T_1 \frac{du}{dy}\frac{\partial u}{\partial p} + V_1 \frac{\partial u}{\partial p}\frac{\partial u}{\partial q}$$

$$= \left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)\frac{du}{dx}\frac{du}{dy}$$

$$= U_1 \frac{du}{dx}\frac{du}{dy};$$

and therefore

$$\frac{du}{dx}\frac{du}{dy} - R\frac{du}{dx}\frac{\partial u}{\partial q} - T\frac{du}{dy}\frac{\partial u}{\partial p} - V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0$$

for the first form of the equation, and

$$R\frac{du}{dx}\frac{\partial u}{\partial q} + T\frac{du}{dy}\frac{\partial u}{\partial p} + V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0$$

for the second form of the equation.

As the quantities R_1, S_1, T_1, U_1, V_1 are skew symmetric in u and v, it is the fact (as may easily be verified) that the same equations, for the respective forms, are satisfied by v also.

For the first form of the equation, we have

$$\left(\frac{du}{dx} - T\frac{\partial u}{\partial p}\right)\left(\frac{du}{dy} - R\frac{\partial u}{\partial q}\right) = (RT+V)\frac{\partial u}{\partial p}\frac{\partial u}{\partial q},$$

and

$$\left(\frac{du}{dx} - T\frac{\partial u}{\partial p}\right)\frac{\partial u}{\partial p} + \left(\frac{du}{dy} - R\frac{\partial u}{\partial q}\right)\frac{\partial u}{\partial q} = -2S\frac{\partial u}{\partial p}\frac{\partial u}{\partial q};$$

and therefore, either

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} - \sigma\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} - \rho\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\},$$

or

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\},$$

where

$$\rho = -S + (S^2 - RT - V)^{\frac{1}{2}} = -S + \theta,$$
$$\sigma = -S - (S^2 - RT - V)^{\frac{1}{2}} = -S - \theta,$$

so that ρ and σ are the roots of the equation

$$\mu^2 + 2\mu S + RT + V = 0.$$

Thus the equations satisfied by u (and by v also) can be replaced by one or other of the above pairs of homogeneous linear equations: in particular cases, both the pairs may be valid.

When the roots of the quadratic are equal, so that $\rho = \sigma = -S$, the equations for u (and for v also) are the single pair

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} + S\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} + S\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\}.$$

For the second form of the equation, we have

$$\left(\frac{\partial u}{\partial q} - \rho_1\frac{\partial u}{\partial p}\right)\left(\frac{\partial u}{\partial q} - \sigma_1\frac{\partial u}{\partial p}\right) = 0,$$

where, as before, ρ_1 and σ_1 are the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0,$$

so that

$$R\rho_1 = S + \alpha, \quad R\sigma_1 = S - \alpha,$$

and

$$\alpha^2 = S^2 - RT.$$

Thus either

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \rho_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \sigma_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\},$$

or

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \sigma_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \rho_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

The equations satisfied by u (and by v also) can be replaced by one or other of these pairs of homogeneous linear equations: in particular cases, both the pairs may be valid.

When the roots of the quadratic are equal, so that $\rho_1 = \sigma_1 = \frac{S}{R}$, the equations for u (and for v also) are the single pair

$$\left.\begin{aligned} R\frac{\partial u}{\partial q} - S\frac{\partial u}{\partial p} = 0 \\ R\frac{du}{dx} + S\frac{du}{dy} + V\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

237. The sets of equations which have been constructed, being (in Monge's method) homogeneous and linear in the differential elements, and (in Boole's method) homogeneous and linear in the first derivatives of an unknown dependent variable, have been obtained on the hypothesis that the differential equation possesses an intermediate integral involving an arbitrary function in its expression. It is important to observe that the equations in Monge's method are equivalent to those in Boole's method, so that the problem of obtaining the integral equivalent of one aggregate is effectively the same as that of obtaining the integral equivalent of the other aggregate.

This remark admits of simple verification for each of the two forms of equation.

When the original equation, supposed to possess an intermediate integral involving an arbitrary function, is of the form

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the equations, subsidiary to the construction of that integral in Monge's method, are the respective aggregates comprised in one or other of the sets

$$\left.\begin{array}{r} dp + T dx + \rho dy = 0 \\ dq + \sigma dx + R dy = 0 \\ dz - p dx - q dy = 0 \end{array}\right\}, \qquad \left.\begin{array}{r} dp + T dx + \sigma dy = 0 \\ dq + \rho dx + R dy = 0 \\ dz - p dx - q dy = 0 \end{array}\right\},$$

where ρ and σ are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0.$$

To construct the intermediate integral of the original equation, we need two integrals

$$u = a, \quad v = b,$$

of one or other of the systems. Let

$$\theta = \text{constant},$$

where θ is a function of x, y, z, p, q, be an integral of the first system; then the relation

$$\frac{\partial\theta}{\partial x} dx + \frac{\partial\theta}{\partial y} dy + \frac{\partial\theta}{\partial z} dz + \frac{\partial\theta}{\partial p} dp + \frac{\partial\theta}{\partial q} dq = 0$$

is satisfied identically in virtue of the equations in that system. Substituting in that relation the values of dz, dp, dq as given by the system in terms of dx and dy, we have

$$\left(\frac{\partial\theta}{\partial x} + p\frac{\partial\theta}{\partial z} - T\frac{\partial\theta}{\partial p} - \sigma\frac{\partial\theta}{\partial q}\right) dx + \left(\frac{\partial\theta}{\partial y} + q\frac{\partial\theta}{\partial z} - \rho\frac{\partial\theta}{\partial p} - R\frac{\partial\theta}{\partial q}\right) dy = 0,$$

that is,

$$\left(\frac{d\theta}{dx} - T\frac{\partial\theta}{\partial p} - \sigma\frac{\partial\theta}{\partial q}\right) dx + \left(\frac{d\theta}{dy} - \rho\frac{\partial\theta}{\partial p} - R\frac{\partial\theta}{\partial q}\right) dy = 0.$$

In the absence of any relation between dx and dy, we have

$$\left.\begin{array}{r} \dfrac{d\theta}{dx} - T\dfrac{\partial\theta}{\partial p} - \sigma\dfrac{\partial\theta}{\partial q} = 0 \\ \dfrac{d\theta}{dy} - \rho\dfrac{\partial\theta}{\partial p} - R\dfrac{\partial\theta}{\partial q} = 0 \end{array}\right\},$$

which are the equations of the first system in Boole's method to be satisfied by the quantities u and v that are needed for the construction of the intermediate integral

$$u = f(v).$$

Similarly, the equations of the second system in Monge's method lead to the equations of the second system in Boole's method.

When the two systems in Monge's method merge into a single system owing to the condition

$$S^2 = RT + V,$$

which gives equal roots for the quadratic in μ, the two systems in Boole's method merge into a single system owing to the same condition; and the single system in Monge's method then leads to the single system in Boole's method.

Again, when the original equation, supposed to possess an intermediate integral involving an arbitrary function, is of the form

$$Rr + 2Ss + Tt = V,$$

the equations, subsidiary to the construction of that integral in Monge's method, are the respective aggregates comprised in one or other of the sets

$$\left.\begin{aligned} dp + \sigma_1 dq - \frac{V}{R}\,dx = 0 \\ dy - \rho_1 dx = 0 \\ dz - (p + \rho_1 q)\,dx = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + \rho_1 dq - \frac{V}{R}\,dx = 0 \\ dy - \sigma_1 dx = 0 \\ dz - (p + \sigma_1 q)\,dx = 0 \end{aligned}\right\},$$

where ρ_1 and σ_1 are the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0.$$

To construct the intermediate integral of the original equation, we need two integrals

$$u = a, \quad v = b,$$

of one or other of the two systems. Let

$$\phi = \text{constant},$$

where ϕ is a function of x, y, z, p, q, be an integral of the first system; then the relation

$$\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz + \frac{\partial\phi}{\partial p}\,dp + \frac{\partial\phi}{\partial q}\,dq = 0$$

is satisfied identically in virtue of the equations in that system. Substituting in that relation the value of dp in terms of dq and dx, as well as the values of dy and dz as given by the system, we have

$$\left\{\frac{\partial \phi}{\partial x} + \rho_1 \frac{\partial \phi}{\partial y} + (p + \rho_1 q)\frac{\partial \phi}{\partial z} + \frac{V}{R}\frac{\partial \phi}{\partial p}\right\} dx + \left(\frac{\partial \phi}{\partial q} - \sigma_1 \frac{\partial \phi}{\partial p}\right) dp = 0,$$

that is,

$$\left(\frac{d\phi}{dx} + \rho_1 \frac{d\phi}{dy} + \frac{V}{R}\frac{\partial \phi}{\partial p}\right) dx + \left(\frac{\partial \phi}{\partial q} - \sigma_1 \frac{\partial \phi}{\partial p}\right) dp = 0.$$

In the absence of any relation between dx and dp alone, we have

$$\left.\begin{aligned} \frac{\partial \phi}{\partial q} - \sigma_1 \frac{\partial \phi}{\partial p} &= 0 \\ \frac{d\phi}{dx} + \rho_1 \frac{d\phi}{dy} + \frac{V}{R}\frac{\partial \phi}{\partial p} &= 0 \end{aligned}\right\},$$

which are the equations in one of the systems in Boole's method to be satisfied by the quantities u and v that are needed for the construction of the intermediate integral

$$u = f(v).$$

Similarly, the equations of the second system in Monge's method lead to the equations of the alternative system in Boole's method.

When the two systems in Monge's method merge into a single system owing to the condition

$$S^2 = RT,$$

which gives equal roots for the quadratic in μ, the two systems in Boole's method also merge into a single system owing to the same condition; and the single system in Monge's method then leads to the single system in Boole's method.

If for either form of equation, there are two distinct intermediate integrals derivable by Monge's method, they are derivable also by Boole's method: for, in each method, both subsidiary systems are then valid.

The primitive of the original equation is derived from the intermediate integral or integrals, whether obtained by the one method or the other.

Ex. 1. Integrate the equation

$$2pqyr+(p^2y+qx)s+xpt=p^2q(rt-s^2)+xy,$$

by obtaining a general intermediate integral.

Comparing the form with

$$rt-s^2+Rr+2Ss+Tt=V,$$

we have

$$R=-\frac{2y}{p},\quad 2S=-\frac{y}{q}-\frac{x}{p^2},\quad T=-\frac{x}{pq},\quad V=-\frac{xy}{p^2q};$$

thus the critical quadratic is

$$\mu^2-\mu\left(\frac{y}{q}+\frac{x}{p^2}\right)+\frac{xy}{p^2q}=0,$$

so that

$$\rho,\ \sigma=\frac{x}{p^2},\ \frac{y}{q},$$

in either arrangement.

The two systems of equations in Monge's method are, firstly,

$$\left.\begin{aligned} dp-\frac{x}{pq}\,dx+\frac{x}{p^2}\,dy&=0\\ dq+\frac{y}{q}\,dx-\frac{2y}{p}\,dy&=0\\ dz-p\,dx-q\,dy&=0 \end{aligned}\right\},$$

which do not possess an integrable combination: and, secondly,

$$\left.\begin{aligned} dp-\frac{x}{pq}\,dx+\frac{y}{q}\,dy&=0\\ dq+\frac{x}{p^2}\,dx-2\frac{y}{p}\,dy&=0\\ dz-p\,dx-q\,dy&=0 \end{aligned}\right\}.$$

Two integrals of the latter are easily seen to be

$$p^2q-\tfrac{1}{2}x^2=a,\quad pq-\tfrac{1}{2}y^2=b;$$

and therefore an intermediate integral is

$$p^2q-\tfrac{1}{2}x^2=f(pq-\tfrac{1}{2}y^2),$$

where f is an arbitrary function. It is the only intermediate integral of the original equation.

When we proceed by Boole's method, one set of equations is

$$\left.\begin{aligned} \frac{d\theta}{dx}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{y}{q}\frac{\partial\theta}{\partial q}&=0\\ \frac{d\theta}{dy}-\frac{x}{p^2}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}&=0 \end{aligned}\right\},$$

which do not possess a common integral; and the other set of subsidiary equations is

$$\left.\begin{aligned}\frac{d\theta}{dx}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0\\ \frac{d\theta}{dy}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0\end{aligned}\right\},$$

which do possess common integrals. When expressed in full, the latter equations are

$$\theta_1=\frac{\partial\theta}{\partial x}+p\frac{\partial\theta}{\partial z}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0,$$

$$\theta_2=\frac{\partial\theta}{\partial y}+q\frac{\partial\theta}{\partial z}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0;$$

in order that these may coexist, we must have

$$(\theta_1,\ \theta_2)=0,$$

that is

$$\left(\frac{y}{q}-\frac{x}{p^2}\right)\frac{\partial\theta}{\partial z}=0,$$

and therefore the equations are

$$\vartheta_1=\frac{\partial\theta}{\partial x}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0,$$

$$\vartheta_2=\frac{\partial\theta}{\partial y}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0,$$

$$\vartheta_3=\frac{\partial\theta}{\partial z}=0.$$

We now have

$$(\vartheta_1,\ \vartheta_2)=0,\quad (\vartheta_1,\ \vartheta_3)=0,\quad (\vartheta_2,\ \vartheta_3)=0;$$

thus the set of these three equations of the first order is a complete Jacobian system in involution. As five variables x, y, z, p, q occur in this Jacobian system, it possesses two integrals algebraically independent of one another: by the ordinary processes explained in Chapter IV in the preceding volume, these are found to be

$$p^2q-\tfrac{1}{2}x^2,\quad pq-\tfrac{1}{2}y^2.$$

Thus, as before, the intermediate integral of the original equation is

$$p^2q-\tfrac{1}{2}x^2=f(pq-\tfrac{1}{2}y^2),$$

where f is an arbitrary function.

When this equation, as an equation of the first order, is integrated, it will give a primitive of the original equation. The subsidiary system in Charpit's method does not appear to offer integrable combinations in finite terms, when f remains a quite arbitrary function. Evidently

$$p^2q-\tfrac{1}{2}x^2=a$$

is a particular intermediate integral, a being constant; it leads to a particular primitive

$$z=a''+\frac{1}{a'}\int(\tfrac{1}{2}x^2+a)^{\frac{1}{2}}\,dx+\frac{a'^2}{y},$$

where a, a', a'' are arbitrary constants. Evidently

$$pq-\tfrac{1}{2}y^2=c$$

is another particular intermediate integral, c being constant, (which, however, is incompatible with the preceding particular intermediate integral); it leads to a particular primitive

$$z=c''+c'x+\frac{1}{c'}(cy+\tfrac{1}{6}y^3),$$

where c, c', c'' are arbitrary constants. Other particular primitives can be obtained by taking other particular forms of f in the general intermediate integral.

In the two particular primitives that have been given, three arbitrary constants occur. We shall hereafter (Chapter XIX) see how, by a method due to Imschenetsky, it is possible to generalise a primitive of such an equation of the second order that contains three arbitrary constants.

Ex. 2. Integrate the equation

$$q(1+q)\,r-(1+p+q+2pq)\,s+p(1+p)\,t=0.$$

Here

$$U=0,\quad V=0,\quad R=q(1+q),\quad 2S=-(1+p+q+2pq),\quad T=p(1+p);$$

the quadratic in μ for this case is

$$q(1+q)\,\mu^2+(1+p+q+2pq)\,\mu+p(1+p)=0,$$

and therefore

$$\rho_1,\ \sigma_1=-\frac{p}{1+q},\quad -\frac{1+p}{q}.$$

One subsidiary system is

$$\left.\begin{aligned}\phi_1&=\frac{\partial\phi}{\partial q}+\frac{p}{1+q}\frac{\partial\phi}{\partial p}=0\\ \phi_2&=\frac{d\phi}{dx}-\frac{1+p}{q}\frac{d\phi}{dy}=0\end{aligned}\right\}.$$

The latter, in full, is

$$\phi_2=\frac{\partial\phi}{\partial x}-\frac{1+p}{q}\frac{\partial\phi}{\partial y}-\frac{\partial\phi}{\partial z}=0.$$

In order that $\phi_1=0$ and $\phi_2=0$ may coexist, we must have

$$0=(\phi_1,\ \phi_2)=\frac{1+p+q}{q^2(1+q)}\frac{\partial\phi}{\partial y}=0,$$

that is, we have three equations

$$\phi_1=\frac{\partial\phi}{\partial q}+\frac{p}{1+q}\frac{\partial\phi}{\partial p}=0,$$

$$\phi_2'=\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial z}=0,$$

$$\phi_3=\frac{\partial\phi}{\partial y}=0:$$

and these are easily proved to form a complete Jacobian system. They therefore possess two independent common integrals, which can be taken in the form

$$\frac{1+q}{p}, \quad x+z;$$

hence an intermediate integral is

$$\frac{1+q}{p}=f(x+z),$$

where f denotes an arbitrary function.

Similarly, the other subsidiary system

$$\left.\begin{aligned}\frac{\partial\phi}{\partial q}+\frac{1+p}{q}\frac{\partial\phi}{\partial p}=0\\ \frac{d\phi}{dx}-\frac{p}{1+q}\frac{d\phi}{dy}=0\end{aligned}\right\}$$

is found to lead to an intermediate integral

$$\frac{1+p}{q}=g(y+z),$$

where g denotes an arbitrary function.

The primitive can be derived, after a quadrature, by combining the two distinct intermediate integrals. When we take these in the form

$$x+z=\phi(u)=\phi\left(\frac{1+q}{p}\right),$$

$$y+z=\psi(v)=\psi\left(\frac{1+p}{q}\right),$$

we have

$$(1+p+q)\,dz=p\,(dz+dx)+q\,(dz+dy),$$

that is,

$$(1+u)(1+v)\,dz=(v+1)\,\phi'(u)\,du+(u+1)\,\psi'(v)\,dv.$$

Hence

$$\begin{aligned}z&=\int\frac{\phi'(u)}{1+u}\,du+\int\frac{\psi'(v)}{1+v}\,dv\\ &=\Phi(u)+\Psi(v)\\ &=F(x+z)+G(y+z),\end{aligned}$$

where F and G are arbitrary functions.

Ex. 3. Integrate the following equations :—

(i) $q(r+t)+ps=0$;

(ii) $z(r-t)-p^2+q^2=0$;

(iii) $x^2r-y^2t=xp-yq+xy$;

(iv) $(r-t)\,xy-s\,(x^2-y^2)=qx-py$;

(v) $(r-s)\,x=(t-s)\,y$;

(vi) $x^2r+2xys+y^2t=f(xp+yq)$;

(vii) $q^2r-2pqs+p^2t=0$;

(viii) $q^2r-2pqs+p^2t=\dfrac{(p+q)^2(p-q)}{y-x}$;

(ix) $q(1+q)r+(p+q+2pq)s+p(1+p)t=0$;

(x) $(b+cq)^2r-2(a+cp)(b+cq)s+(a+cp)^2t=0$;

(xi) $rt-s^2=1$;

(xii) $y(rt-s^2)+qr+(p+x)s+yt+q=0$;

(xiii) $z(1+q^2)r-2pqzs+z(1+p^2)t+1+p^2+q^2+z^2(rt-s^2)=0$;

(xiv) $xy(rt-s^2)-xqr-ypt+pq=0$;

(xv) $p^2q^2(rt-s^2)+q^2r+4pqs+p^2t=1$;

(xvi) $rt-s^2+q\left(\dfrac{q}{z}-\dfrac{1}{y}\right)r-2\dfrac{pq}{z}s+p\left(\dfrac{p}{z}-\dfrac{1}{x}\right)t+\dfrac{pq}{xyz}(z-px-qy)=0.$

General Method for the Intermediate Integral (if any) of any Equation.

238. Alike in Monge's method and in Boole's method, the differential equation of the second order has been supposed to be constructed by the elimination of an arbitrary function from an intermediate integral of given type. We have seen that an intermediate integral, in the form of an equation of the first order involving two arbitrary constants, leads (on the elimination of these constants) to an equation of the second order (§ 180). It is a characteristic property of the process of elimination that the nature of the eliminated magnitudes is ignored; moreover, the eliminant bears no explicit recognisable trace of the sources from which it came. The process is not reversible; if the effect is to be reversed, definite methods must be devised for the purpose.

Now when an equation of the second order is actually given without any indication of its origin, the preceding methods due to Monge and to Boole respectively may happen to be applicable: but the argument adopted for their construction cannot be used to prove that the methods are applicable, because the mode of construction of the differential equation is not revealed by the equation itself. The tests of applicability must be obtained otherwise: they are provided by assigning the conditions that the subsidiary equations in Monge's method possess two integrable combinations and (what are effectively the same relations) the conditions that the subsidiary equations in Boole's method possess two independent integrals. If the conditions are not satisfied, neither of the methods as stated leads to an intermediate integral of the type $u=f(v)$, where f is an arbitrary function: such an intermediate integral is not possessed.

In selecting the equations of the type

$$U(rt-s^2)+Rr+2Ss+Tt=V,$$

regard was paid to the facts, that it was deduced from an intermediate integral of the assumed form and that an equation must be of that type in order to possess such an intermediate integral: but it was not proved (as, indeed, it cannot be proved) either that such an equation unconditionally possesses an intermediate integral of the assigned form or that the intermediate integral of that form is the only kind of intermediate integral which can lead to an equation of the particular type. Accordingly, before proceeding to the discussion of the significance and even of the coexistence of the subsidiary equations, we shall obtain them in a different manner; and the process will shew their organic connection with the original equation.

Consider, more generally, any differential equation of the second order

$$f(x, y, z, p, q, r, s, t)=0;$$

and suppose that it possesses an intermediate integral of the first order

$$u(x, y, z, p, q)=0,$$

or (what is the same thing) that it is compatible with such an equation of the first order: no assumption is made as to the character of u. Then the equation $f=0$ arises from some association of the two equations

$$\frac{du}{dx}+\frac{\partial u}{\partial p}r+\frac{\partial u}{\partial q}s=0,$$

$$\frac{du}{dy}+\frac{\partial u}{\partial p}s+\frac{\partial u}{\partial q}t=0,$$

either by the elimination of some arbitrary quantity or by some combination of the two equations made at will: the equation of the second order, being compatible with the equation of the first order, is compatible with the two derivatives of the latter, and therefore the three equations, which involve r, s, t, are not independent of one another. Consequently, when we proceed to resolve them to obtain expressions for r, s, t, these expressions must be evanescent, that is, on the hypothesis that an equation $u=0$ is compatible with the original equation. When we assign the conditions that the expressions for r, s, t shall be evanescent (which can be secured by substituting for r and t in terms of s from the two derivatives, and making the resulting form of $f=0$ evanescent as an equation for s), we shall have a number of relations that involve the derivatives of u. Thus the quantity u, as to which no assumption has been made save that $u=0$ is an equation of the first order and that therefore both the quantities $\frac{\partial u}{\partial p}$ and $\frac{\partial u}{\partial q}$ do not vanish, satisfies a number of partial differential equations of the first order. *Any common integral of these equations, which involves p or q or both, is an intermediate integral of the original equation.* But, as is known, a simultaneous system of equations of the first order does not unconditionally possess common integrals; in the present instance, the conditions for the possession of common integrals or a common integral are conditions that $f=0$ shall possess an intermediate integral.

Moreover, if the system of equations which must be satisfied by u should possess more than one integral, the relation of these common integrals to one another must be investigated, particularly in connection with the intermediate integral which then is possessed by $f=0$.

Ex. 1. Obtain an intermediate integral, if any, of the equation

$$(sq-tp)^2=(rt-s^2)(sp-rq).$$

Writing

$$u_x=\frac{du}{dx},\quad u_y=\frac{du}{dy},\quad u_p=\frac{\partial u}{\partial p},\quad u_q=\frac{\partial u}{\partial q},$$

we eliminate r and t by means of

$$u_x+ru_p+su_q=0,$$
$$u_y+su_p+tu_q=0,$$

and we make the resulting equation in s evanescent. In order that this may be the fact, we must have

$$\left.\begin{aligned}\frac{u_p^2}{u_q}(pu_p+qu_q)^2&=(pu_p+qu_q)(u_qu_y+u_pu_x)\\ 2pu_y\frac{u_p^2}{u_q}(pu_p+qu_q)&=(pu_p+qu_q)u_xu_y+qu_x(u_qu_y+u_pu_x)\\ p^2u_y^2\frac{u_p^2}{u_q}&=qu_x^2u_y\end{aligned}\right\},$$

which are the simultaneous equations for the determination of u if it exists. We have first to resolve these equations algebraically.

I. We may have

$$pu_p + qu_q = 0, \quad u_x = 0, \quad u_y = 0:$$

that is,

$$p\frac{\partial u}{\partial p} + q\frac{\partial u}{\partial q} = 0, \quad \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0.$$

The conditions of coexistence require that

$$\frac{\partial u}{\partial z} = 0;$$

and then the most general integral is

$$u = \phi\left(\frac{p}{q}\right),$$

where ϕ is arbitrary. Thus there is an intermediate integral

$$\phi\left(\frac{p}{q}\right) = 0,$$

that is,

$$p - aq = 0,$$

where a is an arbitrary constant: but it is very special, for it satisfies the three equations

$$sq - tp = 0, \quad rt - s^2 = 0, \quad sp - rq = 0.$$

II. We may have

$$pu_p + qu_q = 0,$$

and u_x, u_y not zero: we find

$$u_x = -qu_p,$$

$$pu_y - qu_x = 0:$$

that is,

$$p\frac{\partial u}{\partial p} + q\frac{\partial u}{\partial q} = 0, \quad \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + q\frac{\partial u}{\partial p} = 0, \quad p\frac{\partial u}{\partial y} - q\frac{\partial u}{\partial x} = 0.$$

The Jacobian conditions of coexistence require that

$$\frac{\partial u}{\partial y} = 0,$$

by combining the second and third; the third equation then gives

$$\frac{\partial u}{\partial x} = 0.$$

By combining the first and second, we find

$$\frac{\partial u}{\partial z} = 0;$$

and the second equation then gives

$$\frac{\partial u}{\partial p} = 0,$$

so that the first now is

$$\frac{\partial u}{\partial q} = 0.$$

No intermediate integral is thus provided.

III. Assuming that pu_p+qu_q is not zero, as the alternative has been discussed, we find that the three equations are satisfied in virtue of two only, viz.

$$u_x=p\frac{u_p^2}{u_q},\quad u_y=q\frac{u_p^2}{u_q},$$

provided pu_p+2qu_q is not zero. We shall assume this latter condition satisfied for the moment and, later, we shall consider the alternative.

Writing $x, y, z, p, q=x_1, x_2, x_3, x_4, x_5$, we have the two equations in the form

$$f_1=p_1+x_4p_3-x_4\frac{p_4^2}{p_5}=0,$$

$$f_2=p_2+x_5p_3-x_5\frac{p_4^2}{p_5}=0;$$

the Jacobian condition of coexistence is

$$0=(f_1, f_2)=-\left(p_3-\frac{p_4^2}{p_5}\right)\left(x_4\frac{p_4^2}{p_5^2}+2x_5\frac{p_4}{p_5}\right).$$

The second factor on the right-hand side is

$$=\frac{p_4}{p_5^2}(x_4p_4+2x_5p_5)$$

$$=\frac{p}{q^2}\ (pu_p+2qu_q),$$

which does not vanish; consequently

$$p_3-\frac{p_4^2}{p_5}=0,$$

and the equations now are

$$p_3p_5-p_4^2=0,\quad p_1=0,\quad p_2=0.$$

These are a complete Jacobian system: the complete integral of the system is

$$u=k+ax_3+bx_4+cx_5,$$

where k, a, b, c are constants such that

$$ac=b^2.$$

Thus

$$u=k+ax_3+bx_4+\frac{b^2}{a}x_5$$

$$=k+az\ +bp\ +\frac{b^2}{a}q,$$

so that the intermediate integral provided by $u=0$ can be taken in the form

$$z+\alpha p+\alpha^2q+\beta=0,$$

where α and β are a couple of arbitrary constants.

IV. We have to consider the possibilities of the relation

$$pu_p + 2qu_q = 0.$$

We then find that, concurrently with this relation, all the equations are satisfied by

$$u_x = -2qu_p, \quad u_y = -2\frac{q^2}{p}u_p,$$

so that the equations for u are

$$p\frac{\partial u}{\partial p} + 2q\frac{\partial u}{\partial q} = 0,$$

$$\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + 2q\frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + 2\frac{q^2}{p}\frac{\partial u}{\partial p} = 0.$$

The Jacobian condition of coexistence of the first and second requires that

$$\frac{\partial u}{\partial x} = 0;$$

and the condition for the first and third requires that

$$\frac{\partial u}{\partial z} = 0.$$

Hence $\frac{\partial u}{\partial p} = 0$, $\frac{\partial u}{\partial q} = 0$, $\frac{\partial u}{\partial y} = 0$: no intermediate integral is thus provided.

Hence the given equation has

$$p - aq = 0,$$

$$z + ap + a^2q + \beta = 0,$$

for intermediate integrals.

Ex. 2. Prove that all the surfaces, satisfying the equation

$$(sq - tp)^2 = (rt - s^2)(sp - rq),$$

and touching the cone $x^2 + y^2 = (z + a)^2$ along the circle $x^2 + y^2 = a^2$, $z = 0$, are given by equating to zero the c-discriminant of

$$(z + cp + c^2q)^2 - c^2(1 + c^2).$$

(Math. Trip., Part II, 1904.)

Ex. 3. Obtain an integral of the equation

$$z(rt - s^2)^2 - (tp^2 - 2spq + rq^2)(rt - s^2) + (tp - sq)(sp - rq) = 0,$$

by constructing an intermediate integral

$$z = ap + bq + ab.$$

Does any other intermediate integral exist?

Ex. 4. Obtain an intermediate integral of the equation

$$z(rt - s^2) = q^2r - 2pqs + p^2t,$$

in the form

$$z = ap + bq,$$

where a and b are arbitrary constants.

Ex. 5. Prove that the equation

$$\left(x - a\frac{pt-qr}{rt-s^2}\right)^2 + \left(y - a\frac{qr-ps}{rt-s^2}\right)^2 = 1$$

has an intermediate integral of the form

$$px + qy - (a+1)z = p\cos\alpha + q\sin\alpha + b,$$

where α and b are arbitrary constants, and a is a constant.

Explain the result when $a=0$.

Application of the General Method to Special Equations.

239. We proceed to apply the process, just indicated, to the equations

$$rt - s^2 + Rr + 2Ss + Tt = V,$$
$$Rr + 2Ss + Tt = V,$$

which have been considered in the earlier sections of this chapter. The immediate object is the determination of an intermediate integral if such an integral exists, no assumption being made as to the character of such an integral or as to its implicit influence (if any) upon the quantities R, S, T, V, which are supposed to be functions of x, y, z, p, q.

Taking the first of the two forms and assuming that an intermediate integral, if it exists, is an equation

$$u(x, y, z, p, q) = 0,$$

we eliminate r and t from the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

by means of the derivatives of the intermediate integral, which are

$$\frac{du}{dx} + \frac{\partial u}{\partial p} r + \frac{\partial u}{\partial q} s = 0,$$

$$\frac{du}{dy} + \frac{\partial u}{\partial p} s + \frac{\partial u}{\partial q} t = 0;$$

and then the eliminant, as an equation in s, is made evanescent. The conditions for evanescence are

$$\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q} - R\left(\frac{\partial u}{\partial q}\right)^2 + 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} - T\left(\frac{\partial u}{\partial p}\right)^2 = 0,$$

$$\frac{du}{dx}\frac{du}{dy} - R\frac{du}{dx}\frac{\partial u}{\partial q} - T\frac{du}{dy}\frac{\partial u}{\partial p} - V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0;$$

and these are the differential equations to be satisfied by u. Conversely, if these two equations do possess a common integral u which involves p and q, then $u=0$ is an intermediate integral of the original equation: the proof of this converse is an immediate inference from the analysis, taken in reverse course.

Now these equations are exactly the same equations as occur in Boole's method, there deduced upon a more extended assumption: hence, using the algebraical resolution before obtained (§ 236) so as to have the equivalent equations linear in the derivatives of u, we see that any common integral of the two equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \sigma\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \rho\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\},$$

or any common integral of the two equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\},$$

where ρ and σ are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0,$$

is an intermediate integral of the original equation.

Further, if the first pair of equations have a common integral u_1, and if the second pair have a common integral u_2, then the intermediate integrals

$$u_1 = 0, \quad u_2 = 0,$$

coexist. For the Jacobian condition of their coexistence is

$$[u_1, u_2] = 0,$$

that is,

$$\frac{du_1}{dx}\frac{\partial u_2}{\partial p} - \frac{du_2}{dx}\frac{\partial u_1}{\partial p} + \frac{du_1}{dy}\frac{\partial u_2}{\partial q} - \frac{du_2}{dy}\frac{\partial u_1}{\partial q} = 0;$$

on substitution from the two sets of equations for $\frac{du_1}{dx}$ and $\frac{du_1}{dy}$, $\frac{du_2}{dx}$ and $\frac{du_2}{dy}$, respectively, this relation is satisfied identically.

Moreover, we can take the most general forms of u_1 and u_2 that are admissible: thus, if the first system should possess two independent integrals (no matter how particular) represented by v_1 and w_1, we take

$$u_1 = F(v_1, w_1) = 0,$$

or

$$v_1 = f(w_1),$$

where F and f are arbitrary functions: and similarly for u_2. For our immediate purpose, however, the form of the intermediate integral (if any) is less important than the property that it is an integral common to two homogeneous linear equations of the first order, belonging to one or other of the two systems.

When the roots of the quadratic are equal, so that the equation is of the form

$$rt - s^2 + Rr + 2Ss + Tt + RT - S^2 = 0,$$

any intermediate integral is an integral of the equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} + S\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} + S\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\};$$

and conversely.

240. Similarly, for the equation

$$Rr + 2Ss + Tt = V,$$

if

$$u(x, y, z, p, q) = 0$$

is an intermediate integral, a corresponding process shews that u satisfies the equations

$$\left.\begin{aligned} R\left(\frac{\partial u}{\partial q}\right)^2 - 2S\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} + T\left(\frac{\partial u}{\partial p}\right)^2 &= 0 \\ R\frac{du}{dx}\frac{\partial u}{\partial q} + T\frac{du}{dy}\frac{\partial u}{\partial p} + V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\}.$$

Let ρ_1 and σ_1 be the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0;$$

then the equations for u, when resolved, lead either to the system

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \rho_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \sigma_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\},$$

or to the system

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \sigma_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \rho_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

These equations are exactly the same as the equations which occur in Boole's method: but they are now obtained merely on the assumption of the existence of an intermediate integral and without any assumption as to its form.

When the roots of the quadratic are equal, there is only a single system: it is

$$\left.\begin{aligned} R\frac{\partial u}{\partial q} - S\frac{\partial u}{\partial p} = 0 \\ R\frac{du}{dx} + S\frac{du}{dy} + V\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

As for the former case, so for the present case, if u_1 is an integral common to the equations in the first system, and if u_2 is an integral common to the equations in the second system, then the equations

$$u_1 = 0, \quad u_2 = 0,$$

coexist: for the Jacobian condition of coexistence is satisfied identically.

We proceed from the intermediate integral or intermediate integrals in order to obtain a primitive. If there are two intermediate integrals

$$u_1 = 0, \quad u_2 = 0,$$

we resolve these equations with respect to p and q; the values so obtained are substituted in

$$dz = pdx + qdy,$$

and quadrature then leads to a primitive. If there is only a single intermediate integral

$$u = 0,$$

it is regarded as a partial equation of the first order: its integral, obtained by any of the customary processes, is a primitive of the original equation of the second order.

Three Integrals common to the Subsidiary System.

241. It thus appears that the determination of an intermediate integral (if any) of the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

is bound up with the determination of a common integral (if any) of the equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \rho\,\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \sigma\,\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\},$$

or of a common integral (if any) of the equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \sigma\,\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \rho\,\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\}.$$

Now these equations are homogeneous and linear: and there are perfectly definite processes for determining whether the equations in such a system do possess a common integral and, if so, what is the number of algebraically independent integrals which they do possess.

Consider the first system of equations. We take

$$\rho = -S + \theta, \quad \sigma = -S - \theta,$$

where

$$\theta^2 = S^2 - RT - V;$$

and we write the equations in the form

$$\left.\begin{aligned} \Delta(u) = \frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\,\frac{\partial u}{\partial q} &= 0 \\ \Delta'(u) = \frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z} - \sigma\,\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\}.$$

As there are five variables x, y, z, p, q which can occur in u, and as there are two equations initially, four different cases can arise; there can be three independent common integrals, or two, or one, or none, according to the number of equations in the system when it is rendered complete.

In order that the two equations $\Delta(u)=0$, $\Delta'(u)=0$, may coexist, the Jacobian condition

$$(\Delta,\ \Delta')=0$$

must be satisfied, u now being regarded as the dependent variable and x, y, z, p, q as the independent variables. But

$$(\Delta,\ \Delta')=(\sigma-\rho)\frac{\partial u}{\partial z}+\{\Delta'(T)-\Delta(\sigma)\}\frac{\partial u}{\partial p}+\{\Delta'(\rho)-\Delta(R)\}\frac{\partial u}{\partial q};$$

and the right-hand side, which manifestly does not vanish in virtue of $\Delta(u)=0$ and $\Delta'(u)=0$, still must vanish. This requirement can be satisfied in one of two ways: the right-hand side may vanish identically: or, if not vanishing identically, it provides a new non-identical equation when equated to zero. We take the two cases separately.

When $(\Delta,\ \Delta')=0$ is satisfied identically, we have

$$\rho=\sigma,$$

as a first condition. The roots of the quadratic are equal, so that there is only a single subsidiary system: the common value of the equal roots is $-S$. The other conditions are

$$\Delta'(T)-\Delta(\sigma)=0,\quad \Delta'(\rho)-\Delta(R)=0;$$

and therefore the equations

$$\left.\begin{aligned}\Delta(u)&=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-T\frac{\partial u}{\partial p}+S\frac{\partial u}{\partial q}=0\\ \Delta'(u)&=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial p}-R\frac{\partial u}{\partial q}=0\end{aligned}\right\}$$

are such that

$$(\Delta,\ \Delta')=0,$$

provided

$$\Delta'(T)+\Delta(S)=0,\quad \Delta'(S)+\Delta(R)=0:$$

that is, *when these conditions are satisfied, the equations* $\Delta=0$, $\Delta'=0$ *are a complete Jacobian system*, and therefore they possess three independent integrals in common. Let these three independent integrals be u_1, u_2, u_3, obtainable by any of the methods given in Chapter IV in the preceding volume, where u_1, u_2, u_3 are functions of x, y, z, p, q. Now

$$[u_1, u_2] = \frac{du_1}{dx}\frac{\partial u_2}{\partial p} - \frac{du_2}{dx}\frac{\partial u_1}{\partial p} + \frac{du_1}{dy}\frac{\partial u_2}{\partial q} - \frac{du_2}{dy}\frac{\partial u_1}{\partial q}$$
$$= 0,$$

on substitution; and similarly

$$[u_1, u_3] = 0, \quad [u_2, u_3] = 0.$$

Consequently, the equations

$$u_1 = a, \quad u_2 = b, \quad u_3 = c,$$

where a, b, c are arbitrary constants, coexist: and the quantities u_1, u_2, u_3 are independent of one another, so that the three equations can be resolved with respect to any three of the variables. Let this resolution be effected with respect to z, p, q; then we have

$$z = f(x, y, a, b, c),$$
$$p = g(x, y, a, b, c),$$
$$q = h(x, y, a, b, c),$$

and these values of p and q are the derivatives of z.

Thus, with the conditions as satisfied, an integral involving three arbitrary constants has been obtained. This integral can be generalised so as to involve two arbitrary functions: and the generalised form is given by

$$\left.\begin{aligned} b = \phi(a), \quad c = \psi(a) \\ z = f(x, y, a, b, c) \\ 0 = \frac{df}{da} \end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions, and a, b, c are to be eliminated. The origin of this generalisation is to be found in Ampère's method, which will be expounded later: meanwhile, the statement can be verified as follows.

We first need the relations connected with the fact that

$$z = f, \quad p = g, \quad q = h,$$

constitute an integral of the original equation. These three equations are the equivalent of the three equations

$$u_1 = a, \quad u_2 = b, \quad u_3 = c,$$

so that the quantities z, p, q in these three equations are such that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q,$$

$$\frac{\partial p}{\partial x} = r, \quad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = s, \quad \frac{\partial q}{\partial y} = t.$$

When we substitute in each of the three equations

$$u_\mu = \text{constant},$$

for $\mu = 1, 2, 3$, the values $z = f$, $p = g$, $q = h$, we have identities; and therefore

$$\frac{\partial u_\mu}{\partial x} + p\frac{\partial u_\mu}{\partial z} + r\frac{\partial u_\mu}{\partial p} + s\frac{\partial u_\mu}{\partial q} = 0,$$

or, taking account of one of the differential equations satisfied by u_μ, we have

$$(r + T)\frac{\partial u_\mu}{\partial p} + (s - S)\frac{\partial u_\mu}{\partial q} = 0.$$

As this holds for $\mu = 1, 2, 3$, and as u_1, u_2, u_3 are independent of one another, we have

$$r + T = 0, \quad s - S = 0.$$

Similarly, from

$$\frac{\partial u_\mu}{\partial y} + q\frac{\partial u_\mu}{\partial z} + s\frac{\partial u_\mu}{\partial p} + t\frac{\partial u_\mu}{\partial q} = 0,$$

we have

$$s - S = 0, \quad t + R = 0.$$

Consequently,

$$-T = \frac{\partial^2 f}{\partial x^2}, \quad S = \frac{\partial^2 f}{\partial x \partial y}, \quad -R = \frac{\partial^2 f}{\partial y^2};$$

and

$$V = S^2 - RT = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}.$$

With these values, the equation

$$rt - s^2 + Rr + 2Ss + Tt = V = S^2 - RT$$

is satisfied by

$$z = f(x, y, a, b, c).$$

Passing now to the equations of the generalised integral, we have

$$\frac{df}{da}=0,$$

and therefore

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)+\frac{d^2f}{da^2}\frac{\partial a}{\partial x}=0,\quad \frac{\partial}{\partial y}\left(\frac{df}{da}\right)+\frac{d^2f}{da^2}\frac{\partial a}{\partial y}=0\,;$$

thus

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)=-\frac{d^2f}{da^2}\frac{\partial a}{\partial x},$$

$$\frac{\partial}{\partial y}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial y}\right)=-\frac{d^2f}{da^2}\frac{\partial a}{\partial y}.$$

Now from the equation

$$z=f,$$

we have

$$p=\frac{\partial f}{\partial x}+\frac{df}{da}\frac{\partial a}{\partial x}=\frac{\partial f}{\partial x},$$

$$q=\frac{\partial f}{\partial y}+\frac{df}{da}\frac{\partial a}{\partial y}=\frac{\partial f}{\partial y}.$$

Again,

$$r=\frac{\partial^2 f}{\partial x^2}+\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)\frac{\partial a}{\partial x}=\frac{\partial^2 f}{\partial x^2}-\frac{d^2f}{da^2}\left(\frac{\partial a^2}{\partial x}\right),$$

$$s=\frac{\partial^2 f}{\partial x\partial y}+\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)\frac{\partial a}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}-\frac{d^2f}{da^2}\frac{\partial a}{\partial x}\frac{\partial a}{\partial y},$$

$$t=\frac{\partial^2 f}{\partial y^2}+\frac{d}{da}\left(\frac{\partial f}{\partial y}\right)\frac{\partial a}{\partial y}=\frac{\partial^2 f}{\partial y^2}-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial y}\right)^2.$$

On the right-hand sides, the partial derivatives of f with regard to x and y are taken on the hypothesis that a is constant: thus we may put

$$\frac{\partial^2 f}{\partial x^2}=-T,\quad \frac{\partial^2 f}{\partial x\partial y}=S,\quad \frac{\partial^2 f}{\partial y^2}=-R,$$

and therefore

$$r=-T-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial x}\right)^2,$$

$$s=\quad S-\frac{d^2f}{da^2}\frac{\partial a}{\partial x}\frac{\partial a}{\partial y},$$

$$t=-R-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial y}\right)^2.$$

Consequently,

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT,$$

on substituting these values of r, s, t: that is, the original differential equation is satisfied by

$$\left.\begin{aligned} z &= f\{x, y, a, \phi(a), \psi(a)\} \\ 0 &= \frac{df}{da} \end{aligned}\right\},$$

which thus is an integral involving two arbitrary functions. The conditions for the existence of the integral are that the quantities R, S, T, being functions of x, y, z, p, q, shall satisfy the relations

$$\Delta(R) + \Delta'(S) = 0, \quad \Delta(S) + \Delta'(T) = 0,$$

identically, where

$$\Delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - T\frac{\partial}{\partial p} + S\frac{\partial}{\partial q},$$

$$\Delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + S\frac{\partial}{\partial p} - R\frac{\partial}{\partial q}.$$

Ex. 1. Verify the converse, viz. that the equation given by the elimination of a between

$$z = f\{x, y, a, \phi(a), \psi(a)\}, \quad \frac{df}{da} = 0,$$

satisfies an equation of the second order, for which the two conditions are satisfied and for which

$$V = S^2 - RT.$$

Ex. 2. A surface is defined as the locus of the family of curves

$$f(x, y, z, a) = 0, \quad g(x, y, z, a) = 0$$

where a is a parameter; shew that z, regarded as a function of x and y along this surface, satisfies the partial equations

$$Ap + Bq - C = 0,$$

$$A^2r + 2ABs + B^2t = H,$$

where

$$A = J\left(\frac{f, g}{y, z}\right), \quad B = J\left(\frac{f, g}{z, x}\right), \quad C = J\left(\frac{f, g}{x, y}\right),$$

$$H = \delta A - p\delta B - q\delta C,$$

and

$$\delta = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial B} + C\frac{\partial}{\partial C}.$$

Give a geometrical interpretation of the two partial equations: and find the condition or conditions which must be satisfied by quantities A', B', H', in order that the equation

$$A'^2r+2A'B's+B'^2t=H'$$

may possess an integral of the foregoing type.

242. It is not difficult to construct equations of the type just discussed. The equation is

$$rt-s^2+Rr+2Ss+Tt=S^2-RT;$$

and the quantities R, S, T must satisfy the equations

$$\Delta(R)+\Delta'(S)=0, \quad \Delta(S)+\Delta'(T)=0.$$

Hence S may be assumed arbitrarily; and then R and T are given by the two simultaneous equations

$$\left.\begin{aligned}\Delta(R)&=-\Delta'(S)\\ \Delta'(T)&=-\Delta(S)\end{aligned}\right\}.$$

A. The simplest set of cases occurs when

$$S=a,$$

where a is a constant: then $\Delta'(S)=0$, $\Delta(S)=0$; so that R and T are given by the two equations

$$\left.\begin{aligned}\frac{\partial R}{\partial x}+p\frac{\partial R}{\partial z}-T\frac{\partial R}{\partial p}+a\frac{\partial R}{\partial q}=0\\ \frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-R\frac{\partial T}{\partial q}=0\end{aligned}\right\}.$$

Individual forms are easily obtainable.

I. Let

$$R=b,$$

where b is a constant; the first equation is satisfied identically, and T then is any integral of

$$\frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-b\frac{\partial T}{\partial q}=0,$$

so that we can take

$$T=F(x,\ p-ay,\ q+by,\ z-qy-\tfrac{1}{2}by^2),$$

where F is any function of its arguments. As R, S, T, are now known, the differential equation is known.

When we proceed to obtain the primitive by the method in the text, we have to obtain three integrals common to

$$\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-T\frac{\partial u}{\partial p}+a\frac{\partial u}{\partial q}=0,$$

$$\Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+a\frac{\partial u}{\partial p}-b\frac{\partial u}{\partial q}=0.$$

The latter is the equation defining T; hence it has the four independent integrals

$$v_1 = x,$$
$$v_2 = p - ay,$$
$$v_3 = q + by,$$
$$v_4 = z - qy - \tfrac{1}{2}by^2,$$

and any function of these will satisfy $\Delta'(u) = 0$. Let

$$u = g(v_1, v_2, v_3, v_4)$$

be a function of them satisfying $\Delta(u) = 0$: then

$$\frac{\partial g}{\partial v_1} - \frac{\partial g}{\partial v_2} F + \frac{\partial g}{\partial v_3} a + \frac{\partial g}{\partial v_4} v_2 = 0.$$

To obtain the form of g, we construct the subsidiary equations

$$\frac{dv_1}{1} = \frac{dv_2}{-F} = \frac{dv_3}{a} = \frac{dv_4}{v_2}.$$

Let

$$w_1 = \alpha, \quad w_2 = \beta, \quad w_3 = \gamma,$$

be three independent integrals of these equations, where α, β, γ are constants: then, if we eliminate p and q between the three equations

$$w_1 = \alpha, \quad w_2 = \beta, \quad w_3 = \gamma,$$

leading to an equation

$$H(x, y, z, \alpha, \beta, \gamma) = 0,$$

the integral of the differential equation

$$rt - s^2 + br + 2as - tF = a^2 + bF$$

is given by

$$\left.\begin{array}{c} H\{x, y, z, \alpha, \phi(\alpha), \psi(\alpha)\} = 0 \\ \dfrac{dH}{d\alpha} = 0 \end{array}\right\}.$$

A particular case of this form, viz.

$$(p + q + r)(1 + t) = (1 - s)^2$$

is given by Imschenetsky*: it is obtained by taking

$$a = 1, \quad b = 1, \quad F = v_2 + v_3,$$

in what precedes. The quantities w_1, w_2, w_3 arise in the integrals of

$$\frac{dv_2}{dv_1} = -v_2 - v_3, \quad \frac{dv_3}{dv_1} = 1, \quad \frac{dv_4}{dv_1} = v_2;$$

we easily find

$$v_3 = v_1 + \alpha,$$
$$v_2 = \beta e^{-v_1} - v_1 - \alpha + 1,$$
$$v_4 = \gamma - \beta e^{-v_1} - \tfrac{1}{2}v_1^2 - (\alpha - 1)v_1.$$

* At p. 299 of his frequently quoted memoir.

Substituting for v_1, v_2, v_3, v_4, and eliminating p and q, we have

$$z=\gamma-\beta e^{-x}+x-\tfrac{1}{2}(x-y)^2-a(x-y).$$

The generalised integral is

$$\left.\begin{aligned} z&=\phi(a)-e^{-x}\psi(a)+x-\tfrac{1}{2}(x-y)^2-a(x-y)\\ 0&=\phi'(a)-e^{-x}\psi'(a)-x+y \end{aligned}\right\}.$$

Both results are given by Imschenetsky.

II. Still keeping $S=a$, let

$$R=f(y,\ q-ax),$$

where f denotes any function at our disposal. Then the condition

$$\Delta(R)=-\Delta'(S)=0$$

is satisfied identically; and the equation for T is

$$\frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-f(y,\ q-ax)\frac{\partial T}{\partial q}=0.$$

Then

$$T=G(x,\ p-ay,\ \theta,\ \zeta),$$

where G is any function at our disposal, and

$$\theta=\theta(y,\ t)=\text{constant},\quad \zeta=z-axy-\phi(y,\ t)=\text{constant},$$

are two independent integrals of the ordinary equations

$$\frac{dt}{dy}=-f(y,\ t),$$

$$\frac{dz}{dy}=t+ax,$$

in which x is parametric and t denotes $q-ax$. As S, R, T are known, the form of the differential equation is given explicitly.

In order to obtain the primitive by the method in the text, we have to obtain three integrals common to

$$\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-G\frac{\partial u}{\partial p}+a\frac{\partial u}{\partial q}=0,$$

$$\Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+a\frac{\partial u}{\partial p}-f\frac{\partial u}{\partial q}=0.$$

The latter is the equation defining T: we know four independent integrals in the form

$$v_1=x,$$

$$v_2=p-ay,$$

$$v_3=\theta,$$

$$v_4=\zeta;$$

and any functional combination of these will satisfy the second equation for u. Let

$$u=g(v_1,\ v_2,\ v_3,\ v_4)$$

be a functional combination satisfying the first equation ; then, as

$$\Delta(v_1)=1, \quad \Delta(v_2)=-T=-G(v_1, v_2, v_3, v_4),$$
$$\Delta(v_3)=0, \quad \Delta(v_4)=v_2,$$

the equation for g is

$$\frac{\partial g}{\partial v_1}-G\frac{\partial g}{\partial v_2}+v_2\frac{\partial g}{\partial v_4}=0.$$

One integral evidently is

$$v_3:$$

let

$$w_1=\beta, \quad w_2=\gamma,$$

be two independent integrals of the ordinary equations

$$dv_1=\frac{dv_2}{-G}=\frac{dv_4}{v_2},$$

in which v_3 is parametric. An integral of the original equation is given by eliminating p and q between

$$v_3=a, \quad w_1=\beta, \quad w_2=\gamma:$$

and it can be generalised in the usual way.

As an example, let

$$R=q-ax=t;$$

to determine T, we integrate the equations

$$\frac{dt}{dy}=-t, \quad \frac{dz}{dy}=t+ax,$$

so that

$$\theta=te^y, \quad \zeta=z-axy+t,$$

and then

$$T=G(x, p-ay, te^y, z-axy+t),$$

where G is at our disposal, being any function whatever of its arguments.

In particular, let

$$T=p-ay,$$

so that the differential equation is

$$rt-s^2+(q-ax)r+2as+(p-ay)t=a^2-(p-ay)(q-ax).$$

To obtain a primitive, we have

$$(q-ax)e^y=v_3=a;$$

and we have to integrate

$$\frac{dv_2}{dv_1}=-G=-v_2, \quad \frac{dv_4}{dv_1}=v_2.$$

Thus

$$w_1=v_2e^{v_1}=\beta, \quad w_3=v_4+\beta e^{-v_1}=\gamma;$$

the primitive is given by eliminating p and q between

$$(q-ax)e^y=a,$$
$$(p-ay)e^x=\beta,$$
$$z-axy+q-ax=\gamma-\beta e^{-x},$$

that is, it is

$$z = axy - ae^{-y} - \beta e^{-x} + \gamma.$$

The generalised integral comes by taking

$$\beta = \phi(\gamma), \quad a = \psi(\gamma).$$

III. Still keeping $S = a$, write

$$\delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + a\frac{\partial}{\partial q},$$

$$\delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + a\frac{\partial}{\partial p};$$

then the equations for R and T are

$$\delta R = T\frac{\partial R}{\partial p},$$

$$\delta' T = R\frac{\partial T}{\partial q}.$$

If $\frac{\partial R}{\partial p}$ is not zero, then

$$T = \frac{\delta R}{\frac{\partial R}{\partial p}},$$

and therefore

$$\delta'\left(\frac{\delta R}{\frac{\partial R}{\partial p}}\right) = R\frac{\partial}{\partial q}\left(\frac{\delta R}{\frac{\partial R}{\partial p}}\right),$$

which is a differential equation for R of the second order involving five independent variables. The general primitive of this equation does not appear to be obtainable: but special integrals can be obtained.

B. Another set of cases is given by

$$R = 0, \quad T = 0,$$

provided S can be determined so as to satisfy the equations

$$\Delta(S) = 0, \quad \Delta'(S) = 0,$$

which in the present instance are

$$\Delta = \frac{\partial S}{\partial x} + p\frac{\partial S}{\partial z} + S\frac{\partial S}{\partial q} = 0,$$

$$\Delta' = \frac{\partial S}{\partial y} + q\frac{\partial S}{\partial z} + S\frac{\partial S}{\partial p} = 0.$$

The condition of coexistence is that

$$[\Delta, \Delta'] = 0:$$

it is easy to verify that

$$[\Delta, \Delta'] = p\Delta - q\Delta',$$

and therefore the condition is satisfied.

Suppose that S is determined by an equation

$$\sigma(x, y, z, p, q, S)=0:$$

then

$$\frac{\partial \sigma}{\partial x}+\frac{\partial \sigma}{\partial S}\frac{\partial S}{\partial x}=0,$$

and so for other derivatives of S: thus the equations for σ become

$$\left.\begin{aligned}\frac{\partial \sigma}{\partial x}+p\frac{\partial \sigma}{\partial z}+S\frac{\partial \sigma}{\partial q}=0\\ \frac{\partial \sigma}{\partial y}+q\frac{\partial \sigma}{\partial z}+S\frac{\partial \sigma}{\partial p}=0\end{aligned}\right\}.$$

Proceeding in the usual manner, we find that there are four independent integrals of these two equations, viz.

$$S,\quad p-yS,\quad q-xS,\quad z-xp-yq+xyS:$$

and therefore the most general value of σ is

$$\sigma=\Phi(S,\ p-yS,\ q-xS,\ z-px-qy+xyS),$$

where Φ is a completely arbitrary function. But S is given by $\sigma=0$, that is, S is determined by the equation

$$\Phi(S,\ p-yS,\ q-xS,\ z-px-qy+xyS)=0.$$

The differential equation to be integrated is

$$rt=(s-S)^2;$$

and if u_1, u_2, u_3 be three independent integrals of the two equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial q}=0\\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial p}=0\end{aligned}\right\},$$

an integral of the differential equation is obtained by eliminating p and q between

$$u_1=\alpha,\quad u_2=\beta,\quad u_3=\gamma.$$

As an example, take

$$S=p-yS,$$

so that

$$S=\frac{p}{1+y};$$

the quantities u_1, u_2, u_3 are found to be

$$u_1=\frac{p}{1+y},\quad u_2=q-\frac{px}{1+y},\quad u_3=z-qy-\frac{px}{1+y};$$

and the integral is

$$z=\alpha x(1+y)+y\beta+\gamma.$$

Ex. Integrate the equations :—

(i) $rt-s^2+br+2as-a^2=(b+t)f''(x)$;

(ii) $rt-s^2+br+2as-a^2=\dfrac{b+t}{(p-ay)^n}$,

where n is constant ;

(iii) $rt-s^2+br+2as-a^2=\dfrac{b+t}{(q+by)^m}$,

where m is constant ;

(iv) $rt-s^2+br+2as-a^2=c(b+t)(z-qy-\frac{1}{2}y^2)$,

where c is constant ;

(v) $rt-s^2+br+2as-a^2=\kappa\dfrac{q+by}{p-ay}(b+t)$,

where κ is constant ;

(vi) $rt-s^2+br+2as-a^2=\lambda(p-ay)(z-qy-\frac{1}{2}y^2)(b+t)$,

where λ is constant ;

(vii) $rt-s^2+(q-ax)r+2as+tf''(x)=a^2-(q-ax)f''(x)$;

(viii) $rt-s^2+(q-ax)r+2as-a^2=(cx+ay-p)(t+q-ax)$;

(ix) $rt-s^2+rf(x)+2as+tg(y)=a^2-f(x)g(y)$;

(x) $rt=\left(s-\dfrac{q}{1+x}\right)^2$;

(xi) $rt=\left(s-\dfrac{xp+yq-z}{xy-1}\right)^2$;

(xii) $rt=\left(s-\dfrac{pq}{z}\right)^2$.

Two Integrals common to the Subsidiary System.

243. It was seen that the two equations $\Delta(u)=0$, $\Delta'(u)=0$, coexist only if the equation

$$(\Delta, \Delta')=0$$

is satisfied. From its form, it clearly cannot be satisfied in virtue of $\Delta(u)=0$, $\Delta'(u)=0$; and we have discussed the case in which it is satisfied identically. It remains to discuss the case in which it is a new equation and for which therefore not all the three quantities

$$\sigma-\rho,\quad \Delta'(T)-\Delta(\sigma),\quad \Delta'(\rho)-\Delta(R),$$

vanish.

We shall deal with only some of the possibilities in detail: for the present purpose, we shall assume that no one of the three quantities vanishes, so that, writing

$$P = \frac{\Delta'(T) - \Delta(\sigma)}{\rho - \sigma}, \quad Q = \frac{\Delta'(\rho) - \Delta(R)}{\rho - \sigma},$$

the new equation has the form

$$\Delta''(u) = \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0.$$

It is easy to see that, if the system of equations in u is to possess only two independent integrals (so as to justify the assumptions in Monge's method and in Boole's method as regards the origin of the equation), the quantity $\rho - \sigma$ must not vanish. Assume the contrary, so that $\rho = \sigma$: then, for our present purpose, we cannot have both the quantities $\Delta'(T) + \Delta(S)$, $\Delta'(S) + \Delta(R)$, equal to zero. Let the former be different from zero, and write

$$-\mu = \frac{\Delta'(S) + \Delta(R)}{\Delta'(T) + \Delta(S)};$$

the new equation is

$$\nabla(u) = \frac{\partial u}{\partial p} + \mu\frac{\partial u}{\partial q} = 0.$$

The equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \nabla(u) = 0,$$

are to be a complete system if they possess two independent integrals: consequently the equations

$$(\Delta, \Delta') = 0, \quad (\Delta, \nabla) = 0, \quad (\Delta', \nabla) = 0,$$

must be satisfied in virtue of the equations of the system. Now

$$(\Delta, \Delta') = \{\Delta'(T) + \Delta(S)\}\nabla(u) = 0,$$

$$(\Delta, \nabla) = \frac{\partial u}{\partial z} - \nabla(T)\frac{\partial u}{\partial p} + \{\nabla(S) - \Delta(\mu)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \nabla) = \mu\frac{\partial u}{\partial z} + \nabla(S)\frac{\partial u}{\partial p} - \{\nabla(R) + \Delta'(\mu)\}\frac{\partial u}{\partial q}:$$

it is obvious that the quantities (Δ, ∇), (Δ', ∇), do not vanish in virtue of $\Delta = 0$, $\Delta' = 0$, $\nabla = 0$. The three equations are not a complete system: they cannot possess two independent integrals.

Accordingly, we are justified (for our immediate purpose) in assuming that $\rho - \sigma$ does not vanish.

We thus have three equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

and these equations are to coexist: consequently, the relations

$$(\Delta, \Delta') = 0, \quad (\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0,$$

must be satisfied, either identically, or in virtue of the equations of the system, or as new equations. Now

$$(\Delta, \Delta') = (\sigma - \rho)\, \Delta''(u) = 0,$$

thus providing no new condition. Also

$$(\Delta, \Delta'') = P\frac{\partial u}{\partial z} - \{\Delta(P) - \Delta''(T)\}\frac{\partial u}{\partial p} - \{\Delta(Q) - \Delta''(\rho)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \Delta'') = Q\frac{\partial u}{\partial z} - \{\Delta'(P) - \Delta''(\sigma)\}\frac{\partial u}{\partial p} - \{\Delta'(Q) - \Delta''(R)\}\frac{\partial u}{\partial q}.$$

These equations manifestly do not vanish identically; if they vanish in virtue of the equations of the system, we evidently must have

$$(\Delta, \Delta'') = P\Delta''(u), \quad (\Delta', \Delta'') = Q\Delta''(u),$$

and therefore we must have

$$\left.\begin{aligned} \Omega_1 &= P^2 - \Delta(P) + \Delta''(T) = 0 \\ \Omega_2 &= PQ - \Delta(Q) + \Delta''(\rho) = 0 \\ \Omega_3 &= PQ - \Delta'(P) + \Delta''(\sigma) = 0 \\ \Omega_4 &= Q^2 - \Delta'(Q) + \Delta''(R) = 0 \end{aligned}\right\}.$$

Taking account of the relations between the operators $\Delta, \Delta', \Delta''$, viz.

$$\begin{aligned} \Delta\Delta' - \Delta'\Delta &= (\sigma - \rho)\, \Delta'', \\ \Delta\Delta'' - \Delta''\Delta &= P\Delta'', \\ \Delta'\Delta'' - \Delta''\Delta' &= Q\Delta'', \end{aligned}$$

we find

$$\left.\begin{aligned} \Delta'\Omega_1 - \Delta\Omega_3 &= Q\Omega_1 + P\Omega_2 - 2P\Omega_3 \\ \Delta'\Omega_2 - \Delta\Omega_4 &= P\Omega_4 + Q\Omega_3 - 2Q\Omega_2 \end{aligned}\right\}.$$

Accordingly, the four relations may be really equivalent to only two relations: and they are the conditions that the system of equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

shall be a complete system.

Suppose that the conditions are satisfied and that therefore the three equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

are a complete system: they involve five variables x, y, z, p, q, and they therefore possess two independent integrals in common. Let

v and w be these common integrals, taken as simply as possible: then the most general integral is of the form

$$u = F(v, w),$$

where F is an arbitrary function. But

$$u = 0$$

is an equation of the first order compatible with the original differential equation; hence an intermediate integral is

$$F(v, w) = 0,$$

that is,

$$v = f(w),$$

where f is an arbitrary function.

But this is precisely the result obtained by Monge's method and by Boole's method on the assumption, made for the analysis in each method, that the appropriate conditions (there left undetermined) are satisfied. Consequently we have the theorem:

The equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

where $S^2 - RT - V$ is not zero, possesses an intermediate integral of the form

$$v = f(w),$$

provided the relations

$$P^2 - \Delta(P) + \Delta''(T) = 0, \quad PQ - \Delta'(P) + \Delta''(\sigma) = 0,$$
$$PQ - \Delta(Q) + \Delta''(\rho) = 0, \quad Q^2 - \Delta'(Q) + \Delta''(R) = 0,$$

are satisfied; and these relations may be equivalent to only two conditions. The quantities ρ and σ are the (unequal) roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0;$$

the operators Δ and Δ' are given by

$$\Delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - T\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial q};$$

$$\Delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} - \sigma\frac{\partial}{\partial p} - R\frac{\partial}{\partial q};$$

the quantities P and Q are

$$P = \frac{\Delta'(T) - \Delta(\sigma)}{\rho - \sigma}, \quad Q = \frac{\Delta'(\rho) - \Delta(R)}{\rho - \sigma},$$

and the operator Δ'' *is given by*

$$\Delta'' = \frac{\partial}{\partial z} - P\frac{\partial}{\partial p} - Q\frac{\partial}{\partial q}.$$

The preceding result is the definite establishment of the theorem as to the possession of an intermediate integral, involving in its expression an arbitrary function.

No discrimination has been made between the two unequal roots of the quadratic equation: an intermediate integral will be possessed, if the essential conditions are satisfied for either arrangement of the two roots.

If the essential conditions are satisfied for each of the arrangements of the roots, then each arrangement of the roots leads to an intermediate integral involving an arbitrary function. These intermediate integrals coexist, according to an earlier theorem (§ 233). The construction of the primitive is then a matter of mere quadrature of the relation

$$dz = pdx + qdy,$$

after substitution of the values of p and q derived from the simultaneous intermediate integrals. To such equations we shall recur later.

The more frequent case arises when the essential conditions are satisfied for one, but not for both, of the arrangements of the roots. We then have one intermediate integral: the construction of the primitive requires the integration of that equation of the first order, and it will appear (from a theorem of Ampère's which will presently be proved) that, in this integration, the equations connected with the unsatisfying arrangement of the roots of the quadratic occur.

Ex. 1. One very simple case arises when

$$P=0, \quad Q=0.$$

The operator Δ'' is merely $\frac{\partial}{\partial z}$: one of the equations for the intermediate integral is

$$\frac{\partial u}{\partial z}=0,$$

so that z does not occur explicitly in the integral: and the (four) conditions are

$$\frac{\partial T}{\partial z}=0, \quad \frac{\partial \rho}{\partial z}=0, \quad \frac{\partial \sigma}{\partial z}=0, \quad \frac{\partial R}{\partial z}=0.$$

As ρ and σ are unequal to one another, these conditions require that R, S, T, V do not explicitly involve z. We therefore take R, S, T, V free from z; and then, as both P and Q vanish, we have

$$\Delta'(T)=\Delta(\sigma), \quad \Delta'(\rho)=\Delta(R),$$

where now

$$\left.\begin{aligned}\Delta &=\frac{\partial}{\partial x}-T\frac{\partial}{\partial p}-\rho\frac{\partial}{\partial q}\\ \Delta' &=\frac{\partial}{\partial y}-\sigma\frac{\partial}{\partial p}-R\frac{\partial}{\partial q}\end{aligned}\right\}.$$

Four quantities, being functions of x, y, p, q, are at our disposal, subject to the limitation that ρ and σ are not equal to one another.

Thus ρ and σ can be assumed arbitrarily: the quantities R and T are determined by means of the equations

$$\Delta'(T)=\Delta(\sigma), \quad \Delta'(\rho)=\Delta(R):$$

and S and V are given by the equations

$$2S=-\rho-\sigma,$$

$$V=\rho\sigma-RT.$$

A special case will suffice as an illustration. Let

$$\rho=ap, \quad \sigma=cq:$$

then

$$\frac{\partial T}{\partial y}-cq\frac{\partial T}{\partial p}-R\frac{\partial T}{\partial q}=-acp,$$

$$\frac{\partial R}{\partial x}-T\frac{\partial R}{\partial p}-ap\frac{\partial R}{\partial q}=-acq.$$

Without attempting to obtain the general values of R and T, we note that the values

$$R=\lambda p, \quad T=\mu q,$$

satisfy the equations, provided

$$\lambda\mu=ac.$$

Thus the equation

$$rt-s^2+\lambda pr-(ap+cq)s+\mu qt=0,$$

that is, the equation

$$(r+\mu q)(t+\lambda p)=(s+ap)(s+cq),$$

where $\lambda\mu=ac$, possesses an intermediate integral involving an arbitrary function.

The construction of the intermediate integral is left as an exercise.

Ex. 2. Prove that the equation

$$(xp+yq)(rt-s^2)+aq^2r+(a+c)pq+cp^2t=0,$$

where

$$\frac{1}{a}+\frac{1}{c}=2,$$

has an intermediate integral involving an arbitrary function: and obtain the integral.

Ampère's Theorem on an Intermediate Integral.

244. Before proceeding to consider the properties of equations which possess two intermediate integrals each involving an arbitrary function, one characteristic property of equations possessing only a single intermediate integral may be noticed here. It was first obtained by Ampère, and it is as follows:—

When the differential equation possesses an intermediate integral involving an arbitrary function, so that the qualifying conditions are satisfied for one of the two systems of subsidiary equations, then the Charpit relations leading to the integration of the intermediate integral include the other system of subsidiary equations.

Denote by v and w the independent integrals of the subsidiary system

$$\Delta(u) = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0,$$

$$\Delta'(u) = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0,$$

$$\Delta''(u) = \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0,$$

the system being complete: then the intermediate integral can be taken in the form

$$F(v, w) = 0,$$

where F is any arbitrary function; and the three equations are satisfied when F is substituted for u. Now $F=0$ is an equation of the first order; to integrate it, we form the Charpit subsidiary equations (§ 68)

$$\frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial q}} = \frac{dz}{p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}} = \frac{-dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{-dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}},$$

and we need to obtain some integral of these equations. When we substitute for the denominators of the last two fractions, we have the modified set

$$\frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial q}} = \frac{dz}{p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}} = \frac{-dp}{T\frac{\partial F}{\partial p} + \rho\frac{\partial F}{\partial q}} = \frac{-dq}{\sigma\frac{\partial F}{\partial p} + R\frac{\partial F}{\partial q}};$$

and therefore the system includes the equations

$$\left.\begin{aligned} dz &= p\,dx + q\,dy \\ -dp &= T\,dx + \rho\,dy \\ -dq &= \sigma\,dx + R\,dy \end{aligned}\right\}.$$

Let U, = constant, be an integral of this set; then U satisfies the equations

$$\frac{\partial U}{\partial x} + p\frac{\partial U}{\partial z} - T\frac{\partial U}{\partial p} - \sigma\frac{\partial U}{\partial q} = 0,$$

$$\frac{\partial U}{\partial y} + q\frac{\partial U}{\partial z} - \rho\frac{\partial U}{\partial p} - R\frac{\partial U}{\partial q} = 0.$$

These are the equations of the other subsidiary system: hence Ampère's proposition.

We have supposed (though the supposition does not affect the preceding analysis and is postulated solely as providing the least favourable circumstances) that the qualifying conditions are not satisfied for this alternative subsidiary system. On that hypothesis, two independent integrals of this subsidiary system do not exist, for otherwise they would lead to an additional intermediate integral: but the subsidiary system may possess one integral, and that integral is one of the integrals of the Charpit equations. Now this is precisely what is required for the integration of the intermediate integral: we need one integral of those equations, which shall be distinct from $F = 0$ and shall involve p or q.

It therefore appears that, if only a single integral of the alternative subsidiary system can be obtained, say

$$U = c,$$

it can be combined with the intermediate integral

$$F(v, w) = 0$$

so as to give values of p and q which, when substituted in the relation

$$dz = p\,dx + q\,dy,$$

make that relation exact. Quadrature of this exact equation leads to the primitive.

If the alternative system should lead to an intermediate integral

$$G(U_1, U_2) = 0,$$

where G is an arbitrary function, it is clear that special primitives are derivable from any of the combinations

$$\left.\begin{array}{l} v=a \\ U_1=\alpha \end{array}\right\}, \quad \left.\begin{array}{l} w=b \\ U_1=\alpha \end{array}\right\}, \quad \left.\begin{array}{l} v=a \\ U_2=\beta \end{array}\right\}, \quad \left.\begin{array}{l} w=b \\ U_2=\beta \end{array}\right\}.$$

Ex. 1. Consider the equation

$$rt-s^2+pr-(p+q)s+qt=0.$$

The equation for μ is

$$\mu^2-\mu(p+q)+pq=0,$$

so that

$$\rho,\ \sigma=p,\ q.$$

With the assignment

$$\rho=p, \quad \sigma=q,$$

the equations leading to the intermediate integral (if any) are

$$\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

$$\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0.$$

The condition of coexistence is found to be

$$\frac{\partial u}{\partial z}=0\,;$$

and this equation, together with

$$\frac{\partial u}{\partial x}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

$$\frac{\partial u}{\partial y}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

makes a complete Jacobian system. Two independent integrals are easily found to be

$$p^2-q^2, \quad (p+q)\,e^{x+y},$$

so that an intermediate integral is

$$(p+q)\,e^{x+y}=f(p^2-q^2),$$

where f is an arbitrary function.

With the assignment

$$\rho=q, \quad \sigma=p,$$

the integrals leading to an intermediate integral (if any) are

$$\frac{\partial U}{\partial x}+p\frac{\partial U}{\partial z}-q\frac{\partial U}{\partial p}-q\frac{\partial U}{\partial q}=0,$$

$$\frac{\partial U}{\partial y}+q\frac{\partial U}{\partial z}-p\frac{\partial U}{\partial p}-p\frac{\partial U}{\partial q}=0.$$

The condition of coexistence is found to be

$$\frac{\partial U}{\partial z}-\frac{\partial U}{\partial p}-\frac{\partial U}{\partial q}=0\,;$$

this equation, together with

$$\frac{\partial U}{\partial x}+(p-q)\frac{\partial U}{\partial z}=0,$$

$$\frac{\partial U}{\partial y}-(p-q)\frac{\partial U}{\partial z}=0,$$

makes a complete Jacobian system. Two independent integrals are easily found to be

$$z+p-(p-q)(x-y), \quad p-q,$$

so that an intermediate integral is

$$z+p-(p-q)(x-y)=g(p-q),$$

where g is an arbitrary function.

It is easy to verify that

$$p-q, \quad z+p-(p-q)(x-y),$$

are integrals of the Charpit system subsidiary to the integration of

$$(p+q)e^{x+y}=f(p^2-q^2),$$

and that

$$p^2-q^2, \quad (p+q)e^{x+y},$$

are integrals of the Charpit system subsidiary to the integration of

$$z+p-(p-q)(x-y)=g(p-q).$$

The construction of various primitives is left as an exercise.

Corresponding properties and limitations belong to the equation

$$Rr+2Ss+Tt=V;$$

the equations for the determination of an intermediate integral (if any) are

$$\Delta'(u)=\frac{\partial u}{\partial q}-\rho\frac{\partial u}{\partial p}=0,$$

$$\Delta(u)=\frac{du}{dx}+\sigma\frac{du}{dy}+\frac{V}{R}\frac{\partial u}{\partial p}=0,$$

where ρ and σ are the roots of the quadratic

$$R\mu^2-2S\mu+T=0:$$

and, when the roots of the quadratic are unequal, there are two sets of equations in u corresponding to the two assignments of the unequal roots.

The development of the properties and limitations follows exactly the development in the case of the equation

$$rt-s^2+Rr+2Ss+Tt=V,$$

and so need not be given in detail: we subjoin some of the results.

Ex. 2. Prove that, if the equation possesses three independent intermediate integrals of the form

$$u_1=a_1, \quad u_2=a_2, \quad u_3=a_3,$$

then

$$RT=S^2.$$

Ex. 3. Shew that, if the equation

$$r+2Ss+S^2t=V$$

possesses three independent intermediate integrals, then S is determined by an equation

$$f(S, \ p+qS, \ x, \ y, \ z)=0,$$

where f is any arbitrary function (which must, however, involve S): and that V is determined by an equation

$$\frac{\partial V}{\partial q}-S\frac{\partial V}{\partial p}+V\frac{\partial S}{\partial p}+\frac{\partial S}{\partial x}+p\frac{\partial S}{\partial z}+S\frac{\partial S}{\partial y}+qS\frac{\partial S}{\partial z}=0.$$

Prove that, if

$$S=-\frac{a+p}{b+q},$$

then

$$V=(a+p)\,g\left(\frac{a+p}{b+q}, \ x, \ y, \ z\right),$$

where g is any function of its arguments; and determine the form of V when

$$S=F(x, \ y, \ z).$$

Ex. 4. Integrate the equations :—

(i) $$r-2\frac{a+p}{b+q}s+\left(\frac{a+p}{b+q}\right)^2 t=\lambda\,(b+q)\,z,$$

where λ is a constant ;

(ii) $$r+2zs+z^2t=(z-q)\,(p+qz).$$

Ex. 5. Shew that, if the equation

$$r+2Ss+Tt=V$$

possesses an intermediate integral involving an arbitrary function, and if S^2-T is not zero ; also, if

$$\Delta \ =\frac{\partial}{\partial x}+\sigma\frac{\partial}{\partial y}+(p+\sigma q)\frac{\partial}{\partial z}+V\frac{\partial}{\partial p},$$

$$\Delta' =\frac{\partial}{\partial q}-\rho\frac{\partial}{\partial p},$$

$$\Delta''=(1+q\lambda)\frac{\partial}{\partial z}+\lambda\frac{\partial}{\partial y}+P\frac{\partial}{\partial p},$$

where ρ and σ (in one or other of the two possible assignments) are the roots of the quadratic

$$\mu^2-2S\mu+T=0,$$

and where the quantities λ and P are given by

$$\lambda = \frac{\Delta'(\sigma)}{\sigma - \rho}, \quad P = \frac{\Delta'(V) + \Delta(\rho)}{\sigma - \rho};$$

then the necessary and sufficient conditions for the possession of the specified integral are that the equations

$$\left.\begin{aligned} \Delta''(\sigma) - \Delta(\lambda) &= 0 \\ \Delta''(\rho) + \Delta'(P) &= \lambda P \\ \Delta''(V) - \Delta(P) &= 0 \\ \Delta'(\lambda) &= \lambda^2 \end{aligned}\right\}$$

are satisfied.

Can these equations be equivalent to only two independent conditions?

Ex. 6. Shew that the equation

$$r - sp\left(q + \frac{1}{q}\right) + tp^2 = pf(pq),$$

where f is any function, possesses an intermediate integral: and integrate the equation

$$r - sp\left(q + \frac{1}{q}\right) + tp^2 = p(p^2q^2 - 1).$$

One Integral common to the Subsidiary System.

245. In § 243, it was seen that the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

possesses an intermediate integral of the type contemplated by Monge and by Boole, if the system of equations denoted by

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

is a complete Jacobian system: and the necessary conditions were duly set out, together with the limitation that the quadratic

$$\mu^2 + 2\mu S + RT + V = 0$$

should have unequal roots. If, however, in any given case the quadratic has equal roots: or if only some, but not all, of the essential conditions are satisfied for each of the unequal roots: then the system of three equations is not a complete Jacobian system. In either case, the three equations do not possess two independent integrals: and consequently the original equation possesses no intermediate integral that involves an arbitrary function.

Suppose then, that, the three equations are not a complete Jacobian system; any further analytical developments have no significance in connection with the problem as initiated by Monge and by Boole. They, however, do possess significance for the problem as propounded in § 238: for there we are concerned with equations which possess an intermediate integral of any kind whatever, there being no limitation and no requirement as to its functional character. An example was given in which an equation, definitely not of the postulated form, possessed an intermediate integral involving two arbitrary constants: and it is easy to see that an equation

$$u(x, y, z, p, q, a, b) = 0,$$

where a and b are arbitrary constants, can be an intermediate integral of an appropriate equation

$$F(x, y, z, p, q, r, s, t) = 0.$$

The matter will be sufficiently illustrated by briefly continuing the development of the analysis, which is connected with the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

on the hypothesis that the equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

do not constitute a complete Jacobian system. Two cases have to be considered, according as the quadratic

$$\mu^2 + 2\mu S + RT + V = 0$$

does not, or does, possess equal roots.

Assuming that the roots of the quadratic are unequal, so that $\rho - \sigma$ does not vanish, we have the three equations in the form

$$\left.\begin{aligned} \Delta(u) &= \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0 \\ \Delta'(u) &= \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0 \\ \Delta''(u) &= \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0 \end{aligned}\right\},$$

where

$$P = \frac{\Delta'(T) - \Delta(\sigma)}{\rho - \sigma}, \qquad Q = \frac{\Delta'(\rho) - \Delta(R)}{\rho - \sigma},$$

and

$$(\Delta, \Delta') = -(\rho - \sigma)\, \Delta''.$$

Also

$$(\Delta, \Delta'') = P\frac{\partial u}{\partial z} - \{\Delta(P) - \Delta''(T)\}\frac{\partial u}{\partial p} - \{\Delta(Q) - \Delta''(\rho)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \Delta'') = Q\frac{\partial u}{\partial z} - \{\Delta'(P) - \Delta''(\sigma)\}\frac{\partial u}{\partial p} - \{\Delta'(Q) - \Delta''(R)\}\frac{\partial u}{\partial q};$$

so that $(\Delta, \Delta') = 0$ in virtue of the equations retained, and we must have

$$(\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0,$$

also satisfied because the equations are retained.

Now the case, next in importance after those which already have been discussed, is that in which there is only a single integral common to the system. As there are five independent variables x, y, z, p, q in the system, the complete system will involve four linearly independent equations: hence the two new equations will effectively add one extra equation to the set $\Delta = 0$, $\Delta' = 0$, $\Delta'' = 0$. As derivatives with regard to x and y do not occur in Δ and Δ', it follows that a linear relation connects Δ'', (Δ, Δ''), (Δ', Δ''); hence

$$\Theta = \begin{vmatrix} P^2 - \Delta\ (P) + \Delta''(T), & PQ - \Delta\ (Q) + \Delta''(\rho) \\ PQ - \Delta'(P) + \Delta''\ (\sigma), & Q^2 - \Delta'(Q) + \Delta''(R) \end{vmatrix} = 0.$$

Any one of the four constituents in this determinantal form of Θ may vanish, though no one need vanish; if one does vanish, then $\Theta = 0$ will be satisfied by making one other vanish. But not all four constituents can vanish: for then we have

$$(\Delta, \Delta'') = P\Delta'',$$

$$(\Delta', \Delta'') = Q\Delta'',$$

and the complete system would consist of three equations, thus leading to the preceding case.

Moreover, as there is a linear relation connecting Δ'', (Δ, Δ''), (Δ', Δ''), there is effectively one new equation, so that the system has become

$$\Delta = 0, \quad \Delta' = 0, \quad \Delta'' = 0, \quad \Delta''' = 0.$$

The conditions

$$(\Delta, \Delta') = 0, \quad (\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0,$$

are satisfied; we shall assume that the further necessary conditions

$$(\Delta, \Delta''') = 0, \quad (\Delta', \Delta''') = 0, \quad (\Delta'', \Delta''') = 0,$$

also are satisfied; the system is then complete, and it possesses one integral. Consequently, *the subsidiary system has one integral common to all its equations if, when the roots of the quadratic are unequal, certain relations are satisfied: one of these is that the equation*

$$\Theta = 0$$

should be satisfied, without all the constituents in the determinantal form of Θ *vanishing.* Let u denote this common integral: then

$$u = a,$$

where a is an arbitrary constant, is an intermediate integral of the differential equation.

Next, let the roots of the quadratic be equal, so that the equation is

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT,$$

or, what is the same thing,

$$(r + T)(t + R) = (s - S)^2:$$

then $\rho = \sigma = -S$, and

$$\Delta(u) = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} + S\frac{\partial u}{\partial q} = 0,$$

$$\Delta'(u) = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + S\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0.$$

Now both the quantities A and B, where

$$A = \Delta'(T) + \Delta(S), \quad B = \Delta'(S) + \Delta(R),$$

do not vanish, for then $\Delta = 0$ and $\Delta' = 0$ would be a complete system; suppose that A does not vanish, and take

$$-\mu = \frac{B}{A} = \frac{\Delta'(S) + \Delta(R)}{\Delta'(T) + \Delta(S)}.$$

Then we have an equation

$$\nabla(u) = \frac{\partial u}{\partial p} + \mu\frac{\partial u}{\partial q} = 0,$$

and

$$(\Delta, \Delta') = \{\Delta'(T) + \Delta(S)\}\nabla.$$

Also, as before, we have

$$(\Delta, \nabla) = \frac{\partial u}{\partial z} - \nabla(T)\frac{\partial u}{\partial p} + \{\nabla(S) - \Delta(\mu)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \nabla) = \mu\frac{\partial u}{\partial z} + \nabla(S)\frac{\partial u}{\partial p} - \{\nabla(R) + \Delta'(\mu)\}\frac{\partial u}{\partial q};$$

so that $(\Delta, \Delta') = 0$ in virtue of $\nabla = 0$: and we must have

$$(\Delta, \nabla) = 0, \quad (\Delta', \nabla) = 0,$$

if $\nabla = 0$ is to coexist with $\Delta = 0$, $\Delta' = 0$.

For the present purpose, the complete Jacobian system is to contain four linearly independent equations; and therefore as derivatives with regard to x and y occur only in Δ and Δ', there must be one linear relation connecting ∇, (Δ, ∇), (Δ', ∇). The necessary and sufficient condition is easily found to be

$$\Phi = \mu^2\nabla(T) + 2\mu\nabla(S) - \mu\Delta(\mu) + \Delta'(\mu) + \nabla(R) = 0.$$

Moreover, we then have effectively one new equation, so that the system can be taken in a form

$$\Delta = 0, \quad \Delta' = 0, \quad \nabla = 0, \quad \Delta'' = 0.$$

The conditions

$$(\Delta, \Delta') = 0, \quad (\Delta, \nabla) = 0, \quad (\Delta', \nabla) = 0,$$

are satisfied, in virtue of the system; we shall assume that the further necessary conditions

$$(\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0, \quad (\nabla, \Delta'') = 0,$$

also are satisfied. The system then is complete, and so it possesses one integral. Hence *the subsidiary system of the equation*

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT$$

has one integral common to its equations if the relation

$$\Phi = 0$$

is satisfied, as well as certain other relations. Let v be this common integral: then

$$v = c,$$

where c is an arbitrary constant, is an intermediate integral of the differential equation.

Note. In the case of both these intermediate integrals

$$u = a, \quad v = c,$$

the *intermediate integral leads, not to a single equation, but to two equations of the second order*

$$\left.\begin{array}{l} u_x + u_p r + u_q s = 0 \\ u_y + u_p s + u_q t = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} v_x + v_p r + v_q s = 0 \\ v_y + v_p s + v_q t = 0 \end{array}\right\};$$

from each of these, the single equation can be compounded.

Ex. 1. The equation

$$r + 2Ss + S^2 t = V$$

possesses an intermediate integral, involving an arbitrary constant in the form $u(x, y, z, p, q) = a$, but not involving an arbitrary function: prove that, if

$$\Delta = \frac{\partial}{\partial x} + S\frac{\partial}{\partial y} + (p + qS)\frac{\partial}{\partial z} + V\frac{\partial}{\partial p},$$

$$\Delta' = \frac{\partial}{\partial q} - S\frac{\partial}{\partial p},$$

$$\Delta'' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + L\frac{\partial}{\partial p},$$

where

$$L = \frac{\Delta(S) + \Delta'(V)}{\Delta'(S)},$$

sufficient conditions are :—

(i) that $\Delta'(S)$ shall not vanish,

(ii) that the relation

$$L\Delta''(S) + \Delta(L) = \Delta''(V)$$

must be satisfied. Are these conditions necessary?

Ex. 2. The equation

$$r + 2Ss + Tt = V$$

possesses an intermediate integral involving an arbitrary constant in the form $u(x, y, z, p, q) = a$, but not involving an arbitrary function; and the roots, ρ and σ, of the quadratic $\mu^2 - 2S\mu + T = 0$ are unequal. Also, let

$$\Delta = \frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + (p + \sigma q)\frac{\partial}{\partial z} + V\frac{\partial}{\partial p},$$

$$\Delta' = \frac{\partial}{\partial q} - \rho\frac{\partial}{\partial p},$$

$$\Delta'' = (1 + q\lambda)\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial y} + P\frac{\partial}{\partial p},$$

where

$$\lambda = \frac{\Delta'(\sigma)}{\sigma - \rho}, \quad P = \frac{\Delta'(V) + \Delta(\rho)}{\sigma - \rho}.$$

Prove that the necessary and sufficient condition is that the equation

$$\Psi = \begin{vmatrix} \Delta(\lambda) - \Delta''(\sigma), & \Delta(P) - \Delta''(V) \\ \Delta'(\lambda) - \lambda^2, & \Delta'(P) + \Delta''(\rho) - \lambda P \end{vmatrix} = 0$$

shall be satisfied, without the vanishing of the four constituents in the determinant.

Ex. 3. Integrate (so far as to obtain an intermediate integral involving an arbitrary constant) the equations:—

(i) $rt - s^2 + (z - px - qy)(r - s) - pq(s - t) = 0$;

(ii) $r + 2Ss + S^2t = z - qy$,

where S has the forms

(*a*) $S = qe^x - y$,

(*b*) $S = qe^{-x} + y$,

(*c*) $S = (q^2 + y^2)^{\frac{1}{2}}$.

No Integral common to the Subsidiary System.

246. Finally, it may happen that the condition

$$\Theta = 0$$

is not satisfied for the equation

$$rt - s^2 + Rr + 2Ss + Tt = V;$$

then the system can be replaced by

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

The possibility of the method has depended on the assumption that some intermediate integral exists, so that $\frac{\partial u}{\partial p}$ and $\frac{\partial u}{\partial q}$ do not vanish together. In the present circumstances, therefore, the equation does not possess an intermediate integral.

Similarly, when the condition

$$\Phi = 0$$

is not satisfied for the equation

$$(r + T)(t + R) = (s - S)^2,$$

no intermediate integral exists. Likewise, the equation

$$r + 2Ss + S^2t = V$$

possesses no intermediate integral when the condition

$$L\Delta''(S) + \Delta(L) = \Delta''(V)$$

is not satisfied, nor the equation

$$r + 2Ss + Tt = V$$

when the condition

$$\Psi = 0$$

is not satisfied.

In each of these cases, when the last condition for the existence of an intermediate integral involving only one arbitrary constant is not satisfied, we are led to the conclusion that the intermediate integral does not exist because of the equations

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

It is, however, not to be inferred that no integral of the form

$$u(x, y, z) = 0$$

exists; for our analysis has depended upon a non-zero value for $\frac{\partial u}{\partial p}$ or $\frac{\partial u}{\partial q}$ or both. All that can be inferred is that it is useless to proceed to the integration of the equation by attempting to determine an intermediate integral. Some other method for the integration of the equation must be devised, which does not depend upon the use of any supposed intermediate integral and which must avoid any assumption of this character. Such a method is the process constructed by Ampère, to which accordingly we shall now proceed.

Ex. 1. Integrate completely the equation

$$(1 - xy)\{xr - (1 + xy)s + yt\} + (1 - x)p + (1 - y)q = 0. \qquad \text{(Boehm.)}$$

[The primitive is

$$z = \phi(xe^y) + \psi(ye^x).]$$

Ex. 2. Obtain an integral of the equation

$$2xqr + (yp - 2xq)s + 2ypt + 2xy(rt - s^2) + 3pq = 0$$

in the form

$$z = ax^\lambda + by^\mu + c,$$

where a, b, c, λ, μ are arbitrary constants subject to the single relation

$$2\lambda\mu + 1 = 0. \qquad \text{(Dixon.)}$$

Ex. 3. Shew that, if the equation

$$r - \lambda^2 t + \mu = 0,$$

where λ and μ can be functions of x, y, z, p, q, possesses two intermediate integrals, then λ must be of the form

$$\frac{a + 2bp + cp^2}{f + 2gq + hq^2},$$

where the six functions a, b, c, f, g, h of the variables x, y, z are subject to the relation

$$b^2 - ac = g^2 - fh,$$

and that μ must be of the form

$$\mu = \lambda^2 (F + 2Gq + Hq^2) + A + 2Bp + Cp^2,$$

where A, B, C, F, G, H are functions of x, y, z.

Obtain the complete expressions when a, b, c, f, g, h are constants: and construct the primitives of the equations so determined. (Kapteyn.)

