THEORY OF ERRORS AND LEAST SQUARES

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THEORY OF ERRORS AND LEAST SQUARES

A TEXTBOOK FOR COLLEGE STUDENTS AND RESEARCH WORKERS

BY

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PREFACE

THERE are few branches of mathematics which have wider applicability to general scientific work than the Theory of Errors, and few mathematical implements which are capable of greater usefulness to the research worker than the Method of Least Squares. Yet, for some reason, students are rarely given opportunity to acquire facility in these lines, the result being that too many of our scientists and engineers go about their work without such equipment. It would be almost impossible to enumerate the variety of ways in which the ideas relating to these subjects adapt themselves to even such simple bits of quantitative work as the chemist or the surveyor is daily called upon to do. And it is difficult for the writer to imagine how an elaborate research in any of the exact sciences can be carried on at all, without the constant application of these principles throughout both the preliminary and the final stages of the work. The satisfaction to be gained from the application of the theory of *precision* alone is well worth all the time necessary to acquire these subjects. Add to this the fact that the theory of error distribution has direct theoretical bearing upon certain very important laws and problems of physics, chemistry, astronomy, and even of biology, and the reasons for students' having opportunity to attain the elements of the subject become still more emphatic.

This small volume embodies the material used by the writer as lecture notes during the past twelve years. It is intended as a presentation of the Theory of Errors and Least Squares in such a simple and concise form as to be useful, not only as a textbook for undergraduates, but as a handy reference which any research worker can read through in an evening or so and then put into immediate practice.

It will be noticed that the illustrative examples and problems are drawn from various branches of science, suggesting the wide range of possible application. No attempt is made, of course, at an exhaustive treatment in such small compass. Some of the special methods employed by expert computers, often included in larger works, have been purposely omitted. For the convenience of the student, and in order not to interrupt the thread of the subject, a few of the more complicated mathematical discussions have been set apart in the Appendix and referred to at the appropriate places. It is not intended that they shall be omitted from the course when using the book as a text, though the casual reader may get along very well without them.

The writer wishes to express his appreciation to the numerous friends who have kindly given aid by way of furnishing data for the illustrative examples, or otherwise. Where material has been taken from other works, due credit has been given for the same.

L. D. W.

CEDAR RAPIDS, IOWA,
December, 1915.

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THEORY OF ERRORS AND LEAST SQUARES

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CHAPTER I

ON MEASUREMENT

1. Definition of Measurement. — To *measure* a quantity is to determine by any means, direct or indirect, its ratio to the unit employed in expressing the value of that quantity. Thus, in measuring a line, we find that it is a certain number of times as long as the foot or the centimeter, and this number is said to be its *value* in feet or centimeters.

This definition must be clearly understood to be independent of whatever process is used in the measurement. We could measure the area of a polygonal piece of sheet iron in two ways: either by measuring its sides and angles and computing its area by geometry, or by weighing it and comparing its weight to that of a square piece with unit side. Either of these processes is a true measurement of the area, though neither is a *direct* measurement.

2. Indirect Measurement. — Indeed, with the exception of one kind of magnitude, very few measurements are direct. By this is meant that we do not, in general, apply the unit of measure directly to the magnitude to be measured. This is done commonly only in the case of length. We can, in measuring a line, apply the yardstick directly along the line and determine in this way how many times greater one is than the other. But we cannot take a lamp in one hand and a standard candle in the other and determine the candle-power of the lamp in any such direct manner.

So far is the above mentioned principle true, that, as a matter of fact, nearly every kind of measurement is made to depend, in practice, upon measurements of length. This will be clear from a number of illustrations.

Angles are measured, not by applying the wedge-like degree as a unit, but by measuring the *length* of the arc laid off on a curved linear scale, or by measuring the *lengths* of straight lines connected with the angle and computing the latter from its trigonometric functions.

Time is measured, not by counting the minutes and seconds in the interval, but by observing the motion of the clock hands over a curvilinear scale called the dial, marked off in spaces of equal *length;* or by noting the *lengths* marked off on the chronograph record by a pen point which is given a lateral jerk electrically at the beginning and end of the interval. Every magnitude measured off on a dial is finally referred to *length,* as exemplified by pressure gauges, gas meters, electric meters, aneroid barometers, etc.

Temperature is measured off as a *length* on the stem of the thermometer.

Atmospheric pressure is measured, and even expressed, in *inches* or *centimeters* of mercury.

Weight is measured, in the final adjustment, by the position of a slide or rider on a *linear* scale, or in refined work by the position of the balance pointer at equilibrium, the sensibility of the balance being known. The common spring balance