Ex. 4. Integrate the equation

$$qr + (zq - p) s - pzt = 0. \qquad \text{(Goursat.)}$$

Ex. 5. Shew that, if the equation

$$r = \lambda^2 t,$$

where λ is a function of x and y only, possesses an intermediate integral of the form $f(u, v) = 0$, in which f denotes an arbitrary function, the quantity λ must satisfy an equation of the second order: and prove that the most general value of λ which satisfies this equation is given by the elimination of a between the equations

$$\left.\begin{aligned} \lambda \{x - \phi(a)\} &= y - \psi(a) \\ y \{x - \phi(a)\} &= -a + x\psi(a) \end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions. (Goursat.)

Supplementary Note.

In § 180 a warning was given that the aggregate of the usual classes of integrals, which occur in the solution of various individual equations of the second order, and of the Cauchy integrals which are proved to exist for widely comprehensive classes of equations of the second order, does not necessarily exhaust all the integrals that are possessed. Other integrals may exist which, as in the case (§ 34) of corresponding integrals of equations of the first order, are not included among the kinds of integrals there specified.

An illustration of the warning can be given in the case of an equation

$$Rr + 2Ss + Tt = V.$$

Even when it possesses two independent intermediate general integrals

$$f(u, v) = 0, \quad g(u', v') = 0,$$

and, *a fortiori*, when it possesses only a single intermediate general integral, it cannot be proved (and it is not in fact true) that the arbitrary function f or the arbitrary function g can be chosen so that any other intermediate integral is uniquely obtained.

For example, let

$$\vartheta(x, y, z, p, q) = 0$$

be any intermediate integral of the equation in question, supposed to possess a general intermediate integral

$$f(u, v) = 0.$$

Because the terms $rt - s^2$ are absent from the differential equation, we have (as in § 236)

$$J\left(\frac{u, v}{p, q}\right) = 0,$$

as may easily be verified by eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ between the equations

$$\frac{\partial f}{\partial u}\left(\frac{du}{dx} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q}\right) + \frac{\partial f}{\partial v}\left(\frac{dv}{dx} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q}\right) = 0,$$

$$\frac{\partial f}{\partial u}\left(\frac{du}{dy} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q}\right) + \frac{\partial f}{\partial v}\left(\frac{dv}{dy} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q}\right) = 0.$$

Hence the two equations

$$u(x, y, z, p, q) = u, \quad v(x, y, z, p, q) = v,$$

cannot be resolved so as to express p and q in terms of x, y, z, u, v. But in the absence of any conditions upon R, S, T, V, other than those in Ex. 5, § 244, which secure the existence of u and v as integrals of the subsidiary equations, we can conceive the two equations resolved so as to express any other two of the arguments in terms of the remainder: say

$$x = g(z, p, q, u, v), \quad y = h(z, p, q, u, v).$$

Let these expressions be substituted in ϑ, provided $\vartheta = 0$ is not a singularity of u or of v: and let the modified form of the intermediate integral be

$$\vartheta(x, y, z, p, q) = \theta(z, p, q, u, v) = 0;$$

then, if the general intermediate integral is to become the given intermediate integral $\theta = 0$, the arguments z, p, q should not occur in θ, the necessary and sufficient conditions being that the relations

$$\frac{\partial \theta}{\partial z} = 0, \quad \frac{\partial \theta}{\partial p} = 0, \quad \frac{\partial \theta}{\partial q} = 0,$$

should be satisfied identically.

Now, from the intermediate integral in the form $\theta = 0$, we have

$$\frac{\partial \theta}{\partial z} p + \frac{\partial \theta}{\partial p} r + \frac{\partial \theta}{\partial q} s + \frac{\partial \theta}{\partial u}\left(\frac{du}{dx} + \frac{\partial u}{\partial p} r + \frac{\partial u}{\partial q} s\right) + \frac{\partial \theta}{\partial v}\left(\frac{dv}{dx} + \frac{\partial v}{\partial p} r + \frac{\partial v}{\partial q} s\right) = 0,$$

$$\frac{\partial \theta}{\partial z} q + \frac{\partial \theta}{\partial p} s + \frac{\partial \theta}{\partial q} t + \frac{\partial \theta}{\partial u}\left(\frac{du}{dy} + \frac{\partial u}{\partial p} s + \frac{\partial u}{\partial q} t\right) + \frac{\partial \theta}{\partial v}\left(\frac{dv}{dy} + \frac{\partial v}{\partial p} s + \frac{\partial v}{\partial q} t\right) = 0.$$

As in § 236, we denote the (unequal) roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0$$

by ρ and σ. Multiplying the second of the derivatives of $\theta = 0$ by σ, and adding to the first, we have the combined relation

$$\begin{aligned}\frac{\partial \theta}{\partial z}(p + \sigma q) &+ \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) \\ &+ \frac{\partial \theta}{\partial u}\left\{\frac{du}{dx} + \sigma \frac{du}{dy} + \frac{\partial u}{\partial p}(r + \sigma s) + \frac{\partial u}{\partial q}(s + \sigma t)\right\} \\ &+ \frac{\partial \theta}{\partial v}\left\{\frac{dv}{dx} + \sigma \frac{dv}{dy} + \frac{\partial v}{\partial p}(r + \sigma s) + \frac{\partial v}{\partial q}(s + \sigma t)\right\} = 0,\end{aligned}$$

which, on taking account of the subsidiary partial equations of the first order satisfied by u and by v, becomes

$$\begin{aligned}\frac{\partial \theta}{\partial z}(p + \sigma q) &+ \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) \\ &+ \frac{1}{R}\left(\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial \theta}{\partial v}\frac{\partial v}{\partial p}\right)\{-V + Rr + 2Ss + Tt\} = 0.\end{aligned}$$

The values of r, s, t consistent with $\theta = 0$ satisfy the original differential equation; hence the relation

$$\frac{\partial \theta}{\partial z}(p + \sigma q) + \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) = 0$$

must be satisfied, in association with the original differential equation and with the intermediate integral

$$\theta(z, p, q, u, v) = 0,$$

together with derivatives from the intermediate integral. It is clear that the conditions

$$\frac{\partial\theta}{\partial z} = 0, \quad \frac{\partial\theta}{\partial p} = 0, \quad \frac{\partial\theta}{\partial q} = 0,$$

cannot be assigned as identical equations merely as necessary consequences of the relation just obtained: and therefore we conclude that the general intermediate integral, when it is possessed, is not completely (though it may be largely) comprehensive of all the intermediate integrals that may be possessed.

A fortiori, a similar doubt extends as to the comprehensiveness of a general primitive.

Two examples will suffice.

Ex. 1. The equation

$$x^2r - y^2t = (px + qy - z)^2$$

has an intermediate general integral

$$f\left(ye^{\frac{-1}{px+qy-z}}, \, xy\right) = 0.$$

It also possesses a special intermediate integral

$$\vartheta = px + qy - z = 0.$$

Manifestly no form of the function f can be devised which will change $f = 0$ into $\vartheta = 0$: and even the preliminary transformation of ϑ into θ, as in the text, cannot be accomplished, for $\vartheta = 0$ provides an essential singularity of one of the arguments of f.

Ex. 2. The equation

$$x(r+s) - y(s+t) = p + q - xy$$

has an intermediate general integral

$$f(u, v) = 0,$$

where

$$u = x(p + q - xy),$$

$$v = y(p + q - xy),$$

and the value of σ in the text is unity. It also possesses a special intermediate integral

$$\vartheta = p + q - xy = 0.$$

Effecting upon ϑ the transformation which changes it into θ, we find

$$\theta = p + q - \rho,$$

where ρ is given in terms of p, q, u, v by the equation

$$(p+q-\rho)^2 \rho - uv = 0.$$

Thus θ does not involve z: but it does involve p and q: and we therefore do not have

$$\frac{\partial\theta}{\partial p} = 0, \quad \frac{\partial\theta}{\partial q} = 0,$$

satisfied identically, that is, the intermediate general integral does not comprehend the intermediate special integral.

The equation of condition comes to be (with the value of σ)

$$\frac{\partial\theta}{\partial p}(r+s) + \frac{\partial\theta}{\partial q}(s+t) = 0:$$

it is easy to verify that the quantities $\frac{\partial\theta}{\partial p}$, $\frac{\partial\theta}{\partial q}$, while not vanishing identically, do vanish in virtue of $\theta = 0$: and so the equation of condition is satisfied.

CHAPTER XVII.

Ampère's Method applied to Equations of the Second Order in Two Independent Variables.

The method, which Ampère constructed for the integration of partial differential equations, is contained in the two important memoirs presented* to the Institute of France in 1814. The application of the method to equations of the first order is now relatively unimportant, owing to the subsequent discovery of other methods of treating such equations. The application to equations of the second order is still of fundamental importance. The memoirs seem more complicated than they are in fact, the principal cause being the cumbrous character of the notation.

Reference, in this connection, should also be made to the valuable memoir by Imschenetsky†, who was among the earliest writers to indicate the importance of Ampère's researches on the subject.

Limitations on the Integral.

247. The methods given by Monge and by Boole were applied by them to equations of a restricted form: and the assumption that an intermediate integral exists is essential to the practical success of their methods. As has been seen in the last chapter, equations of that restricted form occur which do not satisfy all the conditions needed to justify the assumption; and accordingly it follows that those methods are of limited application.

The method devised by Ampère, though explained only for equations of the second order involving two independent variables, and illustrated mostly by application to equations of the forms

* They are contained in the *Journal de l'École Polytechnique*; one of them, in *Cahier* XVII (1815), pp. 549—611, deals with Ampère's general theory; the other, in *Cahier* XVIII (1819), pp. 1—188, contains the application of the theory to particular equations.

† *Grunert's Archiv*, t. LIV (1872), pp. 209—360.

discussed by Monge and subsequently by Boole, can be extended to equations in any number of independent variables and of any order: moreover, it neither makes nor requires any initial assumption as to the character of the equation or to the existence of an intermediate integral. Further, when the method proves effective for the practical construction of the primitive of a given equation, the primitive is not necessarily provided by means of a single explicit equation between the variables in finite form: but it need hardly be remarked that generality of the equations to be treated by a method is more important than simplicity of form in the primitive.

In the present chapter we shall deal with equations of the second order involving two independent variables; and we shall begin with the general equation

$$f(x, y, z, p, q, r, s, t)=0,$$

where a sufficiently wide class of equations will be provided by supposing that f is merely polynomial in r, s, t. The integral provided by Cauchy's theorem contains two arbitrary functions which may have definite arguments: we shall assume that the arguments are definite. The arguments may be different from one another, or they may be the same as one another; and derivatives of the arbitrary functions with respect to their arguments may occur. We shall assume that the highest derivative of an arbitrary function, which occurs in the integral system, is of finite order and that the integral system is free from partial quadratures which essentially cannot be performed. Lastly, we shall assume that the occurrence of the derivatives of the arbitrary functions is of such a character that (§ 181) the formation of the derivatives of z of successively increasing orders introduces derivatives of the arbitrary functions of successively increasing orders*: but there is no assumption as to the existence of an intermediate integral.

AMPÈRE'S METHOD.

248. Ampère's method is based upon a transformation of the independent variables as the stage of initial departure. Let a new independent variable α be introduced; it is not made determinate until the effect of the transformation is being considered. This

* This aggregate of conditions should be compared with the aggregate of conditions in § 221, where, however, it is specified that the integral shall be given by a single equation resoluble with regard to the dependent variable.

variable α may be a function of both variables x and y, though it may involve not more than one of the variables: let it be used to make x and α the independent variables so that, in the least restricted circumstances, y is a function of x and α. Denoting partial differentiations by $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta \alpha}$ when x and α are the independent variables, we have

$$\frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta \alpha} d\alpha = du$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y}\left(\frac{\delta y}{\delta x} dx + \frac{\delta y}{\delta \alpha} d\alpha\right),$$

for any function u; and therefore

$$\frac{\delta u}{\delta x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\delta y}{\delta x}, \qquad \frac{\delta u}{\delta \alpha} = \frac{\partial u}{\partial y}\frac{\delta y}{\delta \alpha}.$$

Consequently,

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}, \qquad \frac{\delta z}{\delta \alpha} = q\frac{\delta y}{\delta \alpha},$$

$$\frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \qquad \frac{\delta p}{\delta \alpha} = s\frac{\delta y}{\delta \alpha},$$

$$\frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}, \qquad \frac{\delta q}{\delta \alpha} = t\frac{\delta y}{\delta \alpha}.$$

Then, keeping the value of t given by

$$t = \frac{\delta q}{\delta \alpha} \div \frac{\delta y}{\delta \alpha},$$

we have

$$s = \frac{\delta q}{\delta x} - t\frac{\delta y}{\delta x},$$

$$r = \frac{\delta p}{\delta x} - \frac{\delta q}{\delta x}\frac{\delta y}{\delta x} + t\left(\frac{\delta y}{\delta x}\right)^2.$$

Let these values of r and s be substituted in $f = 0$, when f will become a polynomial in t: let this polynomial be arranged in powers of t, so that the equation then is

$$P_0 + P_1 t + \ldots + P_n t^n = 0,$$

where the original degree of f, as a polynomial in r, s, t, is n at least.

Thus far, the quantity α is quite unrestricted, and so it is completely at our disposal: let it be chosen to be the (as yet unknown) argument of one of the arbitrary functions in the integral system. Now, by the hypothesis concerning the character of the integral system, the quantities p and q contain derivatives of that arbitrary function of one order higher than those which occur in the integral system. When we change the independent variables so that they become x and α, the partial derivatives of p and q with regard to x contain only the same derivatives of the arbitrary function as do p and q: while the partial derivative of q with regard to α (which does occur in the transformed equation, being introduced by t, while the partial derivative of p with regard to α does not occur there) contains a derivative of the arbitrary function of α, that is of order higher by one unit than the derivatives occurring in p, q, $\frac{\delta p}{\delta x}$, $\frac{\delta q}{\delta x}$. Hence, in the transformed equation, the quantity t contains a derivative of the arbitrary function of α of one order higher than the derivatives that occur in $P_0, P_1, \ldots, P_n$. Moreover, the differential equation is to be satisfied identically in connection with the integral system, and this must take place whether the independent variables be x and y or x and α. Taking account of the successive degrees of that highest derivative which occurs in t alone, we see that the equation can be satisfied only if

$$P_n = 0, \quad P_{n-1} = 0, \ \ldots, \ P_1 = 0, \quad P_0 = 0.$$

We have seen (§ 186) that $\frac{\delta y}{\delta x}$, which is the derivative of y on the supposition that α is constant, also satisfies the equation

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial r}\left(\frac{\delta y}{\delta x}\right)^2 = 0.$$

249. Thus there is a number of simultaneous equations. If these equations are consistent with one another, and with the original equation $f = 0$, regard being paid to the relations between derivatives with reference to the old independent variables and the new, then the equation can possess an integral system of the specified type. The quantities $P_n, P_{n-1}, \ldots, P_1, P_0$ contain z, p, q, and the derivatives $\frac{\delta y}{\delta x}, \frac{\delta p}{\delta x}, \frac{\delta q}{\delta x}$: also

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}:$$

and therefore the system of equations contains no derivatives with regard to α, so that it can be regarded as a simultaneous system of ordinary equations.

If the arguments of the two arbitrary functions in the integral system are the same, the preceding discussion is complete as regards the inferences to be drawn from the occurrence of the highest argument of an arbitrary function: the inference is merely duplicated by taking each of the arbitrary functions in turn.

If the arguments of the two arbitrary functions in the integral system are different, let the other argument be denoted by β. A corresponding discussion of the equations, after making x and β the independent variables, leads to the same set of equations involving only derivatives with regard to x.

Hence the system, if it is self-consistent, applies to both arguments α and β when they are distinct from one another. As in § 186, they then are distinct integrals of the equation

$$\frac{\partial f}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial s}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial r}\left(\frac{\partial u}{\partial y}\right)^2 = 0.$$

We thus have a number of equations

$$P_n = 0, \quad P_{n-1} = 0, \ \ldots, \ P_1 = 0, \quad P_0 = 0,$$

which are to be consistent with one another and with the original equation in virtue of

$$\frac{\delta p}{\delta x} = r + s\,\frac{\delta y}{\delta x}, \qquad \frac{\delta q}{\delta x} = s + t\,\frac{\delta y}{\delta x}.$$

These equations, other than the original equation, involve x, y, z, p, q, $\frac{\delta y}{\delta x}$, $\frac{\delta p}{\delta x}$, $\frac{\delta q}{\delta x}$; and we also have

$$\frac{\delta z}{\delta x} = p + q\,\frac{\delta y}{\delta x}.$$

Hence there cannot be more than three essentially independent equations: otherwise, it would be possible to eliminate $\frac{\delta y}{\delta x}$, $\frac{\delta p}{\delta x}$, $\frac{\delta q}{\delta x}$, and to obtain a relation between x, y, z, p, q alone, involving no arbitrary constant. There is no guarantee that such a relation is an intermediate integral of the original equation: even if it is an intermediate integral, it contains no arbitrary element, and therefore it is of the nature of a special integral.

We put aside such special integrals when they exist; and therefore we cannot have more than three independent equations in the subsidiary Ampère system.

The system contains partial derivatives with regard to x alone, these being taken on the supposition that an argument of an arbitrary function is constant in their formation.

The system may be irresoluble in the sense that no simpler system or systems can be formed which are equivalent to it: the inference is that there is only a single argument common to the two arbitrary functions in the general integral, the equation

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial r}\left(\frac{\delta y}{\delta x}\right)^2 = 0$$

then having equal roots for $\dfrac{\delta y}{\delta x}$.

The system may be resoluble in the sense that it can be replaced by simpler systems, which are its equivalent. There cannot be more than two such systems, one to be associated with an argument α, the other to be associated with an argument β; in this case, the preceding quadratic has unequal roots.

Whether there be one system or whether there be two systems, the first object is to obtain some integral of a system. When that integral is obtained, the arbitrary element is made either the argument (or some arbitrary function of the argument) associated with the system.

If there is only a single system, further integrals of the system are required, the arbitrary elements being made arbitrary functions of the one argument: the simultaneous integrals provide an integral of the original equation.

If there are two systems, and an integral of each has been obtained, then the arguments α and β are made the independent variables for a new set of equations. Two of these equations are

$$\left.\begin{aligned}\frac{\partial z}{\partial \alpha} &= p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha}\\ \frac{\partial z}{\partial \beta} &= p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\end{aligned}\right\};$$

the others of the set are selected from the respective systems, to express relations among some of the partial derivatives of x, y,

p, q, when the respective arguments are constant. This new set of equations is to be treated as a set of simultaneous equations to express, ultimately, x, y, z in terms of α and β: the equations, which give these expressions, constitute an integral of the original differential equation.

The method is illustrated by the following examples, which correspond to various cases.

Ex. 1. Consider the equation*

$$(r-pt)^2=q^2rt.$$

The arguments of the arbitrary function are the same as one another only if the relation

$$4\frac{\partial f}{\partial t}\frac{\partial f}{\partial r}=\left(\frac{\partial f}{\partial s}\right)^2$$

is satisfied concurrently with the equation, that is, if

$$4p+q^2=0.$$

Putting this possibility on one side for the present, we assume that the arguments are different from one another: and we denote them by α and β.

Denoting partial derivatives with regard to x, when α and x are the independent variables, by z', y', p', q', and substituting

$$r=p'-q'y'+ty'^2$$

in the equation, it becomes

$$\{p'-q'y'+t(y'^2-p)\}^2=q^2t(p'-q'y'+ty'^2);$$

and therefore, if the integral system is of the specified finite type, we must have

$$(y'^2-p)^2=q^2y'^2,$$

$$2(p'-q'y')(y'^2-p)=q^2(p'-q'y'),$$

$$(p'-q'y')^2=0,$$

which are satisfied by

$$p'-q'y'=0,$$

$$(y'^2-p)^2=q^2y'^2.$$

These two equations are consistent with one another: and, in virtue of the equation

$$r=p'-q'y'+ty'^2,$$

they are consistent with the original equation. The integral system can therefore be of the specified type.

We require an integral combination of the system, taken with

$$z'=p+qy'.$$

* It is discussed by Ampère in his second memoir, p. 48.

Now we have

$$\left.\begin{aligned} p'-q'y'&=0\\ y'^2-p&=\pm qy'\end{aligned}\right\},$$

and therefore

$$2y'y''-q'y'=\pm(qy''+q'y').$$

Taking the lower sign, we have

$$(2y'+q)\,y''=0;$$

hence* we can take

$$y''=0;$$

and therefore, as the derivatives with regard to x are taken on the supposition that a is constant, we can write

$$y'=a,$$
$$y=ax+\phi(a),$$

where ϕ is an arbitrary function. Also, with this value, we have

$$a^2+qa-p=0;$$

and

$$p'-aq'=0,$$

that is,

$$p-aq=\psi(a)$$
$$=a^2,$$

from the preceding equation.

Next, we take the equations when x and β are the independent variables. Denoting by z_1, y_1, p_1, q_1 the derivatives with regard to x when β is constant, we still have

$$z_1=p+qy_1,$$
$$p_1-q_1y_1=0,$$
$$y_1^2-p=\pm qy_1:$$

but as the lower sign was taken in the former case, we must take the upper sign now, or the two sets of equations will be identical. Hence our equations are

$$\left.\begin{aligned} z_1&=p+qy_1\\ p_1-q_1y_1&=0\\ y_1^2-p&=qy_1\end{aligned}\right\},$$

when β is constant.

Eliminating y_1 between the last two equations, we have

$$\left(\frac{p_1}{q_1}\right)^2-p=q\frac{p_1}{q_1},$$

that is,

$$\left(\frac{dp}{dq}\right)^2-p=q\frac{dp}{dq},$$

or, writing

$$p=Y,\quad q=X,\quad \frac{dp}{dq}=P,$$

* The value $y'=-\tfrac{1}{2}q$ leads to the temporarily excluded case.

we have an equation in the form

$$Y = P^2 - XP,$$

one of the recognised forms of equations of the first order that are simply integrable. Proceeding in the regular manner, we find

$$(3X - 2P)^2 P = \text{constant}$$
$$= \beta,$$

say, for β has been assumed constant throughout the integrations. Thus

$$(3q - 2y_1)^2 y_1 = \beta$$

is an integrable combination of the system of equations. But

$$y_1^2 - qy_1 = p = a^2 + aq,$$

so that

$$(y_1 + a)(y_1 - a - q) = 0,$$

and therefore

$$y_1 = -a, \text{ or } y_1 = a + q;$$

also

$$a = \tfrac{1}{2}\{-q \pm (q^2 + 4p)^{\frac{1}{2}}\},$$

so that the first pair of values of y_1 are

$$\tfrac{1}{2}\{q \mp (q^2 + 4p)^{\frac{1}{2}}\},$$

and the second pair of values of y_1 are

$$\tfrac{1}{2}\{q \pm (q^2 + 4p)^{\frac{1}{2}}\},$$

that is, the same as the first pair. Hence, without loss of generality, we may take

$$y_1 = -a;$$

and the equations expressing p and q in terms of a and β are

$$(3q + 2a)^2 a = -\beta,$$
$$a^2 + aq - p = 0;$$

while we have

$$y = ax + \phi(a),$$
$$z' = p + qa,$$
$$z_1 = p - qa.$$

Now make x, y, z functions of a and β. Since

$$\frac{\partial y}{\partial x}, \text{ when } a \text{ is constant} = a,$$

$$\frac{\partial y}{\partial x}, \ldots\ldots \beta \ldots\ldots\ldots\ldots\ldots = -a,$$

we have, when a and β are made the variables

$$\frac{\partial y}{\partial a} = -a\frac{\partial x}{\partial a},$$

$$\frac{\partial y}{\partial \beta} = a\frac{\partial x}{\partial \beta}.$$

For convenience, take a^2 in place of a, and $-\beta^2$ in place of β: our equations are

$$\frac{\partial y}{\partial a} = -a^2 \frac{\partial x}{\partial a},$$

$$\frac{\partial y}{\partial \beta} = \quad a^2 \frac{\partial x}{\partial \beta},$$

$$\frac{\partial z}{\partial a} = \quad p\frac{\partial x}{\partial a} + q\frac{\partial y}{\partial a},$$

$$\frac{\partial z}{\partial \beta} = \quad p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}.$$

Also, the values of p and q are given by

$$(3q + 2a^2)\, a^2 = \beta^2,$$

$$a^4 + a^2 q = p,$$

so that

$$q = -\tfrac{2}{3}a^2 + \tfrac{1}{3}\frac{\beta}{a}, \quad p = \tfrac{1}{3}a^4 + \tfrac{1}{3}a\beta.$$

We have

$$\frac{\partial}{\partial a}\left(a^2 \frac{\partial x}{\partial \beta}\right) = \frac{\partial^2 y}{\partial a \partial \beta} = \frac{\partial}{\partial \beta}\left(-a^2 \frac{\partial x}{\partial a}\right),$$

so that

$$a\frac{\partial^2 x}{\partial a \partial \beta} + \frac{\partial x}{\partial \beta} = 0.$$

Hence

$$a\frac{\partial x}{\partial \beta} = \theta''(\beta),$$

and therefore

$$x = \frac{1}{a}\theta'(\beta) + \psi(a),$$

where θ and ψ are arbitrary so far as this equation is concerned. Also,

$$dy = -a^2 \frac{\partial x}{\partial a} da + a^2 \frac{\partial x}{\partial \beta} d\beta$$

$$= \{\theta'(\beta) - a^2\psi'(a)\}\, da + a\theta''(\beta)\, d\beta,$$

and therefore

$$y = a\theta'(\beta) - \int a^2\psi'(a)\, da.$$

But, with the changed value of a, we have (from the former relation)

$$y = a^2 x + \phi(a^2)$$

$$= a\theta'(\beta) + a^2\psi(a) + \phi(a^2).$$

Taking a new arbitrary function χ, such that

$$\psi(a) = a\chi'(a) + 2\chi(a),$$

then

$$\int a^2\psi'(a)\, da = \int \{a^3\chi''(a) + 3a^2\chi'(a)\}\, da$$

$$= a^3\chi'(a);$$

and then

$$\left.\begin{aligned} y &= a\theta'(\beta) - a^3\chi'(a) \\ x &= \frac{1}{a}\theta'(\beta) + a\chi'(a) + 2\chi(a) \end{aligned}\right\}.$$

Lastly,

$$\begin{aligned} dz &= \left(p\frac{\partial x}{\partial a} + q\frac{\partial y}{\partial a}\right)da + \left(p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\right)d\beta \\ &= (p - qa^2)\frac{\partial x}{\partial a}da + (p + qa^2)\frac{\partial x}{\partial \beta}d\beta \\ &= (-a^2\theta' + a^5\chi'' + 3a^4\chi')\,da + (-\tfrac{1}{3}a^3 + \tfrac{2}{3}\beta)\,\theta''\,d\beta\,; \end{aligned}$$

and therefore

$$z = \tfrac{1}{3}(2\beta - a^3)\,\theta'(\beta) - \tfrac{2}{3}\theta(\beta) + \int (a^5\chi'' + 3a^4\chi')\,da.$$

Taking

$$a\chi'(a) + 2\chi(a) = \mu(a),$$

then

$$a^3\chi'(a) = \int a^2\mu'(a)\,da,$$

and we have

$$\left.\begin{aligned} x &= \frac{1}{a}\theta'(\beta) + \mu(a) \\ y &= a\theta'(\beta) - \int a^2\mu'(a)\,da \\ z &= \tfrac{1}{3}(2\beta - a^3)\,\theta'(\beta) - \tfrac{2}{3}\theta(\beta) + \int a^4\mu'(a)\,da \end{aligned}\right\}.$$

We can effect the apparent partial quadratures after taking a new arbitrary function $\sigma(a)$, such that

$$\mu(a) = \frac{d^4\sigma(a)}{da^4}.$$

The integral system then is of the desired form.

Note 1. We have put the relation

$$q^2 + 4p = 0$$

on one side in the preceding investigation. Now suppose

$$q^2 + 4p = 0,$$

so that, differentiating, we have

$$r = -\tfrac{1}{2}qs, \quad s = -\tfrac{1}{2}qt,$$

and therefore

$$r = \tfrac{1}{4}q^2t = -pt,$$

which are consistent with the original differential equation. Thus the relation

$$q^2 + 4p = 0$$

is of the nature of a special integral: we have

$$z = -\tfrac{1}{4}c^2x + cy + a,$$

where c and a are arbitrary constants.

Note 2. The original equation

$$(r-pt)^2=q^2rt$$

can be resolved into the two equations

$$r-\tfrac{1}{2}t\{2p+q^2+q(4p+q^2)^{\frac{1}{2}}\}=0,$$

and

$$r-\tfrac{1}{2}t\{2p+q^2-q(4p+q^2)^{\frac{1}{2}}\}=0.$$

Now (with the earlier significance of a) we had

$$a^2+aq-p=0,$$

$$y=ax+\phi(a),$$

and therefore

$$2y+qx-x(4p+q^2)^{\frac{1}{2}}=\phi\{(4p+q^2)^{\frac{1}{2}}-q\},$$

which is of the nature of an intermediate integral. When it is differentiated, it leads to

$$2r=t\{2p+q^2-q(4p+q^2)^{\frac{1}{2}}\},$$

the sign of the radical being the same as in the intermediate integral. Thus the first branch of the differential equation has

$$2y+qx+x(4p+q^2)^{\frac{1}{2}}=\phi\{-(4p+q^2)^{\frac{1}{2}}-q\},$$

for an intermediate integral; and the second branch has

$$2y+qx-x(4p+q^2)^{\frac{1}{2}}=\phi\{(4p+q^2)^{\frac{1}{2}}-q\},$$

for an intermediate integral.

It will be noticed that the method of integration adopted nowhere uses these integrals.

Ex. 2. As another illustration, consider Ampère's* process of integration applied to

$$(1+q^2)r-2pqs+(1+p^2)t=0,$$

which is the equation of minimal surfaces.

The arguments of the arbitrary functions are the same only if

$$1+p^2+q^2=0.$$

We may put this relation† on one side, as before: it is not inconsistent with the differential equation, but it leads only to a trivial integral: and we shall therefore assume that $1+p^2+q^2$ does not vanish.

* Second memoir, p. 82.

† It arises also, (see § 340), as a subsidiary equation in the integration

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

being connected with the characteristics.

Keeping the same notation as in the last example by writing θ' for $\frac{\delta\theta}{\delta x}$, when a is constant, and θ_1 for $\frac{\delta\theta}{\delta x}$, when β is constant, we substitute

$$s=q'-ty', \quad r=p'-q'y'+ty'^2,$$

in the equation ; and then we make it evanescent, quà relation in t. Thus

$$\left.\begin{aligned}(1+q^2)(p'-q'y')-2pqq' &=0\\ (1+q^2)y'^2+2pqy'+1+p^2&=0\end{aligned}\right\},$$

together with

$$z'=p+qy',$$

are our equations : and, as in the general case, these equations are satisfied also by p_1, q_1, y_1. When we write

$$w=(1+p^2+q^2)^{\frac{1}{2}},$$

the resolved equivalents can be taken

$$\left.\begin{aligned}y'+\frac{pq-iw}{1+q^2} \qquad &=0\\ p'-\frac{pq+iw}{1+q^2}\,q'&=0\\ z'-\frac{p+iqw}{1+q^2} \qquad &=0\end{aligned}\right\},$$

being the system when a is constant ; and

$$\left.\begin{aligned}y_1+\frac{pq+iw}{1+q^2} \qquad &=0\\ p_1-\frac{pq-iw}{1+q^2}\,q_1&=0\\ z_1-\frac{p-iqw}{1+q^2} \qquad &=0\end{aligned}\right\},$$

being the system when β is constant. We need an integral equation for each system.

Now the second equation

$$p'-\frac{pq+iw}{1+q^2}\,q'=0$$

in the first system, that is,

$$\frac{dp}{dq}-\frac{pq+iw}{1+q^2}=0,$$

can be expressed as a Clairaut equation : and its integral is

$$\frac{pq+iw}{1+q^2}=\text{constant},$$

so that we take

$$\frac{pq+iw}{1+q^2}=a.$$

Similarly, the second equation in the other system has an integral

$$\frac{pq - iw}{1+q^2} = \text{constant},$$

and we take

$$\frac{pq - iw}{1+q^2} = \beta.$$

We now proceed to make α and β the independent variables, and we note that

$$y' = -\beta, \quad y_1 = -\alpha:$$

that is, as y' is the value of $\frac{\delta y}{\delta x}$, when α is constant, and y_1 is the value of $\frac{\delta y}{\delta x}$, when β is constant, we have

$$\frac{\partial y}{\partial \beta} = -\beta \frac{\partial x}{\partial \beta}, \quad \frac{\partial y}{\partial \alpha} = -\alpha \frac{\partial x}{\partial \alpha}.$$

Hence

$$\frac{\partial}{\partial \alpha}\left(\beta \frac{\partial x}{\partial \beta}\right) = -\frac{\partial^2 y}{\partial \alpha \, \partial \beta} = \frac{\partial}{\partial \beta}\left(\alpha \frac{\partial x}{\partial \alpha}\right),$$

and therefore

$$\frac{\partial^2 x}{\partial \alpha \, \partial \beta} = 0.$$

Consequently, we can take

$$x = \phi'(\alpha) + \psi'(\beta),$$

where ϕ and ψ are arbitrary functions; and then

$$\frac{\partial y}{\partial \alpha} = -\alpha\phi''(\alpha), \quad \frac{\partial y}{\partial \beta} = -\beta\psi''(\beta),$$

so that

$$y = \phi(\alpha) - \alpha\phi'(\alpha) + \psi(\beta) - \beta\psi'(\beta).$$

Now

$$dz = \left(p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha}\right) d\alpha + \left(p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\right) d\beta$$

$$= (p - \alpha q)\frac{\partial x}{\partial \alpha} d\alpha + (p - \beta q)\frac{\partial x}{\partial \beta} d\beta$$

$$= (p - \alpha q)\,\phi''(\alpha)\, d\alpha + (p - \beta q)\,\psi''(\beta)\, d\beta.$$

Also,

$$\alpha = \frac{pq + iw}{1+q^2},$$

so that

$$\{(1+q^2)\alpha - pq\}^2 = -1 - p^2 - q^2,$$

that is,

$$(1+q^2)\alpha^2 - 2\alpha pq + 1 + p^2 = 0,$$

and therefore

$$p - \alpha q = i(1+\alpha^2)^{\frac{1}{2}}.$$

Similarly

$$p - \beta q = i(1+\beta^2)^{\frac{1}{2}};$$

and so

$$z = i\int (1+a^2)^{\frac{1}{2}}\,\phi''(a)\,da + i\int (1+\beta^2)^{\frac{1}{2}}\,\psi''(\beta)\,d\beta.$$

This equation, with

$$x = \phi'(a) + \psi'(\beta),$$
$$y = \phi(a) - a\phi'(a) + \psi(\beta) - \beta\psi'(\beta),$$

constitutes the integral of the equation.

This is the form given by Legendre and by Ampère; the apparent partial quadrature can be removed by taking

$$\phi(a) = (1+a^2)^{\frac{3}{2}}\,\Phi'(a),$$
$$\psi(\beta) = (1+\beta^2)^{\frac{3}{2}}\,\Psi'(\beta).$$

Ex. 3. Obtain the integral of the preceding equation in Monge's form, viz.

$$x = a + \beta,$$
$$y = f(a) + g(\beta),$$
$$z = i\int \{1+f'^2(a)\}^{\frac{1}{2}}\,da + i\int \{1+g'^2(\beta)\}^{\frac{1}{2}}\,d\beta,$$

where f and g are arbitrary.

Ex. 4. Verify that, if

$$p = \frac{uv-1}{u+v}, \quad iq = \frac{uv+1}{u+v},$$

then

$$a = i\frac{v^2+1}{v^2-1}, \quad \beta = -i\frac{u^2+1}{u^2-1}.$$

Apply these values to obtain the integral of the preceding equation in Weierstrass's form*, viz.

$$\left.\begin{aligned} x &= (1-v^2)V'' + 2vV' - 2V + (1-u^2)U'' + 2uU' - 2U \\ y &= i\{(1+v^2)V'' - 2vV' + 2V - (1+u^2)U'' + 2uU' - 2U\} \\ z &= 2vV'' - 2V' + 2uU'' - 2U' \end{aligned}\right\},$$

where U is any arbitrary function of u, and V is any arbitrary function of v.

Discuss the limitations for an integral that is entirely real.

Ex. 5. Consider the equation†

$$st + x(rt - s^2)^2 = 0.$$

When we substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

and make the resulting equation evanescent in t, we find

$$xq'^2 = 0,$$
$$q' - 2xq'^2(p' + qy') = 0,$$
$$x(p' + q'y')^2 - y' = 0.$$

* This is the integrated form which lends itself most readily to the discussion of minimal surfaces.

† Discussed by Ampère in his first memoir, p. 608: see *ante*, § 183, Ex. 2.

These give

$$q'=0, \quad xp'^2-y'=0,$$

which are consistent with one another and with the original equation.

We take

$$q=\phi'(a),$$

where ϕ is an arbitrary function: then

$$\frac{\delta p}{\delta a}+y'\frac{\delta q}{\delta a}=(s+ty')\frac{\delta y}{\delta a}$$
$$=q'\frac{\delta y}{\delta a}$$
$$=0,$$

and therefore

$$\frac{\delta p}{\delta a}+x\left(\frac{\delta p}{\delta x}\right)^2\phi''(a)=0.$$

Hence

$$p=-k^2\phi'(a)+2kx^{\frac{1}{2}},$$

where k is a constant: and so

$$y'=xp'^2=k^2,$$

and

$$y=k^2x+\theta(a),$$

where θ is an arbitrary function. Now

$$\frac{\delta z}{\delta a}=q\frac{\delta y}{\delta a}$$
$$=\phi'(a)\,\theta'(a),$$

and

$$\frac{\delta z}{\delta x}=p+qy'$$
$$=2kx^{\frac{1}{2}};$$

hence

$$z=\tfrac{4}{3}kx^{\frac{3}{2}}+\chi(a)$$
$$=\tfrac{4}{3}kx^{\frac{3}{2}}+F(y-k^2x),$$

which is the most general integral that can thus be obtained, F denoting an arbitrary function, and k an arbitrary constant.

Ampère's Method applied to Special Equations.

250. The preceding examples give some indication of Ampère's method: it is of interest to apply it to the particular equations

$$rt-s^2+Rr+2Ss+Tt=V,$$
$$Rr+2Ss+Tt=V,$$

the initial assumption now being that the integral system is finite in form and free from partial quadratures. The subsidiary equations are substantially the same as in the methods of Monge and of Boole: but they now bear a new significance.

We write

$$y',\ z',\ p',\ q',\ \text{for}\ \frac{\delta y}{\delta x},\ \frac{\delta z}{\delta x},\ \frac{\delta p}{\delta x},\ \frac{\delta q}{\delta x},$$

the derivatives being taken on the supposition that α is constant: and, if there be two different arguments α and β of arbitrary functions, those derivatives (taken on the supposition that β is constant) are denoted by y_1, z_1, p_1, q_1, respectively. According to the method, we substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

in the equation; and we make the modified equation evanescent as a relation involving t. When applied to the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the process leads to the two equations

$$\left.\begin{aligned} p' + q'y' + Ry'^2 - 2Sy' + T = 0 \\ R(p' - q'y') + 2Sq' - q'^2 - V = 0 \end{aligned}\right\}.$$

Multiplying the first of these by R and subtracting the second from the product, we have

$$q'^2 + 2Rq'y' + R^2y'^2 - 2Sq' - 2RSy' + RT + V = 0,$$

that is,

$$(q' + Ry')^2 - 2S(q' + Ry') + RT + V = 0.$$

Denoting by μ, as before, a root of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0,$$

we have

$$q' + Ry' + \mu = 0.$$

The first equation then gives

$$\begin{aligned} p' + T &= 2Sy' - q'y' - Ry'^2 \\ &= (2S + \mu)\,y' \\ &= -\nu y', \end{aligned}$$

if ν denote the other root of the quadratic.

Let the roots of the quadratic be unequal, being then denoted by ρ and σ. We have, as usual, a couple of linear systems: one of them can be associated with the argument α, the other with the argument β; thus

$$\left.\begin{aligned} p' + \rho y' + T &= 0 \\ q' + Ry' + \sigma &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + Ry_1 + \rho &= 0 \\ z_1 - qy_1 - p &= 0 \end{aligned}\right\},$$

are the linear systems equivalent to the two equations.

When the roots of the quadratic are equal, then

$$\rho = \sigma = -S,$$
$$S^2 = RT + V;$$

there is only a single system

$$\left.\begin{aligned} p' - Sy' + T &= 0 \\ q' + Ry' - S &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\},$$

equivalent to the two equations*.

The equations in form are substantially the same as those which occur in Monge's method: and, as was proved, the latter are equivalent to those which occur in Boole's method. Here, however, they have been obtained without the assumptions upon which

* It has been assumed that there are two arguments or only one, according as the roots of the quadratic are unequal or are equal. As a matter of fact, there are two arguments or only one, according (§ 186) as the equation

$$4\frac{\partial f}{\partial r}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial s}\right)^2,$$

that is, as the equation

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT,$$

is not satisfied or is satisfied, that is, according as the relation

$$S^2 - RT = V$$

does not hold or does hold. These are the conditions which justify the assumption.

those methods are based; and they now are definitely connected with the arguments of the arbitrary functions which are to be found in the integral system.

Construction of the Primitive.

251. The actual process of proceeding to the integral system has already (§ 249) been indicated in general terms.

In the first place, let there be two linear systems, connected with α and β respectively. Suppose that an integral of the system

$$\left.\begin{aligned} p' + \rho y' + T &= 0 \\ q' + R y' + \sigma &= 0 \\ z' - q y' - p &= 0 \end{aligned}\right\}$$

has been obtained: we make the arbitrary element in that integral equal to α. (If another distinct integral of the same system can be obtained, we make the arbitrary element equal to $\phi(\alpha)$, where ϕ is an arbitrary function: the elimination of α will lead to an intermediate integral of the original equation. This, however, is not the most general case: and it is not necessary for the success of the method.) Similarly, suppose that an integral of the system

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + R y_1 + \rho &= 0 \\ z_1 - q y_1 - p &= 0 \end{aligned}\right\}$$

has been obtained: we make the arbitrary element in that integral equal to β. (If a second integral of this system can be obtained, it leads to another intermediate integral of the original equation; but, again, this is not the general case, and it is not necessary for the success of the method.)

We now make α and β the independent variables. As p', q', y', z' are the derivatives of p, q, y, z with regard to x, when α is constant, we have

$$\frac{\partial p}{\partial \beta} = p' \frac{\partial x}{\partial \beta}, \quad \frac{\partial q}{\partial \beta} = q' \frac{\partial x}{\partial \beta}, \quad \frac{\partial y}{\partial \beta} = y' \frac{\partial x}{\partial \beta}, \quad \frac{\partial z}{\partial \beta} = z' \frac{\partial x}{\partial \beta};$$

and similarly for p_1, q_1, y_1, z_1. Hence we have

$$\frac{\partial p}{\partial \beta} + \rho \frac{\partial y}{\partial \beta} + T \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial q}{\partial \beta} + R \frac{\partial y}{\partial \beta} + \sigma \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial z}{\partial \beta} - q \frac{\partial y}{\partial \beta} - p \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial p}{\partial \alpha} + \sigma \frac{\partial y}{\partial \alpha} + T \frac{\partial x}{\partial \alpha} = 0,$$

$$\frac{\partial q}{\partial \alpha} + R \frac{\partial y}{\partial \alpha} + \rho \frac{\partial x}{\partial \alpha} = 0,$$

$$\frac{\partial z}{\partial \alpha} - q \frac{\partial y}{\partial \alpha} - p \frac{\partial x}{\partial \alpha} = 0,$$

which, however, are not six independent equations because of the integral combinations of the former sets that have been used. To obtain the integral system of the original equation of the second order, we have to integrate this set of simultaneous equations of the first order: it need hardly be said that this later integration is facilitated by a knowledge of further integrals (if any) of the original subsidiary systems, though such knowledge is not necessary for the purpose.

We have already (§ 244) seen that, if one system of subsidiary equations leads to an intermediate integral, and if we proceed to the integration of that integral regarded as an equation of the first order, the Charpit equations which are subsidiary for the latter purpose involve the other system of Ampère equations. Later (§ 254) we shall obtain a general property (established by Lie and by Darboux, independently of one another) characteristic of the equations, which possess an intermediate integral arising from each of the Ampère systems.

In the next place, let there be only a single linear system of subsidiary equations: it is

$$\left.\begin{aligned} p' - Sy' + T = 0 \\ q' + Ry' - S = 0 \\ z' - qy' - p = 0 \end{aligned}\right\}.$$

When integrals connected with this system are obtained, the arbitrary elements in the integrals are made equal to α, $\phi(\alpha)$,

$\psi(\alpha)$, where ϕ and ψ are arbitrary functions; and it has already been proved (§ 241) that, if p and q be eliminated from these integrals so as to leave a relation

$$z = F\{x, y, \alpha, \phi(\alpha), \psi(\alpha)\},$$

then the general integral of the original equation is given by the elimination of α between the two equations

$$\left.\begin{aligned} z &= F \\ 0 &= \frac{dF}{d\alpha} \end{aligned}\right\}.$$

252. When the equation to be integrated is

$$Rr + 2Ss + Tt = V,$$

it can similarly be shewn that, if the quadratic equation

$$R\xi^2 - 2S\xi + T = 0$$

has unequal roots λ and μ, there are two sets of subsidiary equations

$$\left.\begin{aligned} y' &= \mu \\ Rp' + \lambda q' &= V \\ p + q\mu &= z' \end{aligned}\right\}, \qquad \left.\begin{aligned} y_1 &= \lambda \\ Rp_1 + \mu q_1 &= V \\ p + q\lambda &= z_1 \end{aligned}\right\},$$

leading to equations

$$\left.\begin{aligned} \frac{\partial y}{\partial \beta} &= \mu \frac{\partial x}{\partial \beta} \\ R\frac{\partial p}{\partial \beta} + \lambda \frac{\partial q}{\partial \beta} &= V\frac{\partial x}{\partial \beta} \\ p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta} &= \frac{\partial z}{\partial \beta} \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{\partial y}{\partial \alpha} &= \lambda \frac{\partial x}{\partial \alpha} \\ R\frac{\partial p}{\partial \alpha} + \mu \frac{\partial y}{\partial \alpha} &= V\frac{\partial x}{\partial \alpha} \\ p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha} &= \frac{\partial z}{\partial \alpha} \end{aligned}\right\}.$$

If, however, the quadratic has equal roots, so that the equation can be taken in the form

$$r + 2Ss + S^2t = V,$$

there is only a single system of subsidiary equations, viz.

$$\left.\begin{aligned} y' &= S \\ p' + Sq' &= V \\ p + Sq &= z' \end{aligned}\right\}.$$

The method of proceeding in the respective cases is the same as in the corresponding cases for the equation

$$rt - s^2 + Rr + 2Ss + Tt = V.$$

Ex. 1. Let Ampère's method be applied to

$$r-t=2\frac{p}{x},$$

which does not possess an intermediate integral.

Pursuing the usual process, the first form of the subsidiary equations is

$$y'^2-1=0,$$

$$p'-q'y'=2\frac{p}{x}.$$

Clearly, there are two systems: we have

$$\left.\begin{aligned} y'&=1\\ p'-q'&=2\frac{p}{x}\\ p+q&=z'\end{aligned}\right\} \qquad \left.\begin{aligned} y_1&=-1\\ p_1+q_1&=2\frac{p}{x}\\ p-q&=z_1\end{aligned}\right\}.$$

An integral of the first system is given by

$$y-x=\text{constant}=\alpha,$$

and one of the second system is given by

$$y+x=\text{constant}=\beta,$$

so that

$$2x=\beta-\alpha.$$

The equations of the first system are

$$\frac{\partial p}{\partial\beta}-\frac{\partial q}{\partial\beta}=2\frac{p}{x}\frac{\partial x}{\partial\beta}=\frac{p}{x},$$

$$\frac{\partial z}{\partial\beta}=(p+q)\frac{\partial x}{\partial\beta}=\tfrac{1}{2}(p+q);$$

and those of the second system are

$$\frac{\partial p}{\partial\alpha}+\frac{\partial q}{\partial\alpha}=2\frac{p}{x}\frac{\partial x}{\partial\alpha}=-\frac{p}{x},$$

$$\frac{\partial z}{\partial\alpha}=(p-q)\frac{\partial x}{\partial\alpha}=-\tfrac{1}{2}(p-q).$$

Eliminating q between the equations

$$\frac{\partial p}{\partial\beta}-\frac{\partial q}{\partial\beta}=\frac{p}{x},\quad \frac{\partial p}{\partial\alpha}+\frac{\partial q}{\partial\alpha}=-\frac{p}{x},$$

we have

$$2\frac{\partial^2 p}{\partial\alpha\,\partial\beta}=\frac{\partial}{\partial\alpha}\left(\frac{p}{x}\right)-\frac{\partial}{\partial\beta}\left(\frac{p}{x}\right)$$

$$=\frac{1}{x}\frac{\partial p}{\partial\alpha}-\frac{1}{x}\frac{\partial p}{\partial\beta}+\frac{p}{x^2},$$

and therefore

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\left(\frac{p}{x}\right)=\frac{1}{x}\frac{\partial^2 p}{\partial\alpha\,\partial\beta}-\frac{1}{2x^2}\frac{\partial p}{\partial\alpha}+\frac{1}{2x^2}\frac{\partial p}{\partial\beta}-\frac{p}{2x^3}=0,$$

so that

$$\frac{p}{x}=2\phi''(\alpha)+2\psi''(\beta),$$

or

$$p=(\beta-\alpha)\{\phi''(\alpha)+\psi''(\beta)\},$$

where ϕ and ψ are arbitrary functions. Also, with this value of p, we have

$$\frac{\partial q}{\partial \alpha}=-\frac{p}{x}-\frac{\partial p}{\partial \alpha}=-\phi''(\alpha)-\psi''(\beta)-(\beta-\alpha)\phi'''(\alpha),$$

$$\frac{\partial q}{\partial \beta}=-\frac{p}{x}+\frac{\partial p}{\partial \beta}=-\phi''(\alpha)-\psi''(\beta)+(\beta-\alpha)\psi'''(\beta);$$

and therefore

$$q=-2\phi'(\alpha)-2\psi'(\beta)-(\beta-\alpha)\{\phi''(\alpha)-\psi''(\beta)\}.$$

Lastly,

$$\frac{\partial z}{\partial \alpha}=-\tfrac{1}{2}(p-q)=-\phi'(\alpha)-\psi'(\beta)-(\beta-\alpha)\phi''(\alpha),$$

$$\frac{\partial z}{\partial \beta}=\ \tfrac{1}{2}(p+q)=-\phi'(\alpha)-\psi'(\beta)+(\beta-\alpha)\psi''(\beta);$$

and therefore

$$\begin{aligned} z &= -2\phi(\alpha)-2\psi(\beta)-(\beta-\alpha)\{\phi'(\alpha)-\psi'(\beta)\} \\ &= \ f(y-x)+g(y+x)+xf'(y-x)-xg'(y+x), \end{aligned}$$

where f and g are arbitrary functions of their arguments. This is the primitive of the original equation.

Ex. 2. As an example in which there is only one argument for the arbitrary functions, so that there is only a single system, consider the equation already (Ex. 2, § 186) discussed, viz.

$$(b+q)^2 r-2(a+p)(b+q)s+(a+p)^2 t=0.$$

When we substitute

$$s=q'-ty', \quad r=p'-q'y'+ty'^2,$$

and make the resulting equation evanescent as regards t, we have

$$(b+q)^2(p'-q'y')-2(a+p)(b+q)q'=0,$$

$$\{(b+q)y'+a+p\}^2=0:$$

and therefore, neglecting the trivial forms

$$b+q=0, \quad a+p=0,$$

which lead to a trivial primitive, we take

$$(b+q)(p'-q'y')-2(a+p)q'=0,$$

$$(b+q)y'+a+p=0.$$

Substituting for y' from the latter into the former, we have

$$(b+q)p'-(a+p)q'=0,$$

that is,

$$\begin{aligned} \frac{a+p}{b+q} &= \text{constant} \\ &= -\alpha, \end{aligned}$$

as the quantity a is not yet specified. Then

$$y' = -\frac{a+p}{b+q}$$
$$= a,$$

so that

$$y = ax + \phi(a),$$

where ϕ is an arbitrary function. Also

$$z' = p + qy'$$
$$= p + qa$$
$$= -(a + ba),$$

and therefore

$$z = -x(a + ba) + \psi(a),$$

where ψ is another arbitrary function. Thus the integral system is

$$\left.\begin{array}{l} y = ax + \phi(a) \\ z = -x(a+ba) + \psi(a) \end{array}\right\};$$

and it is easy to see that it can be exhibited by means of a single equation in any of the three forms

$$x = y\theta(ax + by + z) + \phi(ax + by + z),$$
$$y = xf(ax + by + z) + g(ax + by + z),$$
$$z = xF(ax + by + z) + G(ax + by + z),$$

where all the functional symbols imply arbitrary functions.

Significance of the Subsidiary Equations.

253. We have already indicated that the significance of the subsidiary equations, which formally are the same for all the methods, is wider in Ampère's method than in the methods of Monge and of Boole. The relation between the distinct uses of the subsidiary equations can be exhibited in a different manner as follows, it being assumed for this purpose that the original equation of the second order has an intermediate integral. Let this integral be supposed to occur in connection with the subsidiary system

$$\left.\begin{array}{l} p' + \rho y' + T = 0 \\ q' + Ry' + \sigma = 0 \\ z' - qy' - p = 0 \end{array}\right\},$$

expressed in Ampère's form, the derivatives with regard to x being taken on the assumption that α is constant. Let

$$u(x, y, z, p, q) = \text{constant}, \quad v(x, y, z, p, q) = \text{constant},$$

be two integrable combinations of the foregoing equations.

In Monge's method, we at once construct the intermediate integral

$$u = \phi(v);$$

in order to obtain the primitive, we proceed to integrate this equation, regarded as a partial equation of the first order.

In Ampère's method, remembering that differentiations with regard to x are effected on the hypothesis that α is kept constant in the subsidiary equations considered, we take

$$u = \alpha, \quad v = \phi(\alpha),$$

where ϕ is any arbitrary function. Instead of considering these equations as partial equations of the first order, we regard them merely as two equations connecting x, y, z, p, q, α; the last five quantities are functions of x and β, and their derivatives with regard to x on the supposition that β is constant satisfy the alternative subsidiary system of equations in the form

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + R y_1 + \rho &= 0 \\ z_1 - q y_1 - p &= 0 \end{aligned}\right\}.$$

Moreover,

$$\alpha_1 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y_1 + \frac{\partial u}{\partial z} z_1 + \frac{\partial u}{\partial p} p_1 + \frac{\partial u}{\partial q} q_1,$$

$$\alpha_1 \phi'(\alpha) = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} y_1 + \frac{\partial v}{\partial z} z_1 + \frac{\partial v}{\partial p} p_1 + \frac{\partial v}{\partial q} q_1.$$

We thus have seven equations in all. Among these seven equations, let p, q, p_1, q_1 be eliminated: when the elimination has been effected, there remain three equations involving

$$x;\ y, y_1;\ z, z_1;\ \alpha, \alpha_1.$$

These three equations, in three dependent variables y, z, α, are integrated as an ordinary system of equations; in their integral, there will occur three quantities A, B, C, which are arbitrary constants so far as the integration is concerned.

Now the derivatives y_1, z_1, α_1 have been formed on the supposition that β is constant: hence we take

$$A = \beta, \quad B = \psi(\beta), \quad C = \chi(\beta),$$

where ψ and χ are arbitrary functions. Apparently, there now are three arbitrary functions in the three integral equations connecting x, y, z, α, β; and therefore, when α and β are eliminated among the three equations so as to give a primitive between x, y, z, the primitive would appear to contain three arbitrary functions, instead of merely two in accordance with Cauchy's theorem. The explanation is that we still have to take account of derivation with regard to β: thus

$$z_1 = \frac{dz}{dx} = q\frac{dy}{dx} + p, \qquad \frac{dz}{d\beta} = q\frac{dy}{d\beta},$$

and therefore

$$\frac{d}{dx}\left(q\frac{dy}{d\beta}\right) = \frac{d}{d\beta}\left(q\frac{dy}{dx} + p\right),$$

that is,

$$\begin{aligned}\frac{dp}{d\beta} &= \frac{dq}{dx}\frac{dy}{d\beta} - \frac{dq}{d\beta}\frac{dy}{dx} \\ &= q_1\frac{dy}{d\beta} - y_1\frac{dq}{d\beta}.\end{aligned}$$

From the equations

$$u = \alpha, \quad v = \phi(\alpha),$$

we have p and q as functions of x, y, z, α, which become functions of $x, \beta, \psi(\beta), \chi(\beta)$, on using the integrals of the second set of subsidiary equations. When these values are substituted in the foregoing relation, which must be satisfied identically, one of the three functional forms in the integral system is expressible in terms of the other two: that is, the primitive equation contains only the necessary two arbitrary functional forms.

Note. The mode, in which derivation with regard to β is taken into account, need not necessarily be that which precedes: thus it might be more convenient in practice to substitute directly in the relation

$$\frac{dz}{d\beta} = q\frac{dy}{d\beta}.$$

Ex. 1. Consider the equation

$$(q+yt)(r+1) = s(ys-p-x).$$

Arranged so that the coefficient of $rt-s^2$ is unity, the equation is

$$rt - s^2 + \frac{q}{y}r + \frac{p+x}{y}s + t = -\frac{q}{y}.$$

The critical quadratic is

$$\mu^2 + \frac{p+x}{y}\mu = 0;$$

the roots are unequal, and therefore there are two distinct systems of subsidiary equations; and neither of the systems can possess three integrable combinations. The two systems are respectively

$$\left.\begin{array}{ll} p' - \dfrac{p+x}{y} y' + 1 = 0 \\ q' + \dfrac{q}{y} y' \qquad = 0 \\ z' - qy' \quad - p = 0 \end{array}\right\}, \quad \left.\begin{array}{ll} p_1 \qquad\qquad + 1 = 0 \\ q_1 + \dfrac{q}{y} y_1 - \dfrac{p+x}{y} = 0 \\ z_1 - qy_1 - p \qquad = 0 \end{array}\right\}.$$

The former system possesses the two integrals

$$\frac{p+x}{y} = \text{constant}, \quad qy = \text{constant};$$

and the latter system possesses the two integrals

$$p + x = \text{constant}, \quad qy - x(p+x) = \text{constant}.$$

When the Monge method of using these integrals is adopted, the equation

$$\frac{p+x}{y} = \phi(qy),$$

where ϕ is an arbitrary function, is an intermediate integral: and the equation

$$qy - x(p+x) = \psi(p+x),$$

where ψ is an arbitrary function, is another intermediate integral. The two integrals coexist.

Though it is not possible to resolve the two equations for p and q, yet quadrature of the relation

$$dz = p\,dx + q\,dy$$

becomes simple, on the introduction of new variables

$$qy = a, \quad p + x = \beta.$$

The intermediate integrals give

$$y = \frac{\beta}{\phi(a)},$$

$$x = \frac{a - \psi(\beta)}{\beta},$$

and therefore

$$d(z + \tfrac{1}{2}x^2) = \beta\,dx + \frac{a}{y}\,dy$$

$$= \left\{1 - a\frac{\phi'(a)}{\phi(a)}\right\} da + \left\{\frac{\psi(\beta)}{\beta} - \psi'(\beta)\right\} d\beta.$$

When quadrature is effected, the explicit form being obtained most easily by the introduction of new functions f and g such that

$$\phi(a) = e^{f'(a)}, \quad \psi(\beta) = \beta g'(\beta),$$

we have the primitive given by the equations

$$\left.\begin{aligned} x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \\ \tfrac{1}{2}x^2 + z &= a + f(a) - af'(a) + g(\beta) - \beta g'(\beta) \end{aligned}\right\}.$$

If, still adopting Monge's use of the integrals of a subsidiary system, we proceed to integrate

$$\frac{p+x}{y} = \phi(qy),$$

regarded as a partial equation of the first order, we find

$$p + x = a$$

as an integral of the Charpit subsidiary equations: and then

$$q = \frac{1}{y}\phi^{-1}\left(\frac{a}{y}\right).$$

Hence

$$\begin{aligned} z &= \int (p\,dx + q\,dy) \\ &= ax - \tfrac{1}{2}x^2 + \int \frac{1}{y}\phi^{-1}\left(\frac{a}{y}\right) dy + b \\ &= ax - \tfrac{1}{2}x^2 + \chi\left(\frac{a}{y}\right) + b, \end{aligned}$$

where χ is a new arbitrary function and b is a constant. To obtain the general primitive, we take $b = \omega(a)$: and we have that primitive in the form

$$\left.\begin{aligned} z &= ax - \tfrac{1}{2}x^2 + \chi\left(\frac{a}{y}\right) + \omega(a) \\ 0 &= x + \frac{1}{y}\chi'\left(\frac{a}{y}\right) + \omega'(a) \end{aligned}\right\}.$$

Now consider the use of the integrals of the subsidiary systems in Ampère's method. We take

$$\frac{p+x}{y} = \alpha, \quad qy = \phi(\alpha).$$

The equations of the subsidiary system associated with β are

$$\left.\begin{aligned} p_1 + 1 &= 0 \\ q_1 y + q y_1 - (p+x) &= 0 \\ z_1 - q y_1 - p &= 0 \end{aligned}\right\},$$

the derivatives with regard to x being taken on the hypothesis that β is constant. We have

$$\frac{d}{dx}(y\alpha) = \frac{d}{dx}(p+x) = p_1 + 1 = 0,$$

so that

$$\begin{aligned} y\alpha &= \text{constant, when } \beta \text{ is constant,} \\ &= \beta, \end{aligned}$$

say. Again,

$$\begin{aligned}\phi'(a)\,a_1 &= qy_1 + yq_1\\ &= p + x\\ &= ya\\ &= \beta,\end{aligned}$$

so that

$$\phi(a) = \beta x + \psi(\beta),$$

where ψ is an arbitrary function of β. Again,

$$y = \frac{\beta}{a},$$

so that

$$y_1 = -\frac{\beta}{a^2}\,a_1,$$

and therefore

$$\begin{aligned}z_1 &= qy_1 + p\\ &= -\frac{\phi(a)}{a}\,a_1 + \beta - x;\end{aligned}$$

consequently,

$$z = \beta x - \tfrac{1}{2}x^2 - \Phi(a) + \theta(\beta),$$

where θ is arbitrary, and

$$\phi(a) = a\Phi'(a).$$

We thus have three arbitrary functions θ, ψ, Φ : one of them is dependent upon the other two.

To determine this dependence, we substitute from the equations

$$\begin{aligned}z &= \beta x - \tfrac{1}{2}x^2 - \Phi(a) + \theta(\beta),\\ y &= \frac{\beta}{a},\\ q &= \frac{a}{\beta}\,\phi(a),\end{aligned}$$

in the relation

$$\frac{dz}{d\beta} = q\,\frac{dy}{d\beta}.$$

It gives

$$x - \Phi'(a)\frac{da}{d\beta} + \theta'(\beta) = \frac{a}{\beta}\,\phi(a)\left\{\frac{1}{a} - \frac{\beta}{a^2}\,\frac{da}{d\beta}\right\},$$

so that, taking account of the relation between ϕ and Φ', we have

$$\begin{aligned}x + \theta'(\beta) &= \frac{1}{\beta}\,\phi(a)\\ &= \frac{a}{\beta}\,\Phi'(a),\end{aligned}$$

which gives the required relation. Hence the primitive is

$$\left.\begin{aligned}z &= \beta x - \tfrac{1}{2}x^2 - \Phi\left(\frac{\beta}{y}\right) + \theta(\beta)\\ 0 &= x \qquad - \frac{1}{y}\,\Phi'\left(\frac{\beta}{y}\right) + \theta'(\beta)\end{aligned}\right\},$$

involving two arbitrary functions.

Ex. 2. Compare the Monge method and the Ampère method, by detailed reference to the equation

$$2pqyr + (p^2y + qx)s + xpt - p^2q(rt - s^2) = xy.$$

Lie's Theorem on Equations of the Second Order, possessing Two Intermediate Integrals.

254. In the case of the equations which have just been considered, the process of integration is materially simplified when either of the subsidiary systems leads to an intermediate integral: in particular, we know that, when each of these systems leads to an intermediate integral, the two integrals can be treated as coexistent; and quadrature then will suffice to lead to the primitive. *When an equation of the second order has this property of possessing two independent intermediate integrals, it is reducible to the form*

$$s = 0,$$

by contact transformations. This theorem, which is due to Lie*, can be proved as follows. Let

$$F(u_1, v_1) = 0, \quad G(u_2, v_2) = 0,$$

be the intermediate integrals: as we know (§ 239), these coexist for all functional forms of F and G, and therefore

$$[F, G] = 0.$$

Consequently, taking $F = u_1$ and v_1, $G = u_2$ and v_2, all in turn, we have

$$[u_1, u_2] = 0, \quad [v_1, u_2] = 0, \quad [u_1, v_2] = 0, \quad [u_2, v_2] = 0.$$

(It may be remarked incidentally that these equations verify Ampère's theorem that u_2 and v_2 are integrals of the equations subsidiary to the integration of $F = 0$, regarded as an equation of the first order.) To compare them with the equations of contact transformation, we write

$$v_1 = X, \quad v_2 = Y;$$

and they then become

$$[u_1, u_2] = 0, \quad [X, u_2] = 0, \quad [Y, u_1] = 0, \quad [X, Y] = 0.$$

* *Arch. f. Math. og Nat.*, t. II (1877), pp. 1—9. It was afterwards discovered independently by Darboux: see §§ 38, 39, of his memoir quoted on p. 302 hereafter.

Because the last equation is satisfied, it follows from the theory of contact transformations that a function Z exists, which satisfies

$$[Z, X]=0, \quad [Z, Y]=0,$$

and which is functionally distinct from u_1 and u_2, because of the number of independent variables involved. Further, we know, from the same theory, that it is possible to determine functions P and Q such that the relation

$$dZ-PdX-QdY=\rho\,(dz-pdx-qdy)$$

is satisfied identically, where ρ is a non-vanishing quantity not dependent upon the differential elements. Suppose that these quantities Z, P, Q are known: and effect, alike upon the differential equation and its intermediate integrals, the contact transformation which replaces the variables x, y, z, p, q by X, Y, Z, P, Q; and suppose that, in consequence of this transformation, u_1 and u_2 become U_1 and U_2 respectively, so that the intermediate integrals become

$$F(U_1, X)=0, \quad G(U_2, Y)=0,$$

these being intermediate integrals of the transformed equation. The equation

$$[Y, u_1]=0$$

thus becomes

$$[Y, U_1]=0,$$

that is,

$$\frac{\partial U_1}{\partial Q}=0,$$

so that U_1 is independent of Q. Similarly, the equation

$$[X, u_2]=0$$

becomes

$$[X, U_2]=0,$$

that is,

$$\frac{\partial U_2}{\partial P}=0,$$

so that U_2 is independent of P. Also, the relation

$$[u_1, u_2]=0$$

is replaced by

$$[U_1, U_2]=0:$$

when this is expressed in full, account being taken of the established facts that U_1 and U_2 are independent of Q and of P respectively, we have

$$-\left(\frac{\partial U_2}{\partial X} + P\frac{\partial U_2}{\partial Z}\right)\frac{\partial U_1}{\partial P} + \left(\frac{\partial U_1}{\partial Y} + Q\frac{\partial U_1}{\partial Z}\right)\frac{\partial U_2}{\partial Q} = 0,$$

so that

$$\frac{\dfrac{\partial U_2}{\partial X} + P\dfrac{\partial U_2}{\partial Z}}{\dfrac{\partial U_2}{\partial Q}} = \frac{\dfrac{\partial U_1}{\partial Y} + Q\dfrac{\partial U_1}{\partial Z}}{\dfrac{\partial U_1}{\partial P}}.$$

Because U_2 is independent of P, the first of the fractions is a linear function of P; and because U_1 is independent of Q, the second of the fractions is a linear function of Q. Moreover, the relation is satisfied identically, and therefore each fraction is of the form

$$APQ + BP + CQ + D,$$

where A, B, C, D do not involve P or Q: thus

$$\left.\begin{aligned} \frac{\partial U_1}{\partial Y} &= (BP + D)\frac{\partial U_1}{\partial P} \\ \frac{\partial U_1}{\partial Z} &= (AP + C)\frac{\partial U_1}{\partial P} \\ \frac{\partial U_1}{\partial Q} &= 0 \end{aligned}\right\},\qquad \left.\begin{aligned} \frac{\partial U_2}{\partial X} &= (CQ + D)\frac{\partial U_2}{\partial Q} \\ \frac{\partial U_2}{\partial Z} &= (AQ + B)\frac{\partial U_2}{\partial Q} \\ \frac{\partial U_2}{\partial P} &= 0 \end{aligned}\right\}.$$

Now U_1 is to be functionally distinct from X, because u_1 and v_1 are functionally distinct from one another; and U_2 is to be functionally distinct from Y, because u_2 and v_2 are functionally distinct from one another. Hence the equations for U_1 must be a complete Jacobian system as they stand; and likewise those for U_2 must be a complete Jacobian system as they stand.

The Jacobi-Poisson condition that the equations for U_1 should be complete is

$$\frac{\partial U_1}{\partial P}\left\{P\left(\frac{\partial B}{\partial Z} - \frac{\partial A}{\partial Y}\right) + \frac{\partial D}{\partial Z} - \frac{\partial C}{\partial Y} + AD - BC\right\} = 0;$$

and therefore, as A, B, C, D do not involve P, we must have

$$\frac{\partial B}{\partial Z} - \frac{\partial A}{\partial Y} = 0,$$

$$\frac{\partial D}{\partial Z} - \frac{\partial C}{\partial Y} = BC - AD$$

Assuming these conditions satisfied, we have

$$U_1 = KP + L,$$

where K and L are independent of P and Q, and

$$A = \frac{1}{K}\frac{\partial K}{\partial Z}, \quad B = \frac{1}{K}\frac{\partial K}{\partial Y}, \quad \frac{\partial L}{\partial Y} = KD, \quad \frac{\partial L}{\partial Z} = KC,$$

the conditions for the coexistence of these equations being satisfied. Similarly, we have

$$U_2 = MQ + N,$$

where M and N are independent of P and Q. Consequently, the two intermediate integrals of the transformed equation are

$$F(KP + L, X) = 0, \quad G(MQ + N, Y) = 0;$$

and they may be taken in the forms

$$KP + L - \xi = 0, \quad MQ + N - \eta = 0,$$

where ξ is an arbitrary function of X, and η of Y.

The condition for coexistence of these two equations, viz.

$$[KP + L - \xi, \; MQ + N - \eta] = 0,$$

now becomes

$$-K\left\{Q\frac{\partial M}{\partial X} + \frac{\partial N}{\partial X} + P\left(Q\frac{\partial M}{\partial Z} + \frac{\partial N}{\partial Z}\right)\right\}$$
$$+ M\left\{P\frac{\partial K}{\partial Y} + \frac{\partial L}{\partial Y} + Q\left(P\frac{\partial K}{\partial Z} + \frac{\partial L}{\partial Z}\right)\right\} = 0,$$

and it is satisfied identically; hence

$$M\frac{\partial K}{\partial Z} - K\frac{\partial M}{\partial Z} = 0,$$

$$M\frac{\partial L}{\partial Z} - K\frac{\partial M}{\partial X} = 0,$$

$$M\frac{\partial K}{\partial Y} - K\frac{\partial N}{\partial Z} = 0,$$

$$M\frac{\partial L}{\partial Y} - K\frac{\partial N}{\partial X} = 0.$$

The first of these relations gives

$$M = \lambda K,$$

where λ is a function of X and Y only, not involving Z. From the second, using this result, we have

$$\lambda \frac{\partial L}{\partial Z} = \frac{\partial M}{\partial X} = \frac{\partial (\lambda K)}{\partial X},$$

and therefore

$$\frac{\partial L}{\partial Z} = \frac{\partial K}{\partial X} + K \frac{\partial (\log \lambda)}{\partial X}.$$

From the third, we have

$$\lambda \frac{\partial K}{\partial Y} = \frac{\partial N}{\partial Z},$$

and from the fourth

$$\lambda \frac{\partial L}{\partial Y} = \frac{\partial N}{\partial X}:$$

hence

$$\begin{aligned} \frac{\partial}{\partial X}\left(\lambda \frac{\partial K}{\partial Y}\right) &= \frac{\partial}{\partial Z}\left(\lambda \frac{\partial L}{\partial Y}\right) \\ &= \lambda \frac{\partial^2 L}{\partial Z \partial Y} \\ &= \lambda \left\{\frac{\partial^2 K}{\partial X \partial Y} + \frac{\partial K}{\partial y}\frac{\partial (\log \lambda)}{\partial X} + K \frac{\partial^2 (\log \lambda)}{\partial X \partial Y}\right\}, \end{aligned}$$

and therefore

$$\frac{\partial^2 (\log \lambda)}{\partial X \partial Y} = 0,$$

so that

$$\lambda = \frac{\xi_1}{\eta_1},$$

where ξ_1 is any function of X only, and η_1 is any function of Y only. Taking a new function Θ of X, Y, Z, such that

$$K\xi_1 = \frac{\partial \Theta}{\partial Z},$$

we have

$$L\xi_1 = \frac{\partial \Theta}{\partial X},$$

$$N\eta_1 = \frac{\partial \Theta}{\partial Y};$$

and so

$$KP + L - \xi = \frac{1}{\xi_1}\left(\frac{\partial \Theta}{\partial Z}\frac{\partial Z}{\partial X} + \frac{\partial \Theta}{\partial X}\right) - \xi = 0,$$

$$\lambda KQ + N - \eta = \frac{1}{\eta_1}\left(\frac{\partial \Theta}{\partial Z}\frac{\partial Z}{\partial Y} + \frac{\partial \Theta}{\partial Y}\right) - \eta = 0.$$

As a last transformation, we take a new dependent variable ζ (as may be done arbitrarily in contact transformations) such that

$$\zeta = \Theta;$$

and then the intermediate integrals are

$$\frac{\partial \zeta}{\partial X} = \xi\xi_1 = \text{a function of } X \text{ only,}$$

$$\frac{\partial \zeta}{\partial Y} = \eta\eta_1 = \text{a function of } Y \text{ only,}$$

so that

$$\frac{\partial^2 \zeta}{\partial X \partial Y} = 0,$$

thus establishing Lie's theorem that, if an equation of the second order possesses two independent intermediate integrals, it can be reduced to a form

$$s = 0,$$

by means of contact transformations.

Ex. 1. In Ex. 1, § 253, it was proved that the equation

$$(q + yt)(r + 1) = s(ys - p - x)$$

possesses two intermediate integrals; and that the primitive, obtained by the Monge method, could be represented by the equations

$$\left.\begin{aligned} x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \\ z + \tfrac{1}{2}x^2 &= a + f(a) - af'(a) + g(\beta) - \beta g'(\beta) \end{aligned}\right\}.$$

The quantity a is the X of the preceding investigation, and the quantity β is the Y: and, in particular,

$$qy = a, \quad p + x = \beta.$$

The contact transformation in question is

$$\left.\begin{aligned} z + \tfrac{1}{2}x^2 &= Z \\ p + x &= \beta = Y \\ qy &= a = X \\ x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \end{aligned}\right\},$$

where the forms of f and g are arbitrary: the values of P and Q are

$$P = 1 - af''(a) \qquad Q = -\beta g''(\beta):$$

the relation

$$dz - p\,dx - q\,dy = dZ - P\,dX - Q\,dY$$

is satisfied, the value of Z being

$$Z = a + f(a) - af'(a) + g(\beta) - \beta g'(\beta):$$

and

$$\frac{\partial^2 Z}{\partial X\,\partial Y} = 0.$$

Ex. 2. Obtain two intermediate integrals of the equation

$$Ar + Bs + Ct = 0,$$

where

$$A = pqx^2 + (1+q^2)\,xy, \quad C = -pqy^2 - (1+p^2)\,xy,$$
$$B = (1+q^2)\,y^2 - (1+p^2)\,x^2;$$

construct the contact transformations which change it into the equation

$$s' = 0;$$

and hence derive a primitive.

By means of the intermediate integrals, devise a geometrical interpretation of the equation and its primitive. (Goursat.)

Ex. 3. Surfaces (due to Monge) have one system of their lines of curvature situated upon concentric spheres: construct the partial differential equation of the second order satisfied by such surfaces. Prove that this equation possesses two intermediate integrals: and obtain the contact transformation which changes it into the equation $s' = 0$.

CHAPTER XVIII.

Darboux's Method, and other Methods, for Equations of the Second Order in Two Independent Variables.

The first considerable extension (as distinct from improvements) in the methods of solving partial equations of the second order, effected after the publication of Ampère's memoirs, was obtained by Darboux in a memoir published* in 1870. Previous methods, such as those of Monge and Ampère, had mainly or entirely sought for the primitive through the construction of a compatible equation of lower order; and, as has appeared from the preceding discussions, such a process is not always applicable. Using Ampère's ideas, and combining with them the spirit of Jacobi's method of solving an equation of the first order by associating with it a new equation of its own order, Darboux devised a method which, for wide classes of equations, can lead to a primitive. The central feature is the construction of an equation or equations of the second order, or even of order higher than the second, which are compatible with a given equation of the second order: in order to derive these equations, it is necessary to integrate one or more subsidiary systems. These systems have two forms, which are equivalent to one another: in one form, the equations are homogeneous and linear in the differential elements; in the other, the equations are homogeneous and linear in the first derivatives of the unknown function. To such systems, the Jacobian method of integration can be applied: it has the advantage of indicating the conditions which must be satisfied, if the process is to be effective.

Moreover, Darboux's method is progressive: that is to say, when the tests shew that no equations of a particular order can be associated with a given equation, then it can be applied equally to obtain (if that be possible) equations of the next higher order which are compatible with the given equation. Accordingly, it is effective for all equations of the second order when their primitive can be expressed in finite terms, whether by means of a single integral equation or by means of a number of simultaneous integral equations.

* *Ann. de l'Éc. Norm. Sup.*, t. VII (1870), pp. 163—173.

Further, it can be applied to equations of any order in two independent variables.

Since the publication of Darboux's investigations, many other memoirs upon the subject have appeared. Among them, special mention should be made of those by Hamburger*, Winckler†, König‡, and Sersawy§: references to other writers will be found in these memoirs. A historical summary of the methods devised by various writers for obtaining equations, which are the same as, or are equivalent to, Darboux's equations, is given by Speckman‖: Goursat's discussion¶ of the matter may be consulted with advantage; and a memoir by Sonin** should be consulted.

255. The substantial difference between the main aim of general methods, devised for the integration of partial equations of the first order, and the main aim of such methods, as are expounded in the immediately preceding chapters for the integration of partial equations of the second order, is of significance and importance. In the case of equations of the first order, the aim of general methods such as those devised by Charpit and by Jacobi is to construct equations that can be associated with the original equation: and all the admissible equations thus constructed are themselves of the first order, as is the original equation. In the case of equations of the second order, the aim of the methods that have been expounded is the construction of equations that are compatible with the original equation: in the methods of Monge and of Boole, the admissible equations (when they exist) are of the first order, being thus of order lower than the original equation: in the method of Ampère, the admissible equations may be of the first order and may be of no order at all: in no case, has the order of the associated equation or equations been the same as, or higher than, that of the original equation.

As regards the details of the methods applied to the construction of equations which are associated, or are compatible, with equations already propounded, there is the superficial resemblance that all of them, in so far as they involve inverse

* *Crelle*, t. LXXXI (1876), pp. 271—280; *ib.*, t. XCIII (1882), pp. 201—214.

† *Wien. Ber.*, t. LXXXVIII (1883), pp. 7—74; *ib.*, t. LXXXIX (1884), pp. 614—624.

‡ *Math. Ann.*, t. XXIV (1884), pp. 465—536.

§ *Wien. Denkschr.*, t. XLIX (1884), pp. 1—104.

‖ *Arch. Néerl.*, t. XXVII (1894), pp. 303—354.

¶ *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, ch. VI, VII.

** *Math. Ann.*, t. XLIX (1897), pp. 417—447: it was first published, in Russian, in 1874.

integrational operations, demand such operations in only the first degree. The reason of the resemblance, such as it is, lies in the facts, that even moderately general inverse operations can be effected only if they are of the first degree, and that the feasible operations of the second degree are exceedingly limited in scope. The resemblance, therefore, has nothing to do with the orders of the equations concerned and is due solely to exiguity of facility with inverse operations: it needs no further comment.

There is one outstanding difference between a system of subsidiary equations used in connection with an equation of the first order and such a system used in connection with an equation of the second order. All the equations in both kinds of systems are of the type called ordinary: they seek a provisional expression of all the variables in terms of a single variable. When the original equation is of the first order, the number of equations in the subsidiary system is equal to the number of variables provisionally regarded as dependent; when the original equation is of the second order, the number of equations in the subsidiary system (when it is effective for its purpose) is one less than the number of variables provisionally regarded as dependent. As has been seen in the discussion of the subsidiary systems, retaining the significance given them most widely by Ampère's method, this excess by one unit cannot be used to give an arbitrary provisional value to one of the dependent variables; it arises from the latency of the argument of the arbitrary function or functions, which occur in the primitive of the equations of the second order.

The outlook beyond these considerations, applied to the simplest case when an equation of the second order in two independent variables is propounded for solution, suggests two questions. On the one hand, is it possible to further the construction of a primitive by associating, with the original equation, an equation involving partial derivatives of order higher than the first? On the other hand, is it possible, by proceeding to derivatives of higher order, to construct a subsidiary system (which, presumably, shall be ordinary) that is complete?

Moreover, when the discussion is not restricted to equations in only two independent variables but extends to those involving any number, a further question will arise as to whether a subsidiary system (if the method of subsidiary equations is then of any effective use) will be, not merely complete or incomplete, but ordinary or partial. Putting this question on one side for the present, as well as cognate questions that are easy enough to propound, we proceed to consider the two earlier questions. The first discussion, which these questions received, is contained in a memoir by Darboux*; since the publication of that memoir, they have received much attention, especially in regard to particular equations.

Cauchy's Method, restated and discussed after Darboux.

256. It has already been seen, in Chapter VI of the preceding volume during the exposition of Cauchy's method for an original equation of the first order, that the subsidiary equations deduced by the process of changing the independent variables are a complete ordinary system. Let the same process, which was first suggested by Ampère, be applied to an equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0.$$

Denoting $\dfrac{\partial f}{\partial x}$ by X, and so for the other derivatives of f, we have the total differential equation equivalent to $f=0$ in the form

$$X\,dx + Y\,dy + Z\,dz + P\,dp + Q\,dq + R\,dr + S\,ds + T\,dt = 0.$$

The independent variables x and y are changed to x and u, where u is left to be determined and is not a function of x alone. As usual, we have

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u},$$

$$\frac{\partial p}{\partial x} = r + s\frac{\partial y}{\partial x}, \quad \frac{\partial p}{\partial u} = s\frac{\partial y}{\partial u},$$

$$\frac{\partial q}{\partial x} = s + t\frac{\partial y}{\partial x}, \quad \frac{\partial q}{\partial u} = t\frac{\partial y}{\partial u};$$

three of these are equations which, in the new derivatives, involve derivation with regard to x alone. Moreover,

$$\frac{\partial}{\partial u}\left(p + q\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(q\frac{\partial y}{\partial u}\right),$$

* *Ann. de l'Éc. Norm. Sup.*, 1re Sér., t. VII (1870), pp. 163—173; it is reproduced as Note X at the end of volume IV of his *Théorie générale des surfaces*.

and therefore

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial q}{\partial u}\frac{\partial y}{\partial x};$$

and, similarly,

$$\frac{\partial r}{\partial u} = \frac{\partial s}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial s}{\partial u}\frac{\partial y}{\partial x},$$

$$\frac{\partial s}{\partial u} = \frac{\partial t}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial t}{\partial u}\frac{\partial y}{\partial x}.$$

Substituting the values of the differential elements, which occur in the total differential form of the original equation, and remembering that dx and du are independent, we have

$$Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + \ldots + T\frac{\partial t}{\partial u} = 0,$$

as one of the equations: and when in this equation we insert the values of all the derivatives with respect to u, expressed in terms of $\frac{\partial y}{\partial u}$ and $\frac{\partial t}{\partial u}$, we find

$$\left\{Y + Zq + Ps + Qt + R\left(\frac{\partial s}{\partial x} - \frac{\partial t}{\partial x}\frac{\partial y}{\partial x}\right) + S\frac{\partial t}{\partial x}\right\}\frac{\partial y}{\partial u} + \left\{R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T\right\}\frac{\partial t}{\partial u} = 0.$$

The variable u is at our disposal; let it be so chosen that, when y is expressed as a function of x and u, the equation

$$R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T = 0$$

is satisfied; then, as $\frac{\partial y}{\partial u}$ is not zero, we also have

$$Y + Zq + Ps + Qt + R\left(\frac{\partial s}{\partial x} - \frac{\partial t}{\partial x}\frac{\partial y}{\partial x}\right) + S\frac{\partial t}{\partial x} = 0.$$

When the coefficient of $\frac{\partial t}{\partial x}$ is modified by means of the immediately preceding equation, we have

$$Y + Zq + Ps + Qt + R\frac{\partial s}{\partial x} + T\frac{\frac{\partial t}{\partial x}}{\frac{\partial y}{\partial x}} = 0;$$

and thus there are two other equations involving derivatives with regard to x alone.

The remaining equation, derived from the total differential form of the original equation, is

$$X + Y\frac{\partial y}{\partial x} + Z\frac{\partial z}{\partial x} + P\frac{\partial p}{\partial x} + Q\frac{\partial q}{\partial x} + R\frac{\partial r}{\partial x} + S\frac{\partial s}{\partial x} + T\frac{\partial t}{\partial x} = 0,$$

another equation involving derivatives with regard to x alone. From the earlier equation, we have

$$Y\frac{\partial y}{\partial x} + Zq\frac{\partial y}{\partial x} + Ps\frac{\partial y}{\partial x} + Qt\frac{\partial y}{\partial x} + R\frac{\partial s}{\partial x}\frac{\partial y}{\partial x} + T\frac{\partial t}{\partial x} = 0;$$

subtracting this from the equation just obtained, and using the other equations already constructed, we have

$$X + Zp + Pr + Qs + R\frac{\partial r}{\partial x} + T\frac{\dfrac{\partial s}{\partial x}}{\dfrac{\partial y}{\partial x}} = 0,$$

which will be used to replace the immediately preceding equation.

We thus have six equations which involve no derivatives with regard to u but only derivatives with regard to x alone; and these are all the equations of this character which can be constructed among these quantities. The dependent variables, being unknown functions of x and u, are y, z, p, q, r, s, t, seven in number; and so the subsidiary system of six ordinary equations (leaving arbitrary constants to be made arbitrary functions of u) cannot completely determine the seven dependent variables. In the case of an original equation of the first order, the subsidiary system thus constructed was sufficient to determine the dependent variables involved: so that, for equations of the second order, there is a relative diminution in the efficiency of the subsidiary system.

257. In the investigations of Monge and Boole, reasons (which have been explained) led to the consideration of the equations

$$Ar + 2Bs + Ct + rt - s^2 = D,$$
$$Ar + 2Bs + Ct = D,$$

and of no others, the quantities A, B, C, D not involving r, s, t; it so happens that, when these equations are submitted to the preceding process, the subsidiary system is simplified very considerably in form. The equation

$$R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T = 0,$$

for the former equation, becomes

$$(A+t)\left(\frac{\partial y}{\partial x}\right)^2 - 2(B-s)\frac{\partial y}{\partial x} + C + r = 0,$$

that is,

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C + t\left(\frac{\partial y}{\partial x}\right)^2 + 2s\frac{\partial y}{\partial x} + r = 0,$$

or, using the initial equations connected with the change of independent variables, we have

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x}\frac{\partial y}{\partial x} = 0.$$

Again, using the equations

$$\frac{\partial p}{\partial x} = r + s\frac{\partial y}{\partial x}, \quad \frac{\partial q}{\partial x} = s + t\frac{\partial y}{\partial x},$$

to remove r and s from the equation

$$Ar + 2Bs + Ct + rt - s^2 = D,$$

we find that, in consequence of the combination $rt - s^2$, there is no term in t^2 and, in consequence of the subsidiary equation just obtained, there is no term in t; the result is

$$A\left(\frac{\partial p}{\partial x} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial x}\right) + 2B\frac{\partial q}{\partial x} - \left(\frac{\partial q}{\partial x}\right)^2 = D,$$

another equation of the subsidiary system. And we always have

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}.$$

There thus are three equations in the subsidiary system*; it involves four dependent variables, viz. y, z, p, q.

Again, for the equation $Ar + 2Bs + Ct = D$, the subsidiary system is similarly obtainable in the form

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C = 0,$$

$$A\left(\frac{\partial p}{\partial x} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial x}\right) + 2B\frac{\partial q}{\partial x} = D,$$

with

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x},$$

* It can be resolved so as to acquire the form given in earlier chapters: the resolution is irrelevant to the present discussion.

again a set of three equations in four dependent variables, viz. y, z, p, q.

Now though, in the case of both of these equations, the subsidiary system is ineffective for the complete determination of y, z, p, q, it may happen that integrable combinations can be constructed for special instances of those equations: examples have occurred freely in preceding chapters. In that event, the arbitrary constants that occur in the integrated combinations are to be regarded as functions of the other variable u, arbitrary so far as concerns the subsidiary system of derivatives with regard to x: but the arbitrary functions are subject to the equations

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial u}, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u}.$$

The quantities r, s, t have disappeared from the subsidiary system belonging to the less special type and, in their disappearance, they have removed three equations.

Similar remarks apply when the subsidiary system belonging to the equation $f=0$ admits of integrable combinations; the arbitrary functions of u, into which the arbitrary constants in the integrated combinations are changed, are subject to the two preceding equations, as well as to the equations

$$\frac{\partial r}{\partial u} = \frac{\partial s}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial x}\frac{\partial s}{\partial u}, \quad \frac{\partial p}{\partial u} = s\frac{\partial y}{\partial u},$$

$$\frac{\partial s}{\partial u} = \frac{\partial t}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial x}\frac{\partial t}{\partial u}, \quad \frac{\partial q}{\partial u} = t\frac{\partial y}{\partial u}.$$

258. It thus follows that, when we restrict ourselves to derivatives of the second order during the construction of the particular kind of subsidiary system, the number of equations in the system is one less than the number of dependent variables which it contains. It is natural to inquire whether, by proceeding to derivatives of higher orders, it is possible at any stage to construct a complete subsidiary system involving derivatives with regard to x alone. The answer has been given by Darboux: it is to the negative effect, for *the number of equations involving derivatives of x alone is always less by unity than the number of dependent variables which are to be determined.* This theorem of Darboux's can be verified for the next succeeding order as follows:

and the course of the verification will shew how the unit deficiency is maintained in succeeding orders.

Denoting by α, β, γ, δ, the derivatives of z, which are of the third order with regard to the original variables x and y, and taking account of the change of independent variables effected in the method, we have

$$\frac{\partial r}{\partial x} = \alpha + \beta \frac{\partial y}{\partial x}, \qquad \frac{\partial r}{\partial u} = \beta \frac{\partial y}{\partial u},$$

$$\frac{\partial s}{\partial x} = \beta + \gamma \frac{\partial y}{\partial x}, \qquad \frac{\partial s}{\partial u} = \gamma \frac{\partial y}{\partial u},$$

$$\frac{\partial t}{\partial x} = \gamma + \delta \frac{\partial y}{\partial x}, \qquad \frac{\partial t}{\partial u} = \delta \frac{\partial y}{\partial u};$$

these provide three new equations, involving derivatives of x alone, towards the amplified subsidiary system. Also, as before, we have

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial \beta}{\partial u}\frac{\partial y}{\partial x},$$

$$\frac{\partial \beta}{\partial u} = \frac{\partial \gamma}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial \gamma}{\partial u}\frac{\partial y}{\partial x},$$

$$\frac{\partial \gamma}{\partial u} = \frac{\partial \delta}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial \delta}{\partial u}\frac{\partial y}{\partial x}.$$

Now from the equation $f = 0$, we have

$$f_1 = R\alpha + S\beta + T\gamma + U = 0,$$

$$f_2 = R\beta + S\gamma + T\delta + V = 0,$$

where U and V do not involve α, β, γ, δ: these equations are the complete derivatives of $f = 0$ with regard to the old independent variables. Forming the total derivative of

$$f_1 = R\alpha + S\beta + T\gamma + U = 0,$$

and introducing the new independent variables, we have

$$R\frac{\partial \alpha}{\partial u} + S\frac{\partial \beta}{\partial u} + T\frac{\partial \gamma}{\partial u} + \Theta = 0,$$

$$R\frac{\partial \alpha}{\partial x} + S\frac{\partial \beta}{\partial x} + T\frac{\partial \gamma}{\partial x} + \Phi = 0,$$

where Θ involves linearly the derivatives of x, y, z, p, q, r, s, t with regard to u. The latter equation belongs to the amplified system.

Substituting in the former so that no derivatives with regard to u survive except $\frac{\partial y}{\partial u}$ and $\frac{\partial \delta}{\partial u}$, we find that it takes a form

$$\left\{R\left(\frac{\partial y}{\partial x}\right)^3 - S\left(\frac{\partial y}{\partial x}\right)^2 + T\frac{\partial y}{\partial x}\right\}\frac{\partial \delta}{\partial u} + \Theta_1\frac{\partial y}{\partial u} = 0,$$

where Θ_1 involves derivatives with regard to x only. The coefficient of $\frac{\partial \delta}{\partial u}$ vanishes on account of an earlier equation; and $\frac{\partial y}{\partial u}$ is not zero, so that the relation is

$$\Theta_1 = 0,$$

thus providing another equation for the amplified system. Consequently, the equation

$$f_1 = R\alpha + S\beta + T\gamma + U = 0$$

provides two new equations for the amplified system: they can be denoted by

$$\frac{\partial f_1}{\partial x} = 0, \qquad \Theta_1 = 0.$$

Similarly, the equation

$$f_2 = R\beta + S\gamma + T\delta + V = 0$$

provides two new equations for the amplified system: they can be denoted by

$$\frac{\partial f_2}{\partial x} = 0, \qquad \Theta_2 = 0.$$

It therefore appears as if the completely amplified system contains seven more members than the system for the second order, while it involves only four new dependent variables $\alpha, \beta, \gamma, \delta$, in addition to the old dependent variables; and it might therefore be imagined that the unit deficiency, which marked the old system, is more than supplied by the new system. But this is not, in fact, the case: for the amplified system of thirteen equations, involving the eleven dependent variables $y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$, contains four dependent equations when $f=0$ is retained. Thus, in the old system, a linear combination of the equations leads to

$$df = 0;$$

and therefore one of the six can be rejected, because the equation

$$f = 0$$

is retained. Again, a linear combination of

$$\frac{\partial f_1}{\partial x} = 0, \qquad \Theta_1 = 0,$$

together with equations from the old system, leads to the equation

$$df_1 = 0;$$

and therefore one of these two new equations can be rejected because the equation $f_1 = 0$, with other equations of the system, is a necessary consequence of the retained equation $f = 0$. Similarly, a linear combination of

$$\frac{\partial f_2}{\partial x} = 0, \qquad \Theta_2 = 0,$$

together with equations from the old system, leads to the equation

$$df_2 = 0;$$

and therefore one of these two new equations can be rejected because the equation $f_2 = 0$, with other equations of the system, is a necessary consequence of the retained equation $f = 0$. Lastly, a bilinear combination of

$$\frac{\partial f_1}{\partial x} = 0, \quad \Theta_1 = 0, \quad \frac{\partial f_2}{\partial x} = 0, \quad \Theta_2 = 0,$$

together with equations from the old system, leads to the equation

$$d^2 f = 0,$$

which is a necessary consequence of the retained equation $f = 0$; and therefore one more of the new equations can be rejected. Hence the amplified system of ordinary equations subsidiary to $f = 0$, when derivatives of the third order are introduced, contains ten independent members (including $f = 0$) and involves eleven dependent variables: the unit deficiency, characteristic of the subsidiary system for derivatives of the second order, is characteristic of the subsidiary system for derivatives of the third order.

And so for the orders, in increasing succession: the result, as stated in Darboux's theorem, applies to all orders.

Ex. Verify Darboux's theorem for the equations

$$Ar + 2Bs + Ct + rt - s^2 = D,$$

$$Ar + 2Bs + Ct = D,$$

in the case of derivatives of the third order, the quantities A, B, C, D not involving r, s, t.

259. It thus appears that, at no stage in the succession of increasing orders, can we construct a subsidiary system which shall be complete if the original equation is of the second order. But it may happen that, just as the incomplete system in Monge's method and that same system (in a wider significance) in Ampère's method offer integrable combinations, so in the process indicated the incomplete subsidiary system may at some stage offer integrable combinations. If the subsidiary system for derivatives of the second order should offer no integrable combinations, it may happen that the subsidiary system for derivatives of the third order will do so; if not, then the subsidiary system for derivatives of the fourth order may do so: and so on.

Suppose that the incomplete subsidiary system for derivatives of order n offers integrable combinations

$$F = \text{constant}, \quad G = \text{constant}.$$

The constant quantities have their quality on the hypothesis that u is constant and, subject to this hypothesis, they are unrestricted so far as the subsidiary system is concerned; hence we may take

$$F = \phi(u), \quad G = \psi(u).$$

The forms of ϕ and ψ may be limited by the other subsidiary equations which involve derivatives with regard to u: whatever their forms may be, the preceding equations are consistent with the subsidiary system which itself is consistent with the original equation. When we eliminate u, we have an equation

$$V = 0,$$

which is compatible with the given equation: it is an equation of order n; and if either ϕ or ψ is arbitrary, while ϕ and ψ are independent of one another, the new equation $V = 0$ involves an arbitrary function.

We thus have an indication of a method of obtaining equations compatible with a given equation and so, as will be seen almost at once, of proceeding to the integration of the given equation: the method is associated with the name of Darboux. It is not of compelling effect, for its success depends upon contingencies that cannot be controlled: but its operation manifestly is wider than the operation of the methods previously expounded.

Darboux's Method for constructing Compatible Equations.

260. Having thus been led to the inference that, in favouring circumstances, an equation

$$V = 0$$

of order higher than the first, say of order n, may be compatible with, and not independent of, a given equation

$$f = 0$$

of the second order, we naturally desire to have the means of constructing V: one method, due to Darboux, is as follows. Take all the derivatives of $f = 0$, with regard to both independent variables, of all orders up to and including those of order $n - 1$; among these will be n equations, which involve the $n + 2$ derivatives of z which are of order $n + 1$. Take the two first derivatives of $V = 0$, supposed to be of order n: these give two equations also involving the $n + 2$ derivatives of z which are of order $n + 1$: so that, in all, there are $n + 2$ equations in these $n + 2$ derivatives. The equation $V = 0$ is not merely compatible with $f = 0$ and therefore with derivatives of $f = 0$, but also it is not independent of $f = 0$ and therefore of derivatives of $f = 0$; hence the $n + 2$ equations are not independent of one another. Consequently, when they are resolved so as to express the values of the $n + 2$ highest derivatives of z, the values so obtained must be indeterminate; and thus there will be at least two conditions* which, as they involve the first derivatives of V, are a set of simultaneous partial equations of the first order for the determination of V. If they possess a common integral (and the tests, as to the possession of a common integral by a set of simultaneous equations of the first order in a single dependent variable, are known), then an equation compatible with the original equation can be constructed.

* Darboux points out that, if the $n + 2$ equations are independent of one another so that the $n + 2$ derivatives of z of order $n + 1$ can be obtained from them, then all derivatives of z of order higher than n can be expressed in terms of derivatives of order not higher than n. Having obtained these, we should then (by the process of successive quadratures) obtain a value of z which contains only a limited number of arbitrary constants at most and therefore could not imply the existence of an arbitrary function: an integral of the type which he requires would not then be given. Accordingly, the $n + 2$ equations must not be independent of one another.

It may be noted that the method in § 238 is the special case of the above method when $n = 1$.

The new equations which may thus be obtainable are of two kinds. If it should happen that the equations for V possess only a single integral, say V_1, then the new equation is of the form

$$V_1 = a,$$

where a is an arbitrary constant. If it should happen that the equations for V possess simultaneous integrals, say V_1 and V_2, then the new equation is of the form

$$\phi(V_1, V_2) = 0,$$

where ϕ is an arbitrary function. If it should happen that the equations for V possess two sets of simultaneous integrals, say V_1 and V_2, V_3 and V_4, there are two new equations of the form

$$\phi(V_1, V_2) = 0, \quad \psi(V_3, V_4) = 0,$$

where ϕ and ψ are arbitrary functions; this is the most effective case. Moreover, the equations for V cannot possess more than two sets of integrals, because they are quadratic in form. And it may happen that the equations for V possess no integrals: we should then proceed to the next higher order.

As regards the use to be made of the new equations thus obtained, consider the most effective case, when there are two new equations

$$\phi(V_1, V_2) = 0, \quad \psi(V_3, V_4) = 0,$$

which, for the present, we shall assume* to be compatible with one another. These equations are of order n, so that they are two equations among the $n+1$ derivatives of z of that order. When we take all derivatives of $f=0$ of order $n-2$, there result $n-1$ equations involving the derivatives of z of order n: so that there are, in all, $n+1$ equations in the same number of those derivatives, and they therefore suffice to express those derivatives in terms of the derivatives which are of lower order. When substitution takes place in the last of the $n-2$ sets of differential relations of the type

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$
$$dr = \alpha\,dx + \beta\,dy, \quad ds = \beta\,dx + \gamma\,dy, \quad dt = \gamma\,dx + \delta\,dy,$$
$$\vdots$$

we have quadratures to effect: when these are effected, the final primitive will, in some form or other, involve the two arbitrary functions ϕ and ψ introduced by the new equations.

* The assumption is justified in § 265.

261. The simplest case of course occurs when the equation

$$f(x, y, z, p, q, r, s, t) = 0$$

possesses an intermediate integral, being an equation of the first order: this possibility has been fully discussed under the methods of Monge, Boole, and Ampère. We proceed to the alternative case when the equation $f=0$ possesses no intermediate integral. In order to consider whether an equation

$$V = V(x, y, z, p, q, r, s, t) = 0$$

coexists with $f=0$, though it is not resoluble into $f=0$, we must, after the preceding explanations, form the equations

$$\left.\begin{aligned} 0 &= \frac{df}{dx} + \alpha\frac{\partial f}{\partial r} + \beta\frac{\partial f}{\partial s} + \gamma\frac{\partial f}{\partial t} \\ 0 &= \frac{df}{dy} + \beta\frac{\partial f}{\partial r} + \gamma\frac{\partial f}{\partial s} + \delta\frac{\partial f}{\partial t} \\ 0 &= \frac{dV}{dx} + \alpha\frac{\partial V}{\partial r} + \beta\frac{\partial V}{\partial s} + \gamma\frac{\partial V}{\partial t} \\ 0 &= \frac{dV}{dy} + \beta\frac{\partial V}{\partial r} + \gamma\frac{\partial V}{\partial s} + \delta\frac{\partial V}{\partial t} \end{aligned}\right\},$$

where

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q},$$

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q},$$

and so for $\frac{dV}{dx}$ and $\frac{dV}{dy}$; and then we express the conditions that the values of α, β, γ, δ, furnished by these four equations, are indeterminate. The necessary conditions are

$$\left\| \begin{matrix} \frac{dV}{dx}, & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t}, & 0 \\ \frac{dV}{dy}, & 0, & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t} \\ \frac{df}{dx}, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t}, & 0 \\ \frac{df}{dy}, & 0, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \end{matrix} \right\| = 0,$$

which in effect are equivalent to two conditions formally independent of one another: and each of these conditions is of the second degree in the derivatives of V.

The condition

$$\begin{vmatrix} \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t}, & 0 \\ 0\ , & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t} \\ \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t}, & 0 \\ 0\ , & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \end{vmatrix} = 0$$

is easily seen to be

$$\left(\frac{\partial f}{\partial r}\right)^2 \left(\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r}\right) \left(\frac{\partial V}{\partial t} - \mu \frac{\partial V}{\partial s} + \mu^2 \frac{\partial V}{\partial r}\right) = 0,$$

where λ and μ are the roots of the quadratic

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0;$$

so that, assuming (as will be assumed) that $\frac{\partial f}{\partial r}$ is not zero, we have either

$$\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r} = 0,$$

or

$$\frac{\partial V}{\partial t} - \mu \frac{\partial V}{\partial s} + \mu^2 \frac{\partial V}{\partial r} = 0.$$

Next, the other independent condition can be taken in the form

$$\begin{vmatrix} \frac{dV}{dx}, & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t} \\ \frac{dV}{dy}, & 0\ , & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s} \\ \frac{df}{dx}, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \\ \frac{df}{dy}, & 0\ , & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s} \end{vmatrix} = 0.$$

With this condition, we associate either of the two equations that replace the earlier condition, say

$$\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r} = 0;$$

and we notice that

$$\frac{\partial f}{\partial t} = \lambda\mu \frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial s} = (\lambda + \mu) \frac{\partial f}{\partial r}.$$

Expanding the condition, using these relations, and removing a factor

$$\frac{\partial f}{\partial s}\frac{\partial V}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial V}{\partial s},$$

(which must not vanish because, taken in conjunction with

$$\frac{\partial V}{\partial t} - \theta \frac{\partial V}{\partial s} + \theta^2 \frac{\partial V}{\partial r} = 0,$$

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0,$$

its vanishing would imply that f and V are not functionally independent of one another, quà functions of r, s, t), the equation reduces to

$$\left(\frac{dV}{dx} + \mu \frac{dV}{dy}\right)\frac{\partial f}{\partial r} - \frac{\partial V}{\partial r}\frac{df}{dx} - \frac{1}{\lambda}\frac{\partial V}{\partial t}\frac{df}{dy} = 0.$$

Accordingly, there are two linear systems for the determination of V; they are

$$\left.\begin{aligned} &\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r} = 0 \\ &\frac{dV}{dx} + \mu \frac{dV}{dy} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial f}{\partial r}}\frac{df}{dx} - \mu \frac{\frac{\partial V}{\partial t}}{\frac{\partial f}{\partial t}}\frac{df}{dy} = 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} &\frac{\partial V}{\partial t} - \mu \frac{\partial V}{\partial s} + \mu^2 \frac{\partial V}{\partial r} = 0 \\ &\frac{dV}{dx} + \lambda \frac{dV}{dy} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial f}{\partial r}}\frac{df}{dx} - \lambda \frac{\frac{\partial V}{\partial t}}{\frac{\partial f}{\partial t}}\frac{df}{dy} = 0 \end{aligned}\right\},$$

respectively, where λ and μ are the roots of the critical quadratic

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0.$$

The two linear systems become one and the same when this quadratic has equal roots.

These results can be obtained quite simply by assigning the conditions that one of the four equations involving α, β, γ, δ is a linear combination of the other three. In order that this condition may be satisfied, quantities ρ, μ, τ must exist such that

$$\left.\begin{aligned} \frac{dV}{dx} + \mu \frac{dV}{dy} + \tau \frac{df}{dx} + \rho \frac{df}{dy} &= 0 \\ \frac{\partial V}{\partial r} + \tau \frac{\partial f}{\partial r} &= 0 \\ \frac{\partial V}{\partial s} + \mu \frac{\partial V}{\partial r} + \tau \frac{\partial f}{\partial s} + \rho \frac{\partial f}{\partial r} &= 0 \\ \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial s} + \tau \frac{\partial f}{\partial t} + \rho \frac{\partial f}{\partial s} &= 0 \\ \mu \frac{\partial V}{\partial t} + \rho \frac{\partial f}{\partial t} &= 0 \end{aligned}\right\} ;$$

when we substitute in the third equation the value of $\frac{\partial V}{\partial r}$ as given by the second equation, and in the fourth equation the value of $\frac{\partial V}{\partial t}$ as given by the fifth equation, and then eliminate $\frac{\partial V}{\partial s}$ between the two equations thus modified, we have (on removing a non-zero factor $\mu\tau - \rho$) the equation

$$\mu^2 \frac{\partial f}{\partial r} - \mu \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = 0.$$

Let μ and λ be the roots of this equation : then

$$-\tau \frac{\partial f}{\partial r} = \frac{\partial V}{\partial r},$$

$$-\mu \frac{\partial V}{\partial t} = \rho \frac{\partial f}{\partial t} = \rho\mu\lambda \frac{\partial f}{\partial r},$$

giving values of ρ and τ which change the first equation to the form

$$\left(\frac{dV}{dx} + \mu \frac{dV}{dy}\right) \frac{\partial f}{\partial r} = \frac{\partial V}{\partial r} \frac{df}{dx} + \frac{1}{\lambda} \frac{\partial V}{\partial t} \frac{df}{dy},$$

agreeing with the foregoing result. Similarly for the other equation.

Each of the two systems, so obviously similar to the subsidiary systems in the earlier methods, is homogeneous and linear in the derivatives of V; and therefore it is only necessary to apply the tests, already established for such sets of simultaneous equations as are homogeneous and linear of the first degree, in order to determine whether either set or both sets can possess a common integral or common integrals.

One or two results are immediately obtained. In the first place, we know that, if u be an argument of an arbitrary function in the primitive of $f=0$, and if y be expressed as a function of x and u, then

$$\frac{\partial f}{\partial r}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{\partial f}{\partial s}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial t} = 0,$$

where $\frac{\partial y}{\partial x}$ is formed on the supposition that u is constant; thus $\frac{\partial y}{\partial x}$ is the μ of the preceding investigation, and we have

$$\frac{\partial V}{\partial r}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{\partial V}{\partial s}\frac{\partial y}{\partial x} + \frac{\partial V}{\partial t} = 0.$$

In other words, *the characteristic equation for the argument of an arbitrary function in the primitive of $f=0$ has the same form for any equation of the second order that is compatible with $f=0$.*

In the next place, if

$$u = \text{constant}, \quad v = \text{constant},$$

are two distinct integrals belonging to either of the two systems, then

$$\phi(u, v) = 0,$$

where ϕ is arbitrary, is also an integral of the system. This property obviously follows from the fact that each of the equations in the system is homogeneous and linear.

In the third place, it is clear that all the equations for V are satisfied by taking

$$V = f,$$

which accordingly will be an integrable combination for each of the systems. If, therefore, either system is to furnish an integral of the form

$$\phi(u, v) = 0,$$

which is functionally distinct from $f=0$, the system must provide three integrable combinations, viz.

$$f=0, \quad u = \text{constant}, \quad v = \text{constant}:$$

in other words, when the system for V is made complete in the Jacobian sense, it must possess three independent integrals. The variables which can enter into the expression for V are x, y, z, p, q, r, s, t, being eight in number; and therefore, when the system is made complete, it cannot contain more than five independent equations, all of these being linear and homogeneous in the derivatives of V.

Ex. 1. Consider the equation

$$f=r-t-n\frac{p}{x}=0,$$

where n is a constant. The equation is not integrable by Monge's method: it is integrable in finite terms after change of the variables, by Laplace's method, when n is an even integer.

We have

$$\frac{\partial f}{\partial r}=1, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=-1,$$

so that the characteristic equation for the argument of an arbitrary function in the primitive is

$$\theta^2-1=0:$$

hence there are two systems, which belong to the arrangements $\lambda=1$ and $\mu=-1$, $\lambda=-1$ and $\mu=1$, respectively.

Take the arrangement $\lambda=1$, $\mu=-1$. Since

$$\frac{df}{dx}=-n\frac{r}{x}+n\frac{p}{x^2}, \quad \frac{df}{dy}=-n\frac{s}{x},$$

the equations for V (if it exists) are

$$\Delta_1(V)=\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+\frac{\partial V}{\partial t}=0,$$

$$\Delta_2(V)=\frac{dV}{dx}-\frac{dV}{dy}+n\left(\frac{r}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial r}+n\frac{s}{x}\frac{\partial V}{\partial t}$$

$$=\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+(p-q)\frac{\partial V}{\partial z}+(r-s)\frac{\partial V}{\partial p}+(s-t)\frac{\partial V}{\partial q}+n\left(\frac{r}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial r}+n\frac{s}{x}\frac{\partial V}{\partial t}=0.$$

In order that these two equations may have a common integral, they must satisfy the Jacobian condition

$$(\Delta_1, \Delta_2)=0,$$

which, not being satisfied in virtue solely of $\Delta_1=0$ and $\Delta_2=0$, gives a new equation

$$\frac{n}{x}\left(\frac{\partial V}{\partial r}-\frac{\partial V}{\partial t}\right)+2\left(\frac{\partial V}{\partial p}-\frac{\partial V}{\partial q}\right)=0.$$

We combine this with $\Delta_1=0$, and we write

$$\nabla_1=2\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+2\frac{x}{n}\frac{\partial V}{\partial p}-2\frac{x}{n}\frac{\partial V}{\partial q}=0,$$

$$\nabla_2=2\frac{\partial V}{\partial t}-\frac{\partial V}{\partial s}-2\frac{x}{n}\frac{\partial V}{\partial p}+2\frac{x}{n}\frac{\partial V}{\partial q}=0;$$

and then, substituting in $\Delta_2=0$ the value of $\frac{\partial V}{\partial r}$ from $\nabla_1=0$ and the value of $\frac{\partial V}{\partial t}$ from $\nabla_2=0$, we have

$$\nabla_3=\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+(p-q)\frac{\partial V}{\partial z}+\frac{p}{x}\frac{\partial V}{\partial p}+\left(r-t-\frac{p}{x}\right)\frac{\partial V}{\partial q}+\tfrac{1}{2}n\left(\frac{r+s}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial s}=0.$$

Then

$$(\nabla_1,\ \nabla_2)=0,$$

$$(\nabla_1,\ \nabla_3)=\nabla_4=0,$$

$$(\nabla_2,\ \nabla_3)=-\nabla_4=0,$$

where

$$\nabla_4=4\frac{x}{n}\frac{\partial V}{\partial z}+2\frac{\partial V}{\partial q}+\frac{n-2}{2x}\frac{\partial V}{\partial s}=0.$$

Further,

$$(\nabla_1,\ \nabla_4)=0,$$

$$(\nabla_2,\ \nabla_4)=0,$$

$$(\nabla_3,\ \nabla_4)=\frac{4}{n}\frac{\partial V}{\partial z}-\frac{n-2}{2x^2}\frac{\partial V}{\partial s};$$

and therefore we take

$$\Delta_4=\frac{4}{n}\frac{\partial V}{\partial z}-\frac{n-2}{2x^2}\frac{\partial V}{\partial s}=0,$$

$$\Delta_5=\frac{\partial V}{\partial q}+\frac{n-2}{2x}\frac{\partial V}{\partial s}=0,$$

so that

$$\nabla_4=x\Delta_4+2\Delta_5,$$

and consequently the equation $\nabla_4=0$ can now be omitted.

At the present stage, our equations are

$$\nabla_1=0,\quad \nabla_2=0,\quad \nabla_3=0,\quad \Delta_4=0,\quad \Delta_5=0;$$

they are such that

$$(\nabla_1,\ \nabla_2)=0,\quad (\nabla_1,\ \nabla_3)=0,\quad (\nabla_1,\ \Delta_4)=0,\quad (\nabla_1,\ \Delta_5)=0;$$

$$(\nabla_2,\ \nabla_3)=0,\quad (\nabla_2,\ \Delta_4)=0,\quad (\nabla_2,\ \Delta_5)=0,\quad (\Delta_4,\ \Delta_5)=0;$$

and

$$(\nabla_3,\ \Delta_4)=\tfrac{1}{4}(n-2)(n+4)\frac{1}{x^3}\frac{\partial V}{\partial s},$$

$$(\nabla_3,\ \Delta_5)=\tfrac{1}{4}n\Delta_4-\tfrac{1}{8}(n-2)(n+4)\frac{1}{x^2}\frac{\partial V}{\partial s}.$$

We already have five equations in the system; and this is the greatest number which it can contain, if it is to provide the proper number of integrals:

hence both (∇_3, Δ_4) and (∇_3, Δ_5) must vanish, either identically or in virtue of the equations already retained. This condition can be satisfied only if

$$(n-2)(n+4)=0.$$

We take $n=2$, so that the original equation is

$$r-t-2\frac{p}{x}=0;$$

and then a complete system of equations for V is constituted by

$$\nabla_1=0,\quad \nabla_2=0,\quad \nabla_3=0,\quad \Delta_4=0,\quad \Delta_5=0.$$

These can be replaced by an equivalent linear combination of the five, in the form

$$\frac{\partial V}{\partial z}=0,\quad \frac{\partial V}{\partial q}=0,$$

$$2\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+x\frac{\partial V}{\partial p}=0,\quad 2\frac{\partial V}{\partial t}-\frac{\partial V}{\partial s}-x\frac{\partial V}{\partial p}=0,$$

$$\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+\frac{p}{x}\frac{\partial V}{\partial p}+\left(\frac{r+s}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial s}=0;$$

and we obtain three independent integrals of this complete set in the form

$$r-t-2\frac{p}{x},\quad x+y,$$

$$\frac{r+2s+t}{x}+\frac{1}{x}\left(r-t-2\frac{p}{x}\right).$$

The original differential equation is

$$r-t-2\frac{p}{x}=0;$$

hence the integral, provided by the system, is

$$\frac{r+2s+t}{x}=\text{arbitrary function of } x+y$$

$$=-4f'''(x+y),$$

where f is arbitrary.

Similarly dealing with the system for the arrangement $\lambda=-1$ and $\mu=1$, we find an integral

$$\frac{r-2s+t}{x}=4g'''(y-x).$$

Hence, treating the original equation and the two deduced integrals as simultaneous equations to determine r, s, t, we find

$$\left.\begin{aligned} r &= \frac{p}{x}-xf'''+xg''' \\ s &= \quad\;\; -xf'''-xg''' \\ t &= -\frac{p}{x}-xf'''+xg''' \end{aligned}\right\}.$$

They satisfy the conditions

$$\frac{\partial r}{\partial y}=\frac{\partial s}{\partial x},\quad \frac{\partial s}{\partial y}=\frac{\partial t}{\partial x},$$

so that they are appropriate values for quadratures of

$$dp=r\,dx+s\,dy,$$
$$dq=s\,dx+t\,dy.$$

Effecting the quadratures, we find

$$\frac{p}{x}=-f''-g'',$$
$$q=-x(f''-g'')+f'+g';$$

when these values of p and q are substituted in

$$dz=p\,dx+q\,dy,$$

and quadrature is effected, we have

$$\begin{aligned} z&=f+g-x(f'-g') \\ &=f(x+y)+g(y-x)-x\{f'(x+y)-g'(y-x)\}, \end{aligned}$$

which is the primitive of the equation

$$r-t=2\frac{p}{x};$$

it involves two arbitrary functions f and g.

Note. If n is neither 2 nor -4 in value, then the system has no integral of the type required: and it is found that then the other system also has no such integral. In that event, we cannot obtain equations of the second order which can be associated with $f=0$: if the method is to be effective, it could then be so, only when we proceed to construct equations of higher orders.

Ex. 2. Integrate similarly the equation

$$r-t+4\frac{p}{x}=0.$$

Ex. 3. Shew that the equation

$$s=f(z)$$

cannot be integrated by Darboux's method, when the quantity

$$f(z)f''(z)-f'^2(z)$$

is different from zero. Discuss the cases when this quantity vanishes.

(Lie.)

Ex. 4. Can the equation

$$rt-s^2=c(1+p^2+q^2)^2,$$

where c is a non-vanishing constant, be integrated by Darboux's method?

(Lie.)

Integration of Simultaneous Equations of the Second Order.

262. In the preceding method of dealing with an equation of the second order in two independent variables, when the equation is known not to possess an intermediate integral, it appears that the process leads (for some classes of equations) to the construction of an integral in the form of an equation of the second order, which can be associated with the original equation: and that, in particular, when two integrals of such a form can be constructed, the derivation of the primitive is then merely a matter of quadratures. If, however, only one such integral is obtained, then quadratures will not be sufficient; and the use of the integral for the derivation of the primitive has still to be developed. The position thus created raises a similar question as to the determination of the integral (or integrals) common to two compatible equations of the second order in two independent variables.

Taking the later question indicated as the more general, we denote the two equations by

$$f=0, \quad f'=0,$$

where f and f' are given functions of x, y, z, p, q, r, s, t. Let

$$u=a,$$

where u is another (unknown) function of the same variables, and where a is a constant, be an equation which is compatible with $f=0$ and $f'=0$ and is algebraically independent of them. The three equations usually suffice to give values of r, s, t in terms of x, y, z, p, q: the conditions, that the values so obtained are the second derivatives of z, are that the relations

$$\frac{dr}{dy}=\frac{ds}{dx}, \qquad \frac{ds}{dy}=\frac{dt}{dx},$$

should be satisfied, where

$$\left.\begin{aligned} \frac{d}{dx} &= \frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q} \\ \frac{d}{dy} &= \frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q} \end{aligned}\right\}.$$

Now, from the two given equations and the assumed equation, we have

$$\frac{\partial f}{\partial r}\frac{dr}{dx}+\frac{\partial f}{\partial s}\frac{ds}{dx}+\frac{\partial f}{\partial t}\frac{dt}{dx}=-\frac{df}{dx},$$

$$\frac{\partial f'}{\partial r}\frac{dr}{dx}+\frac{\partial f'}{\partial s}\frac{ds}{dx}+\frac{\partial f'}{\partial t}\frac{dt}{dx}=-\frac{df'}{dx},$$

$$\frac{\partial u}{\partial r}\frac{dr}{dx}+\frac{\partial u}{\partial s}\frac{ds}{dx}+\frac{\partial u}{\partial t}\frac{dt}{dx}=-\frac{du}{dx};$$

and therefore

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{ds}{dx}=-J\left(\frac{f, f', u}{r, x, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{dt}{dx}=-J\left(\frac{f, f', u}{r, s, x}\right).$$

Similarly, we have

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{dr}{dy}=-J\left(\frac{f, f', u}{y, s, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{ds}{dy}=-J\left(\frac{f, f', u}{r, y, t}\right).$$

When we use the relations of condition, we have the equations

$$J\left(\frac{f, f', u}{r, x, t}\right)=J\left(\frac{f, f', u}{y, s, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, x}\right)=J\left(\frac{f, f', u}{r, y, t}\right).$$

Conversely, these two equations are sufficient to secure the two relations of condition.

The two equations thus obtained for u are homogeneous and linear of the first order. When we write

$$\frac{df}{dx}=X,\quad \frac{df}{dy}=Y,\quad \frac{\partial f}{\partial r}=R,\quad \frac{\partial f}{\partial s}=S,\quad \frac{\partial f}{\partial t}=T,$$

$$\frac{df'}{dx}=X',\quad \frac{df'}{dy}=Y',\quad \frac{\partial f'}{\partial r}=R',\quad \frac{\partial f'}{\partial s}=S',\quad \frac{\partial f'}{\partial t}=T',$$

the equations for u take the form

$$\begin{aligned}\Delta_1(u)=\frac{\partial u}{\partial r}(XT'-TX')&+\frac{\partial u}{\partial s}(YT'-TY')\\ &+\frac{\partial u}{\partial t}(RX'-XR'+SY'-YS')\\ &+\frac{du}{dx}(TR'-RT')+\frac{du}{dy}(TS'-ST')=0,\end{aligned}$$

$$\Delta_2(u) = \frac{\partial u}{\partial t}(RY' - YR') + \frac{\partial u}{\partial s}(RX' - XR')$$
$$+ \frac{\partial u}{\partial r}(XS' - SX' + YT' - TY')$$
$$+ \frac{du}{dx}(SR' - RS') + \frac{du}{dy}(TR' - RT') = 0.$$

Here u is a function of the eight variables x, y, z, p, q, r, s, t. If the simultaneous equations $\Delta_1 = 0$, $\Delta_2 = 0$, require other n equations in order to make them a complete Jacobian system, which then will consist of $2 + n$ equations, the equations possess $6 - n$ common integrals. Two such integrals are provided by $u = f$, $u = f'$; and so the equations possess $4 - n$ common integrals, which are algebraically distinct from f and f'.

The method manifestly is effective, if n is less than four; and the conditions for the coexistence of $\Delta_1 = 0$, $\Delta_2 = 0$, are conditions that involve the derivatives of f and f'. Consequently, the conditions that $n < 4$, so that the completed Jacobian system contains not more than five equations, are effectively the conditions that the two equations $f = 0$, $f' = 0$, are compatible with one another. If $f' = 0$ has been constructed by Darboux's method so as to be associated with $f = 0$, the conditions are satisfied; but if $f' = 0$ be given as a new equation, independent of any indication as to the mode of its construction, it is clear that $f' = 0$ cannot be taken arbitrarily.

The simplest case occurs when $n = 0$, so that $\Delta_1 = 0$, $\Delta_2 = 0$, are then a complete Jacobian system of themselves. In that case, there are four common integrals, say, u_1, u_2, u_3, u_4; hence we have six equations in all, viz.

$$f = 0, \quad f' = 0,$$
$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4.$$

When p, q, r, s, t are eliminated among these equations, we have a relation between z, x, y, which involves four arbitrary constants.

The least simple case, when the method is effective, occurs for $n = 1$. We then have three equations

$$f = 0, \quad f' = 0, \quad u = a,$$

which suffice to determine r, s, t as functions of the other five variables; when these are substituted in

$$dz = pdx + qdy,$$
$$dp = rdx + sdy,$$
$$dq = sdx + tdy,$$

and the quadratures are effected, we again obtain a primitive involving four arbitrary constants*.

Note. It should, however, be remarked that the two equations $\Delta_1 = 0$, $\Delta_2 = 0$, represent only a single equation, if we retain the equations which can be deduced by Darboux's method applied to the equations $f = 0, f' = 0$. From the results of § 261, it follows that, when a quantity θ is chosen so as to satisfy the equation

$$T - \theta S + \theta^2 R = 0,$$

(with the preceding notation), then, because $f' = 0$ is compatible with $f = 0$, the equations

$$T' - \theta S' + \theta^2 R' = 0,$$
$$T'Y - TY' = \theta (RX' - R'X),$$

must be satisfied for one or other of the two values of θ. Consider the expression $\Delta_1 - \theta\Delta_2$. In this expression, the coefficient of $\dfrac{\partial u}{\partial r}$ is

$$\begin{aligned} & XT' - TX' - \theta (XS' - SX' + YT' - TY') \\ &= X (T' - \theta S' + \theta^2 R') - X' (T - \theta S + \theta^2 R) \\ &= 0; \end{aligned}$$

the coefficient of $\dfrac{\partial u}{\partial s}$

$$= YT' - Y'T - \theta (RX' - R'X) = 0;$$

the coefficient of $\dfrac{\partial u}{\partial t}$

$$\begin{aligned} &= RX' - XR' + SY' - YS' - \theta (RY' - YR') \\ &= \frac{1}{\theta} Y (T' - \theta S' + \theta^2 R') - \frac{1}{\theta} Y' (T - \theta S + \theta^2 R) \\ &= 0; \end{aligned}$$

* The investigation is due to Vályi, *Crelle*, t. XCV (1883), pp. 99—101. Some notes by Bianchi, *Atti d. Reale Acc. d. Lincei*, Ser. 4[ta], t. II, 2° Sem., (1886), pp. 218—223, 237—241, 307—310, and a memoir by von Weber, *Münch. Sitzungsb.*, t. XXV (1895—6), pp. 101—113; may also be consulted. See also Goursat's treatise, quoted on p. 303, chapter VI.

the coefficients of $\frac{du}{dx}$ and of $\frac{du}{dy}$ are easily seen to vanish. Hence

$$\Delta_1 = \theta \Delta_2,$$

in virtue of the equations connecting $f=0$ and $f'=0$ in Darboux's method.

It thus appears that the Vályi process does not add to the theory: and indeed, analysis similar to that which precedes will, when applied to the equations in Darboux's method expressing the conditions of coexistence of

$$f=0, \quad u=0,$$

lead to Vályi's equations. The importance of the results rather lies in the fact that, when two simultaneous equations of the second order

$$f=0, \quad f'=0,$$

are given, the common primitive (if any) can be obtained by integrating simultaneous equations of the first order, followed (if need be) by quadratures.

263. In the preceding investigation, the conditions of compatibility have been assumed to be satisfied, although no attempt has been made to construct these conditions. Some, at least, of them can be obtained by constructing the successive derivatives of the equations. Let the two equations $f=0$ and $f'=0$ be supposed resolved, so as to express r and t explicitly in a form

$$r + \theta(x, y, z, p, q, s) = 0, \quad t + \phi(x, y, z, p, q, s) = 0.$$

Denoting the four third derivatives of z by α, β, γ, δ, and writing

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \theta\frac{\partial}{\partial p} + s\frac{\partial}{\partial q},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} - \phi\frac{\partial}{\partial q},$$

we have

$$\alpha + \beta\frac{\partial\theta}{\partial s} + \frac{d\theta}{dx} = 0,$$

$$\beta + \gamma\frac{\partial\theta}{\partial s} + \frac{d\theta}{dy} = 0,$$

$$\gamma + \beta\frac{\partial\phi}{\partial s} + \frac{d\phi}{dx} = 0,$$

$$\delta + \gamma\frac{\partial\phi}{\partial s} + \frac{d\phi}{dy} = 0.$$

These four equations determine values of α, β, γ, δ, unless the quantity

$$1-\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s}$$

vanishes.

Assuming, in the first place, that this quantity does not vanish, the values of α, β, γ, δ are definitely expressible in terms of the derivatives of θ and ϕ; and they must obey the relations

$$\frac{d\alpha}{dy}+\frac{\partial\alpha}{\partial s}\gamma=\frac{d\beta}{dx}+\frac{\partial\beta}{\partial s}\beta,$$

$$\frac{d\beta}{dy}+\frac{\partial\beta}{\partial s}\gamma=\frac{d\gamma}{dx}+\frac{\partial\gamma}{\partial s}\beta,$$

$$\frac{d\gamma}{dy}+\frac{\partial\gamma}{\partial s}\gamma=\frac{d\delta}{dx}+\frac{\partial\delta}{\partial s}\beta.$$

When the values of α, β, γ, δ are substituted, we shall have relations affecting the quantities θ and ϕ alone: these must be satisfied. Assuming that the necessary conditions are satisfied, so that definite proper values for the third derivatives of z are known, then all the succeeding derivatives can be constructed. Taking their values for initial values $x=a$, $y=b$, we can construct a Cauchy integral; and the only unassigned (and therefore arbitrary) quantities, which occur in the expression of the integral, are the initial values of z, p, q, s. In other words, we should then expect an integral common to the two equations and involving four arbitrary constants.

Ex. Are the equations

$$r+sX=0, \quad t+sY=0,$$

compatible, where X is a function of x only, and Y is a function of y only?

In this case, the critical quantity is $XY-1$, which does not vanish; consequently, the derivatives of the third order can be obtained. They are

$$\alpha=-sX', \quad \beta=0, \quad \gamma=0, \quad \delta=-sY';$$

and it is easy to verify that these values do satisfy the conditions. Consequently, the equations are compatible; and they possess an integral

$$\begin{aligned} z=a+bx+cy+Axy \\ -A\left(\frac{x^2}{2!}X_0+\frac{x^3}{3!}X_0'+\frac{x^4}{4!}X_0''+\ldots\right) \\ -A\left(\frac{y^2}{2!}Y_0+\frac{y^3}{3!}Y_0'+\frac{y^4}{4!}Y_0''+\ldots\right), \end{aligned}$$

involving the four arbitrary constants a, b, c, A, the quantities X_0, X_0', ..., Y_0, Y_0', ..., denoting the values of X, X', ..., Y, Y', ..., when $x=0$, $y=0$.

Next, suppose that the relation

$$1-\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s}=0$$

is satisfied: then the two derivatives of θ and the two derivatives of ϕ, which involve the quantities α, β, γ, δ, are equivalent to only three equations involving these quantities, together with the equation

$$\frac{d\theta}{dy}-\frac{\partial\theta}{\partial s}\frac{d\phi}{dx}=0.$$

A procedure, similar to that adopted in the case of the first hypothesis, will serve to settle the question as to whether the new equations

$$1-\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s}=0, \qquad \frac{d\theta}{dy}-\frac{\partial\theta}{\partial s}\frac{d\phi}{dx}=0,$$

are compatible with one another and with the original equations: we shall assume that all the necessary conditions are satisfied, and that all the equations are therefore compatible with one another. There are various possibilities.

It may happen that each of the new equations is an equation of only the first order in the derivatives of z. They are compatible with one another and with the original equations: hence they determine p and q, and consequently all derivatives of z, in terms of x, y, z. The resulting primitive, common to all the equations, involves one arbitrary constant: this may be regarded as an arbitrarily assigned value of z for initial values of x and y.

It may happen that only one of the new equations is an equation of the first order, the other three equations being compatible with it and with one another. That equation can be regarded as determining (say) p in terms of x, y, z, q: the other equations, and their derivatives, determine all the derivatives of z in terms of these same quantities. The resulting primitive, common to all the equations, involves two arbitrary constants: they may be regarded as the values arbitrarily assigned to z and q for initial values of x and y.

It may happen that one or other of the new equations, while not an equation of the first order, is a new equation of the second order compatible with, yet algebraically independent of, the original equations: in that case, s is the only derivative of the second order which it can involve. This last equation, in conjunction with the original equations, will serve to determine r, s, t, in terms of x, y, z, p, q: and then all the derivatives of z are expressible in terms of the same quantities. The resulting primitive, common to all the equations, involves three arbitrary constants: they may be regarded as the values arbitrarily assigned to z, p, q, for initial values of x and y.

Lastly, it may happen that each of the new equations is satisfied identically, so that the only surviving significant equations are the original two equations: such a pair of equations is said, after Lie*, to be *in involution*. They do not suffice for the determination of r, s, t, in terms of x, y, z, p, q: one of these three quantities can have an arbitrary initial value assigned to it, and the other two then are determinate. Again, there are only three independent derived equations in α, β, γ, δ, and they do not suffice for the determination of these four magnitudes in terms of x, y, z, p, q, r, s, t: one of these magnitudes can have an arbitrary initial value assigned to it, and the other three then are determinate. Similarly for the derivatives of the fourth order: denoting the four deduced equations of the third order by $A=0$, $B=0$, $C=0$, $D=0$, so that $C=0$ is a consequence of the other three, we have

$$\frac{dA}{dx}=0,$$

$$\frac{dA}{dy}=0=\frac{dB}{dx},$$

$$\frac{dB}{dy}=0,$$

$$\frac{dD}{dy}=0,$$

as four independent equations involving the five derivatives of the fourth order. We do not have

$$\frac{dC}{dx}=0,\quad \frac{dC}{dy}=0,$$

as new independent equations, because $C=0$ is a dependent equation; and, as

$$\frac{dD}{dx}=\frac{dC}{dy},$$

* *Leipz. Ber.*, t. XLVII (1895), p. 73.

we do not have

$$\frac{dD}{dx}=0,$$

as a new independent equation. The four independent equations do not suffice for the determination of the five derivatives in question: one of them can have an arbitrary initial value assigned to it, and the other four then are determinate. And so for all the orders in succession: each of them allows an arbitrary initial value. Hence, when we construct an integral by means of a doubly-infinite power-series, the integral so obtained will involve an unlimited number of constants*.

Ex. 1. Consider the equations †

$$f=r-q=0,\quad f'=t-p=0.$$

Thus

$$\frac{df}{dx}=-s,\quad \frac{df}{dy}=-t,\quad \frac{df'}{dx}=-r,\quad \frac{df'}{dy}=-s;$$

and so the equations for u are

$$\Delta_1=\frac{du}{dx}+s\frac{\partial u}{\partial r}+t\frac{\partial u}{\partial s}+r\frac{\partial u}{\partial t}=0,$$

$$\Delta_2=\frac{du}{dy}+t\frac{\partial u}{\partial r}+r\frac{\partial u}{\partial s}+s\frac{\partial u}{\partial t}=0.$$

The Jacobian condition of coexistence, viz.

$$(\Delta_1,\ \Delta_2)=0,$$

is satisfied identically: thus $\Delta_1=0$ and $\Delta_2=0$ are a complete Jacobian system. Accordingly, they have six common integrals: two of these are f and f': hence other four, algebraically independent of f and f', are required. When any one of the customary methods of integration is adopted, it leads to four integrals

$$u_1=s-z,$$
$$u_2=\tfrac{1}{3}(r+s+t)\,e^{-x-y},$$
$$u_3=\tfrac{1}{3}(r+\omega^2 s+\omega t)\,e^{-\omega x-\omega^2 y},$$
$$u_4=\tfrac{1}{3}(r+\omega s+\omega^2 t)\,e^{-\omega^2 x-\omega y},$$

where ω is an imaginary cube-root of unity. Eliminating p, q, r, s, t between the six equations

$$f=0,\quad f'=0,$$
$$u_\mu=a_\mu,\qquad (\mu=1,\ 2,\ 3,\ 4),$$

* For further discussion of simultaneous equations in involution, reference may be made to Lie's memoir quoted on p. 295, and to chapter VI in Goursat's treatise quoted on p. 303.

† This example is given by Vályi (*l.c.*).

we have a primitive in the form

$$z+a_1=a_2e^{x+y}+\omega a_3e^{\omega x+\omega^2 y}+\omega^2 a_4 e^{\omega^2 x+\omega y}$$
$$=c_2e^{x+y}+c_3e^{\omega x+\omega^2 y}+c_4e^{\omega^2 x+\omega y},$$

on changing the constants.

Ex. 2. Obtain an integral common to the equations

$$r-t=0,$$
$$rt-s^2=a(x^2-y^2),$$

in the form

$$z=\alpha x+\beta y+\gamma+b(x+y)^3+c(x-y)^3,$$

where

$$144bc=a.$$

Ex. 3. Obtain an integral common to the equations

$$r-t=0,$$
$$zs=pq+2c^2z^3,$$

in the form

$$z=\wp\{c(x+y)+\alpha\}-\wp\{c(x-y)+\beta\},$$

where α, β, and the two invariants of the elliptic functions are four arbitrary constants. Verify also that

$$z_1=4\frac{h^2}{c^2}\{e^{h(x+y)+\alpha}-e^{h(x+y)-\alpha}\}^{-2},$$

$$z_2=4\frac{k^2}{c^2}\{e^{k(x-y)+\beta}-e^{k(x-y)-\beta}\}^{-2},$$

where h, α, k, β are arbitrary constants, also are integrals; and further that, when $h=k$, then z_1-z_2 is an integral. (Bourlet.)

Ex. 4. Prove that the two equations

$$3r+s^3=0, \quad st=1,$$

possess a common primitive: and obtain this primitive in its most general form. (Goursat.)

Ex. 5. Prove that the equations

$$\frac{r}{u^2}=\frac{s}{-u}=t,$$

where u is a function of x, y, z, p, q, satisfying the equation

$$u\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}+(p+uq)\frac{\partial u}{\partial z}=0,$$

possess a primitive, representing developable surfaces: and investigate the properties of the surfaces which satisfy the equations. (Goursat.)

Ex. 6. Obtain a complete integral of the equations

$$\frac{r}{1+p^2}=\frac{s}{pq}=\frac{t}{1+q^2}$$

in the form

$$x^2+y^2+z^2+2a_1x+2a_2y+2a_3z=a_4.$$

Do the equations possess other integrals? (Goursat.)

Ex. 7. In connection with the two equations

$$\left.\begin{array}{r} r+f(x,\ y,\ z,\ p,\ q,\ s,\ t)=0 \\ u(x,\ y,\ z,\ p,\ q,\ s,\ t)=a \end{array}\right\},$$

where a is a constant, let quantities Δ, D_2, D_3, D_4 be defined as follows :—

$$\Delta=\left(\frac{\partial u}{\partial t}\right)^2-\frac{\partial f}{\partial s}\frac{\partial u}{\partial s}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial t}\left(\frac{\partial u}{\partial s}\right)^2,$$

$$-D_2=\frac{df}{dy}\left(\frac{\partial u}{\partial t}\right)^2-\frac{du}{dx}\left(\frac{\partial f}{\partial s}\frac{\partial u}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial u}{\partial s}\right)-\frac{du}{dy}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t},$$

$$-D_3=-\frac{df}{dy}\frac{\partial u}{\partial s}\frac{\partial u}{\partial t}+\frac{du}{dx}\frac{\partial u}{\partial t}+\frac{du}{dy}\frac{\partial f}{\partial t}\frac{\partial u}{\partial s},$$

$$-D_4=\frac{df}{dy}\left(\frac{\partial u}{\partial s}\right)^2-\frac{du}{dx}\frac{\partial u}{\partial s}+\frac{du}{dy}\left(\frac{\partial u}{\partial t}-\frac{\partial f}{\partial s}\frac{\partial u}{\partial s}\right),$$

where

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}:$$

prove that a quantity R exists, such that

$$\Delta\frac{dD_3}{dx}-D_3\frac{d\Delta}{dx}-\left(\Delta\frac{dD_2}{dy}-D_2\frac{d\Delta}{dy}\right)=R\frac{\partial u}{\partial t},$$

$$\Delta\frac{dD_4}{dx}-D_4\frac{d\Delta}{dx}-\left(\Delta\frac{dD_3}{dy}-D_3\frac{d\Delta}{dy}\right)=R\frac{\partial u}{\partial s}.$$

Shew that any integral u of the equation $R=0$, which does not make Δ vanish, leads to a complete integral of the equation $r+f=0$ involving five parameters.

Prove that the two equations

$$r+f=0,\quad u=a,$$

are a system in involution, if

$$\Delta=0,$$

and if, at the same time, another condition (which obtain) is satisfied: the integrals of the equation $r+f=0$ are then governed by the theorem in the text.

Discuss the integral or integrals of the equation $r+f=0$, if $\Delta=0$ (but not the other condition for involution) is satisfied. (König.)

Note. The memoir by König, from which the foregoing results are taken and which has been quoted already (p. 303), discusses also the case, when an equation of order higher than the second is compatible with an equation $r+f=0$.

HAMBURGER'S METHOD.

264. The method, usually associated with the name of Darboux, is not the only process of constructing equations of the second order (when this can be done) compatible with a given equation: an equivalent set of equations, in differential elements rather than in differential coefficients, is provided by Hamburger's application*, of his process (as explained in Chapter XI of the preceding volume) of solving a number of simultaneous equations of the first order in the same number of dependent variables, to the integration of equations of order higher than the first. In particular, consider a general equation

$$f = f(x, y, z, p, q, r, s, t) = 0;$$

the first stage of the problem is to determine two other equations

$$u = a, \quad v = b,$$

where u and v are functions of r, s, t, and of the other variables, such that u, v, f, being algebraically independent, provide values of r, s, t, which make the equations

$$\left.\begin{aligned} dp &= r\,dx + s\,dy \\ dq &= s\,dx + t\,dy \\ dz &= p\,dx + q\,dy \end{aligned}\right\}$$

an integrable system. With the preceding notation, we have

$$R\frac{\partial r}{\partial x} + S\frac{\partial s}{\partial x} + T\frac{\partial t}{\partial x} = -X,$$

$$R\frac{\partial r}{\partial y} + S\frac{\partial s}{\partial y} + T\frac{\partial t}{\partial y} = -Y;$$

and we also have

$$\begin{aligned} dr &= \frac{\partial r}{\partial x}dx + \frac{\partial s}{\partial x}dy, \\ ds &= \frac{\partial s}{\partial x}dx + \frac{\partial t}{\partial x}dy = \frac{\partial r}{\partial y}dx + \frac{\partial s}{\partial y}dy, \\ dt &= \qquad\qquad\qquad\qquad\;\; \frac{\partial s}{\partial y}dx + \frac{\partial t}{\partial y}dy. \end{aligned}$$

Following the method of dealing with simultaneous equations of the first order, we construct two linear combinations of these

* *Crelle*, t. XCIII (1882), pp. 188—214.

relations in differential elements, one of them involving derivatives of r, s, t, with regard to x only, and the other of them involving derivatives of the same quantities with regard to y only: they are

$$\lambda_1 dr + \lambda_2 ds = \frac{\partial r}{\partial x}\lambda_1 dx + \frac{\partial s}{\partial x}(\lambda_2 dx + \lambda_1 dy) + \frac{\partial t}{\partial x}\lambda_2 dy,$$

$$\lambda_1 ds + \lambda_2 dt = \frac{\partial r}{\partial y}\lambda_1 dx + \frac{\partial s}{\partial y}(\lambda_2 dx + \lambda_1 dy) + \frac{\partial t}{\partial y}\lambda_2 dy,$$

whatever be the values of λ_1 and λ_2. In connection with these relations, and having regard to the preceding complete derivatives of f with regard to x and to y, we construct the subsidiary equations

$$\frac{\lambda_1 dx}{R} = \frac{\lambda_2 dx + \lambda_1 dy}{S} = \frac{\lambda_2 dy}{T} = \frac{\lambda_1 dr + \lambda_2 ds}{-X} = \frac{\lambda_1 ds + \lambda_2 dt}{-Y}.$$

The equality of the first three fractions determines values of $\frac{dy}{dx}$ and of $\frac{\lambda_2}{\lambda_1}$ for the subsidiary system. Taking

$$dy = \mu dx,$$

we have

$$\lambda_1 (S - \mu R) = \lambda_2 R,$$

$$\lambda_2 (T - \mu S) = -\mu\lambda_1 T,$$

$$= -\mu\lambda_2 \frac{RT}{S - \mu R},$$

which, on the removal of a non-zero factor $\lambda_2 S$, gives

$$\mu^2 R - \mu S + T = 0:$$

and then

$$\frac{\lambda_2}{\lambda_1} = \frac{S}{R} - \mu = \lambda,$$

where λ is the other root of the quadratic. Hence we have

$$\left.\begin{aligned} R dr + (S - \mu R)\, ds &= -X dx \\ R ds + (S - \mu R)\, dt &= -Y dx \\ dy &= \mu dx \\ dz &= (p + \mu q)\, dx \\ dp &= (r + \mu s)\, dx \\ dq &= (s + \mu t)\, dx \end{aligned}\right\},$$

where μ is a root of the quadratic

$$\mu^2 R - \mu S + T = 0.$$

Let $u =$ constant be an integral equivalent of this subsidiary system, so that $du = 0$ is a linear combination of the set of equations: then

$$\left(\frac{du}{dx} + \mu \frac{du}{dy}\right) dx + \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt$$
$$= \alpha \{R dr + (S - \mu R) ds + X dx\}$$
$$+ \beta \{R ds + (S - \mu R) dt + Y dx\};$$

and therefore

$$\left.\begin{aligned} \frac{du}{dx} + \mu \frac{du}{dy} &= \alpha X + \beta Y \\ \frac{\partial u}{\partial r} &= \alpha R \\ \frac{\partial u}{\partial s} &= \alpha (S - \mu R) + \beta R \\ \frac{\partial u}{\partial t} &= \beta (S - \mu R) = \beta \frac{T}{\mu} \end{aligned}\right\}.$$

When α and β are eliminated among these four equations, two relations survive; they are

$$\left.\begin{aligned} \lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} &= 0 \\ \frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R} \frac{\partial u}{\partial r} - \mu \frac{Y}{T} \frac{\partial u}{\partial t} &= 0 \end{aligned}\right\},$$

which (§ 261) are the equations given by Darboux's method as characteristic of a quantity u, where

$$u = \text{constant}$$

is an equation of the second order that can be associated with the given equation $f = 0$. Also, when the quadratic

$$\mu^2 R - \mu S + T = 0$$

has unequal roots, we have the two sets of equations that occur in Darboux's method, on taking the values of μ in turn.

Remembering the relation between the subsidiary system in differential elements, which arises in Monge's method of constructing an intermediate integral of the first order (when it exists), and the subsidiary system in differential coefficients, which arises in Boole's method of achieving the same aim, we see that there is a corresponding relation between the two sets of equations for constructing a compatible integral in the form of an equation of the second order. The integration of the equations

$$\left.\begin{aligned} \lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0 \\ \frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R}\frac{\partial u}{\partial r} - \mu \frac{Y}{T}\frac{\partial u}{\partial t} = 0 \end{aligned}\right\}$$

is equivalent to the quadrature of the system

$$\left.\begin{aligned} \frac{dx}{1} = \frac{dy}{\mu} = \frac{dz}{p+\mu q} = \frac{dp}{r+\mu s} = \frac{dq}{s+\mu t} \\ = \frac{dr+\lambda ds}{-\frac{X}{R}} = \frac{ds+\lambda dt}{-\frac{Y}{R}} \end{aligned}\right\},$$

where λ and μ are the roots of the quadratic

$$\mu^2 R - \mu S + T = 0.$$

265. It was assumed (§ 260) that, if each of the two systems of equations possesses an integral involving r, s, t, the two integrals can be combined with $f=0$ to furnish values of r, s, and t, which make the quadratures possible: the assumption can be established as follows. Let μ_1 and μ_2 be the two roots of the quadratic; and let

$$v_1 = 0, \quad v_2 = 0,$$

be the integrals of the respective systems, such that these two equations and $f=0$ can be resolved so as to express r, s, t in terms of the other variables: then the Jacobian

$$J\left(\frac{f, v_1, v_2}{r, s, t}\right)$$

does not vanish. Writing

$$\frac{dv_1}{dx} = X_1, \quad \frac{\partial v_1}{\partial r} = R_1, \quad \frac{\partial v_1}{\partial s} = S_1, \quad \frac{\partial v_1}{\partial t} = T_1,$$

and so for derivatives of v_2, we have the equations

$$X + R\frac{dr}{dx} + S\frac{ds}{dx} + T\frac{dt}{dx} = 0,$$

$$X_1 + R_1\frac{dr}{dx} + S_1\frac{ds}{dx} + T_1\frac{dt}{dx} = 0,$$

satisfied identically when the resolved values of r, s, t are substituted in $f=0$, $v_1=0$; and therefore

$$X_1R - R_1X + (S_1R - R_1S)\frac{ds}{dx} + (T_1R - R_1T)\frac{dt}{dx} = 0.$$

Similarly,

$$Y_1T - T_1Y + (R_1T - T_1R)\frac{dr}{dy} + (S_1T - T_1S)\frac{ds}{dy} = 0.$$

But on account of the equations satisfied by v_1, we have

$$X_1 - R_1\frac{X}{R} = -\mu_1\left(Y_1 - \frac{Y}{T}T_1\right),$$

so that

$$RX_1 - R_1X = -\frac{R}{T}\mu_1(TY_1 - YT_1)$$

$$= -\frac{1}{\mu_2}(TY_1 - YT_1);$$

consequently,

$$\mu_2(S_1R - R_1S)\frac{ds}{dx} + (R_1T - T_1R)\frac{dr}{dy}$$

$$+ \mu_2(T_1R - R_1T)\frac{dt}{dx} + (S_1T - T_1S)\frac{ds}{dy} = 0.$$

Now

$$\mu_2(S_1R - R_1S) = \mu_2R\{S_1 - (\mu_1 + \mu_2)R_1\}$$

$$= \mu_2R\left(\frac{1}{\mu_2}T_1 - \mu_1R_1\right)$$

$$= RT_1 - TR_1,$$

so that the first line of the last equation is

$$(RT_1 - TR_1)\left(\frac{ds}{dx} - \frac{dr}{dy}\right);$$

and, similarly, the second line is found to be

$$(T_1S - S_1T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right);$$

thus the equation is

$$(RT_1 - TR_1)\left(\frac{ds}{dx} - \frac{dr}{dy}\right) + (T_1S - S_1T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right) = 0.$$

Similarly, we find

$$(RT_2 - TR_2)\left(\frac{ds}{dx} - \frac{dr}{dy}\right) + (T_2S - S_2T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right) = 0.$$

Now

$$\begin{vmatrix} RT_1 - TR_1, & T_1S - S_1T \\ RT_2 - TR_2, & T_2S - S_2T \end{vmatrix} = -T \begin{vmatrix} R, & S, & T \\ R_1, & S_1, & T_1 \\ R_2, & S_2, & T_2 \end{vmatrix};$$

the Jacobian does not vanish and, without loss of generality, we may assume that T does not vanish; hence the two equations can hold only if

$$\frac{ds}{dx} - \frac{dr}{dy} = 0, \quad \frac{dt}{dx} - \frac{ds}{dy} = 0,$$

that is, the conditions of integrability are satisfied.

In this discussion, an assumption obviously is made that the roots of the quadratic differ from one another. When they are the same, so that there is only a single set of equations, and when that single set offers an integrable combination $v = 0$, we then should use the Vályi process for associating a third equation with $v = 0, f = 0$. The consideration of the matter will be resumed in the discussion of the characteristics (Chap. XX.).

The actual construction of the integrals of a subsidiary system, when they are possessed by it, and the use made of the integrals, correspond with the construction and the use in Ampère's method. Suppose that a subsidiary system has two integrals, say

$$v_1 = \text{constant}, \quad v_2 = \text{constant};$$

the equation to be associated with the original equation is of the form

$$v_2 = \phi(v_1),$$

where ϕ is an arbitrary function. Suppose also that the other subsidiary system has two integrals

$$w_1 = \text{constant}, \quad w_2 = \text{constant};$$

then, similarly, we have an equation

$$w_2 = \psi(w_1),$$

where ψ is arbitrary. The three equations

$$f = 0, \quad v_2 = \phi(v_1), \quad w_2 = \psi(w_1),$$

are used to effect the quadratures in

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy, \quad dz = p\,dx + q\,dy;$$

and frequently, it is convenient in practice to replace x and y by v_1 and w_1 as the independent variables.

Ex. 1. Let it be required to integrate the equation

$$r-qs+pt=0.$$

With the preceding notation, we have

$$R=1, \quad S=-q, \quad T=t;$$

thus λ and μ are the roots of the quadratic

$$\theta^2+q\theta+p=0,$$

so that

$$\lambda+\mu=-q, \quad \lambda\mu=p.$$

Also

$$X=rt-s^2, \quad Y=0:$$

thus a subsidiary system, taken in connection with the differential elements, is

$$\frac{dp}{dx}=r+\mu s=(q+\mu)s-pt=-\lambda s-\lambda\mu t,$$

$$\frac{dq}{dx}=s+\mu t,$$

and therefore

$$\frac{dp}{dx}+\lambda\frac{dq}{dx}=0.$$

Hence

$$\frac{dp}{dq}=-\lambda=\tfrac{1}{2}q+(\tfrac{1}{4}q^2-p)^{\frac{1}{2}},$$

and therefore

$$\left(\frac{dp}{dq}\right)^2-q\frac{dp}{dq}=-p,$$

the well-known Clairaut form: we therefore can write

$$\frac{dp}{dq}=\text{constant},$$

so that

$$\lambda=\text{constant}$$

for the system. Also, as $Y=0$, we have

$$ds+\lambda\,dt=0,$$

that is,

$$s+\lambda t=\text{constant};$$

and so an appropriate integral is given by

$$s+\lambda t=\phi(\lambda).$$

Similarly, from the other system, we have

$$s+\mu t=\psi(\mu),$$

where ϕ and ψ are arbitrary. Thus we have

$$p=\lambda\mu, \quad q=-\lambda-\mu,$$

$$t=\frac{\phi(\lambda)-\psi(\mu)}{\lambda-\mu},$$

$$s=\frac{-\mu\phi(\lambda)+\lambda\psi(\mu)}{\lambda-\mu},$$

$$r=qs-pt=\frac{\mu^2\phi(\lambda)-\lambda^2\psi(\mu)}{\lambda-\mu},$$

and therefore

$$rt - s^2 = -\phi(\lambda)\,\psi(\mu).$$

The equations for quadrature are

$$r\,dx + s\,dy = dp = \mu\,d\lambda + \lambda\,d\mu,$$
$$s\,dx + t\,dy = dq = -d\lambda - d\mu;$$

hence

$$(rt - s^2)\,dx = (\mu t + s)\,d\lambda + (\lambda t + s)\,d\mu$$
$$= \psi(\mu)\,d\lambda + \phi(\lambda)\,d\mu,$$

that is,

$$-dx = \frac{d\lambda}{\phi(\lambda)} + \frac{d\mu}{\psi(\mu)}.$$

Similarly,

$$-dy = \frac{\lambda}{\phi(\lambda)}\,d\lambda + \frac{\mu}{\psi(\mu)}\,d\mu;$$

and therefore

$$dz = p\,dx + q\,dy$$
$$= \frac{\lambda^2}{\phi(\lambda)}\,d\lambda + \frac{\mu^2}{\psi(\mu)}\,d\mu.$$

To obtain explicit expressions for x, y, z, we take

$$\phi(\lambda) = \frac{1}{f'''(\lambda)}, \quad \psi(\mu) = \frac{1}{g'''(\mu)},$$

where f and g are arbitrary; and we have

$$\left.\begin{aligned} z &= \lambda^2 f''(\lambda) - 2\lambda f'(\lambda) + 2f(\lambda) + \mu^2 g''(\mu) - 2\mu g'(\mu) + 2g(\mu) \\ -x &= f''(\lambda) + g''(\mu) \\ -y &= \lambda f''(\lambda) - f'(\lambda) + \mu g''(\mu) - g'(\mu) \end{aligned}\right\},$$

which constitute the primitive of the equation.

Ex. 2. Obtain the primitive of the equation

$$qr - ps + t = \frac{p^2 - 4q}{y + a},$$

in the form

$$-\frac{1}{y+a} = f''(\alpha) + g''(\beta),$$

$$-\frac{x}{y+a} = \alpha f''(\alpha) - f'(\alpha) + \beta g''(\beta) - g'(\beta),$$

$$-z - \frac{x^2}{y+a} = \alpha^2 f''(\alpha) - 2\alpha f'(\alpha) + 2f(\alpha) + \beta^2 g''(\beta) - 2\beta g'(\beta) + 2g(\beta).$$

(De Boer.)

Ex. 3. Shew that the equation

$$r + t = zu,$$

where u is a function of x and y different from zero, cannot have an intermediate integral. Find the equation or equations to be satisfied by u, in order that two equations of the second order, of the Darboux type, may be compatible with the given equation; and, assuming the conditions satisfied, obtain the primitive.

Ex. 4. Obtain two equations of the second order, that are compatible with the equation

$$r=f(s),$$

in the form

$$\left.\begin{aligned} t-g(s)=\phi(y) \\ y+xf'(s)=\psi(s) \end{aligned}\right\},$$

where

$$g'(s)f'(s)=1;$$

and construct the primitive. (Goursat.)

Ex. 5. When a surface is referred to its minimal lines as parametric curves, each of the coordinates of any point on the surface satisfies the equation

$$rt-s^2-cqr-apt=b(\lambda-pq)-acpq,$$

where the arc on the surface is given by

$$ds^2=4\lambda\, dx\, dy,$$

p, q, r, s, t are the first and the second derivatives of any one of the coordinates, and where

$$a=\frac{\partial(\log\lambda)}{\partial x}, \quad c=\frac{\partial(\log\lambda)}{\partial y}, \quad b=2\frac{\partial^2(\log\lambda)}{\partial x\,\partial y}:$$

so that the equations of all surfaces deformable into a given surface are thus provided *.

Prove that the differential equation possesses no intermediate integral of the first order: and find the equation which must be satisfied by λ, if equations of the second order exist that are compatible with, but are algebraically independent of, the given equation.

EQUATIONS $f(r, s, t)=0$ INTEGRABLE BY DARBOUX'S METHOD.

266. In an interesting memoir †, De Boer discusses equations of the form

$$f(r, s, t)=0,$$

which admit two compatible equations of the second order derivable through the two subsidiary systems of equations in Darboux's method. The following discussion differs in form from that which is given in the memoir quoted.

Let the given equation $f=0$ be resolved with regard to one of its arguments, say r, so that it has the form

$$r+g(s, t)=0.$$

* Darboux, *Théorie générale des surfaces*, t. III, p. 261.

† *Arch. Néerl.*, t. XXVII (1894), pp. 355—412.

Writing

$$\frac{\partial g}{\partial s} = S, \quad \frac{\partial g}{\partial t} = T,$$

the quadratic, which has λ and μ for its roots, is

$$\kappa^2 - \kappa S + T = 0:$$

these quantities λ and μ are functions of s and t only. The equations for the determination of u are, in general,

$$\lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0,$$

$$\frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R}\frac{\partial u}{\partial r} - \mu \frac{Y}{T}\frac{\partial u}{\partial t} = 0.$$

In the present case,

$$X = 0, \quad Y = 0,$$

for f involves only r, s, t: also, if u contained r explicitly, that variable could be removed by substituting its value $-g(s, t)$. Hence we have equations for u in the form

$$\Delta_1(u) = \frac{\partial u}{\partial r} = 0,$$

$$\Delta_2(u) = \lambda \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0,$$

$$\Theta\,(u) = \frac{du}{dx} + \mu \frac{du}{dy} = 0.$$

Applying the Jacobian tests of coexistence, we must have

$$\Delta_3(u) = (\Delta_1, \Theta) = \frac{\partial u}{\partial p} = 0;$$

and then

$$\Delta_4(u) = (\Delta_3, \Theta) = \frac{\partial u}{\partial z} = 0.$$

Using $\Delta_3 = 0$ and $\Delta_4 = 0$, we can replace Θ by Θ', where

$$\Theta'(u) = \frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} + (s + \mu t)\frac{\partial u}{\partial q} = 0.$$

We have

$$(\Delta_1, \Theta') = 0, \quad (\Delta_1, \Delta_2) = 0, \quad (\Delta_1, \Delta_3) = 0, \quad (\Delta_1, \Delta_4) = 0;$$

$$(\Delta_2, \Delta_3) = 0, \quad (\Delta_2, \Delta_4) = 0, \quad (\Delta_3, \Delta_4) = 0;$$

$$(\Delta_3, \Theta') = 0, \quad (\Delta_4, \Theta') = 0;$$

and it remains to consider (Δ_2, Θ'). We have

$$(\Delta_2, \Theta') = \frac{\partial u}{\partial y} \Delta_2(\mu) + \{t\Delta_2(\mu) + \lambda - \mu\} \frac{\partial u}{\partial q}.$$

Let

$$t + \frac{\lambda - \mu}{\Delta_2(\mu)} = \theta;$$

and take

$$\Delta_5(u) = \frac{\partial u}{\partial y} + \theta \frac{\partial u}{\partial q} = 0,$$

$$\Delta_6(u) = \frac{\partial u}{\partial x} + (s + \mu t - \mu\theta) \frac{\partial u}{\partial q} = 0,$$

so that $(\Delta_2, \Theta') = 0$; and $\Theta' = 0$ can be omitted when $\Delta_5 = 0$ and $\Delta_6 = 0$ are retained. The complete Jacobian system is not to contain more than six members* if there is to be an equation of the type specified by Darboux: for the system must then possess two distinct integrals. We already have six equations, viz.

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0, \quad \Delta_4 = 0, \quad \Delta_5 = 0, \quad \Delta_6 = 0;$$

hence the equations

$$(\Delta_2, \Delta_5) = 0, \quad (\Delta_2, \Delta_6) = 0,$$

must be satisfied, all the other conditions for the system being actually satisfied. Consequently,

$$\Delta_2(\theta) = 0$$

from the former, and

$$(t - \theta)\Delta_2(\mu) - \mu\Delta_2(\theta) + \lambda - \mu = 0;$$

and both these conditions are satisfied by means of the single equation

$$\Delta_2(\theta) = 0,$$

that is,

$$\lambda \frac{\partial \theta}{\partial s} - \frac{\partial \theta}{\partial t} = 0,$$

or

$$\left(\lambda \frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\lambda - \mu}{\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t}}\right) = 1,$$

which is a necessary condition that the selected system should have an integral of the assigned type.

* This appears to differ from the earlier theory: the explanation is that $\Delta_1 = 0$ prevents the original equation from occurring as an integral.

Suppose this condition satisfied. As λ is a function of s and t only, let an integral of

$$\frac{ds}{\lambda} = dt$$

be given by

$$v = v(s, t) = \text{constant}.$$

The one integral of the Jacobian system is

$$v;$$

and another integral is given by

$$q - y\theta - x(s + \mu t - \mu\theta);$$

and therefore an equation that can coexist with the given equation is

$$q - y\theta - x(s + \mu t - \mu\theta) = \phi(v),$$

where ϕ is an arbitrary function.

In order that the same kind of equation, compatible with the original equation, may be provided by the alternative subsidiary system, it is obvious that the corresponding condition (obtained by the interchange of λ and μ) must be satisfied, that is,

$$\left(\mu\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\mu - \lambda}{\mu\frac{\partial\lambda}{\partial s} - \frac{\partial\lambda}{\partial t}}\right) = 1;$$

and then, if

$$\vartheta = t + \frac{\mu - \lambda}{\mu\frac{\partial\lambda}{\partial s} - \frac{\partial\lambda}{\partial t}},$$

the required equation to be associated with the given equation is

$$q - y\vartheta - x(s + \lambda t - \lambda\vartheta) = \psi(w),$$

where ψ is arbitrary, and where

$$w = w(s, t) = \text{constant}$$

is an integral of

$$\frac{ds}{\mu} = dt.$$

We now have three relations, theoretically expressing r, s, t in terms of the other variables in such a way that the equations

$$\left.\begin{aligned} dp &= r\,dx + s\,dy \\ dq &= s\,dx + t\,dy \\ dz &= p\,dx + q\,dy \end{aligned}\right\}$$

are a completely integrable system. In practice, and assuming the equation $f(r, s, t) = 0$ resolved with regard to r, it would obviously be convenient to make s and t (or v and w) the independent variables for the operative quadratures.

It therefore appears that the conditions

$$\left(\lambda \frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\lambda - \mu}{\lambda \dfrac{\partial \mu}{\partial s} - \dfrac{\partial \mu}{\partial t}}\right) = 1,$$

$$\left(\mu \frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\mu - \lambda}{\mu \dfrac{\partial \lambda}{\partial s} - \dfrac{\partial \lambda}{\partial t}}\right) = 1,$$

secure the existence of two equations of the specified type which can be associated with $r + g(s, t) = 0$.

267. In order to discuss the two preceding conditions to be satisfied by the function $g(s, t)$, and in order both to abbreviate the notation and to simplify it, we replace s and t temporarily by x and y. Derivatives of g with regard to its arguments will be denoted by $p, q, r, s, t, \alpha, \beta, \gamma, \delta$: derivatives of λ and of μ with regard to x and y will be denoted by $\lambda_1, \lambda_2, \ldots$, so that

$$\frac{\partial \lambda}{\partial x} = \lambda_1, \quad \frac{\partial^2 \lambda}{\partial x^2} = \lambda_{11}, \quad \frac{\partial \lambda}{\partial y} = \lambda_2, \quad \frac{\partial^2 \lambda}{\partial x \partial y} = \lambda_{12}, \ldots,$$

and similarly for μ.

The first of the two conditions is

$$\frac{\lambda\lambda_1 - \lambda_2}{\lambda\mu_1 - \mu_2} - \frac{\lambda - \mu}{(\lambda\mu_1 - \mu_2)^2}\{\lambda^2\mu_{11} - 2\lambda\mu_{12} + \mu_{22} + \mu_1(\lambda\lambda_1 - \lambda_2)\} = 2,$$

which is easily reduced to

$$(\lambda\lambda_1 - \lambda_2)(\mu\mu_1 - \mu_2) - 2(\lambda\mu_1 - \mu_2)^2 = (\lambda - \mu)(\lambda^2\mu_{11} - 2\lambda\mu_{12} + \mu_{22});$$

and the second of the two conditions similarly reduces to

$$(\mu\mu_1 - \mu_2)(\lambda\lambda_1 - \lambda_2) - 2(\mu\lambda_1 - \lambda_2)^2 = (\mu - \lambda)(\mu^2\lambda_{11} - 2\mu\lambda_{12} + \lambda_{22}).$$

Now

$$\lambda + \mu = S = p, \quad \lambda\mu = T = q;$$

hence

$$\lambda_1 + \mu_1 = r,$$

$$\lambda_2 + \mu_2 = s = \mu\lambda_1 + \lambda\mu_1,$$

$$t = \mu\lambda_2 + \lambda\mu_2,$$

and therefore

$$\left.\begin{aligned}(\lambda-\mu)\lambda_1 &= \lambda r - s\\ (\lambda-\mu)\mu_1 &= -\mu r + s\end{aligned}\right\}, \qquad \left.\begin{aligned}(\lambda-\mu)\lambda_2 &= \lambda s - t\\ (\lambda-\mu)\mu_2 &= -\mu s + t\end{aligned}\right\}.$$

As

$$\lambda\mu_1 - \mu_2 = -(\mu\lambda_1 - \lambda_2),$$

the two left-hand sides of the reduced conditions are the same: their common value is

$$-\frac{A}{(\lambda-\mu)^2},$$

where

$$\begin{aligned}A &= (\lambda^2 r - 2\lambda s + t)(\mu^2 r - 2\mu s + t) + 2\{\lambda\mu r - (\lambda+\mu)s + t\}^2\\ &= 3(qr - ps + t)^2 + (rt - s^2)(p^2 - 4q).\end{aligned}$$

Subtracting the two equations in their reduced forms (and assuming, as has been done throughout, that $\lambda - \mu$ is not zero), we have

$$\lambda^2\mu_{11} + \mu^2\lambda_{11} - 2(\lambda\mu_{12} + \mu\lambda_{12}) + \mu_{22} + \lambda_{22} = 0.$$

When we construct the symmetrical combinations in this equation and substitute, it is found (after some reduction) that this equation takes the form

$$q\alpha - p\beta + \gamma = 2\frac{\lambda_1 - \mu_1}{\lambda - \mu}(qr - ps + t).$$

A first integral of this equation can be at once obtained in the form

$$qr - ps + t = (\lambda - \mu)^2 Y,$$

where, so far, Y is an arbitrary function of y.

Again, adding the equations, we have

$$-\frac{2A}{(\lambda-\mu)^2} = (\lambda-\mu)\{\lambda^2\mu_{11} - \mu^2\lambda_{11} - 2(\lambda\mu_{12} - \mu\lambda_{12}) + \mu_{22} - \lambda_{22}\};$$

constructing the combinations on the right-hand side, substituting, and reducing, we find

$$-\frac{2A}{(\lambda-\mu)^2} = -p(q\alpha - p\beta + \gamma) + 2(q\beta - p\gamma + \delta) + 2U,$$

where

$$U = \frac{qr - ps + t}{(\lambda-\mu)^2}\{(p^2 - 2q)r - 2ps + 2t\}.$$

Noting that

$$q\alpha - p\beta + \gamma = \frac{\partial}{\partial x}(qr - ps + t),$$

$$q\beta - p\gamma + \delta = \frac{\partial}{\partial y}(qr - ps + t) - (rt - s^2),$$

substituting, and reducing, we find that the equation is satisfied, provided

$$Y^2 + Y' = 0,$$

and therefore

$$Y = \frac{1}{y + a},$$

where a is a constant. Also

$$(\lambda - \mu)^2 = p^2 - 4q;$$

and therefore the equation* for the determination of g is

$$qr - ps + t = \frac{p^2 - 4q}{y + a}.$$

The equations, constituting the primitive of this partial equation, have already been given (§ 265, Ex. 2). Taking account of the facts, that we are seeking equations of the form

$$r + g(s, t) = 0,$$

that in the differential equation thus obtained x and y have replaced s and t, and that g is the dependent variable which can therefore be replaced by $-r$, we infer that any differential equation of the second order, such as to admit of two equations of the second order compatible with itself and with one another, is given by the system

$$-\frac{1}{t + a} = f''(\alpha) + g''(\beta),$$

$$-\frac{s}{t + a} = \alpha f''(\alpha) - f'(\alpha) + \beta g''(\beta) - g'(\beta),$$

$$r - \frac{s^2}{t + a} = \alpha^2 f''(\alpha) - 2\alpha f'(\alpha) + 2f(\alpha) + \beta^2 g''(\beta) - 2\beta g'(\beta) + 2g(\beta).$$

Ex. 1. For the preceding equations, prove that

$$\alpha = x + \lambda y, \quad \beta = x + \mu y;$$

deduce the values of p and q, and integrate the equation.

* The case considered by Goursat, t. II, p. 132, is obtained by making Y vanish through an infinite value of a.

Ex. 2. In the preceding investigation, it has been assumed that the critical quadratic

$$\kappa^2 - \kappa S + T = 0$$

has unequal roots, so that there are two subsidiary systems. Discuss the case when the quadratic has equal roots.

Ex. 3. Determine the form of the function k, if the equation

$$r + k(t) = 0$$

is integrable by Darboux's method: and integrate the equation.

In this case,

$$S = 0, \quad T = k'(t);$$

and the equation giving λ and μ is

$$\kappa^2 + k'(t) = 0.$$

Thus

$$\lambda = -\mu,$$

and neither of them involves s. We can proceed, either from the general result just given, or from the original conditions in § 266. It is easy to see that the two conditions are equivalent to one only, viz.

$$\frac{\partial}{\partial t}\left(\frac{2\mu}{\frac{\partial\mu}{\partial t}}\right) = -1,$$

so that

$$\mu(t+a)^2 = 3b,$$

where a and b are constants. Hence

$$k'(t) = -\mu^2$$
$$= -\frac{9b^2}{(t+a)^4},$$

and therefore

$$k(t) = -c + \frac{3b^2}{(t+a)^3},$$

where c is an arbitrary constant. Thus the original differential equation is

$$(r+c)(t+a)^3 = 3b^2.$$

The general investigation gives assistance towards the construction of the primitive. We have

$$w = \text{constant}$$

as an integral of

$$\frac{ds}{\mu} - dt = 0,$$

that is, we can take

$$w = s + \frac{3b}{t+a};$$

and similarly, we have

$$v = \text{constant}$$

as an integral of

$$\frac{ds}{\lambda} - dt = 0,$$

that is, we can take

$$v = s - \frac{3b}{t+a}.$$

Hence

$$s = \tfrac{1}{2}(w+v), \quad \frac{3b}{t+a} = \tfrac{1}{2}(w-v);$$

and

$$r + c = \frac{3b^2}{(t+a)^3} = \frac{1}{72b}(w-v)^3.$$

Again,

$$\theta = t + \frac{\lambda - \mu}{\lambda \dfrac{\partial \mu}{\partial s} - \dfrac{\partial \mu}{\partial t}} = t + \frac{2\mu}{\dfrac{\partial \mu}{\partial t}} = -a;$$

and, similarly,

$$\vartheta = -a.$$

The two integral equations that can be associated with the given equation are

$$q + ay - xw = V'',$$

$$q + ay - xv = W'',$$

where V and W are arbitrary functions of v and of w respectively; and therefore

$$x = \frac{V'' - W''}{v - w},$$

$$q + ay = \frac{vV'' - wW''}{v - w}.$$

Also,

$$dq = s\,dx + t\,dy,$$

so that

$$dy = \frac{1}{t+a}\{d(q+ay) - s\,dx\},$$

which, on substitution and reduction, gives

$$y = \frac{1}{12b}\{(w-v)(V'' + W'') + 2(V' - W')\}.$$

Further,

$$dp = r\,dx + s\,dy,$$

so that

$$d(p - rx - sy) = -x\,dr - y\,ds;$$

on substitution and reduction, we have

$$p - rx - sy = \frac{1}{12b}\{(v-w)(V' + W') - 2V + 2W\}.$$

We thus have x, y, p, q expressed in terms of v and w; another quadrature, effected on

$$dz = p\,dx + q\,dy$$

after substitution, gives the value of z.

The result agrees with the result given by De Boer.

Ex. 4. Discuss the case when $b = 0$. (De Boer.)

Compatible Equations of Higher Orders derivable by Darboux's Method.

268. Should it be found that neither of the subsidiary systems, constructed with a view to the formation of an equation of the second order to be associated with the original equation, can provide such an equation, then we proceed to use the method for the construction of an equation or equations of higher order which can be associated with

$$f = f(x, y, z, p, q, r, s, t) = 0.$$

As the present argument follows the earlier argument closely, here it will be made quite brief: and it will be restricted to the consideration of equations of the third order.

The derivatives of z of the third order will, as before, be denoted by $\alpha, \beta, \gamma, \delta$: those of the fourth order by $\pi, \rho, \sigma, \tau, \upsilon$, where

$$\pi = \frac{\partial^4 z}{\partial x^4}, \quad \rho = \frac{\partial^4 z}{\partial x^3 \partial y}, \quad \sigma = \frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \tau = \frac{\partial^4 z}{\partial x \partial y^3}, \quad \upsilon = \frac{\partial^4 z}{\partial y^4}.$$

We have

$$0 = \frac{d^2 f}{dx^2} + \frac{\partial f}{\partial r}\pi + \frac{\partial f}{\partial s}\rho + \frac{\partial f}{\partial t}\sigma,$$

$$0 = \frac{d^2 f}{dx\,dy} + \frac{\partial f}{\partial r}\rho + \frac{\partial f}{\partial s}\sigma + \frac{\partial f}{\partial t}\tau,$$

$$0 = \frac{d^2 f}{dy^2} + \frac{\partial f}{\partial r}\sigma + \frac{\partial f}{\partial s}\tau + \frac{\partial f}{\partial t}\upsilon,$$

where $\frac{d^2 f}{dx^2}$ includes the complete second derivative of f with regard to x except the terms involving the fourth derivatives of z, and similarly for $\frac{d^2 f}{dx\,dy}, \frac{d^2 f}{dy^2}$. Let

$$u = u(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0$$

be an equation of the third order which is compatible with the given equation of the second order: then, taking

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} + \alpha\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial s} + \gamma\frac{\partial}{\partial t},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q} + \beta\frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s} + \delta\frac{\partial}{\partial t},$$

we have

$$0=\frac{du}{dx}+\frac{\partial u}{\partial\alpha}\pi+\frac{\partial u}{\partial\beta}\rho+\frac{\partial u}{\partial\gamma}\sigma+\frac{\partial u}{\partial\delta}\tau,$$

$$0=\frac{du}{dy}+\frac{\partial u}{\partial\alpha}\rho+\frac{\partial u}{\partial\beta}\sigma+\frac{\partial u}{\partial\gamma}\tau+\frac{\partial u}{\partial\delta}\upsilon.$$

Thus there are five equations for the determination of values of $\pi, \rho, \sigma, \tau, \upsilon$; as before, because

$$u=0, \quad f=0,$$

are compatible with one another and are not independent of one another, the values provided for the five derivatives by the five derived equations must not be determinate: and therefore

$$\left\|\begin{matrix} \frac{du}{dx}, & \frac{\partial u}{\partial\alpha}, & \frac{\partial u}{\partial\beta}, & \frac{\partial u}{\partial\gamma}, & \frac{\partial u}{\partial\delta}, & 0 \\ \frac{du}{dy}, & 0, & \frac{\partial u}{\partial\alpha}, & \frac{\partial u}{\partial\beta}, & \frac{\partial u}{\partial\gamma}, & \frac{\partial u}{\partial\delta} \\ \frac{d^2f}{dx^2}, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t}, & 0, & 0 \\ \frac{d^2f}{dxdy}, & 0, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t}, & 0 \\ \frac{d^2f}{dy^2}, & 0, & 0, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \end{matrix}\right\|=0.$$

These are equivalent to two independent relations among the derivatives of u.

The two independent relations are resolved into equations that are linear in the derivatives of u; and the process is the same as in § 261, leading here also in general to a couple of subsidiary systems. Let λ and μ be the roots of the critical quadratic

$$\frac{\partial f}{\partial t}-\theta\frac{\partial f}{\partial s}+\theta^2\frac{\partial f}{\partial r}=0;$$

when the equations for u are resolved, we have

$$\left.\begin{aligned} &\frac{\partial u}{\partial\delta}-\lambda\frac{\partial u}{\partial\gamma}+\lambda^2\frac{\partial u}{\partial\beta}-\lambda^3\frac{\partial u}{\partial\alpha}=0 \\ &\frac{du}{dx}+\mu\frac{du}{dy}-\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial f}{\partial r}}\frac{d^2f}{dx^2}-\frac{\frac{\partial u}{\partial\beta}-\lambda\frac{\partial u}{\partial\alpha}}{\frac{\partial f}{\partial r}}\frac{d^2f}{dxdy}-\mu\frac{\frac{\partial u}{\partial\delta}}{\frac{\partial f}{\partial t}}\frac{d^2f}{dy^2}=0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned}&\frac{\partial u}{\partial \delta}-\mu\frac{\partial u}{\partial \gamma}+\mu^2\frac{\partial u}{\partial \beta}-\mu^3\frac{\partial u}{\partial \alpha}=0\\ &\frac{du}{dx}+\lambda\frac{du}{dy}-\frac{\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}}\frac{d^2f}{dx^2}-\frac{\dfrac{\partial u}{\partial \beta}-\mu\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}}\frac{d^2f}{dxdy}-\lambda\frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial t}}\frac{d^2f}{dy^2}=0\end{aligned}\right\},$$

as the two subsidiary systems for the determination of the quantity u. The form of the second equation in each system can be modified.

If the method is to be effective in the sense designed by Darboux, the first subsidiary system must have two independent integrals u_1 and u_2, and the second subsidiary system must likewise have two independent integrals v_1 and v_2. When these requirements are satisfied, then

$$\phi(u_1, u_2)=0, \quad \psi(v_1, v_2)=0,$$

are two equations of the third order compatible with

$$f=0,$$

whatever be the arbitrary functions ϕ and ψ. Also, we have

$$\frac{df}{dx}+\alpha\frac{\partial f}{\partial r}+\beta\frac{\partial f}{\partial s}+\gamma\frac{\partial f}{\partial t}=0,$$

$$\frac{df}{dy}+\beta\frac{\partial f}{\partial r}+\gamma\frac{\partial f}{\partial s}+\delta\frac{\partial f}{\partial t}=0;$$

these two equations, together with $\phi=0$ and $\psi=0$, suffice for the determination of $\alpha, \beta, \gamma, \delta$, in terms of the variables that occur in f. Their values are substituted in the first three relations of the set

$$\left.\begin{aligned}dr&=\alpha dx+\beta dy\\ ds&=\beta dx+\gamma dy\\ dt&=\gamma dx+\delta dy\\ dp&=r dx+s dy\\ dq&=s dx+t dy\\ dz&=p dx+q dy\end{aligned}\right\}.$$

The set then becomes an exactly integrable system: quadratures lead to the primitive, which obviously will contain two arbitrary functions. And, as before, it may be convenient to change the independent variables in the quadratures: thus it may be a practical shortening of the calculations to select u_2 and v_2 for this purpose.

Just as in the case of the Monge method and the Boole method for the construction of an intermediate integral, when the integration of a set of equations in differential elements was equivalent to the integration of a system of equations in differential coefficients of the first order, and (§§ 259, 264) was similarly the case in the construction of a compatible equation of the second order, so here also it is possible to construct a compatible equation of the third order by means of a set of equations in differential elements. The integration of the system of equations

$$\left.\begin{aligned}&\frac{\partial u}{\partial \delta} - \lambda\frac{\partial u}{\partial \gamma} + \lambda^2\frac{\partial u}{\partial \beta} - \lambda^3\frac{\partial u}{\partial \alpha} = 0\\ &\frac{du}{dx} + \mu\frac{du}{dy} - \frac{\frac{\partial u}{\partial \alpha}}{\frac{\partial f}{\partial r}}\frac{d^2 f}{dx^2} - \frac{\frac{\partial u}{\partial \beta} - \lambda\frac{\partial u}{\partial \alpha}}{\frac{\partial f}{\partial r}}\frac{d^2 f}{dxdy} - \mu\frac{\frac{\partial u}{\partial \delta}}{\frac{\partial f}{\partial t}}\frac{d^2 f}{dy^2} = 0\end{aligned}\right\},$$

is equivalent to the integration of the set of equations

$$\left.\begin{aligned}\frac{dx}{1} = \frac{dy}{\mu} &= \frac{dz}{p+\mu q} = \frac{dp}{r+\mu s} = \frac{dq}{s+\mu t}\\ &= \frac{dr}{\alpha+\mu\beta} = \frac{ds}{\beta+\mu\gamma} = \frac{dt}{\gamma+\mu\delta}\\ &= \frac{d\alpha + \lambda d\beta}{-\frac{\frac{d^2 f}{dx^2}}{\frac{\partial f}{\partial r}}} = \frac{d\beta + \lambda d\gamma}{-\frac{\frac{d^2 f}{dxdy}}{\frac{\partial f}{\partial r}}} = \frac{d\gamma + \lambda d\delta}{-\frac{\frac{d^2 f}{dy^2}}{\frac{\partial f}{\partial r}}}\end{aligned}\right\},$$

λ and μ being the roots of the critical quadratic: and similarly for the other system.

The equations in the differential elements here have their obviously simplest form. The equations, which involve the derivatives of u, are capable of a variety of forms: in particular, it is easy to verify that the second equation as given is equivalent to (and can be replaced by) the equation

$$\begin{aligned}\left(\frac{du}{dx} + \mu\frac{du}{dy}\right)\frac{\partial f}{\partial r} = \left(\frac{d^2 f}{dx^2} - \lambda\frac{d^2 f}{dxdy} + \lambda^2\frac{d^2 f}{dy^2}\right)\frac{\partial u}{\partial \alpha}&\\ + \left(\frac{d^2 f}{dxdy} - \lambda\frac{d^2 f}{dy^2}\right)\frac{\partial u}{\partial \beta} + \frac{d^2 f}{dy^2}\frac{\partial u}{\partial \gamma}&,\end{aligned}$$

which, though containing more terms than the other form, is often more convenient in practice.

The subsidiary system of the two initial equations for u contains twelve independent variables. It must be satisfied by three integrals

$$f,\quad \frac{df}{dx}+\alpha\frac{\partial f}{\partial r}+\beta\frac{\partial f}{\partial s}+\gamma\frac{\partial f}{\partial t},\quad \frac{df}{dy}+\beta\frac{\partial f}{\partial r}+\gamma\frac{\partial f}{\partial s}+\frac{\partial f}{\partial t};$$

if it is to possess an integral of the type required by Darboux, it must possess two integrals independent of the three just mentioned. Accordingly, when it is made a complete Jacobian system, that system will consist of seven members.

Ex. 1. Consider the equation

$$r-t-4\frac{p}{x}=0,$$

which has no intermediate integral and does not admit of a compatible equation of the second order of Darboux's type. It can be integrated, after transformation of the independent variables, by the Laplace method; but here it will be considered as an illustration of the Darboux method so as, if possible, to obtain equations of the third order with which it is compatible.

The critical quadratic is

$$\theta^2-1=0,$$

so that there are two subsidiary sets of equations, given by the two assignments of the roots. We have

$$\begin{aligned}\frac{\partial f}{\partial r}&=1,\\ \frac{d^2f}{dx^2}&=-4\frac{a}{x}+8\frac{r}{x^2}-8\frac{p}{x^3},\\ \frac{d^2f}{dx\,dy}&=-4\frac{\beta}{x}+4\frac{s}{x^2},\\ \frac{d^2f}{dy^2}&=-4\frac{\gamma}{x}.\end{aligned}$$

The subsidiary system, given by $\lambda=1$ and $\mu=-1$, is

$$\theta_1(u)=\frac{\partial u}{\partial a}-\frac{\partial u}{\partial \beta}+\frac{\partial u}{\partial \gamma}-\frac{\partial u}{\partial \delta}=0,$$

$$\theta_2(u)=\frac{du}{dx}-\frac{du}{dy}+\left(4\frac{a-\beta+\gamma}{x}-4\frac{2r-s}{x^2}+8\frac{p}{x^3}\right)\frac{\partial u}{\partial a}$$
$$+\left(4\frac{\beta-\gamma}{x}-4\frac{s}{x^2}\right)\frac{\partial u}{\partial \beta}+4\frac{r}{x}\frac{\partial u}{\partial \gamma}=0.$$

We have

$$(\theta_1,\theta_2)=\frac{4}{x}\theta_3(u)=0,$$

where
$$\theta_3(u)=3\frac{\partial u}{\partial a}-2\frac{\partial u}{\partial \beta}+\frac{\partial u}{\partial \gamma}+\tfrac{1}{2}x\left(\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)=0.$$
We also have
$$(\theta_3,\ \theta_2)=18\frac{\partial u}{\partial a}-10\frac{\partial u}{\partial \beta}+4\frac{\partial u}{\partial \gamma}+x\left(\tfrac{9}{2}\frac{\partial u}{\partial r}-\tfrac{7}{2}\frac{\partial u}{\partial s}+\tfrac{1}{2}\frac{\partial u}{\partial t}\right)+x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0.$$
From the last equation, combined with
$$\theta_1(u)=0,\quad \theta_3(u)=0,$$
we can express $\frac{\partial u}{\partial a}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ in terms of the other derivatives: the results are
$$\Delta_1(u)=\frac{\partial u}{\partial a}-\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(3\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-5\frac{\partial u}{\partial t}\right)+\tfrac{1}{2}x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0,$$
$$\Delta_2(u)=\frac{\partial u}{\partial \beta}-3\frac{\partial u}{\partial \delta}+x\left(\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-3\frac{\partial u}{\partial t}\right)+x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0,$$
$$\Delta_3(u)=\frac{\partial u}{\partial \gamma}-3\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(\frac{\partial u}{\partial r}+3\frac{\partial u}{\partial s}-7\frac{\partial u}{\partial t}\right)+\tfrac{1}{2}x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0.$$
Let these values of $\frac{\partial u}{\partial a}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ be substituted in $\theta_2(u)$, and let the resulting form be denoted by $\theta_2'(u)$: then
$$\theta_2'(u)=0.$$
Now
$$(\Delta_1,\ \Delta_2)=0,\quad (\Delta_1,\ \Delta_3)=0,\quad (\Delta_2,\ \Delta_3)=0;$$
and
$$(\Delta_1,\ \theta_2')=-\theta_4,\quad (\Delta_2,\ \theta_2')=-2\theta_4,\quad (\Delta_3,\ \theta_2')=-\theta_4,$$
where
$$\theta_4(u)=\frac{\partial u}{\partial r}-3\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial p}-3x\frac{\partial u}{\partial q}-x^2\frac{\partial u}{\partial z}=0.$$
Further,
$$(\theta_4,\ \theta_2')=6\theta_5,$$
where
$$\theta_5(u)=\frac{\partial u}{\partial q}+x\frac{\partial u}{\partial z}=0;$$
and
$$(\theta_5,\ \theta_2')=-\frac{\partial u}{\partial z}=0.$$

Combining these equations so as to have simple forms, we take them to be
$$\nabla_1(u)=\frac{\partial u}{\partial z}=0,$$
$$\nabla_2(u)=\frac{\partial u}{\partial q}=0,$$
$$\nabla_3(u)=\frac{\partial u}{\partial r}-3\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial p}=0,$$
$$\nabla_4(u)=\frac{\partial u}{\partial a}-\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)=0,$$
$$\nabla_5(u)=\frac{\partial u}{\partial \beta}-3\frac{\partial u}{\partial \delta}+x\frac{\partial u}{\partial s}=0,$$
$$\nabla_6(u)=\frac{\partial u}{\partial \gamma}-3\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(-\frac{\partial u}{\partial r}+3\frac{\partial u}{\partial s}-\frac{\partial u}{\partial t}\right)=0,$$

$$\nabla_7(u) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\left\{-\alpha+\beta+\gamma-\delta+\frac{1}{x}(5r-4s)-\frac{2}{x^2}p\right\}$$
$$+\frac{\partial u}{\partial r}\left(\frac{r}{x}-\frac{2p}{x^2}\right)+\frac{\partial u}{\partial s}\left\{-\alpha-2\beta-\gamma+\frac{1}{x}(2r+3s)-\frac{2}{x^2}p\right\}$$
$$+\frac{\partial u}{\partial \delta}\left\{\frac{4}{x}(\alpha+2\beta+\gamma)-\frac{8}{x^2}(r+s)+\frac{8}{x^3}p\right\}=0.$$

This is a complete Jacobian system: hence it possesses five simultaneous integrals. Three of these five are known, being

$$r-t-4\frac{p}{x},$$
$$\alpha-\gamma-4\frac{r}{x}+4\frac{p}{x^2},$$
$$\beta-\delta-4\frac{s}{x},$$

all of which vanish. Two others, when these results are used, are found to be

$$\frac{1}{x^2}(\alpha+3\beta+3\gamma+\delta), \quad x+y;$$

hence the subsidiary system provides an equation

$$\alpha+3\beta+3\gamma+\delta=x^2\phi(x+y),$$

ϕ being an arbitrary function: this equation is compatible with the original equation.

Similarly, the subsidiary system, given by taking $\lambda=-1$ and $\mu=1$, provides an equation

$$\alpha-3\beta+3\gamma-\delta=x^2\psi(y-x),$$

ψ being an arbitrary function: this equation is compatible with the original equation.

These two equations, together with

$$\alpha-\gamma=4\frac{r}{x}-4\frac{p}{x^2}, \quad \beta-\delta=4\frac{s}{x},$$

give values of α, β, γ, δ, in terms of the other quantities. Substituting them in the differential relations

$$dr=\alpha\,dx+\beta\,dy, \quad ds=\beta\,dx+\gamma\,dy, \quad dt=\gamma\,dx+\delta\,dy,$$

effecting the quadratures, and substituting the deduced values of r and s in

$$dp=r\,dx+s\,dy, \quad dq=s\,dx+t\,dy,$$

and, lastly, substituting the deduced values of p and q in

$$dz=p\,dx+q\,dy,$$

a quadrature leads to the value

$$z=x^2(f''-g'')-3x(f'+g')+3(f-g),$$

where

$$\phi(x+y)=8\frac{\partial^5 f(x+y)}{\partial x^5}, \quad \psi(y-x)=8\frac{\partial^5 g(y-x)}{\partial y^5}.$$

Ex. 2. Integrate the equations:—

$$\text{(i)} \quad r-t-\frac{1}{x}(p-q)=0\,;$$

$$\text{(ii)} \quad x^2r+2xys+y^2(1-x^2)t=0\,;$$

neither of which possesses an intermediate integral.

Ex. 3. Obtain an integral of the equation

$$u^2r-t=0,$$

where u is a function of x and y, satisfying a relation

$$uy+x=f(u),$$

and f is any functional form. (Winckler.)

Ex. 4. Solve the equation

$$r+t+\frac{8z}{(1+x^2+y^2)^2}=0,$$

obtaining the primitive in the form

$$z=f'(u)+g'(v)-\frac{2}{1+x^2+y^2}\{vf(u)+ug(v)\},$$

where $u=x+iy$, $v=x-iy$. (Schwarz.)

Ex. 5. Obtain two equations of the third order, which are compatible with (but are not mere derivatives of) the equation

$$x^{\frac{4}{3}}r-t=0. \qquad \text{(Winckler.)}$$

Ex. 6. Shew that the equation

$$r-t+zf(x)=0$$

possesses two compatible equations of the second order, if

$$\frac{d}{dx}\left(\frac{1}{f}\frac{df}{dx}\right)+f=0\,;$$

and find the equation that must be satisfied by f, if there are two compatible equations of the third order.

Ex. 7. When an equation

$$r+g(s,\ t)=0$$

admits two equations of the third order, compatible with itself and algebraically independent of its derivatives with regard to x and to y, in the forms

$$\phi(u_1,\ u_2)=0, \quad \psi(v_1,\ v_2)=0,$$

the quantities u_1 and u_2 are integrals of one subsidiary system, and the quantities v_1 and v_2 are integrals of the other subsidiary system.

Obtain the conditions, analogous to those in § 266, in order that each of the subsidiary systems may possess two integrals which are not immediately derivable from the given equation; and, by means of these equations, find the suitable forms of the original equation.

In particular, obtain the equations

$$r=f(t),$$

which have the required property.

CHAPTER XIX.

Generalisation of Integrals.

The present chapter is devoted to the problem of connecting the general primitive of an equation of the second order with a primitive, that is either complete or incomplete in the aggregate of parameters which it contains. The problem is of the utmost importance in the case of equations of the first order: on that account, Lagrange attempted it for an equation of the second order, using his method of the variation of parameters for this purpose. Having included too many parameters, he did not attain to a satisfactory result except in special cases; consequently, he set the method aside. Later, Imschenetsky limited the number of parameters and made less restricted conditions in order to secure the generalisation: he obtains a generalisation that is important and, within the limits of analysis which can be effected, is practicable.

The original equation and the generalising equation are, in many instances, connected with one another by means of contact transformations.

269. It was seen in the case of partial differential equations of the first order that, when a complete integral is known, it can be adapted to the derivation of other classes of integrals: the method used for this purpose is the variation of parameters. In the case of partial differential equations of the second order, integrals have been obtained containing a number of arbitrary constants; thus there were complete integrals (§ 180) containing five arbitrary constants, and there were integrals (§ 241) for special types of equations containing three arbitrary constants; and other instances have occurred. Such integrals are not necessarily particular forms of the general integral: and it is natural to inquire whether the method of variation of parameters, applied to such integrals, will lead to the general integral or to any other classes of integrals. For the purposes of the present discussion, we shall assume that there are only two independent variables.

Following Lagrange*, by whom the question was first considered, we begin with a complete integral in the form

$$f(x, y, z, a_1, a_2, a_3, a_4, a_5) = 0,$$

where a_1, a_2, a_3, a_4, a_5 are arbitrary constants: the elimination of these five constants, among the six equations

$$f = 0, \quad p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} = 0, \quad q\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} = 0,$$

$$r\frac{\partial f}{\partial z} + p^2\frac{\partial^2 f}{\partial z^2} + 2p\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial x^2} = 0,$$

$$s\frac{\partial f}{\partial z} + pq\frac{\partial^2 f}{\partial z^2} + q\frac{\partial^2 f}{\partial x \partial z} + p\frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial x \partial y} = 0,$$

$$t\frac{\partial f}{\partial z} + q^2\frac{\partial^2 f}{\partial z^2} + 2q\frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial y^2} = 0,$$

leads in general† to a single differential equation

$$\phi(x, y, z, p, q, r, s, t) = 0,$$

which accordingly has $f = 0$ for a complete integral.

On the analogy of partial differential equations of the first order, we attempt to generalise the complete integral by making the five parameters functions of the variables: this is done most directly by making a_1 and a_2 functions of x and y, and a_3, a_4, a_5 functions of a_1 and a_2. The functions are to be determined by the condition, that the forms of z, p, q, r, s (and therefore, owing to the differential equation, the form of t also) are left unaltered by the change: the passage from $f = 0$ to the differential equation will then be the same as before. Writing

$$\left.\begin{aligned} \frac{d}{da_1} &= \frac{\partial}{\partial a_1} + \frac{\partial a_3}{\partial a_1}\frac{\partial}{\partial a_3} + \frac{\partial a_4}{\partial a_1}\frac{\partial}{\partial a_4} + \frac{\partial a_5}{\partial a_1}\frac{\partial}{\partial a_5} \\ \frac{d}{da_2} &= \frac{\partial}{\partial a_2} + \frac{\partial a_3}{\partial a_2}\frac{\partial}{\partial a_3} + \frac{\partial a_4}{\partial a_2}\frac{\partial}{\partial a_4} + \frac{\partial a_5}{\partial a_2}\frac{\partial}{\partial a_5} \end{aligned}\right\},$$

* *Œuvres complètes*, t. IV, pp. 5—108.

† The conditions are the non-evanescence of certain Jacobians of the left-hand members of the six equations with regard to the constants: the detailed examination of their forms can be omitted, as not pertinent to the immediate discussion.

we see that the values of p and of q are unaltered if

$$\frac{\partial a_1}{\partial x}\frac{df}{da_1}+\frac{\partial a_2}{\partial x}\frac{df}{da_2}=0,$$

$$\frac{\partial a_1}{\partial y}\frac{df}{da_1}+\frac{\partial a_2}{\partial y}\frac{df}{da_2}=0;$$

and therefore, as no limitation upon full generality is imposed by assuming a_1 and a_2 to be independent functions of x and y, we have

$$\frac{df}{da_1}=0,\quad \frac{df}{da_2}=0.$$

When these are satisfied, the values of p and q are given by

$$p\frac{\partial f}{\partial z}+\frac{\partial f}{\partial x}=0,\quad q\frac{\partial f}{\partial z}+\frac{\partial f}{\partial y}=0.$$

Differentiating the first of these equations with respect to x and to y, and introducing the condition that the second derivatives of z are to be the same as before, we find

$$\left(p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}\right)\frac{\partial a_1}{\partial x}+\left(p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}\right)\frac{\partial a_2}{\partial x}=0,$$

$$\left(p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}\right)\frac{\partial a_1}{\partial y}+\left(p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}\right)\frac{\partial a_2}{\partial y}=0;$$

consequently,

$$p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}=0,\quad p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}=0.$$

Similarly treating the second of the equations, we have

$$q\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial y}=0,\quad q\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial y}=0.$$

But the equation

$$\frac{df}{da_1}=0$$

is satisfied identically when the proper values of z and the parameters are substituted: hence

$$p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}+\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2 f}{da_1\,da_2}\frac{\partial a_2}{\partial x}=0,$$

$$q\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial y}+\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial y}+\frac{d^2 f}{da_1\,da_2}\frac{\partial a_2}{\partial y}=0,$$

and therefore

$$\frac{d^2f}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2f}{da_1da_2}\frac{\partial a_2}{\partial x}=0,$$

$$\frac{d^2f}{da_1^2}\frac{\partial a_1}{\partial y}+\frac{d^2f}{da_1da_2}\frac{\partial a_2}{\partial y}=0.$$

These equations give

$$\frac{d^2f}{da_1^2}=0,\quad \frac{d^2f}{da_1da_2}=0,$$

because a_1 and a_2 are independent functions of x and y. Similarly, the equation

$$\frac{df}{da_2}=0$$

leads to the relations

$$\frac{d^2f}{da_1da_2}=0,\quad \frac{d^2f}{da_2^2}=0.$$

We thus have six equations in all, viz.

$$f=0,$$

$$\frac{df}{da_1}=0,\quad \frac{df}{da_2}=0,$$

$$\frac{d^2f}{da_1^2}=0,\quad \frac{d^2f}{da_1da_2}=0,\quad \frac{d^2f}{da_2^2}=0,$$

which are free from p, q, r, s, t; the second and the third contain first derivatives of a_3, a_4, a_5 with regard to a_1 and a_2, and the last three contain second derivatives of the same quantities. Now let x, y, z be eliminated among the six equations: the resulting eliminant is composed of three simultaneous equations of the second order in three dependent variables. The problem, thus provided for the determination of a_3, a_4, a_5 in terms of a_1 and a_2, is more difficult than the original problem, which is the solution of a single equation of the second order in a single dependent variable. Consequently, the derivation of further integrals from the complete integral cannot be regarded as generally possible if attempted by the indicated process.

It is possible that the method may be effective in particular cases: but the course of the analysis must be different. Thus Lagrange takes the equation

$$t=m,$$

where m is a constant (which can be made unity without loss of generality). Obviously the equation

$$z = a_1 + a_2 x + a_3 y + a_4 xy + a_5 (x^2 + my^2),$$

provides an integral ; and then

$$p = a_2 + a_4 y + 2a_5 x,$$
$$q = a_3 + a_4 x + 2ma_5 y.$$

Varying the parameters, and keeping the values of r, s, t unaltered, we have

$$0 = da_1 + x\,da_2 + y\,da_3 + xy\,da_4 + (x^2 + my^2)\,da_5,$$
$$0 = da_2 + y\,da_4 + 2x\,da_5,$$
$$0 = da_3 + x\,da_4 + 2my\,da_5.$$

The last two can be replaced by

$$da_3 + m^{\frac{1}{2}} da_2 + (x + m^{\frac{1}{2}} y)(da_4 + 2m^{\frac{1}{2}} da_5) = 0,$$
$$da_3 - m^{\frac{1}{2}} da_2 + (x - m^{\frac{1}{2}} y)(da_5 - 2m^{\frac{1}{2}} da_5) = 0 ;$$

and the first of them can then be regarded as giving da_1. The first modified equation shews that $da_3 + m^{\frac{1}{2}} da_2$ and $da_4 + 2m^{\frac{1}{2}} da_5$ vanish together, so that $a_3 + m^{\frac{1}{2}} a_2$ and $a_4 + 2m^{\frac{1}{2}} a_5$ are constant together: hence, taking account of their generally variable values, we can write

$$a_3 + m^{\frac{1}{2}} a_2 = \phi(a_4 + 2m^{\frac{1}{2}} a_5),$$

where ϕ is any functional form ; and then

$$x + m^{\frac{1}{2}} y + \phi'(a_4 + 2m^{\frac{1}{2}} a_5) = 0.$$

Similarly, the second modified equation leads to the relations

$$a_3 - m^{\frac{1}{2}} a_2 = \psi(a_4 - 2m^{\frac{1}{2}} a_5),$$
$$x - m^{\frac{1}{2}} y + \psi'(a_4 - 2m^{\frac{1}{2}} a_5) = 0,$$

where ψ is any functional form. Writing

$$x + m^{\frac{1}{2}} y = u, \quad x - m^{\frac{1}{2}} y = v,$$

and inverting the functional forms ϕ' and ψ', we have

$$a_4 + 2m^{\frac{1}{2}} a_5 = g(u), \quad a_4 - 2m^{\frac{1}{2}} a_5 = h(v) ;$$

and then

$$da_3 + m^{\frac{1}{2}} da_2 = -ug'(u)\,du,$$

so that

$$a_3 + m^{\frac{1}{2}} a_2 = -\int ug'(u)\,du.$$

Similarly,

$$a_3 - m^{\frac{1}{2}} a_2 = -\int vh'(v)\,dv.$$

For α_1, we have

$$-d\alpha_1 = -\frac{1}{4m^{\frac{1}{2}}}(u+v)\{ug'(u)\,du - vh'(v)\,dv\}$$
$$-\frac{1}{4m^{\frac{1}{2}}}(u-v)\{ug'(u)\,du + vh'(v)\,dv\}$$
$$+\frac{1}{8m^{\frac{1}{2}}}(u^2-v^2)\{g'(u)\,du + h'(v)\,dv\}$$
$$+\frac{1}{8m^{\frac{1}{2}}}(u^2+v^2)\{g'(u)\,du - h'(v)\,dv\}$$
$$= -\frac{1}{4m^{\frac{1}{2}}}\{u^2g'(u)\,du - v^2h'(v)\,dv\},$$

so that

$$\alpha_1 = \frac{1}{4m^{\frac{1}{2}}}\int u^2g'(u)\,du - \frac{1}{4m^{\frac{1}{2}}}\int v^2h'(v)\,dv.$$

With the values of α_1, α_2, α_3, α_4, α_5 thus obtained, z becomes

$$z = \frac{1}{4m^{\frac{1}{2}}}\left\{\int u^2g'(u)\,du - 2u\int ug'(u)\,du + u^2g(u)\right\}$$
$$-\frac{1}{4m^{\frac{1}{2}}}\left\{\int v^2h'(v)\,dv - 2v\int vh'(v)\,dv + v^2h(v)\right\};$$

writing

$$g(u) = 2m^{\frac{1}{2}}G'''(u), \quad h(v) = -2m^{\frac{1}{2}}H''(v),$$

and effecting the quadratures, we find

$$z = G(u) + H(v)$$
$$= G(x+m^{\frac{1}{2}}y) + H(x-m^{\frac{1}{2}}y).$$

All the conditions are satisfied by keeping ϕ and ψ arbitrary: hence g and h are arbitrary, and therefore also G and H are arbitrary functions.

IMSCHENETSKY'S GENERALISATION.

270. In the preceding example, we have obtained the general integral; but it is manifest that, in the process, the effectiveness depends on the peculiar simplicity of the equations. In general, as already stated, the method of variation of parameters, when applied to equations of the second order, requires the solution of a problem distinctly more difficult than the original problem: on this account, Lagrange described* the method as more curious than useful.

* *L.c.*, p. 101.

Bour applied the method to a number of cases in which the difficulties were overcome: and he placed on record* his opinion that the method would yet be developed. It was reserved for Imschenetsky† to achieve a real generalisation of an integral by the Lagrangian method of variation of parameters, the equation being of the form considered by Monge, Ampère, and Boole.

The real difficulty in the generalisation, which was attempted by Lagrange, lies in the necessity of determining three out of five parameters by means of partial equations of the second order. In the generalisation which was achieved by Imschenetsky, what is required is the determination of one parameter in terms of other two by means of a single equation of the second order: but, instead of using an integral (of the type called complete) involving five arbitrary parameters, he makes an integral, which involves three such parameters, the foundation of the structure of other integrals. It is of course no longer possible to assume that, in the variation of the parameters, each of the derivatives p, q, r, s, t remains unaltered in form: for the immediate purpose, and in the absence of assigned initial conditions, it is sufficient that the partial equation of the second order shall be satisfied. This is precisely the requirement which, in the last resort, is adopted in Imschenetsky's method, the forms of p and q being kept unaltered.

Accordingly, let it be assumed that an integral of the equation of the second order has been obtained in a form

$$z = f(x, y, a, b, c),$$

involving three arbitrary constants: the process by which the integral has been obtained is immaterial. When this value of z is substituted in the equation, the latter is satisfied identically.

Now let a and b be chosen to be independent functions of x and y, subject to the condition that p and q have the same forms as when a and b are parametric; and let c then be chosen such a function of x and y (therefore, also, of a and b) that the differential equation is satisfied. When we write

$$\frac{df}{da} = \frac{\partial f}{\partial a} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial a}, \quad \frac{df}{db} = \frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial b},$$

* *Journ. de l'Éc. Polyt.*, Cah. XXXIX (1862), p. 191.

† *Grunert's Archiv*, t. LIV (1872), ch. IV.

the forms of p and q are unaltered in the changed circumstances, provided

$$\frac{df}{da}\frac{\partial a}{\partial x}+\frac{df}{db}\frac{\partial b}{\partial x}=0,\quad \frac{df}{da}\frac{\partial a}{\partial y}+\frac{df}{db}\frac{\partial b}{\partial y}=0:$$

hence, as a and b are independent functions of x and y, we must have

$$\frac{df}{da}=0,\quad \frac{df}{db}=0,$$

the values of p and q still being given in the forms

$$p=\frac{\partial f}{\partial x},\quad q=\frac{\partial f}{\partial y}.$$

Again,

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 f}{\partial x^2}+\frac{dp}{da}\frac{\partial a}{\partial x}+\frac{dp}{db}\frac{\partial b}{\partial x}=r+h,$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 f}{\partial x\partial y}+\frac{dp}{da}\frac{\partial a}{\partial y}+\frac{dp}{db}\frac{\partial b}{\partial y}$$

$$=\frac{\partial^2 f}{\partial x\partial y}+\frac{dq}{da}\frac{\partial a}{\partial x}+\frac{dq}{db}\frac{\partial b}{\partial x}=s+k,$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 f}{\partial y^2}+\frac{dq}{da}\frac{\partial a}{\partial y}+\frac{dq}{db}\frac{\partial b}{\partial y}=t+l,$$

where r, s, t are the second derivatives of f which satisfy the original differential equation.

Also, because the proper values of a and b, as functions of x and y, satisfy the equations

$$\frac{df}{da}=0,\quad \frac{df}{db}=0,$$

identically, we have

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)+\frac{d^2 f}{da^2}\frac{\partial a}{\partial x}+\frac{d^2 f}{da\,db}\frac{\partial b}{\partial x}=0,$$

and three similar equations. When the proper value of c, as a function of a and b, is substituted in f, the latter becomes a function of x, y, a, b only: and the partial derivative of the modified function with regard to a is the quantity denoted by $\frac{df}{da}$. Hence

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)=\frac{dp}{da},$$

and therefore

$$\frac{d^2f}{da^2}\frac{\partial a}{\partial x}+\frac{d^2f}{da\,db}\frac{\partial b}{\partial x}=-\frac{dp}{da};$$

similarly,

$$\frac{d^2f}{da\,db}\frac{\partial a}{\partial x}+\frac{d^2f}{db^2}\frac{\partial b}{\partial x}=-\frac{dp}{db},$$

$$\frac{d^2f}{da^2}\frac{\partial a}{\partial y}+\frac{d^2f}{da\,db}\frac{\partial b}{\partial y}=-\frac{dq}{da},$$

$$\frac{d^2f}{da\,db}\frac{\partial a}{\partial y}+\frac{d^2f}{db^2}\frac{\partial b}{\partial y}=-\frac{dq}{db}.$$

These equations enable us to express the derivatives of a and of b with respect to x and y in terms of $\frac{dp}{da}, \frac{dp}{db}, \frac{dq}{da}, \frac{dq}{db}$; and they obviously verify the relation

$$\frac{dp}{da}\frac{\partial a}{\partial y}+\frac{dp}{db}\frac{\partial b}{\partial y}=\frac{dq}{da}\frac{\partial a}{\partial x}+\frac{dq}{db}\frac{\partial b}{\partial x}.$$

We find

$$h\Delta=\left(\frac{d^2f}{da^2},\ -\frac{d^2f}{da\,db},\ \frac{d^2f}{db^2}\right)\!\left(\frac{dp}{db},\ \frac{dp}{da}\right)^2,$$

$$k\Delta=\left(\frac{d^2f}{da^2},\ -\frac{d^2f}{da\,db},\ \frac{d^2f}{db^2}\right)\!\left(\frac{dp}{db},\ \frac{dp}{da}\right)\!\left(\frac{dq}{db},\ \frac{dq}{da}\right),$$

$$l\Delta=\left(\frac{d^2f}{da^2},\ -\frac{d^2f}{da\,db},\ \frac{d^2f}{db^2}\right)\!\left(\frac{dq}{db},\ \frac{dq}{da}\right)^2,$$

where

$$\Delta=\left(\frac{d^2f}{da\,db}\right)^2-\frac{d^2f}{da^2}\frac{d^2f}{db^2};$$

also

$$(k^2-hl)\,\Delta=\left(\frac{dp}{da}\frac{dq}{db}-\frac{dp}{db}\frac{dq}{da}\right)^2.$$

Suppose that the differential equation is

$$U\left\{\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2\right\}+R\frac{\partial^2 z}{\partial x^2}+2S\frac{\partial^2 z}{\partial x\partial y}+T\frac{\partial^2 z}{\partial y^2}=V,$$

where U, R, S, T, V do not involve second derivatives; then it is satisfied by

$$z=f(x, y, a, b, c),$$

whether a, b, c be parametric or variable, and therefore

$$U(rt-s^2)+Rr+2Ss+Tt=V,$$

$$U\{(r+h)(t+l)-(s+k)^2\}+R(r+h)+2S(s+k)+T(t+l)=V.$$

Subtracting, we have

$$(R + Ut)h + 2(S - Us)k + (T + Ur)l = U(k^2 - hl):$$

when the preceding values of h, k, l are substituted in this equation, it takes the form

$$R_1 \frac{d^2f}{da^2} - 2S_1 \frac{d^2f}{da\,db} + T_1 \frac{d^2f}{db^2} = V_1,$$

where

$$R_1 = \left(R + Ut,\ S - Us,\ T + Ur\right)\!\left(\frac{dp}{db},\ \frac{dq}{db}\right)^2,$$

$$S_1 = \left(R + Ut,\ S - Us,\ T + Ur\right)\!\left(\frac{dp}{db},\ \frac{dq}{db}\right)\!\left(\frac{dp}{da},\ \frac{dq}{da}\right),$$

$$T_1 = \left(R + Ut,\ S - Us,\ T + Ur\right)\!\left(\frac{dp}{da},\ \frac{dq}{da}\right)^2,$$

$$V_1 = U\left(\frac{dp}{da}\frac{dq}{db} - \frac{dp}{db}\frac{dq}{da}\right)^2,$$

the quantities r, s, and t, in these expressions being the second derivatives of $f(x, y, a, b, c)$, when a, b, c are parametric. Now

$$\frac{d^2f}{da^2} = \frac{\partial^2 f}{\partial a^2} + 2\frac{\partial^2 f}{\partial a \partial c}\frac{\partial c}{\partial a} + \frac{\partial^2 f}{\partial c^2}\left(\frac{\partial c}{\partial a}\right)^2 + \frac{\partial f}{\partial c}\frac{\partial^2 c}{\partial a^2},$$

and so for the others; so that the new equation is linear in the second derivatives of c, and the coefficients of these derivatives involve x, y, z, a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$. The equations

$$z - f = 0,\quad \frac{df}{da} = 0,\quad \frac{df}{db} = 0,$$

determine x, y, and z as functions of a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$: when their values are substituted in the coefficients, the new equation takes the form

$$A\frac{\partial^2 c}{\partial a^2} + 2H\frac{\partial^2 c}{\partial a \partial b} + B\frac{\partial^2 c}{\partial b^2} = F,$$

where A, H, B, F are functions of a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$ only, and do not involve second derivatives of c.

When this equation has been integrated, expressing c as a function of a and b, then the original equation possesses an integral, which results from the elimination of a, b, c among the equations

$$z = f(x, y, a, b, c),$$

$$c = \theta(a, b),$$

$$0 = \frac{\partial f}{\partial a} + \frac{\partial f}{\partial c}\frac{\partial \theta}{\partial a},$$

$$0 = \frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}\frac{\partial \theta}{\partial b}.$$

The possibility of the generalisation thus depends on the integration of the new equation for c: and the form of that equation is affected by the form of f. In practice, it would therefore usually be convenient to take simple forms of f where choice can be exercised; it is unnecessary to aim at securing generality in the form of f, because that generality can be secured through the form of θ when the new equation can be completely integrated.

There are various ways in which an integral involving three parameters can be obtained. Sometimes it is possible to write down such an integral almost by inspection. Again, when the subsidiary systems in the methods of Monge and of Boole, and in the method of Ampère, possess three integrable combinations, in the forms

$$u = a, \quad v = b, \quad w = c,$$

the elimination of p and q leads to an integral involving three parameters: in this particular case, the critical quadratic must have equal roots. Again, if either of the subsidiary systems in question admits only one integrable combination in a form

$$u = a,$$

where u involves p or q or both, this equation, regarded as of the first order and integrated by Charpit's method, will lead to an integral involving three parameters. Further, if each of the subsidiary systems admits only a single integrable combination in a form

$$u = a, \quad v = b,$$

and if these equations can be resolved for p and q, the substitution of these values in

$$dz = p\,dx + q\,dy,$$

followed by a quadrature, gives an integral with the desired three parameters. As will be seen later (§§ 271, 274), some of these possibilities (and they are not exhaustive) are bound up with the form of the equation for the determination of c in terms of a and b.

Ex. 1. Consider the equation

$$(q+yt)(r+1)=(ys-p-x)s.$$

When we proceed to integrate it by Ampère's method, we find that one of the subsidiary systems admits of the integrable combinations

$$\frac{p+x}{y}=\text{constant},\quad qy=\text{constant},$$

and that the other of the subsidiary systems admits of the integrable combinations

$$p+x=\text{constant},\quad qy-x(p+x)=\text{constant}.$$

In order to construct some integral of the original equation, involving three arbitrary constants, we take

$$qy=a,\quad p+x=b;$$

and then, as

$$dz=p\,dx+q\,dy$$

$$=(b-x)\,dx+\frac{a}{y}\,dy,$$

we have

$$z=c+bx-\tfrac{1}{2}x^2+a\log y.$$

This is the integral to be generalised. We have, in the notation of the text,

$$R=q,\quad S=\tfrac{1}{2}(p+x),\quad T=y,\quad U=y;$$

also

$$p=b-x,\quad q=\frac{a}{y},\quad r=-1,\quad s=0,\quad t=-\frac{a}{y^2},$$

so that

$$\frac{dp}{da}=0,\quad \frac{dp}{db}=1,$$

$$\frac{dq}{da}=\frac{1}{y},\quad \frac{dq}{db}=0.$$

Thus

$$R+Ut=0,\quad S-Us=\tfrac{1}{2}(p+x)=\tfrac{1}{2}b,\quad T+Ur=0;$$

and therefore

$$R_1=0,\quad S_1=\tfrac{1}{2}\frac{b}{y},\quad T_1=0,\quad V_1=\frac{1}{y}.$$

Moreover,

$$\frac{d^2f}{da^2}=\frac{\partial^2c}{\partial a^2},\quad \frac{d^2f}{da\,db}=\frac{\partial^2c}{\partial a\,\partial b},\quad \frac{d^2f}{db^2}=\frac{\partial^2c}{\partial b^2};$$

consequently, the equation for c is

$$b\frac{\partial^2c}{\partial a\,\partial b}=-1.$$

Hence

$$\frac{\partial c}{\partial a}=\phi'(a)-\log b,$$

$$c=\phi(a)+\psi(b)-a\log b.$$

Consequently,

$$z=\phi(a)+\psi(b)+bx-\tfrac{1}{2}x^2+a\log\frac{y}{b};$$

and the other equations are

$$0=\phi'(a)+\log\frac{y}{b},$$

$$0=x-\frac{a}{b}+\psi'(b).$$

From the last equation but one, we have

$$\phi'(a)=\log\frac{b}{y},$$

and therefore

$$\phi''(a)=\frac{y}{b}\frac{\partial}{\partial a}\left(\frac{b}{y}\right).$$

Let

$$\phi(a)-a\phi'(a)=-\theta\left(\frac{b}{y}\right),$$

the left-hand side being manifestly some function of $\frac{b}{y}$; then

$$a\phi''(a)=\theta'\left(\frac{b}{y}\right)\frac{\partial}{\partial a}\left(\frac{b}{y}\right),$$

that is,

$$\frac{a}{b}=\frac{1}{y}\theta'\left(\frac{b}{y}\right);$$

hence the equations are

$$\left.\begin{aligned} z&=bx-\tfrac{1}{2}x^2+\psi(b)-\theta\left(\frac{b}{y}\right)\\ 0&=x+\psi'(b)-\frac{1}{y}\theta'\left(\frac{b}{y}\right)\end{aligned}\right\},$$

which constitute a general primitive for the equation.

Ex. 2. The equation

$$r+t=0$$

possesses an integral

$$z=c+ax+by:$$

prove that the equation for c is only a transformation of the original equation, and deduce the customary primitive.

Ex. 3. The equation

$$r+2(q-x)s+(q-x)^2t=q$$

possesses two integrals

$$z=ax+by+\tfrac{1}{2}bx^2+c,$$
$$z=ax+by+\tfrac{1}{2}bx(x-b)+c;$$

generalise each of these.

Ex. 4. The equation

$$r-t=2\frac{p}{x}$$

admits an integral

$$z=c+ay+bx^3:$$

generalise it, so as to obtain the primitive

$$z=\theta(y+x)+\psi(y-x)-x\theta'(y+x)+x\psi'(y-x).$$

271. For the purpose of Imschenetsky's generalisation, it is necessary to have an integral of the given differential equation involving three arbitrary parameters; in order to complete the generalisation, it is necessary to obtain the primitive of the linear equation of the second order satisfied by c. If the three-constant integral has been obtained without the use of any systematic method, say as by mere inspection, the equation for c has no special properties or form. If, however, that integral has been obtained through one of the subsidiary systems in Ampère's method, it is possible to recognise an *a priori* limitation upon the form of the equation satisfied by c. For example, suppose that the subsidiary system associated with the argument α offers an integrable combination of the form

$$\begin{aligned} u(x, y, z, p, q) &= \text{constant} \\ &= \alpha, \end{aligned}$$

in accordance with the Ampère process: and let this equation, of the first order, be integrated by any of the methods leading to a complete integral, which will have a form

$$z = f(x, y, \alpha, a, c).$$

When Imschenetsky's generalising process is applied to this integral so as to determine c in terms of a and α, the equation for the determination of c is linear in

$$\frac{\partial^2 c}{\partial \alpha^2}, \quad \frac{\partial^2 c}{\partial a \partial \alpha}, \quad \frac{\partial^2 c}{\partial a^2}.$$

The arbitrary functions, which occur in the general integral of the original equation, are introduced by the arbitrary functions, which occur in the value of c. Now in one of the arbitrary functions in the required general integral, the argument is known (from the theory of Ampère's method) to be α; hence α must be the argument in one of the arbitrary functions occurring in the completed expression for c. Thus the equation determining c must be satisfied by an expression containing an arbitrary function of α together with, it may be, some of the derivatives of this function. In order that this may be the case, the term in $\frac{\partial^2 c}{\partial \alpha^2}$ must be absent from the equation: otherwise, the equation could not be satisfied by such a value of c, for that term would introduce derivatives of the arbitrary function of order higher than those introduced by any other term.

When there are two subsidiary systems, and when each of them admits an integrable combination of the form

$$u(x, y, z, p, q) = \text{constant} = \alpha,$$
$$v(x, y, z, p, q) = \text{constant} = \beta,$$

respectively, where (by Ampère's theory) α and β are the arguments of the arbitrary functions in the general primitive, we resolve these two equations for p and q, substitute the resolved values in

$$dz = p\,dx + q\,dy,$$

and effect the quadrature: when the equation thus obtained is resolved with regard to z, it becomes

$$z = h(x, y, \alpha, \beta, c).$$

The Imschenetsky method can be applied to generalise this integral: an argument, similar to that in the preceding case, shews that the equation which determines c as a function of α and β is of the form

$$\frac{\partial^2 c}{\partial\alpha\partial\beta} = C,$$

where C is a function of α, β, c, $\frac{\partial c}{\partial\alpha}$, $\frac{\partial c}{\partial\beta}$ at the utmost.

Lastly, in the case of equations having only a single subsidiary system, so that the arbitrary functions in the general primitive have one and the same argument α, suppose that there is an integrable combination

$$g(x, y, z, p, q) = \text{constant}$$
$$= \alpha.$$

Let this be integrated, by Charpit's method or otherwise, leading to a complete integral

$$z = k(x, y, \alpha, a, c).$$

When this integral is generalised by the Imschenetsky process, so as to make c a function of a and α, then the equation of the second order determining c is similarly proved to have the form

$$\frac{\partial^2 c}{\partial a} = K,$$

where K is a function of a, α, c, and $\frac{\partial c}{\partial a}$, but does not involve $\frac{\partial c}{\partial \alpha}$. The absence of the term in $\frac{\partial^2 c}{\partial a \partial \alpha}$ is due to the fact that there is only a single argument in the arbitrary functions; in consequence, such a term would introduce a derivative of higher order than any introduced by other terms.

Hence when the equation in three parameters has been obtained, wholly or partly, from the integrable combinations of the subsidiary systems, the generalising equations are of the forms

$$2H \frac{\partial^2 c}{\partial a \partial b} + B \frac{\partial^2 c}{\partial b^2} = F,$$

with which may be associated

$$A \frac{\partial^2 c}{\partial a^2} + 2H \frac{\partial^2 c}{\partial a \partial b} = F,$$

as arising through the use of the alternative subsidiary system;

$$\frac{\partial^2 c}{\partial a \partial b} = C;$$

$$\frac{\partial^2 c}{\partial a^2} = K;$$

in the respective cases: and F, C, K vanish, if the combination $rt - s^2$ does not occur in the original equation, for then $U = 0$.

Ex. 1. Integrate the equations:—

(i) $x^4 r - 4x^2 qs + 4q^2 t + 2px^3 = 0$;

(ii) $x^2 r + 2x^2 s + \left(x^2 - \frac{b^2}{x^2 q^2}\right) t - 2z = 0$;

(iii) $r + 2qs + (q^2 - x^2) t - q = 0$;

(iv) $x^4 r - 4x^2 qs + 3qt + 2px^3 = 0$;

(v) $zs + \frac{z}{p^2} t + pq = 0$.

(Ampère; Imschenetsky.)

Ex. 2. The equation

$$a \frac{x^2}{y^2} r + b \frac{y^2}{x^2} t + (lx + my + nxy)(rt - s^2 + 1)$$
$$= \frac{1}{x^2 y^2} (lx + my + nxy)(z - px - qy + xy)^2$$

has an integral

$$z = ax + by + cxy;$$

obtain the primitive. (Imschenetsky.)

Ex. 3. Verify that the equation

$$rt - s^2 = (1 + p^2 + q^2)^2$$

is satisfied by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 1;$$

shew how to generalise this integral.

(The differential equation is the equation of surfaces with constant curvature, all the quantities in the equations quoted being real when the curvature is positive. For a full discussion of the properties, reference should be made to the treatises on differential geometry by Darboux and by Bianchi, where full citations of the original authorities will be found).

Generalisation of an Intermediate Integral.

272. The preceding discussion relates to the generalisation of a primitive of the equation, when the primitive is not complete. Similarly, it is possible to generalise a complete intermediate integral. Let

$$u(x, y, z, p, q, a, b) = 0$$

be such an integral, so that the differential equation of the second order is the result of eliminating a and b between the equations

$$u = 0, \quad \frac{du}{dx} = 0, \quad \frac{du}{dy} = 0.$$

The eliminant manifestly will be the same if a and b, instead of remaining parametric, are replaced by functions of x and y, such that

$$\frac{\partial u}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} = 0,$$

$$\frac{\partial u}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial y} = 0.$$

It might be possible that the equations

$$\frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

should satisfy all the conditions necessary for coexistence with $u = 0$ and with the equation of the second order: if these conditions are satisfied, the result of eliminating a and b between

$$u = 0, \quad \frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

would be a special or singular intermediate integral: the equations of the second order would be a very limited class. The alternative is that the relation

$$\frac{\partial (a, b)}{\partial (x, y)} = 0$$

should be satisfied. If this is satisfied identically (and we shall neglect all other cases), a functional relation exists between a and b: let it be

$$b = \phi (a).$$

Also we have

$$\frac{\partial u}{\partial a} da + \frac{\partial u}{\partial b} db = 0,$$

that is,

$$\frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \phi' (a) = 0.$$

The three equations

$$\left.\begin{aligned} u = 0, \quad b = \phi (a) \\ \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \phi' (a) = 0 \end{aligned}\right\}$$

constitute a generalisation of the complete intermediate integral.

Ex. 1. The equation

$$z - \frac{q^2 r - 2pqs + p^2 t}{rt - s^2} + \frac{(sp - rq)(sq - tp)}{(rt - s^2)^2} = 0$$

has a complete intermediate integral

$$z + cp + aq + ac = 0:$$

a generalised form is given by

$$\left.\begin{aligned} z + \ aq \ + (p + a) \phi \ (a) = 0 \\ q + \phi (a) + (p + a) \phi' (a) = 0 \end{aligned}\right\}.$$

Ex. 2. The equation

$$p + q = \lambda + \mu (x + y),$$

where λ and μ are arbitrary constants, is a complete intermediate integral of

$$r = t.$$

Generalising it, we take

$$\lambda = \phi (\mu),$$

and then

$$0 = \phi' (\mu) + x + y,$$

that is, μ is a function of $x + y$: changing the functions, we find

$$p + q = 2F' (x + y).$$

Similarly, from

$$p - q = \alpha + \beta (x - y),$$

where α and β are arbitrary, we find

$$p - q = 2G' (x - y).$$

Hence

$$p = F'(x+y) + G'(x-y),$$
$$q = F'(x+y) - G'(x-y):$$

substituting in

$$dz = p\,dx + q\,dy,$$

we have the customary general primitive

$$z = F(x+y) + G(x-y).$$

Imschenetsky's Method applied to Laplace's Linear Equation.

273. It might be possible to generalise an incomplete primitive of an equation not of the form considered by Imschenetsky; but the analysis connected with even so simple an equation as

$$rt = 1,$$

having an incomplete primitive

$$z = ax + by + \tfrac{1}{2}x^2e^c + \tfrac{1}{2}y^2e^{-c},$$

is enough to suggest that the process would usually be impracticable.

There is, however, one class of equations, which are formally included among those considered and which yet provide little towards the construction of an incomplete primitive: it is the class of Laplace's linear equations

$$s + Ap + Bq + Cz = 0,$$

where A, B, C are functions of x and y only. For general values of A, B, and C, this equation does not possess an intermediate integral: hence there is no simplification to be expected *a priori* in the form of the generalising equation. In order to apply the Imschenetsky method, we take a primitive

$$z = az_1 + bz_2 + cz_3 = f,$$

where initially a, b, c are arbitrary parameters, and z_1, z_2, z_3 obviously are particular integrals of the equation. Applying the detailed results of the method, we have

$$\frac{df}{da} = z_1 + z_3\frac{\partial c}{\partial a} = 0,$$

$$\frac{df}{db} = z_2 + z_3\frac{\partial c}{\partial b} = 0;$$

so that x and y are functions of $\dfrac{\partial c}{\partial a}$ and $\dfrac{\partial c}{\partial b}$ only.

With the earlier notation, we have

$$R=0, \quad S=\tfrac{1}{2}, \quad T=0, \quad U=0, \quad V=-(Ap+Bq+Cz);$$

hence

$$R_1=\frac{dp}{db}\frac{dq}{db},$$

$$2S_1=\frac{dp}{db}\frac{dq}{da}+\frac{dp}{da}\frac{dq}{db},$$

$$T_1=\frac{dp}{da}\frac{dq}{da},$$

$$V_1=0,$$

and the equation for c is

$$R_1\frac{d^2f}{da^2}-2S_1\frac{d^2f}{da\,db}+T_1\frac{d^2f}{db^2}=V_1.$$

But

$$\frac{d^2f}{da^2}=z_3\frac{\partial^2c}{\partial a^2}, \quad \frac{d^2f}{da\,db}=z_3\frac{\partial^2c}{\partial a\,\partial b}, \quad \frac{d^2f}{db^2}=z_3\frac{\partial^2c}{\partial b^2};$$

and therefore the equation for c is

$$\frac{dp}{db}\frac{dq}{db}\frac{\partial^2c}{\partial a^2}-\left(\frac{dp}{db}\frac{dq}{da}+\frac{dp}{da}\frac{dq}{db}\right)\frac{\partial^2c}{\partial a\,\partial b}+\frac{dp}{da}\frac{dq}{da}\frac{\partial^2c}{\partial b^2}=0.$$

Also,

$$\frac{dp}{da}=p_1+p_3\frac{\partial c}{\partial a}, \quad \frac{dp}{db}=p_2+p_3\frac{\partial c}{\partial b},$$

$$\frac{dq}{da}=q_1+q_3\frac{\partial c}{\partial a}, \quad \frac{dq}{db}=q_2+q_3\frac{\partial c}{\partial b};$$

hence, when z_1, z_2, z_3 are known, the coefficients for the differential equation can be regarded as known. It is clear that these coefficients are expressible in terms of the derivatives $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$ alone: and so the differential equation for the determination of c does not explicitly involve a, b, or c.

When this equation is integrated, so as to give c in terms of a and b in a form

$$c=\theta(a,\ b),$$

then the general primitive of the Laplace's equation is given by eliminating a and b between the equations

$$\left.\begin{aligned} z &= z_1 a + z_2 b + z_3 \theta(a,\ b) \\ 0 &= z_1 \quad + z_3 \frac{\partial \theta}{\partial a} \\ 0 &= z_2 \quad + z_3 \frac{\partial \theta}{\partial b} \end{aligned}\right\}.$$

It seems obvious that, the simpler the forms of z_1, z_2, z_3 initially chosen, the less complicated will usually be the details of the generalising analysis.

Ex. Consider the equation

$$s - \frac{p+q}{x+y} = 0.$$

We can take

$$z_1 = xy, \quad z_2 = x - y, \quad z_3 = 1;$$

so that

$$z = axy + b(x-y) + c,$$

and then

$$xy = -\frac{\partial c}{\partial a},$$

$$x - y = -\frac{\partial c}{\partial b}.$$

Also

$$\frac{dp}{da} = y, \quad \frac{dp}{db} = 1,$$

$$\frac{dq}{da} = x, \quad \frac{dq}{db} = -1;$$

substituting these values in the equation for c, we find

$$-\frac{\partial^2 c}{\partial a^2} - (x-y)\frac{\partial^2 c}{\partial a\, \partial b} + xy \frac{\partial^2 c}{\partial b^2} = 0,$$

that is,

$$\frac{\partial^2 c}{\partial a^2} - \frac{\partial c}{\partial b}\frac{\partial^2 c}{\partial a\, \partial b} + \frac{\partial c}{\partial a}\frac{\partial^2 c}{\partial b^2} = 0.$$

A primitive of this equation has already (§ 265, Ex. 1) been given: it is constituted by the three equations

$$\left.\begin{aligned} c &= \lambda^2 f''(\lambda) - 2\lambda f'(\lambda) + 2f(\lambda) + \mu^2 g''(\mu) - 2\mu g'(\mu) + 2g(\mu) \\ -a &= f''(\lambda) + g''(\mu) \\ -b &= \lambda f''(\lambda) - f'(\lambda) + \mu g''(\mu) - g'(\mu) \end{aligned}\right\}.$$

The quantities λ and μ must be identified: we have

$$\begin{aligned}\lambda^2 f'''d\lambda+\mu^2 g'''d\mu &= dc \\ &= \frac{\partial c}{\partial a}da+\frac{\partial c}{\partial b}db \\ &= -\frac{\partial c}{\partial a}(f'''d\lambda+g'''d\mu)-\frac{\partial c}{\partial b}(\lambda f'''d\lambda+\mu g'''d\mu);\end{aligned}$$

and therefore

$$\lambda^2=-\frac{\partial c}{\partial a}-\lambda\frac{\partial c}{\partial b}=xy+\lambda(x-y),$$

$$\mu^2=-\frac{\partial c}{\partial a}-\mu\frac{\partial c}{\partial b}=xy+\mu(x-y).$$

Hence we can take

$$\lambda=x,\quad \mu=-y.$$

Writing

$$f(\lambda)=X,\quad g(\mu)=Y,$$

we have

$$\begin{aligned}c&=x^2X''-2xX'+2X+y^2Y''-2yY'+2Y,\\ -a&=X''+Y'',\\ -b&=xX''-X'-yY''+Y'.\end{aligned}$$

Substituting these values of a, b, c in the equation

$$z=axy+b(x-y)+c,$$

and reducing, we find

$$z=2X+2Y-(x+y)(X'+Y'),$$

which is the general primitive of the original equation.

274. The transformation adopted in § 273 and applied to the preceding example, is easily seen there to be a contact-transformation between the two sets of variables: in fact, we have

$$\begin{aligned}z&=az_1+bz_2+cz_3\\ &=z_3\left(c-a\frac{\partial c}{\partial a}-b\frac{\partial c}{\partial b}\right);\end{aligned}$$

and, in the example, z_3 is unity. This property can be secured in general by an appropriate initial modification of the Laplace equation.

Let z_3 be a particular integral of the equation, and write $z=z_3Z$, so that, if

$$Z_1=\frac{z_1}{z_3},\quad Z_2=\frac{z_2}{z_3},$$

then Z_1, Z_2, 1 are three particular integrals of the transformed equation which is easily found to be

$$S+A'P+B'Q=0,$$

where

$$A' = A + \frac{q_3}{z_3}, \quad B' = B + \frac{p_3}{z_3}.$$

Accordingly, as the term in Z has been removed, we may suppose that our equation initially has the form

$$s + Ap + Bq = 0.$$

We then take

$$z = c + az_1 + bz_2;$$

and the other equations then become

$$0 = \frac{\partial c}{\partial a} + z_1, \quad 0 = \frac{\partial c}{\partial b} + z_2,$$

so that

$$z = c - a\frac{\partial c}{\partial a} - b\frac{\partial c}{\partial b},$$

being the Legendrian contact-transformation.

The direct construction of the generalising equation is simple. Denoting the second derivatives of c with regard to a and b by ρ, σ, τ respectively, and writing

$$\frac{\partial a}{\partial x} = a_x, \quad \frac{\partial a}{\partial y} = a_y,$$

and similarly for b, we have

$$\left.\begin{aligned} -p_1 &= \rho a_x + \sigma b_x \\ -q_1 &= \rho a_y + \sigma b_y \end{aligned}\right\}, \quad \left.\begin{aligned} -p_2 &= \sigma a_x + \tau b_x \\ -q_2 &= \sigma a_y + \tau b_y \end{aligned}\right\},$$

as derivatives of the equations

$$-z_1 = \frac{\partial c}{\partial a}, \quad -z_2 = \frac{\partial c}{\partial b};$$

hence

$$\left.\begin{aligned} (\rho\tau - \sigma^2)\, a_x &= -p_1\tau + p_2\sigma \\ (\rho\tau - \sigma^2)\, a_y &= -q_1\tau + q_2\sigma \\ (\rho\tau - \sigma^2)\, b_x &= p_1\sigma - p_2\rho \\ (\rho\tau - \sigma^2)\, b_y &= q_1\sigma - q_2\rho \end{aligned}\right\}.$$

Again, from the equation

$$z = c + az_1 + bz_2,$$

we have

$$p = ap_1 + bp_2,$$

$$q = aq_1 + bq_2,$$

the terms that arise through the variations of a, b, c vanishing; and then

$$s = as_1 + bs_2 + p_1 a_y + p_2 b_y$$
$$= as_1 + bs_2 - \frac{1}{\rho\tau - \sigma^2}\{p_1 q_1 \tau - (p_1 q_2 + p_2 q_1)\sigma + p_2 q_2 \rho\}.$$

Also,

$$s = -Ap - Bq$$
$$= -a(Ap_1 + Bq_1) - b(Ap_2 + Bq_2)$$
$$= as_1 + bs_2;$$

and therefore

$$p_1 q_1 \tau - (p_1 q_2 + p_2 q_1)\sigma + p_2 q_2 \rho = 0.$$

Moreover,

$$p_1 = \frac{dp}{da}, \quad p_2 = \frac{dp}{db},$$

from the value of p; and similarly

$$q_1 = \frac{dq}{da}, \quad q_2 = \frac{dq}{db},$$

from the value of q. Consequently,

$$\frac{dp}{db}\frac{dq}{db}\frac{\partial^2 c}{\partial a^2} - \left(\frac{dp}{da}\frac{dq}{db} + \frac{dp}{db}\frac{dq}{da}\right)\frac{\partial^2 c}{\partial a\,\partial b} + \frac{dp}{da}\frac{dq}{da}\frac{\partial^2 c}{\partial b^2} = 0,$$

agreeing with the earlier form obtained for the equation which is to determine c.

275. A different way of proceeding is as follows*. Let the equation be taken in the form

$$s + \alpha p + \beta q = 0,$$

where α and β are functions of x and y only, so that $z = 1$ is an integral of the equation. Let z_1 and z_2 be two other integrals of the equation, and assume that

$$z = u + vz_1 + wz_2,$$

where the new unknown quantities u, v, w are to be determined by the condition that the forms of p and q (and therefore, owing to the equation, the form of s also) are the same, when u, v, w are variable, as they would be if u, v, w were parametric. Then

$$p = vp_1 + wp_2,$$
$$q = vq_1 + wq_2,$$

* In connection with this investigation, a memoir by R. Liouville, *Journ. de l'Éc. Polyt.*, Cah. LVI (1886), pp. 7—62, may be consulted.

provided

$$\frac{\partial u}{\partial x} + z_1 \frac{\partial v}{\partial x} + z_2 \frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial u}{\partial y} + z_1 \frac{\partial v}{\partial y} + z_2 \frac{\partial w}{\partial y} = 0 :$$

also

$$s = vs_1 + ws_2,$$

provided

$$p_1 \frac{\partial v}{\partial y} + p_2 \frac{\partial w}{\partial y} = 0,$$

because s can be derived from p, and

$$q_1 \frac{\partial v}{\partial x} + q_2 \frac{\partial w}{\partial x} = 0,$$

because s can be derived from q. Hence

$$\frac{\partial v}{\partial x} = -\frac{q_2}{q_1}\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial y} = -\frac{p_2}{p_1}\frac{\partial w}{\partial y},$$

$$\frac{\partial u}{\partial x} = \left(z_1 \frac{q_2}{q_1} - z_2\right)\frac{\partial w}{\partial x}, \quad \frac{\partial u}{\partial y} = \left(z_1 \frac{p_2}{p_1} - z_2\right)\frac{\partial w}{\partial y};$$

and therefore

$$\frac{\partial}{\partial y}\left(\frac{q_2}{q_1}\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{p_2}{p_1}\frac{\partial w}{\partial y}\right),$$

$$\frac{\partial}{\partial y}\left\{\left(z_1 \frac{q_2}{q_1} - z_2\right)\frac{\partial w}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\left(z_1 \frac{p_2}{p_1} - z_2\right)\frac{\partial w}{\partial y}\right\},$$

apparently two distinct equations of the second order satisfied by w. It is easy, however, to see that the second equation (on the removal of a non-vanishing factor z_1) becomes the same as the first; in fact, they both reduce to the single equation

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{p_2}{q_2}\frac{q_1 t_2 - q_2 t_1}{p_1 q_2 - p_2 q_1}\frac{\partial w}{\partial x} - \frac{q_2}{p_2}\frac{p_1 r_2 - p_2 r_1}{p_1 q_2 - p_2 q_1}\frac{\partial w}{\partial y} = 0,$$

which may be written

$$\frac{\partial^2 w}{\partial x \partial y} + \alpha' \frac{\partial w}{\partial x} + \beta' \frac{\partial w}{\partial y} = 0.$$

When the value of w is known, then

$$v = -\int\left(\frac{q_2}{q_1}\frac{\partial w}{\partial x}\,dx + \frac{p_2}{p_1}\frac{\partial w}{\partial y}\,dy\right),$$

so that v is obtainable by quadrature: and then

$$u = -\int (z_1 dv + z_2 dw),$$

so that u also is obtainable by quadrature.

Ex. 1. Let the method be applied to the equation

$$s = \frac{p+q}{x+y}.$$

Particular integrals are given by

$$z_1 = x - y, \quad z_2 = xy;$$

we therefore take

$$z = u + (x-y)v + xyw.$$

The equations satisfied by u, v, w are

$$\frac{\partial u}{\partial x} + (x-y)\frac{\partial v}{\partial x} + xy\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial u}{\partial y} + (x-y)\frac{\partial v}{\partial y} + xy\frac{\partial w}{\partial y} = 0,$$

$$-\frac{\partial v}{\partial x} + x\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial v}{\partial y} + y\frac{\partial w}{\partial y} = 0.$$

From these we have

$$\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial x}, \quad \frac{\partial u}{\partial x} = -x^2\frac{\partial w}{\partial x},$$

$$\frac{\partial v}{\partial y} = -y\frac{\partial w}{\partial y}, \quad \frac{\partial u}{\partial y} = -y^2\frac{\partial w}{\partial y};$$

and therefore

$$\frac{\partial}{\partial y}\left(x\frac{\partial w}{\partial x}\right) = -\frac{\partial}{\partial x}\left(y\frac{\partial w}{\partial y}\right),$$

$$\frac{\partial}{\partial y}\left(x^2\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(y^2\frac{\partial w}{\partial y}\right),$$

both satisfied in virtue of

$$\frac{\partial^2 w}{\partial x\,\partial y} = 0;$$

and therefore

$$w = X'' + Y'',$$

where X and Y are arbitrary functions of x and of y respectively. Then

$$v = \int (xX'''dx - yY'''dy)$$
$$= xX'' - X' - yY'' + Y',$$
$$u = -\int (x^2X'''dx + y^2Y'''dy)$$
$$= -x^2X'' + 2xX' - 2X - y^2Y'' + 2yY' - 2Y;$$

consequently,

$$z = u + (x-y)v + xyw$$
$$= -2X - 2Y + (x+y)(X' + Y').$$

Ex. 2. Apply the preceding method to the equation

$$s=m\frac{p+q}{x+y},$$

where m is a constant; and using the integrals

$$z_1=x-y, \quad z_2=x^m y^m,$$

prove that the equation for w is

$$\frac{\partial^2 w}{\partial x'\,\partial y'}=\frac{m-1}{x'+y'}\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial y'}\right),$$

where $xx'=1$, $yy'=1$. Hence integrate the equation

$$s=2\frac{p+q}{x+y};$$

and shew how the property can be used to connect the integration of the two equations

$$s=m\frac{p+q}{x+y}, \quad s=(m-2)\frac{p+q}{x+y}.$$

Ex. 3. Shew that, if z_1, z_2, z_3 denote three linearly independent integrals of the equation

$$s+ap+bq+cz=0,$$

where a, b, c are functions of x and y alone, and if three quantities u, v, w are introduced such that

$$z=uz_1+vz_2+wz_3$$

is another integral, which keeps the same forms for p, q, s, whether u, v, w be variable or parametric, then w satisfies an equation

$$\begin{vmatrix} z_1, & p_1, & q_1 \\ z_2, & p_2, & q_2 \\ z_3, & p_3, & q_3 \end{vmatrix}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\begin{vmatrix} z_1, & p_1 \\ z_2, & p_2 \end{vmatrix}}{\begin{vmatrix} z_1, & q_1 \\ z_2, & q_2 \end{vmatrix}}\begin{vmatrix} z_1, & q_1, & t_1 \\ z_2, & q_2, & t_2 \\ z_3, & q_3, & t_3 \end{vmatrix}\frac{\partial w}{\partial x}$$

$$-\frac{\begin{vmatrix} z_1, & q_1 \\ z_2, & q_2 \end{vmatrix}}{\begin{vmatrix} z_1, & p_1 \\ z_2, & p_2 \end{vmatrix}}\begin{vmatrix} z_1, & p_1, & r_1 \\ z_2, & p_2, & r_2 \\ z_3, & p_3, & r_3 \end{vmatrix}\frac{\partial w}{\partial y}=0.$$

Shew how to determine u and v when w is known: and prove that u and v satisfy equations of the same form as the equation satisfied by w. Prove also that, if the original differential equation be of rank n in either of the variables x and y, then the equation for w is of rank $n+1$.

Ex. 4. Prove that, if z_1, z_2, z_3 denote three linearly independent integrals of Laplace's linear equation

$$s+ap+bq+cz=0,$$

no relation, which is homogeneous and of the second order, can subsist among z_1, z_2, z_3 alone.

CHAPTER XX.

Characteristics of Equations of the Second Order: Intermediate Integrals.

The theory of characteristics led to an effective method of integration when applied to equations of the first order; it can be applied also to equations of the second order and, in the application, it indicates the geometrical significance of the possession of intermediate integrals. The following account of these characteristics and of the construction of equations that possess intermediate integrals, not necessarily of the type considered by Monge, is based mainly upon Goursat's memoir*.

276. In order to solve Cauchy's problem for an equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0,$$

in the most general form of that problem, which assigns general functions of x and y as values of z and p when x and y are connected by a given relation $g(x, y) = 0$, we can proceed as follows. Along the curve representing the relation, we have

$$\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy = 0,$$

as a property of the curve: also, because the value of z, say $\phi(x, y)$, is given along the curve, the quantity

$$pdx + qdy, \ = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy,$$

for the elements dx, dy along the curve, is known. Hence, as p is given along the curve, so also q is known along the curve; and therefore Cauchy's problem amounts to the determination of a

* *Acta Math.*, t. XIX (1895), pp. 285—340. Reference also should be made to vol. I, ch. IV, of his treatise already (p. 7) quoted.

surface, satisfying the differential equation, passing through a given curve in the plane of x and y, and touching a given developable surface* along the curve.

Accordingly, at points on the curve, the values of x, y, z, p, q in connection with Cauchy's problem are expressible in terms of a single parameter u: and the values of r, s, t are given by the three equations

$$f=0,$$
$$r\,dx+s\,dy=dp,$$
$$s\,dx+t\,dy=dq.$$

These three equations will suffice for the determination of r, s, t in terms of u, unless the Jacobian of the three equations with respect to the three variables vanishes, that is, unless

$$\begin{vmatrix} R, & S, & T \\ dx, & dy, & 0 \\ 0, & dx, & dy \end{vmatrix}=0,$$

where

$$R=\frac{\partial f}{\partial r},\quad S=\frac{\partial f}{\partial s},\quad T=\frac{\partial f}{\partial t}.$$

When expanded, this equation is

$$R\left(\frac{dy}{dx}\right)^2-S\frac{dy}{dx}+T=0.$$

In the first place, suppose that (save possibly at isolated points, and these we neglect) the last equation is not satisfied: then the three equations determine one, or more than one, set of values of r, s, t in terms of u. We select any one such set, and proceed to consider the derivatives of the third order along the curve.

Denoting these as before by $\alpha, \beta, \gamma, \delta$, and writing

$$\left.\begin{aligned} X&=\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+r\frac{\partial f}{\partial p}+s\frac{\partial f}{\partial q} \\ Y&=\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}+s\frac{\partial f}{\partial p}+t\frac{\partial f}{\partial q} \end{aligned}\right\},$$

* The values of p and q determine the tangent plane: along the given curve, they are functions of a single variable, so that the equation of the tangent planes contains only a single parameter: they thus are enveloped by a developable surface.

we have

$$dr = \alpha\, dx + \beta\, dy,$$
$$ds = \beta\, dx + \gamma\, dy,$$
$$dt = \gamma\, dx + \delta\, dy,$$

along the curve, and

$$X + R\alpha + S\beta + T\gamma = 0,$$
$$Y + R\beta + S\gamma + T\delta = 0,$$

always. Thus there are five equations involving the four quantities $\alpha, \beta, \gamma, \delta$: so that one of them must be dependent upon the others, or there must be a linear relation among them. It is easily obtained; for, on multiplying the fourth by dx, the fifth by dy, and using the first three, we have

$$X\,dx + Y\,dy + R\,dr + S\,ds + T\,dt = 0,$$

that is,

$$df = 0,$$

which is satisfied in virtue of $f = 0$: so that there are only four independent equations. Also, denoting the value of $\frac{dy}{dx}$ along the curve by μ, and eliminating the quantities β and γ from the fourth equation by the first two, we have

$$X + \alpha\left(R - \frac{1}{\mu} S + \frac{1}{\mu^2} T\right) + \frac{dr}{dy} S + \left(\frac{ds}{dy} - \frac{1}{\mu}\frac{dr}{dy}\right) T = 0.$$

The coefficient of α does not vanish, by hypothesis: and therefore α is determinate at the point on the curve in the Cauchy problem. Similarly, the values of β, γ, δ are determinate there.

Similarly for the derivatives of all the orders in succession: each of them is determinate at the point on the curve, as associated with the assigned initial conditions: the only requirement is that

$$R\mu^2 - S\mu + T$$

does not vanish generally along the curve.

Accordingly, let the function z be developed in a power-series in $x - a$, $y - b$, where a, b is a point on the curve; as all the derivatives of z at a, b are known, all the coefficients in the series are known. Under certain conditions, which do not substantially concern us here, the series can be proved to converge: the function which it represents is an integral of the equation: and we merely obtain Cauchy's theorem again.

277. In the next place, consider a curve C for which the equation

$$R\left(\frac{dy}{dx}\right)^2 - S\frac{dy}{dx} + T = 0$$

is everywhere satisfied. Reviewing the past analysis, we see that the three equations, which involve r, s, t, do not suffice for the determination of those three quantities: one of the quantities can be taken arbitrarily, and then the other two are determinate. Similarly, the equations involving α, β, γ, δ do not suffice for the determination of those four quantities: one of them can be taken arbitrarily, and then the other three are determinate. The same holds for all the other orders in succession. Now suppose that

$$z = \phi(x, y)$$

is an integral of the differential equation: it represents a surface. At all points on this surface, z and all its derivatives are functions of x and y, and so also are R, S, T: hence the equation

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$$

defines two families of curves upon the surface, except when the relation

$$S^2 - 4RT = 0$$

is satisfied identically, in which case it defines only a single family. When there are two families of such curves, then one curve of each family (and, in general, only one curve of each family) passes through a point on the surface: and the directions of the two curves through the point, one from each family, are different from one another unless the point lies upon the locus

$$S^2 - 4RT = 0, \quad z = \phi(x, y).$$

Consider now the equations that are satisfied along C. Everywhere upon the surface we have

$$\begin{aligned} X + R\alpha + S\beta + T\gamma &= 0, \\ \alpha\,dx + \beta\,dy &= dr, \\ \beta\,dx + \gamma\,dy &= ds, \\ \gamma\,dx + \delta\,dy &= dt, \end{aligned}$$

and therefore everywhere on the surface, we have

$$X + \alpha\left(R - \frac{1}{\mu}S + \frac{1}{\mu^2}T\right) + \frac{dr}{dy}S + \left(\frac{ds}{dy} - \frac{1}{\mu}\frac{dr}{dy}\right)T = 0,$$

μ being $\dfrac{dy}{dx}$ and, for full variation of values of dx and dy, giving all directions through the point. In particular, select a direction giving a curve C through the point, so that

$$R\mu^2 - S\mu + T = 0;$$

the coefficient of α vanishes, and the coefficient of $\dfrac{dr}{dy}$ is $S - \dfrac{1}{\mu} T$, that is, it is $R\mu$; hence

$$X + \frac{ds}{dy} T + R\mu \frac{dr}{dy} = 0,$$

which can be written in the form

$$X\, dxdy + R drdy + T dsdx = 0.$$

Similarly, from

$$Y + R\beta + S\gamma + T\delta = 0,$$

we find

$$Y\, dxdy + R dsdy + T dtdx = 0.$$

It therefore follows that the equations

$$\left.\begin{array}{c} f = 0 \\ dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy \\ Rdy^2 - Sdxdy + Tdx^2 = 0 \\ X\,dxdy + Rdrdy + Tdsdx = 0 \\ Y\,dxdy + Rdsdy + Tdt\,dx = 0 \end{array}\right\}$$

are satisfied along the curve C. Apparently there are seven equations: but the relation

$$df = 0$$

is satisfied in virtue of the last six equations, so that $f = 0$ can be regarded as an equation not independent of the last six equations.

The aggregate of these equations determines a *characteristic* of the equation

$$f = 0.$$

They involve eight quantities x, y, z, p, q, r, s, t, and no quantities of order higher than the second: and being ordinary relations among differential elements, they determine seven of the quantities in terms of the eighth. But the number of independent equations is only six: one of the seven quantities can be arbitrarily assigned, and the other six are then determinate, their expressions (and therefore also the characteristic) being affected by the form of the arbitrary assignment.

Hence every integral of the differential equation, when that integral is regarded as a surface, is a locus of characteristic curves; and a given differential equation usually has two systems of characteristics though, for equations of specialised form, the two systems may coalesce into a single system.

Let x be selected as the independent variable for a characteristic of $f=0$: and let λ, μ be the roots of the quadratic

$$R\theta^2 - S\theta + T = 0,$$

so that

$$S = R(\lambda+\mu), \quad T = R\lambda\mu.$$

Then the equations of one characteristic, after a slight transformation, become

$$\left.\begin{aligned} \frac{dy}{dx} &= \lambda \\ \frac{dz}{dx} &= p + \lambda q \\ \frac{dp}{dx} &= r + \lambda s \\ \frac{dq}{dx} &= s + \lambda t \\ \frac{dr}{dx} + \mu\frac{ds}{dx} &= -\frac{X}{R} \\ \frac{ds}{dx} + \mu\frac{dt}{dx} &= -\frac{Y}{R} \end{aligned}\right\};$$

the similar equations of the other characteristic are given by the interchange of λ and μ in the preceding equations.

278. In particular, let the equation be

$$rt - s^2 + Ar + 2Bs + Ct = D;$$

so that

$$R = A + t, \quad S = 2(B - s), \quad T = C + r,$$

A, B, C, D not involving derivatives of the second order: the quadratic in $dy : dx$ becomes

$$(A+t)\,dy^2 - 2(B-s)\,dxdy + (C+r)\,dx^2 = 0,$$

that is,

$$A\,dy^2 - 2B\,dxdy + C\,dx^2 + dxdp + dydq = 0.$$

Again, substituting in the original equation the values

$$r = \frac{dp}{dx} - \lambda s, \quad t = \frac{1}{\lambda}\left(\frac{dq}{dx} - s\right),$$

given in the equations of the characteristic, and taking account of the differential relation just stated, together with

$$\frac{dy}{dx} = \lambda,$$

we find

$$A\,dp\,dy + C\,dq\,dx + dp\,dq - D\,dx\,dy = 0.$$

These two equations are the equations that occur in the subsidiary systems of Monge and of Ampère. Using the modified forms there adopted, we denote by ρ and σ the roots of the quadratic

$$m^2 + 2mB + AC + D = 0,$$

which has equal roots only when the quadratic

$$R\theta^2 - S\theta + T = 0$$

also has equal roots; and we find that the foregoing two equations lead to the pair

$$\left.\begin{array}{l} dp + C\,dx + \rho\,dy = 0 \\ dq + \sigma\,dx + A\,dy = 0 \end{array}\right\},$$

and to another pair obtained by interchanging ρ and σ. Three other equations of the characteristic become

$$dz - p\,dx - q\,dy = 0,$$

$$dp - r\,dx - s\,dy = 0,$$

$$dq - s\,dx - t\,dy = 0;$$

and the remaining two are the same as before.

Now three equations of this set, viz.

$$\left.\begin{array}{l} dp + C\,dx + \rho\,dy = 0 \\ dq + \sigma\,dx + A\,dy = 0 \\ dz - p\,dx - q\,dy = 0 \end{array}\right\},$$

involve only x, y, z, p, q: any set of values, which can satisfy these three equations, determines a *characteristic of the first order*. It is sufficient to have a set of values satisfying the equations, and there is no necessity to have an intermediate integral: but if the given differential does possess an intermediate integral in any form, that integral will, of itself, determine a characteristic of the first order.

The aggregate of all the equations determines *characteristics of the second order.* The relations of the characteristics of the two orders and, in particular, of an intermediate integral to the characteristics of the second order, appear as follows.

We know that, in connection with a given equation, there are six independent equations in the system which determines the characteristics. When there is a characteristic of the first order, it is determined by three of these equations; and therefore three equations remain for the characteristics of the second order, which accordingly are sufficient for the determination of r, s, t without limitations or conditions. It therefore follows that the characteristics of the second order include those (if any) of the first order.

But, further, let the quantities x, y, z, p, q, as connected with a characteristic of the first order when it exists, be expressed in terms of a parameter α; this evidently is possible, in connection with Ampère's theory. We have

$$r\frac{dx}{d\alpha} = \frac{dp}{d\alpha} - s\frac{dy}{d\alpha},$$

$$\frac{dr}{d\alpha}\left(\frac{dx}{d\alpha}\right)^2 = \frac{dx}{d\alpha}\left(\frac{d^2p}{d\alpha^2} - \frac{ds}{d\alpha}\frac{dy}{d\alpha} - s\frac{d^2y}{d\alpha^2}\right) - \left(\frac{dp}{d\alpha} - s\frac{dy}{d\alpha}\right)\frac{d^2x}{d\alpha^2},$$

$$t\frac{dy}{d\alpha} = \frac{dq}{d\alpha} - s\frac{dx}{d\alpha};$$

and the equation

$$X\,dx\,dy + R\,dr\,dy + T\,ds\,dx = 0$$

becomes

$$X\frac{dx}{d\alpha}\frac{dy}{d\alpha} + R\frac{dr}{d\alpha}\frac{dy}{d\alpha} + T\frac{ds}{d\alpha}\frac{dx}{d\alpha} = 0.$$

Replacing X, R, T by their values, and substituting for r, $\frac{dr}{d\alpha}$, t from the preceding relations, we find

$$\frac{ds}{d\alpha} = \frac{A_1s^2 + A_2s + A_3}{A_4s + A_5}$$
$$= g(s, \alpha),$$

where A_1, A_2, A_3, A_4, A_5 are functions of α. The integral of this equation involves an arbitrary constant: it determines s, and the preceding relations give the values of r and t. Hence, when an equation

$$rt - s^2 + Ar + 2Bs + Ct = D$$

possesses a characteristic of the first order, the equations of the characteristic of the second order, which includes that of the first order, involve an arbitrary constant.

279. Returning to the consideration of the general case, we have seen that an integral system passes through the curve C along which the equation

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$

is everywhere satisfied. There then is a want of determinateness in the equations

$$f = 0, \quad dp - rdx - sdy = 0, \quad dq - sdx - tdy = 0,$$

as regards the derivation of sets of values of r, s, t: instead of assuming r arbitrarily, so that s and t can then be regarded as determinate, we derive from the three equations an infinitude of values, and we can regard them as a continuous system. In this aspect, we have an infinitude of integral surfaces which are themselves a continuous system; and as these surfaces have the same values of x, y, z, p, q along the curve C, they touch one another along the curve. Hence, when we substitute values of r and t, derived from

$$dp - rdx - sdy = 0, \quad dq - sdx - tdy = 0,$$

and expressed in terms of s, in the equation $f = 0$, the changed form of the last equation must, quà equation in s, become evanescent: the coefficients of all powers of s must therefore vanish. There will thus result a number of simultaneous relations in x, y, z, p, q, dx, dy, dp, dq, which are homogeneous in the last four quantities. As the modified equation would otherwise have determined s, it would have contained at least two terms; so that there cannot be fewer than two conditions. On the other hand, the relations are homogeneous in dx, dy, dp, dq: hence there cannot be more than three independent relations.

Among the several cases, which thus are possibilities for an equation $f = 0$, the following may be noted.

(i) The relations may constitute an inconsistent system, which accordingly cannot be satisfied. The equation does not then possess an infinitude of integral surfaces having contact of the first order along the curve C. The quantities r, s, t are determinate; and two integral surfaces have contact of the second order along the characteristics.

(ii) The relations may be such as to yield an equation or equations involving x, y, z, p, q only.

(iii) The relations may be such as not to yield any equation free from differential elements: the preceding explanations shew that the number of independent relations in the set is either two or three.

Geometrical Interpretation.

280. The difference between the various cases can be illustrated through a geometrical interpretation introduced by Goursat. He regards r, s, t as the coordinates of a point in space, and x, y, z, p, q as parametric quantities: then the equation $f=0$ represents a surface. Denoting current coordinates by ρ, σ, τ, we have the tangent plane to the surface $f=0$ at the point r, s, t given by

$$(\rho - r) R + (\sigma - s) S + (\tau - t) T = 0;$$

and a parallel plane through the origin is

$$\rho R + \sigma S + \tau T = 0.$$

The original equation has two characteristics in general: but they coalesce into one if

$$S^2 = 4RT.$$

The envelope of the second plane, when this condition is satisfied, is

$$\rho\tau - \sigma^2 = 0,$$

which is a cone: so that the two characteristics coalesce into one, when the tangent planes of the surface $f=0$ are parallel to the tangent planes of the cone

$$rt - s^2 = 0.$$

Again, when we take

$$\frac{dp}{dx} = \xi, \quad \frac{dq}{dx} = \eta, \quad \frac{dy}{dx} = m,$$

so that ξ, η, m are parametric quantities, the equations

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

become

$$r = -ms + \xi, \quad s = -mt + \eta,$$

which are the equations of a line parallel to

$$r = -ms, \quad s = -mt,$$

and the latter is a generator of the cone

$$rt - s^2 = 0.$$

To find the intersections of the line with the surface $f = 0$, we substitute

$$r = -ms + \xi, \quad t = -\frac{1}{m}(s - \eta),$$

in the equation. In general, we have a set of values of s thus given: but if it happens that the equation is evanescent after the substitution, the line lies entirely in the surface $f = 0$, which therefore possesses generators parallel to those of the cone $rt - s^2 = 0$.

When the surface $f = 0$ is perfectly unconditioned, it obviously will not possess this special property: we have the first of the preceding cases (§ 279).

When the surface $f = 0$ is not quite arbitrary, but is such that certain conditions among its parameters x, y, z, p, q are satisfied, the special property can be possessed: we have the second of the preceding cases.

When the surface $f = 0$ is not quite arbitrary, the property may be possessed for appropriate values, or sets of values, of m, ξ, η. We then have the third of the preceding cases: it contains a couple of sub-cases.

In the first of these sub-cases, there are two relations between x, y, z, p, q, ξ, η, m. Hence there is a simple infinitude of sets of values for ξ, η, m, so that the surface contains a simple infinitude of generators: and these are parallel to the generators of the cone

$$rt - s^2 = 0,$$

which therefore is an asymptotic cone of the ruled surface (or scroll) represented by $f = 0$.

In the second of the sub-cases, there are three relations between x, y, z, p, q, ξ, η, m. We then have a finite number of sets of values for ξ, η, m, so that the surface represented by $f=0$ contains only a finite number of straight lines; it is not a ruled surface.

When the first sub-case arises, in which there are two relations, these may be taken in the form

$$G_1(x, y, z, p, q, \xi, \eta, m) = 0,$$
$$G_2(x, y, z, p, q, \xi, \eta, m) = 0;$$

owing to them, the equations

$$f=0, \quad r = -ms + \xi, \quad s = -mt + \eta,$$

do not determine r, s, t definitely. If then functions x, y, z, p, q of a single variable can be so chosen that the equations

$$G_1 = 0, \quad G_2 = 0, \quad dz = pdx + qdy,$$

hold, r, s, t are not determinate. In these circumstances, two integrals of the differential equation can have the same values of x, y, z, p, q, but r, s, t will not necessarily be the same; the contact of the integrals along the characteristic cannot generally be of the second order but is generally of the first order.

Although the equations are satisfied for one of the roots m of the quadratic

$$m^2R - mS + T = 0,$$

they are not usually satisfied for the other root: then, for that other root, r, s, t are determinate; and so two integrals, having contact of only the first order along one characteristic, will usually have contact of the second order along the other. If, however, the quadratic has equal roots, there is only one characteristic: and the contact of two integrals is only of the first order.

In both of these cases, the surface represented by $f=0$ has an infinitude of generators parallel to those of the cone $rt-s^2=0$: in the former, the surface is a scroll; in the latter, it is developable.

Classification of Equations according to Characteristics.

281. Returning now to the differential equations of the second order, we can classify them according to their characteristics.

One class of equations is composed of those which possess two different characteristics of the second order. Two integrals (when x, y, z are regarded as coordinates) have contact of the second order at least along both of the characteristics.

Another class of equations is composed of those which, when x, y, z, p, q are regarded as parametric, represent scrolls (ruled undevelopable surfaces) having $rt - s^2 = 0$ for an asymptotic cone. There are two distinct characteristics; two integrals have contact of only the first order along one of the characteristics and contact of the second order along the other.

Another class of equations is composed of those which, when x, y, z, p, q are regarded as parametric, represent developable surfaces having their tangent planes parallel to the tangent planes of the cone $rt - s^2 = 0$. There is a single characteristic: two integrals have contact of only the first order along that characteristic.

Another class of equations is composed of those which are linear in r, s, t, $rt - s^2$, of the form

$$f = Ar + 2Bs + Ct + K(rt - s^2) - D = 0.$$

When K is not zero, the scroll $f = 0$ has generators parallel to those of $rt - s^2 = 0$. Usually $f = 0$ has two systems of generators distinct from one another; there then are two systems of characteristics, and two integrals have contact of only the first order along each of them. But when the relation

$$B^2 - AC = DK$$

is satisfied, so that $f = 0$ has the form

$$(Kr + C)(Kt + A) = (Ks - B)^2,$$

and the surface is a cone, the same as $rt - s^2 = 0$ and similarly placed, there is only one system of generators: there is a single characteristic, and two integrals have contact of only the first order along it.

When K is zero, the surface $f=0$ is a plane: through any point of it, there are generally two (real or imaginary) straight lines in it parallel to generators of the cone $rt-s^2=0$; in that case, there are two characteristics, and two integrals have contact of only the first order along each of them. But if the plane is parallel to a tangent plane to the cone, then through a point in it only one line can be drawn parallel to a generator; there is a single characteristic, and two integrals have contact of only the first order along that characteristic.

It thus appears that, it a relation exists between x, y, z, p, q free from differential elements, the differential equation possesses at least one characteristic of the first order: though, conversely, it is not the fact that, even if the equation possesses a characteristic of the first order, some relation exists between x, y, z, p, q, which is free from differential elements. When such a relation does exist in a form, represented by

$$u(x, y, z, p, q)=0,$$

so that all its integrals possessing any arbitrary element (that is, integrals other than singular or special) satisfy the equation of the second order, it is called an intermediate integral. Thus the investigation of the characteristics of the first order involves the construction of intermediate integrals, if any such exist.

A method has already been given (in Chapter XVI) for the construction of intermediate integrals, if any, of a propounded equation of the second order: and that method is effective for any such equation, for it gives the tests that are necessary and sufficient to secure the existence of the intermediate integral. We need not, therefore, deal further with this question of constructing the intermediate integrals (if any) of a given equation.

Equations having Intermediate Integrals.

282. But the association of intermediate integrals with characteristics of the first order suggests an inquiry into the classes of equations that do possess intermediate integrals. After the classification of equations of the second order which has been adopted, it manifestly is unnecessary to consider any of the equations in the first of the selected classes in § 281: and, after the full discussion of equations of the form

$$rt - s^2 + Ar + 2Bs + Ct = D,$$
$$Ar + 2Bs + Ct = D,$$

given in Chapter XVI, it is unnecessary to give further consideration to any of the equations in the last of those selected classes. We need only therefore consider equations of those classes which possess a characteristic of the first order.

There are two modes of constructing such equations: and, when regard is paid to the association of intermediate integrals with characteristics of the first order, the two modes are equivalent to one another.

A characteristic of the first order, if it exists, arises by assigning the conditions that the equations

$$f = 0, \quad dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

must be inadequate for the precise determination of r, s, t. The number of conditions, independent of one another, may be two or may be three; if, taken concurrently with the relation

$$dz = p\,dx + q\,dy,$$

they are satisfied by any relation

$$u(x, y, z, p, q) = 0,$$

which is independent of differential elements, that relation is an intermediate integral. Now the equations of the characteristic of $u = 0$, regarded as an equation of the first order, are (§ 94)

$$\frac{dx}{u_p} = \frac{dy}{u_q} = \frac{dz}{pu_p + qu_q} = \frac{dp}{-u_x} = \frac{dq}{-u_y},$$

where

$$u_x = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}, \quad u_y = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z},$$

and u_p, u_q are the derivatives of u with regard to p and q: and this characteristic must be the characteristic of f which is of the first order. The equations

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

are transformed by the coexistent equations of the characteristic of u into the equations

$$-u_x = ru_p + su_q, \quad -u_y = su_p + tu_q:$$

and the conditions now are that these two equations, together with $f=0$, must be inadequate for the precise determination of r, s, t.

The latter process is the alternative method of proceeding to an intermediate integral, and it is the method adopted in §§ 238—244. In order to be effective, there must be either two relations or three relations, involving the derivatives of u and independent of one another: and only those equations of the second order can have intermediate integrals when the application of the process leads to two relations or to three relations which, independent of one another, can be satisfied by a common value of u.

After these explanations, we shall adopt the latter process so far as the analysis is concerned: we shall therefore make the equations

$$f=0, \quad u_x + ru_p + su_q = 0, \quad u_y + su_p + tu_q = 0,$$

inadequate for the precise determination of r, s, t: and this indeterminateness will require either two independent relations or three independent relations.

First Case.

283. In the first place, suppose that the requirement for indeterminateness in the equations, so far as r, s, t are concerned, leads to a couple of algebraically independent conditions. Let these conditions be resolved for u_x and u_y, so that they have a form

$$\left.\begin{aligned} u_x + g(x, y, z, p, q, u_p, u_q) = 0 \\ u_y + h(x, y, z, p, q, u_p, u_q) = 0 \end{aligned}\right\},$$

where the functions g and h are homogeneous, of order unity, in u_p and u_q. We require integrals u of these two simultaneous partial equations of the first order: as the number of variables, which can occur in u, is five, the number of independent integrals can be three, two, one, or none.

We proceed by the Jacobian method. It is convenient to change the notation: we take

$$x,\ y,\ z,\ p,\ q = x_1,\ x_2,\ x_3,\ x_4,\ x_5,$$

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial z},\ \frac{\partial u}{\partial p},\ \frac{\partial u}{\partial q} = p_1,\ p_2,\ p_3,\ p_4,\ p_5:$$

and then the equations are

$$F_1 = p_1 + x_4 p_3 + g(x_1, x_2, x_3, x_4, x_5, p_4, p_5) = 0,$$
$$F_2 = p_2 + x_5 p_3 + h(x_1, x_2, x_3, x_4, x_5, p_4, p_5) = 0.$$

The Poisson-Jacobi condition of coexistence, which is $(F_1, F_2) = 0$, must be satisfied: expressed in full, it is

$$\frac{\partial h}{\partial x_1} - \frac{\partial g}{\partial x_2} + x_4 \frac{\partial h}{\partial x_3} - x_5 \frac{\partial g}{\partial x_3}$$
$$+ \frac{\partial g}{\partial p_4}\frac{\partial h}{\partial x_4} - \left(p_3 + \frac{\partial g}{\partial x_4}\right)\frac{\partial h}{\partial p_4} + \frac{\partial g}{\partial p_5}\left(p_3 + \frac{\partial h}{\partial x_5}\right) - \frac{\partial h}{\partial p_5}\frac{\partial g}{\partial x_5} = 0.$$

Now this equation cannot be satisfied in virtue of $F_1 = 0$ or $F_2 = 0$, for it contains neither p_1 nor p_2: hence either it is an identity on account of the forms of g and h, or it is a new equation.

When it is an identity, the equations $F_1 = 0$ and $F_2 = 0$ are a complete Jacobian system; and then they possess three common integrals independent of one another. In that event, noting that neither g nor h contains p_3, we have

$$\frac{\partial g}{\partial p_5} = \frac{\partial h}{\partial p_4},$$

$$\frac{\partial h}{\partial x_1} - \frac{\partial g}{\partial x_2} + x_4 \frac{\partial h}{\partial x_3} - x_5 \frac{\partial g}{\partial x_3} + J\left(\frac{g,\ h}{p_4,\ x_4}\right) + J\left(\frac{g,\ h}{p_5,\ x_5}\right) = 0.$$

The former of these shews that there is some function F, such that

$$g = \frac{\partial F}{\partial p_4}, \quad h = \frac{\partial F}{\partial p_5};$$

and F consists of two parts, one of them homogeneous of the second order in p_4 and p_5, the other of them not involving p_4 or p_5. Writing

$$p_5 = m p_4,$$

we can take

$$F = p_4^2 \psi(x_1, x_2, x_3, x_4, x_5, m) + G$$
$$= p_4^2 \psi + G,$$

where G does not involve p_4 or m; and then

$$g(x_1, x_2, x_3, x_4, x_5, p_4, p_5) = 2p_4\psi - p_5 \frac{\partial \psi}{\partial m},$$

$$h(x_1, x_2, x_3, x_4, x_5, p_4, p_5) = p_4 \frac{\partial \psi}{\partial m}.$$

The equations of the characteristic of u are

$$\frac{dx}{p_4} = \frac{dy}{p_5} = \frac{dp}{-p_1 - x_4 p_3} = \frac{dq}{-p_2 - x_5 p_3}$$

$$= \frac{dp}{2p_4\psi - p_5 \frac{\partial\psi}{\partial m}} = \frac{dq}{p_4 \frac{\partial\psi}{\partial m}},$$

so that

$$dy = m\,dx,$$

$$dp = 2\psi\,dx - \frac{\partial\psi}{\partial m}\,dy,$$

$$dq = \frac{\partial\psi}{\partial m}\,dx.$$

In order to obtain the differential equation, we have

$$dp = r\,dx + s\,dy = (r + sm)\,dx,$$

$$dq = s\,dx + t\,dy = (s + tm)\,dx,$$

and therefore

$$r + sm = 2\psi - m\frac{\partial\psi}{\partial m},$$

$$s + tm = \frac{\partial\psi}{\partial m}.$$

When we eliminate m, we have the required equation: or, what is the same thing, *the required equation is given by the elimination of m between the equations*

$$\left.\begin{aligned} r + 2sm + tm^2 &= 2\psi \\ s + tm &= \frac{\partial\psi}{\partial m} \end{aligned}\right\}.$$

(These equations, on the geometrical illustration of § 280, represent a developable surface: the result may be compared with the results before obtained). The function ψ does not involve $r, s,$ or t: and it has to satisfy a condition represented by the second relation between g and h. When the forms obtained for g and h are substituted in that relation, it takes the form*

$$2\frac{\partial^2\psi}{\partial m^2}\left(m\frac{\partial\psi}{\partial x_4} - \frac{\partial\psi}{\partial x_5}\right) + \frac{\partial^2\psi}{\partial m\partial x_5}\frac{\partial\psi}{\partial m} + \frac{\partial^2\psi}{\partial m\partial x_4}\left(2\psi - m\frac{\partial\psi}{\partial m}\right)$$

$$+ (x_4 + m x_5)\frac{\partial^2\psi}{\partial m\,\partial x_3} + m\frac{\partial^2\psi}{\partial m\partial x_2} + \frac{\partial^2\psi}{\partial m\partial x_1} - 2\frac{\partial\psi}{\partial m}\frac{\partial\psi}{\partial x_4} - 2x_5\frac{\partial\psi}{\partial x_3} - 2\frac{\partial\psi}{\partial x_2} = 0,$$

* The equation differs from Goursat's form (*Acta Math.*, t. XIX, p. 322) by the omission of one term.

or, on restoring the old variables, it is

$$2\frac{\partial^2\psi}{\partial m^2}\left(m\frac{\partial\psi}{\partial p}-\frac{\partial\psi}{\partial q}\right)+\frac{\partial^2\psi}{\partial m\partial q}\frac{\partial\psi}{\partial m}+\frac{\partial^2\psi}{\partial m\partial p}\left(2\psi-m\frac{\partial\psi}{\partial m}\right)$$

$$+(p+mq)\frac{\partial^2\psi}{\partial m\partial z}+m\frac{\partial^2\psi}{\partial m\partial y}+\frac{\partial^2\psi}{\partial m\partial x}-2\frac{\partial\psi}{\partial m}\frac{\partial\psi}{\partial p}-2q\frac{\partial\psi}{\partial z}-2\frac{\partial\psi}{\partial y}=0.$$

Any integral of this equation gives a possible form for ψ, and so determines a differential equation of the second order.

In this case, the equations $F_1=0$ and $F_2=0$ have three common integrals functionally independent of one another: let them be θ_1, θ_2, θ_3. We resolve the equations

$$F_1=0,\quad F_2=0,\quad \theta_1=a',\quad \theta_2=b',\quad \theta_3=c',$$

for p_1, p_2, p_3, p_4, p_5: we substitute in

$$du=\sum_{n=1}^{5}p_n dx_n,$$

integrate, and divide out by a homogeneous constant: and we then have a relation

$$u(x,\ y,\ z,\ p,\ q,\ a,\ b,\ c)=0,$$

which is an intermediate integral of the constructed differential equation of the second order. One constant, say c, is additive.

284. The primitive of the differential equation of the second order is obtained simply, on the basis of a general proposition due to Goursat. Let

$$u=0$$

denote the intermediate integral; and let a, b denote the two constants in u which are not merely additive: then Goursat's theorem is that, *if p and q be eliminated between the equations*

$$u=0,\quad \frac{\partial u}{\partial a}=\alpha,\quad \frac{\partial u}{\partial b}=\beta,$$

where α and β are arbitrary constants, the eliminant is a primitive of the equation of the second order. The proof is as follows. We have

$$F_1=p_1+x_4p_3+g(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5)=0,$$

$$F_2=p_2+x_5p_3+h(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5)=0;$$

so that

$$\frac{\partial p_1}{\partial a} + x_4 \frac{\partial p_3}{\partial a} + \frac{\partial g}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial g}{\partial p_5}\frac{\partial p_5}{\partial a} = 0,$$

$$\frac{\partial p_2}{\partial a} + x_5 \frac{\partial p_3}{\partial a} + \frac{\partial h}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial h}{\partial p_5}\frac{\partial p_5}{\partial a} = 0.$$

Now let the Poisson-Jacobi combinant of u and $\frac{\partial u}{\partial a}$ be constructed: thus

$$\left[u, \frac{\partial u}{\partial a}\right] = (p_1 + x_4 p_3)\frac{\partial p_4}{\partial a} - p_4\left(\frac{\partial p_1}{\partial a} + x_4\frac{\partial p_3}{\partial a}\right)$$
$$+ (p_2 + x_5 p_3)\frac{\partial p_5}{\partial a} - p_5\left(\frac{\partial p_2}{\partial a} + x_5\frac{\partial p_3}{\partial a}\right)$$
$$= -g\frac{\partial p_4}{\partial a} + p_4\left(\frac{\partial g}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial g}{\partial p_5}\frac{\partial p_5}{\partial a}\right)$$
$$-h\frac{\partial p_5}{\partial a} + p_5\left(\frac{\partial h}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial h}{\partial p_5}\frac{\partial p_5}{\partial a}\right)$$
$$= \frac{\partial p_4}{\partial a}\left(-g + p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial h}{\partial p_4}\right) + \frac{\partial p_5}{\partial a}\left(-h + p_4\frac{\partial g}{\partial p_5} + p_5\frac{\partial h}{\partial p_5}\right)$$
$$= \frac{\partial p_4}{\partial a}\left(-g + p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial g}{\partial p_5}\right) + \frac{\partial p_5}{\partial a}\left(-h + p_4\frac{\partial h}{\partial p_4} + p_5\frac{\partial h}{\partial p_5}\right),$$

because of the relation

$$\frac{\partial g}{\partial p_5} = \frac{\partial h}{\partial p_4}.$$

Also g and h are homogeneous of the first order in p_4 and p_5, so that

$$g = p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial g}{\partial p_5}, \quad h = p_4\frac{\partial h}{\partial p_4} + p_5\frac{\partial h}{\partial p_5};$$

consequently,

$$\left[u, \frac{\partial u}{\partial a}\right] = 0.$$

Similarly, we can prove that

$$\left[u, \frac{\partial u}{\partial b}\right] = 0,$$

$$\left[\frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}\right] = 0:$$

hence the equations

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta,$$

coexist. The elimination of p and q between them leads to a relation between $x, y, z, a, b, c, \alpha, \beta$, which is consistent with all of them: it is a complete primitive.

Goursat also shews that the general primitive of the equation of the second order can be deduced from a knowledge of the intermediate integral. Denoting the two non-additive constants in u by a and b as before, consider the equations

$$u=0, \quad b=\phi(a), \quad \frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\phi'(a)=0.$$

Resolving the last two equations for a and b, and substituting the deduced values in $u, \frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}$, let us denote the results by u_1, u_2, u_3 respectively. Then

$$\begin{aligned}\frac{\partial u_1}{\partial x_n}&=\frac{\partial u}{\partial x_n}+\left\{\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\phi'(a)\right\}\frac{\partial a}{\partial x_n}\\&=\frac{\partial u}{\partial x_n},\\ \frac{\partial u_2}{\partial x_n}&=\frac{\partial^2 u}{\partial x_n\partial a}+\left\{\frac{\partial^2 u}{\partial a^2}+\frac{\partial^2 u}{\partial a\partial b}\phi'(a)\right\}\frac{\partial a}{\partial x_n},\end{aligned}$$

for $n=1, 2, 3, 4, 5$; also

$$0=\frac{\partial^2 u}{\partial x_n\partial a}+\frac{\partial^2 u}{\partial x_n\partial b}\phi'(a)+\left\{\frac{\partial^2 u}{\partial a^2}+2\frac{\partial^2 u}{\partial a\partial b}\phi'(a)+\frac{\partial^2 u}{\partial b^2}\phi'^2(a)+\frac{\partial u}{\partial b}\phi''(a)\right\}\frac{\partial a}{\partial x_n}.$$

When the value of $\frac{\partial a}{\partial x_n}$ given by the last equation is substituted in $\frac{\partial u_2}{\partial x_n}$, we have

$$\frac{\partial u_2}{\partial x_n}=\lambda\frac{\partial^2 u}{\partial x_n\partial a}+\mu\frac{\partial^2 u}{\partial x_n\partial b},$$

where λ and μ do not change for the different values of n. Hence, as $\frac{\partial u_1}{\partial x_n}=\frac{\partial u}{\partial x_n}$, we have

$$\begin{aligned}[u_1, u_2]&=\lambda\left[u_1, \frac{\partial u}{\partial a}\right]+\mu\left[u_1, \frac{\partial u}{\partial b}\right]\\&=\lambda\left[u, \frac{\partial u}{\partial a}\right]+\mu\left[u, \frac{\partial u}{\partial b}\right]\\&=0;\end{aligned}$$

and, similarly,

$$[u_1, u_3]=0,$$
$$[u_2, u_3]=0.$$

Thus we obtain a primitive by the elimination of p and q between the three equations

$$u_1 = 0, \quad u_2 = \alpha, \quad u_3 = \beta,$$

or (what is the same thing) by the elimination of p, q, a, b between the five equations

$$u = 0, \quad \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \phi'(a) = 0, \quad b = \phi(a), \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta.$$

Eliminating p and q first, we have relations

$$\theta(x, y, z, a, b, \alpha, \beta) = 0,$$
$$b = \phi(a), \quad \alpha + \beta\phi'(a) = 0,$$

from which to eliminate a and b. Replacing α by a new constant α', such that $\alpha' + \beta\phi'(a) = 0$, we have a primitive given by

$$\theta\{x, y, z, a, \phi(a), -\beta\phi'(a), \beta\} = 0;$$

and when we replace β by $\chi(a)$, where χ is an arbitrary function, the general primitive of the constructed equation of the second order is given by

$$\left.\begin{aligned} \theta\{x, y, z, a, \phi(a), -\chi(a)\phi'(a), \chi(a)\} = 0 \\ \frac{\partial\theta}{\partial a} + \frac{\partial\theta}{\partial\phi}\phi'(a) + \frac{\partial\theta}{\partial\phi'}\phi''(a) + \frac{\partial\theta}{\partial\chi}\chi'(a) = 0 \end{aligned}\right\},$$

involving two arbitrary functions.

285. The equation for ψ is of the second order, being linear in the derivatives of that order; but the construction of the quantity ψ is made difficult, on account of the number of independent variables with respect to which the derivatives are taken. In the absence of the knowledge of the most general integral, we cannot construct the aggregate of equations of the type under consideration: but, by obtaining special values of ψ satisfying the equation, we can construct special equations or even special classes of equations. One or two examples will suffice.

Ex. 1. It is obvious that the equation for ψ is satisfied by taking ψ to be a function of m only: ψ can be an arbitrary function of m. The differential equation then occurs as the eliminant of

$$\left.\begin{aligned} r + 2sm + tm^2 = 2\psi \\ s + tm = \frac{\partial\psi}{\partial m} \end{aligned}\right\},$$

or, what is the same thing, it is obtained by equating to zero the m-discriminant of

$$r+2sm+tm^2-2\psi.$$

The equations, satisfied by the quantity u connected with an intermediate integral, are

$$F_1=p_1+x_4p_3+2p_4\psi\left(\frac{p_5}{p_4}\right)-p_5\psi'\left(\frac{p_5}{p_4}\right)=0,$$

$$F_2=p_2+x_5p_3+p_4\psi'\left(\frac{p_5}{p_4}\right)=0;$$

and these two equations are a complete system. Obviously

$$(F_1,\ p_r)=0,\quad (F_2,\ p_r)=0,$$

for $r=1, 2, 3$; so that p_1, p_2, p_3 can be taken as the three integrals that are independent of one another. We then have

$$p_1=a,\quad p_2=b,\quad p_3=c,\quad F_1=0,\quad F_2=0;$$

thus

$$p_4(2\psi-m\psi')=-a-cx_4,$$

$$p_4\psi'=-b-cx_5,$$

and so m is determined by the relation

$$2\frac{\psi}{\psi'}-m=\frac{a+cx_4}{b+cx_5}=\mu,$$

say. Now

$$du=\sum_{n=1}^{5}p_n dx_n,$$

and therefore

$$d(ax_1+bx_2+cx_3-u)=\frac{b+cx_5}{\psi'}(dx_4+m\,dx_5).$$

Changing the independent variables on the right-hand side to m and x_5, we have

$$a+cx_4=(b+cx_5)\mu,$$

and therefore

$$c\,dx_4=c\mu\,dx_5+(b+cx_5)\mu' dm;$$

also

$$\mu+m=2\frac{\psi}{\psi'}.$$

Consequently,

$$d(ax_1+bx_2+cx_3-u)=\frac{b+cx_5}{\psi'}\left\{(\mu+m)\,dx_5+\frac{b+cx_5}{c}\mu' dm\right\}$$

$$=\frac{1}{c}\,d\left\{(b+cx_5)^2\frac{\psi}{\psi'^2}\right\},$$

on reduction: and therefore

$$ax_1+bx_2+cx_3-u=\frac{\psi}{c\psi'^2}(b+cx_5)^2+A.$$

Now the intermediate integral is $u=0$; hence, resuming the variables x, y, z, p, q, and making c equal to unity (which involves no loss of generality), we have

$$z+ax+by-(b+q)^2\frac{\psi}{\psi'^2}-k=0,$$

where k is an arbitrary additive constant: the argument m of ψ, which is an arbitrary function, is given by

$$2\frac{\psi}{\psi'}-m=\frac{p+a}{q+b}.$$

The equation thus obtained, account being taken of the value of m, is an intermediate integral of the equation of the second order, represented by

$$\text{Discr.}_m\,(r+2sm+tm^2-2\psi)=0.$$

The equation, which determines m in connection with the intermediate integral, shews that m is a function of p and q: hence, when we construct the equations of the characteristic of

$$z+ax+by-(b+q)^2\frac{\psi}{\psi'^2}-k=0,$$

one of them is

$$\frac{-dp}{a+p}=\frac{-dq}{b+q}.$$

An integral of this equation is

$$\frac{p+a}{q+b}=\gamma,$$

where γ is a constant: hence, for the primitive, m is a constant. Combining this equation with the intermediate integral, we have

$$p+a=\gamma\,(z+ax+by-k)^{\frac{1}{2}}\,\psi'\psi^{-\frac{1}{2}},$$
$$q+b=\ \ (z+ax+by-k)^{\frac{1}{2}}\,\psi'\psi^{-\frac{1}{2}};$$

substituting these values of p and q in the relation

$$dz=p\,dx+q\,dy,$$

and effecting the quadrature, we have

$$2\,(z+ax+by-k)^{\frac{1}{2}}=(\gamma x+y+\delta)\,\psi'\psi^{-\frac{1}{2}},$$

where the argument m of ψ now is given by

$$2\frac{\psi}{\psi'}-m=\gamma.$$

This equation is a complete primitive, involving five arbitrary constants a, b, k, γ, δ.

To obtain the general primitive, let

$$p+a=v\,(2\psi-m\psi'),$$

where v is a new magnitude: then

$$q+b=v\psi',$$

and

$$u=z+ax+by-k-v^2\psi=0.$$

Now

$$1=\frac{\partial v}{\partial a}(2\psi-m\psi')+v(\psi'-m\psi'')\frac{\partial m}{\partial a},$$

$$0=\frac{\partial v}{\partial a}\psi'+v\psi''\frac{\partial m}{\partial a}:$$

hence

$$\frac{\partial u}{\partial a}=x-2v\frac{\partial v}{\partial a}\psi-v^2\psi'\frac{\partial m}{\partial a}$$

$$=x-v;$$

and

$$\frac{\partial u}{\partial b}=y-vm,$$

similarly obtained. Accordingly, after the general theory, we eliminate m and v between the equations

$$z+ax+by-k-v^2\psi=0,$$

$$x-\ \ v\ =\alpha,$$

$$y-mv=\beta,$$

leading to a relation

$$\theta(x, y, z, a, b, \alpha, \beta)=0.$$

The general primitive is given by

$$\left.\begin{array}{c}\theta\{x, y, z, a, \phi(a), -\phi'(a)\chi(a), \chi(a)\}=0\\ \dfrac{d\theta}{da}=0\end{array}\right\}.$$

Ex. 2. It is natural to enquire whether ψ can have the form

$$\psi=Am^2+2Bm+C,$$

where A, B, C involve only x, y, z, p, q. If it is possible, the differential equation of the second order is the eliminant of

$$r+2sm+tm^2=2Am^2+4Bm+2C,$$

$$s+tm=2Am+2B,$$

that is, it is

$$(r-2C)(t-2A)=(s-2B)^2.$$

On the present hypothesis, there are to be three integrals of the subsidiary system: we, therefore, have the problem of § 241.

The preceding analysis shews that the possibility will be realised, if ψ satisfies the differential equation of the second order. That this may be the case, we find (on actual substitution of the supposed value of ψ) that the relation

$$\Theta_1 m^3+\Theta_2 m^2+\Theta_3 m+\Theta_4=0$$

must be satisfied identically, where Θ_1, Θ_2, Θ_3, Θ_4 do not contain m; and therefore we must have

$$\Theta_1=0,\quad \Theta_2=0,\quad \Theta_3=0,\quad \Theta_4=0.$$

On effecting the calculations, it appears that the equations

$$\Theta_1=0,\quad \Theta_2=0,$$

must be satisfied, both identically: the equation $\Theta_3=0$ becomes

$$\left(\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+2B\frac{\partial}{\partial q}+2C\frac{\partial}{\partial p}\right)A=\left(\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+2A\frac{\partial}{\partial q}+2B\frac{\partial}{\partial p}\right)B,$$

which, on noting that $2C$, $2B$, $2A=R$, S, T, is the old relation

$$\Delta R=\Delta' S;$$

and the equation $\Theta_4=0$ similarly leads to the old relation

$$\Delta S=\Delta' T.$$

The equation, subject to these conditions, has already been fully discussed.

Ex. 3. Obtain the conditions which must be satisfied, in order that ψ may (if possible) have either of the forms

$$\frac{Am+B}{Cm+D},\quad Am^3+3Bm^2+3Cm+D,$$

where A, B, C, D are functions of x, y, z, p, q only.

Ex. 4. Discuss the equations that arise when ψ has either of the following forms, each of which satisfies the equation of the second order:

(i), $\psi=am^2p^n$, where a and n are arbitrary;

(ii), $\psi=am+a^2f(q)+b$, where a and b are arbitrary, and f is an arbitrary function.

286. We now pass to the consideration of the equations

$$F_1=0,\quad F_2=0,$$

when the condition $(F_1, F_2)=0$, which must be satisfied and cannot be satisfied in virtue of either of the preceding equations, is not an identity; consequently, it is a new equation, say

$$F_3=Ap_3+B=0,$$

where

$$A=\frac{\partial g}{\partial p_5}-\frac{\partial h}{\partial p_4},$$

and

$$B=\frac{\partial h}{\partial x_1}-\frac{\partial g}{\partial x_2}+x_4\frac{\partial h}{\partial x_3}-x_5\frac{\partial g}{\partial x_3}+J\left(\frac{g,\ h}{p_4,\ x_4}\right)+J\left(\frac{g,\ h}{p_5,\ x_5}\right).$$

If the number of independent integrals of the original system $F_1=0$, $F_2=0$, is to be two, then $F_1=0$, $F_2=0$, $F_3=0$, must be a complete Jacobian system.

If, in these circumstances, it were possible that A should vanish identically, the equation $F_3=0$ would become

$$B=0.$$

Now B is homogeneous of the first order in p_4 and p_5: hence it can be resolved into a number of equations of the type

$$p_4 + \mu p_5 = 0,$$

where μ is a function of x_1, x_2, x_3, x_4, x_5; and the set of equations would be

$$\begin{aligned} F_1 &= p_1 + x_4 p_3 + p_5 G = 0, \\ F_2 &= p_2 + x_5 p_3 + p_5 H = 0, \\ F_3 &= \qquad\quad p_4 + p_5 \mu = 0, \end{aligned}$$

where G, H, μ are functions of x_1, x_2, x_3, x_4, x_5 only. Also

$$(F_1, F_3) = p_5 \left(\frac{\partial \mu}{\partial x_1} + x_4 \frac{\partial \mu}{\partial x_3} + G \frac{\partial \mu}{\partial x_5}\right) - \left(p_3 + p_5 \frac{\partial G}{\partial x_4}\right) - \mu p_5 \frac{\partial G}{\partial x_5}.$$

The right-hand side must vanish: it cannot vanish in virtue of $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, because p_3 occurs only in the term $-p_3$, while p_1 and p_2 do not occur; and it does not vanish identically. Hence it is a new equation, and therefore the set of three equations is not complete. It therefore follows that, when $(F_1, F_2) = 0$ provides a new equation, the quantity A is not zero.

Two cases occur for consideration when A is not zero, according as B does or does not vanish.

SECOND CASE.

287. When B vanishes, the equation $F_3 = 0$ becomes $p_3 = 0$: when this is used to modify the other equations, the system is

$$\begin{aligned} F_1 &= p_1 + p_4 G\,(x_1, x_2, x_3, x_4, x_5, m) = 0, \\ F_2 &= p_2 + p_4 H\,(x_1, x_2, x_3, x_4, x_5, m) = 0, \\ F_3 &= p_3 \qquad\qquad\qquad\qquad\qquad\quad = 0, \end{aligned}$$

where

$$p_5 = m p_4.$$

As the system is to be complete, the necessary conditions must be satisfied. From $(F_1, F_3) = 0$, we have

$$\frac{\partial G}{\partial x_3} = 0;$$

from $(F_2, F_3) = 0$, we have

$$\frac{\partial H}{\partial x_3} = 0;$$

and from $(F_1, F_2) = 0$, we have (after an easy reduction)

$$\frac{\partial H}{\partial x_1} - \frac{\partial G}{\partial x_2} + G\frac{\partial H}{\partial x_4} - H\frac{\partial G}{\partial x_4} + m\frac{\partial(G, H)}{\partial(x_4, m)} - \frac{\partial(G, H)}{\partial(x_5, m)} = 0.$$

It follows that both F and G are explicitly free from x_3: and the last equation shews, that either G or H may be arbitrarily assumed (subject to the non-occurrence of x_3), and that the other is determinable as an integral of an equation of the first order, though the explicit form of this integral will be affected by the form adopted for the other quantity. Thus, assuming G assigned, the equation is

$$\frac{\partial H}{\partial x_1} + \left(G - m\frac{\partial G}{\partial m}\right)\frac{\partial H}{\partial x_4} + \frac{\partial G}{\partial m}\frac{\partial H}{\partial x_5} + \left(m\frac{\partial G}{\partial x_4} - \frac{\partial G}{\partial x_5}\right)\frac{\partial H}{\partial m}$$
$$= H\frac{\partial G}{\partial x_4} + \frac{\partial G}{\partial x_2};$$

for the determination of H, we should require integrals of the system

$$dx_1 = \frac{dx_4}{G - m\dfrac{\partial G}{\partial m}} = \frac{dx_5}{\dfrac{\partial G}{\partial m}} = \frac{dm}{m\dfrac{\partial G}{\partial x_4} - \dfrac{\partial G}{\partial x_5}} = \frac{dH}{H\dfrac{\partial G}{\partial x_4} + \dfrac{\partial G}{\partial x_2}}.$$

Ex. 1. A set of cases is provided by taking

$$G = \frac{\partial\phi}{\partial x_1}, \quad H = \frac{\partial\phi}{\partial x_2},$$

where ϕ is any function of x_1, x_2, m. The equations are

$$F_3 = p_3 = 0,$$
$$F_1 = p_1 + p_4\frac{\partial\phi}{\partial x_1} = 0,$$
$$F_2 = p_2 + p_4\frac{\partial\phi}{\partial x_2} = 0.$$

Evidently

$$(F_r, p_4) = 0, \quad (F_r, p_5) = 0,$$

for $r = 1, 2, 3$; hence p_4 and p_5 are two independent integrals of the system. Accordingly, we take

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0, \quad p_4 = a, \quad p_5 = b,$$

where a and b are constants; and then

$$du = -a\frac{\partial\phi\left(x_1, x_2, \dfrac{b}{a}\right)}{\partial x_1}dx_1 - a\frac{\partial\phi\left(x_1, x_2, \dfrac{b}{a}\right)}{\partial x_2}dx_2 + a\,dx_4 + b\,dx_5,$$

so that

$$u = -a\phi\left(x_1, x_2, \frac{b}{a}\right) + ax_4 + bx_5 + c.$$

Now $u=0$ is the intermediate integral: hence

$$ax_4+bx_5+c=a\phi\left(x_1,\ x_2,\ \frac{b}{a}\right),$$

or, in the other notation,

$$p+aq+\beta=\phi\,(x,\ y,\ a),$$

where a and β are arbitrary constants.

The equations of the characteristic are

$$\frac{dx}{1}=\frac{dy}{a}=\frac{dp}{\dfrac{\partial\phi}{\partial x}}=\frac{dq}{\dfrac{\partial\phi}{\partial y}},$$

and

$$dp=rdx+sdy,\quad dq=sdx+tdy:$$

consequently,

$$\left.\begin{aligned} r+sa&=\frac{\partial\phi}{\partial x}\\ s+ta&=\frac{\partial\phi}{\partial y}\end{aligned}\right\}.$$

The required differential equation of the second order is given by eliminating a between these equations.

A general intermediate integral is given by the elimination of a between the equations

$$\left.\begin{aligned} p+aq+\psi\,(a)&=\phi\,(x,\ y,\ a)\\ q+\psi'\,(a)&=\frac{\partial\phi}{\partial a}\end{aligned}\right\},$$

ψ being an arbitrary function. To proceed to the primitive, we take p and q in the forms

$$p=\phi-a\,\frac{\partial\phi}{\partial a}+a\psi'-\psi,$$

$$q=\frac{\partial\phi}{\partial a}-\psi';$$

and then the relation

$$pdx+qdy$$

must be an exact differential. That this may be the case, a must be such a function of x and y that

$$\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x},$$

that is,

$$\frac{\partial^2\phi}{\partial x\,\partial a}+\left(\frac{\partial^2\phi}{\partial a^2}-\psi''\right)\frac{\partial a}{\partial x}=\frac{\partial\phi}{\partial y}-a\,\frac{\partial^2\phi}{\partial y\,\partial a}+a\left(\psi''-\frac{\partial^2\phi}{\partial a^2}\right)\frac{\partial a}{\partial y},$$

or, what is the same thing,

$$\frac{\partial a}{\partial x}+a\,\frac{\partial a}{\partial y}=\frac{\dfrac{\partial\phi}{\partial y}-a\,\dfrac{\partial^2\phi}{\partial y\,\partial a}-\dfrac{\partial^2\phi}{\partial x\,\partial a}}{\dfrac{\partial^2\phi}{\partial a^2}-\psi''}=U,$$

say. When the form of ϕ is given, we have an equation for the determination of a: the most general value can be obtained by Lagrange's method of integration. We construct the equations

$$dx = \frac{dy}{a} = \frac{da}{U};$$

if two integrals are

$$\theta(x, y, a) = \text{constant},$$
$$\chi(x, y, a) = \text{constant},$$

the most general value of a is given by the elimination of β between the equations

$$\theta(x, y, a) = \beta, \quad \chi(x, y, a) = f(\beta),$$

where f is arbitrary. We combine these with

$$p = \phi - a\frac{\partial\phi}{\partial a} + a\psi' - \psi, \quad q = \frac{\partial\phi}{\partial a} - \psi';$$

we substitute in

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= \left(p\frac{\partial x}{\partial a} + q\frac{\partial y}{\partial a}\right)da + \left(p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\right)d\beta, \end{aligned}$$

and effect the quadrature: the result is the general primitive of the differential equation, for it involves two arbitrary functions f and ψ.

Ex. 2. A simple example arises by taking

$$\phi(x, y, a) = xya;$$

and then the general intermediate integral is given by

$$\left.\begin{aligned} p + aq + \psi(a) &= xya \\ q + \psi'(a) &= xy \end{aligned}\right\}.$$

Hence

$$p = a\psi' - \psi, \quad q = xy - \psi';$$

and now the argument a of ψ must be such as to make

$$p\,dx + q\,dy$$

an exact differential: hence

$$y - \psi''\frac{\partial a}{\partial x} = a\psi''\frac{\partial a}{\partial y}.$$

The most general value of a satisfying this relation is found by Lagrange's rule: it is easily proved to be the result of eliminating β between the two equations

$$y^2 = 2a\psi' - 2\psi + 2\beta,$$
$$x = f'(\beta) + \int \frac{\psi'' da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

where β is to be regarded as a constant during the quadrature that occurs in x, and f is an arbitrary function.

Again, we have

$$\begin{aligned} d(px - z) &= x\,dp - q\,dy \\ &= x(dp - y\,dy) + \psi' dy \\ &= -x\,d\beta + \psi' dy: \end{aligned}$$

substituting for x and for dy, and rearranging, we have

$$d\{px - z + f(\beta)\} = A\,da + B\,d\beta,$$

where

$$A = \frac{a\psi'\psi''}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

$$B = \frac{\psi'}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}} - \int \frac{\psi'' da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

and, in B, the quantity β during the quadrature is constant, becoming parametric after the quadrature has been effected. We at once verify that

$$\frac{\partial A}{\partial \beta} = \frac{\partial B}{\partial a},$$

so that the right-hand side of the differential relation is a perfect differential: the result of quadrature can be expressed in the form

$$\int \frac{a\psi'\psi'' da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

β being kept constant during the integration expressed in the result.

Gathering together the various equations, we have

$$p = a\psi' - \psi,$$
$$q = xy - \psi',$$
$$y^2 = 2a\psi' - 2\psi + 2\beta,$$
$$x = f'(\beta) + \int \frac{\psi'' da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$
$$z - px = f(\beta) - \int \frac{a\psi'\psi'' da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

where the two quadratures (in which β is to be kept unvarying) can be regarded as explicitly possible. The differential equation of the second order, of which this aggregate of equations constitutes the general primitive, is given by eliminating a between the equations

$$\left.\begin{aligned} r + sa &= ya \\ s + ta &= xa \end{aligned}\right\},$$

that is, it is

$$rt - s^2 - rx + sy = 0.$$

Its general intermediate integral is given by

$$p + aq + \psi(a) = xya, \quad q + \psi'(a) = xy,$$

that is,

$$p = f(q - xy).$$

Ex. 3. Obtain a general intermediate integral of the equation

$$st + x(rt - s^2)^2 = 0,$$

in the form

$$\left.\begin{aligned} p &= 2ax^{\frac{1}{2}} - a^2 q + 2\psi(a) \\ 0 &= x^{\frac{1}{2}} - aq + \psi'(a) \end{aligned}\right\};$$

and deduce the primitive. (Ampère; Goursat.)

Ex. 4. Integrate the equation

$$(rt-s^2)(sx-yt)+(sy-rx)^2=0.$$

Ex. 5. Another class of equations having intermediate integrals is provided by assuming

$$G=\frac{\partial F(x_1, x_2, m)}{\partial x_1},$$

$$H=\frac{\partial F(x_1, x_2, m)}{\partial x_2}+H';$$

when we denote m by x_6, and write

$$\frac{\partial H'}{\partial x_n}=p_n',$$

the equations to determine H' are

$$p_1'+\left(G-x_6\frac{\partial G}{\partial x_6}\right)p_4'+\frac{\partial G}{\partial x_6}p_5'=0,$$

$$p_3'=0.$$

The two equations are a complete Jacobian system: there are six independent variables in the construction of H', viz., $x_1, x_2, x_3, x_4, x_5, x_6$: and therefore there are four algebraically independent integrals common to the two equations. These are easily found to be

$$x_2,\quad x_4-F+x_6\frac{\partial F}{\partial x_6},\quad x_5-\frac{\partial F}{\partial x_6},\quad x_6;$$

and therefore we can take

$$\begin{aligned} H' &= \phi\left(x_2,\ x_4-F+x_6\frac{\partial F}{\partial x_6},\ x_5-\frac{\partial F}{\partial x_6},\ x_6\right) \\ &= \phi\left(y,\ p-F+m\frac{\partial F}{\partial m},\ q-\frac{\partial F}{\partial m},\ m\right), \end{aligned}$$

where ϕ is any arbitrary function of its four possible arguments.

With this value of H', the conditions for the completeness of the system

$$\left.\begin{aligned} F_1&=p_1+p_4G=0 \\ F_2&=p_2+p_4H=0 \\ F_3&=p_3\qquad\ \ =0 \end{aligned}\right\}$$

are satisfied: it therefore possesses two common integrals, which can be deduced by the usual processes, and the forms of which will be affected by the form of H'.

Let $u=0$ be the most general intermediate integral; its characteristic is

$$\frac{dx}{-p_4}=\frac{dy}{-p_5}=\frac{dp}{p_1}=\frac{dq}{p_2},$$

so that

$$dy=m\,dx,$$
$$dp=G\,dx,$$
$$dq=H\,dx:$$

hence

$$\left.\begin{aligned} r+sm=G&=\frac{\partial F(x,\ y,\ m)}{\partial x} \\ s+tm=H&=\frac{\partial F(x,\ y,\ m)}{\partial y}+H' \end{aligned}\right\}.$$

The elimination of m between these two equations leads to the required equation of the second order.

Third Case.

288. Passing to the alternative case, suppose that B does not vanish; then, as B is homogeneous of the first order in p_4 and p_5, it can be expressed in the form

$$p_4 h\,(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ m);$$

and so the system of equations is transformable to

$$\left.\begin{aligned} F_1 &= p_1 + p_4 f\,(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ m) = 0 \\ F_2 &= p_2 + p_4 g\,(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ m) = 0 \\ F_3 &= p_3 + p_4 h\,(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ m) = 0 \end{aligned}\right\},$$

where, as before,

$$p_5 = mp_4.$$

The system is to be complete, and the necessary conditions must be satisfied. From $(F_1,\ F_2)=0$, we have

$$\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2} + f\frac{\partial g}{\partial x_4} - g\frac{\partial f}{\partial x_4} - m\frac{\partial\,(f,\ g)}{\partial\,(m,\ x_4)} + \frac{\partial\,(f,\ g)}{\partial\,(m,\ x_5)} = 0:$$

from $(F_2,\ F_3)=0$, we have

$$\frac{\partial h}{\partial x_2} - \frac{\partial g}{\partial x_3} + g\frac{\partial h}{\partial x_4} - h\frac{\partial g}{\partial x_4} - m\frac{\partial\,(g,\ h)}{\partial\,(m,\ x_4)} + \frac{\partial\,(g,\ h)}{\partial\,(m,\ x_5)} = 0:$$

and from $(F_3,\ F_1)=0$, we have

$$\frac{\partial f}{\partial x_3} - \frac{\partial h}{\partial x_1} + h\frac{\partial f}{\partial x_4} - f\frac{\partial h}{\partial x_4} - m\frac{\partial\,(h,f)}{\partial\,(m,\ x_4)} + \frac{\partial\,(h,f)}{\partial\,(m,\ x_5)} = 0:$$

and each of these three conditions must be satisfied identically. They may be regarded as three simultaneous equations of the first order, for the determination of $f,\ g,\ h$.

Ex. 1. One simple set of cases is easily obtained. Let $G(x_1, x_2, x_3, m)$ denote any function of x_1, x_2, x_3, m: then the three conditions are satisfied by taking

$$f=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_1},$$

$$g=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_2},$$

$$h=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_3}.$$

Assuming these to be the values of f, g, h, we find

$$(F_r, p_4)=0, \quad (F_r, p_5)=0,$$

for $r=1, 2, 3$: so that p_4 and p_5 are independent integrals of the system. We therefore take

$$F_1=0, \quad F_2=0, \quad F_3=0, \quad p_4=a, \quad p_5=b;$$

and then, as

$$du=\sum_{n=1}^{5} p_n dx_n,$$

we have

$$u=-aF\left(x_1, x_2, x_3, \frac{b}{a}\right)+ap_4+bp_5+c.$$

The intermediate integral is $u=0$; hence, returning to the earlier notation, it is

$$p=F(x, y, z, a)-aq+\beta.$$

The characteristic of this equation of the first order is given by

$$\frac{dx}{1}=\frac{dy}{a}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}},$$

that is, by

$$dy=a\,dx,$$

$$r\,dx+s\,dy=\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)dx,$$

$$s\,dx+t\,dy=\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)dx;$$

and therefore the differential equation of the second order is obtained by eliminating a between the two equations

$$\left.\begin{aligned} r+sa&=\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z} \\ s+ta&=\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z} \end{aligned}\right\},$$

where F denotes $F(x, y, z, m)$.

The general intermediate integral is given by

$$\left.\begin{aligned} p&=F(x, y, z, a)-aq+\theta(a) \\ 0&=\frac{\partial F}{\partial a}-q+\theta'(a) \end{aligned}\right\},$$

where $\theta(a)$ is an arbitrary function. From this equation, we can proceed to the primitive by a process similar to that in § 287, Ex. 1.

Ex. 2. Obtain an intermediate integral of the equation

$$(rt-s^2)^2 z-(rt-s^2)(q^2r-2pqs+p^2t)+q^2rs-pq(rt+s^2)+p^2st=0,$$

in the form

$$abz+ap+bq=1;$$

and find the general primitive.

Ex. 3. Obtain an intermediate integral of the equation

$$q^2rs+pq(rt+s^2)+p^2st+pq=qxr+(px+yq)s+pyt,$$

in the form

$$pq-ax-by+ab=0;$$

and find the general primitive.

Ex. 4. Integrate the equation

$$(rt-s^2)^2=(qr-ps)^2+(qs-pt)^2.$$

Remaining Cases.

289. The remaining case, in which $F_1=0$ and $F_2=0$ can be a complete system only if two other equations are associated with them, is comparatively unimportant. There is only a single integral of that complete system: and so the intermediate integral, which is being sought, contains only a single constant. When resolved with respect to this constant, the intermediate integral has the form

$$v=v(x,\ y,\ z,\ p,\ q)=a.$$

Let this equation of the first order be integrated so as to obtain its complete primitive, which will contain two additional constants and can have a form

$$f(x,\ y,\ z,\ a,\ b,\ c)=0.$$

It may be possible to generalise this by means of Imschenetsky's process and to construct a general integral involving two arbitrary functions.

The significance of the integral and of the various equations is limited by the fact that, in reality, the intermediate integral leads to a couple of equations of the second order, viz.

$$v_x+rv_p+sv_q=0,\quad v_y+sv_p+tv_q=0,$$

neither of which contains any arbitrary element: the integrals cannot be regarded as belonging to either equation alone.

290. We now have to deal with the alternative, left over from § 282, in which the requirement, that the equations

$$f=0, \quad u_x+ru_p+su_q=0, \quad u_y+su_p+tu_q=0,$$

shall be indeterminate so far as concerns r, s, t, leads to three algebraically independent relations. These relations are homogeneous in u_x, u_y, u_p, u_q; when resolved, they yield a set (or a number of sets) of relations of the type

$$\left.\begin{aligned} u_x+u_q\rho=0 \\ u_y+u_q\sigma=0 \\ u_p+u_q\tau=0 \end{aligned}\right\},$$

where ρ, σ, τ are functions of x, y, z, p, q alone.

It may be at once noted that $f=0$ is not the only equation of the second order which arises in connection with the intermediate integral: in fact,

$$\left.\begin{aligned} \rho+r\tau-s=0 \\ \sigma+s\tau-t=0 \end{aligned}\right\}$$

are a couple of equations, with which $f=0$ is not inconsistent and which actually are of the second order. Hence the integral, when it exists, cannot be regarded as belonging to either equation alone; and this alternative case accordingly does not demand further consideration.

Ex. 1. Consider the equation

$$rt-b^2x^2t^2+py-qs=0,$$

propounded by Ampère at the end of the first of the memoirs quoted in Chapter XVII.

The necessary conditions for the possession of an intermediate integral are three: viz.

$$\frac{u_xu_y}{u_pu_q}+py-b^2x^2\frac{u_y^2}{u_q^2}=0,$$

$$\frac{u_y}{u_p}+\frac{u_x}{u_q}-q-2\frac{b^2x^2u_pu_y}{u_q^2}=0,$$

$$1-b^2x^2\frac{u_p^2}{u_q^2}=0.$$

These resolve themselves into the two sets

$$u_x=\left(q-\frac{py}{q}\right)u_q,$$

$$u_y=-\epsilon\frac{py}{bxq}u_q,$$

$$u_p=\epsilon\frac{1}{bx}u_q:$$

where ϵ is ± 1; and it is easy to prove that, in addition to the given equation, these require

$$py - qs = \epsilon bxqt.$$

The original equation can only have integrals in finite form free from partial quadratures when they satisfy the last equation also (see § 183, Ex. 3).

The discussion of these simultaneous equations of the second order by the method of Vályi (§ 263) is left as an exercise.

Ex. 2. The equations

$$r + f(x, y, z, p, q, s) = 0, \quad t + g(x, y, z, p, q, s) = 0,$$

are a system in involution (§ 263): shew that they possess a common characteristic, the equations of which are

$$dx = \frac{dy}{\frac{\partial f}{\partial s}} = \frac{dz}{p + q\frac{\partial f}{\partial s}} = \frac{dp}{-f + s\frac{\partial f}{\partial s}} = \frac{dq}{s - g\frac{\partial f}{\partial s}} = \frac{ds}{-\frac{df}{dy}}.$$

Indicate a mode of deducing an integral surface from the integrated equations of the characteristic. (Lie; Goursat.)

CHAPTER XXI.

General Transformation of Equations of the Second Order.

As indicated at the beginning of the present chapter, the general theory of the transformation of partial equations of order higher than the first has been only slightly developed. Such developments as have been effected are concerned with equations of the second order in two independent variables: and they are chiefly associated with geometrical properties or interpretations. The discussion in the present chapter relates to matters that had their origin in some investigations by Bäcklund, upon simultaneous partial equations of the first order in two dependent variables*, and upon the theory of the transformation of surfaces in ordinary space†. For the contents of the chapter, reference may be made to these memoirs by Bäcklund, to Goursat's discussions‡ of the matter, and to a thesis§ by Clairin where other references are given.

Processes of Transformation for Equations of the Second Order.

291. Before concluding the discussion of equations of the second order in one dependent variable and two independent variables, it is natural to consider those processes of transformation which can be used in connection with such equations. We have seen how Lie's theory of contact-transformations not merely elucidates the consideration of equations of the first order but actually provides a method of constructing various classes of

* *Math. Ann.*, t. XVII (1880), pp. 285—328.

† *Math. Ann.*, t. XIX (1882), pp. 387—422.

‡ In Chapter IX of his "Leçons sur l'intégration des équations aux dérivées partielles du second ordre," and in a memoir, *Annales de Toulouse*, 2e Sér., t. IV (1902), pp. 299—340.

§ *Ann. de l'École Norm. Sup.*, 3e Sér., t. XIX (1902), Supplément.

integrals of such an equation. Again, Laplace's process of solving a linear equation of the second order (as expounded in Chapter XIII) is really a process of transformation: and other cases have arisen in which transformations, perhaps depending upon the form of particular equations or particular classes of equations, have been used. Such instances raise obvious questions as to what is the most general type of these transformations and as to how far they are, or can be made, effective in throwing fresh light either upon the construction of the integrals or upon the general theory.

It may at once be stated that such results, as have been obtained, do not constitute a theory of nearly the same important range for equations of the second order as does Lie's theory of contact-transformations for equations of the first order: that, in their present development, they are connected with equations in only two independent variables: and that they apply only to equations of the Monge-Ampère form and, even so, not to the most general equations of that form. A comparatively brief sketch of the investigations already achieved will suffice to give an indication of their range and their significance.

Perhaps the simplest mode of initiating the discussion is to propound the question as one concerned with the transformation of surfaces in ordinary space. In accordance with the now familiar notions of Lie's theory, let x, y, z, p, q denote an element of any surface, and x', y', z', p', q' denote an element of any other surface. In order to connect the two elements completely though not uniquely, it is necessary (but not unconditionally sufficient) to have five distinct equations connecting the two sets of variables. But, as each set of five variables is to define an element of a surface, we have the Pfaffian relations

$$dz = p\,dx + q\,dy, \quad dz' = p'dx' + q'dy';$$

and therefore, if we are seeking the equations which are sufficient to determine an element of one surface corresponding to any given element of the other, we need take only four distinct equations, provided regard is paid to the Pfaffian relation. Accordingly, we shall take four equations of a form

$$F_n(x, y, z, p, q, x', y', z', p', q') = 0,$$

for $n = 1, 2, 3, 4$; and it will be assumed that the four equations are independent of one another.

A critical Relation.

292. These four equations do undoubtedly make surface-elements in the different spaces correspond with one another: no conditions are required. But if, instead of securing the correspondence of elements, it is desired that the equations should secure that a surface or surfaces in one space should correspond with a surface or surfaces in the other, an inquiry into the circumstances of the correspondence is needed: and it is conceivable that conditions may emerge limiting the correspondence. On the other hand, there should survive a generality in the results which arises from a different cause; for when two surfaces correspond, an unlimited number must correspond arising through the application, to both surfaces, of general contact-transformations. The latter generality will so far be discounted by declaring that, for the present purpose, surfaces transformable into one another by Lie's contact-transformations are equivalent to one another.

As the four equations are independent of one another, we shall consider them resoluble so as to express four out of the five variables x', y', z', p', q', in terms of the remaining quantities that occur. The preceding explanations shew that contact-transformations can be applied to both elements without affecting the immediate issue; we may therefore suppose that the four equations have been resolved so as to express x', y', p', q', in forms

$$\left.\begin{aligned} x' &= X(x, y, z, p, q, z') = X \\ y' &= Y(x, y, z, p, q, z') = Y \\ p' &= P(x, y, z, p, q, z') = P \\ q' &= Q(x, y, z, p, q, z') = Q \end{aligned}\right\}.$$

A simpler form would arise, if X, Y, P, Q were explicitly independent of z'; for the present, however, the more general form will be used.

We have to determine the surface or surfaces in the set x, y, z, p, q which, through the above equations, lead to corresponding surfaces in the set x', y', z', p', q'. In order that the latter set of variables, when varying continuously, may determine a surface, the relation

$$dz' - p'dx' - q'dy' = 0$$

must be satisfied. When the values of x', y', p', q' are substituted and the terms in the same differentials are collected, this relation becomes

$$A\,dz' + (B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0,$$

where

$$A = P\frac{\partial X}{\partial z'} + Q\frac{\partial Y}{\partial z'} - 1,$$

$$B = P\left(\frac{\partial X}{\partial x} + p\frac{\partial X}{\partial z}\right) + Q\left(\frac{\partial Y}{\partial x} + p\frac{\partial Y}{\partial z}\right),$$

$$C = P\left(\frac{\partial X}{\partial y} + q\frac{\partial X}{\partial z}\right) + Q\left(\frac{\partial Y}{\partial y} + q\frac{\partial Y}{\partial z}\right),$$

$$D = P\frac{\partial X}{\partial p} + Q\frac{\partial Y}{\partial p},$$

$$E = P\frac{\partial X}{\partial q} + Q\frac{\partial Y}{\partial q}.$$

Now the integral equivalent of the differential relation is to consist of a single equation; hence the relation must be integrable. The necessary and sufficient condition of integrability is

$$A\left\{\frac{d}{dy}(B + Dr + Es) - \frac{d}{dx}(C + Ds + Et)\right\}$$

$$+ (B + Dr + Es)\left\{\frac{\partial}{\partial z'}(C + Ds + Et) - \frac{dA}{dy}\right\}$$

$$+ (C + Ds + Et)\left\{\frac{dA}{dx} - \frac{\partial}{\partial z'}(B + Dr + Es)\right\} = 0,$$

where

$$\frac{dA}{dx} = \frac{\partial A}{\partial x} + p\frac{\partial A}{\partial z} + r\frac{\partial A}{\partial p} + s\frac{\partial A}{\partial q},$$

$$\frac{dA}{dy} = \frac{\partial A}{\partial y} + q\frac{\partial A}{\partial z} + s\frac{\partial A}{\partial p} + t\frac{\partial A}{\partial q},$$

and similarly for the derivatives of the other quantities. It is easy to see that, in this equation, the derivatives of z of the third order disappear: and the condition becomes

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0,$$

where

$$U = A\left(\frac{\partial D}{\partial q} - \frac{\partial E}{\partial p}\right) + E\left(\frac{\partial A}{\partial p} - \frac{\partial D}{\partial z'}\right) + D\left(\frac{\partial E}{\partial z'} - \frac{\partial A}{\partial q}\right),$$

and so for other coefficients: the quantities U, R, S, T, V can involve x, y, z, p, q, z', but they do not involve derivatives of z which are of the second order.

293. In the first place, suppose that z' does occur in the conditional equation; then the equation provides a value of z' in terms of the other quantities that occur, viz., x, y, z, p, q, r, s, t. Let this value be substituted in the initial form

$$A dz' + (B + Dr + Es)\, dx + (C + Ds + Et)\, dy = 0$$

of the differential relation, which may be written

$$\alpha dz' + \beta dx + \gamma dy = 0,$$

more briefly; the result of substitution is

$$\left(\alpha \frac{dz'}{dx} + \beta\right) dx + \left(\alpha \frac{dz'}{dy} + \gamma\right) dy = 0,$$

where $\dfrac{dz'}{dx}$ and $\dfrac{dz'}{dy}$ are the complete derivatives of the value of z'. Consequently, z satisfies the equations

$$\alpha \frac{dz'}{dx} + \beta = 0, \quad \alpha \frac{dz'}{dy} + \gamma = 0,$$

which are two equations of the third order.

These two equations, while distinct from one another, are compatible with one another in virtue of the original condition of integrability. For, when complete derivatives with regard to y and to x are taken of the two equations respectively, they give

$$\alpha \frac{d^2z'}{dxdy} + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \alpha}{\partial z'} \frac{dz'}{dy}\right) \frac{dz'}{dx} + \frac{\partial \beta}{\partial y} + \frac{\partial \beta}{\partial z'} \frac{dz'}{dy} = 0,$$

$$\alpha \frac{d^2z'}{dxdy} + \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial z'} \frac{dz'}{dx}\right) \frac{dz'}{dy} + \frac{\partial \gamma}{\partial x} + \frac{\partial \gamma}{\partial z'} \frac{dz'}{dx} = 0;$$

subtracting, and substituting from the equations for the first derivatives of z' that have remained, we find

$$\alpha \left(\frac{\partial \beta}{\partial y} - \frac{\partial \gamma}{\partial x}\right) + \beta \left(\frac{\partial \gamma}{\partial z'} - \frac{\partial \alpha}{\partial y}\right) + \gamma \left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial z'}\right) = 0,$$

which is the condition of integrability, known to be satisfied.

Hence the two equations have integrals in common. To determine the amount of arbitrary element that occurs in the most general common integral, we can proceed on the basis of the existence-theorems as discussed in Chapter II. Most simply, let initial values $\phi_0(y)$, $\phi_1(y)$, $\phi_2(y)$, when $x = a$, be assigned to z, p, r, respectively; then the values of $\frac{\partial^3 z}{\partial x \partial y^2}$ and $\frac{\partial^3 z}{\partial y^3}$, when $x = a$, can be regarded as known, and the two equations of the third order can then be regarded as determining $\frac{\partial^3 z}{\partial x^3}$ and $\frac{\partial^3 z}{\partial x^2 \partial y}$, that is, as determining $\frac{\partial^3 z}{\partial x^3}$ and $\frac{d\phi_2}{dy}$, when $x = a$. Eliminating $\frac{\partial^3 z}{\partial x^3}$ between them, we have an ordinary differential equation of the first order for the determination of ϕ_2: the integral of such an equation contains a single arbitrary element. Consequently, when values are assigned to z and p, the two differential equations possess a simple infinitude of common integrals. Taking the initial conditions more generally as in §§ 24, 178, we can say that the two equations of the third order determine a simple infinitude of surfaces, which satisfy them and which touch a given developable along a given curve.

To each such surface of elements x, y, z, p, q, there corresponds a single z'-surface; for z' is known in terms of z and its derivatives, that is, z' is known as a function of x and y.

Hence, on the present assumption, there is a simple infinitude of z-surfaces touching a given developable along a given curve; to each of them there corresponds a single z'-surface.

294. In the second place, suppose that z' does not occur in the conditional equation

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0;$$

the latter equation is then of the Monge-Ampère form and it serves to determine z. When a value of z satisfying this equation is substituted in the relation

$$Adz' + (B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0,$$

the latter equation is integrable by a single integral equation which involves an arbitrary constant. We thus obtain a simple infinitude of surfaces in the set x', y', z', p', q', corresponding to a single surface in the set x, y, z, p, q: and the single surface is an integral of an equation of the second order.

This result will always arise when the four equations of the transformation either do not involve z' explicitly, or can be changed, by a contact transformation, so as not to involve z' explicitly: when either condition is satisfied, we shall clearly have the simplest comprehensive set of transformations. The character of the Monge-Ampère equation, which has been obtained, will be discussed almost immediately.

It is to be remarked that, in both the preceding hypotheses as to the form of the conditional equation, the quantity A is tacitly assumed to be different from zero. The general value of A is

$$P\frac{\partial X}{\partial z'} + Q\frac{\partial Y}{\partial z'} - 1:$$

and the tacit assumption will be justified, if neither X nor Y involves z'. If however A should vanish, the conditional relation becomes

$$(B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0,$$

and the condition of integrability can be taken to be

$$\frac{\partial}{\partial z'}\left(\frac{C + Ds + Et}{B + Dr + Es}\right) = 0:$$

but the resulting integrated equation will be ineffective, because the arbitrary quantity which arises in the quadrature is a constant, so that the equation will not involve z'.

The relation can, however, be satisfied by taking

$$B + Dr + Es = 0,$$
$$C + Ds + Et = 0;$$

these two conditions lead, on the elimination of z', to an equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0:$$

and then, when z is determined so as to be an integral of this equation, either of the conditions will serve to determine z' in terms of x, y, z, p, q. Usually this determination of z' will be unique, so that then a single surface in the set x', y', z', p', q' corresponds to each surface in the set x, y, z, p, q, arising as an integral of the equation $f = 0$.

As the equation $f = 0$ arises from the elimination of z' between two equations that are linear in r, s, t, it is not an equation of quite general form: when we regard r, s, t as the non-homogeneous coordinates of a point in a new space, the equation represents a ruled surface having generators parallel to those of the cone $rt - s^2 = 0$, so that the differential equation $f = 0$ possesses a system of characteristics of the first order.

Ex. Shew that, when the four equations

$$F_n=0$$

are not resolved so as to express x', y', p', q' in terms of the other variables, the equation of the second order (which is the condition of integrability for the differential relation $dz'=p'dx'+q'dy'$) can be expressed in the form

$$(12)[34]+(13)[42]+(14)[23]+(34)[12]+(42)[13]+(23)[14]=0,$$

where

$$[ij]=\left(\frac{\partial F_i}{\partial x'}+p'\frac{\partial F_i}{\partial z'}\right)\frac{\partial F_j}{\partial p'}-\left(\frac{\partial F_j}{\partial x'}+p'\frac{\partial F_j}{\partial z'}\right)\frac{\partial F_i}{\partial p'}$$
$$+\left(\frac{\partial F_i}{\partial y'}+q'\frac{\partial F_i}{\partial z'}\right)\frac{\partial F_j}{\partial q'}-\left(\frac{\partial F_j}{\partial y'}+q'\frac{\partial F_j}{\partial z'}\right)\frac{\partial F_i}{\partial q'},$$

$$(ij)=\frac{dF_i}{dx}\frac{dF_j}{dy}-\frac{dF_j}{dx}\frac{dF_i}{dy},$$

and where

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}.$$

(Bäcklund, Darboux.)

Bäcklund Transformations.

295. It thus appears that, when the transformations

$$F_n(x, y, z, p, q, x', y', z', p', q')=0,$$

for $n=1, 2, 3, 4$, are used so as to construct a surface or surfaces in the set x', y', z', p', q', which correspond with a surface or surfaces in the set x, y, z, p, q, there are certain cases when the variable z is an integral of a partial differential equation of the second order having the form of the equations considered by Monge and by Ampère. Conversely, if we seek to construct a surface or surfaces in the set x, y, z, p, q, which, under the same transformations, shall correspond with a surface or surfaces in the set x', y', z', p', q', it may happen that the variable z' is also an integral of a similar partial differential equation of the second order.

When the variables z and z' thus separately satisfy partial equations of the second order, these equations can be regarded as transformable into one another by the four relations

$$F_n(x, y, z, p, q, x', y', z', p', q')=0,$$

for $n = 1, 2, 3, 4$. This set of four relations is usually called a *Bäcklund transformation*, because the properties were first discussed in some of Bäcklund's investigations*.

As the transformations in question change, into one another, surfaces which are integrals of equations of the second order or (what is the same thing, when the geometrical link is dropped) transform equations of the second order into one another, it is important to obtain the limitations and the restrictions (if any) upon the equations, other than that of belonging to the Monge-Ampère type. Thus two questions arise for discussion at the outset. One is to determine whether a given Monge-Ampère type admits a Bäcklund transformation: the other question is the construction of Bäcklund transformations in general.

296. It is easy to see that there are several kinds of Bäcklund transformations†, discriminated according to the correspondence of the surfaces.

One kind of transformation arises when a single surface in one set of elements corresponds to only a single surface in the other set, and conversely. As there is only a single surface in x', y', z', p', q', the quadrature of the relation

$$dz' = p'dx' + q'dy',$$

after it has been duly modified, does not occur for operation: and the condition

$$A = P\frac{\partial X}{\partial z'} + Q\frac{\partial Y}{\partial z'} - 1 = 0$$

is therefore satisfied. The equivalent condition, associated with the transformations in the form

$$F_n = 0,$$

for $n = 1, 2, 3, 4$, is that the Jacobian

$$\left(\frac{F_1, F_2, F_3, F_4}{x', y', p', q'}\right)$$

should vanish, the x'-derivatives being $\dfrac{\partial F}{\partial x'} + p'\dfrac{\partial F}{\partial z'}$; and similarly for the y'-derivatives.

* These are contained in the two memoirs in volumes XVII and XIX of the *Mathematische Annalen*, which have already (p. 425 of this volume) been quoted.

† The following classification is due to Clairin, *Ann. de l'Éc. Norm.*, 3e Sér., t. XIX (1902), Supplément, p. 15.

Another kind of transformation arises, when a single surface in one set of elements corresponds to a single surface in the other set, while to the single surface in the latter set there corresponds a simple infinitude of surfaces in the former.

A third kind of transformation arises when, to each surface in either set of elements, there corresponds a simple infinitude of surfaces in the other.

Ex. 1. Prove that if two surfaces, arising as integrals of equations of the second order, correspond to one another under a Bäcklund transformation, their characteristics also correspond to one another. (Goursat.)

Ex. 2. Prove that the equations

$$z=\frac{\partial w}{\partial x'},\quad z^2+(x-y)^2pq=2\frac{\partial w}{\partial y'},\quad \frac{2z}{x-y}+p-q=q',\quad \frac{2}{x-y}=p',$$

where w is a function of x' and y' only, constitute a Bäcklund transformation; and obtain the equations of the second order satisfied by z and z' respectively. (Cosserat, Goursat.)

Ex. 3. Discuss the transformation constituted by the equations

$$z=\frac{\partial w}{\partial x'},\quad z^2+(x-y)^2pq=2\frac{\partial w}{\partial y'},\quad \frac{x+y}{x-y}=p',\quad px-qy+\frac{x+y}{x-y}z=q',$$

where w is a function of x' and y' only. (Goursat.)

297. A question next arises as to the degree of generality possessed by the equations of the Monge-Ampère form, which arise in connection with a Bäcklund transformation: does a perfectly general equation of that form necessarily admit such a transformation? It will be seen that the answer is in the negative.

It has appeared that the equations of the transformation may be taken in the form

$$x'=X,\quad y'=Y,\quad p'=P,\quad q'=Q,$$

where X, Y, P, Q are functions of x, y, z, p, q, and of z'. All the analysis is much more complicated when z' occurs explicitly in these functions: we shall therefore be content to deal with those transformations of the equations which are such that z' does not occur explicitly in the functions X, Y, P, Q.

Proceeding to consider in detail the simplest form of the original equations when they do not explicitly involve z', so that the equation

$$U(rt-s^2)+Rr+2Ss+Tt+V=0$$

does not explicitly involve z', we see at once that

$$A = -1;$$

and we easily find the coefficients U, R, S, T, V, in the preceding equations, to be

$$\left.\begin{aligned}
U &= \left(\frac{X, P}{p, q}\right) + \left(\frac{Y, Q}{p, q}\right) = \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p} \\
R &= \left(\frac{X, P}{p, y}\right) + \left(\frac{Y, Q}{p, y}\right) = \frac{dD}{dy} - \frac{\partial C}{\partial p} \\
T &= \left(\frac{P, X}{q, x}\right) + \left(\frac{Q, Y}{q, x}\right) = \frac{\partial B}{\partial q} - \frac{dE}{dx} \\
V &= \left(\frac{P, X}{y, x}\right) + \left(\frac{Q, Y}{y, x}\right) = \frac{dB}{dy} - \frac{dC}{dx} \\
2S &= \left(\frac{X, P}{q, y}\right) + \left(\frac{P, X}{p, x}\right) + \left(\frac{Y, Q}{q, y}\right) + \left(\frac{Q, Y}{p, x}\right) \\
&= \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy}
\end{aligned}\right\},$$

where, as usual,

$$\left(\frac{X, P}{p, q}\right) = \frac{\partial X}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial X}{\partial q}\frac{\partial P}{\partial p},$$

$$\left(\frac{X, P}{p, y}\right) = \frac{\partial X}{\partial p}\frac{dP}{dy} - \frac{dX}{dy}\frac{\partial P}{\partial p},$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z},$$

and so for the other combinations. The quantities B, C, D, E are the forms of the coefficients of the equation of § 292, simplified by the non-occurrence of z', so that they are

$$\left.\begin{aligned}
B &= P\frac{dX}{dx} + Q\frac{dY}{dx} \\
C &= P\frac{dX}{dy} + Q\frac{dY}{dy} \\
D &= P\frac{\partial X}{\partial p} + Q\frac{\partial Y}{\partial p} \\
E &= P\frac{\partial X}{\partial q} + Q\frac{\partial Y}{\partial q}
\end{aligned}\right\},$$

the equations of transformation themselves being

$$\left.\begin{aligned} x' &= X(x, y, z, p, q) = X \\ y' &= Y(x, y, z, p, q) = Y \\ p' &= P(x, y, z, p, q) = P \\ q' &= Q(x, y, z, p, q) = Q \end{aligned}\right\}.$$

The new form of the relation, by hypothesis, does not involve z': therefore either it is identically satisfied, or it is an equation of the second order for the determination of z.

Significance of the Critical Relation.

298. In the first place, suppose that the conditional relation is identically satisfied: in that case, each coefficient in the relation must vanish. As U then vanishes, we have

$$\frac{\partial D}{\partial q} - \frac{\partial E}{\partial p} = 0;$$

and therefore we may take

$$D = \frac{\partial \theta}{\partial p}, \quad E = \frac{\partial \theta}{\partial q},$$

where, so far as D and E are concerned, θ can be any function of x, y, z, p, q. Next, because R vanishes, we have

$$\frac{\partial C}{\partial p} = \frac{dD}{dy} = \left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z}\right)\frac{\partial \theta}{\partial p}$$
$$= \frac{\partial}{\partial p}\left(\frac{\partial \theta}{\partial y} + q\frac{\partial \theta}{\partial z}\right);$$

and therefore

$$C = \frac{\partial \theta}{\partial y} + q\frac{\partial \theta}{\partial z} + \phi(x, y, z, q)$$
$$= \frac{d\theta}{dy} + \phi(x, y, z, q),$$

where, so far as C is concerned, ϕ is an arbitrary function of the arguments indicated. Similarly, from the vanishing of T, we find

$$B = \frac{d\theta}{dx} + \psi(x, y, z, p),$$

where, so far as B is concerned, ψ is an arbitrary function. When these values are substituted in the expression for S, and when account is taken of the fact that S vanishes, we find

$$\frac{\partial \psi}{\partial p} = \frac{\partial \phi}{\partial q};$$

as q cannot occur on the left-hand side nor p on the right, we clearly have

$$\phi = qH + K, \quad \psi = pH + L,$$

where, so far as concerns S, θ, ϕ, the quantities H, K, L, are functions of the variables x, y, z only. Lastly, because V vanishes, we have

$$\frac{d\psi}{dy} = \frac{d\phi}{dx},$$

that is,

$$p\frac{\partial H}{\partial y} + pq\frac{\partial H}{\partial z} + \frac{\partial L}{\partial y} + q\frac{\partial L}{\partial z}$$
$$= q\frac{\partial H}{\partial x} + pq\frac{\partial H}{\partial z} + \frac{\partial K}{\partial x} + p\frac{\partial K}{\partial z}.$$

Hence

$$\frac{\partial H}{\partial y} = \frac{\partial K}{\partial z}, \quad \frac{\partial L}{\partial z} = \frac{\partial H}{\partial x}, \quad \frac{\partial K}{\partial x} = \frac{\partial L}{\partial y};$$

and therefore some function I of x, y, z exists, such that

$$H = \frac{\partial I}{\partial z}, \quad K = \frac{\partial I}{\partial y}, \quad L = \frac{\partial I}{\partial x}.$$

Consequently,

$$\phi = qH + K = \frac{dI}{dy}, \quad \psi = pH + L = \frac{dI}{dx};$$

and therefore, if

$$\Theta = \theta + I,$$

we have

$$B = \frac{d\Theta}{dx}, \quad C = \frac{d\Theta}{dy}, \quad D = \frac{\partial\Theta}{\partial p}, \quad E = \frac{\partial\Theta}{\partial q},$$

so that

$$Bdx + Cdy + Ddp + Edq$$
$$= d\Theta - \frac{\partial\Theta}{\partial z}(dz - pdx - qdy).$$

Again,

$$p'dx' + q'dy'$$
$$= Bdx + Cdy + Ddp + Edq + \left(P\frac{\partial X}{\partial z} + Q\frac{\partial Y}{\partial z}\right)(dz - pdx - qdy),$$

and therefore

$$d\Theta - p'dx' - q'dy' = \left(\frac{\partial \Theta}{\partial z} - P\frac{\partial X}{\partial z} - Q\frac{\partial Y}{\partial z}\right)(dz - pdx - qdy).$$

Accordingly, we take

$$z' = \Theta;$$

and this equation, together with the four initial equations

$$x' = X, \quad y' = Y, \quad p' = P, \quad q' = Q,$$

defines a contact-transformation which, as is known, changes every surface into some other surface.

It has already been declared that surfaces, which are transformable into one another by contact-transformations, are to be regarded as equivalent to one another for the present purpose. Consequently, the case, when the equation

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0$$

is identically satisfied, provides no new or independent transformations.

299. In the next place, suppose that the relation is satisfied, though not identically; then it is an equation of the Monge-Ampère form which serves to determine z. We proceed to shew that this equation is not perfectly general: that is, we cannot assume that any postulated equation of the form can be associated with a Bäcklund transformation.

To justify this statement, two forms of equation will be considered, according as the term in $rt - s^2$ is present or is absent. First, suppose the term in $rt - s^2$ to be present: then the equation may be taken

$$rt - s^2 + ar + bs + ct + e = 0,$$

where a, b, c, e are functions of x, y, z, p, q. If this equation is effectively the same as the equation which arises in the discussion of the Bäcklund transformations, a quantity μ must exist, such that

$$U = \mu, \quad R = \mu a, \quad 2S = \mu b, \quad T = \mu c, \quad V = \mu e.$$

The equations for the determination of μ, B, C, D, E are

$$\mu = \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p},$$

$$\mu a = \frac{dD}{dy} - \frac{\partial C}{\partial p},$$

$$\mu b = \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy},$$

$$\mu c = \frac{\partial B}{\partial q} - \frac{dE}{dx},$$

$$\mu e = \frac{dB}{dy} - \frac{dC}{dx},$$

five equations for five quantities. Let a new variable θ, an unknown function of x, y, z, p, q, be introduced by the relation

$$E = \frac{\partial \theta}{\partial q};$$

and let other unknown quantities be introduced by the relations

$$B = \frac{d\theta}{dx} + B',$$

$$C = \frac{d\theta}{dy} + C',$$

$$D = \frac{\partial \theta}{\partial p} + D'.$$

The foregoing equations become

$$\mu = \frac{\partial D'}{\partial q},$$

$$\mu a = \frac{dD'}{dy} - \frac{\partial C'}{\partial p},$$

$$\mu b = \frac{\partial B'}{\partial p} - \frac{\partial C'}{\partial q} - \frac{dD'}{dx},$$

$$\mu c = \frac{\partial B'}{\partial q},$$

$$\mu e = \frac{dB'}{dy} - \frac{dC'}{dx}.$$

We still have five equations, but they now involve explicitly only four unknown quantities μ, B', C', D': and θ remains undetermined. Manifestly, all the equations cannot be satisfied simultaneously unless certain conditions are fulfilled: and therefore an arbitrarily postulated Monge-Ampère equation of the assumed form does not arise (and cannot be made to arise) through a Bäcklund transformation.

Next, suppose that the postulated Monge-Ampère equation is devoid of the term in $rt - s^2$, so that it has the form

$$ar + bs + ct + e = 0.$$

When we proceed as in the preceding case, the quantities B, C, D, E, and an unknown multiplier λ, must satisfy the equations

$$0 = \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p},$$

$$\lambda a = \frac{dD}{dy} - \frac{\partial C}{\partial p},$$

$$\lambda b = \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy},$$

$$\lambda c = \frac{\partial B}{\partial q} - \frac{dE}{dx},$$

$$\lambda e = \frac{dB}{dy} - \frac{dC}{dx}.$$

The first of these is satisfied by taking

$$D = \frac{\partial \theta}{\partial p}, \qquad E = \frac{\partial \theta}{\partial q},$$

where θ can be any function of x, y, z, p, q; and then, when we write

$$B = \frac{d\theta}{dx} + B',$$

$$C = \frac{d\theta}{dy} + C',$$

the remaining equations are

$$\lambda a = -\frac{\partial C'}{\partial p},$$

$$\lambda b = \frac{\partial B'}{\partial p} - \frac{\partial C'}{\partial q},$$

$$\lambda c = \frac{\partial B'}{\partial q},$$

$$\lambda e = \frac{dB'}{dy} - \frac{dC'}{dx}.$$

We thus have four equations for the determination of the three quantities λ, B', C', the variable θ being left arbitrary: the equations cannot be satisfied unconditionally.

Hence any arbitrarily assumed Monge-Ampère equation cannot be associated with a Bäcklund transformation.

Ex. 1. Implicit functions P and Q, of p and of q respectively, are defined by the equations

$$P-1=e^{p-P}, \quad Q-1=e^{q-Q};$$

prove that the relations

$$\left.\begin{aligned} p &= e^{-(x+y)p'+\frac{z}{x+y}} - (x+y)p' + \frac{z}{x+y} + 1 \\ q &= e^{z'+\frac{z}{x+y}} + z' + \frac{z}{x+y} + 1 \end{aligned}\right\}$$

constitute a Bäcklund transformation connecting the equations

$$(x+y)s=PQ,$$

$$s'+\frac{p'}{x+y}=\frac{z'}{(x+y)^2}.$$

Discuss the relation of the integrals of the equations. (Clairin.)

Ex. 2. Prove that the equations

$$s'=0, \quad s+qe^{-z}=0,$$

are transformed into one another by the Bäcklund transformation

$$p=p'+e^{-z}, \quad q=e^{z'-z}.$$ (Clairin.)

Ex. 3. Prove that the equation

$$p-z\frac{s}{q}=f\left(x, \frac{s}{q}\right)$$

can be changed to the equation $s'=0$, by a Bäcklund transformation: and obtain the transformation. (Clairin.)

Application to the Linear Equation.

300. Consider the linear equation*

$$s+\alpha p+\beta q+\gamma z=0,$$

where α, β, γ are functions of x and y only, so as to construct the Bäcklund transformations (if any) of the form

$$x'=x, \quad y'=y, \quad p'=P, \quad q'=Q,$$

which can be associated with the equation.

* The equation, in this connection, is discussed by Goursat, in the memoir quoted on p. 425.

With these assumptions as to the form of the transformation, we at once have (from § 292) the values of B, C, D, E in the form

$$B = P, \quad C = Q, \quad D = 0, \quad E = 0.$$

Thus the quantity θ in the preceding investigation is a function of x, y, and z only; also

$$\frac{\partial C'}{\partial p} = 0, \quad \frac{\partial B'}{\partial q} = 0,$$

so that

$$B' = \text{function of } x, y, z, p = \phi(x, y, z, p),$$

$$C' = \text{function of } x, y, z, q = \psi(x, y, z, q).$$

We then have

$$P = \frac{d\theta}{dx} + \phi(x, y, z, p),$$

$$Q = \frac{d\theta}{dy} + \psi(x, y, z, q):$$

and the equations, to be satisfied by the functions ϕ and ψ, are

$$\left.\begin{aligned} \lambda &= \frac{\partial\phi}{\partial p} - \frac{\partial\psi}{\partial q} \\ \lambda(\alpha p + \beta q + \gamma z) &= \frac{d\phi}{dy} - \frac{d\psi}{dx} \end{aligned}\right\},$$

where λ is the (unknown) multiplier: these equations being satisfied consistently with the limitations imposed on the forms of ϕ and ψ.

Writing

$$\mu = \lambda(\alpha p + \beta q + \gamma z),$$

we can take the equations as a set for the determination of ϕ, in the form

$$\frac{\partial\phi}{\partial p} = \frac{\partial\psi}{\partial q} + \lambda,$$

$$\frac{d\phi}{dy} = \frac{d\psi}{dx} + \mu,$$

$$\frac{\partial\phi}{\partial q} = 0.$$

The Jacobian conditions of coexistence of these three equations are

$$0 = \frac{d}{dy}\left(\frac{\partial \psi}{\partial q}\right) + \frac{d\lambda}{dy} - \frac{\partial \mu}{\partial p} - \frac{\partial \psi}{\partial z},$$

$$0 = \frac{\partial^2 \psi}{\partial q^2} + \frac{\partial \lambda}{\partial q},$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial q}\left(\frac{d\psi}{dx}\right) + \frac{\partial \mu}{\partial q};$$

in the first of these, regard has been paid to the fact that ψ is independent of p. In order that the last of them may coexist with the three equations which involve derivatives of ϕ, we similarly have

$$0 = \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial q}\right) + \frac{\partial \lambda}{\partial z} - \frac{\partial^2}{\partial p \partial q}\left(\frac{d\psi}{dx}\right) - \frac{\partial^2 \mu}{\partial p \partial q},$$

$$0 = \frac{\partial}{\partial z}\left(\frac{d\psi}{dx}\right) + \frac{\partial \mu}{\partial z} - \frac{d}{dy}\left\{\frac{\partial}{\partial q}\left(\frac{d\psi}{dx}\right)\right\} - \frac{d}{dy}\left(\frac{\partial \mu}{\partial q}\right),$$

$$0 = \frac{\partial^2}{\partial q^2}\left(\frac{d\psi}{dx}\right) + \frac{\partial^2 \mu}{\partial q^2}.$$

We thus obtain five equations involving derivatives of ψ only: combining them with

$$\frac{\partial \psi}{\partial p} = 0,$$

which is satisfied in virtue of the form of ψ, we see that the necessary and sufficient conditions for the coexistence of the whole set are

$$\frac{\partial^2 \lambda}{\partial p \partial q} = 0,$$

$$\frac{d}{dx}\left(\frac{\partial \lambda}{\partial q}\right) = \frac{\partial^2 \mu}{\partial q^2},$$

$$\frac{d}{dy}\left(\frac{\partial \lambda}{\partial p}\right) = \frac{\partial^2 \mu}{\partial p^2},$$

$$\frac{\partial \lambda}{\partial z} = \frac{\partial^2 \mu}{\partial p \partial q},$$

$$\frac{d^2 \lambda}{dx dy} = \frac{d}{dx}\left(\frac{\partial \mu}{\partial p}\right) + \frac{d}{dy}\left(\frac{\partial \mu}{\partial q}\right) - \frac{\partial \mu}{\partial z}.$$

The form of λ is already known, being

$$\lambda = \frac{\partial \phi}{\partial p} - \frac{\partial \psi}{\partial q},$$

where ϕ is explicitly independent of q, and ψ of p: this form satisfies the first of these equations. Also

$$\mu = \lambda (\alpha p + \beta q + \gamma z),$$

where α, β, γ are functions of x and y only.

When these values are substituted in the fourth of these equations, we have

$$\frac{\partial^2 \phi}{\partial p \partial z} - \beta \frac{\partial^2 \phi}{\partial p^2} = \frac{\partial^2 \psi}{\partial q \partial z} - \alpha \frac{\partial^2 \psi}{\partial q^2}$$
$$= \frac{\partial^2 u}{\partial z^2},$$

where u is any function of x, y, z alone, because ϕ is independent of q and ψ of p. Hence

$$\phi = p \frac{\partial u}{\partial z} + \beta u + v + f(x, y, \xi),$$

$$\psi = q \frac{\partial u}{\partial z} + \alpha u + w + g(x, y, \eta),$$

where

$$\xi = p + \beta z, \quad \eta = q + \alpha z,$$

v and w are any functions of x, y, z, and f and g are any functions of their arguments. With these values, the value of λ is given by

$$\lambda = \frac{\partial f}{\partial \xi} - \frac{\partial g}{\partial \eta}.$$

When this value of λ and the corresponding value of μ are substituted in the third of the equations, we find

$$\left\{\alpha \xi + \left(\gamma - \alpha\beta - \frac{\partial \beta}{\partial y}\right) z\right\} \frac{\partial^3 f}{\partial \xi^3} + 2\alpha \frac{\partial^2 f}{\partial \xi^2} = \frac{\partial^3 f}{\partial y \partial \xi^2}.$$

We assume that the coefficient of z, which is one of the invariants of the linear equation, does not vanish: and then the preceding equation for f leads to the two equations

$$\frac{\partial^3 f}{\partial \xi^3} = 0,$$

$$\frac{\partial^3 f}{\partial y \partial \xi^2} = 2\alpha \frac{\partial^2 f}{\partial \xi^2}.$$

Hence

$$\frac{\partial f}{\partial \xi} = l + m\xi,$$

where

$$\frac{\partial m}{\partial y} = 2\alpha m,$$

and l, m are functions of x and y only.

Proceeding similarly with the second of the equations, and assuming that $\frac{\partial \alpha}{\partial x} + \alpha\beta - \gamma$ (which is another invariant of the linear equation) does not vanish, we find

$$\frac{\partial g}{\partial \eta} = l' - m'\eta,$$

where

$$\frac{\partial m'}{\partial x} = 2\beta m',$$

and l', m' are functions of x and y only.

The value of λ now is given by

$$\begin{aligned}\lambda &= l + m\xi - l' + m'\eta \\ &= mp + m'q + (m\beta + m'\alpha)z + n,\end{aligned}$$

say, where n is a function of x and y only, and

$$\frac{\partial m}{\partial y} = 2\alpha m, \quad \frac{\partial m'}{\partial x} = 2\beta m'.$$

When this value of λ is substituted in the remaining equation, viz. in

$$\frac{d^2\lambda}{dx\,dy} = \frac{d}{dx}\left(\frac{\partial \mu}{\partial p}\right) + \frac{d}{dy}\left(\frac{\partial \mu}{\partial q}\right) - \frac{\partial \mu}{\partial z},$$

we find

$$\begin{aligned}&\frac{\partial^2}{\partial x \partial y}(m\beta + m'\alpha) \\ &= \frac{\partial}{\partial x}(m\gamma + m\alpha\beta + m'\alpha^2) + \frac{\partial}{\partial y}(m'\gamma + m'\alpha\beta + m\beta^2) - 2\gamma(m\beta + m'\alpha),\end{aligned}$$

which must be satisfied in order that the term in z may vanish; and

$$\frac{\partial^2 n}{\partial x \partial y} = \frac{\partial (n\alpha)}{\partial x} + \frac{\partial (n\beta)}{\partial y} - n\gamma,$$

in order that the terms independent of z, p, q may vanish: the terms in p and in q respectively are unconditionally evanescent.

Denoting the two invariants of the original equation by h and k, where

$$h = \frac{\partial \alpha}{\partial x} + \alpha\beta - \gamma, \quad k = \frac{\partial \beta}{\partial y} + \alpha\beta - \gamma,$$

and taking account of the equations satisfied by m and m', the first of the two preceding equations can be changed to the form

$$\frac{\partial (mk)}{\partial x} + \frac{\partial (m'h)}{\partial y} = 2\alpha m'h + 2\beta mk.$$

It is to be noticed that the equation satisfied by n is the adjoint of the original equation.

There are thus three equations in the two unknown quantities m and m': consequently, some condition must be satisfied. Let

$$\alpha = \frac{\partial A}{\partial y}, \quad \beta = \frac{\partial B}{\partial x},$$

where A can have as an arbitrary additive term any function of x, and B can have as an arbitrary additive term any function of y: then

$$m = Xe^{2A}, \quad m' = Ye^{2B},$$

where X and Y are arbitrary functions of x and of y respectively, which may vanish but will not vanish in the most general case. Assuming that X and Y do not vanish, and absorbing the X into e^{2A} and the Y into e^{2B}, we can take

$$m = e^{2A}, \quad m' = e^{2B}.$$

Substituting these values in the third equation which involves m and m', we have

$$e^{2A}\left(\frac{\partial k}{\partial x} + 2k\frac{\partial A}{\partial x} - 2\beta k\right) + e^{2B}\left(\frac{\partial h}{\partial y} + 2h\frac{\partial B}{\partial y} - 2\alpha h\right) = 0.$$

This relation involving the coefficients in the equation

$$s + \alpha p + \beta q + \gamma z = 0$$

must be satisfied, if the equation is to be capable of a Bäcklund transformation.

It is to be noted that the three equations can be simultaneously satisfied by taking X and Y both zero, so that m and m' vanish. In that case, we have

$$\lambda = n,$$

where n, a function of x and y only, is an integral of the adjoint equation; and the equations for ϕ and ψ become

$$\frac{\partial \phi}{\partial p} - \frac{\partial \psi}{\partial q} = n,$$

$$\frac{d\phi}{dy} - \frac{d\psi}{dx} = n(\alpha p + \beta q + \gamma z).$$

It is easy to verify that possible values of ϕ and ψ are given by

$$\left.\begin{aligned} \phi &= up + z\left(\frac{\partial u'}{\partial x} + nb\right) + \frac{d\theta}{dx} \\ \psi &= u'q + z\left(\frac{\partial u}{\partial y} - na\right) + \frac{d\theta}{dy} \end{aligned}\right\},$$

where

$$u - u' = n,$$

and θ is any function of x, y, z; and that then the transformation is given by

$$x' = x, \quad y' = y, \quad p' = \phi, \quad q' = \psi,$$

$$\theta + zu - z' = \int\left\{\left(z\frac{\partial n}{\partial x} - znb\right)dx + (nq + zna)\,dy\right\}.$$

The explicit value of z' is known as soon as the values of z and n are known.

Ex. Verify that a Bäcklund transformation of the preceding equation is given by

$$p' = \left(nb - \frac{\partial n}{\partial x}\right)z, \quad q' = -nq - naz;$$

and that another is given by

$$p' = -np - nbz, \quad q' = \left(na - \frac{\partial n}{\partial y}\right)z.$$

(Goursat.)

301. The process adopted by Goursat for a determination of quantities ϕ and ψ, which are to satisfy the equations

$$\lambda = \frac{\partial \phi}{\partial p} - \frac{\partial \psi}{\partial q},$$

$$\mu = \lambda(\alpha p + \beta q + \gamma z) = \frac{d\phi}{dy} - \frac{d\psi}{dx},$$

is as follows, on the assumption that λ is known. Take two particular functions $\phi_1(x, y, z, p)$ and $\psi_1(x, y, z, q)$, such that

$$\frac{\partial \phi_1}{\partial p} - \frac{\partial \psi_1}{\partial q} = \lambda;$$

then

$$\frac{\partial(\phi-\phi_1)}{\partial p}=\frac{\partial(\psi-\psi_1)}{\partial q}=U,$$

where U is any function only of x, y, z, so that

$$\phi=\phi_1+Up+V,\quad \psi=\psi_1+Uq+W,$$

where V and W are functions only of x, y, z. When these values are substituted in the second equation, we have

$$\frac{\partial V}{\partial y}-\frac{\partial W}{\partial x}+p\left(\frac{\partial U}{\partial y}-\frac{\partial W}{\partial z}\right)+q\left(\frac{\partial V}{\partial z}-\frac{\partial U}{\partial x}\right)=\rho,$$

where

$$\rho=\mu-\frac{d\phi_1}{dy}+\frac{d\psi_1}{dx}.$$

Evidently ρ is a linear function of p and q; and

$$\frac{\partial U}{\partial y}-\frac{\partial W}{\partial z}=\frac{\partial\rho}{\partial p},$$

$$\frac{\partial V}{\partial z}-\frac{\partial U}{\partial x}=\frac{\partial\rho}{\partial q},$$

$$\frac{\partial V}{\partial y}-\frac{\partial W}{\partial x}=\rho-p\frac{\partial\rho}{\partial p}-q\frac{\partial\rho}{\partial q},$$

ρ being determined by the equation

$$\frac{\partial^2\rho}{\partial x\partial p}+\frac{\partial^2\rho}{\partial y\partial q}=\frac{\partial\rho}{\partial z}-p\frac{\partial^2\rho}{\partial p\partial z}-q\frac{\partial^2\rho}{\partial q\partial z},$$

or, what is the same thing, by the equation

$$\frac{d}{dx}\left(\frac{\partial\rho}{\partial p}\right)+\frac{d}{dy}\left(\frac{\partial\rho}{\partial q}\right)-\frac{\partial\rho}{\partial z}=0.$$

When the value of ρ is substituted, we have

$$\frac{d}{dx}\left(\frac{\partial\mu}{\partial p}\right)+\frac{d}{dy}\left(\frac{\partial\mu}{\partial q}\right)-\frac{\partial\mu}{\partial z}=\frac{d^2}{dxdy}\left(\frac{\partial\phi_1}{\partial p}-\frac{\partial\psi_1}{\partial q}\right)$$

$$=\frac{d^2\lambda}{dxdy},$$

one of the equations already (§ 300) known as one to be satisfied. Assuming the value of μ to be such that this equation is satisfied, we see that the three equations in U, V, W are equivalent to a couple only, so that one of the three quantities U, V, W can be taken arbitrarily. Goursat takes

$$U=0;$$

and then

$$V = \int_{z_0}^{z} \frac{\partial \rho}{\partial q}\, dz + f(x, y),$$

$$W = -\int_{z_0}^{z} \frac{\partial \rho}{\partial p}\, dz + g(x, y),$$

provided

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \rho - p\frac{\partial \rho}{\partial p} - q\frac{\partial \rho}{\partial q} - \int_{z_0}^{z} \left(\frac{\partial^2 \rho}{\partial x \partial p} + \frac{\partial^2 \rho}{\partial y \partial q}\right) dz$$

$$= \left[\rho - p\frac{\partial \rho}{\partial p} - q\frac{\partial \rho}{\partial q}\right]_{z=z_0}.$$

The values of V and W (and therefore of ϕ and ψ) will be known, if f and g are any two functions satisfying this last relation.

When the particular form

$$\mu = \lambda(\alpha p + \beta q + \gamma z)$$

is taken, so that λ is an integral of the equation adjoint to

$$s + \alpha p + \beta q + \gamma z = 0,$$

and therefore can be regarded as a function of x and y only, we have first to obtain particular functions ϕ_1 and ψ_1, such that

$$\frac{\partial \phi_1}{\partial p} - \frac{\partial \psi_1}{\partial q} = \lambda.$$

Obviously we can take

$$\phi_1 = \tfrac{1}{2}\lambda p, \quad \psi_1 = -\tfrac{1}{2}\lambda q;$$

and then

$$\rho = \lambda(\alpha p + \beta q + \gamma z) - \tfrac{1}{2}p\frac{\partial \lambda}{\partial y} - \tfrac{1}{2}q\frac{\partial \lambda}{\partial x}.$$

Making $z_0 = 0$, we have

$$V = \int_{0}^{z} \frac{\partial \rho}{\partial q}\, dz + f(x, y) = \left(\lambda\beta - \tfrac{1}{2}\frac{\partial \lambda}{\partial x}\right) z + f,$$

$$W = -\int_{0}^{z} \frac{\partial \rho}{\partial p}\, dz + g(x, y) = -\left(\lambda\alpha - \tfrac{1}{2}\frac{\partial \lambda}{\partial y}\right) z + g,$$

and

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \left[\rho - p\frac{\partial \rho}{\partial p} - q\frac{\partial \rho}{\partial q}\right]_{z=0}$$

$$= 0,$$

so that we can take

$$f = \frac{\partial H}{\partial x}, \quad g = \frac{\partial H}{\partial y},$$

where H is an arbitrary function of x and y. The values of ϕ and ψ are

$$\left.\begin{aligned} p' = \phi = \quad \tfrac{1}{2}\lambda p + \left(\lambda\beta - \tfrac{1}{2}\frac{\partial\lambda}{\partial x}\right) z + \frac{\partial H}{\partial x} \\ q' = \psi = -\tfrac{1}{2}\lambda q - \left(\lambda\alpha - \tfrac{1}{2}\frac{\partial\lambda}{\partial y}\right) z + \frac{\partial H}{\partial y} \end{aligned}\right\};$$

earlier investigations shew that we can add $\frac{d\theta}{dx}$ to p, and $\frac{d\theta}{dy}$ to q, where θ is any function of x, y, z.

Simultaneous Equations of the First Order.

302. In connection with these investigations upon the transformation of equations of the second order, it is worth while considering another set of investigations on lines initiated by Bäcklund. Reference was made to them in the introductory note to Chapter XI; they are concerned with the discussion of simultaneous equations of the first order, the number of dependent variables being equal to the number of equations. The simplest case arises when there are two simultaneous equations involving two dependent variables: denoting the latter by z and z', and their derivatives by p, q, p', q', respectively, we may take the equations in the form

$$\left.\begin{aligned} f(x, y, z, z', p, q, p', q') = 0 \\ g(x, y, z, z', p, q, p', q') = 0 \end{aligned}\right\}.$$

When these two equations $f=0$ and $g=0$, can be resolved algebraically for (say) p and q, the condition being that the Jacobian of f and g with regard to p and q does not vanish identically, the resolution leads to equations

$$\left.\begin{aligned} p = f_1(x, y, z, z', p', q') \\ q = f_2(x, y, z, z', p', q') \end{aligned}\right\}.$$

Now we must have

$$\frac{dp}{dy} = \frac{dq}{dx},$$

where the respective derivatives of p and q are complete: hence

$$\frac{\partial f_1}{\partial y} + f_2\frac{\partial f_1}{\partial z} + q'\frac{\partial f_1}{\partial z'} + s'\frac{\partial f_1}{\partial p'} + t'\frac{\partial f_1}{\partial q'}$$
$$= \frac{\partial f_2}{\partial x} + f_1\frac{\partial f_2}{\partial z} + p'\frac{\partial f_2}{\partial z'} + r'\frac{\partial f_2}{\partial p'} + s'\frac{\partial f_2}{\partial q'},$$

being a relation which is linear in r', s', t', and which, in general, involves x, y, z, z', p', q'. When z occurs, we can conceive this relation resolved so as to express z, in terms of the independent variables, of z', and of the derivatives of z' up to the second order inclusive; when the value of z so given is substituted in the given equations (or in their resolved equivalents), they become two equations of the third order for the determination of z'. From the general theory we know that, unless the original equations cannot be resolved with respect to p and p', or with respect to q and q', they possess common integrals; consequently, the two equations of the third order which are satisfied by z' must be compatible with one another, and they must therefore lead to a value or values of z' which involve (or may involve) arbitrary elements. To each such value of z', there corresponds the resolved value of z as given above.

If, however, it should happen that the relation, which expresses the condition

$$\frac{dp}{dy}=\frac{dq}{dx},$$

should be explicitly free from z, then it becomes a single equation for z', of the second order and linear. When a value of z' has been obtained satisfying this equation of the second order, and when it is substituted in the equations expressing p and q, then a process of quadrature leads to a value of z which involves an arbitrary constant. We may therefore infer that an infinitude of integrals z will correspond to a single integral z'.

Note. Exceptional cases arise when the relation, which expresses the condition

$$\frac{dp}{dy}=\frac{dq}{dx},$$

does not involve r', s', t'; that this may happen, we must have

$$\frac{\partial f_2}{\partial p'}=0, \quad \frac{\partial f_1}{\partial q'}=0, \quad \frac{\partial f_1}{\partial p'}=\frac{\partial f_2}{\partial q'}.$$

If the relation in this special event should involve z, then z' satisfies two equations of the second order. If it should not involve z, then z' satisfies a single equation of the first order; and the relations between the integrals z and z' are the same as in the more general event.

303. When the two equations $f=0$, $g=0$, cannot be resolved algebraically for p and q, because the Jacobian of f and g with regard to p and q vanishes, either identically or in virtue of the two equations, then the elimination of p between the two equations compels the elimination of q also: the result of the elimination is an equation

$$g(x, y, z, z', p', q')=0.$$

If g involves z, we can imagine $g=0$ resolved with regard to z; and then the original equations can be replaced by a set (or by a number of sets) of equations of the form

$$f(x, y, z, z', p, q, p', q')=0,$$
$$z-g_1(x, y, z', p', q')=0.$$

Substituting for z, we usually have an equation of the second order for the determination of z'; to every integral of that equation, there corresponds a single value of z.

These are the results which arise from resolution of the two original equations with regard to p and q. The equations can equally be resolved with regard to p' and q', or the resolution can equally fail: and there will be corresponding relations between the integrals z' and z. We thus have the same three kinds of pairs of equations as arise in the Bäcklund transformations; they are as follows:—

(i) the equations may be such that an integral z' corresponds to a single integral, and conversely:

(ii) the equations may be such that a single integral of one equation corresponds to a single integral of the other, while a simple infinitude of integrals of that other corresponds to a single integral of the first:

(iii) the equations may be such that to each integral of either equation there corresponds a simple infinitude of integrals of the other.

In each case, the simple infinitude of integrals arises through the presence of an arbitrary parameter: and the equations are equations of the second order.

Ex. 1. Of the first case, the general type is such that the two initial equations can be expressed in the form

$$z'=F(x, y, z, p, q),$$
$$z=G(x, y, z', p', q'),$$

where F and G are explicit functions: the elimination of either of the dependent variables leads to an equation of the second order for the determination of the other.

The best known instance is associated with Laplace's linear equation, which is

$$s+ap+bq+cz=0,$$

a, b, c being functions of x and y only. When we take

$$z'=q+az, \quad z''=p+bz,$$

we find, with the notation of Chapter XIII,

$$hz=p'+bz', \quad kz=q''+az'':$$

then z' and z'' satisfy the respective equations

$$s'+a'p'+b'q'+c'z'=0,$$
$$s''+a''p''+b''q''+c''z''=0,$$

with definite values of a', b', c', a'', b'', c''. A single value of z and a single value of z' correspond uniquely to one another: likewise for a single value of z and a single value of z''. Hence a single value of z' and a single value of z'' correspond uniquely to one another through the medium of the unique corresponding z: but the analytical expression of the correspondence between z' and z'' is of a different character, for it involves derivatives of the second order.

Ex. 2. In the equation

$$as+bp+\psi(x, y, z, q, t)=0,$$

the coefficients a and b are functions of x, y, z, q; shew that, if

$$z'=\phi(x, y, z, q),$$

where ϕ is any integral of

$$b\frac{\partial\phi}{\partial q}-a\frac{\partial\phi}{\partial z}=0,$$

z' satisfies an equation of the second order. Shew also that each integral of either equation corresponds to only a single integral of the other.

(Teixeira.)

Ex. 3. Of the second kind of correspondence in the text, a type is represented by a couple of equations

$$z=f(x, y, z', p', q'),$$
$$0=g(x, y, p, q, z', p', q'),$$

these equations being supposed resoluble with regard to p' and q', in a form

$$p'=F(x, y, z, p, q, z'),$$
$$q'=G(x, y, z, p, q, z').$$

It is clear that z' satisfies an equation of the second order: if the relation

$$\frac{dF}{dy}=\frac{dG}{dx}$$

does not contain z' (and this will be assumed to be the fact), then z also satisfies an equation of the second order. To each integral z' there corresponds one, and only one, integral z: to each integral z, there corresponds a simple infinitude of integrals z'.

The simplest set of cases is given by the equations

$$z=f(x, y, p', q'),$$
$$0=g(x, y, p, q, p', q'),$$

the Jacobian of f and g with respect to p' and q' being supposed not to vanish.

Ex. 4. Of the third kind of correspondence in the text, a type is represented by a couple of equations

$$f(x, y, p, q, p', q')=0,$$
$$g(x, y, p, q, p', q')=0,$$

when they can be resolved with respect to p and q, and also with respect to p' and q': the sufficient condition is that neither of the Jacobians

$$J\left(\frac{f, g}{p, q}\right), \quad J\left(\frac{f, g}{p', q'}\right),$$

should vanish.

It is easy to see that z and z' both satisfy equations of the second order. To each integral z', there corresponds a simple infinitude of integrals z; and to each integral z, there corresponds a simple infinitude of integrals z'.

Ex. 5. Discuss the character of the correspondence of the integrals of the equations

$$p+p'=\sin(z-z'),$$
$$q-q'=\sin(z+z'),$$

being equations connected with surfaces of constant curvature. Obtain integrals z and z', which are functions of x^2+y^2 only; and interpret the results. (Bianchi; Darboux.)

Ex. 6. Shew that every equation $f(x, y, z, p, q, r, s, t)=0$ of the second order which, under the transformation

$$z'=q,$$

leads to an equation of the second order for the determination of z', can be brought to the form

$$X_1 r+X_2 p+X_3 z+g(x, y, q, s, t)=0,$$

where X_1, X_2, X_3 are any functions of x alone; and discuss the correspondence of the integrals of the two equations. (Goursat.)

Ex. 7. Shew that, if z satisfies a linear equation

$$Ar+2Bs+Ct+Dp+Eq+Fz=0,$$

and if a new quantity u be defined by the relation

$$u=\alpha p+\beta q+\gamma z,$$

where A, B, C, D, E, F, a, β, γ are functions of x and y only, then, when u satisfies an equation of the second order, $u=0$ must be satisfied by two distinct integrals of the original equation, provided $A\beta^2-2Ba\beta+Ca^2$ does not vanish.

Obtain the condition, necessary to secure that u satisfies an equation of the second order, when $A\beta^2-2Ba\beta+Ca^2$ does vanish. (Goursat.)

Ex. 8. Given two equations

$$p'=az'+\beta z+\gamma p+\delta q+\eta,$$
$$q'=a'z'+\beta'z+\gamma'p+\delta'q+\eta',$$

where all the coefficients are functions of x and y only; prove that, if z is to satisfy an equation of the second order resulting from the elimination of z', it is necessary and sufficient that

$$\frac{\partial a}{\partial y}=\frac{\partial a_1}{\partial x}.$$

When this relation is satisfied, and when $\gamma\delta'-\gamma'\delta=0$, shew that the elimination of z leads to an equation of the second order for z': and obtain the conditions for this result, when $\gamma\delta'-\gamma'\delta$ does not vanish.

Discuss the correspondence of the integrals. (Goursat.)

CHAPTER XXII.

Equations of the Third and Higher Orders, in Two Independent Variables.

The present chapter is only a brief outline of the application of the preceding theory and the various preceding methods (where they can be applied) to equations of order higher than the second; and, with the fewest exceptions, these applications are made to equations of the third order only.

The earlier sections are devoted to the discussion of equations of the third order having an intermediate integral of the second order. On this topic, a paper by Tanner* may be consulted: the method of exposition adopted is different from Tanner's, and is the extension of the method expounded in Chapter XVI. Some results relating to equations of the third order had previously been given by Falk†, who also gave some results, mostly formal, relating to equations of order n. Reference also may be made to the treatise by Natani‡, and to the investigations by Hamburger§ on equations of order higher than the second.

304. The methods, that have been expounded, were devised in connection with equations of the second order. They can be applied, with the appropriate modifications, to equations of order higher than the second: and some instances will now be given. As however the development does not seem to lead to any new kinds of properties, but is concerned with details of a kind already familiar in the equations of the second order, and as no new method of constructing integrals has been devised for equations of higher orders, we shall give only a brief discussion of this part of the subject.

* *Proc. Lond. Math. Soc.*, vol. VIII (1877), pp. 229—261.

† *Acta Ups.*, t. VIII (1872), pp. 1—40.

‡ *Die höhere Analysis*, quoted at the beginning of Chap. XI.

§ *Crelle*, t. XCIII (1882), pp. 201—214.

With the notation already adopted, the derivatives of z with regard to x and y of the third order will be denoted by $\alpha, \beta, \gamma, \delta$. The differential equation of the third order can then be taken in the form

$$f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0;$$

and, for the purposes of discussion, the equation will be supposed polynomial in the derivatives of highest order.

We naturally begin with those equations which possess an intermediate integral or which admit an equation of lower order compatible with themselves. The simplest form in which such an integral will occur is

$$\theta(u, v) = 0,$$

where θ is an arbitrary functional form, while u and v are definite functions of x, y, z, p, q, r, s, t: the corresponding differential equation is

$$E(\alpha\gamma - \beta^2) + F(\alpha\delta - \beta\gamma) + G(\beta\delta - \gamma^2) \\ + A\alpha + B\beta + C\gamma + D\delta + H = 0,$$

where A, B, C, D, E, F, G, H do not involve derivatives of the third order. This form is a first condition that an equation of the third order should have an intermediate integral of the specified type; yet the necessary form is only one among other conditions. Another simple form of intermediate integral is

$$g(x, y, z, p, q, r, s, t, a, b) = 0,$$

where a and b are arbitrary constants: the elimination of a and b between the equations

$$g = 0, \quad \frac{dg}{dx} = 0, \quad \frac{dg}{dy} = 0,$$

leads to an equation of the third order having $g = 0$ for an intermediate integral. And other cases can be constructed when other types of intermediate integral are postulated.

To construct the classes of equations that have intermediate integrals of a form

$$u(x, y, z, p, q, r, s, t) = 0,$$

without any specification of the element or elements of generality in u, we use the property that the three equations

$$f = 0,$$
$$u_x + \alpha u_r + \beta u_s + \gamma u_t = 0,$$
$$u_y + \beta u_r + \gamma u_s + \delta u_t = 0,$$

where

$$\left.\begin{aligned} u_x &= \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} \\ u_y &= \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} \end{aligned}\right\},$$

are not independent of one another; so that, regarded as three equations from which two of the four quantities α, β, γ, δ can be eliminated, they will provide an evanescent eliminant. Suppose that we use the second and the third of the equations to express α and δ in terms of β and γ, the coefficients in the expressions being homogeneous of order zero in the derivatives of u. Let these expressions be substituted in $f=0$, where f is a polynomial in the third derivatives: after substitution and collection of terms, there would be three terms at least in an arbitrary equation, viz. a term in β, a term in γ, and a term free from β and γ. The result of the substitution must, on the hypothesis adopted, lead to an evanescent form: hence the coefficient of each term in β and γ is to be evanescent, and the vanishing of each such coefficient gives a condition to be satisfied by u. Consequently, there will be three conditions at least. Each of the conditions must involve u_x, u_y, u_r, u_s, u_t homogeneously, and therefore there cannot be more than four independent conditions. We therefore have two cases to consider:—

(i) when there are three independent conditions for u;

(ii) when there are four independent conditions for u.

When u is known, or when we infer that u can be determined, then the differential equation possesses an intermediate integral.

Note. It has been assumed that the number of independent conditions is not two. If there were only two, we could consider them resolved for u_x and u_y in the form

$$u_x + g(u_r, u_s, u_t) = 0, \quad u_y + h(u_r, u_s, u_t) = 0:$$

the differential equation should then result from the elimination of the ratios $u_r : u_s : u_t$ from

$$\alpha u_r + \beta u_s + \gamma u_t = g, \quad \beta u_r + \gamma u_s + \delta u_t = h,$$

and there are, in general, too few equations for the performance of the elimination.

A general Class of Equations.

305. To illustrate the working by a particular case, consider the equation

$$E(\alpha\gamma-\beta^2)+F(\alpha\delta-\beta\gamma)+G(\beta\delta-\gamma^2)$$
$$+A\alpha+B\beta+C\gamma+D\delta+H=0,$$

which may have an intermediate integral.

According to the preceding argument, we take

$$\alpha=-\beta\frac{u_s}{u_r}-\gamma\frac{u_t}{u_r}-\frac{u_x}{u_r},$$

$$\delta=-\beta\frac{u_r}{u_t}-\gamma\frac{u_s}{u_t}-\frac{u_y}{u_t};$$

we substitute these values of α and δ in the equation, and we then make the resulting equation evanescent so far as regards the determination of β and γ.

The terms in β^2, $\beta\gamma$, γ^2 disappear in virtue of a single relation

$$Eu_t-Fu_s+Gu_r=0.$$

The term in β disappears in virtue of the relation

$$E\frac{u_y}{u_r}+F\frac{u_x}{u_t}-A\frac{u_s}{u_r}-D\frac{u_r}{u_t}+B=0,$$

the preceding relation being used to simplify the form. Similarly, from the disappearance of the term in γ, we have

$$F\frac{u_y}{u_r}+G\frac{u_x}{u_t}-A\frac{u_t}{u_r}-D\frac{u_s}{u_t}+C=0;$$

and lastly, the aggregate of terms independent of β and γ gives the relation

$$F\frac{u_xu_y}{u_ru_t}-A\frac{u_x}{u_r}-D\frac{u_y}{u_t}+H=0.$$

There are apparently four equations.

From the second and the third of these relations, we find (also using the first relation)

$$(EG-F^2)(Fu_x-Du_r)=(-AFG+BF^2-CFE+DE^2)\,u_t,$$
$$(EG-F^2)(Fu_y-Au_t)=(AG^2-BFG+CF^2-DEF)\,u_r;$$

and the fourth relation can be written

$$(Fu_x - Du_r)(Fu_y - Au_t) = (AD - FH)\, u_r u_t.$$

Consequently

$$(AFG - BF^2 + CFE - DE^2)(AG^2 - BFG + CF^2 - DEF)$$
$$= (EG - F^2)^2 (FH - AD),$$

a relation between the coefficients of the original equation which must be satisfied. When satisfied, it renders the fourth relation for u a mere identity in the presence of the other relations; also, it can be regarded as determining H, on the removal of an obviously superfluous factor F.

Following Tanner*, we select one class of equations determined by the property that $EG - F^2$ does not vanish: and this class will be composed of two sub-classes, according as F does not or does vanish. The subsidiary equations are

$$\left.\begin{aligned} Fu_x - Du_r - \theta u_t &= 0 \\ Fu_y - \theta' u_r - Au_t &= 0 \\ Fu_s - Gu_r - Eu_t &= 0 \end{aligned}\right\},$$

when F and $EG - F^2$ do not vanish, and the quantities θ and θ' are

$$\left.\begin{aligned} (EG - F^2)\,\theta &= - AFG + BF^2 - CFE + DE^2 \\ (EG - F^2)\,\theta' &= AG^2 - BFG + CF^2 - DEF \end{aligned}\right\};$$

obviously

$$\theta\theta' = AD - FH.$$

But, when F does vanish while neither E nor G vanishes, the subsidiary equations are

$$\left.\begin{aligned} Eu_t + Gu_r &= 0 \\ Eu_y &= -\left(D\frac{E}{G} + B\right) u_r + Au_s \\ Eu_x &= \left(A\frac{G}{E} + C\right) u_r + D\frac{E}{G} u_s \end{aligned}\right\},$$

while the value of H is easily found to be

$$H = \frac{A^2G + ACE}{E^2} + \frac{D^2E + BDG}{G^2};$$

the fourth relation then is an identity.

* *Proc. L. M. S.*, vol. VIII (1877), p. 237.

For either of the sub-classes of equations, we have a system of homogeneous linear equations of the first order in one dependent variable. This system can be treated in the ordinary way, by being made a complete Jacobian system. The most useful case arises when the completed system contains six equations so that, as there are eight variables, the system will have two algebraically independent integrals: if these be u_1 and u_2, the intermediate integral is

$$f(u_1, u_2) = 0,$$

where f is an arbitrary function. If the completed system contains seven equations, the only intermediate integral is of the form

$$u = a;$$

and that integral leads to two equations of the third order, not to one only. If the completed system contains eight equations, there is no intermediate integral.

Ex. 1. Consider the equation

$$a\gamma - \beta^2 + a\delta - \beta\gamma - (\beta\delta - \gamma^2) = 0.$$

Here $E=1$, $F=1$, $G=-1$, and all the other coefficients vanish. The necessary relation between the set of coefficients is satisfied: and so, formally, we have an instance of the first sub-class. Also

$$\theta = 0, \quad \theta' = 0;$$

thus the subsidiary equations are

$$u_x = 0, \quad u_y = 0, \quad u_s - u_r + u_t = 0.$$

Taken in full, these are

$$\Delta_1 = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} = 0,$$

$$\Delta_2 = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} = 0,$$

$$\Delta_3 = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0.$$

We have

$$(\Delta_1, \Delta_2) = 0,$$

$$(\Delta_1, \Delta_3) = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} = 0,$$

$$(\Delta_2, \Delta_3) = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0,$$

by the Jacobian conditions of coexistence: hence

$$\frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

With these equations, we have now

$$\Delta_1' = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} = 0,$$

$$\Delta_2' = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0:$$

also

$$\left(\Delta_1', \frac{\partial u}{\partial p}\right) = \frac{\partial u}{\partial z} = 0,$$

$$\left(\Delta_2', \frac{\partial u}{\partial q}\right) = \frac{\partial u}{\partial z} = 0.$$

Hence our equations are the complete Jacobian system

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0,$$

$$\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0.$$

Two independent integrals are $r - s$, $s + t$: hence an intermediate integral of the original equation is

$$f(r - s, \ s + t) = 0.$$

Ex. 2. Obtain intermediate integrals of the equations :—

(i) $\alpha\delta - \beta\gamma = 0$;

(ii) $t(\alpha\gamma - \beta^2) - s(\alpha\delta - \beta\gamma) + r(\beta\delta - \gamma^2) = 0$;

(iii) $t(\alpha\gamma - \beta^2) - r(\beta\delta - \gamma^2) + st\alpha - t(r+1)\beta + s(r+1)\gamma - r^2\delta = rt - s^2$.

(Tanner.)

306. Another class of these equations is the class determined by the condition

$$EG - F^2 = 0,$$

while E, F, G do not vanish together. Obviously both E and G cannot vanish, for then F also would vanish: and the case $E = 0$, $F = 0$, and $G \gtrless 0$, is obtainable from the case $G = 0$, $F = 0$, and $E \gtrless 0$, by interchange of variables. Hence there are two sub-classes in the present class:

(i) E, F, G all different from zero:

(ii) $E = 0$, $F = 0$, and G different from zero.

First, take the case when no one of the quantities E, F, G vanishes: then we may take

$$F = mE, \quad G = m^2E,$$

as fulfilling the general condition, m being not zero. The equations for θ and θ' then give

$$Am^3 - Bm^2 + Cm - D = 0,$$

as the necessary relation among the coefficients. In order to obtain the equations for u, we revert to the original relations of § 305, the first of which now is

$$u_t - mu_s + m^2 u_r = 0.$$

The second is

$$\frac{u_y}{u_r} + m\frac{u_x}{u_t} = \frac{A}{E}\frac{u_s}{u_r} + \frac{D}{E}\frac{u_r}{u_t} - \frac{B}{E},$$

while the third is

$$m\frac{u_y}{u_r} + m^2\frac{u_x}{u_t} = \frac{A}{E}\frac{u_t}{u_r} + \frac{D}{E}\frac{u_s}{u_t} - \frac{C}{E};$$

and these two are equivalent to one another, in virtue of the first relation and of the condition satisfied among the coefficients. The third relation is

$$(Fu_x - Du_r)(Fu_y - Au_t) = (AD - FH)\, u_r u_t.$$

Resolving the modified relations so as to obtain u_x, u_y, u_s in terms of u_r and u_t, we have

$$\left.\begin{aligned} u_x &= \frac{D}{mE}\,u_r + \frac{\theta}{m^2E}\,u_t \\ u_y &= \frac{\beta}{mE}\,u_r + \frac{A}{mE}\,u_t \\ u_s &= mu_r \quad + \frac{1}{m}\,u_t \end{aligned}\right\},$$

where θ is either root of the quadratic equation

$$\theta^2 - (m^2A - mB)\,\theta + m\,(AD - FH) = 0,$$

and where

$$\beta = \frac{m}{\theta}(AD - FH),$$

β being the other root of the quadratic equation.

In general, the quadratic has distinct roots: and thus there can be two distinct systems. In the most favourable combination, each of the systems would lead to an intermediate integral; and we should then have two intermediate integrals.

If, however, the condition

$$m\,(mA - B)^2 = 4\,(AD - FH)$$

is satisfied, the quadratic has equal roots: there is only a single system, and there cannot then be more than a single intermediate integral.

In either event, the system of simultaneous equations is treated in the usual manner.

Ex. 1. Consider the equation

$$\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2=0.$$

We have $E=1$, $F=1$, $G=1$: it thus is an example of the preceding case.

Also, there is only a single subsidiary system for u, because the quadratic equation is

$$\theta^2=0.$$

This subsidiary system is easily found to be

$$u_x=0, \quad u_y=0,$$

$$u_r-u_s+u_t=0:$$

two independent integrals are

$$r+s, \quad s+t;$$

and therefore an intermediate integral of the original equation is

$$f(r+s, \ s+t)=0.$$

A primitive of the equation is easily constructed.

Ex. 2. Integrate the equations :—

(i) $$\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2 + \frac{1}{y}(s+t)(\alpha+\beta)+\frac{1}{x}(r+s)(\gamma+\delta)+\frac{1}{xy}(r+s)(s+t)=0;$$

(ii) $$\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2+a(\alpha+\beta)+b(\beta+\gamma)=0,$$

where, in the latter, a and b are constants. (Tanner.)

Ex. 3. Obtain an intermediate integral of the equation

$$r^2(\beta\delta-\gamma^2)+rs(\beta\gamma-\alpha\delta)+s^2(\alpha\gamma-\beta^2)=0.$$

307. Next, take the case when $E=0$, $F=0$, and G is different from zero. The first of the relations in § 305 becomes

$$u_r=0,$$

so that u cannot involve r; and we therefore must reinvestigate from the beginning. Our differential equation is

$$G(\beta\delta-\gamma^2)+A\alpha+B\beta+C\gamma+D\delta+H=0;$$

and it is presumed to have an intermediate integral

$$u=u(x, y, z, p, q, s, t)=0.$$

Consequently, when we proceed from the relations

$$u_x + \beta u_s + \gamma u_t = 0,$$
$$u_y + \gamma u_s + \delta u_t = 0,$$

to eliminate the derivatives of the third order from the given equation, the result must be evanescent: hence

$$G\frac{u_x u_y}{u_s u_t} - B\frac{u_x}{u_s} - D\frac{u_y}{u_t} + H = 0,$$
$$G\left(\frac{u_x}{u_t} + \frac{u_y}{u_s}\right) - B\frac{u_t}{u_s} + C - D\frac{u_s}{u_t} = 0,$$
$$A = 0,$$

together, of course, with

$$u_r = 0.$$

We then resolve these equations: and we easily find that they are equivalent to the set

$$\left.\begin{aligned} u_r &= 0 \\ Gu_x - Du_s - \theta u_t &= 0 \\ Gu_y - \theta' u_s - Bu_t &= 0 \end{aligned}\right\},$$

where θ and θ' are the roots of the quadratic

$$\mu^2 + C\mu + BD - GH = 0.$$

Ex. 1. In the case of the equation

$$\beta\delta - \gamma^2 + t\beta - 2s\gamma + r\delta + rt - s^2 = 0,$$

we have

$$G = 1, \quad B = t, \quad C = -2s, \quad D = r, \quad H = rt - s^2;$$

hence

$$\theta = \theta' = s,$$

and the subsidiary equations for u are

$$\left.\begin{aligned} u_x - ru_s - su_t &= 0 \\ u_y - su_s - tu_t &= 0 \\ u_r &= 0 \end{aligned}\right\}.$$

When the system is rendered complete, it is found to possess two independent integrals

$$s + p, \quad t + q;$$

and therefore an intermediate integral exists in the form

$$f(s+p,\ t+q) = 0,$$

where f is an arbitrary function.

Ex. 2. Obtain an intermediate integral of the equation

$$pq(\beta\delta - \gamma^2) + qst\beta - (pst + qrt)\gamma + s^2p\delta - st(rt - s^2) = 0.$$

308. Next, we consider equations, for which $E=0$, $F=0$, $G=0$, and which therefore are linear of the form

$$A\alpha + B\beta + C\gamma + D\delta + H = 0.$$

Proceeding as usual to eliminate α and δ from the equation by means of the derivatives

$$u_x + \alpha u_r + \beta u_s + \gamma u_t = 0, \quad u_y + \beta u_r + \gamma u_s + \delta u_t = 0,$$

of the supposed intermediate integral $u=0$, and making the resulting equation evanescent as a relation in β and γ, we find

$$A\frac{u_s}{u_r} + D\frac{u_r}{u_t} - B = 0,$$

$$A\frac{u_t}{u_r} + D\frac{u_s}{u_t} - C = 0,$$

$$A\frac{u_x}{u_r} + D\frac{u_y}{u_t} - H = 0,$$

as the necessary and sufficient conditions. Taking

$$D\frac{u_r}{u_t} = A\mu,$$

the first of these relations gives

$$\frac{u_s}{u_r} = \frac{B}{A} - \mu;$$

and the second then becomes

$$\frac{D}{\mu} + A\mu\left(\frac{B}{A} - \mu\right) - C = 0,$$

that is,

$$A\mu^3 - B\mu^2 + C\mu - D = 0.$$

Let l, m, n be the roots of this cubic: then the three relations can be replaced by the set

$$\left.\begin{aligned} u_t &= mnu_r \\ u_s &= (m+n)\,u_r \\ A\,(u_x + lu_y) &= Hu_r \end{aligned}\right\};$$

and when the roots of the cubic are unequal, there are three such systems.

If $D=0$ and A is not zero, the inference merely is that one of the roots of the cubic is zero; and the corresponding subsidiary systems are simplified in form. If $A=0$ and D is not zero, we interchange the variables: and we then have the preceding case.

If $A=0$ and $D=0$, it is simplest to reinvestigate the subsidiary equations from the beginning. The coefficients B and C cannot vanish simultaneously: we shall assume that B does not vanish. Then the equation

$$B\beta + C\gamma + H = 0$$

is to be an inference from

$$u_x + \alpha u_r + \beta u_s + \gamma u_t = 0, \quad u_y + \beta u_r + \gamma u_s + \delta u_t = 0,$$

and therefore quantities λ and μ must exist such that

$$\begin{aligned} H &= \lambda u_x + \mu u_y, \\ B &= \lambda u_s + \mu u_r, \\ C &= \lambda u_t + \mu u_s, \\ 0 &= \lambda u_r, \\ 0 &= \mu u_t : \end{aligned}$$

consequently, the subsidiary systems are

$$\left.\begin{aligned} u_r = 0, \quad u_t &= 0 \\ H u_s = B u_x &+ C u_y \end{aligned}\right\},$$

$$\left.\begin{aligned} C u_r &= B u_s \\ C u_y &= H u_s \\ u_t &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} B u_t &= C u_s \\ B u_x &= H u_s \\ u_r &= 0 \end{aligned}\right\}.$$

In every case, we have a system or systems of subsidiary equations for the determination of u. Each of the equations is homogeneous and linear in the derivatives of u; and they can be treated by the customary Jacobian process of integration.

Ex. 1. Consider the equation

$$xy(\beta - \gamma) + xr - 2(x-y)s - yt + 2p - 2q = 0.$$

Here

$$A = 0 = D, \quad B = xy, \quad C = -xy,$$

$$H = xr - 2(x-y)s - yt + 2p - 2q.$$

The first of the subsidiary systems is

$$\frac{\partial u}{\partial r}=0,$$

$$\frac{\partial u}{\partial t}=0,$$

$$H\frac{\partial u}{\partial s}-xy\,(u_x-u_y)=0.$$

The Poisson-Jacobi conditions of coexistence of the first and third, and of the second and third, give

$$-xy\frac{\partial u}{\partial p}+x\frac{\partial u}{\partial s}=0,\quad xy\frac{\partial u}{\partial q}-y\frac{\partial u}{\partial s}=0:$$

so that we have

$$\frac{\partial u}{\partial r}=0,\quad \frac{\partial u}{\partial t}=0,\quad \frac{\partial u}{\partial s}-y\frac{\partial u}{\partial p}=0,\quad \frac{\partial u}{\partial s}-x\frac{\partial u}{\partial q}=0\;;$$

and, using these, the other equation is

$$(sy-sx+2p-2q)\frac{\partial u}{\partial s}-xy\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+xy\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0.$$

The Poisson-Jacobi conditions of coexistence of the last equation with the third and with the fourth of the modified system are satisfied in virtue of

$$\frac{\partial u}{\partial s}-xy\frac{\partial u}{\partial z}=0,$$

regard being had to all the equations. The system can be replaced by

$$\frac{\partial u}{\partial r}=0,\quad \frac{\partial u}{\partial t}=0,$$

$$y\frac{\partial u}{\partial p}=\frac{\partial u}{\partial s},\quad x\frac{\partial u}{\partial q}=\frac{\partial u}{\partial s},\quad xy\frac{\partial u}{\partial z}=\frac{\partial u}{\partial s},$$

$$xy\left(\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial s}(sy-sx+p-q)=0\;;$$

and, in this form, it is easily seen to be a complete Jacobian system. Consequently, it possesses two algebraically independent integrals: two such are

$$xys+yq+xp+z,\quad x+y\;;$$

and therefore an intermediate integral of the original equation is

$$xys+yq+xp+z=g''(x+y),$$

where g is an arbitrary function.

Proceeding in the same way with the second of the subsidiary systems, we find an intermediate integral in the form

$$xy\,(r-s)+(2y-x)\,p-yq-z=h''(x),$$

where h is an arbitrary function.

And similarly proceeding from the third of the subsidiary systems, we find an intermediate integral in the form

$$xy(t-s)-xp+(2x-y)q-z=k''(y),$$

where k is an arbitrary function.

Each one of these three intermediate integrals admits of integration. Further, they can be treated as existing simultaneously: the proof is simple. Either by integration of one of the integrals, or by quadrature that is based upon all three of them, we find a primitive in the form

$$xyz=h(x)+k(y)+g(x+y).$$

Ex. 2. Integrate the equations:—

(i) $x^3a+3x^2y\beta+3xy^2\gamma+y^3\delta+2(x^2r+2xys+y^2t)=0$;

(ii) $a+3u\beta+3u^2\gamma+u^3\delta=0$,

where u is given by the equation

$$r+2su+tu^2=0. \qquad \text{(Falk.)}$$

Ex. 3. Obtain an intermediate integral, involving an arbitrary function, and obtain further (as far as possible) the primitive, of the following equations:—

(i) $t(a\gamma-\beta^2)-s(a\delta-\beta\gamma)+r(\beta\delta-\gamma^2)=0$;

(ii) $q^2(a\gamma-\beta^2)+pq(a\delta-\beta\gamma)+p^2(\beta\delta-\gamma^2)$
$+2qsta+2(pst-qs^2-qrt)\beta$
$+2prs\delta+2(qrs-ps^2-prt)\gamma=(rt-s^2)^2$;

(iii) $(s-t)^2(a\gamma-\beta^2)+(s-r)(s-t)(a\delta-\beta\gamma)+(s-r)^2(\beta\gamma-\delta^2)=0$;

(iv) $$a\gamma-\beta^2+\frac{r}{a-s}(a\delta-\beta\gamma)+\left(\frac{r}{a-s}\right)^2(\beta\gamma-\delta^2)$$
$$+\frac{\lambda r}{a-s}a+\left\{\frac{\lambda r^2}{(a-s)^2}-\frac{\mu r}{a-s}\right\}\beta-\frac{\mu r^2}{(a-s)^2}\gamma=0,$$

where a, λ, μ are constants;

(v) $a-3c\beta+3c^2\gamma-c^3\delta=0$,

where c is a constant;

(vi) $(xs+yt)\beta-(xr+ys)\gamma=\left(\frac{x}{q}-\frac{y}{p}\right)s(rt-s^2)$;

(vii) $(qs-pt)\beta-(qr-ps)\gamma=s(rt-s^2)$;

(viii) $\beta=s\gamma$. (Tanner.)

Ex. 4. Denoting the derivatives of z of the fourth order with respect to x and y by ι, κ, λ, μ, ν, prove that, if the equation

$$A(\iota\lambda-\kappa^2)+B(\iota\mu-\kappa\lambda)+C(\iota\nu-\kappa\mu)+D(\kappa\mu-\lambda^2)+E(\kappa\nu-\lambda\mu)+F(\lambda\nu-\mu^2)$$
$$=I\iota+K\kappa+L\lambda+M\mu+N\nu+H,$$

where $A, \ldots, F, I, \ldots, N, H$ are functions of $x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$, possesses an intermediate integral

$$\phi(u, v)=0,$$

where ϕ is arbitrary, and where u and v are definite functions of z and its derivatives up to the third order inclusive, and of x and y, then the relation

$$AF-BE+CD=0$$

must be satisfied, as well as two other relations which involve only these coefficients $A, \ldots, H$ in the differential equation.

Assuming the conditions satisfied, prove that u and v are independent integrals of a system

$$\left.\begin{aligned} Cw_x - Nw_\alpha + \theta w_\delta &= 0 \\ Cw_y + \phi w_\alpha - Iw_\delta &= 0 \\ Cw_\beta - Ew_\alpha - Aw_\delta &= 0 \\ Cw_\gamma - Fw_\alpha - Bw_\delta &= 0 \end{aligned}\right\},$$

where

$$(AF-C^2)\,\theta = -KC^2 + MAC - NAB + IEC,$$
$$(AF-C^2)\,\phi = KCF - MC^2 + NBC - IEF.$$

Equations having Intermediate Integrals.

309. Proceeding now with the equation of general form and dealing first with the case when there are three equations for the determination of u, where $u=0$ is the supposed intermediate integral, we may suppose that these equations (which are homogeneous of order zero in u_x, u_y, u_r, u_s, u_t) are resolved so as to express three of these quantities, say u_x, u_y, u_s, in terms of the other two. Let

$$x, y, z, p, q, r, s, t = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8,$$

respectively; and let

$$\frac{\partial u}{\partial x_i} = p_i,$$

for $i = 1, \ldots, 8$; then the three equations can be taken in the form

$$\Delta = p_1 + x_4 p_3 + x_6 p_4 + x_7 p_5 + h(x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta' = p_2 + x_5 p_3 + x_7 p_4 + x_8 p_5 + k(x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta'' = p_7 \qquad\qquad + l(x_1, \ldots, x_8, p_6, p_8) = 0,$$

where each of the functions h, k, l is homogeneous of the first order in p_6 and p_8.

As $\Delta = 0$, $\Delta' = 0$, $\Delta'' = 0$, are simultaneous equations of the first order in one dependent variable, they must satisfy the Poisson-Jacobi conditions of coexistence

$$(\Delta, \Delta') = 0, \quad (\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0.$$

The condition $(\Delta, \Delta'') = 0$, when expressed in full, is

$$\frac{\partial l}{\partial x_1} + x_4 \frac{\partial l}{\partial x_3} + x_6 \frac{\partial l}{\partial x_4} + x_7 \frac{\partial l}{\partial x_5} - \left(p_5 + \frac{\partial h}{\partial x_7}\right) + \frac{\partial (h, l)}{\partial (p_6, x_6)} + \frac{\partial (h, l)}{\partial (p_8, x_8)} = 0;$$

and the condition $(\Delta', \Delta'') = 0$, when expressed in full, is

$$\frac{\partial l}{\partial x_2} + x_5 \frac{\partial l}{\partial x_3} + x_7 \frac{\partial l}{\partial x_4} + x_8 \frac{\partial l}{\partial x_5} - \left(p_4 + \frac{\partial k}{\partial x_7}\right) + \frac{\partial (k, l)}{\partial (p_6, x_6)} + \frac{\partial (k, l)}{\partial (p_8, x_8)} = 0.$$

Before using the other condition, it is convenient to use these two relations: they clearly are independent of the three equations already obtained, and so they are new equations expressing p_5 and p_4 respectively in terms of the other quantities. Let

$$\Delta_4 = p_4 + f_4 (x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta_5 = p_5 + f_5 (x_1, \ldots, x_8, p_6, p_8) = 0:$$

substituting these values, let Δ become ∇, where

$$\nabla = p_1 + x_4 p_3 + h' (x_1, \ldots, x_8, p_6, p_8) = 0.$$

Now we must have

$$(\nabla, \Delta_4) = 0:$$

expressed in full, the condition is

$$\frac{\partial f_4}{\partial x_1} + x_4 \frac{\partial f_4}{\partial x_3} - \left(p_3 + \frac{\partial h'}{\partial x_4}\right) + \frac{\partial (h', f_4)}{\partial (p_6, x_6)} + \frac{\partial (h', f_4)}{\partial (p_8, x_8)} = 0.$$

This again is obviously a new equation: and it expresses p_3 in terms of the other quantities. Inserting this value, and gathering together the various equations, we have

$$\left.\begin{aligned}
0 = \Delta_1 &= p_1 + f_1 (x_1, \ldots, x_8, p_6, p_8) = p_1 + p_8 \phi_1 (x_1, \ldots, x_8, m) \\
0 = \Delta_2 &= p_2 + f_2 (x_1, \ldots, x_8, p_6, p_8) = p_2 + p_8 \phi_2 (x_1, \ldots, x_8, m) \\
0 = \Delta_3 &= p_3 + f_3 (x_1, \ldots, x_8, p_6, p_8) = p_3 + p_8 \phi_3 (x_1, \ldots, x_8, m) \\
0 = \Delta_4 &= p_4 + f_4 (x_1, \ldots, x_8, p_6, p_8) = p_4 + p_8 \phi_4 (x_1, \ldots, x_8, m) \\
0 = \Delta_5 &= p_5 + f_5 (x_1, \ldots, x_8, p_6, p_8) = p_5 + p_8 \phi_5 (x_1, \ldots, x_8, m) \\
0 = \Delta_7 &= p_7 + f_7 (x_1, \ldots, x_8, p_6, p_8) = p_7 + p_8 \phi_7 (x_1, \ldots, x_8, m)
\end{aligned}\right\},$$

where

$$m = \frac{p_6}{p_8},$$

all the functions $f_1, f_2, f_3, f_4, f_5, f_7$ being homogeneous of the first order in p_6 and p_8. Thus

$$\frac{\partial f_a}{\partial p_6} = \frac{\partial \phi_a}{\partial m}, \quad \frac{\partial f_a}{\partial p_8} = \phi_a - m\frac{\partial \phi_a}{\partial m};$$

and so the Poisson-Jacobi conditions of coexistence, being

$$(\Delta_i, \Delta_j) = 0,$$

take the forms

$$\frac{\partial \phi_j}{\partial x_i} - \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_i}{\partial m}\frac{\partial \phi_j}{\partial x_6} - \frac{\partial \phi_j}{\partial m}\frac{\partial \phi_i}{\partial x_6}$$

$$+ \left(\phi_i - m\frac{\partial \phi_i}{\partial m}\right)\frac{\partial \phi_j}{\partial x_8} - \left(\phi_j - m\frac{\partial \phi_j}{\partial m}\right)\frac{\partial \phi_i}{\partial x_8} = 0,$$

on the removal of a superfluous factor p_8.

The above system of equations will be regarded as complete, in order that there may be an intermediate integral; hence these relations, for all the combinations $i, j = 1, 2, 3, 4, 5, 7$, must be satisfied. It is then obvious that they can only be satisfied identically.

Suppose that functions $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_7$ are known, satisfying the foregoing conditions: then the Jacobian system is complete, and it possesses integrals satisfying all the equations, that is, there is an intermediate integral. As, however, the equations are no longer necessarily linear and homogeneous in the derivatives of u, we cannot declare that the intermediate integral necessarily involves an arbitrary functional form, though it will involve some arbitrary element. Let it be denoted by

$$u = 0;$$

then, as

$$mp_8 = p_6,$$

we have

$$m\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}.$$

The equation of the third order is given by the elimination of the arbitrary element and of m between the equations

$$\left.\begin{aligned} m\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial r} \\ \text{and} \qquad \alpha m + \beta\phi_7 + \gamma + \phi_1 + p\phi_3 + r\phi_4 + s\phi_5 &= 0 \\ \beta m + \gamma\phi_7 + \delta + \phi_2 + q\phi_3 + s\phi_4 + t\phi_5 &= 0 \end{aligned}\right\}.$$

The development of the analysis follows a course similar to that adopted (in Chapter XX) for the corresponding questions relating to equations of the second order in two independent variables.

Ex. 1. Prove that the equation

$$\alpha^2\delta^2 + \alpha\gamma^3 + \beta^3\delta = 3\alpha\beta\gamma\delta$$

possesses an intermediate integral, in the form of an equation of the second order involving two independent arbitrary constants. Obtain this integral: and deduce a primitive.

Ex. 2. Obtain intermediate integrals of the equations:—

(i) $\alpha\delta - \beta\gamma = 0$;

(ii) $$(\beta\delta-\gamma^2)^2 r + (\beta\gamma - \alpha\delta)(\beta\delta-\gamma^2) s + (\alpha\gamma-\beta^2)(\beta\delta-\gamma^2) t = (\beta\gamma-\alpha\delta)(\alpha\gamma-\beta^2);$$

(iii) $$(\alpha\delta-\beta\gamma)\{(\alpha\delta-\beta\gamma) r + 2(\beta^2-\alpha\gamma) s\} + \{(\alpha\delta-\beta\gamma) t + 2(\gamma^2-\beta\delta) s\}^2 = 0.$$

310. Next dealing with the case where four equations arise in the process of § 304 for the determination of u, we may suppose these equations resolved for the four ratios $u_x : u_y : u_r : u_s : u_t$, in a form

$$\left.\begin{aligned} u_x + u_s\theta_1 &= 0 \\ u_y + u_s\theta_2 &= 0 \\ u_r + u_s\theta_3 &= 0 \\ u_t + u_s\theta_4 &= 0 \end{aligned}\right\},$$

where θ_1, θ_2, θ_3, θ_4 can be functions of all the eight variables, subject of course to the necessary conditions of coexistence of the four equations.

It is, however, comparatively unnecessary to discuss the detailed development of this case: for even when the conditions are satisfied, so that an intermediate integral exists, that integral leads to two equations of the third order and not to only a single equation. In fact, these equations are

$$\left.\begin{aligned}\theta_1 + \alpha\theta_3 + \beta + \gamma\theta_4 = 0\\ \theta_2 + \beta\theta_3 + \gamma + \delta\theta_4 = 0\end{aligned}\right\};$$

we shall not further consider the case.

Ex. Obtain intermediate integrals of the equations:—

(i) $$\frac{a\gamma-\beta^2}{r}=\frac{a\delta-\beta\gamma}{2s}=\frac{\beta\delta-\gamma^2}{t};$$

(ii) $$\left.\begin{aligned}(ta+r\gamma)(t\beta+r\delta)=4rt\beta\gamma\\ t(a\gamma-\beta^2)-r(\beta\delta-\gamma^2)=0\end{aligned}\right\}.$$

Ampère's Method applied to Equations of the Third Order.

311. When a given equation of the third order does not possess an intermediate integral, in the form of an equation of the second order involving an arbitrary element, so that the preceding method does not apply, we still may be able to proceed to a primitive by applying Ampère's method, as used for equations of the second order, or Darboux's method, for the construction of compatible equations of the third or of higher order.

Using an extension of Ampère's method, we denote by u the argument of any one of the three arbitrary functions that occur in the integral equivalent, supposed to be free from partial quadratures: and we change the independent variables from x and y to x and u, on the assumption that u is not a function of x alone. Adopting the notation of Chapter XVII and denoting derivatives with regard to x and u by $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta u}$, we have the former relations, viz.

$$\frac{\delta z}{\delta x}=p+q\frac{\delta y}{\delta x},\quad \frac{\delta z}{\delta u}=q\frac{\delta y}{\delta u},$$

$$\frac{\delta p}{\delta x}=r+s\frac{\delta y}{\delta x},\quad \frac{\delta p}{\delta u}=s\frac{\delta y}{\delta u},$$

$$\frac{\delta q}{\delta x}=s+t\frac{\delta y}{\delta x},\quad \frac{\delta q}{\delta u}=t\frac{\delta y}{\delta u},$$

as well as the further relations

$$\frac{\delta r}{\delta x} = \alpha + \beta \frac{\delta y}{\delta x}, \quad \frac{\delta r}{\delta u} = \beta \frac{\delta y}{\delta u},$$

$$\frac{\delta s}{\delta x} = \beta + \gamma \frac{\delta y}{\delta x}, \quad \frac{\delta s}{\delta u} = \gamma \frac{\delta y}{\delta u},$$

$$\frac{\delta t}{\delta x} = \gamma + \delta \frac{\delta y}{\delta x}, \quad \frac{\delta t}{\delta u} = \delta \frac{\delta y}{\delta u}.$$

Keeping the value of δ as given by the last equation, viz.,

$$\delta = \frac{\delta t}{\delta u} \div \frac{\delta y}{\delta u},$$

we have

$$\gamma = \frac{\delta t}{\delta x} - \delta \frac{\delta y}{\delta x},$$

$$\beta = \frac{\delta s}{\delta x} - \frac{\delta t}{\delta x}\frac{\delta y}{\delta x} + \delta \left(\frac{\delta y}{\delta x}\right)^2,$$

$$\alpha = \frac{\delta r}{\delta x} - \frac{\delta s}{\delta x}\frac{\delta y}{\delta x} + \frac{\delta t}{\delta x}\left(\frac{\delta y}{\delta x}\right)^2 - \delta \left(\frac{\delta y}{\delta x}\right)^3.$$

Let these values of α, β, γ be substituted in the given equation

$$f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0,$$

which will be supposed to be a polynomial in the derivatives of the third order. After the substitution, f will become a polynomial in δ alone; thus the equation, arranged in powers of δ, acquires a form

$$Q_0 + Q_1\delta + \ldots + Q_m\delta^m = 0,$$

where the original degree of f as a polynomial in α, β, γ, δ is m at least.

Now the equation is to be satisfied identically when the proper value of z is substituted. In that value, there occur an arbitrary function of u and its derivatives up to finite order; and these derivatives occur in p, q, r, s, t. Now, in the derivative $\frac{\delta t}{\delta u}$, the order of the highest derivative of the arbitrary function is greater than the order of the derivatives of the arbitrary function which occur in any of the quantities $Q_0, \ldots, Q_m$: that is, in the transformed equation, the quantity δ contains higher derivatives of the arbitrary function than occur elsewhere. The transformed equation must be satisfied identically in connection with the integral system:

when account is taken of the successive powers of δ, it is easy to see that the requirement as to the equation can be fulfilled, only if

$$Q_m = 0, \quad Q_{m-1} = 0, \ldots, Q_1 = 0, \quad Q_0 = 0.$$

Further, having regard to the preceding values of α, β, γ substituted in $f = 0$, we see that the equation

$$\frac{\partial f}{\partial \delta} - \frac{\partial f}{\partial \gamma}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial \beta}\left(\frac{\delta y}{\delta x}\right)^2 - \frac{\partial f}{\partial \alpha}\left(\frac{\delta y}{\delta x}\right)^3 = 0$$

must be satisfied: but it is not additional to the other equations, being satisfied in virtue of them and of the subsidiary equations. Now $\frac{\delta y}{\delta x}$ is given by the relation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\delta y}{\delta x} = 0;$$

and therefore

$$\frac{\partial f}{\partial \delta}\left(\frac{\partial u}{\partial y}\right)^3 + \frac{\partial f}{\partial \gamma}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial u}{\partial x} + \frac{\partial f}{\partial \beta}\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right)^3 = 0,$$

which accordingly is an equation satisfied by the argument of any arbitrary function that occurs in the integral equivalent of the given equation, on the hypothesis adopted as to the character of that equivalent.

Reverting to the earlier relations, we see that they give a number of simultaneous equations. If these equations are consistent with one another, and are also consistent with $f = 0$, regard being paid to the relations between the derivatives relative to the old independent variables and the new, then the original equation can possess an integral system of the specified type. The quantities $Q_0, Q_1, \ldots, Q_m$ contain z, p, q, r, s, t, and also the derivatives of these with regard to x: and we also have

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}, \quad \frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \quad \frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}.$$

Thus the system of equations contains no derivatives with regard to u: it can be regarded as a system of simultaneous ordinary equations.

Ex. 1. Consider the equation

$$\beta - \gamma = \frac{r - t}{x + y}.$$

When we substitute the values of β and γ in terms of δ, we have two equations after the application of the preceding process. One of these equations is

$$\left(\frac{\delta y}{\delta x}\right)^2 + \frac{\delta y}{\delta x} = 0:$$

as this is a degenerate form of the cubic, the arguments of the three arbitrary functions that occur in the integral equivalent are

$$y, \quad x, \quad x+y.$$

The remaining equation is

$$\frac{\delta s}{\delta x} - \frac{\delta t}{\delta x}\frac{\delta y}{\delta x} - \frac{\delta t}{\delta x} = \frac{r-t}{x+y}.$$

Taking the argument u, where

$$u = x+y,$$

we have this equation in the form

$$u\frac{\delta s}{\delta x} = r - t,$$

where u is constant in derivation with respect to x. Also,

$$\frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x} = r - s,$$

$$\frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x} = s - t,$$

in the present case; thus

$$\frac{\delta p}{\delta x} + \frac{\delta q}{\delta x} = r - t,$$

and therefore

$$u\frac{\delta s}{\delta x} = \frac{\delta p}{\delta x} + \frac{\delta q}{\delta x},$$

so that, as u is parametric, we have

$$us = p + q + \text{constant}.$$

The constant on the right-hand side is subject to the constancy of u: let it be

$$u\theta''(u) - 2\theta'(u),$$

where θ is any arbitrary function. Thus

$$us = p + q + u\theta''(u) - 2\theta'(u),$$

and therefore

$$s = \frac{p+q}{x+y} + \theta''(x+y) - \frac{2}{x+y}\theta'(x+y),$$

which is an intermediate integral.

The primitive is

$$z = X + Y - \tfrac{1}{2}(x+y)(X' + Y') + \theta(x+y),$$

where X and Y are arbitrary functions of x and of y respectively.

Ex. 2. Integrate the equation

$$\beta+\gamma=\frac{r+2s+t}{x+y};$$

and obtain a primitive of

$$(x-2cy)^2(c\beta+\gamma)=(x-2cy)\,t+2cq$$

in the form

$$z=X+2c\{Y+\theta(x-cy)\}-(x-2cy)\{Y'-c\theta'(x-cy)\},$$

where X, Y, $\theta(x-cy)$ are arbitrary functions of x, of y, and of $x-cy$ respectively.

Darboux's Method applied to Equations of the Third Order.

312. It is natural to consider the extension of Darboux's method, as explained in Chapter XVIII, to equations of order higher than the second. When it appears that an equation

$$f=f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta)=0$$

has no intermediate integral in the form of an equation of the second order, the method seeks to obtain a new equation of the third order, say

$$u=u(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta)=0,$$

which may coexist with $f=0$, though it is not resoluble into $f=0$. Let the derivatives of z of the fourth order be denoted by ι, κ, λ, μ, ν: and write

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}+\alpha\frac{\partial}{\partial r}+\beta\frac{\partial}{\partial s}+\gamma\frac{\partial}{\partial t},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}+\beta\frac{\partial}{\partial r}+\gamma\frac{\partial}{\partial s}+\delta\frac{\partial}{\partial t}.$$

Then, in accordance with the earlier explanations, we assign the conditions that the equations

$$\left.\begin{aligned}
0&=\frac{df}{dx}+\iota\frac{\partial f}{\partial\alpha}+\kappa\frac{\partial f}{\partial\beta}+\lambda\frac{\partial f}{\partial\gamma}+\mu\frac{\partial f}{\partial\delta}\\
0&=\frac{df}{dy}\qquad+\kappa\frac{\partial f}{\partial\alpha}+\lambda\frac{\partial f}{\partial\beta}+\mu\frac{\partial f}{\partial\gamma}+\nu\frac{\partial f}{\partial\delta}\\
0&=\frac{du}{dx}+\iota\frac{\partial u}{\partial\alpha}+\kappa\frac{\partial u}{\partial\beta}+\lambda\frac{\partial u}{\partial\gamma}+\mu\frac{\partial u}{\partial\delta}\\
0&=\frac{du}{dy}\qquad+\kappa\frac{\partial u}{\partial\alpha}+\lambda\frac{\partial u}{\partial\beta}+\mu\frac{\partial u}{\partial\gamma}+\nu\frac{\partial u}{\partial\delta}
\end{aligned}\right\}$$

are not linearly independent of one another. That this may be the case, quantities l, m, n must exist such that

$$\begin{aligned}
\frac{du}{dx} + l\frac{du}{dy} - m\frac{df}{dx} - n\frac{df}{dy} &= 0,\\
\frac{\partial u}{\partial \alpha} \qquad\quad - m\frac{\partial f}{\partial \alpha} \qquad\quad &= 0,\\
\frac{\partial u}{\partial \beta} + l\frac{\partial u}{\partial \alpha} - m\frac{\partial f}{\partial \beta} - n\frac{\partial f}{\partial \alpha} &= 0,\\
\frac{\partial u}{\partial \gamma} + l\frac{\partial u}{\partial \beta} - m\frac{\partial f}{\partial \gamma} - n\frac{\partial f}{\partial \beta} &= 0,\\
\frac{\partial u}{\partial \delta} + l\frac{\partial u}{\partial \gamma} - m\frac{\partial f}{\partial \delta} - n\frac{\partial f}{\partial \gamma} &= 0,\\
l\frac{\partial u}{\partial \delta} \qquad\quad - n\frac{\partial f}{\partial \delta} &= 0.
\end{aligned}$$

Now the quantity $n - lm$ cannot be zero: for if it were, we should have

$$\frac{\partial u}{\partial \alpha} = m\frac{\partial f}{\partial \alpha}, \quad \frac{\partial u}{\partial \beta} = m\frac{\partial f}{\partial \beta}, \quad \frac{\partial u}{\partial \gamma} = m\frac{\partial f}{\partial \gamma}, \quad \frac{\partial u}{\partial \delta} = m\frac{\partial f}{\partial \delta},$$

and u would not be functionally independent of f, so far as concerns α, β, γ, δ. Multiply the second equation by l^4, the third by $-l^3$, the fourth by l^2, the fifth by $-l$, and add all these to the sixth: then we have

$$(n - ml)\left(l^3\frac{\partial f}{\partial \alpha} - l^2\frac{\partial f}{\partial \beta} + l\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \delta}\right) = 0,$$

so that l is a root of the equation

$$\theta^3\frac{\partial f}{\partial \alpha} - \theta^2\frac{\partial f}{\partial \beta} + \theta\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \delta} = 0.$$

Let ρ, σ, τ be the three roots of this cubic, and let

$$l = \tau;$$

we shall assume that ρ, σ, τ are unequal. Again, multiply those equations in order by $\rho^3, -\rho^2, \rho, -1, \frac{1}{\rho}$ respectively, and add: we have

$$\left(1 - \frac{l}{\rho}\right)\left(\rho^3\frac{\partial u}{\partial \alpha} - \rho^2\frac{\partial u}{\partial \beta} + \rho\frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta}\right) = 0,$$

and l is not equal to ρ; so that

$$\rho^3 \frac{\partial u}{\partial \alpha} - \rho^2 \frac{\partial u}{\partial \beta} + \rho \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} = 0.$$

Similarly,

$$\sigma^3 \frac{\partial u}{\partial \alpha} - \sigma^2 \frac{\partial u}{\partial \beta} + \sigma \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} = 0.$$

Also

$$m = \frac{\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial \alpha}},$$

$$n = l \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial \delta}} = \frac{1}{\rho\sigma} \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial \alpha}};$$

and therefore the first equation is

$$\left(\frac{du}{dx} + \tau \frac{du}{dy}\right) \frac{\partial f}{\partial \alpha} = \frac{df}{dx} \frac{\partial u}{\partial \alpha} + \frac{1}{\rho\sigma} \frac{df}{dy} \frac{\partial u}{\partial \delta}.$$

With each arrangement of the roots of

$$\theta^3 \frac{\partial f}{\partial \alpha} - \theta^2 \frac{\partial f}{\partial \beta} + \theta \frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \delta} = 0,$$

we have three equations satisfied by u; one integral obviously is $u = f$. If they possess an integral, which is distinct from f and involves some of the derivatives α, β, γ, δ, then

$$u = 0$$

is a new equation independent of, and compatible with, $f = 0$. The test, as to whether they do or do not possess such an integral, is obtained as usual: the set of partial equations of the first order in u is made a complete Jacobian system. If when thus completed, the system contains n equations, it possesses $11 - n$ new integrals: for there are twelve variables that can occur, and f is certainly an integral.

It may happen that one distribution of the roots of the cubic may provide a system which possesses new integrals, and that another distribution does not. The most favourable case occurs when three integrals are provided: the least favourable case occurs when no integrals are obtained. Moreover, when $n = 9$ for any system, being the value of n which often occurs when the method proves effective, there are two integrals, say u_1 and u_2; the most general integral is then $\phi(u_1, u_2)$, where ϕ is arbitrary: and the equation compatible with $f=0$ is

$$\phi(u_1, u_2) = 0.$$

When the process leads to no such integral, then we attempt to find an equation of the fourth order compatible with, but not composed of, the equations

$$\frac{df}{dx} = 0, \qquad \frac{df}{dy} = 0,$$

the complete derivatives of $f=0$.

In all cases, the subsidiary equations in this extension of Darboux's method are homogeneous and linear of the first order.

313. Instead of subsidiary equations which are linear and homogeneous partial equations of the first order, we can obtain a subsidiary system in differential elements as follows. Let

$$X = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q} + \alpha\frac{\partial f}{\partial r} + \beta\frac{\partial f}{\partial s} + \gamma\frac{\partial f}{\partial t},$$

$$Y = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q} + \beta\frac{\partial f}{\partial r} + \gamma\frac{\partial f}{\partial s} + \delta\frac{\partial f}{\partial t},$$

$$A = \frac{\partial f}{\partial \alpha}, \quad B = \frac{\partial f}{\partial \beta}, \quad C = \frac{\partial f}{\partial \gamma}, \quad D = \frac{\partial f}{\partial \delta};$$

then, because of the equation

$$f = 0,$$

we have

$$A\frac{\partial \alpha}{\partial x} + B\frac{\partial \beta}{\partial x} + C\frac{\partial \gamma}{\partial x} + D\frac{\partial \delta}{\partial x} = -X,$$

$$A\frac{\partial \alpha}{\partial y} + B\frac{\partial \beta}{\partial y} + C\frac{\partial \gamma}{\partial y} + D\frac{\partial \delta}{\partial y} = -Y.$$

Now the system

$$dr = \alpha dx + \beta dy, \quad dp = r dx + s dy,$$
$$ds = \beta dx + \gamma dy, \quad dq = s dx + t dy,$$
$$dt = \gamma dx + \delta dy, \quad dz = p dx + q dy,$$

is to be perfectly integrable; hence, among other relations, we must have

$$\frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y}, \quad \frac{\partial \gamma}{\partial x} = \frac{\partial \beta}{\partial y}, \quad \frac{\partial \delta}{\partial x} = \frac{\partial \gamma}{\partial y},$$

the derivatives with regard to x and y being complete. Hence

$$d\alpha = \frac{\partial \alpha}{\partial x} dx + \frac{\partial \beta}{\partial x} dy,$$

$$d\beta = \frac{\partial \beta}{\partial x} dx + \frac{\partial \gamma}{\partial x} dy = \frac{\partial \alpha}{\partial y} dx + \frac{\partial \beta}{\partial y} dy,$$

$$d\gamma = \frac{\partial \gamma}{\partial x} dx + \frac{\partial \delta}{\partial x} dy = \frac{\partial \beta}{\partial y} dx + \frac{\partial \gamma}{\partial y} dy,$$

$$d\delta \qquad\qquad\qquad\qquad = \frac{\partial \gamma}{\partial y} dx + \frac{\partial \delta}{\partial y} dy.$$

In accordance with Hamburger's method of procedure (§§ 167, 173), we form the combinations

$$\lambda_1 d\alpha + \lambda_2 d\beta + \lambda_3 d\gamma = \frac{\partial \alpha}{\partial x} \lambda_1 dx + \frac{\partial \beta}{\partial x} (\lambda_2 dx + \lambda_1 dy) + \frac{\partial \gamma}{\partial x} (\lambda_3 dx + \lambda_2 dy) + \frac{\partial \delta}{\partial x} \lambda_3 dy,$$

$$\lambda_1 d\beta + \lambda_2 d\gamma + \lambda_3 d\delta = \frac{\partial \alpha}{\partial y} \lambda_1 dx + \frac{\partial \beta}{\partial y} (\lambda_2 dx + \lambda_1 dy) + \frac{\partial \gamma}{\partial y} (\lambda_3 dx + \lambda_2 dy) + \frac{\partial \delta}{\partial y} \lambda_3 dy;$$

and then, comparing these with the two first derivatives of $f = 0$, we construct the linear equations

$$\frac{\lambda_1 dx}{A} = \frac{\lambda_2 dx + \lambda_1 dy}{B} = \frac{\lambda_3 dx + \lambda_2 dy}{C} = \frac{\lambda_3 dy}{D}$$

$$= \frac{\lambda_1 d\alpha + \lambda_2 d\beta + \lambda_3 d\gamma}{-X} = \frac{\lambda_1 d\beta + \lambda_2 d\gamma + \lambda_3 d\delta}{-Y}.$$

The equality of the first four fractions determines τ and the ratios $\lambda_1 : \lambda_2 : \lambda_3$, where

$$dy = \tau dx.$$

Writing each of the fractions as equal to $J dx$, we have

$$\lambda_1 = JA,$$
$$\lambda_2 = J(B - \tau A),$$
$$\lambda_3 = J(C - \tau B + \tau^2 A),$$
$$\tau\lambda_3 = JD:$$

hence τ is determined by the cubic

$$A\tau^3 - B\tau^2 + C\tau - D = 0,$$

the significance of which will appear later. Using these values of $\lambda_1, \lambda_2, \lambda_3$, we further have

$$A\,d\alpha + (B - \tau A)\,d\beta + \frac{1}{\tau} D d\gamma = -X\,dx,$$

$$A\,d\beta + (B - \tau A)\,d\gamma + \frac{1}{\tau} D d\delta = -Y dx,$$

together with

$$\begin{aligned} dz &= (p + \tau q)\,dx, & dr &= (\alpha + \tau\beta)\,dx, \\ dp &= (r + \tau s)\,dx, & ds &= (\beta + \tau\gamma)\,dx, \\ dq &= (s + \tau t)\,dx, & dt &= (\gamma + \tau\delta)\,dx. \end{aligned}$$

The first two of these equations can be modified. Let τ denote any root of the cubic equation, and let the other two roots be denoted by ρ and σ: then

$$B - \tau A = (\rho + \sigma)\,A,$$

$$\frac{1}{\tau} D = \rho\sigma A,$$

and so the first two equations become

$$d\alpha + (\rho + \sigma)\,d\beta + \rho\sigma\,d\gamma = -\frac{X}{A}\,dx,$$

$$d\beta + (\rho + \sigma)\,d\gamma + \rho\sigma\,d\delta = -\frac{Y}{A}\,dx.$$

Whichever form be adopted, we have a system of equations linear in the differential elements; and permutation of the roots of the cubic, when these are unequal, gives three such systems.

What is desired, in every case, is an integrable combination of the equations. The following process leads to the subsidiary equations in Darboux's method.

314. Let du be a linear combination of the equations of a system which is an exact differential: then multipliers $\lambda_1, \ldots, \lambda_9$ must exist such that the relation

$$\begin{aligned} du = \;& \lambda_1\,(dy - \tau dx) + \lambda_2\,\{dz - (p + \tau q)\,dx\} \\ & + \lambda_3\,\{dp - (r + \tau s)\,dx\} + \lambda_4\,\{dq - (s + \tau t)\,dx\} \\ & + \lambda_5\,\{dr - (\alpha + \tau\beta)\,dx\} + \lambda_6\,\{ds - (\beta + \tau\gamma)\,dx\} \\ & + \lambda_7\,\{dt - (\gamma + \tau\delta)\,dx\} + \lambda_8\left\{d\alpha + (\rho + \sigma)\,d\beta + \rho\sigma\,d\gamma + \frac{X}{A}\,dx\right\} \\ & + \lambda_9\left\{d\beta + (\rho + \sigma)\,d\gamma + \rho\sigma\,d\delta + \frac{Y}{A}\,dx\right\} \end{aligned}$$

holds identically, these multipliers being free from differential elements. We at once have

$$\lambda_1 = \frac{\partial u}{\partial y}, \quad \lambda_2 = \frac{\partial u}{\partial z},$$

$$\lambda_3 = \frac{\partial u}{\partial p}, \quad \lambda_4 = \frac{\partial u}{\partial q},$$

$$\lambda_5 = \frac{\partial u}{\partial r}, \quad \lambda_6 = \frac{\partial u}{\partial s}, \quad \lambda_7 = \frac{\partial u}{\partial t};$$

also

$$\frac{\partial u}{\partial \alpha} = \lambda_8,$$

$$\frac{\partial u}{\partial \beta} = \lambda_8 (\rho + \sigma) + \lambda_9,$$

$$\frac{\partial u}{\partial \gamma} = \lambda_8 \rho\sigma + \lambda_9 (\rho + \sigma),$$

$$\frac{\partial u}{\partial \delta} = \lambda_9 \rho\sigma;$$

and

$$\frac{\partial u}{\partial x} = \lambda_9 \frac{Y}{A} + \lambda_8 \frac{X}{A} - \lambda_5 (\alpha + \tau\beta) - \lambda_6 (\beta + \tau\gamma) - \lambda_7 (\gamma + \tau\delta)$$
$$- \lambda_3 (r + \tau s) - \lambda_4 (s + \tau t) - \lambda_2 (p + \tau q) - \lambda_1 \tau.$$

Hence

$$\left.\begin{aligned} \frac{du}{dx} + \tau \frac{du}{dy} - \frac{X}{A}\frac{\partial u}{\partial \alpha} - \frac{Y}{A\rho\sigma}\frac{\partial u}{\partial \delta} = 0 \\ \rho^3 \frac{\partial u}{\partial \alpha} - \rho^2 \frac{\partial u}{\partial \beta} + \rho \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} = 0 \\ \sigma^3 \frac{\partial u}{\partial \alpha} - \sigma^2 \frac{\partial u}{\partial \beta} + \sigma \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} = 0 \end{aligned}\right\}.$$

These are the partial differential equations in Darboux's method; they can be used to determine integrable combinations (if any) of the subsidiary system in the differential elements. One such combination is f: it is ineffective for our purpose, because $f = 0$ is the original equation: and so what is required is some other combination.

Moreover, by permuting the roots of the cubic

$$A\tau^3 - B\tau^2 + C\tau - D = 0,$$

we shall have three systems; and we proceed to obtain integrable combinations (if any) other than f belonging to each of the systems. The most favourable case occurs *when each of the systems gives an integrable combination involving* $\alpha, \beta, \gamma, \delta$: *if these be*

$$u = 0, \quad v = 0, \quad w = 0,$$

they can be combined with $f = 0$, *so as to express* $\alpha, \beta, \gamma, \delta$ *in terms of the other variables: and the construction of the primitive is then merely a matter of quadratures.* The proof is the same as for former similar propositions.

Ex. 1. Consider the equation

$$\alpha + a\beta - \gamma - a\delta = \frac{2}{x}(r + as),$$

where a is a constant unequal to 1 or -1. The critical cubic is

$$\theta^3 - a\theta^2 - \theta + a = 0;$$

the roots of this cubic are $-1, +1, a$: and therefore the arguments of the arbitrary functions in the primitive are

$$y + x, \quad y - x, \quad y - ax.$$

Taking $l, m, n = -1, 1, a$ in some order, we have the equations subsidiary to an intermediate integral of the second order (if it exists) in the form

$$[u_x] + l\,[u_y] - \frac{2}{x}(r + as)\,u_r = 0,$$

$$u_s - (m + n)\,u_r = 0,$$

$$u_t - mn = 0,$$

where

$$[u_x] = u_x + pu_z + ru_p + su_q,$$

$$[u_y] = u_y + qu_z + su_p + tu_q.$$

It is not difficult to prove that these equations do not possess a common integral; hence there is no intermediate integral.

We must therefore seek to construct some equation or equations in $\alpha, \beta, \gamma, \delta$, which are compatible with, but not resoluble into, the given equation. Taking the preceding method, we shall have three subsidiary systems, corresponding respectively to the three arrangements

(i) $\tau = \quad a; \quad \rho, \sigma = \quad 1, \ -1;$

(ii) $\tau = \quad 1; \quad \rho, \sigma = -1, \quad a;$

(iii) $\tau = -1; \quad \rho, \sigma = \quad 1, \quad a.$

The subsidiary system for the distribution (i), when made a complete Jacobian system, is found to contain ten equations; it thus possesses two integrals. One of these must be

$$\alpha + a\beta - \gamma - a\delta - \frac{2}{x}(r + as),$$

which vanishes owing to the original equation: the other is

$$y-ax,$$

which is not useful for our purpose. The distribution (i) therefore leads to no new equation.

The subsidiary system for the distribution (ii), when made a complete Jacobian system, is found to contain nine equations: it thus possesses two integrals, in addition to the vanishing integral

$$a+a\beta-\gamma-a\delta-\frac{2}{x}(r+as).$$

These two integrals are obtainable in the forms

$$\frac{1}{x}\{a-2\beta+\gamma+a(\beta-2\gamma+\delta)\},\quad y-x.$$

Accordingly, the distribution (ii) provides a new equation which can be taken in the form

$$a-2\beta+\gamma+a(\beta-2\gamma+\delta)=4xf'''(y-x),$$

where f is an arbitrary function.

Similarly, the distribution (iii) provides a new equation which can be taken in the form

$$a+2\beta+\gamma+a(\beta+2\gamma+\delta)=4xg'''(y+x),$$

where g is an arbitrary function.

We thus have two new equations compatible with, but not resoluble into, the original equation. When they are treated as simultaneous equations, they give

$$\begin{aligned} a+a\beta &= \frac{1}{x}(r+as)+xf'''+xg''', \\ \beta+a\gamma &= \qquad\qquad -xf'''+xg''', \\ \gamma+a\delta &= -\frac{1}{x}(r+as)+xf'''+xg'''. \end{aligned}$$

The construction of the primitive depends upon quadratures in the first instance. We have

$$\begin{aligned} dr+a\,ds &= (a+a\beta)\,dx+(\beta+a\gamma)\,dy \\ &= (r+as)\frac{dx}{x}+x(dg'''-df'''); \end{aligned}$$

and therefore

$$\frac{r+as}{x}=g''-f''.$$

Next,

$$\begin{aligned} ds+a\,dt &= (\beta+a\gamma)\,dx+(\gamma+a\delta)\,dy \\ &= x(dg''+df'')-\frac{r+as}{x}\,dy \\ &= x\,dg''-g''dy+x\,df''+f''dy; \end{aligned}$$

and therefore

$$s+at=xg''-g'+xf''+f'.$$

Consequently, we have

$$dp + a\,dq = (r + as)\,dx + (s + at)\,dy$$
$$= x\,dg'' - g'dy + x\,df'' + f'dy,$$

and therefore

$$p + aq = xg' - g + xf' + f.$$

This is an equation of the first order: integrating it by the usual process, we have

$$z = \psi(y - ax)$$
$$+ x\theta'(y + x) - \frac{2+a}{1+a}\theta(y + x),$$
$$+ x\phi'(y - x) + \frac{2-a}{1-a}\phi(y - x),$$

where

$$(1 + a)\,\theta' = g, \quad (a - 1)\,\phi' = f;$$

and, in this last form, θ, ϕ, ψ are the three arbitrary functions in the primitive.

Ex. 2. Obtain the primitives of the equations:—

(i) $\alpha + \beta - \gamma - \delta = \frac{2}{x}(r + s)$;

(ii) $\alpha - \beta - \gamma + \delta = \frac{2}{x}(r - s)$.

Equations of the nth order.

315. After the preceding investigations dealing with equations of the third order in two independent variables, it will be sufficient to state the results of a similar type which appertain to equations of order n also in two independent variables. We write

$$p_{l,m} = \frac{\partial^{l+m} z}{\partial x^l \partial y^m},$$

and we assume the equation to be

$$f(x, y, z, p_{10}, p_{01}, \ldots, p_{n,0}, p_{n-1,1}, \ldots, p_{1,n-1}, p_{0,n}) = 0.$$

The general primitive of such an equation, in whatever form it occurs, involves n arbitrary functions, each of a single argument: if α denote any of these arguments, supposed to be a quantity depending upon both x and y, then the derivative of y with regard to x on the supposition that α is constant, being given by the relation

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{\delta y}{\delta x} = 0,$$

satisfies the equation

$$\frac{\partial f}{\partial p_{n,0}}\left(\frac{\delta y}{\delta x}\right)^n - \frac{\partial f}{\partial p_{n-1,1}}\left(\frac{\delta y}{\delta x}\right)^{n-1} + \ldots + (-1)^{n-1}\frac{\partial f}{\partial p_{1,n-1}}\frac{\delta y}{\delta x} + (-1)^n \frac{\partial f}{\partial p_{0,n}} = 0,$$

while α satisfies the equivalent equation

$$\frac{\partial f}{\partial p_{n,0}}\left(\frac{\partial \alpha}{\partial x}\right)^n + \frac{\partial f}{\partial p_{n-1,1}}\left(\frac{\partial \alpha}{\partial x}\right)^{n-1}\frac{\partial \alpha}{\partial y} + \ldots + \frac{\partial f}{\partial p_{0,n}}\left(\frac{\partial \alpha}{\partial y}\right)^n = 0.$$

Ex. 1. Shew that, if the equation

$$f(x, y, z, p_{1,0}, p_{0,1}, \ldots, p_{n,0}, \ldots, p_{0,n}) = 0$$

possesses an intermediate integral in the form of an equation

$$\phi(u, v) = 0,$$

where ϕ is an arbitrary function, and u and v are definite functions of x, y, z and of all the derivatives of z up to order $n-1$ inclusive, then $f=0$ is of the form

$$\Sigma A_{l,l'}(p_{n-l,n-m}p_{n-l',n-m'} - p_{n-l+1,n-m-1}p_{n-l'-1,n-m'+1}) + \Sigma B_l p_{n-l,n-m} + C = 0,$$

where

$$l+m=n, \quad l'+m'=n,$$

and the coefficients $A_{l,l'}$, B_l, C do not involve any derivatives of order n.

When the particular equation of order n possesses an intermediate integral in the form of an equation

$$\phi(u, v) = 0$$

of order $n-1$, where ϕ is an arbitrary function, shew that the coefficients $A_{l,l'}$ identically satisfy relations

$$A_{l,l'}A_{k,k'} - A_{l,k}A_{l',k'} + A_{l,k'}A_{l',k} = 0,$$

for all values of l and l' different from k and k'. Obtain other identical relations which must be satisfied by the coefficients $A_{l,l'}$, when the given equation has an intermediate integral.

Ex. 2. Shew that, if the equation in the preceding example does not necessarily possess an intermediate integral in the form of an equation of lower order, while it does admit the existence of a compatible equation

$$g(x, y, z, p_{10}, p_{01}, \ldots, p_{n,0}, \ldots, p_{0,n}) = 0,$$

which is of order n and is not resoluble into $f=0$, then g satisfies one of the sets of equations

$$\frac{dg}{dx} + \tau_i\frac{dg}{dy} = \frac{X}{P_n}\frac{\partial g}{\partial p_{n,0}} + \tau_i\frac{Y}{P_0}\frac{\partial g}{\partial p_{0,n}},$$

$$\sum_{\lambda=0}^{n}(-1)^{n-\lambda}\tau_j^{\,n-\lambda}\frac{\partial g}{\partial p_{n-\lambda,\lambda}} = 0,$$

where j has all the values in 1, ..., n other than i, and the sets of equations are varied by giving the values 1, ..., n to i in succession: and where

$$P_n = \frac{\partial f}{\partial p_{n,0}}, \quad P_0 = \frac{\partial f}{\partial p_{0,n}}, \quad X = \frac{df}{dx}, \quad Y = \frac{df}{dy},$$

while also $\dfrac{d}{dx}$ and $\dfrac{d}{dy}$, as applied to f and to g, imply complete derivation with regard to x and to y respectively through all derivatives of z up to those of order $n-1$ inclusive: and where, lastly, τ_i is any one of the roots of the equation

$$\sum_{\lambda=0}^{n} (-1)^{n-\lambda} \tau_j^{\,n-\lambda} \frac{\partial f}{\partial p_{n-\lambda,\lambda}} = 0.$$

CHAPTER XXIII.

Equations of the Second Order in more than Two Independent Variables, having an Intermediate Integral.

As indicated in the opening sentences, the aim of this chapter is the extension, to equations involving a number of independent variables greater than two, of the methods of Monge, Boole, and Goursat, which are applicable to equations that involve only two independent variables and possess an intermediate integral. The results are given, and even the notation is specially devised, for the case when the number of independent variables is three: but many of the results can obviously be generalised to the case when the number of independent variables is n, though it has seemed unnecessary to state them explicitly.

Much of the material of the chapter is derived from a memoir by the author*; reference may also be made to memoirs by Tanner†, Sersawy‡, von Weber§, Vivanti||, and Coulon¶.

316. The preceding discussions have shewn that the theory of partial equations of the first order in one dependent variable and any number of independent variables can be regarded as complete. It is in a much slighter degree that the same claim can be made as regards equations of order higher than the first when there are

* *Camb. Phil. Trans.*, t. xvi (1898), pp. 191—218: other references are there given.

† *Proc. Lond. Math. Soc.*, t. vii (1876), pp. 43—60, 75—90, *ib.*, t. ix (1878), pp. 41—61, 76—90.

‡ *Wien. Denkschr.*, t. xlix (1885), pp. 1—104; many results are stated for n variables.

§ *Math. Ann.*, t. xlvii (1896), pp. 230—262.

|| *Math. Ann.*, t. xlviii (1897), pp. 474—513.

¶ "Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode des caractéristiques," (*Thèse*, Hermann, Paris, 1902), where other references also will be found.

two independent variables. Still, methods have been given which suffice for the integration of large classes of equations; most of them depend upon some subsidiary equations, in which all the magnitudes involved are temporarily held to be functions of one of the independent variables. Among the methods thus devised for equations of the second order, those associated with the names of Monge and of Boole presuppose (if they are to be effective) that the equation is of a special form and that some of its elements satisfy certain appropriate conditions: the most general methods are those due to Ampère and to Darboux respectively, being effective when the integral is expressible in finite terms without partial quadratures. There is also (§ 238) a method, intermediate in generality between these two classes of processes: it deals with equations of the second order (or of higher orders) in two independent variables which possess an intermediate integral, not of the particular type considered by Monge and by Boole.

When we proceed to equations of more extensive type, the natural generalisation is to be found in an increase in the number of independent variables: and such equations occur in various branches of mathematical physics, involving three independent variables (as in the theory of space-potential) or four independent variables (as in the theory of the conduction of heat in a solid body, and in the theory of propagation of vibrations in three dimensions). These equations are of a very special form, and very special analysis is needed for the full development of the particular solutions: but their occurrence challenges a consideration of equations of general form, to which the individually special methods are quite inapplicable.

Accordingly, in this chapter, we shall discuss general equations of the second order which involve three independent variables: the restriction of the number of independent variables to three is made for the sake of brevity: and, in spite of the notation adopted for special service in the equations considered, it is not difficult to see that many of the properties can be extended to equations in any number of independent variables.

The three independent variables are denoted by x, y, z: the dependent variable is denoted by v, and its first and second derivatives are denoted according to the scheme:—

$$\frac{\partial v}{\partial x}=l,\qquad \frac{\partial v}{\partial y}=m,\qquad \frac{\partial v}{\partial z}=n,$$

$$\frac{\partial^2 v}{\partial x^2}=a,\qquad \frac{\partial^2 v}{\partial y^2}=b,\qquad \frac{\partial^2 v}{\partial z^2}=c,$$

$$\frac{\partial^2 v}{\partial y\partial z}=f,\quad \frac{\partial^2 v}{\partial z\partial x}=g,\quad \frac{\partial^2 v}{\partial x\partial y}=h;$$

and then the general differential equation of the second order can be represented by

$$F(x, y, z, v, l, m, n, a, b, c, f, g, h)=0.$$

317. It will be convenient, in the first place, to consider (so as to set on one side as being definite) those equations which have an intermediate integral; and the discussion will be limited to those equations of the second order which are the sole consequence, in that order, of the intermediate integral. Let this integral, supposed to exist, have the form

$$u(x, y, z, v, l, m, n)=0;$$

and, in accordance with earlier notations, write

$$u_x=\frac{\partial u}{\partial x}+l\frac{\partial u}{\partial v},\qquad u_l=\frac{\partial u}{\partial l},$$

$$u_y=\frac{\partial u}{\partial y}+m\frac{\partial u}{\partial v},\qquad u_m=\frac{\partial u}{\partial m},$$

$$u_z=\frac{\partial u}{\partial z}+n\frac{\partial u}{\partial v},\qquad u_n=\frac{\partial u}{\partial n}.$$

Then we have

$$u_x+au_l+hu_m+gu_n=0,$$

$$u_y+hu_l+bu_m+fu_n=0,$$

$$u_z+gu_l+fu_m+cu_n=0.$$

Owing to the hypothesis of an intermediate integral, the equation $F=0$ is to be satisfied in virtue of these three equations, that is, when we resolve the three equations for any three of the second derivatives (say for a, b, c) and substitute the deduced values in $F=0$, the latter must become evanescent: and therefore the coefficients of the various combinations of f, g, h must vanish. This requirement provides a number of relations that are homogeneous in the quantities $u_x, u_y, u_z, u_l, u_m, u_n$, so that there cannot be more than five such relations; the actual number less than five will depend upon the equation itself. Each relation is a partial equation of the first order: and u is provided by the common integral (if any) of the system of simultaneous equations.

As regards the actual number of relations, it is easy to see that, for such equations as are amenable to the method, there are generally three relations at least. After substitution for a, b, c, has taken place, the modified equation is of the form

$$T + Pf + Qg + Rh + \ldots = 0;$$

in order that it may be evanescent, we must have

$$\ldots,\quad R = 0,\quad Q = 0,\quad P = 0,\quad T = 0.$$

If these were equivalent to only one relation, the equivalence would arise through the occurrence of a vanishing factor common to all the expressions $\ldots, R, Q, P, T$: let this factor be

$$u_x - \theta(u_y, u_z, u_l, u_m, u_n),$$

where θ is homogeneous of the first order in u_y, u_z, u_l, u_m, u_n. We then have three equations

$$\theta + au_l + hu_m + gu_n = 0,$$
$$u_y + hu_l + bu_m + fu_n = 0,$$
$$u_z + gu_l + fu_m + cu_n = 0,$$

involving five quantities homogeneously, and the elimination of these quantities is to lead to the equation of the second order: such elimination is not possible in general.

If the equations were equivalent to only two relations, they may be taken in the form

$$\phi(u_x, u_y, u_z, u_l, u_m, u_n) = 0,$$
$$\psi(u_x, u_y, u_z, u_l, u_m, u_n) = 0,$$

where ϕ and ψ are homogeneous: the original equation is to be derivable by the elimination of the six derivatives of u between these two equations and the other three

$$\left.\begin{aligned} u_x + au_l + hu_m + gu_n = 0 \\ u_y + hu_l + bu_m + fu_n = 0 \\ u_z + gu_l + fu_m + cu_n = 0 \end{aligned}\right\};$$

and this elimination is not possible in general.

The respective eliminations might be possible for very particular cases: we shall put them on one side as being too special. Accordingly, we conclude in general that, when an equation of the second order possesses an intermediate integral, the number of algebraically independent relations determining the quantity u in this method of proceeding is either three, or four, or five.

Note 1. In testing whether a given equation possesses an intermediate integral, it might be convenient to eliminate f, g, h, rather than a, b, c, by means of the equations derived from $u=0$: all that is necessary, in order to obtain the appropriate relations, is to use the three derived equations in order to eliminate three of the second derivatives of v from the given equation.

Note 2. As our object is rather to indicate the method which is of general effectiveness than to apply it in an exhaustive discussion of all inclusible cases, we shall make an initial limitation.

If there were four algebraically independent relations determining the quantity u, these could have a form

$$\alpha_i(u_x, u_y, u_z, u_l, u_m, u_n) = 0,$$

for $i = 1, 2, 3, 4$. Each of these is homogeneous of order unity in the six quantities $u_x, u_y, u_z, u_l, u_m, u_n$. Hence, when we proceed to the equation of the second order, we have to eliminate the two ratios $u_l : u_m : u_n$ between the four equations

$$\alpha_i(-au_l - hu_m - gu_n, -hu_l - bu_m - fu_n, \\ -gu_l - fu_m - cu_n, u_l, u_m, u_n) = 0,$$

for $i = 1, 2, 3, 4$: that is, two equations will appear in the eliminant, which accordingly will consist of two differential equations of the second order.

Similarly, if there were five algebraically independent relations determining the quantity u, there would result three simultaneous differential equations of the second order.

Both these cases will be left on one side: the method will be expounded sufficiently for the most important case, when the relations lead to only a single differential equation of the second order.

Case when the Three Relations are a Complete Jacobian System.

318. Dealing therefore with the case when the number of algebraically independent relations is three, we imagine them resolved with regard to three of the quantities u_x, u_y, u_z, u_l, u_m, u_n, choosing the first three by preference. The relations then have the form

$$L = u_x + \lambda(u_l, u_m, u_n) = 0,$$
$$M = u_y + \mu(u_l, u_m, u_n) = 0,$$
$$N = u_z + \nu(u_l, u_m, u_n) = 0,$$

where λ, μ, ν are homogeneous of the first order in u_l, u_m, u_n, and otherwise may involve the variables v, x, y, z, l, m, n. When we write

$$v, x, y, z, l, m, n = x_1, x_2, x_3, x_4, x_5, x_6, x_7,$$

and

$$\frac{\partial u}{\partial x_i} = p_i,$$

for $i = 1, \ldots, 7$, the equations for u are

$$\left.\begin{aligned} L &= x_5 p_1 + p_2 + \lambda(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \\ M &= x_6 p_1 + p_3 + \mu(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \\ N &= x_7 p_1 + p_4 + \nu(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \end{aligned}\right\}.$$

These equations must satisfy the necessary Poisson-Jacobi conditions for coexistence: that is, the relations

$$(L, M) = 0, \quad (M, N) = 0, \quad (N, L) = 0,$$

must be satisfied, either identically, or in virtue of the equations $L = 0$, $M = 0$, $N = 0$, or as new equations in the system. Now the relation

$$(L, M) = p_1\left(\frac{\partial \mu}{\partial p_5} - \frac{\partial \lambda}{\partial p_6}\right) + x_6\frac{\partial \lambda}{\partial x_1} - x_5\frac{\partial \mu}{\partial x_1} + \frac{\partial \lambda}{\partial x_3} - \frac{\partial \mu}{\partial x_2} + \sum_{i=5}^{7}\frac{\partial(\lambda, \mu)}{\partial(x_i, p_i)} = 0$$

manifestly cannot be satisfied in virtue of $L = 0$, $M = 0$, $N = 0$, for it does not involve either p_2, p_3, or p_4; hence it is either an identity or a new equation.

In order that the relation may be an identity, the term in p_1 must vanish by itself, for p_1 does not occur elsewhere; hence, as a first condition, we have

$$\frac{\partial \mu}{\partial p_5} = \frac{\partial \lambda}{\partial p_6}.$$

Similarly, if $(M, N) = 0$, and $(N, L) = 0$ are identities, we have

$$\frac{\partial \nu}{\partial p_6} = \frac{\partial \mu}{\partial p_7}, \quad \frac{\partial \lambda}{\partial p_7} = \frac{\partial \nu}{\partial p_5}.$$

Other relations, connected with the remaining terms in the three identities, will have to be satisfied: assuming them satisfied, we see that (on the hypothesis adopted) the three equations constitute a complete Jacobian system. The relations, connecting the derivatives of λ, μ, ν with respect to p_5, p_6, p_7, shew that a function Θ exists, such that

$$\lambda = \frac{\partial \Theta}{\partial p_5}, \quad \mu = \frac{\partial \Theta}{\partial p_6}, \quad \nu = \frac{\partial \Theta}{\partial p_7}.$$

This function Θ consists of two parts: the first is a quantity, homogeneous of the second order in p_5, p_6, p_7, and involving the variables $x_1, \ldots, x_7$: the second is a quantity independent of p_5, p_6, p_7. Let

$$p_5 = \rho p_7, \quad p_6 = \sigma p_7;$$

then Θ may be taken in the form

$$\Theta = p_7^2\, \theta\, (x_1, \ldots, x_7, \rho, \sigma) + X,$$

where X is the additive part of Θ independent of p_5, p_6, p_7, and where now there is no restriction upon the form of θ so far as regards homogeneity. Clearly

$$\lambda = p_7 \frac{\partial \theta}{\partial \rho},$$

$$\mu = p_7 \frac{\partial \theta}{\partial \sigma},$$

$$\nu = p_7 \left(2\theta - \rho \frac{\partial \theta}{\partial \rho} - \sigma \frac{\partial \theta}{\partial \sigma}\right),$$

the derivatives of X not appearing in λ, μ, ν.

Substituting these values of λ and μ in the remaining terms of $(L, M) = 0$ and removing a factor p_7, which is common to all the terms after the substitution has been effected, we find

$$\begin{aligned} & x_6 \frac{\partial^2 \theta}{\partial x_1 \partial \rho} - x_5 \frac{\partial^2 \theta}{\partial x_1 \partial \sigma} + \frac{\partial^2 \theta}{\partial x_3 \partial \rho} - \frac{\partial^2 \theta}{\partial x_2 \partial \sigma} \\ & + \frac{\partial^2 \theta}{\partial x_5 \partial \rho} \frac{\partial^2 \theta}{\partial \rho \partial \sigma} - \frac{\partial^2 \theta}{\partial x_5 \partial \sigma} \frac{\partial^2 \theta}{\partial \rho^2} \\ & + \frac{\partial^2 \theta}{\partial x_6 \partial \rho} \frac{\partial^2 \theta}{\partial \sigma^2} - \frac{\partial^2 \theta}{\partial x_6 \partial \sigma} \frac{\partial^2 \theta}{\partial \rho \partial \sigma} \\ & + \frac{\partial^2 \theta}{\partial x_7 \partial \rho} \left(\frac{\partial \theta}{\partial \sigma} - \rho \frac{\partial^2 \theta}{\partial \rho \partial \sigma} - \sigma \frac{\partial^2 \theta}{\partial \sigma^2}\right) - \frac{\partial^2 \theta}{\partial x_7 \partial \sigma} \left(\frac{\partial \theta}{\partial \rho} - \rho \frac{\partial^2 \theta}{\partial \rho^2} - \sigma \frac{\partial^2 \theta}{\partial \rho \partial \sigma}\right) = 0. \end{aligned}$$

When substitution of the values of μ and ν takes place in the remaining terms of $(M, N) = 0$, a similar equation of the second order arises: and another equation of the second order is provided by the remaining terms of $(N, L) = 0$.

These three equations must be satisfied by θ: when any integral common to all three is known, we have the means of constructing the corresponding equation of the second order possessing an intermediate integral. For

$$u_x + au_l + hu_m + gu_n = 0,$$

that is,

$$\lambda + ap_5 + hp_6 + gp_7 = 0,$$

and therefore

$$-\frac{\partial\theta}{\partial\rho} + a\rho + h\sigma + g = 0.$$

Similarly,

$$-\frac{\partial\theta}{\partial\sigma} + h\rho + b\sigma + f = 0,$$

and

$$-2\theta + \rho\frac{\partial\theta}{\partial\rho} + \sigma\frac{\partial\theta}{\partial\sigma} + g\rho + f\sigma + c = 0;$$

the latter, in connection with the other two, can be replaced by

$$-2\theta + a\rho^2 + 2h\rho\sigma + b\sigma^2 + 2g\rho + 2f\sigma + c = 0.$$

Eliminating ρ and σ between the equations or (what is the same thing) equating to zero the discriminant of the quantity on the left-hand side of the last equation, we have an equation of the second order possessing an intermediate integral as required.

As regards the intermediate integral u itself, it is determined by the three equations which form a complete Jacobian system. This system involves seven independent variables, and therefore it possesses four algebraically independent integrals; let these be u_1, u_2, u_3, u_4. We proceed from the equations

$$L = 0, \quad M = 0, \quad N = 0,$$

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

and resolve these for $p_1, \ldots, p_7$; substituting in

$$du = \sum_{i=1}^{7} p_i dx_i,$$

effecting the quadrature, and dividing out by one of the constants a_1, a_2, a_3, a_4, we have

$$\frac{1}{a_4} u = \phi(x_1, \ldots, x_7, a, b, c) + a',$$

where a, b, c are three arbitrary constants, and a' is an additive constant. As $u = 0$ is the intermediate integral, we can take the latter in the form

$$\phi(x_1, \ldots, x_7, a, b, c) + a' = 0,$$

that is,

$$\phi(v, x, y, z, l, m, n, a, b, c) + a' = 0,$$

which is an equation of the first order.

319. To obtain the primitive of the differential equation constructed with $\phi + a' = 0$ as an intermediate integral, we might proceed to construct the primitive of the equation of the first order: but the theory of § 284 can be generalised, so as to allow the primitive to be constructed merely by operations of elimination. When we substitute

$$u = \phi + a'$$

in the differential equations $L = 0$, $M = 0$, $N = 0$, these are satisfied identically: hence

$$x_5 \frac{\partial p_1}{\partial a} + \frac{\partial p_2}{\partial a} + \frac{\partial \lambda}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \lambda}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \lambda}{\partial p_7}\frac{\partial p_7}{\partial a} = 0,$$

$$x_6 \frac{\partial p_1}{\partial a} + \frac{\partial p_3}{\partial a} + \frac{\partial \mu}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \mu}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \mu}{\partial p_7}\frac{\partial p_7}{\partial a} = 0,$$

$$x_7 \frac{\partial p_1}{\partial a} + \frac{\partial p_4}{\partial a} + \frac{\partial \nu}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \nu}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \nu}{\partial p_7}\frac{\partial p_7}{\partial a} = 0.$$

Now let the Poisson-Jacobi combination for u and $\frac{\partial u}{\partial a}$ be constructed: it is

$$\begin{aligned}\left[u, \frac{\partial u}{\partial a}\right] = \; & (p_2 + x_5 p_1)\frac{\partial p_5}{\partial a} - p_5\left(\frac{\partial p_2}{\partial a} + x_5\frac{\partial p_1}{\partial a}\right) \\ & + (p_3 + x_6 p_1)\frac{\partial p_6}{\partial a} - p_6\left(\frac{\partial p_3}{\partial a} + x_6\frac{\partial p_1}{\partial a}\right) \\ & + (p_4 + x_7 p_1)\frac{\partial p_7}{\partial a} - p_7\left(\frac{\partial p_4}{\partial a} + x_7\frac{\partial p_1}{\partial a}\right)\end{aligned}$$

$$= -\lambda \frac{\partial p_5}{\partial a} + p_5 \left(\frac{\partial \lambda}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \lambda}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \lambda}{\partial p_7} \frac{\partial p_7}{\partial a} \right)$$

$$- \mu \frac{\partial p_6}{\partial a} + p_6 \left(\frac{\partial \mu}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \mu}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \mu}{\partial p_7} \frac{\partial p_7}{\partial a} \right)$$

$$- \nu \frac{\partial p_7}{\partial a} + p_7 \left(\frac{\partial \nu}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \nu}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \nu}{\partial p_7} \frac{\partial p_7}{\partial a} \right)$$

$$= \frac{\partial p_5}{\partial a} \left(-\lambda + p_5 \frac{\partial \lambda}{\partial p_5} + p_6 \frac{\partial \mu}{\partial p_5} + p_7 \frac{\partial \nu}{\partial p_5} \right)$$

$$+ \frac{\partial p_6}{\partial a} \left(-\mu + p_5 \frac{\partial \lambda}{\partial p_6} + p_6 \frac{\partial \mu}{\partial p_6} + p_7 \frac{\partial \nu}{\partial p_6} \right)$$

$$+ \frac{\partial p_7}{\partial a} \left(-\nu + p_5 \frac{\partial \lambda}{\partial p_7} + p_6 \frac{\partial \mu}{\partial p_7} + p_7 \frac{\partial \nu}{\partial p_7} \right)$$

$$= \frac{\partial p_5}{\partial a} \left(-\lambda + p_5 \frac{\partial \lambda}{\partial p_5} + p_6 \frac{\partial \lambda}{\partial p_6} + p_7 \frac{\partial \lambda}{\partial p_7} \right)$$

$$+ \frac{\partial p_6}{\partial a} \left(-\mu + p_5 \frac{\partial \mu}{\partial p_5} + p_6 \frac{\partial \mu}{\partial p_6} + p_7 \frac{\partial \mu}{\partial p_7} \right)$$

$$+ \frac{\partial p_7}{\partial a} \left(-\nu + p_5 \frac{\partial \nu}{\partial p_5} + p_6 \frac{\partial \nu}{\partial p_6} + p_7 \frac{\partial \nu}{\partial p_7} \right),$$

on account of the relations between the derivatives of λ, μ, ν with regard to p_5, p_6, p_7. As λ, μ, ν are homogeneous of the first order in p_5, p_6, p_7, the coefficients of $\frac{\partial p_5}{\partial a}$, $\frac{\partial p_6}{\partial a}$, $\frac{\partial p_7}{\partial a}$ vanish separately; hence

$$\left[u, \frac{\partial u}{\partial a} \right] = 0.$$

Similarly, we can prove that

$$\left[u, \frac{\partial u}{\partial b} \right] = 0, \quad \left[u, \frac{\partial u}{\partial c} \right] = 0.$$

Corresponding analysis leads to relations

$$\left[\frac{\partial u}{\partial a}, \frac{\partial u}{\partial b} \right] = 0, \quad \left[\frac{\partial u}{\partial b}, \frac{\partial u}{\partial c} \right] = 0, \quad \left[\frac{\partial u}{\partial c}, \frac{\partial u}{\partial a} \right] = 0.$$

It therefore follows that the conditions of coexistence of the equations

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta, \quad \frac{\partial u}{\partial c} = \gamma,$$

are satisfied. The elimination of l, m, n among these four equations leads to a relation between v, x, y, z, a', a, b, c, α, β, γ, which is consistent with all of them: it is a complete primitive.

The last proposition shews how to deduce the complete primitive of the equations under consideration from the intermediate integral when the latter is known. The general primitive can also be deduced from that integral; the result can be established by analysis precisely analogous to that in § 284 used for the establishment of the corresponding result in the case of two variables. The mode of deduction is as follows:

Let a, b, c be the three non-additive constants in u; and let the result of eliminating l, m, n between

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta, \quad \frac{\partial u}{\partial c} = \gamma,$$

be denoted by

$$g(x, y, z, v, a, b, c, \alpha, \beta, \gamma) = 0.$$

Then the general primitive is given by the elimination of a and b between the three equations

$$g\left\{x, y, z, v, a, b, \phi(a, b), -\frac{\partial \phi}{\partial a}\chi(a, b), -\frac{\partial \phi}{\partial b}\chi(a, b), \chi(a, b)\right\} = 0,$$

$$\frac{dg}{da} = 0, \quad \frac{dg}{db} = 0,$$

ϕ and χ being arbitrary functions.

In the present case, when the subsidiary equations for u possess four algebraically independent integrals, we can construct the complete primitive and the general primitive by direct operations effected upon the intermediate integral.

The three equations which θ must satisfy are complicated in form; and they involve a larger number of variables than the equations under our consideration, while at the same time they are of the second order. Consequently, we can hardly expect, at the present stage, to obtain the most general function θ which satisfies the equations: one or two examples will suffice to illustrate the theory.

Ex. 1. It is not difficult to verify that the three equations are satisfied, when θ is any function of ρ and σ involving no other variables: let such a value of θ be chosen. In that case, which merely is the generalisation of the case considered by Goursat (§ 284, Ex. 1), the equation of the second order is obtained by equating the discriminant of the equation

$$-2\theta + a\rho^2 + 2h\rho\sigma + b\sigma^2 + 2g\rho + 2f\sigma + c = 0$$

to zero, so that the equation will be of the form

$$F(a,\ b,\ c,\ f,\ g,\ h) = 0,$$

involving derivatives of the second order only.

The intermediate integral $u=0$ depends upon the three equations

$$F_1 = 0 = p_2 + x_5 p_1 + p_7 \frac{\partial\theta}{\partial\rho},$$

$$F_2 = 0 = p_3 + x_6 p_1 + p_7 \frac{\partial\theta}{\partial\sigma},$$

$$F_3 = 0 = p_4 + x_7 p_1 + p_7 \left(2\theta - \rho\frac{\partial\theta}{\partial\rho} - \sigma\frac{\partial\theta}{\partial\sigma}\right).$$

This is a complete Jacobian system, and therefore it possesses four algebraically independent integrals. The simpler these are taken, the better: for we know how to obtain the complete primitive and the general primitive from the intermediate integral, if only the last should contain the proper number of arbitrary constants. Now it is clear that

$$(F_1,\ p_i) = 0, \quad (F_2,\ p_i) = 0, \quad (F_3,\ p_i) = 0,$$

for $i = 1, 2, 3, 4$: and therefore we may take p_1, p_2, p_3, p_4 as the four common integrals.

To determine u, we proceed from the equations

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0,$$
$$p_1 = a_1, \quad p_2 = a_2, \quad p_3 = a_3, \quad p_4 = a_4;$$

we resolve them with regard to $p_1, \ldots, p_7$, and we substitute in

$$du = \sum_{i=1}^{7} p_i\, dx_i.$$

Hence

$$d(u - a_1x_1 - a_2x_2 - a_3x_3 - a_4x_4) = p_7(\rho\, dx_5 + \sigma\, dx_6 + dx_7),$$

where ρ, σ, and p_7 are given by

$$\left.\begin{aligned} p_7\frac{\partial\theta}{\partial\rho} &= -a_2 - a_1x_5 \\ p_7\frac{\partial\theta}{\partial\sigma} &= -a_3 - a_1x_6 \\ p_7\left(2\theta - \rho\frac{\partial\theta}{\partial\rho} - \sigma\frac{\partial\theta}{\partial\sigma}\right) &= -a_4 - a_1x_7 \end{aligned}\right\}.$$

The right-hand side of the equation giving du must be an exact differential: let it be denoted by dU, and (for the evaluation of U) let the independent variables be changed from x_5, x_6, x_7, to ρ, σ, x_7. Taking

$$\frac{\partial\theta}{\partial\rho} = \theta_1, \quad \frac{\partial\theta}{\partial\sigma} = \theta_2, \quad 2\theta - \rho\theta_1 - \sigma\theta_2 = \Delta,$$

we have

$$p_7 = -\frac{1}{\Delta}(a_4 + a_1 x_7),$$

$$\frac{\theta_1}{\Delta} = \frac{a_2 + a_1 x_5}{a_4 + a_1 x_7},$$

$$\frac{\theta_2}{\Delta} = \frac{a_3 + a_1 x_6}{a_4 + a_1 x_7},$$

so that

$$a_1\,dx_5 = a_1 \frac{\theta_1}{\Delta}\,dx_7 + (a_4 + a_1 x_7)\,d\frac{\theta_1}{\Delta},$$

$$a_1\,dx_6 = a_1 \frac{\theta_2}{\Delta}\,dx_7 + (a_4 + a_1 x_7)\,d\frac{\theta_2}{\Delta}:$$

consequently,

$$-dU = \frac{a_4 + a_1 x_7}{\Delta^2}\,2\theta\,dx_7 + \frac{(a_4 + a_1 x_7)^2}{a_1 \Delta}\left\{\rho d\frac{\theta_1}{\Delta} + \sigma d\frac{\theta_2}{\Delta}\right\}$$

$$= \frac{\theta}{\Delta^2}\,2\,(a_4 + a_1 x_7)\,dx_7 + \frac{(a_4 + a_1 x_7)^2}{a_1}\,d\left(\frac{\theta}{\Delta^2}\right),$$

on reduction; hence

$$-dU = d\left\{\frac{\theta\,(a_4 + a_1 x_7)^2}{a_1 \Delta^2}\right\},$$

and consequently

$$u = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 - \frac{(a_4 + a_1 x_7)^2}{a_1}\frac{\theta}{\Delta^2} + a_5.$$

Now $u=0$ is the intermediate integral: hence, dividing throughout by a_1, and changing the constants and the variables, we can take it in the form

$$v + ax + by + cz - (c+n)^2 \frac{\theta}{\Delta^2} = a',$$

where θ and Δ are known functions of ρ and σ, and where l and m are given by the equations

$$\frac{\theta_1}{a+l} = \frac{\theta_2}{b+m} = \frac{\Delta}{c+n}:$$

in other words, *the intermediate integral is given by the elimination of* ρ *and* σ *between the equations*

$$\left.\begin{aligned} v + ax + by + cz - (c+n)^2 \frac{\theta}{\Delta^2} = a' \\ \frac{\theta_1}{a+l} = \frac{\theta_2}{b+m} = \frac{\Delta}{c+n} \end{aligned}\right\}.$$

To obtain a complete primitive, we need the values of

$$\frac{\partial u}{\partial a},\quad \frac{\partial u}{\partial b},\quad \frac{\partial u}{\partial c}.$$

For this purpose, let a new quantity ξ be introduced by the definition

$$a + l = \theta_1 \xi;$$

then

$$b + m = \theta_2 \xi,$$

$$c + n = \Delta \xi;$$

and also

$$u=v+ax+by+cz-\theta\xi^2-a'.$$

We have

$$1=\theta_1\frac{\partial\xi}{\partial a}+\xi\left(\theta_{11}\frac{\partial\rho}{\partial a}+\theta_{12}\frac{\partial\sigma}{\partial a}\right),$$

$$0=\theta_2\frac{\partial\xi}{\partial a}+\xi\left(\theta_{12}\frac{\partial\rho}{\partial a}+\theta_{22}\frac{\partial\sigma}{\partial a}\right),$$

$$0=\Delta\,\frac{\partial\xi}{\partial a}+\xi\left(\Delta_1\,\frac{\partial\rho}{\partial a}+\Delta_2\,\frac{\partial\sigma}{\partial a}\right),$$

the suffixes 1 and 2 implying derivation with regard to ρ and σ respectively: multiplying the first of these by ρ, the second by σ, and adding to the third, we have

$$\rho=2\theta\frac{\partial\xi}{\partial a}+\xi\left(\theta_1\,\frac{\partial\rho}{\partial a}+\theta_2\frac{\partial\sigma}{\partial a}\right).$$

Now

$$\frac{\partial u}{\partial a}=x-2\theta\xi\frac{\partial\xi}{\partial a}-\xi^2\left(\theta_1\,\frac{\partial\rho}{\partial a}+\theta_2\frac{\partial\sigma}{\partial a}\right)$$

$$=x-\rho\xi;$$

and, similarly,

$$\frac{\partial u}{\partial b}=y-\sigma\xi,$$

$$\frac{\partial u}{\partial c}=z-\xi.$$

Hence, by the general theory, we eliminate ρ, σ, ξ among the equations

$$\begin{aligned} v+ax+by+cz-\theta\xi^2&=a', \\ x-\rho\xi&=\alpha, \\ y-\sigma\xi&=\beta, \\ z-\xi&=\gamma, \end{aligned}$$

the constant a' being unessential. The complete* primitive is of the form

$$g(x,\ y,\ z,\ v-a',\ a,\ b,\ c,\ \alpha,\ \beta,\ \gamma)=0.$$

The *general primitive* is obtained by the elimination of all the constants between the equations

$$\left.\begin{aligned} &g=0, \quad \frac{dg}{da}=0, \quad \frac{dg}{db}=0 \\ &c=\phi(a,\ b), \quad \gamma=\chi(a,\ b) \\ &\alpha=-\chi(a,\ b)\frac{\partial\phi}{\partial a}, \quad \beta=-\chi(a,\ b)\frac{\partial\phi}{\partial b} \end{aligned}\right\}.$$

* It is the most complete primitive thus obtainable. But it is not the complete primitive in the customary sense; for it contains only seven, not nine, arbitrary constants.

Ex. 2. Let it be required to find those equations of the specified type, which are provided by taking

$$2\theta = A\rho^2 + 2H\rho\sigma + B\sigma^2 + 2G\rho + 2F\sigma + C,$$

where A, H, B, G, F, C involve v, x, y, z, l, m, n, but no other variable quantities.

For such values of these magnitudes as satisfy the conditions, the differential equation of the second order is given by equating to zero the discriminant of the equation

$$(a-A)\rho^2 + 2(h-H)\rho\sigma + (b-B)\sigma^2 + 2(g-G)\rho + 2(f-F)\sigma + c - C = 0:$$

consequently, it is

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix} = 0.$$

The equations for the intermediate integral are

$$p_2 + x_5 p_1 + p_7(A\rho + H\sigma + G) = 0,$$

that is,

$$p_2 + x_5 p_1 + Ap_5 + Hp_6 + Gp_7 = 0,$$

with

$$p_3 + x_6 p_1 + Hp_5 + Bp_6 + Fp_7 = 0,$$
$$p_4 + x_7 p_1 + Gp_5 + Fp_6 + Cp_7 = 0.$$

Let these be denoted by

$$\Delta(u) = \left(\frac{\partial}{\partial x} + l\frac{\partial}{\partial v} + A\frac{\partial}{\partial l} + H\frac{\partial}{\partial m} + G\frac{\partial}{\partial n}\right)u = 0,$$

$$\Delta'(u) = \left(\frac{\partial}{\partial y} + m\frac{\partial}{\partial v} + H\frac{\partial}{\partial l} + B\frac{\partial}{\partial m} + F\frac{\partial}{\partial n}\right)u = 0,$$

$$\Delta''(u) = \left(\frac{\partial}{\partial z} + n\frac{\partial}{\partial v} + G\frac{\partial}{\partial l} + F\frac{\partial}{\partial m} + C\frac{\partial}{\partial n}\right)u = 0.$$

As these constitute a complete system, the Poisson-Jacobi relations

$$(\Delta, \Delta') = 0, \quad (\Delta', \Delta'') = 0, \quad (\Delta'', \Delta) = 0,$$

must be satisfied without the introduction of any new equations for u. The necessary and sufficient conditions* are

$$\left.\begin{aligned} \Delta' A &= \Delta H \\ \Delta'' A &= \Delta G \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta'' B &= \Delta' F \\ \Delta B &= \Delta' H \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta C &= \Delta'' G \\ \Delta' C &= \Delta'' F \end{aligned}\right\},$$

$$\Delta F = \Delta' G = \Delta'' H:$$

we shall assume that they are satisfied.

In these circumstances, the system possesses four algebraically independent integrals: let them be u_1, u_2, u_3, u_4. Then the equations

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

* They agree with the conditions, otherwise obtained, in a paper by the author, *Camb. Phil. Trans.*, vol. XVI (1898), p. 198.

can be treated as coexistent, when we revert to the older variables and take

$$v,\ x,\ y,\ z = x_1,\ x_2,\ x_3,\ x_4,$$
$$l,\ m,\ n = x_5,\ x_6,\ x_7.$$

For

$$\begin{aligned}[u_1,\ u_2] = &\left(\frac{\partial u_1}{\partial x} + l\,\frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial l} - \left(\frac{\partial u_2}{\partial x} + l\,\frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial l} \\ &+ \left(\frac{\partial u_1}{\partial y} + m\,\frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial m} - \left(\frac{\partial u_2}{\partial y} + m\,\frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial m} \\ &+ \left(\frac{\partial u_1}{\partial z} + n\,\frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial n} - \left(\frac{\partial u_2}{\partial z} + n\,\frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial n} \\ = 0,&\end{aligned}$$

on substituting from the equations which are satisfied by u_1 and u_2; and, similarly,

$$[u_i,\ u_j] = 0,$$

for all the combinations $i, j = 1, 2, 3, 4$. As we have four equations

$$u_1 = a_1,\quad u_2 = a_2,\quad u_3 = a_3,\quad u_4 = a_4,$$

which are coexistent with one another, and as the quantities u_1, u_2, u_3, u_4 are functionally independent of one another, it follows that l, m, n can be eliminated among the four equations: and the eliminant is of the form

$$g(v,\ x,\ y,\ z,\ a_1,\ a_2,\ a_3,\ a_4) = 0,$$

which is a primitive involving four arbitrary constants.

The primitive thus obtained can be modified so as to give the general primitive. When we take

$$a_3 = \phi(a_1,\ a_2),\quad a_4 = \psi(a_1,\ a_2),$$

where ϕ and ψ are arbitrary functions, then *the equation, which results from the elimination of a_1 and a_2 between the equations*

$$\left.\begin{aligned} g(v,\ x,\ y,\ z,\ a_1,\ a_2,\ \phi,\ \psi) &= 0 \\ \frac{\partial g}{\partial a_1} + \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial a_1} + \frac{\partial g}{\partial \psi}\frac{\partial \psi}{\partial a_1} &= 0 \\ \frac{\partial g}{\partial a_2} + \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial a_2} + \frac{\partial g}{\partial \psi}\frac{\partial \psi}{\partial a_2} &= 0 \end{aligned}\right\},$$

furnishes the general primitive of the equation of the second order. This statement can be established easily by verifying that the value of v thus given does actually satisfy the equation. Suppose the integral equations resolved, so as to express v explicitly: the system then becomes

$$\left.\begin{aligned} v &= k(x,\ y,\ z,\ a_1,\ a_2,\ \phi,\ \psi) \\ 0 &= \frac{\partial k}{\partial a_1} + \frac{\partial k}{\partial \phi}\frac{\partial \phi}{\partial a_1} + \frac{\partial k}{\partial \psi}\frac{\partial \psi}{\partial a_1} = \frac{dk}{da_1} \\ 0 &= \frac{\partial k}{\partial a_2} + \frac{\partial k}{\partial \phi}\frac{\partial \phi}{\partial a_2} + \frac{\partial k}{\partial \psi}\frac{\partial \psi}{\partial a_2} = \frac{dk}{da_2} \end{aligned}\right\},$$

and the quantities a_1, a_2, as assigned by the last two equations, are functions of x, y, z.

The first derivatives of v, as determined by the system, are

$$l=\frac{\partial k}{\partial x}+\frac{dk}{da_1}\frac{\partial a_1}{\partial x}+\frac{dk}{da_2}\frac{\partial a_2}{\partial x}=\frac{\partial k}{\partial x},$$

and, similarly,

$$m=\frac{\partial k}{\partial y},\quad n=\frac{\partial k}{\partial z};$$

so that, in form, the first derivatives are the same, whether a_1 and a_2 be parametric or variable.

As regards the second derivatives of v, we have

$$\begin{aligned}a&=\frac{\partial l}{\partial x}\\ &=\frac{\partial^2 k}{\partial x^2}+\frac{\partial^2 k}{\partial a_1\partial x}\frac{\partial a_1}{\partial x}+\frac{\partial^2 k}{\partial a_2\partial x}\frac{\partial a_2}{\partial x};\end{aligned}$$

but from $\dfrac{dk}{da_1}=0$, $\dfrac{dk}{da_2}=0$, it follows that

$$0=\frac{\partial^2 k}{\partial a_1\partial x}+\frac{d^2 k}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2 k}{da_1 da_2}\frac{\partial a_2}{\partial x},$$

$$0=\frac{\partial^2 k}{\partial a_2\partial x}+\frac{d^2 k}{da_1 da_2}\frac{\partial a_1}{\partial x}+\frac{d^2 k}{da_2^2}\frac{\partial a_2}{\partial x};$$

and therefore

$$a=\frac{\partial^2 k}{\partial x^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)^2,$$

where

$$\alpha,\ \beta,\ \gamma=\frac{d^2 k}{da_1^2},\ \frac{d^2 k}{da_1 da_2},\ \frac{d^2 k}{da_2^2},$$

respectively. Similarly,

$$h=\frac{\partial^2 k}{\partial x\partial y}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right),$$

$$g=\frac{\partial^2 k}{\partial x\partial z}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right),$$

$$b=\frac{\partial^2 k}{\partial y^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right)^2,$$

$$f=\frac{\partial^2 k}{\partial y\partial z}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right),$$

$$c=\frac{\partial^2 k}{\partial z^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right)^2.$$

Now, in connection with the equations

$$u_i=a_i,$$

for $i=1, 2, 3, 4$, we have

$$\frac{\partial u_i}{\partial x}+l\frac{\partial u_i}{\partial v}+a\frac{\partial u_i}{\partial l}+h\frac{\partial u_i}{\partial m}+g\frac{\partial u_i}{\partial n}=0;$$

and (regard being paid to the two notations), we have

$$\frac{\partial u_i}{\partial x}+l\frac{\partial u_i}{\partial v}+A\frac{\partial u_i}{\partial l}+H\frac{\partial u_i}{\partial m}+G\frac{\partial u_i}{\partial n}=0;$$

hence, when the quantities a_1, a_2, a_3, a_4 are constant,

$$(a-A)\frac{\partial u_i}{\partial l}+(h-H)\frac{\partial u_i}{\partial m}+(g-G)\frac{\partial u_i}{\partial n}=0,$$

for $i=1, 2, 3, 4$. Not all the quantities

$$J\left(\frac{u_1,\ u_2,\ u_3,\ u_4}{l,\ m,\ n}\right)$$

vanish: hence, when a_1, a_2, a_3, a_4 are constant,

$$A=a=\frac{\partial^2 k}{\partial x^2},$$

$$H=h=\frac{\partial^2 k}{\partial x\partial y},$$

$$G=g=\frac{\partial^2 k}{\partial x\partial z}.$$

Similarly,

$$B=b=\frac{\partial^2 k}{\partial y^2},$$

$$F=f=\frac{\partial^2 k}{\partial y\partial z},$$

$$C=c=\frac{\partial^2 k}{\partial z^2},$$

when a_1, a_2, a_3, a_4 are constant quantities. Thus the equations for a, b, c, f, g, h, when a_1, a_2, a_3, a_4 are made variable, acquire the forms

$$a-A=-\left(\alpha,\ \beta,\ \gamma\between\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)^2,$$

$$f-F=-\left(\alpha,\ \beta,\ \gamma\between\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\between\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right);$$

and so for the others. If, therefore, the differential equation

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix}=0$$

is to be satisfied when a_1, a_2, a_3, a_4 are variable, we must have the determinant

$$\begin{vmatrix} (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2)^2, & (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\eta_1,\ \eta_2), & (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\zeta_1,\ \zeta_2) \\ (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\eta_1,\ \eta_2), & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2)^2, & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2\between\zeta_1,\ \zeta_2) \\ (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\zeta_1,\ \zeta_2), & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2\between\zeta_1,\ \zeta_2), & (\alpha,\ \beta,\ \gamma\between\zeta_1,\ \zeta_2)^2 \end{vmatrix},$$

equal to zero, where ξ_1, η_1, ζ_1 are the derivatives of a_1, and ξ_2, η_2, ζ_2 are those of a_2; as the determinant is the product of

$$\begin{vmatrix} \xi_1, & \xi_2, & 0 \\ \eta_1, & \eta_2, & 0 \\ \zeta_1, & \zeta_2, & 0 \end{vmatrix} \text{ and } \begin{vmatrix} \alpha\xi_1+\beta\xi_2, & \beta\xi_1+\gamma\xi_2, & 0 \\ \alpha\eta_1+\beta\eta_2, & \beta\eta_1+\gamma\eta_2, & 0 \\ \alpha\zeta_1+\beta\zeta_2, & \beta\zeta_1+\gamma\zeta_2, & 0 \end{vmatrix},$$

it vanishes identically.

Hence the proposition is valid.

Note. Having obtained the general primitive, we need not now concern ourselves as to intermediate integrals: but it must be noticed that the construction of the general primitive depends upon the possibility of eliminating l, m, n, between the equations

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

with the result of giving a primitive that involves the four arbitrary constants.

If this elimination is not possible, or if the eliminant does not possess the assumed form, then we must proceed otherwise. It is easy to see that not more than two of the quantities u_1, u_2, u_3, u_4 can be free from the variables l, m, n; for if three of them, say u_1, u_2, u_3, are functions of x, y, z, v only, we should have

$$l\frac{\partial u_1}{\partial v} + \frac{\partial u_1}{\partial x} + \lambda = 0,$$

$$l\frac{\partial u_2}{\partial v} + \frac{\partial u_2}{\partial x} + \lambda = 0,$$

$$l\frac{\partial u_3}{\partial v} + \frac{\partial u_3}{\partial x} + \lambda = 0,$$

where $\lambda = \lambda(x_1, \ldots, x_7, 0, 0, 0)$: and corresponding equations hold for derivatives with regard to y and to z. These equations imply that

$$u_1 - u_2 = \omega(u_2 - u_3),$$

where ω is a functional form: the integrals u_1, u_2, u_3 are not then independent. Accordingly, at least two of the four quantities, say u_3 and u_4, involve some of the variables l, m, n; hence

$$u_3 = \phi(u_1, u_2),$$
$$u_4 = \psi(u_1, u_2),$$

are general intermediate integrals, ϕ and ψ being arbitrary functions. On account of the relations

$$[u_i, u_j] = 0, \qquad (i, j = 1, 2, 3, 4),$$

these intermediate integrals coexist: and the primitive can be obtained by integrating either of them, or by integrating both of them as a simultaneous system: the general integral of either, regarded as an equation of the first order, leads to a primitive.

Moreover, it will be found that a knowledge of the form of u_4 is of substantial assistance in the integration of the equations subsidiary to the integration of

$$u_3 = \phi(u_1, u_2);$$

and, similarly, with the knowledge of u_3 in relation to the integration of

$$u_4 = \psi(u_1, u_2):$$

these results are easily established by considering the characteristic of each of these equations of the first order.

Ex. 3. Prove that the equation

$$x^2z\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} - x^2n(ab-h^2) - 2z(xl-my)(bc-f^2) + 4xzm(ch-fg)$$
$$= 4xmnh - 2n(xl-my)b + 4m^2zc - 4m^2n,$$

has an integral

$$v = a_1x^3 + a_2x^2y + a_3z^2 + a_4\,;$$

and deduce the general primitive. (Vivanti.)

Ex. 4. Verify that the conditions in the preceding discussion for the equation

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix} = 0$$

are satisfied, (i), when

$$A=\lambda,\quad B=\mu,\quad F=a,\quad G=\beta,\quad H=\gamma,\quad C=\begin{vmatrix} l, & m, & n \\ \lambda, & \gamma, & \beta \\ \gamma, & \mu, & a \end{vmatrix},$$

where λ, μ, a, β, γ are constants: (ii) also, when

$$A=\frac{1+l^2}{v},\quad B=\frac{1+m^2}{v},\quad C=\frac{1+n^2}{v},$$
$$F=\frac{mn}{v},\quad G=\frac{nl}{v},\quad H=\frac{lm}{v}\,;$$

(iii) also, when*

$$A=\frac{l}{x},\quad B=\frac{m}{y},\quad C=\frac{n}{z},\quad F=0,\quad G=0,\quad H=0.$$

Obtain the general primitive in the respective cases.

Cases when the Three Relations are not a Complete System.

320. We still have to consider the case in which the relations

$$(L, M)=0,\quad (M, N)=0,\quad (N, L)=0,$$

are satisfied, though not identically; then they are equations additional to $L=0$, $M=0$, $N=0$. They are of the form

$$\nu_1p_1+\nu_2=0,$$
$$\lambda_1p_1+\lambda_2=0,$$
$$\mu_1p_1+\mu_2=0,$$

* This third example is due to Tanner, *Proc. L. M. S.*, t. VII (1876), p. 89.

where

$$\nu_1 = \frac{\partial \mu}{\partial p_5} - \frac{\partial \lambda}{\partial p_6}, \quad \lambda_1 = \frac{\partial \nu}{\partial p_6} - \frac{\partial \mu}{\partial p_7}, \quad \mu_1 = \frac{\partial \lambda}{\partial p_7} - \frac{\partial \nu}{\partial p_5}.$$

The most important case arises when they are equivalent to only a single additional equation: and this can occur in three kinds of ways, viz.

(i) two of the conditions may be satisfied identically, and the remaining condition then gives the new equation:

(ii) one of the conditions may be satisfied identically, and the other two give new equations which are equivalent to one another:

(iii) no one of the conditions may be satisfied identically, but the three are equivalent to one another.

Let the new equation be $P = 0$. Then the relations

$$(P, L) = 0, \quad (P, M) = 0, \quad (P, N) = 0,$$

must also be satisfied, either identically or in virtue of the equations of the system

$$L = 0, \quad M = 0, \quad N = 0, \quad P = 0.$$

We shall assume that this requirement is actually met without the association of other new equations. The system is a complete Jacobian system; as it involves seven variables, it possesses three algebraically independent integrals, a set of which may be denoted by u_1, u_2, u_3.

We then resolve the equations

$$u_1 = a_1', \quad u_2 = a_2', \quad u_3 = a_3',$$

together with the four equations of the complete system, so as to give the values of $p_1, \ldots, p_7$, the quantities a_1', a_2', a_3' being constants. The values are substituted in

$$du = \sum_{i=1}^{7} p_i dx_i;$$

and quadrature is effected, giving an equation of the form

$$u = \omega(v, x, y, z, l, m, n, a_1', a_2', a_3') + a_4',$$

where a_4' is an arbitrary constant. Now $u = 0$ is the intermediate integral: adopting this value of u, and dividing out by one of the arbitrary constants, say by a_3', we have the intermediate integral in a form

$$\omega(v, x, y, z, l, m, n, a_1, a_2) + a_3 = 0.$$

This may be called a *complete intermediate integral,* as it contains the greatest number of arbitrary independent constants which generally can be eliminated from the equations

$$\omega + a_3 = 0, \quad \frac{d\omega}{dx} = 0, \quad \frac{d\omega}{dy} = 0, \quad \frac{d\omega}{dz} = 0:$$

the eliminant is the differential equation required.

A general intermediate integral, obtained in the usual manner from the complete intermediate integral, is given by the elimination of a_1 and a_2 between

$$\left.\begin{aligned} \omega(v, x, y, z, l, m, n, a_1, a_2) + \phi(a_1, a_2) = 0 \\ \frac{\partial\omega}{\partial a_1} + \frac{\partial\phi}{\partial a_1} = 0 \\ \frac{\partial\omega}{\partial a_2} + \frac{\partial\phi}{\partial a_2} = 0 \end{aligned}\right\},$$

where ϕ is an arbitrary constant.

In order to proceed to the primitive, we integrate either of the intermediate integrals as an equation of the first order: its general integral will be a general primitive of the differential equation of the second order.

This primitive has been obtained from the set of equations $L = 0$, $M = 0$, $N = 0$, which may be only one of several sets of equations deduced from the original conditions. When there are other sets, each of them must be discussed: and each may lead to a primitive. The various primitives are so many branches of the final primitive.

Generalisation of Monge's Equation.

321. One of the simplest classes of equations is constituted by the generalisation of equations which belong to the type considered by Monge. Let θ, ϕ, ψ denote three algebraically independent functions of v, x, y, z, l, m, n: then an equation of the first order is given by

$$F(\theta, \phi, \psi) = 0,$$

where F is an arbitrary function. Also, let

$$\frac{d}{dx} = \frac{\partial}{\partial x} + l\frac{\partial}{\partial v} + a\frac{\partial}{\partial l} + h\frac{\partial}{\partial m} + g\frac{\partial}{\partial n},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + m\frac{\partial}{\partial v} + h\frac{\partial}{\partial l} + b\frac{\partial}{\partial m} + f\frac{\partial}{\partial n},$$

$$\frac{d}{dz} = \frac{\partial}{\partial z} + n\frac{\partial}{\partial v} + g\frac{\partial}{\partial l} + f\frac{\partial}{\partial m} + c\frac{\partial}{\partial n}:$$

then, in order to construct an equation of the second order which has $F = 0$ for an intermediate integral (and which therefore will be of the class under consideration), it is sufficient to eliminate the derivatives of F between the equations

$$\frac{\partial F}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dx} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dx} = 0,$$

$$\frac{\partial F}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dy} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dy} = 0,$$

$$\frac{\partial F}{\partial \theta}\frac{d\theta}{dz} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dz} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dz} = 0.$$

Obviously the equation is

$$\begin{vmatrix} \frac{d\theta}{dx}, & \frac{d\phi}{dx}, & \frac{d\psi}{dx} \\ \frac{d\theta}{dy}, & \frac{d\phi}{dy}, & \frac{d\psi}{dy} \\ \frac{d\theta}{dz}, & \frac{d\phi}{dz}, & \frac{d\psi}{dz} \end{vmatrix} = 0,$$

which, when expanded in full, is

$$\begin{aligned} D\Delta + PA + QB + RC + 2SF + 2TG + 2UH& \\ + Ia + Jb + Kc + 2Lf + 2Mg + 2Nh &= W, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2,$$

$$A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2,$$

$$F = gh - af, \quad G = hf - bg, \quad H = fg - ch,$$

and the coefficients are various combinations of the derivatives of θ, ϕ, ψ. As there are fourteen coefficients in the equation, all dependent upon θ, ϕ, ψ, it is manifest that a considerable number of relations among them must be satisfied.

In particular,

$$D = J\left(\frac{\theta, \phi, \psi}{l, m, n}\right), \quad W = J\left(\frac{\theta, \phi, \psi}{x, y, z}\right).$$

Now the form of equation thus obtained is the only possible form when an intermediate integral of the assumed functional form exists: but an equation of that form does not necessarily possess such an intermediate integral, for (as we have indicated) certain conditions must be satisfied. The conditions may be obtained as follows.

322. Assuming that the equation of the second order has an intermediate integral

$$u(v, x, y, z, l, m, n) = 0,$$

and having regard to its relation to the equation, we know that when the equations

$$u_x + au_l + hu_m + gu_n = 0,$$
$$u_y + hu_l + bu_m + fu_n = 0,$$
$$u_z + gu_l + fu_m + cu_n = 0,$$

are used to eliminate three derivatives of the second order (say a, b, c) from

$$D\Delta + PA + QB + RC + 2SF + 2TG + 2UH$$
$$+ Ia + Jb + Kc + 2Lf + 2Mg + 2Nh = W,$$

the resulting equation must be evanescent. Now

$$a = -\frac{u_m u_n}{u_l}\left(\frac{u_x}{u_m u_n} + \frac{g}{u_m} + \frac{h}{u_n}\right):$$

writing

$$X = \frac{u_x}{u_m u_n}, \quad Y = \frac{u_y}{u_n u_l}, \quad Z = \frac{u_z}{u_l u_m},$$

$$\phi = \frac{f}{u_l}, \quad \gamma = \frac{g}{u_m}, \quad \eta = \frac{h}{u_n},$$

we have

$$a = -\frac{u_m u_n}{u_l}(X \quad + \gamma + \eta);$$

similarly,

$$b = -\frac{u_n u_l}{u_m}(Y + \phi \qquad + \eta),$$

$$c = -\frac{u_l u_m}{u_n}(Z + \phi + \gamma \qquad).$$

Again,

$$\begin{aligned} A &= bc - f^2 \\ &= u_l^2 \{YZ + \phi(Y+Z) + \gamma Y + \eta Z + \phi\gamma + \gamma\eta + \eta\phi\}, \end{aligned}$$

with similar expressions for B and C: also,

$$\begin{aligned} F &= gh - af \\ &= u_m u_n (X\phi + \phi\gamma + \gamma\eta + \eta\phi), \end{aligned}$$

with similar expressions for G and H. Lastly,

$$\begin{aligned} \Delta = -u_l u_m u_n \{XYZ &+ (\phi\gamma + \gamma\eta + \eta\phi)(X + Y + Z) \\ &+ \phi(XY + XZ) + \gamma(YX + YZ) + \eta(ZX + ZY)\}. \end{aligned}$$

Substituting these values in the given equation, and equating to zero the coefficients of the various combinations of ϕ, γ, η, so that the modified equation is evanescent, we have various relations.

The coefficient of $\phi\gamma + \gamma\eta + \eta\phi$ yields the relation

$$\begin{aligned} -Du_l u_m u_n (X + Y + Z) + Pu_l^2 + Qu_m^2 + Ru_n^2 \\ + 2Su_m u_n + 2Tu_n u_l + 2Uu_l u_m = 0. \end{aligned}$$

The coefficients of ϕ, of γ, and of η, yield the relations

$$\begin{aligned} -Du_l u_m u_n (XY + XZ) - J\frac{u_n u_l}{u_m} - K\frac{u_l u_m}{u_n} + 2Lu_l \\ + P(Y + Z)u_l^2 + QXu_m^2 + RXu_n^2 + 2SXu_m u_n = 0, \end{aligned}$$

$$\begin{aligned} -Du_l u_m u_n (YX + YZ) - I\frac{u_m u_n}{u_l} - K\frac{u_l u_m}{u_n} + 2Mu_m \\ + PYu_l^2 + Q(X + Z)u_m^2 + RYu_n^2 + 2TYu_n u_l = 0, \end{aligned}$$

$$\begin{aligned} -Du_l u_m u_n (ZX + ZY) - I\frac{u_m u_n}{u_l} - J\frac{u_n u_l}{u_m} + 2Nu_n \\ + PZu_l^2 + QZu_m^2 + R(X + Y)u_n^2 + 2UZu_l u_m = 0, \end{aligned}$$

respectively; and the terms, independent of ϕ, γ, η, yield the relation

$$\begin{aligned} -Du_l u_m u_n XYZ - IX\frac{u_m u_n}{u_l} - JY\frac{u_n u_l}{u_m} - KZ\frac{u_l u_m}{u_n} \\ + PYZu_l^2 + QZXu_m^2 + RXYu_n^2 = W. \end{aligned}$$

Apparently, there are five relations involving the quantities u_x, u_y, u_z, u_l, u_m, u_n: they can, under conditions, be reduced to a set (or to sets) of three equations.

The forms of these relations (and the remembrance of the subsidiary equations in Boole's method in the case of two independent variables) suggest the homogeneous linear forms

$$\left.\begin{aligned} Du_x &= Pu_l + \gamma' u_m + \beta u_n \\ Du_y &= \gamma u_l + Qu_m + \alpha' u_n \\ Du_z &= \beta' u_l + \alpha u_m + Ru_n \end{aligned}\right\}:$$

in these at this stage, we shall regard α, α', β, β', γ, γ' as six quantities to be determined and as independent of u_l, u_m, u_n.

The first of the preceding five relations is then satisfied identically, if

$$\left.\begin{aligned} \gamma + \gamma' &= 2U \\ \beta + \beta' &= 2T \\ \alpha + \alpha' &= 2S \end{aligned}\right\},$$

which accordingly will be regarded as three equations for the determination of the six quantities.

The second of the five relations is satisfied identically, if

$$\beta\beta' = PR - DJ,$$

$$\gamma\gamma' = PQ - DK,$$

$$\beta\gamma + \beta'\gamma' = 2(LD + SP);$$

the third is satisfied identically if, further,

$$\alpha\alpha' = QR - DI,$$

$$\alpha\gamma + \alpha'\gamma' = 2(MD + TQ);$$

the fourth is satisfied identically if, further,

$$\alpha\beta + \alpha'\beta' = 2(ND + UR);$$

and the fifth is satisfied identically if, further,

$$\alpha\beta\gamma + \alpha'\beta'\gamma' = 2PQR - PID - QJD - RKD - D^2W.$$

These equations for α, α', β, β', γ, γ' imply relations among the coefficients of the differential equation, which must be satisfied if it is to possess an intermediate integral of the assumed type.

Moreover, these linear forms in the derivatives of u secure that the equation is satisfied: for, substituting from the equation

$$u_x + au_l + hu_m + gu_n = 0,$$

in the equation

$$Du_x = Pu_l + \gamma' u_m + \beta u_n,$$

we have

$$(aD+P)\,u_l + (hD+\gamma')\,u_m + (gD+\beta)\,u_n = 0;$$

with two similar equations. Eliminating u_l, u_m, u_n, we have

$$\begin{vmatrix} aD+P, & hD+\gamma', & gD+\beta \\ hD+\gamma, & bD+Q, & fD+\alpha' \\ gD+\beta', & fD+\alpha, & cD+R \end{vmatrix} = 0;$$

when this determinant is expanded, and the relations among the quantities α, α', β, β', γ, γ' are used, the initial equation is reproduced.

323. Accordingly, we shall assume that the appropriate relations among the coefficients of the given equation are satisfied; so that α, α', β, β', γ, γ' are determinate, and have one set (or several sets) of values. The equations for u then have the form

$$\Delta_1(u) = \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} - \frac{1}{D}\left(P\frac{\partial u}{\partial l} + \gamma'\frac{\partial u}{\partial m} + \beta\frac{\partial u}{\partial n}\right),$$

$$\Delta_2(u) = \frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} - \frac{1}{D}\left(\gamma\frac{\partial u}{\partial l} + Q\frac{\partial u}{\partial m} + \alpha'\frac{\partial u}{\partial n}\right),$$

$$\Delta_3(u) = \frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} - \frac{1}{D}\left(\beta'\frac{\partial u}{\partial l} + \alpha\frac{\partial u}{\partial m} + R\frac{\partial u}{\partial n}\right),$$

on the assumption that D is not zero. They must satisfy the Poisson-Jacobi conditions

$$(\Delta_1, \Delta_2) = 0, \quad (\Delta_2, \Delta_3) = 0, \quad (\Delta_3, \Delta_1) = 0;$$

these are

$$0 = \frac{\gamma-\gamma'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_2\left(\frac{P}{D}\right) - \Delta_1\left(\frac{\gamma}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_2\left(\frac{\gamma'}{D}\right) - \Delta_1\left(\frac{Q}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_2\left(\frac{\beta}{D}\right) - \Delta_1\left(\frac{\alpha'}{D}\right)\right\}\frac{\partial u}{\partial n},$$

$$0 = \frac{\alpha-\alpha'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_3\left(\frac{\gamma}{D}\right) - \Delta_2\left(\frac{\beta'}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_3\left(\frac{Q}{D}\right) - \Delta_2\left(\frac{\alpha}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_3\left(\frac{\alpha'}{D}\right) - \Delta_2\left(\frac{R}{D}\right)\right\}\frac{\partial u}{\partial n},$$

$$0 = \frac{\beta-\beta'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_1\left(\frac{\beta'}{D}\right) - \Delta_3\left(\frac{P}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_1\left(\frac{\alpha}{D}\right) - \Delta_3\left(\frac{\gamma'}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_1\left(\frac{R}{D}\right) - \Delta_3\left(\frac{\beta}{D}\right)\right\}\frac{\partial u}{\partial n},$$

respectively. They clearly are not satisfied in virtue of

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0.$$

If they are satisfied identically, then

$$\alpha = \alpha' = S, \quad \beta = \beta' = T, \quad \gamma = \gamma' = U,$$

$$\left.\begin{aligned} \Delta_2\left(\frac{P}{D}\right) &= \Delta_1\left(\frac{S}{D}\right) \\ \Delta_3\left(\frac{P}{D}\right) &= \Delta_1\left(\frac{T}{D}\right) \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta_1\left(\frac{Q}{D}\right) &= \Delta_2\left(\frac{U}{D}\right) \\ \Delta_3\left(\frac{Q}{D}\right) &= \Delta_2\left(\frac{S}{D}\right) \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta_1\left(\frac{R}{D}\right) &= \Delta_3\left(\frac{T}{D}\right) \\ \Delta_2\left(\frac{R}{D}\right) &= \Delta_3\left(\frac{S}{D}\right) \end{aligned}\right\},$$

$$\Delta_1\left(\frac{S}{D}\right) = \Delta_2\left(\frac{T}{D}\right) = \Delta_3\left(\frac{U}{D}\right);$$

and the differential equation is

$$\begin{vmatrix} aD + P, & hD + U, & gD + T \\ hD + U, & bD + Q, & fD + S \\ gD + T, & fD + S, & cD + R \end{vmatrix} = 0.$$

The case has already been discussed (§ 319, Ex. 2). We shall assume the alternative hypothesis, that the Poisson-Jacobi conditions are not satisfied identically: they therefore provide a new equation or new equations. We shall suppose that they provide one new equation in one or other of the three kinds of ways above indicated: let it be

$$\Delta_4 = 0.$$

The presence of this additional equation requires that the additional Poisson-Jacobi conditions

$$(\Delta_1, \Delta_4) = 0, \quad (\Delta_2, \Delta_4) = 0, \quad (\Delta_3, \Delta_4) = 0,$$

shall be satisfied: we shall assume that they are satisfied without the association of any new equations. The system of equations

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0, \quad \Delta_4 = 0,$$

is complete; as it involves seven independent variables, it possesses three algebraically independent integrals which may be denoted by u_1, u_2, u_3. Then

$$\phi(u_1, u_2, u_3) = 0,$$

where ϕ is any arbitrary function of its arguments, is an integral of the system: it manifestly also is a general intermediate integral of the original equation.

This integral is provided by a subsidiary system associated with one set of values of $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$. It might happen that another intermediate integral is provided by the subsidiary system associated with a different set of values of $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$, the additional conditions of course being supposed to be satisfied: let it be denoted by

$$\psi(u_1', u_2', u_3') = 0,$$

where ψ is arbitrary, and u_1', u_2', u_3' are the three algebraically independent integrals of the new subsidiary system.

It is easy to assign the circumstances which allow these two intermediate integrals (if obtainable) to be treated as simultaneous. The quantities u_1, u_2, u_3 are integrals of the first system, each equation in which is homogeneous and linear in the derivatives of u: hence ϕ is also an integral, and we have

$$\left.\begin{aligned} D\phi_x &= P\phi_l + \gamma'\phi_m + \beta\phi_n \\ D\phi_y &= \gamma\phi_l + Q\phi_m + \alpha'\phi_n \\ D\phi_z &= \beta'\phi_l + \alpha\phi_m + R\phi_n \end{aligned}\right\}.$$

Similarly, as u_1', u_2', u_3' are integrals of an alternative system, ψ also is an integral of that system which may be taken to have the form

$$\left.\begin{aligned} D\psi_x &= P\psi_l + \Gamma'\psi_m + B\psi_n \\ D\psi_y &= \Gamma\psi_l + Q\psi_m + A'\psi_n \\ D\psi_z &= B'\psi_l + A\psi_m + R\psi_n \end{aligned}\right\},$$

where

A, A' are either α, α'; or α', α:

B, B' .... β, β'; or β', β:

Γ, Γ' .... γ, γ'; or γ', γ:

the set of first alternatives throughout giving the system for ϕ. In order that the equations

$$\phi = 0, \quad \psi = 0,$$

may coexist, the relation

$$[\phi, \psi] = 0$$

must be satisfied, that is, we must have

$$\phi_x\psi_l - \psi_x\phi_l + \phi_y\psi_m - \psi_y\phi_m + \phi_z\psi_n - \psi_z\phi_n = 0.$$

Substituting for $\phi_x, \phi_y, \phi_z, \psi_x, \psi_y, \psi_z$ from the equations in which they respectively occur and collecting terms, we have

$$\begin{aligned}(\gamma' - \Gamma)\,\phi_m\psi_l + (\gamma - \Gamma')\,\psi_m\phi_l \\ + (\alpha' - A)\,\phi_n\psi_m + (\alpha - A')\,\psi_n\phi_m \\ + (\beta' - B)\,\phi_l\psi_n + (\beta - B')\,\psi_l\phi_n = 0.\end{aligned}$$

Evidently this is satisfied identically, when

$$\begin{aligned} A = \alpha', \quad A' = \alpha, \\ B = \beta', \quad B' = \beta, \\ \Gamma = \gamma', \quad \Gamma' = \gamma: \end{aligned}$$

and therefore we have the result:

If all the conditions for the possession of three algebraically independent integrals be satisfied for each of the systems

$$\left.\begin{aligned} D\phi_x &= P\phi_l + \gamma'\phi_m + \beta\phi_n \\ D\phi_y &= \gamma\phi_l + Q\phi_m + \alpha'\phi_n \\ D\phi_z &= \beta'\phi_l + \alpha\phi_m + R\phi_n \end{aligned}\right\}, \qquad \left.\begin{aligned} D\psi_x &= P\psi_l + \gamma\psi_m + \beta'\psi_n \\ D\psi_y &= \gamma'\psi_l + Q\psi_m + \alpha\psi_n \\ D\psi_z &= \beta\psi_l + \alpha'\psi_m + R\psi_n \end{aligned}\right\},$$

then any intermediate integral of the differential equation provided by the first system can be associated with any intermediate integral provided by the second system.

Moreover, when we proceed to integrate either intermediate integral as an equation of the first order so as to obtain the primitive, the subsidiary equations for that integration include the equations subsidiary to the construction of an integral of the other system. When any integrals of the other system are known, they can be used to simplify the integration that leads to the primitive.

Other theorems, analogous to the corresponding theorems for equations in two independent variables, can similarly be obtained. We shall not enter into the further development of the details connected with the type of equation under consideration: the method has been sufficiently outlined to allow of application to any particular equation.

Ex. 1. Integrate the equation

$$\begin{vmatrix} a+a_1, & h+a_6, & g+a_5 \\ h+a_6, & b+a_2, & f+a_4 \\ g+a_5, & f+a_4, & c+a_3 \end{vmatrix} + a_7^2\,(c+a_3) = 0,$$

where all the quantities $a_1, \ldots, a_7$ are constants, shewing that, when certain conditions of inequality are satisfied, there are two subsidiary systems.

Ex. 2. Integrate the equation

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & v \end{vmatrix} = 0.$$

The Linear Equation.

324. In the preceding illustrations of the theory, it has been assumed that D is not zero. When D is zero, it is simpler, in any particular case, to re-apply the process from the beginning than to modify the equations of the more general form. By way of illustration, we shall re-apply it to the equation

$$Aa + 2Hh + Bb + 2Gg + 2Ff + Cc = K,$$

where the quantities A, B, C, F, G, H, K are functions of x, y, z, v, l, m, n, and do not involve any derivative of the second order.

Substituting from the equations

$$u_x + au_l + hu_m + gu_n = 0,$$

$$u_y + hu_l + bu_m + fu_n = 0,$$

$$u_z + gu_l + fu_m + cu_n = 0,$$

for a, b, c in terms of f, g, h, and making the transformed equation evanescent, we have

$$2F - B\frac{u_n}{u_m} - C\frac{u_m}{u_n} = 0,$$

$$2G - A\frac{u_n}{u_l} - C\frac{u_l}{u_n} = 0,$$

$$2H - A\frac{u_m}{u_l} - B\frac{u_l}{u_m} = 0,$$

$$K + A\frac{u_x}{u_l} + B\frac{u_y}{u_m} + C\frac{u_z}{u_n} = 0,$$

apparently four equations. But, on writing

$$u_l = \theta u_n, \quad u_m = \phi u_n,$$

the first three equations become

$$2F - \frac{B}{\phi} - C\phi = 0,$$

$$2G - \frac{A}{\theta} - C\theta = 0,$$

$$2H - A\frac{\phi}{\theta} - B\frac{\theta}{\phi} = 0:$$

hence

$$4(GH - AF) = A^2\frac{\phi}{\theta^2} + BC\frac{\theta^2}{\phi} - AC\phi - AB\theta$$

$$= \left(A\frac{\phi}{\theta} - B\frac{\theta}{\phi}\right)\left(\frac{A}{\theta} - C\theta\right)$$

$$= 4(H^2 - AB)^{\frac{1}{2}}(G^2 - AC)^{\frac{1}{2}};$$

and therefore, assuming that A does not vanish, we have

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.$$

Thus one purely algebraical relation among the coefficients A, B, C, F, G, H must be satisfied*: and then the equations for u are three in number, viz.

$$\left.\begin{aligned} u_l &= \theta u_n \\ u_m &= \phi u_n \\ \frac{A}{\theta}u_x + \frac{B}{\phi}u_y + Cu_z + Ku_n &= 0 \end{aligned}\right\},$$

with possibly two values for θ and possibly two values for ϕ.

Now that the equations for u have been obtained, the same method as before can be used for the construction of the intermediate integral (when it exists) and for the consequent derivation of the primitive.

Ex. 1. As a single application, let it be required to find the general primitive of

$$x^2a + 2xyh + y^2b + 2xzg + 2yzf + z^2c = 0.$$

The condition

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0$$

* It was first given by Euler, though not from the point of view of this method of integration: *Inst. Calc.*, t. III, p. 448.

is obviously satisfied: and the equations for u are easily found to consist of the single system

$$\left.\begin{aligned}\frac{u_l}{x}=\frac{u_m}{y}=\frac{u_n}{z}\\ xu_x+yu_y+zu_z=0\end{aligned}\right\}.$$

This is a complete Jacobian system as it stands: hence it possesses four algebraically independent integrals, and these can be taken in the form

$$\frac{y}{x},\ \frac{z}{x},\ \frac{lx+my+nz}{x},\ v-(lx+my+nz).$$

Here, no one of the quantities u_1, u_2, u_3, u_4 involves derivatives of u with regard to x, y, z, v, l, m, n. We adopt the method explained in the Note at the end of Ex. 2 in § 319: we have two general intermediate integrals

$$\frac{lx+my+nz}{x}=\phi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

$$v-(lx+my+nz)=\psi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

where ϕ and ψ are arbitrary functions of their arguments. The general primitive of the original equation is at once given by treating these equations as simultaneous, which is known to be permissible: that primitive is

$$v=x\phi\left(\frac{y}{x},\ \frac{z}{x}\right)+\psi\left(\frac{y}{x},\ \frac{z}{x}\right).$$

Ex. 2. Integrate the equation

$$x^2a+2xyh+y^2b+2xzg+2yzf+z^2c=\alpha(xl+ym+zn)+\beta v,$$

where α and β are constants.

Ex. 3. Integrate the equation

$$7a+2b+c+9h+8g+3f=\frac{l+m+n}{x+y-z}. \qquad \text{(Vivanti.)}$$

Ex. 4. Deduce the primitive of the equation

$$l^2(bc-f^2)+2lm(fg-ch)+m^2(ac-g^2)$$
$$+2ln(fh-bg)+2mn(gh-af)+n^2(ab-h^2)=0,$$

by applying a contact-transformation to the equation in the preceding Ex. 1; or otherwise integrate the equation.

Subsidiary System in Differential Elements.

325. The preceding investigation depends upon the integration of simultaneous partial differential equations of the first order; and this integration, as usual, ultimately depends upon the integration of a system or of systems of simultaneous ordinary equations. There is an alternative method of proceeding, which uses ordinary equations as directly subsidiary to the construction of the intermediate integral when it exists: they are the equations of the characteristics of the first order. The two methods bear to one another the same relation as do the corresponding methods applied to equations of the second order in two independent variables.

It will be sufficient for the present purpose if the method is applied to the equation

$$\begin{aligned} D\Delta + PA + QB + RC + 2SF + 2TG + 2UH \\ + Ia + Jb + Kc + 2Lf + 2Mg + 2Nh = W, \end{aligned}$$

already (§ 322) considered by the other method. Writing

$$\begin{aligned} dl &= a\,dx + h\,dy + g\,dz, \\ dm &= h\,dx + b\,dy + f\,dz, \\ dn &= g\,dx + f\,dy + c\,dz, \end{aligned}$$

and substituting, in the differential equation, values of a, b, c derived from these equations in the form

$$\begin{aligned} a &= \frac{dy\,dz}{dx}\left(\frac{dl}{dy\,dz} - \frac{g}{dy} - \frac{h}{dz}\right), \\ b &= \frac{dz\,dx}{dy}\left(\frac{dm}{dx\,dz} - \frac{f}{dx} - \frac{h}{dz}\right), \\ c &= \frac{dx\,dy}{dz}\left(\frac{dn}{dx\,dy} - \frac{f}{dx} - \frac{g}{dy}\right), \end{aligned}$$

we obtain the equations of the characteristic by making the resulting equation evanescent*. Hence, as

$$\begin{aligned} A &= bc - f^2 \\ &= \left\{\frac{dm\,dn}{dx\,dz\,dy\,dz} - \frac{f}{dx}\left(\frac{dm}{dx\,dz} + \frac{dn}{dx\,dy}\right) - \frac{g}{dy}\frac{dm}{dx\,dz} - \frac{h}{dz}\frac{dn}{dx\,dy}\right. \\ &\qquad \left. + \frac{fg}{dx\,dy} + \frac{gh}{dy\,dz} + \frac{fh}{dx\,dz}\right\} dx^2, \end{aligned}$$

with similar expressions for B and C, and

$$\begin{aligned} F &= gh - af \\ &= dy\,dz\left(\frac{fg}{dx\,dy} + \frac{gh}{dy\,dz} + \frac{fh}{dx\,dz} - \frac{f}{dx}\frac{dl}{dy\,dz}\right), \end{aligned}$$

* This is only a statement as to actual results: the argument is similar to that in the case of two variables (§ 233) and need not be repeated here.

with similar expressions for G and H, and

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

$$= \left\{ \frac{dl}{dydz}\frac{dm}{dzdx}\frac{dn}{dxdy} \right.$$

$$+ \left(\frac{fg}{dxdy} + \frac{gh}{dydz} + \frac{fh}{dxdz}\right)\left(\frac{dl}{dydz} + \frac{dm}{dzdx} + \frac{dn}{dxdy}\right)$$

$$- \frac{f}{dx}\frac{dl}{dydz}\left(\frac{dm}{dzdx} + \frac{dn}{dxdy}\right)$$

$$- \frac{g}{dy}\frac{dm}{dxdz}\left(\frac{dl}{dydz} + \frac{dn}{dxdy}\right)$$

$$\left. - \frac{h}{dz}\frac{dn}{dxdy}\left(\frac{dl}{dydz} + \frac{dm}{dxdz}\right)\right\} dxdydz,$$

we have

$$D(dxdl + dydm + dzdn)$$
$$+ Pdx^2 + Qdy^2 + Rdz^2 + 2Sdydz + 2Tdxdz + 2Udxdy = 0,$$

from the evanescence of the terms in fg, gh, hf:

$$Ddl(dydm + dzdn) + Pdx(dydm + dzdn)$$
$$+ dl(Qdy^2 + 2Sdydz + Rdz^2) + dx(Jdz^2 - 2Ldydz + Kdy^2) = 0,$$

from the evanescence of the term in f:

$$Ddm(dzdn + dxdl) + Qdy(dzdn + dxdl)$$
$$+ dm(Rdz^2 + 2Tdzdx + Pdx^2) + dy(Kdx^2 - 2Mdxdz + Idz^2) = 0,$$

from the evanescence of the term in g:

$$Ddn(dxdl + dydm) + Rdz(dxdl + dydm)$$
$$+ dn(Pdx^2 + 2Udxdy + Qdy^2) + dz(Idy^2 - 2Ndxdy + Jdx^2) = 0,$$

from the evanescence of the term in h: and

$$Ddldmdn + Idldydz + Jdmdzdx + Kdndxdy$$
$$+ Pdmdndx + Qdndldy + Rdldmdz = Wdxdydz,$$

being the aggregate of the remaining terms.

Proceeding as before, we reduce these five equations to three, conditionally on certain relations among the coefficients being satisfied*: these relations are sufficient to secure the possibility of determining six quantities α, α', β, β', γ, γ', such that

$$\left.\begin{aligned}\alpha+\alpha'&=2S\\ \alpha\alpha'&=QR-DI\end{aligned}\right\},\quad \left.\begin{aligned}\beta+\beta'&=2T\\ \beta\beta'&=PR-DJ\end{aligned}\right\},\quad \left.\begin{aligned}\gamma+\gamma'&=2U\\ \gamma\gamma'&=PQ-DK\end{aligned}\right\},$$

$$\begin{aligned}\beta\gamma+\beta'\gamma'&=2(LD+SP),\\ \gamma\alpha+\gamma'\alpha'&=2(MD+TQ),\\ \alpha\beta+\alpha'\beta'&=2(ND+UR),\\ \alpha\beta\gamma+\alpha'\beta'\gamma'&=2PQR-PID-QJD-RKD-D^2W.\end{aligned}$$

We shall assume the conditions satisfied. When the differential relations are resolved, we have

$$\left.\begin{aligned}Ddl+Pdx+\gamma dy+\beta' dz&=0\\ Ddm+\gamma' dx+Qdy+\alpha dz&=0\\ Ddn+\beta dx+\alpha' dy+Rdz&=0\\ dv-ldx-mdy-ndz&=0\end{aligned}\right\},$$

as the equations of the characteristic of the first order.

It is easy to see that, if

$$du(x, y, z, v, l, m, n)=0$$

is a linear integrable combination of these equations, then

$$\begin{aligned}Du_x&=Pu_l+\gamma' u_m+\beta u_n,\\ Du_y&=\gamma u_l+Qu_m+\alpha' u_n,\\ Du_z&=\beta' u_l+\alpha u_m+Ru_n,\end{aligned}$$

being the former set (§ 322) of equations for u.

Ex. 1. When the coefficients D, P, Q, R, S, T, U vanish, so that the equation has the linear form

$$Ia+Jb+Kc+2Lf+2Mg+2Nh=W,$$

the preceding equations are evanescent: prove that the equations of the characteristic of the first order are

$$\left.\begin{aligned}dy=\theta\,dx,\quad dz=\phi\,dx\\ Idl+\frac{J}{\theta}dm+\frac{K}{\phi}dn=Wdx\end{aligned}\right\},$$

* They are due to the fact that all the coefficients are certain functional combinations of the derivatives of three quantities, as explained in § 321; the explanation is similarly set out in Vivanti's memoir, quoted at the beginning of this chapter (p. 490).

where

$$I\theta^2 - 2N\theta + J = 0, \quad I\phi^2 - 2M\phi + K = 0;$$

and prove that

$$\begin{vmatrix} I, & N, & M \\ N, & J, & L \\ M, & L, & K \end{vmatrix} = 0,$$

in order that there may be a characteristic of the first order.

(See, for comparison, § 324.)

Ex. 2. Obtain the equations of the characteristics of the first order of the equation

$$PA + QB + RC + 2SF + 2TG + 2UH + Ia + Jb + Kc + 2Lf + 2Mg + 2Nh = W,$$

obtaining the preliminary algebraic relations among the coefficients which must hold if there is to be a characteristic of the first order.

Ex. 3. Integrate the equation

$$a - h - g + f = 0.$$

Ex. 4. A dependent variable z is a function of four independent variables x_1, x_2, x_3, x_4; and the derivatives of the first order and the second order are denoted by p_i, for $i = 1, 2, 3, 4$, and by p_{ij}, for $i, j = 1, 2, 3, 4$. Shew that, if the equation

$$\sum_{i=1}^{4} \sum_{j=1}^{4} A_{ij}\, p_{ij} = U$$

possesses an intermediate integral of the first order, where the coefficients A_{ij} are functions of the variables and the first derivatives only, and where $A_{ij} = A_{ji}$, then the minor of every term in the diagonal of the determinant

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14} \\ A_{21}, & A_{22}, & A_{23}, & A_{24} \\ A_{31}, & A_{32}, & A_{33}, & A_{34} \\ A_{41}, & A_{42}, & A_{43}, & A_{44} \end{vmatrix} = 0$$

vanishes: and obtain the equations of the characteristic of the first order.

Shew that the conditions are satisfied for the equation

$$\sum_{i=1}^{4} \sum_{j=1}^{4} x_i x_j\, p_{ij} = U,$$

where U involves $z, x_1, x_2, x_3, x_4, p_1\ p_2, p_3, p_4$ at the utmost: and construct the primitive in the cases

(i) $U = 0$, (ii) $U = az$, (iii) $U = c(x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4)$,

where a and c are constants.

CHAPTER XXIV.

Equations of the Second Order in Three Independent Variables, not necessarily having an Intermediate Integral.

The present chapter, like the preceding chapter, is devoted to the extension, to equations involving a number of independent variables greater than two, of methods applicable to equations in only two independent variables. As before, the results are given and the notation is specially devised for equations in three independent variables: but many of the results can obviously be generalised to the case when the number of independent variables is n, though it has not seemed necessary to state them in the general form.

As regards the range of the chapter, no assumption is made (as was done in the preceding chapter) that an intermediate integral exists: and the particular methods, generalised from equations in two independent variables, are those of Ampère and of Darboux. The chapter is mainly based upon a memoir by the author*.

Some illustrations of the theory, in the case of n independent variables, are to be found in another memoir by the author†: they belong to the theory of symmetrical algebra.

Moreover, it is to be understood that only the general theory of the partial equations is considered: there is no attempt to construct and coordinate the properties of particular equations, however important these may be in mathematical physics‡. Similarly, there is no discussion of the integrals of particular equations as determined by so-called boundary conditions§.

* *Phil. Trans.*, vol. 191 (1898), pp. 1—86.

† *Camb. Phil. Trans.*, vol. xvi (1898), pp. 291—325.

‡ Such equations, together with applications to mathematical physics, are discussed in Weber's edition (in two volumes, 1900) of Riemann's lectures *Die partiellen Differentialgleichungen der mathematischen Physik.*

§ Full references will be found in Sommerfeld's article on this part of the subject, *Encyclopädie der mathematischen Wissenschaften*, t. ii, pp. 504—570.

326. We now proceed to discuss the possibility of obtaining an integral of an equation

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

when no assumption is made concerning the existence of an intermediate integral. If a method of any generality can be devised, it should implicitly or explicitly lead to an intermediate integral (if any such exists).

By Cauchy's theorem, the equation usually possesses an integral which is determined by the value of v and of one of its first derivatives for an assigned relation between x, y, z. Regarding this relation as the equation of a surface, we can consider that Cauchy's theorem gives the values of v and l over the surface. If the surface be

$$S(x, y, z) = 0,$$

then, for all variations subject to the relation

$$\frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial z} dz = 0,$$

we know the value of

$$l dx + m dy + n dz;$$

but l is known over the surface, hence m and n are known over the surface. Thus v, l, m, n can be regarded as known all over the surface, in connection with Cauchy's theorem.

Now consider the higher derivatives at points on the surface. Denoting the derivatives of z with regard to x and to y along the surface by p and q, we have

$$\begin{aligned} dl &= a dx + h dy + g dz = (a + pg) dx + (h + qg) dy, \\ dm &= h dx + b dy + f dz = (h + pf) dx + (b + qf) dy, \\ dn &= g dx + f dy + c dz = (g + pc) dx + (f + qc) dy, \end{aligned}$$

along the surface so that, as l, m, n are known everywhere on the surface, the quantities on the right-hand sides of the equations

$$a + pg = \frac{dl}{dx}, \quad h + qg = \frac{dl}{dy},$$

$$h + pf = \frac{dm}{dx}, \quad b + qf = \frac{dm}{dy},$$

$$g + pc = \frac{dn}{dx}, \quad f + qc = \frac{dn}{dy},$$

are known along the surface. It is obvious that, if the values thus given are to be consistent with one another, we must have

$$\frac{dl}{dy}+p\frac{dn}{dy}=\frac{dm}{dx}+q\frac{dn}{dx}:$$

consequently, the equations are equivalent to five only, so that they can determine five of the quantities a, b, c, f, g, h, in terms of the remaining one, say

$$f=\frac{dn}{dy}-qc,$$

$$b=\frac{dm}{dy}-q\frac{dn}{dy}+q^2c,$$

$$g=\frac{dn}{dx}-pc,$$

$$a=\frac{dl}{dx}-p\frac{dn}{dx}+p^2c,$$

$$h=\frac{dm}{dx}-p\frac{dn}{dy}+pqc=\frac{dl}{dy}-q\frac{dn}{dx}+pqc.$$

We also have

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h)=0,$$

so that there are six equations to determine the six derivatives of the second order. These generally will be sufficient for the purpose; and therefore the six derivatives can be regarded as known along the surface.

Similarly, the derivatives of v of all orders can be regarded as known along the surface, being determinable in the same manner as a, b, c, f, g, h. Then, taking any point x_0, y_0, z_0 on the surface as an initial point, provided only that it is an ordinary point for the equation as given, we can expand v as a series of powers of $x-x_0$, $y-y_0$, $z-z_0$, the coefficients in which can be regarded as known. The convergence of the series can be established as in the proof of Cauchy's general theorem; and we then have the integral as established by that theorem.

This conclusion, however, is justified only if the six equations do actually determine a set or sets of values of a, b, c, f, g, h: it will fail of establishment, if sets of values are not determinately derivable. Such a result occurs for instance when $\phi=0$ becomes evanescent on the substitution of the values of a, b, f, g, h in terms of c: one condition then is

$$p^2 \frac{\partial \phi}{\partial a} + pq \frac{\partial \phi}{\partial h} + q^2 \frac{\partial \phi}{\partial b} - p \frac{\partial \phi}{\partial g} - q \frac{\partial \phi}{\partial f} + \frac{\partial \phi}{\partial c} = 0,$$

but this, of course, is only one among a number of such equations. Moreover, even when the integral has been obtained, it is found only in the form of infinite power-series: consequently, it is desirable to possess other methods of proceeding to an integral.

Extension of Ampère's Method.

327. One such method, applicable to equations whose integrals are expressible in finite form without partial quadratures, is to be found in a generalisation of Ampère's method devised in connection with equations involving only two independent variables. Let u be an argument in an arbitrary function occurring in the integral; and let the independent variables be changed from x, y, z to x, y, u, on the supposition that u is not independent of z. Then

$$\begin{aligned} l\,dx + m\,dy + n\,dz &= dv \\ &= \frac{dv}{dx} dx + \frac{dv}{dy} dy + \frac{dv}{du} du: \end{aligned}$$

hence, if p and q denote the derivatives of z with regard to x and to y respectively when u is constant, we have

$$\left.\begin{aligned} l + np &= \frac{dv}{dx} \\ m + nq &= \frac{dv}{dy} \\ n \frac{\partial z}{\partial u} &= \frac{dv}{du} \end{aligned}\right\}.$$

Similarly,

$$\left.\begin{aligned} a + gp &= \frac{dl}{dx} \\ h + gq &= \frac{dl}{dy} \\ g \frac{\partial z}{\partial u} &= \frac{dl}{du} \end{aligned}\right\}, \quad \left.\begin{aligned} h + fp &= \frac{dm}{dx} \\ b + fq &= \frac{dm}{dy} \\ f \frac{\partial z}{\partial u} &= \frac{dm}{du} \end{aligned}\right\}, \quad \left.\begin{aligned} g + cp &= \frac{dn}{dx} \\ f + cq &= \frac{dn}{dy} \\ c \frac{\partial z}{\partial u} &= \frac{dn}{du} \end{aligned}\right\}.$$

As before, we have

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx},$$

a universal relation; and then, using the first two equations out of each of the three sets in order to express a, b, f, g, h in terms of c, we have

$$f = \frac{dn}{dy} - qc,$$

$$b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2c,$$

$$g = \frac{dn}{dx} - pc,$$

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2c,$$

$$h = \frac{dm}{dx} - p\frac{dn}{dy} + pqc = \frac{dl}{dy} - q\frac{dn}{dx} + pqc,$$

and we take

$$c = \frac{\dfrac{dn}{du}}{\dfrac{dz}{du}}, \quad n = \frac{\dfrac{dv}{du}}{\dfrac{dz}{du}}.$$

When these values of a, b, f, g, h are substituted in

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

it becomes an equation involving c: and when we suppose (as now will be assumed) that ϕ is a polynomial function of its various arguments a, b, c, f, g, h, of order μ', the transformed equation in c can be arranged in powers of c. It will take a form

$$E_0 + E_1c + \ldots + E_\mu c^\mu = 0,$$

where μ is not greater than μ', and may be less than μ' if particular combinations of the derivatives of the second order occur in ϕ.

When the value of v is substituted, the equation must be satisfied identically. Now, in the expression

$$\frac{dn}{du} \div \frac{dz}{du},$$

being the value of c, the u-derivative of the arbitrary function which has u for an argument is of order higher than that of any other u-derivative contained in any of the quantities which make up $E_0, E_1, \ldots, E_\mu$: hence the equation can be satisfied identically, only if

$$E_\mu = 0, \quad E_{\mu-1} = 0, \ldots, E_1 = 0, \quad E_0 = 0.$$

If $\mu = \mu'$, then the equation $E_\mu = 0$ is the equivalent of the equation

$$p^2 \frac{\partial \phi}{\partial a} + pq \frac{\partial \phi}{\partial h} + q^2 \frac{\partial \phi}{\partial b} - p \frac{\partial \phi}{\partial g} - q \frac{\partial \phi}{\partial f} + \frac{\partial \phi}{\partial c} = 0;$$

if $\mu < \mu'$, this equation still is satisfied in virtue of the equations $E_\mu = 0, \ldots, E_0 = 0$: in either case, it is satisfied. Now, as p and q are the derivatives of z with regard to x and y when u is constant, we have

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} = 0:$$

substituting for p and q, we have

$$A \left(\frac{\partial u}{\partial x}\right)^2 + H \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + B \left(\frac{\partial u}{\partial y}\right)^2 + G \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + F \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + C \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

where

$$A, B, C, F, G, H = \frac{\partial \phi}{\partial a}, \frac{\partial \phi}{\partial b}, \frac{\partial \phi}{\partial c}, \frac{\partial \phi}{\partial f}, \frac{\partial \phi}{\partial g}, \frac{\partial \phi}{\partial h}.$$

This relation must therefore be satisfied by an argument of an arbitrary function which occurs in the primitive of the original equation.

Characteristic Invariant.

328. But, further, this relation is invariantive for all changes of the independent variables. Suppose them changed according to the law

$$x' = \xi(x, y, z),$$

$$y' = \eta(x, y, z),$$

$$z' = \zeta(x, y, z);$$

and let $l', m', n', a', b', c', f', g', h'$ be the respective derivatives of v with regard to the new variables. Then

$$l, m, n = \left(\begin{vmatrix} \xi_x, & \eta_x, & \zeta_x \\ \xi_y, & \eta_y, & \zeta_y \\ \xi_z, & \eta_z, & \zeta_z \end{vmatrix} \right)(l', m', n'),$$

$$a = (a', b', c', f', g', h')(\xi_x, \eta_x, \zeta_x)^2 + \ldots,$$

$$h = (a', b', c', f', g', h')(\xi_x, \eta_x, \zeta_x)(\xi_y, \eta_y, \zeta_y) + \ldots,$$

$$\vdots$$

the omitted terms represented by $+ \ldots$ being terms which involve derivatives of the first order only. Applying these transformations, let $\phi(v, x, y, z, l, m, n, a, b, c, f, g, h)$ become $\phi'(v, x', y', z', l', m', n', a', b', c', f', g', h')$; then

$$A' = \frac{\partial \phi'}{\partial a'} = A\xi_x^2 + H\xi_x\xi_y + B\xi_y^2 + G\xi_x\xi_z + F\xi_y\xi_z + C\xi_z^2,$$

$$H' = \frac{\partial \phi'}{\partial h'} = 2A\xi_x\eta_x + H(\xi_x\eta_y + \eta_x\xi_y) + 2B\xi_y\eta_y$$
$$+ G(\xi_x\eta_z + \xi_z\eta_x) + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z,$$

and so on: hence, if

$$u(x, y, z) = u'(x', y', z'),$$

so that

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} = \left(\begin{vmatrix} \xi_x, & \eta_x, & \zeta_x \\ \xi_y, & \eta_y, & \zeta_y \\ \xi_z, & \eta_z, & \zeta_z \end{vmatrix} \right)\left(\frac{\partial u'}{\partial x'}, \frac{\partial u'}{\partial y'}, \frac{\partial u'}{\partial z'}\right),$$

we have

$$A\left(\frac{\partial u}{\partial x}\right)^2 + H\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \ldots = A'\left(\frac{\partial u'}{\partial x'}\right)^2 + H'\frac{\partial u'}{\partial x'}\frac{\partial u'}{\partial y'} + \ldots,$$

after substitution and collection of terms.

The alleged property is thus established: on account of the property, the relation satisfied by u is called the *characteristic invariant* of the given differential equation.

329. The equations

$$E_0 = 0, \quad E_1 = 0, \ldots, E_\mu = 0,$$

are resolved into sets as simple as possible: and then we seek integrable combinations, as many as possible, of each particular set. In effecting the integrations, it is to be borne in mind that the quantity u is latent: and in order to take account of it, the equations

$$\frac{1}{n}\frac{dv}{du}=\frac{1}{g}\frac{dl}{du}=\frac{1}{f}\frac{dm}{du}=\frac{1}{c}\frac{dn}{du}=\frac{dz}{du}$$

(which, except for one part of the discussion, have been left on one side) must be used and be satisfied. When all the operations have been completed and limitations upon functional forms as required by the equations have been imposed, a number of relations will result.

The importance of the process lies in the fact that *the relations thus obtained satisfying all these subsidiary equations constitute an integral of the original equation.* For suppose that, in the expression which v acquires from the relations, we consider u eliminated in favour of z: then

$$\begin{aligned}\frac{dv}{dx}dx+\frac{dv}{dy}dy+\frac{dv}{du}du &= dv\\ &=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}\left(p\,dx+q\,dy+\frac{dz}{du}du\right),\end{aligned}$$

whence

$$\frac{dv}{du}=\frac{\partial v}{\partial z}\frac{dz}{du},$$

$$\frac{dv}{dx}=\frac{\partial v}{\partial z}p+\frac{\partial v}{\partial x},$$

$$\frac{dv}{dy}=\frac{\partial v}{\partial z}q+\frac{\partial v}{\partial y}.$$

Comparing these with the equations of the system that led to the integral relation, we have

$$l=\frac{\partial v}{\partial x},\quad m=\frac{\partial v}{\partial y},\quad n=\frac{\partial v}{\partial z}.$$

Next, take a quantity c defined by the equation

$$c=\frac{\dfrac{dn}{du}}{\dfrac{dz}{du}},$$

so that, in virtue of the integral system, its value is determinate. Since the equations

$$\frac{dv}{dx}=l+np,\quad \frac{dv}{du}=n\frac{dz}{du},$$

are satisfied by the integral system (for they are members of the subsidiary system), we have

$$\frac{d}{du}(l + np) = \frac{d}{dx}\left(n\frac{dz}{du}\right),$$

and therefore

$$\frac{dl}{du} + p\frac{dn}{du} = \frac{dn}{dx}\frac{dz}{du},$$

the terms $n\dfrac{d^2z}{dxdu}$ on both sides cancelling: consequently,

$$\begin{aligned}\frac{dl}{du} &= \frac{dn}{dx}\frac{dz}{du} - p\frac{dn}{du}\\ &= \left(\frac{dn}{dx} - pc\right)\frac{dz}{du}\\ &= g\frac{dz}{du}.\end{aligned}$$

Similarly, from

$$\frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n\frac{dz}{du},$$

we can prove that the integral relations lead to

$$\frac{dm}{du} = f\frac{dz}{du}.$$

Again, we have

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}dx + \frac{\partial^2 v}{\partial x\partial y}dy + \frac{\partial^2 v}{\partial x\partial z}\left(p\,dx + q\,dy + \frac{dz}{du}du\right)\\ = d\left(\frac{\partial v}{\partial x}\right)\\ = dl\\ = \frac{dl}{dx}dx + \frac{dl}{dy}dy + \frac{dl}{du}du,\end{aligned}$$

and therefore

$$\frac{dl}{du} = \frac{dz}{du}\frac{\partial^2 v}{\partial x\partial z},$$

$$\frac{dl}{dx} = p\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial^2 v}{\partial x^2},$$

$$\frac{dl}{dy} = q\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial^2 v}{\partial x\partial y}.$$

Comparing these with the former equations, we have

$$a = \frac{\partial^2 v}{\partial x^2}, \quad h = \frac{\partial^2 v}{\partial x\partial y}, \quad g = \frac{\partial^2 v}{\partial x\partial z}.$$

Similarly, we find

$$b=\frac{\partial^2 v}{\partial y^2},\quad f=\frac{\partial^2 v}{\partial y\partial z},\quad c=\frac{\partial^2 v}{\partial z^2}.$$

Again, when we take the combination

$$E_0+cE_1+\ldots+c^\mu E_\mu=0,$$

and eliminate from this equation the derivatives of l, m, n with regard to x and y by means of the other equations, we have

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h)=0:$$

and the quantities $l, m, n, a, b, c, f, g, h$ are such that

$$l,\ m,\ n=\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z},$$

$$a,\ b,\ c,\ f,\ g,\ h=\frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^2 v}{\partial y\partial z},\ \frac{\partial^2 v}{\partial z\partial x},\ \frac{\partial^2 v}{\partial x\partial y}:$$

that is, v is an integral of the original differential equation. We therefore may summarise the result as follows:

When an equation of the second order $\phi=0$ is transformed into

$$E_0+E_1c+\ldots+E_\mu c^\mu=0,$$

by the elimination of a, b, f, g, h through the equations

$$a=\frac{dl}{dx}-p\frac{dn}{dx}+p^2c,\quad b=\frac{dm}{dy}-q\frac{dn}{dy}+q^2c,$$

$$g=\frac{dn}{dx}-pc,\quad f=\frac{dn}{dy}-qc,$$

$$h=\frac{dm}{dx}-p\frac{dn}{dy}+pqc=\frac{dl}{dy}-q\frac{dn}{dx}+pqc,$$

and when, in the integral equivalent of the simultaneous system

$$\left.\begin{array}{c} E_0=0,\quad E_1=0,\ \ldots,\ E_\mu=0 \\ \dfrac{dm}{dx}-p\dfrac{dn}{dy}=\dfrac{dl}{dy}-q\dfrac{dn}{dx} \\ \dfrac{dv}{dx}=l+np,\quad \dfrac{dv}{dy}=m+nq \\ \dfrac{dz}{dx}=p,\quad \dfrac{dz}{dy}=q \end{array}\right\},$$

all arbitrary constants are made functions of u, and all arbitrary functions are made to involve u, subject to the equation

$$\frac{dv}{du} = n\frac{dz}{du},$$

the value of v so provided is an integral of the original differential equation.

The integer μ is never less than unity: the equation

$$\frac{dm}{dx} - p\frac{dn}{dy} = \frac{dl}{dy} - q\frac{dn}{dx}$$

is a functional consequence of the two equations

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq,$$

being a necessity to their coexistence: thus there are at least six equations for the determination of five quantities l, m, n, v, z as functions of x and y, the quantity u being latent throughout these equations. There is no perfectly general process, the universal application of which is subject only to the difficulties of quadrature, that can be applied to the system: and indeed, no such process can be expected which does not essentially involve the hypothesis made as to the character of the integral—that it is expressible in finite terms without partial quadratures.

When an integral equivalent of the system has been obtained, the relation of the argument u to the other variables in that equivalent can be settled by means of the equation

$$\frac{dv}{du} = n\frac{dz}{du},$$

which must be satisfied identically. The arbitrary elements that then survive will indicate how far the integral can be regarded as coinciding with the integral in Cauchy's theorem.

A detailed example will shew the working of the preceding theory in connection with the explanations.

Ex. Consider the equation

$$b = f - g + h,$$

which, as a matter of fact, does possess intermediate integrals. The characteristic invariant is

$$-pq + q^2 - p + q = 0,$$

that is,

$$(p-q)(q+1)=0;$$

so that two sets of subsidiary equations arise for consideration: they are given by

$$p-q=0, \quad q+1=0.$$

When we substitute the values of b, f, g, h, given in terms of c by the subsidiary equations, and take account of the characteristic invariant, we find that (in the general notation) $\mu=1$, that $E_1=0$ is the characteristic invariant, and that the equation $E_0=0$ is

$$\frac{dm}{dy}+(p-q-1)\frac{dn}{dy}+\frac{dn}{dx}-\frac{dm}{dx}=0.$$

First, let

$$p-q=0,$$

that is, on integration

$$z=\theta(x+y, u),$$

where θ is an arbitrary function. Also, using the relation $p-q=0$, the preceding subsidiary equation is

$$\frac{d(m-n)}{dy}-\frac{d(m-n)}{dx}=0,$$

so that

$$m-n=\psi(x+y, u),$$

where ψ is an arbitrary function: hence, taking account of the preceding equation that defines z, we have

$$m-n=g(x+y, z),$$

where g is an arbitrary function. This equation is manifestly an intermediate integral: when integrated, it leads to a primitive

$$v=G(x+y, z)+H(y+z, x),$$

where G and H are arbitrary functions.

Next, let

$$q+1=0,$$

that is, on integration

$$y+z=\vartheta(x, u'),$$

where ϑ is an arbitrary function, and u' is a new argument. The other subsidiary equation is

$$\frac{dm}{dy}-\frac{dm}{dx}+\frac{dn}{dx}+p\frac{dn}{dy}=0:$$

hence, in connection with the universal equation

$$\frac{dm}{dx}-p\frac{dn}{dy}-\frac{dl}{dy}+q\frac{dn}{dx}=0,$$

together with the relation $q+1=0$, we have

$$\frac{dm}{dy}-\frac{dl}{dy}=0,$$

so that

$$l-m=\chi(x, u'),$$

where χ is an arbitrary function. Taking account of the preceding equation that defines u', we have

$$l-m=k(y+z,\ x),$$

where k is an arbitrary function. Again, this equation is an intermediate integral: when integrated, it leads to the same primitive

$$v=G(x+y,\ z)+H(y+z,\ x)$$

as before.

Application of Darboux's Method.

330. When it happens that no integrable combination of the preceding subsidiary equations is obtainable, we pass from the preceding generalisation of Ampère's method for equations in two independent variables to a generalisation of Darboux's method, whereby we seek to obtain equations (if any) that are compatible with a given equation and are not of lower order.

When the method is applied to an equation of the second order

$$\phi(v,\ x,\ y,\ z,\ l,\ m,\ n,\ a,\ b,\ c,\ f,\ g,\ h)=0,$$

we need to take account of derivatives of the third order. Let

$$\alpha_0=\frac{\partial^3 v}{\partial x^3},\qquad \alpha_1=\frac{\partial^3 v}{\partial x^2\partial z},\qquad \alpha_2=\frac{\partial^3 v}{\partial x\partial z^2},\qquad \alpha_3=\frac{\partial^3 v}{\partial z^3},$$

$$\beta_0=\frac{\partial^3 v}{\partial x^2\partial y},\quad \beta_1=\frac{\partial^3 v}{\partial x\partial y\partial z},\quad \beta_2=\frac{\partial^3 v}{\partial y\partial z^2},$$

$$\gamma_0=\frac{\partial^3 v}{\partial x\partial y^2},\quad \gamma_1=\frac{\partial^3 v}{\partial y^2\partial z},$$

$$\delta_0=\frac{\partial^3 v}{\partial y^3},$$

so that

$$\left.\begin{aligned}
da&=\alpha_0 dx+\beta_0 dy+\alpha_1 dz\\
dh&=\beta_0 dx+\gamma_0 dy+\beta_1 dz\\
db&=\gamma_0 dx+\delta_0 dy+\gamma_1 dz\\
dg&=\alpha_1 dx+\beta_1 dy+\alpha_2 dz\\
df&=\beta_1 dx+\gamma_1 dy+\beta_2 dz\\
dc&=\alpha_2 dx+\beta_2 dy+\alpha_3 dz
\end{aligned}\right\}.$$

The equation $\phi = 0$ must be satisfied identically when the appropriate value of v is substituted: hence, writing

$$X = \frac{\partial \phi}{\partial x} + l\frac{\partial \phi}{\partial v} + a\frac{\partial \phi}{\partial l} + h\frac{\partial \phi}{\partial m} + g\frac{\partial \phi}{\partial n},$$

$$Y = \frac{\partial \phi}{\partial y} + m\frac{\partial \phi}{\partial v} + h\frac{\partial \phi}{\partial l} + b\frac{\partial \phi}{\partial m} + f\frac{\partial \phi}{\partial n},$$

$$Z = \frac{\partial \phi}{\partial z} + n\frac{\partial \phi}{\partial v} + g\frac{\partial \phi}{\partial l} + f\frac{\partial \phi}{\partial m} + c\frac{\partial \phi}{\partial n},$$

and keeping the former notation for derivatives of ϕ with regard to a, b, c, f, g, h, we have

$$\left.\begin{aligned} X + A\alpha_0 + H\beta_0 + B\gamma_0 + G\alpha_1 + F\beta_1 + C\alpha_2 = 0 \\ Y + A\beta_0 + H\gamma_0 + B\delta_0 + G\beta_1 + F\gamma_1 + C\beta_2 = 0 \\ Z + A\alpha_1 + H\beta_1 + B\gamma_1 + G\alpha_2 + F\beta_2 + C\alpha_3 = 0 \end{aligned}\right\},$$

in connection with the value of v provided by the integral sought.

Now suppose that the independent variables are changed from x, y, z to x, y, u, where the only assumption made at the moment is that u certainly involves z: then, keeping the former significance for p and q, the derivatives of a, b, c, f, g, h with regard to x, y, u are given by the equations

$$\frac{da}{dx} = \alpha_0 + \alpha_1 p, \quad \frac{da}{dy} = \beta_0 + \alpha_1 q, \quad \frac{da}{du} = \alpha_1 \frac{dz}{du},$$

$$\frac{dh}{dx} = \beta_0 + \beta_1 p, \quad \frac{dh}{dy} = \gamma_0 + \beta_1 q, \quad \frac{dh}{du} = \beta_1 \frac{dz}{du},$$

$$\frac{db}{dx} = \gamma_0 + \gamma_1 p, \quad \frac{db}{dy} = \delta_0 + \gamma_1 q, \quad \frac{db}{du} = \gamma_1 \frac{dz}{du},$$

$$\frac{dg}{dx} = \alpha_1 + \alpha_2 p, \quad \frac{dg}{dy} = \beta_1 + \alpha_2 q, \quad \frac{dg}{du} = \alpha_2 \frac{dz}{du},$$

$$\frac{df}{dx} = \beta_1 + \beta_2 p, \quad \frac{df}{dy} = \gamma_1 + \beta_2 q, \quad \frac{df}{du} = \beta_2 \frac{dz}{du},$$

$$\frac{dc}{dx} = \alpha_2 + \alpha_3 p, \quad \frac{dc}{dy} = \beta_2 + \alpha_3 q, \quad \frac{dc}{du} = \alpha_3 \frac{dz}{du}.$$

The following relations, derived from these equations, obviously subsist among the derivatives of a, b, c, f, g, h with regard to x and y, viz.

$$\frac{dh}{dx} + q\frac{dg}{dx} = \frac{da}{dy} + p\frac{dg}{dy},$$

$$\frac{db}{dx} + q\frac{df}{dx} = \frac{dh}{dy} + p\frac{df}{dy},$$

$$\frac{df}{dx} + q\frac{dc}{dx} = \frac{dg}{dy} + p\frac{dc}{dy},$$

each of which is free from the derivatives of the third order. Again, the equations can be used to express all but one of the derivatives of the third order in terms of that one: expressing them in terms of β_1, we find

$$\alpha_0 = \frac{da}{dx} - \frac{p}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) - \beta_1\frac{p^2}{q},$$

$$\alpha_1 = \frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + \beta_1\frac{p}{q},$$

$$\alpha_2 = \frac{1}{q}\frac{dq}{dy} - \beta_1\frac{1}{q},$$

$$\alpha_3 = \frac{1}{p}\frac{dc}{dx} - \frac{1}{pq}\frac{dg}{dy} + \beta_1\frac{1}{pq},$$

$$\beta_0 = \frac{dh}{dx} - \beta_1 p,$$

$$\beta_2 = \frac{1}{p}\frac{df}{dx} - \beta_1\frac{1}{p},$$

$$\gamma_0 = \frac{dh}{dy} - \beta_1 q,$$

$$\gamma_1 = \frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right) + \beta_1\frac{q}{p},$$

$$\delta_0 = \frac{db}{dy} - \frac{q}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right) - \beta_1\frac{q^2}{p}.$$

Other expressions are obtainable: but they are equivalent to this set, in virtue of the earlier three relations. Moreover, the value of β_1 is taken to be

$$\frac{dh}{du} \div \frac{dz}{du}.$$

When the values of the derivatives of v of the third order, as expressed in terms of β_1, are substituted in the three equations which arise as first derivatives of $\phi = 0$, we have

$$X + A\left\{\frac{da}{dx} - \frac{p}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right)\right\} + H\frac{dh}{dx} + B\frac{dh}{dy}$$
$$+ G\frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + C\frac{1}{q}\frac{dq}{dy} - \frac{\Delta}{q}\beta_1 = 0,$$

$$Y + A\frac{dh}{dx} + H\frac{dh}{dy} + B\left\{\frac{db}{dy} - \frac{q}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right)\right\}$$
$$+ F\frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right) + C\frac{1}{p}\frac{df}{dx} - \frac{\Delta}{p}\beta_1 = 0,$$

$$Z + A\frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + B\frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right)$$
$$+ G\frac{1}{q}\frac{dq}{dy} + F\frac{1}{p}\frac{df}{dx} + C\left(\frac{1}{p}\frac{dc}{dx} - \frac{1}{pq}\frac{dg}{dy}\right) + \frac{\Delta}{pq}\beta_1 = 0,$$

where

$$\Delta = Ap^2 + Hpq + Bq^2 - Gp - Fq + C;$$

and, in each of the three equations, β_1 has the previously mentioned value.

331. At this stage, there are two distinct courses of reasoning which, while leading to the same subsidiary equations, give divided significance to the equations.

According to the considerations in the first of these courses, we use the freedom, given by the fact that the new variable u has remained arbitrary, to impose a condition: we suppose u to be chosen so that $\Delta = 0$, that is, we have

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0.$$

The term in β_1 disappears from the three equations: and these equations, after slight modifications, can be made to assume the forms

$$E_1 = X + A\left(\frac{da}{dx} - p\frac{dg}{dx}\right) + H\left(\frac{dh}{dx} - p\frac{dg}{dy}\right) + B\left(\frac{dh}{dy} - q\frac{dg}{dy}\right)$$
$$+ G\frac{dg}{dx} + F\frac{dg}{dy} = 0,$$

$$E_2 = Y + A\left(\frac{dh}{dx} - p\frac{df}{dx}\right) + H\left(\frac{db}{dx} - p\frac{df}{dy}\right) + B\left(\frac{db}{dy} - q\frac{df}{dy}\right)$$
$$+ G\frac{df}{dx} + F\frac{df}{dy} = 0,$$

$$E_3 = Z + A\left(\frac{dg}{dx} - p\frac{dc}{dx}\right) + H\left(\frac{df}{dx} - p\frac{dc}{dy}\right) + B\left(\frac{df}{dy} - q\frac{dc}{dy}\right) + G\frac{dc}{dx} + F\frac{dc}{dy} = 0.$$

According to the considerations in the second course of reasoning, we assume that the equations of the integral can be expressed in finite form and that the variable u is an argument of an arbitrary function in the integral. Then, as $\frac{dh}{du}$ in the value of β_1 involves a differentiation with regard to u higher than any which occurs in any other term, and as the equations are to be satisfied identically, the term in β_1 must disappear in and by itself from each equation: hence

$$\Delta = 0,$$

being the same conclusion as before. The remaining parts of the equations must also vanish: they have already been given in the forms

$$E_1 = 0, \quad E_2 = 0, \quad E_3 = 0.$$

332. The quantities, which occur in these simultaneous subsidiary equations and which have to be determined for our present purpose, are eleven in number, viz., $a, b, c, f, g, h, l, m, n, v, z$: they are functions of x, y, u. Omitting initially those equations in which derivatives with regard to u occur, the eleven quantities are to be functions of x and y. The constants, which arise in the integration, are made functions of u; and arbitrary functions, which are introduced, involve u: the forms of the functions must be such that the equations containing derivatives of u are satisfied.

The equations for the determination of the eleven unknown quantities are simultaneous partial equations of the first order. Among them, we have

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq,$$

$$\frac{dl}{dx} = a + gp, \quad \frac{dl}{dy} = h + gq,$$

$$\frac{dm}{dx} = h + fp, \quad \frac{dm}{dy} = b + fq,$$

$$\frac{dn}{dx} = g + cp, \quad \frac{dn}{dy} = f + cq,$$

which are equivalent to seven in all, because the relation

$$\frac{dl}{dy}+p\frac{dn}{dy}=\frac{dm}{dx}+q\frac{dn}{dx}$$

is identically satisfied by the foregoing values of the derivatives of l, m, n. Moreover, the relations

$$\frac{dh}{dx}+q\frac{dg}{dx}=\frac{da}{dy}+p\frac{dg}{dy},$$

$$\frac{db}{dx}+q\frac{df}{dx}=\frac{dh}{dy}+p\frac{df}{dy},$$

$$\frac{df}{dx}+q\frac{dc}{dx}=\frac{dg}{dy}+p\frac{dc}{dy},$$

are deducible from the preceding relations by substituting in

$$\frac{d}{dy}\left(\frac{dl}{dx}\right)=\frac{d}{dx}\left(\frac{dl}{dy}\right),\quad \frac{d}{dy}\left(\frac{dm}{dx}\right)=\frac{d}{dx}\left(\frac{dm}{dy}\right),\quad \frac{d}{dy}\left(\frac{dn}{dx}\right)=\frac{d}{dx}\left(\frac{dn}{dy}\right):$$

so that they are not independent equations. Consequently, on the score of this set of equations, there are seven which are independent of one another: they are a universal set, belonging to all equations of the type under consideration.

The remaining equations in the system belong specially to the equation $\phi=0$: they are

$$\Delta=0,\quad E_1=0,\quad E_2=0,\quad E_3=0,$$

being four in number.

Hence the tale of independent equations in the subsidiary system is eleven, the same as the number of quantities to be determined at this stage.

The original equation $\phi=0$ is an integral of the system. For the effective solution of the system, ten other integrals would be required: in particular cases, the process can be appreciably shortened.

Ex. 1. Consider the equation

$$a+f-g-h+\frac{2l-m-n}{y+z}=0,$$

which has no intermediate integral. Here

$$\Delta=p^2-pq+p-q=0,$$

so that we have two cases to consider, viz.

$$p+1=0,\quad p-q=0.$$

The first of these gives

$$y=\text{function of } x+z,$$

when the appropriate argument u is constant: the second gives

$$z=\text{function of } x+y,$$

when the other appropriate argument u is constant.

The three equations $E_1=0$, $E_2=0$, $E_3=0$, are

$$\frac{da}{dx}-\frac{dh}{dx}-(p+1)\left(\frac{dg}{dx}-\frac{dg}{dy}\right)+\frac{2a-h-g}{y+z}=0,$$

$$\frac{dh}{dx}-\frac{db}{dx}-(p+1)\left(\frac{df}{dx}-\frac{df}{dy}\right)+\frac{2h-b-f}{y+z}-\frac{2l-m-n}{(y+z)^2}=0,$$

$$\frac{dg}{dx}-\frac{df}{dx}-(p+1)\left(\frac{dc}{dx}-\frac{dc}{dy}\right)+\frac{2g-f-c}{y+z}-\frac{2l-m-n}{(y+z)^2}=0.$$

First, let

$$p+1=0:$$

then, from the first and the third of the equations, we have

$$\frac{d}{dx}(a-h-g+f)+\frac{2a-h-g-2g+f+c}{y+z}+\frac{2l-m-n}{(y+z)^2}=0.$$

Now

$$\begin{aligned}2a-h-g-2g+f+c&=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)(2l-m-n)\\&=\left(\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}\right)(2l-m-n)\\&=\frac{d}{dx}(2l-m-n),\end{aligned}$$

and

$$\frac{d}{dx}\left(\frac{1}{y+z}\right)=-p\frac{1}{(y+z)^2}=\frac{1}{(y+z)^2};$$

hence

$$\frac{d}{dx}(a-h-g+f)+\frac{d}{dx}\left(\frac{2l-m-n}{y+z}\right)=0.$$

We thus recover the original differential equation which is an integral of the system: the arbitrary function, which otherwise would enter, is made definite by the equation: in fact,

$$a-h-g+f+\frac{2l-m-n}{y+z}=0.$$

When this integral is used to eliminate $2l-m-n$ from the second equation, the latter becomes

$$\frac{d}{dx}(h-b)+\frac{a+h-b-g}{y+z}=0.$$

Combining this with the first equation, we have

$$\frac{d}{dx}(a-2h+b)+\frac{a-2h+b}{y+z}=0:$$

hence, as $p+1=0$,

$$\frac{d}{dx}\left(\frac{a-2h+b}{y+z}\right)=0.$$

Consequently,

$$\frac{a-2h+b}{y+z}=\text{arbitrary function of } u:$$

and, from $p+1=0$, we have

$$y=\text{function of } x+z,\ u:$$

hence, eliminating u, we have

$$\frac{a-2h+b}{y+z}=\theta(x+z,\ y),$$

where θ is any arbitrary function. Let another arbitrary function χ of $x+z$ and y be chosen, such that

$$\chi=\chi(x+z,\ y)=\chi(\eta,\ y),$$

and

$$\theta=\frac{\partial^3\chi}{\partial\eta^3}-3\frac{\partial^3\chi}{\partial\eta^2\partial y}+3\frac{\partial^3\chi}{\partial\eta\partial y^2}-\frac{\partial^3\chi}{\partial y^3}:$$

then our integral is

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(l-m)=(y+z)\left\{\frac{\partial^3\chi}{\partial\eta^3}-3\frac{\partial^3\chi}{\partial\eta^2\partial y}+3\frac{\partial^3\chi}{\partial\eta\partial y^2}-\frac{\partial^3\chi}{\partial y^3}\right\}.$$

A first integral of this equation is

$$l-m=(y+z)\left(\frac{\partial^2\chi}{\partial\eta^2}-2\frac{\partial^2\chi}{\partial\eta\partial y}+\frac{\partial^2\chi}{\partial y^2}\right)+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi(x+y,\ z),$$

where ψ is an arbitrary function of its arguments. This is an equation of the first order: but it is not an intermediate integral of the original equation in the ordinary sense of that term, for it contains two arbitrary functions: and the original equation cannot be derived from this equation alone.

Further integration leads to the relation

$$v=(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right)+2\chi+x\psi(x+y,\ z)+\Psi(x+y,\ z),$$

where Ψ is an arbitrary function of its arguments. This equation is of the nature of a primitive: but the number of arbitrary functions is three, being too great* by one unit: and there are various ways of reducing the number. Perhaps the simplest of these ways is to notice that the original equation is symmetrical in y and z, so that the integral can be expected to have the same symmetry. Accordingly, let

$$x+y=\zeta,$$
$$\vartheta=\vartheta(\zeta,\ z);$$

* Their presence is due to the fact that the equation really is the primitive of the equation of the third order

$$\frac{d}{dx}\left(\frac{a-2h+b}{y+z}\right)=0,$$

which is compatible with the given equation.

then, taking

$$\Psi(\zeta, z)+\{\zeta+z-(y+z)\}\psi(\zeta, z)$$
$$=(y+z)\left(\frac{\partial\vartheta}{\partial\zeta}-\frac{\partial\vartheta}{\partial z}\right)+2\vartheta,$$

we have

$$v=2\chi+2\vartheta+(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\frac{\partial\vartheta}{\partial\zeta}-\frac{\partial\vartheta}{\partial z}\right),$$

where χ and ϑ are arbitrary functions.

Next, consider the equation

$$p-q=0:$$

then the three equations of general identity take the form

$$\frac{dh}{dx}+p\frac{dg}{dx}=\frac{da}{dy}+p\frac{dg}{dy},$$
$$\frac{db}{dx}+p\frac{df}{dx}=\frac{dh}{dy}+p\frac{df}{dy},$$
$$\frac{df}{dx}+p\frac{dc}{dx}=\frac{dg}{dy}+p\frac{dc}{dy}.$$

Using these relations to eliminate the terms in p from the equations which are particular to the present example, we have

$$\frac{d}{dx}(a-g)-\frac{d}{dy}(a-g)+\frac{2a-h-g}{y+z}=0,$$
$$\frac{d}{dx}(h-f)-\frac{d}{dy}(h-f)+\frac{2h-b-f}{y+z}-\frac{2l-m-n}{(y+z)^2}=0,$$
$$\frac{d}{dx}(g-c)-\frac{d}{dy}(g-c)+\frac{2g-f-c}{y+z}-\frac{2l-m-n}{(y+z)^2}=0.$$

Proceeding as before, we can recover the original differential equation. When it is used to eliminate the term in $2l-m-n$ from the second and the third of these equations, they become

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(h-f)+\frac{a+h-b-g}{y+z}=0,$$
$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(g-c)+\frac{a-h+g-c}{y+z}=0.$$

From the first of the former three and the second of the latter two, we find

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(a-2g+c)+\frac{a-2g+c}{y+z}=0.$$

But as $p=q$, we have

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(y+z)=-1,$$

and therefore

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)\left(\frac{a-2g+c}{y+z}\right)=0,$$

so that

$$\frac{a-2g+c}{y+z}=\text{arbitrary function of } x+y,\ u',$$

where

$$x+y=\text{arbitrary function of } z,\ u'.$$

Eliminating u', we have

$$\frac{a-2g+c}{y+z}=\mu\,(x+y,\ z),$$

where μ is an arbitrary function of its arguments. Let

$$x+y=\zeta,$$

and introduce a new arbitrary function $\psi'(\zeta, z)$, such that

$$\mu=\frac{\partial^3\psi'}{\partial\zeta^3}-3\,\frac{\partial^3\psi'}{\partial\zeta^2\partial z}+3\,\frac{\partial^3\psi'}{\partial\zeta\partial z^2}-\frac{\partial^3\psi'}{\partial z^3}:$$

then our integral is

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)(l-n)=(y+z)\left\{\frac{\partial^3\psi'}{\partial\zeta^3}-3\,\frac{\partial^3\psi'}{\partial\zeta^2\partial z}+3\,\frac{\partial^3\psi'}{\partial\zeta\partial z^2}-\frac{\partial^3\psi'}{\partial z^3}\right\}.$$

A first integral of this equation is

$$l-n=(y+z)\left(\frac{\partial^2\psi'}{\partial\zeta^2}-2\,\frac{\partial^2\psi'}{\partial\zeta\partial z}+\frac{\partial^2\psi'}{\partial z^2}\right)+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma\,(x+z,\ y),$$

where σ is an arbitrary function of its arguments. This is an equation of the first order: but, for precisely the same reasons as were applied in the earlier case, it is not an intermediate integral of the equation in the ordinary sense of the term.

Further integration leads to the relation

$$v=(y+z)\left(\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial z}\right)+2\psi'+x\sigma\,(x+z,\ y)+\Sigma\,(x+z,\ y),$$

where Σ is an arbitrary function of its arguments. This equation is of the nature of a primitive: as before, the number* of arbitrary functions is too great by one unit. Effecting the necessary reductions by making the integral symmetrical in y and z, because the original equation is symmetrical in those variables, let

$$x+z=\eta$$

then, taking

$$\chi=\chi\,(\eta,\ y);$$

$$\Sigma\,(\eta,\ y)+\{\eta+y-(y+z)\}\,\sigma\,(\eta,\ y)$$

$$=(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right)+2\chi,$$

* The equation is really the primitive of

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)\left(\frac{a-2g+c}{y+z}\right)=0,$$

which is an equation of the third order, compatible with the given equation and having three arbitrary functions in its primitive.

we have

$$v=2\chi+2\psi'+(y+z)\left(\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial z}+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right).$$

Another way of proceeding is as follows. The two equations of the first order are

$$l-m=(y+z)\left(\frac{\partial^2\chi}{\partial\eta^2}-2\frac{\partial^2\chi}{\partial\eta\partial y}+\frac{\partial^2\chi}{\partial y^2}\right)+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi(x+y,\ z),$$

$$l-n=(y+z)\left(\frac{\partial^2\psi'}{\partial\zeta^2}-2\frac{\partial^2\psi'}{\partial\zeta\partial z}+\frac{\partial^2\psi'}{\partial z^2}\right)+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma(x+z,\ y);$$

in order that they may coexist, they must satisfy the Poisson-Jacobi condition which, when developed, gives

$$\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma=0.$$

Taking account of the arguments of the functions and of the fact that the functions are arbitrary, we see that this condition can be satisfied only if

$$-\sigma=\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y},$$

$$-\psi=\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}.$$

Assuming these relations satisfied, we have the two equations of the first order coexisting with one another in the form

$$l-m=(y+z)(\chi_{11}-2\chi_{12}+\chi_{22})+\chi_1-\chi_2-(\psi_1'-\psi_2'),$$

$$l-n=(y+z)(\psi'_{11}-2\psi'_{12}+\psi'_{22})+\psi_1'-\psi_2'-(\chi_1-\chi_2),$$

where

$$\chi_1=\frac{\partial\chi}{\partial\eta},\quad \chi_{11}=\frac{\partial^2\chi}{\partial\eta^2},\quad \chi_2=\frac{\partial\chi}{\partial y},$$

and so on: and

$$\chi=\chi(\eta,\ y)=\chi(x+z,\ y),$$

$$\psi'=\psi(\zeta,\ z)=\psi(x+y,\ z).$$

Even so, neither of the equations is an intermediate integral.

Taken as a pair of equations in a single dependent variable, their common integral can be obtained by any of the regular methods in Chapter IV: it is found to be

$$v=2\chi+2\psi'+(y+z)(\chi_1-\chi_2+\psi_1-\psi_2),$$

in the preceding notation.

Ex. 2. Obtain the primitive of the equation

$$a-h-g+f+\frac{l-n+\beta(l-m)}{a(x+y+z)+y+\beta z}=0,$$

where a and β are constants.

INFLUENCE, UPON THE INTEGRAL, OF THE RESOLUBILITY OF THE CHARACTERISTIC INVARIANT.

333. In the two examples that have been given, the characteristic invariant $\Delta = 0$ has been resoluble into two linear equations: these, taken in turn with other equations of the system, have led to integrable forms. There is, however, no indication of systematic method to be pursued in the quest of such forms: and some systematic method must be devised if the process is to be effective. Before proceeding to the discussion of such a method, it is advisable to indicate a classification of equations of the second order determined by the resolubility or the non-resolubility of $\Delta = 0$ into two linear equations.

When $\Delta = 0$ is resoluble into two linear equations, the subsidiary equations are of Lagrange's linear form so far as concerns derivatives of a, b, c, f, g, h: their integral is such that some combination θ of variables can be an arbitrary function of some other combination χ. But the equations themselves subsist on the condition that some other combination ψ of the variables is constant, and so this combination ψ is latent in the preceding relation. When explicit account is taken of it, the integral has a form

$$\theta = \Theta(\chi, \psi),$$

where Θ is an arbitrary function: this equation can coexist with the original equation. Hence, when the method is effective because the integral of the equation is expressible in finite terms, we infer that, *if $\Delta = 0$ is a resoluble equation, arbitrary functions of two arguments occur in the most general integral equivalent of the original equation in three independent variables.*

The converse also is true: *if an integral relation in three independent variables involves at least one arbitrary function of two distinct arguments and if it is equivalent to a partial differential equation of the second order free from arbitrary functional forms, the characteristic invariant can be resolved into two linear equations.*

Let ξ and η denote two independent functions of x, y, z, so that not more than one of the three quantities

$$\xi_x \eta_y - \xi_y \eta_x, \quad \xi_y \eta_z - \xi_z \eta_y, \quad \xi_z \eta_x - \xi_x \eta_z$$

can vanish. We assume that ξ and η are the arguments of an arbitrary function which occurs in the integral: let it occur in the form

$$v = \Theta\{\ldots, \rho(\xi, \eta), \ldots\},$$

where ρ denotes the derivative of the arbitrary function in Θ which is of the highest order. Then

$$l = \frac{\partial\Theta}{\partial\rho}(\rho_1\xi_x + \rho_2\eta_x) + \text{other terms},$$

with similar expressions for m and n: also,

$$a = \frac{\partial\Theta}{\partial\rho}(\rho_{11}\xi_x^2 + 2\rho_{12}\xi_x\eta_x + \rho_{22}\eta_x^2) + \ldots,$$

$$f = \frac{\partial\Theta}{\partial\rho}\{\rho_{11}\xi_y\xi_z + \rho_{12}(\xi_y\eta_z + \xi_z\eta_y) + \rho_{22}\eta_y\eta_z\} + \ldots,$$

where the unexpressed terms involve derivatives of ρ of order lower than those expressed: and there are similar values for b, g, c, h. By hypothesis, the integral equation is to be equivalent to a partial differential equation of the second order

$$\phi(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0,$$

free from all arbitrary functional forms. Hence, when the preceding values of the derivatives are substituted in this equation which must be satisfied, the terms involving the various combinations of the arbitrary functions must disappear. Thus the highest power of ρ_{11} must disappear of itself: it disappears in the combination $\rho_{11}\dfrac{\partial\Theta}{\partial\rho}$: and the necessary condition is

$$\xi_x^2\frac{\partial\phi}{\partial a} + \xi_x\xi_y\frac{\partial\phi}{\partial h} + \xi_x\xi_z\frac{\partial\phi}{\partial g} + \xi_y^2\frac{\partial\phi}{\partial b} + \xi_y\xi_z\frac{\partial\phi}{\partial f} + \xi_z^2\frac{\partial\phi}{\partial c} = 0,$$

or, in the old notation,

$$A\xi_x^2 + H\xi_x\xi_y + G\xi_x\xi_z + B\xi_y^2 + F\xi_y\xi_z + C\xi_z^2 = 0.$$

Similarly, the highest power of ρ_{12} must disappear: it disappears in the combination $\rho_{12}\dfrac{\partial\Theta}{\partial\rho}$: and the necessary condition is

$$2A\xi_x\eta_x + H(\xi_x\eta_y + \xi_y\eta_x) + G(\xi_x\eta_z + \xi_z\eta_x) + 2B\xi_y\eta_y + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z = 0.$$

Similarly, the highest power of ρ_{22} must disappear, also in a combination $\rho_{22}\dfrac{\partial\Theta}{\partial\rho}$: the necessary condition is

$$A\eta_x^2 + H\eta_x\eta_y + G\eta_x\eta_z + B\eta_y^2 + F\eta_y\eta_z + C\eta_z^2 = 0.$$

Now these conditions give

$$(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)^2$$
$$= (\tfrac{1}{4}H^2 - AB)\,\xi_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\xi_y\xi_z + (\tfrac{1}{4}G^2 - AC)\,\xi_z^2,$$

$$(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)$$
$$= (\tfrac{1}{4}H^2 - AB)\,\xi_y\eta_y + (\tfrac{1}{4}GH - \tfrac{1}{2}AF)(\xi_y\eta_z + \xi_z\eta_y)$$
$$+ (\tfrac{1}{4}G^2 - AC)\,\xi_z\eta_z,$$

and

$$(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)^2$$
$$= (\tfrac{1}{4}H^2 - AB)\,\eta_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\eta_y\eta_z + (\tfrac{1}{4}G^2 - AC)\,\eta_z^2,$$

respectively. Squaring the middle equation, and subtracting the product of the first and third from that square, reducing, and removing a factor A, we have

$$I\,(\xi_y\eta_z - \xi_z\eta_y)^2 = 0,$$

where I is the discriminant of Δ.

Similarly, modifying the equations so as to obtain expressions for the squares and the product of

$$\tfrac{1}{2}H\xi_x + B\xi_y + \tfrac{1}{2}F\xi_z, \quad \tfrac{1}{2}H\eta_x + B\eta_y + \tfrac{1}{2}F\eta_z,$$

and proceeding as before, we should find

$$I\,(\xi_x\eta_z - \xi_z\eta_x)^2 = 0:$$

and a corresponding treatment of the three equations, with reference to

$$\tfrac{1}{2}G\xi_x + \tfrac{1}{2}F\xi_y + C\xi_z, \quad \tfrac{1}{2}G\eta_x + \tfrac{1}{2}F\eta_y + C\eta_z,$$

leads to a relation

$$I\,(\xi_x\eta_y - \xi_y\eta_x)^2 = 0.$$

We know that not more than one of the quantities

$$\xi_y\eta_z - \xi_z\eta_y, \quad \xi_x\eta_z - \xi_z\eta_x, \quad \xi_x\eta_y - \xi_y\eta_x,$$

can vanish: hence we must have

$$I = 0.$$

Consequently, $\Delta = 0$ is resoluble into two linear equations. The proposition is therefore established.

If however ρ, instead of being a function of two arguments ξ and η, is a function of only a single argument u, then we can deduce only a single equation

$$Au_x^2 + Hu_xu_y + Gu_xu_z + Bu_y^2 + Fu_yu_z + Cu_z^2 = 0;$$

and we cannot prove that

$$I = 0.$$

It does not follow that $\Delta = 0$ is not resoluble in particular cases: we cannot affirm that the circumstances in general do provide a resoluble characteristic invariant. Hence *when* $\Delta = 0$ *cannot be resolved into two equations linear in* p *and* q, *we infer that each of the arbitrary functions, which occur in the integral equivalent, has only a single argument.*

It thus appears that equations of the second order in three independent variables, whose integrals can be expressed in finite terms without essential partial quadratures, belong to one or other of two classes, according as the characteristic invariant $\Delta = 0$ can or cannot be resolved into two equations linear in p and q.

Equations having a Resoluble Characteristic Invariant.

334. Consider now more particularly those equations for which $\Delta = 0$ is resoluble into two linear equations or into a repeated linear equation. If a subsidiary system possesses an integrable combination, it is desirable to have a general method of determining the combination: and the method should indicate whether an integrable combination does or does not exist.

Assuming that the discriminant of $\Delta = 0$ vanishes, we first obtain the two linear equations into which the characteristic invariant is resolved. It will be sufficient for our purpose to assume that a occurs in the original equation $\phi = 0$, so that A does not vanish. We have, by $\Delta = 0$,

$$(Ap + \tfrac{1}{2}Hq - \tfrac{1}{2}G)^2 = (\tfrac{1}{4}H^2 - AB)\,q^2 + (AF - \tfrac{1}{2}GH)\,q + \tfrac{1}{4}G^2 - AC;$$

so that, writing

$$\tfrac{1}{4}H^2 - AB = \theta^2,$$

$$AF - \tfrac{1}{2}GH = -2\theta^2\vartheta,$$

and therefore

$$\tfrac{1}{4}G^2 - AC = \theta^2\vartheta^2,$$

on account of the vanishing of the discriminant, we find

$$Ap + \tfrac{1}{2}Hq - \tfrac{1}{2}G = \pm\,\theta\,(q - \vartheta).$$

Thus the two equations equivalent to $\Delta = 0$ are

$$\left.\begin{aligned} Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0 \\ Ap + (\tfrac{1}{2}H + \theta)\,q - (\tfrac{1}{2}G + \theta\vartheta) = 0 \end{aligned}\right\},$$

where

$$\theta^2 = \tfrac{1}{4}H^2 - AB, \qquad \vartheta = \frac{GH - 2AF}{H^2 - 4AB}.$$

Consider the subsidiary system associated with the linear equation

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0.$$

Take the equation $E_1 = 0$ of § 331: it is

$$X + A\frac{da}{dx} + H\frac{dh}{dx} + B\frac{dh}{dy} + G\frac{dg}{dx} + F\frac{dg}{dy}$$
$$- p\left(A\frac{dg}{dx} + H\frac{dg}{dy}\right) - qB\frac{dg}{dy} = 0.$$

Take also the first of the three equations of identity of § 332 in the form

$$\frac{da}{dy} - \frac{dh}{dx} + p\frac{dg}{dy} - q\frac{dg}{dx} = 0.$$

Multiply the latter by $\tfrac{1}{2}H - \theta$, and add to the former. In the resulting equation, the coefficient of $\dfrac{dg}{dx}$

$$\begin{aligned} &= G - Ap - (\tfrac{1}{2}H - \theta)\,q \\ &= \tfrac{1}{2}G + \theta\vartheta; \end{aligned}$$

the coefficient of $\dfrac{dg}{dy}$

$$\begin{aligned} &= F - Hp - Bq + (\tfrac{1}{2}H - \theta)\,p \\ &= F - \{(\tfrac{1}{2}H + \theta)\,p + Bq\} \\ &= - F - \frac{\tfrac{1}{2}H + \theta}{A}\{Ap + (\tfrac{1}{2}H - \theta)\,q\} \\ &= \frac{\tfrac{1}{2}GH - 2\theta^2\vartheta}{A} - \frac{\tfrac{1}{2}H + \theta}{A}(\tfrac{1}{2}G - \theta\vartheta) \\ &= \frac{1}{A}(\tfrac{1}{2}G + \theta\vartheta)(\tfrac{1}{2}H - \theta); \end{aligned}$$

and therefore the terms involving derivatives of g are

$$(\tfrac{1}{2}G + \theta\vartheta)\left\{\frac{dg}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{dg}{dy}\right\}.$$

The terms involving derivatives of a are

$$A\left\{\frac{da}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{da}{dy}\right\};$$

and those involving derivatives of h are

$$(\tfrac{1}{2}H + \theta)\frac{dh}{dx} + B\frac{dh}{dy}$$

$$= (\tfrac{1}{2}H + \theta)\left\{\frac{dh}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{dh}{dy}\right\}.$$

Hence, when we write

$$\delta = \frac{d}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{d}{dy},$$

the equation becomes

$$X + A\,\delta a + (\tfrac{1}{2}H + \theta)\,\delta h + (\tfrac{1}{2}G + \theta\vartheta)\,\delta g = 0.$$

The other equations $E_2 = 0$, $E_3 = 0$, can be similarly treated with the respective relations of identity: and so, connected with the linear equation

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0,$$

we have a system of three subsidiary equations free from p and q in the form

$$\left.\begin{array}{l} X + A\,\delta a + (\tfrac{1}{2}H + \theta)\,\delta h + (\tfrac{1}{2}G + \theta\vartheta)\,\delta g = 0 \\ Y + A\,\delta h + (\tfrac{1}{2}H + \theta)\,\delta b + (\tfrac{1}{2}G + \theta\vartheta)\,\delta f = 0 \\ Z + A\,\delta g + (\tfrac{1}{2}H + \theta)\,\delta f + (\tfrac{1}{2}G + \theta\vartheta)\,\delta c = 0 \end{array}\right\},$$

with the foregoing definition of δ.

Now suppose that

$$u(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is an integrable combination of these equations: then

$$\frac{du}{dx} = 0, \qquad \frac{du}{dy} = 0,$$

that is,

$$\frac{\partial u}{\partial a}\frac{da}{dx}+\ldots+\frac{\partial u}{\partial h}\frac{dh}{dx}+(a+gp)\frac{\partial u}{\partial l}+(h+fp)\frac{\partial u}{\partial m}+(g+cp)\frac{\partial u}{\partial n}$$
$$+(l+np)\frac{\partial u}{\partial v}+\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}=0,$$

$$\frac{\partial u}{\partial a}\frac{da}{dy}+\ldots+\frac{\partial u}{\partial h}\frac{dh}{dy}+(h+gq)\frac{\partial u}{\partial l}+(b+fq)\frac{\partial u}{\partial m}+(f+cq)\frac{\partial u}{\partial n}$$
$$+(m+nq)\frac{\partial u}{\partial v}+\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}=0.$$

Multiply the latter by $\frac{1}{A}(\tfrac{1}{2}H-\theta)$ and add to the former: where p and q occur in the resulting equation, it is in a combination

$$p+\frac{1}{A}(\tfrac{1}{2}H-\theta)\,q,$$

which (on account of the linear equation) we replace by

$$\frac{1}{A}(\tfrac{1}{2}G-\theta\vartheta);$$

the resulting equation then is

$$\frac{\partial u}{\partial a}\delta a+\frac{\partial u}{\partial b}\delta b+\frac{\partial u}{\partial c}\delta c+\frac{\partial u}{\partial f}\delta f+\frac{\partial u}{\partial g}\delta g+\frac{\partial u}{\partial h}\delta h$$
$$+\frac{\partial u}{\partial x}+\frac{1}{A}(\tfrac{1}{2}H-\theta)\frac{\partial u}{\partial y}+\frac{1}{A}(\tfrac{1}{2}G-\theta\vartheta)\frac{\partial u}{\partial z}$$
$$+\left\{l+\frac{m}{A}(\tfrac{1}{2}H-\theta)+\frac{n}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial v}$$
$$+\frac{\partial u}{\partial l}\left\{a+\frac{h}{A}(\tfrac{1}{2}H-\theta)+\frac{g}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}$$
$$+\left\{h+\frac{b}{A}(\tfrac{1}{2}H-\theta)+\frac{f}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial m}$$
$$+\left\{g+\frac{f}{A}(\tfrac{1}{2}H-\theta)+\frac{c}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial n}=0.$$

Substitute from the three earlier equations for δa, δb, δc, in terms of δf, δg, δh; then, as u is an integrable combination of those equations, the modified equation must become evanescent, so that the coefficients of δf, δg, and δh, as well as the aggregate of terms independent of them, must vanish. Thus

$$\left.\begin{aligned}
0 &= \frac{\partial u}{\partial h} - \frac{\frac{1}{2}H+\theta}{A}\frac{\partial u}{\partial a} - \frac{A}{\frac{1}{2}H+\theta}\frac{\partial u}{\partial b} \\
0 &= \frac{\partial u}{\partial g} - \frac{\frac{1}{2}G+\theta\vartheta}{A}\frac{\partial u}{\partial a} - \frac{A}{\frac{1}{2}G+\theta\vartheta}\frac{\partial u}{\partial c} \\
0 &= \frac{\partial u}{\partial f} - \frac{\frac{1}{2}G+\theta\vartheta}{\frac{1}{2}H+\theta}\frac{\partial u}{\partial b} - \frac{\frac{1}{2}H+\theta}{\frac{1}{2}G+\theta\vartheta}\frac{\partial u}{\partial c} \\
0 &= -\frac{X}{A} - \frac{Y}{\frac{1}{2}H+\theta} - \frac{Z}{\frac{1}{2}G+\theta\vartheta} + \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} + a\frac{\partial u}{\partial l} + h\frac{\partial u}{\partial m} + g\frac{\partial u}{\partial n} \\
&\quad + \frac{\frac{1}{2}H-\theta}{A}\left(\frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} + h\frac{\partial u}{\partial l} + b\frac{\partial u}{\partial m} + f\frac{\partial u}{\partial n}\right) \\
&\quad + \frac{\frac{1}{2}G-\theta\vartheta}{A}\left(\frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} + g\frac{\partial u}{\partial l} + f\frac{\partial u}{\partial m} + c\frac{\partial u}{\partial n}\right)
\end{aligned}\right\},$$

being a set of simultaneous equations for u, associated with

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0.$$

This system of four equations for u, each of which is linear of the first order, must be made a complete Jacobian system: when this is done, let it contain M equations. There are thirteen possible arguments for u: hence the system would possess

$$13 - M$$

algebraically independent integrals. Among these integrals must be included

$$\phi\,(a, b, c, f, g, h, l, m, n, v, x, y, z)$$

from the original equation, as well as the two distinct integrals of the set

$$\frac{dx}{A} = \frac{dy}{\frac{1}{2}H-\theta} = \frac{dz}{\frac{1}{2}G-\theta\vartheta},$$

say ξ, η. Putting these on one side, we should have to obtain

$$10 - M$$

algebraically independent integrals of the complete Jacobian system: they would be obtainable by known processes.

If $M = 10$, there is no integrable combination. If $M = 9$, which is a not uncommon case when the method is effective, there is one such integral. Let it be u: then

$$u = \psi\,(\xi, \eta),$$

where ψ is an arbitrary function, is an equation of the second order coexistent with the original equation

$$\phi = 0.$$

335. The preceding system is associated with one of the two linear equations into which $\Delta = 0$ is resoluble. A corresponding system is associated with the other equation

$$Ap + (\tfrac{1}{2}H + \theta)\, q - (\tfrac{1}{2}G + \theta\vartheta) = 0:$$

formally, it can be derived from the preceding system by changing the sign of θ. The process of constructing the integrable combination (if any) is the same as before: when an integral exists, it is given as an integral common to a complete Jacobian system of equations of the first order.

Ex. We shall briefly set out the subsidiary equations that arise in connection with

$$a - h - g + f + \frac{2l - m - n}{y + z} = 0,$$

which has already been considered.

In the case of this equation, $\Delta = 0$ is resoluble: it is

$$(p - q)(p + 1) = 0.$$

First, let the equation

$$p + 1 = 0$$

be taken: here

$$A = 1, \quad H = -1, \quad G = -1:$$

thus

$$\theta = -\tfrac{1}{2}, \quad \vartheta = -1.$$

With the notation of the general investigation, we have

$$\delta = \frac{d}{dx},$$

$$\tfrac{1}{2}H + \theta = -1, \quad \tfrac{1}{2}G + \theta\vartheta = 0:$$

and so three of the subsidiary equations are

$$X + \frac{da}{dx} - \frac{dh}{dx} = 0,$$

$$Y + \frac{dh}{dx} - \frac{db}{dx} = 0,$$

$$Z + \frac{dg}{dx} - \frac{df}{dx} = 0,$$

where

$$X=\frac{2a-h-g}{y+z},$$

$$Y=\frac{2h-b-f}{y+z}-\frac{2l-m-n}{(y+z)^2},$$

$$Z=\frac{2g-f-c}{y+z}-\frac{2l-m-n}{(y+z)^2}.$$

If

$$u=u(a,\ b,\ c,\ f,\ g,\ h,\ l,\ m,\ n,\ v,\ x,\ y,\ z)=0$$

be an integral combination of the above equations, it must happen that

$$\frac{du}{dx}=0$$

(with $p+1=0$) is a linear combination of the three equations: as q is undetermined, it is useless to consider $\frac{du}{dy}=0$. When therefore we substitute

$$\frac{da}{dx}=\frac{dh}{dx}-X,\quad \frac{db}{dx}=\frac{dh}{dx}+Y,\quad \frac{df}{dx}=\frac{dg}{dx}+Z,$$

in the equation

$$\frac{du}{dx}=0,$$

the modified form of the latter must become evanescent. The necessary and sufficient conditions are

$$\frac{\partial u}{\partial c}=0,\quad \frac{\partial u}{\partial f}+\frac{\partial u}{\partial g}=0,\quad \frac{\partial u}{\partial a}+\frac{\partial u}{\partial h}+\frac{\partial u}{\partial b}=0,$$

$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}+(a-g)\frac{\partial u}{\partial l}+(h-f)\frac{\partial u}{\partial m}+(g-c)\frac{\partial u}{\partial n}+(l-n)\frac{\partial u}{\partial v}$$
$$=X\frac{\partial u}{\partial a}-Y\frac{\partial u}{\partial b}-Z\frac{\partial u}{\partial f}.$$

This set of four simultaneous equations must be rendered complete by associating with it all the Poisson-Jacobi conditions which provide new equations. When thus made complete by the ordinary processes, it is equivalent to the set of nine equations

$$\frac{\partial u}{\partial v}=0,\quad \frac{\partial u}{\partial c}=0,\quad -\frac{\partial u}{\partial h}=\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b},$$

$$\frac{1}{2}\frac{\partial u}{\partial l}=-\frac{\partial u}{\partial m}=-\frac{\partial u}{\partial n}=\frac{1}{y+z}\frac{\partial u}{\partial f}=-\frac{1}{y+z}\frac{\partial u}{\partial g}=\frac{1}{y+z}\left(\frac{\partial u}{\partial a}-\frac{\partial u}{\partial b}\right),$$

$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}-\frac{2l-m-n}{(y+z)^2}\frac{\partial u}{\partial a}-\frac{2a-3h+b+f-g}{y+z}\frac{\partial u}{\partial b}=0.$$

This system involves thirteen variables: consequently, it possesses four algebraically independent integrals. They can be taken in the form

$$y,\quad x+z,$$

$$\frac{2a-3h+b+f-g}{y+z}+\frac{2l-m-n}{(y+z)^2},$$

$$a-h-g+f+\frac{2l-m-n}{y+z}.$$

The last is zero, owing to the original differential equation: when it is used to modify the third, the latter can be replaced by

$$\frac{a-2h+b}{y+z}.$$

Consequently, the most general integral of the subsidiary system is

$$\Phi\left(\frac{a-2h+b}{y+z},\ x+z,\ y\right)=0,$$

where Φ is arbitrary: an equivalent form is

$$\frac{a-2h+b}{y+z}=\theta\,(x+z,\ y),$$

where θ is arbitrary.

Next, take the alternative linear equation arising out of $\Delta=0$: it is

$$p-q=0.$$

Proceeding similarly to the construction of a system to be associated with this relation, we find that, if an equation

$$U(a,\ b,\ c,\ f,\ g,\ h,\ l,\ m,\ n,\ v,\ x,\ y,\ z)=0$$

exists simultaneously with the given equation, U is determined by the system

$$\frac{\partial U}{\partial b}=0,\quad \frac{\partial U}{\partial f}+\frac{\partial U}{\partial h}=0,\quad \frac{\partial U}{\partial a}+\frac{\partial U}{\partial g}+\frac{\partial U}{\partial c}=0,$$

$$\frac{\partial U}{\partial x}-\frac{\partial U}{\partial y}+(l-m)\frac{\partial U}{\partial v}+(a-h)\frac{\partial U}{\partial l}+(h-g)\frac{\partial U}{\partial m}+(g-f)\frac{\partial U}{\partial n}$$

$$=X\frac{\partial U}{\partial a}+Y\frac{\partial U}{\partial h}-Z\frac{\partial U}{\partial c},$$

with the earlier values of X, Y, Z. This system is to be made complete: when complete, it is found to possess

$$\frac{a-2g+c}{y+z}=\psi\,(x+y,\ z),$$

where ψ is arbitrary, as its most general integral.

336. When either of the subsidiary systems leads to a new equation of the second order, compatible with the original equation and involving one arbitrary function, it can be used as an equation initially propounded for integration. Frequently it will possess an intermediate integral involving one arbitrary function more than itself, that is, an equation of the first order involving a couple of arbitrary functions: and indeed, it may almost be expected, even though the original equation itself possesses no intermediate integral. Through the desired primitive, the dependent variable v (and therefore also its derivatives) involves a couple of arbitrary functions: hence any combination of v, l, m, n, will generally be expressible in terms of two arbitrary functions; and the preceding equation of the first order is of this type.

The integration of this equation of the first order leads to a primitive, often involving one other arbitrary function: substitution in the differential equation leads to relations between the arbitrary functions which reduce their number to two.

When both the subsidiary systems lead to new equations of the second order, we proceed similarly with each of them.

Examples have already been given.

It may however happen that neither of the subsidiary systems admits of an integral (other than the given equation) of the kind desired: the inference is that no equation of the second order containing only a single arbitrary function can be associated or is compatible, with the given equation. It may however be the case that some new equation of the third order—new, that is to say, in the sense that it is algebraically independent of the derivatives of the given equation—can be associated with the equation, and is such that its expression involves an arbitrary function: and this may occur for each of the two linear equations into which $\Delta = 0$ can be resolved.

Similarly, if there is no new equation of the third order, there may be a new equation of the fourth order of the kind desired, associable with (but algebraically independent of) the given equation, and involving an arbitrary function. And so on, precisely as in Darboux's method (Chapter XVIII) for partial equations of the second order in two independent variables: the object of the process is to find a new equation of finite order, which involves an arbitrary function, and which can be associated with the given equation while it is algebraically independent of the derivatives of the given equation.

We have assumed that the equation possesses no intermediate integral: if it did, the derivatives of that integral of appropriate order would arise, at each of the stages contemplated, as integrals at those stages. If, then, no new equation of finite order is compatible with the given equation*, the method ceases to be effective. In that case, the only result generally attainable at present seems to be that which occurs in the establishment of Cauchy's existence-theorem: the integral certainly contains two arbitrary functions, but its expression (in the form of a converging series) is not finite.

The process will be sufficiently illustrated by the analysis adapted to the construction of a new equation of the third order, independent of the derivatives of the given equation: we shall limit the discussion to that aim.

Compatible Equations of the Third Order constructed by Darboux's Method.

337. Accordingly, we assume that the equation

$$\phi(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is such that no equation of the first order and no equation of the second order can be found, compatible with $\phi = 0$ and involving only one arbitrary function: we have to find the subsidiary system or systems, which will give an equation (if any) of the third order, compatible with $\phi = 0$ and involving one arbitrary function. This new equation, when it exists, must be compatible with, and algebraically independent of, the three derivatives of $\phi = 0$ which, in the notation of § 330, are

$$X + A\alpha_0 + H\beta_0 + B\gamma_0 + G\alpha_1 + F\beta_1 + C\alpha_2 = 0,$$
$$Y + A\beta_0 + H\gamma_0 + B\delta_0 + G\beta_1 + F\gamma_1 + C\beta_2 = 0,$$
$$Z + A\alpha_1 + H\beta_1 + B\gamma_1 + G\alpha_2 + F\beta_2 + C\alpha_3 = 0.$$

The actual details are only an amplification of those given in the earlier cases: and so the explanations can be abbreviated.

When the proper values of v and of the deduced derivatives are substituted in $\phi = 0$, the new form is an identity: so that, when the latter is differentiated with regard to the independent variables, the results are identities. Having regard to the hypothesis

* A simple instance is given by the equation

$$a - h - g + f + \lambda\,\frac{2l - m - n}{y + z} = 0,$$

where λ is a positive constant other than an integer.

that first derivatives of $\phi=0$ have not led to an effective end, we form all the second derivatives: these are

$$\frac{d^2\phi}{dx^2}=0,\quad \frac{d^2\phi}{dy^2}=0,\quad \frac{d^2\phi}{dz^2}=0,$$
$$\frac{d^2\phi}{dy\,dz}=0,\quad \frac{d^2\phi}{dz\,dx}=0,\quad \frac{d^2\phi}{dx\,dy}=0.$$

These equations involve derivatives of v of the fourth order. These derivatives are fifteen in number: let them be denoted by $r_1, \ldots, r_{15}$, according to the scheme

$$r_1=\frac{\partial^4 v}{\partial x^4},\quad r_3=\frac{\partial^4 v}{\partial x^3\partial z},\quad r_6=\frac{\partial^4 v}{\partial x^2\partial z^2},\quad r_{10}=\frac{\partial^4 v}{\partial x\partial z^3},\quad r_{15}=\frac{\partial^4 v}{\partial z^4},$$
$$r_2=\frac{\partial^4 v}{\partial x^3\partial y},\quad r_5=\frac{\partial^4 v}{\partial x^2\partial y\partial z},\quad r_9=\frac{\partial^4 v}{\partial x\partial y\partial z^2},\quad r_{14}=\frac{\partial^4 v}{\partial y\partial z^3},$$
$$r_4=\frac{\partial^4 v}{\partial x^2\partial y^2},\quad r_8=\frac{\partial^4 v}{\partial x\partial y^2\partial z},\quad r_{13}=\frac{\partial^4 v}{\partial y^2\partial z^2},$$
$$r_7=\frac{\partial^4 v}{\partial x\partial y^3},\quad r_{12}=\frac{\partial^4 v}{\partial y^3\partial z},\quad r_{11}=\frac{\partial^4 v}{\partial y^4}.$$

Further, let (XX) denote the portion of $\frac{d^2\phi}{dx^2}$ which is free from derivatives of v of the fourth order, (XY) the corresponding portion of $\frac{d^2\phi}{dx\,dy}$, and so on. Then the six equations are

$$\begin{aligned}
(XX)+Ar_1+Hr_2+Gr_3+Br_4+Fr_5+Cr_6&=0,\\
(XY)+Ar_2+Hr_4+Gr_5+Br_7+Fr_8+Cr_9&=0,\\
(XZ)+Ar_3+Hr_5+Gr_6+Br_8+Fr_9+Cr_{10}&=0,\\
(YY)+Ar_4+Hr_7+Gr_8+Br_{11}+Fr_{12}+Cr_{13}&=0,\\
(YZ)+Ar_5+Hr_8+Gr_9+Br_{12}+Fr_{13}+Cr_{14}&=0,\\
(ZZ)+Ar_6+Hr_9+Gr_{10}+Br_{13}+Fr_{14}+Cr_{15}&=0.
\end{aligned}$$

As before, let the independent variables be changed from x, y, z, to x, y, u, where u is a function of x, y, z, as yet undetermined. Thus z is a function of x, y, u: partial derivatives with regard to the new variables will be denoted by $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{du}$; and, in particular, we write

$$p=\frac{dz}{dx},\quad q=\frac{dz}{dy}.$$

Now

$$\begin{aligned}
d\alpha_0&=r_1dx+r_2dy+r_3dz\\
&=(r_1+r_3p)\,dx+(r_2+r_3q)\,dy+r_3\frac{dz}{du}du,
\end{aligned}$$

and similarly for the others: hence

$$\frac{d\alpha_0}{dx} = r_1 + r_3 p, \quad \frac{d\alpha_0}{dy} = r_2 + r_3 q, \quad \frac{d\alpha_0}{dz} = r_3 \frac{dz}{du},$$

$$\frac{d\alpha_1}{dx} = r_3 + r_6 p, \quad \frac{d\alpha_1}{dy} = r_5 + r_6 q, \quad \frac{d\alpha_1}{dz} = r_6 \frac{dz}{du},$$

$$\frac{d\alpha_2}{dx} = r_6 + r_{10} p, \quad \frac{d\alpha_2}{dy} = r_9 + r_{10} q, \quad \frac{d\alpha_2}{dz} = r_{10} \frac{dz}{du},$$

$$\frac{d\alpha_3}{dx} = r_{10} + r_{15} p, \quad \frac{d\alpha_3}{dy} = r_{14} + r_{15} q, \quad \frac{d\alpha_3}{dz} = r_{15} \frac{dz}{du},$$

$$\frac{d\beta_0}{dx} = r_2 + r_5 p, \quad \frac{d\beta_0}{dy} = r_4 + r_5 q, \quad \frac{d\beta_0}{dz} = r_5 \frac{dz}{du},$$

$$\frac{d\beta_1}{dx} = r_5 + r_9 p, \quad \frac{d\beta_1}{dy} = r_8 + r_9 q, \quad \frac{d\beta_1}{dz} = r_9 \frac{dz}{du},$$

$$\frac{d\beta_2}{dx} = r_9 + r_{14} p, \quad \frac{d\beta_2}{dy} = r_{13} + r_{14} q, \quad \frac{d\beta_2}{dz} = r_{14} \frac{dz}{du},$$

$$\frac{d\gamma_0}{dx} = r_4 + r_8 p, \quad \frac{d\gamma_0}{dy} = r_7 + r_8 q, \quad \frac{d\gamma_0}{dz} = r_8 \frac{dz}{du},$$

$$\frac{d\gamma_1}{dx} = r_8 + r_{13} p, \quad \frac{d\gamma_1}{dy} = r_{12} + r_{13} q, \quad \frac{d\gamma_1}{dz} = r_{13} \frac{dz}{du},$$

$$\frac{d\delta_0}{dx} = r_7 + r_{12} p, \quad \frac{d\delta_0}{dy} = r_{11} + r_{12} q, \quad \frac{d\delta_0}{dz} = r_{12} \frac{dz}{du}.$$

There are twenty equations in the first two columns, involving the fifteen derivatives r; but they are equivalent to only fourteen independent equations in these derivatives, owing to the relations

$$\frac{d\beta_0}{dx} - \frac{d\alpha_0}{dy} = p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx},$$

$$\frac{d\beta_1}{dx} - \frac{d\alpha_1}{dy} = p\frac{d\alpha_2}{dy} - q\frac{d\alpha_2}{dx},$$

$$\frac{d\beta_2}{dx} - \frac{d\alpha_2}{dy} = p\frac{d\alpha_3}{dy} - q\frac{d\alpha_3}{dx},$$

$$\frac{d\gamma_0}{dx} - \frac{d\beta_0}{dy} = p\frac{d\beta_1}{dy} - q\frac{d\beta_1}{dx},$$

$$\frac{d\gamma_1}{dx} - \frac{d\beta_1}{dy} = p\frac{d\beta_2}{dy} - q\frac{d\beta_2}{dx},$$

$$\frac{d\delta_0}{dx} - \frac{d\gamma_0}{dy} = p\frac{d\gamma_1}{dy} - q\frac{d\gamma_1}{dx},$$

which are free from the derivatives r, and also are free from all derivatives with regard to u. The fourteen independent equations can then be used to express all but one of the derivatives r in terms of that one: the simplest forms of expression occur when the one is r_5, or r_8, or r_9: we choose r_5, and we have

$$r_1 = -r_5 \frac{p^2}{q} + \frac{p}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right) + \frac{d\alpha_0}{dx},$$

$$r_2 = -r_5 p + \frac{d\beta_0}{dx},$$

$$r_3 = r_5 \frac{p}{q} - \frac{1}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right),$$

$$r_4 = -r_5 q + \frac{d\beta_0}{dy},$$

$$r_6 = -r_5 \frac{1}{q} + \frac{1}{q}\frac{d\alpha_1}{dy},$$

$$r_7 = -r_5 \frac{q^2}{p} + \frac{q}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{d\gamma_0}{dy},$$

$$r_8 = r_5 \frac{q}{p} - \frac{1}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right),$$

$$r_9 = -r_5 \frac{1}{p} + \frac{1}{p}\frac{d\beta_1}{dx},$$

$$r_{10} = r_5 \frac{1}{pq} - \frac{1}{pq}\frac{d\alpha_1}{dy} + \frac{1}{p}\frac{d\alpha_2}{dx},$$

$$r_{11} = -r_5 \frac{q^3}{p^2} + \frac{q^2}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{q}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right) + \frac{d\delta_0}{dy},$$

$$r_{12} = r_5 \frac{q^2}{p^2} - \frac{q}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) - \frac{1}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right),$$

$$r_{13} = -r_5 \frac{q}{p^2} + \frac{1}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{1}{p}\frac{d\gamma_1}{dx},$$

$$r_{14} = r_5 \frac{1}{p^2} - \frac{1}{p^2}\frac{d\beta_1}{dx} + \frac{1}{p}\frac{d\beta_2}{dx},$$

$$r_{15} = -r_5 \frac{1}{qp^2} + \frac{1}{qp^2}\frac{d\beta_1}{dx} - \frac{1}{qp}\frac{d\beta_2}{dx} + \frac{1}{q}\frac{d\alpha_3}{dy}.$$

When these values of all the derivatives r, in terms of r_5, are substituted in the six equations arising through the second derivatives of $\phi = 0$, and when terms are collected, it appears that the terms in r_5 in the six equations are

$$-r_5\frac{1}{q}\Delta,\quad -r_5\frac{1}{p}\Delta,\quad r_5\frac{1}{pq}\Delta,\quad -r_5\frac{q}{p^2}\Delta,\quad r_5\frac{1}{p^2}\Delta,\quad -r_5\frac{1}{p^2q}\Delta,$$

respectively, where Δ is the same quantity as before. For the same reasons in two different courses of reasoning as in the former case (§ 331), so that they need not be repeated here, the term in r_5 is made to disappear from each of the equations: thus

$$\Delta = Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0,$$

which is the characteristic invariant. And then the six equations, after some simple changes, are

$$(XX) + A\left(\frac{d\alpha_0}{dx} - p\frac{d\alpha_1}{dx}\right) + H\left(\frac{d\beta_0}{dx} - p\frac{d\alpha_1}{dy}\right) + B\left(\frac{d\beta_0}{dy} - q\frac{d\alpha_1}{dy}\right)$$
$$+ G\frac{d\alpha_1}{dx} + F\frac{d\alpha_1}{dy} = 0,$$

$$(XY) + A\left(\frac{d\beta_0}{dx} - p\frac{d\beta_1}{dx}\right) + H\left(\frac{d\gamma_0}{dx} - p\frac{d\beta_1}{dy}\right) + B\left(\frac{d\gamma_0}{dy} - q\frac{d\beta_1}{dy}\right)$$
$$+ G\frac{d\beta_1}{dx} + F\frac{d\beta_1}{dy} = 0,$$

$$(XZ) + A\left(\frac{d\alpha_1}{dx} - p\frac{d\alpha_2}{dx}\right) + H\left(\frac{d\beta_1}{dx} - p\frac{d\alpha_2}{dy}\right) + B\left(\frac{d\beta_1}{dy} - q\frac{d\alpha_2}{dy}\right)$$
$$+ G\frac{d\alpha_2}{dx} + F\frac{d\alpha_2}{dy} = 0,$$

$$(YY) + A\left(\frac{d\gamma_0}{dx} - p\frac{d\gamma_1}{dx}\right) + H\left(\frac{d\delta_0}{dx} - p\frac{d\gamma_1}{dy}\right) + B\left(\frac{d\delta_0}{dy} - q\frac{d\gamma_1}{dy}\right)$$
$$+ G\frac{d\gamma_1}{dx} + F\frac{d\gamma_1}{dy} = 0,$$

$$(YZ) + A\left(\frac{d\beta_1}{dx} - p\frac{d\beta_2}{dx}\right) + H\left(\frac{d\gamma_1}{dx} - p\frac{d\beta_2}{dy}\right) + B\left(\frac{d\gamma_1}{dy} - q\frac{d\beta_2}{dy}\right)$$
$$+ G\frac{d\beta_2}{dx} + F\frac{d\beta_2}{dy} = 0,$$

$$(ZZ) + A\left(\frac{d\alpha_2}{dx} - p\frac{d\alpha_3}{dx}\right) + H\left(\frac{d\beta_2}{dx} - p\frac{d\alpha_3}{dy}\right) + B\left(\frac{d\beta_2}{dy} - q\frac{d\alpha_3}{dy}\right)$$
$$+ G\frac{d\alpha_3}{dx} + F\frac{d\alpha_3}{dy} = 0.$$

As regards the aggregate of differential equations in the system, we have all these new equations, as well as all the equations that arose in the earlier stage. The total number of quantities occurring as dependent variables is 21, made up of v, z, l, m, n, the six derivatives of v of the second order, and the ten derivatives of v of the third order. Of the total system of equations, we definitely know four integrals, viz.

$$\phi,\quad \frac{d\phi}{dx},\quad \frac{d\phi}{dy},\quad \frac{d\phi}{dz},$$

each of which vanishes on account of the initial equation. What is wanted is, if possible, some new integrable combination which is algebraically independent of these four.

338. Thus far the reasoning has not been influenced by the character of the invariant $\Delta=0$, and consequently it applies whether the invariant be resoluble or not. Now suppose that $\Delta=0$ is resoluble into two linear equations: and let one of them be

$$Ap+(\tfrac{1}{2}H-\theta)q-(\tfrac{1}{2}G-\theta\vartheta)=0,$$

where

$$\theta^2=\tfrac{1}{4}H^2-AB,\qquad \vartheta=\frac{GH-2AF}{H^2-4AB}.$$

Then combining the equations of identity with the equations particular to $\phi=0$ in the same way as before in § 334, and writing

$$\delta=\frac{d}{dx}+\frac{\tfrac{1}{2}H-\theta}{A}\frac{d}{dy},$$

we have

$$\begin{aligned}
(XX)+A\,\delta\alpha_0+(\tfrac{1}{2}H+\theta)\,\delta\beta_0+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\alpha_1&=0,\\
(XY)+A\,\delta\beta_0+(\tfrac{1}{2}H+\theta)\,\delta\gamma_0+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\beta_1&=0,\\
(XZ)+A\,\delta\alpha_1+(\tfrac{1}{2}H+\theta)\,\delta\beta_1+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\alpha_2&=0,\\
(YY)+A\,\delta\gamma_0+(\tfrac{1}{2}H+\theta)\,\delta\delta_0+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\gamma_1&=0,\\
(YZ)+A\,\delta\beta_1+(\tfrac{1}{2}H+\theta)\,\delta\gamma_1+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\beta_2&=0,\\
(ZZ)+A\,\delta\alpha_2+(\tfrac{1}{2}H+\theta)\,\delta\beta_2+(\tfrac{1}{2}G+\theta\vartheta)\,\delta\alpha_3&=0,
\end{aligned}$$

as an aggregate of equations included in the subsidiary system belonging to the linear equation that arises out of $\Delta=0$. Now let

$$E(\alpha_0,\ \ldots,\ \delta_0,\ a,\ \ldots,\ h,\ l,\ m,\ n,\ v,\ x,\ y,\ z)=0$$

be an integrable combination of the subsidiary system: then we must have

$$\Sigma \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dx} + \Sigma \frac{\partial E}{\partial a} \frac{da}{dx} + \Sigma \frac{\partial E}{\partial l} \frac{dl}{dx} + \frac{\partial E}{\partial v} (l + np) + \frac{\partial E}{\partial x} + p \frac{\partial E}{\partial z} = 0,$$

$$\Sigma \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dy} + \Sigma \frac{\partial E}{\partial a} \frac{da}{dy} + \Sigma \frac{\partial E}{\partial l} \frac{dl}{dy} + \frac{\partial E}{\partial v} (m + nq) + \frac{\partial E}{\partial y} + q \frac{\partial E}{\partial z} = 0,$$

where the respective summations extend over all the derivatives of the same order as the typical term. Multiplying the former by A, the latter by $\frac{1}{2}H - \theta$, and paying regard to the linear equation in p and q, we have

$$A \Sigma \frac{\partial E}{\partial \alpha_0} \delta\alpha_0 + A\,(E, x) + (\tfrac{1}{2}H - \theta)\,(E, y) + (\tfrac{1}{2}G - \theta\vartheta)\,(E, z) = 0,$$

where

$$(E, x) = \left(\frac{\partial}{\partial x} + l \frac{\partial}{\partial v} + a \frac{\partial}{\partial l} + h \frac{\partial}{\partial m} + g \frac{\partial}{\partial n} \right.$$
$$\left. + \alpha_0 \frac{\partial}{\partial a} + \beta_0 \frac{\partial}{\partial h} + \gamma_0 \frac{\partial}{\partial b} + \alpha_1 \frac{\partial}{\partial g} + \beta_1 \frac{\partial}{\partial f} + \alpha_2 \frac{\partial}{\partial c}\right) E,$$

$$(E, y) = \left(\frac{\partial}{\partial y} + m \frac{\partial}{\partial v} + h \frac{\partial}{\partial l} + b \frac{\partial}{\partial m} + f \frac{\partial}{\partial n} \right.$$
$$\left. + \beta_0 \frac{\partial}{\partial a} + \gamma_0 \frac{\partial}{\partial h} + \delta_0 \frac{\partial}{\partial b} + \beta_1 \frac{\partial}{\partial g} + \gamma_1 \frac{\partial}{\partial f} + \beta_2 \frac{\partial}{\partial c}\right) E,$$

$$(E, z) = \left(\frac{\partial}{\partial z} + n \frac{\partial}{\partial v} + g \frac{\partial}{\partial l} + f \frac{\partial}{\partial m} + c \frac{\partial}{\partial n} \right.$$
$$\left. + \alpha_1 \frac{\partial}{\partial a} + \beta_1 \frac{\partial}{\partial h} + \gamma_1 \frac{\partial}{\partial b} + \alpha_2 \frac{\partial}{\partial g} + \beta_2 \frac{\partial}{\partial f} + \alpha_3 \frac{\partial}{\partial c}\right) E.$$

As $E = 0$ is an integrable combination of the subsidiary system, and as the deduced equation satisfied by E does not contain either p or q, this deduced equation must be a linear combination of the preceding six equations, so that it must be expressible in a form

$$\lambda_1 \{(XX) + \ldots\} + \lambda_2 \{(XY) + \ldots\} + \lambda_3 \{(XZ) + \ldots\}$$
$$+ \lambda_4 \{(YY) + \ldots\} + \lambda_5 \{(YZ) + \ldots\} + \lambda_6 \{(ZZ) + \ldots\} = 0,$$

with appropriate values of the indeterminate multipliers $\lambda_1, \ldots, \lambda_6$. The conditions, necessary and sufficient to secure this result, are

$$0 = \rho^3 \frac{\partial E}{\partial \alpha_0} - \rho^2 \frac{\partial E}{\partial \beta_0} + \rho \frac{\partial E}{\partial \gamma_0} - \frac{\partial E}{\partial \delta_0},$$

$$0 = \sigma^3 \frac{\partial E}{\partial \alpha_0} - \sigma^2 \frac{\partial E}{\partial \alpha_1} + \sigma \frac{\partial E}{\partial \alpha_2} - \frac{\partial E}{\partial \alpha_3},$$

$$0 = \sigma^3 \frac{\partial E}{\partial \delta_0} - \sigma^2 \rho \frac{\partial E}{\partial \gamma_1} + \sigma \rho^2 \frac{\partial E}{\partial \beta_2} - \rho^3 \frac{\partial E}{\partial \alpha_3},$$

$$0 = 2\rho\sigma \frac{\partial E}{\partial \alpha_0} - \sigma \frac{\partial E}{\partial \beta_0} - \rho \frac{\partial E}{\partial \alpha_1} + \frac{\partial E}{\partial \beta_1} - \frac{1}{\rho} \frac{\partial E}{\partial \gamma_1} + \sigma \frac{\partial E}{\partial \delta_0},$$

$$0 = (XX) \frac{\partial E}{\partial \alpha_0} + (XY) \left\{ \frac{\partial E}{\partial \beta_0} - \rho \frac{\partial E}{\partial \alpha_0} \right\} + (XZ) \left\{ \frac{\partial E}{\partial \alpha_1} - \sigma \frac{\partial E}{\partial \alpha_0} \right\}$$

$$+ (YY) \frac{1}{\rho} \frac{\partial E}{\partial \delta_0} + (YZ) \left\{ \frac{1}{\sigma} \frac{\partial E}{\partial \beta_2} - \frac{\rho}{\sigma^2} \frac{\partial E}{\partial \alpha_3} \right\} + (ZZ) \frac{1}{\sigma} \frac{\partial E}{\partial \alpha_3},$$

where

$$\rho A = \tfrac{1}{2} H + \theta, \quad \sigma A = \tfrac{1}{2} G + \theta \vartheta.$$

This system of simultaneous equations of the first order must be made complete through the addition of all new equations which are provided by the Poisson-Jacobi conditions. The number of variables, which can occur in E, is 23: hence, if the complete system contains μ equations, the number of algebraically independent integrals is

$$23 - \mu.$$

Among these integrals will be found:

(i) the quantity ϕ which occurs in the original differential equation $\phi = 0$;

(ii) the three derivatives $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, all of which vanish in virtue of that differential equation;

(iii) the two integrals (say ξ and η) of the equations

$$\frac{dx}{A} = \frac{dy}{\frac{1}{2}H - \theta} = \frac{dz}{\frac{1}{2}G - \theta\vartheta},$$

which are subsidiary to the integration of the linear component of $\Delta = 0$.

Putting these on one side, there remain $17 - \mu$ new algebraically independent integrals: thus μ cannot be greater than 16, if the process is to be effective at this stage. If, when $\mu=16$, the new integral be denoted by u, then

$$u=\psi(\xi, \eta),$$

ψ being an arbitrary function of its arguments, is an equation of the third order that can be associated with the given equation.

Note I. The use made of this equation of the third order is similar to that made of the new equation of the second order in § 336: it can lead to a primitive, the excessive number of arbitrary functions, which it contains initially, being reduced to their proper tale of two by means of the differential equation.

Note II. The preceding analysis is associated with one of the two linear equations which are supposed to be the equivalent of $\Delta=0$: the corresponding set of subsidiary equations for the other linear equation is obtainable by changing the sign of θ throughout.

When this subsidiary system leads to a new equation of the third order, it can be treated in the same way as the preceding equation. If it should happen that each subsidiary system leads to such an equation, and if each such equation leads to an equation of the first order involving a superfluous number of arbitrary functions, the Poisson-Jacobi condition of coexistence affords a means of making the due reduction in the number.

Ex. 1. Apply the preceding process to the equation

$$a-h-g+f+2\frac{2l-m-n}{y+z}=0,$$

obtaining the equations of the third order

$$\alpha_0-3\beta_0+3\gamma_0-\delta_0=(y+z)\,\phi\,(x+z,\ y),$$
$$a_0-3a_1+3a_2-a_3=(y+z)\,\psi\,(x+y,\ z),$$

where ϕ and ψ are arbitrary, these equations being associable with the original equation of the second order.

Construct the general primitive.

Ex. 2. Denoting $F(\alpha, \beta)$ and $G(\alpha', \beta')$ respectively by F and G, where

$$\alpha=x+y,\quad \beta=z,\quad \alpha'=x+z,\quad \beta'=y;$$

also denoting by δ and by δ' respectively the operations

$$\delta=\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta},\quad \delta'=\frac{\partial}{\partial\alpha'}-\frac{\partial}{\partial\beta'};$$

prove that the integral equation

$$v = \sum_{s=0}^{\mu} \left\{ \frac{\mu!\,(2\mu-s)!}{(\mu-s)!\;s!\;2\mu!} (y+z)^s (\delta^s F + \delta'^s G) \right\},$$

where μ is a positive integer, satisfies the partial equation of the second order

$$a - h - g + f + \mu \frac{2l - m - n}{y+z} = 0.$$

Ex. 3. Discuss the equation

$$a - h - g + f + \mu \frac{2l - m - n}{y+z} = 0,$$

where μ is a negative integer: and obtain a primitive in the particular case when $\mu = -1$.

339. When the characteristic invariant $\Delta = 0$ cannot be resolved into linear equations, it is not possible to give so detailed a development of the subsidiary equations as in the preceding sections. We must fall back upon the theorem in § 329 and obtain, if possible, an integral of the subsidiary equations there given, or an integral of the equations in § 337 which hold whether $\Delta = 0$ be resoluble or not.

No general process, at present known, will apply to the simultaneous partial equations in a number of dependent variables: the equations, either in form or in number, do not admit of the application of Hamburger's process; and until some process can be devised, which is generally effective for such systems of equations, each attempt to construct an integral of them must be special to the equation under consideration.

We proceed to one or two examples, selecting equations in mathematical physics for this purpose.

Application to Laplace's Equation.

340. Consider Laplace's equation

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0,$$

which, in the notation of the present chapter, is

$$a + b + c = 0.$$

The characteristic invariant, being

$$p^2 + q^2 + 1 = 0,$$

is irresoluble.

When we substitute

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2 c, \quad b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2 c,$$

in the differential equation and make the resulting form evanescent quà equation in c, we have

$$\frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} = 0,$$

$$p^2 + q^2 + 1 = 0:$$

also, we have generally

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n\frac{dz}{du},$$

$$\frac{dm}{dx} + q\frac{dn}{dx} = \frac{dl}{dy} + p\frac{dn}{dy};$$

these are the aggregate of the subsidiary equations. Now a primitive of the equation

$$p^2 + q^2 + 1 = 0$$

is

$$z + ix\cos\alpha + iy\sin\alpha = \beta,$$

where α and β are constants: but the unknown variable u is constant throughout the system; hence we take

$$z + ix\cos u + iy\sin u = f(u),$$

where f is arbitrary, so far as this equation is concerned.

Instead of proceeding at once with the other equations, we utilise the fact, given by the general theory, that u will be an argument of an arbitrary function: and, for this purpose, we make x, y, u the independent variables in the differential equation. Let

$$\tau = f'(u) + ix\sin u - iy\cos u,$$

where obviously

$$\tau\frac{\partial u}{\partial z} = 1;$$

then

$$l = \frac{dv}{dx} + \frac{i\cos u}{\tau}\frac{dv}{du},$$

$$m = \frac{dv}{dy} + \frac{i\sin u}{\tau}\frac{dv}{du},$$

$$n = \frac{1}{\tau}\frac{dv}{du},$$

$$a = \frac{d^2v}{dx^2} + 2\frac{i\cos u}{\tau}\frac{d^2v}{dx\,du} - \frac{\cos^2 u}{\tau^2}\frac{d^2v}{du^2}$$
$$+ \frac{dv}{du}\left\{2\frac{\cos u \sin u}{\tau^2} + (f'' + ix\cos u + iy\sin u)\frac{\cos^2 u}{\tau^3}\right\},$$

$$b = \frac{d^2v}{dy^2} + 2\frac{i\sin u}{\tau}\frac{d^2v}{dy\,du} - \frac{\sin^2 u}{\tau^2}\frac{d^2v}{du^2}$$
$$+ \frac{dv}{du}\left\{-2\frac{\cos u \sin u}{\tau^2} + (f'' + ix\cos u + iy\sin u)\frac{\sin^2 u}{\tau^3}\right\},$$

$$c = \frac{1}{\tau^2}\frac{d^2v}{du^2} - \frac{1}{\tau^3}\frac{dv}{du}(f'' + ix\cos u + iy\sin u).$$

Hence, substituting in the equation

$$a + b + c = 0,$$

we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{2i}{\tau}\left(\cos u\frac{d^2v}{dx\,du} + \sin u\frac{d^2v}{dy\,du}\right) = 0.$$

It is clear that the equation will be satisfied by taking v equal to any function of u: we proceed to obtain integrals expressible in terms of u and τ. Writing

$$v = \theta(u, \tau),$$

we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = -\frac{d^2\theta}{d\tau^2},$$

$$\cos u\frac{d^2v}{dx\,du} + \sin u\frac{d^2v}{dy\,du} = i\frac{d\theta}{d\tau};$$

the preceding equation becomes

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{\tau}\frac{d\theta}{d\tau} = 0.$$

Hence

$$\theta = F(u) + \frac{G(u)}{\tau},$$

where F and G are arbitrary functions. Consequently, a primitive of Laplace's equation

$$\nabla^2 v = 0$$

is given by

$$v = F(u) + \frac{G(u)}{\tau},$$

where u is determined by the equation

$$z + ix\cos u + iy\sin u = f(u);$$

the quantity τ is given by

$$\tau = f'(u) + ix \sin u - iy \cos u,$$

and F and G are arbitrary functions.

When we proceed with the subsidiary equations, which are constructed on the supposition that the partial derivations are effected with a constant u, and when we use the integral of $\Delta = 0$ in the form

$$z + ix \cos u + iy \sin u = f(u),$$

so that

$$p = -i \cos u, \quad q = -i \sin u,$$

the latter quantities being therefore constant in the subsidiary equations, we infer the further relation

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) = 0.$$

Hence there is some function ξ of x and y (and possibly involving u), such that

$$l - np = \frac{d\xi}{dy}, \quad m - nq = -\frac{d\xi}{dx}.$$

Moreover, we have

$$l + np = \frac{dv}{dx}, \quad m + nq = \frac{dv}{dy};$$

consequently,

$$2l = \frac{dv}{dx} + \frac{d\xi}{dy},$$

$$2m = \frac{dv}{dy} - \frac{d\xi}{dx},$$

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy}, \quad 2nq = \frac{dv}{dy} + \frac{d\xi}{dx}.$$

The last two equations give

$$q\left(\frac{dv}{dx} - \frac{d\xi}{dy}\right) = p\left(\frac{dv}{dy} + \frac{d\xi}{dx}\right),$$

so that

$$\frac{d}{dx}(qv - p\xi) = \frac{d}{dy}(pv + q\xi).$$

Hence some function w of x and y (and possibly involving u) exists, such that

$$pv + q\xi = -\frac{dw}{dx},$$

$$qv - p\xi = -\frac{dw}{dy},$$

and therefore

$$v = p\frac{dw}{dx} + q\frac{dw}{dy}, \quad \xi = q\frac{dw}{dx} - p\frac{dw}{dy}.$$

Thus, from the equation

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy},$$

we have

$$2n = \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2};$$

and therefore, as

$$\frac{dv}{du} = n\frac{dz}{du},$$

while

$$\frac{dz}{du} = f'(u) + ix\sin u - iy\cos u = \tau,$$

we have

$$2\frac{d}{du}\left(p\frac{dw}{dx} + q\frac{dw}{dy}\right) = \tau\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right).$$

This is the equation limiting the form of w: if its general integral were known, the general value of v could be deduced.

Ex. 1. Let u be defined as a function of x, y, z, by means of the relation

$$z + ix\cos u + iy\sin u = f(u),$$

where f is arbitrary; and write

$$\tau = f'(u) + ix\sin u - iy\cos u.$$

Prove that a primitive of the equation

$$a + b + c + \kappa^2 v = 0,$$

where κ is a constant, is given by

$$\tau v = e^{\kappa(x\sin u - y\cos u)}\phi(u) + e^{-\kappa(x\sin u - y\cos u)}\psi(u),$$

ϕ and ψ being arbitrary functions.

Ex. 2. Prove that a primitive of the equation

$$b + c = \mu h,$$

where μ is a constant, cannot be obtained in finite terms when free from partial quadratures: and obtain a primitive in the form

$$v = F + 4\frac{x}{\mu}\frac{\partial^2 F}{\partial\xi\partial\eta} + \left(4\frac{x}{\mu}\right)^2\frac{1}{2!}\frac{\partial^4 F}{\partial\xi^2\partial\eta^2} + \dots,$$

where $F = F(\xi, \eta)$, and

$$\xi = z + iy, \quad \eta = z - iy.$$

Ex. 3. All the preceding results, by a slight change of notation, can be modified so as to give corresponding results for the equation

$$\frac{d^2v}{dt^2}=\kappa^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$

κ being a constant. Let a quantity u be defined by the equation

$$x\cos u+y\sin u-\kappa t=f(u),$$

where u is arbitrary, and let

$$\theta=f'(u)+x\sin u-y\cos u:$$

then a primitive of the equation is given by

$$v=F(u)+\frac{G(u)}{\theta},$$

where F and G are arbitrary functions.

A primitive, in the shape of a definite integral, is easily obtainable in the form

$$v=\int_0^{2\pi} g\,(x\cos u+y\sin u-\kappa t,\ u)\,du,$$

where g is an arbitrary function of its two arguments.

The form of integral, which is most useful in the applications to mathematical physics, arises by taking

$$ve^{cti}=w,$$

where w is independent of t: we then have

$$\kappa^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=c^2 w.$$

The detailed interest of the integral is then bound up with the initial (or boundary) conditions; the general form of the latter equation can be constructed in accordance with the results of Chapter XIV.

WHITTAKER'S INTEGRAL OF LAPLACE'S EQUATION.

341. An integral, in a form involving partial quadratures, has been obtained by Whittaker*: it is sufficiently general to include all uniform integrals, which have no singularities for finite values of the variables.

Taking the origin as a point of reference (if it were not, we should merely write $x-x_0$, $y-y_0$, $z-z_0$ in place of x, y, z), we note that the quantity

$$(z+ix\cos\alpha+iy\sin\alpha)^\mu$$

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is an integral of the differential equation, α being any constant. When μ is a whole number, this integral can be represented in the form

$$\sum_{r=0}^{\mu} g_r \cos r\alpha + \sum_{s=1}^{\mu} h_s \sin s\alpha,$$

where the highest power of z in g_r is $z^{\mu-r}$ and in h_s is $z^{\mu-s}$: also, g_r is an even function of y, and h_s is an odd function of y; so that no linear relation can exist among the $2\mu+1$ quantities $g_0, \ldots, g_\mu$, $h_1, \ldots, h_\mu$. Moreover, as α is arbitrary, each of these $2\mu+1$ quantities is an integral of the equation; hence $g_0, \ldots, g_\mu$, $h_1, \ldots, h_\mu$ are $2\mu+1$ linearly independent integrals of the equation, each of them being homogeneous polynomials in the variables of order μ.

Let an integral, which is regular at the origin and has no singularities in domains round the origin, be expanded in a series of powers, say

$$v = \sum_{\mu=0} v_\mu,$$

which converges uniformly: here, v_μ represents the aggregate of terms that are homogeneous in the variables of order μ. In order that v may satisfy the equation

$$a + b + c = 0,$$

it is clear that v_μ must satisfy the equation also for all values of μ: thus

$$a_\mu + b_\mu + c_\mu = 0.$$

The number of terms in v_μ, taken in the most general form, is

$$\tfrac{1}{2}(\mu+1)(\mu+2),$$

which accordingly is the number of arbitrary constants in v_μ taken arbitrarily: the number of terms in $a_\mu + b_\mu + c_\mu$ is

$$\tfrac{1}{2}(\mu-1)\mu,$$

which accordingly is the number of relations among these arbitrary constants required to secure that v_μ may satisfy the equation; hence the number of constants left undetermined is

$$\begin{aligned}&\tfrac{1}{2}(\mu+1)(\mu+2) - \tfrac{1}{2}(\mu-1)\mu\\ &= 2\mu+1,\end{aligned}$$

which accordingly is the number of linearly independent integrals of the equation in the form of homogeneous polynomials of order μ. But this is precisely the number of the linearly independent quantities $g_0, \ldots, g_\mu, h_1, \ldots, h_\mu$, and these are of the same type. Thus the quantity v_μ in its most general form, when it contains the $2\mu+1$ constants, is a linear function of $g_0, \ldots, g_\mu, h_1, \ldots, h_\mu$.

Now, since

$$(z + ix\cos\alpha + iy\sin\alpha)^\mu = \sum_{r=0}^{\mu} g_r \cos r\alpha + \sum_{s=1}^{\mu} h_s \sin s\alpha,$$

we have

$$g_r = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu \cos r\alpha\, d\alpha,$$

$$h_s = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu \sin s\alpha\, d\alpha\,;$$

and therefore

$$v_\mu = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu\, \theta_\mu(\alpha)\, d\alpha,$$

where

$$\theta_\mu(\alpha) = \sum_{r=0}^{\mu} \beta_r \cos r\alpha + \sum_{s=1}^{\mu} \gamma_s \sin s\alpha,$$

$\beta_0, \ldots, \beta_\mu, \gamma_1, \ldots, \gamma_\mu$ being arbitrary constants. Consequently,

$$\begin{aligned} v &= v_0 + v_1 + v_2 + \ldots \\ &= \frac{1}{\pi}\int_0^{2\pi} \Sigma\, \{(z + ix\cos\alpha + iy\sin\alpha)^\mu\, \theta_\mu(\alpha)\}\, d\alpha \\ &= \int_0^{2\pi} f(z + ix\cos\alpha + iy\sin\alpha,\ \alpha)\, d\alpha, \end{aligned}$$

where f is an arbitrary function of its two arguments, regular in $z + ix\cos\alpha + iy\sin\alpha$, and periodic in α.

This is Whittaker's integral of the equation. It is easily seen to be equivalent to the Cauchy integral which, for the present purpose, has initial conditions such that

$$v = \sum_{m=0}\sum_{n=0} c_{m,n} x^m y^n,$$

$$\frac{\partial v}{\partial z} = \sum_{m=0}\sum_{n=0} k_{m,n} x^m y^n,$$

when $z = 0$: hence we should have

$$\sum_{n=0}^{\mu} c_{\mu-n,n} x^{\mu-n} y^n = \frac{i^\mu}{\pi}\int_0^{2\pi} (x\cos\alpha + y\sin\alpha)^\mu\ \theta_\mu(\alpha)\, d\alpha,$$

$$\sum_{n=0}^{\mu-1} k_{\mu-1-n,n} x^{\mu-1-n} y^n = \mu\frac{i^{\mu-1}}{\pi}\int_0^{2\pi} (x\cos\alpha + y\sin\alpha)^{\mu-1}\, \theta_\mu(\alpha)\, d\alpha.$$

There are $2\mu+1$ constants in θ; these suffice to determine the $\mu+1$ constants c and the μ constants k, and conversely.

Taking $x=\rho\cos\phi$, $y=\rho\sin\phi$, the first equation gives

$$\sum_{n=0}^{\mu} c_{\mu-n,n}\cos^{\mu-n}\phi\sin^n\phi$$

$$=\frac{i^\mu}{\pi}\int_0^{2\pi}\cos^\mu(\phi-\alpha)\,\theta_\mu(\alpha)\,d\alpha$$

$$=\frac{i^\mu}{2^{\mu-1}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{\mu!}{p!(\mu-p)!}\cos\{(\mu-2p)(\phi-\alpha)\}\,d\alpha$$

$$=\frac{i^\mu}{2^{\mu-1}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{\mu!}{p!(\mu-p)!}\{\cos(\mu-2p)\phi\cos(\mu-2p)\alpha$$
$$+\sin(\mu-2p)\phi\sin(\mu-2p)\alpha\}\,d\alpha;$$

consequently,

$$\frac{i^\mu}{2^{\mu-1}}\frac{\mu!}{p!(\mu-p)!}\beta_{\mu-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu}c_{\mu-n,n}\int_0^{2\pi}\cos^{\mu-n}\phi\sin^n\phi\cos(\mu-2p)\phi\,d\phi,$$

$$\frac{i^\mu}{2^{\mu-1}}\frac{\mu!}{p!(\mu-p)!}\gamma_{\mu-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu}c_{\mu-n,n}\int_0^{2\pi}\cos^{\mu-n}\phi\sin^n\phi\sin(\mu-2p)\phi\,d\phi.$$

Similarly, the second equation gives

$$\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\cos^{\mu-1-n}\phi\sin^n\phi$$

$$=\frac{\mu i^{\mu-1}}{\pi}\int_0^{2\pi}\cos^{\mu-1}(\phi-\alpha)\,\theta_\mu(\alpha)\,d\alpha$$

$$=\frac{\mu i^{\mu-1}}{2^{\mu-2}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{(\mu-1)!}{p!(\mu-1-p)!}$$
$$\{\cos(\mu-1-2p)\phi\cos(\mu-1-2p)\alpha$$
$$+\sin(\mu-1-2p)\phi\sin(\mu-1-2p)\alpha\}\,d\alpha;$$

consequently,

$$\frac{\mu i^{\mu-1}}{2^{\mu-2}}\frac{(\mu-1)!}{p!(\mu-1-p)!}\beta_{\mu-1-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\int_0^{2\pi}\cos^{\mu-1-n}\phi\sin^n\phi\cos(\mu-1-2p)\phi\,d\phi,$$

$$\frac{\mu i^{\mu-1}}{2^{\mu-2}}\frac{(\mu-1)!}{p!(\mu-1-p)!}\gamma_{\mu-1-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\int_0^{2\pi}\cos^{\mu-1-n}\phi\sin^n\phi\sin(\mu-1-2p)\phi\,d\phi.$$

All the constants in $\theta_\mu(\alpha)$ can therefore be obtained so as to satisfy the assigned conditions.

Ex. 1. Writing

$$z=r\cos\theta,\quad x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,$$

prove that the harmonic integrals, usually denoted by

$$r^n P_n^m(\cos\theta)\cos m\phi,\quad r^n P_n^m(\cos\theta)\sin m\phi,$$

are numerical multiples of

$$\int_0^{2\pi}(z+ix\cos a+iy\sin a)^n\cos ma\,da,$$

$$\int_0^{2\pi}(z+ix\cos a+iy\sin a)^n\sin ma\,da,$$

respectively. (Whittaker.)

Ex. 2. With the same notation as in the last question, prove that the integrals, usually denoted by

$$e^{kz}J_m(k\rho)\cos m\phi,\quad e^{kz}J_m(k\rho)\sin m\phi,$$

are numerical multiples of

$$\int_0^{2\pi}e^{k(z+ix\cos a+iy\sin a)}\cos ma\,da,$$

$$\int_0^{2\pi}e^{k(z+ix\cos a+iy\sin a)}\sin ma\,da,$$

respectively. (Whittaker.)

Ex. 3. Integrals of the differential equation are known in forms

$$F(x+iy),\quad G(x+iz);$$

determine the forms of the function f, so that these may be included in the Whittaker integral.

Ex. 4. Prove that an equation of any order p in the form

$$F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)v=0,$$

where $F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$ is a symbolical homogeneous polynomial of order p, can be satisfied by an integral

$$v=\int\phi(\xi x+\eta y+\zeta z,\ t)\,dt=0,$$

where ξ, η, ζ are coordinates of a point on the curve

$$F(\xi,\ \eta,\ \zeta)=0,$$

expressed in terms of a parameter t. Are there any limitations upon the path of integration with respect to t? (Bateman.)

Ex. 5. Prove that all uniform integrals of the equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial t^2},$$

which have no singularities for finite values of the variables, are included in the form

$$v=\int_0^{2\pi}\int_0^{\pi} f(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta+t,\ \theta,\ \phi)\,d\theta\,d\phi,$$

where f is an arbitrary function of its three arguments.

Deduce integrals of the equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+v=0. \qquad \text{(Whittaker.)}$$

Ex. 6. Prove that all uniform integrals of the equation in the preceding example, which have no singularities for finite values of the variables, are included in the form

$$v=\int_0^{2\pi} f(x\cos\theta+y\sin\theta+iz,\ x\sin\theta-y\cos\theta+t,\ \theta)\,d\theta.$$

(Bateman.)

Ex. 7. Expressing the homogeneous coordinates $\xi,\ \eta,\ \zeta,\ \tau$, of any point on a surface

$$F(x,\ y,\ z,\ t)=0,$$

where F is a homogeneous polynomial in x, y, z, t of order p, in terms of two parameters θ and ϕ, prove that the relation

$$v=\iint f(\xi x+\eta y+\zeta z+\tau t,\ \theta,\ \phi)\,d\theta\,d\phi,$$

the integral being taken over any part of the surface $F=0$, and the function f being arbitrary, is a primitive of the equation

$$F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ \frac{\partial}{\partial t}\right)v=0$$

of order p. Prove that, if $F=0$ be a ruled surface, the primitive can be expressed in terms of a single quadrature. (Bateman.)

Ex. 8. Obtain an integral of the equation

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}=0$$

in the form

$$\theta=\left\{\frac{\kappa}{r}(xq'-yp')+2\frac{\kappa'}{r}(yp-qx)\right\}f(u)+(a-xp'-yq'-zr')f'(u),$$

where p, q, r are arbitrary functions of a variable u, subject to the single relation

$$p^2+q^2+r^2=0,$$

where

$$p'^2+q'^2+r'^2=-\kappa,$$

and where dashes denote differentiation with regard to u.

(Bromwich.)

Ex. 9. Quantities ξ, η, ζ are defined in terms of a quantity s by the relation

$$a\xi+\beta\eta+\gamma\zeta=\begin{vmatrix} a_1s+b_1, & a_2s+b_2, & a_3s+b_3 \\ a\ \ , & \beta\ \ , & \gamma \\ x\ \ , & y\ \ , & z \end{vmatrix},$$

where a, β, γ are arbitrary parameters: also, Δ denotes the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ x\ , & y\ , & z \end{vmatrix},$$

the quantities a and b being constants; and $f(\xi, \eta, \zeta)$, $\psi(\xi, \eta, \zeta)$, are homogeneous polynomials in ξ, η, ζ, of orders n and $n-2$ respectively. Prove that the magnitude u, as defined by the relation

$$u=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(\xi, \eta, \zeta)\,\Delta}{f(\xi, \eta, \zeta)}\,ds,$$

satisfies the equation

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)u=0.$$

Apply this result to obtain an integral of the equation

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^4 u}{\partial z^4}=0. \qquad \text{(Fredholm.)}$$

As has been hinted in the course of this, the concluding, Part of the present work, and as must have become obvious from the accounts given in this volume, the theory of partial differential equations, which are of order higher than the first, is not nearly so fully developed as is the theory of equations of the first order. Indeed, many of the methods are tentative: and though sometimes it has been possible to assign adequate tests for their success, their limitations remain in general. If I might change one word in a remark of a great historian, I would say of such equations that "the [mathematician] may applaud the importance and variety "of his subject; but, while he is conscious of his own imperfections, "he must often accuse the deficiency of his materials."

In these circumstances, it has become necessary, to an even greater extent than in the earlier Parts of this work, to make a selection from among the topics available for discussion and to impose limitations upon the extent of the treatment. My purpose has been to indicate the organic character of such methods as have been devised rather than to record as many formal results as possible; and I have preferred to expound the processes only for equations of the lowest orders involving the smallest numbers of independent variables. Were the methods quite general and the theory approximately complete, a discussion of equations of order n involving m independent variables would undoubtedly be necessary; but, as these conditions are not realised, I have acted on my opinion that, in the present state of the subject, adequate exposition can be made by reference to the simplest comprehensive classes of equations. Usually, the increase in order from one to two, or from two to three, and the increase in the number of independent variables from two to three, introduce the essential difficulties that impede advance; when these have been solved, the way often is clear for further increase in either direction. It is only by the solution of new essential difficulties that substantial progress is made.

The selection of topics discussed may be deemed inadequate by some readers: the omissions may be deemed arbitrary by others. The choice has, of course, been made deliberately. My desire has been to give a continuous exposition of those portions of the subject, which not only seem to me to be the most important but also bear some promise of leading into paths of knowledge that will be trodden by investigators in days yet to come.

INDEX TO PART IV.

(Part IV occupies volumes v and vi of the present work. The figures refer to the pages in these volumes.

The Table of Contents at the beginning of volume v and of volume vi may also be consulted.)

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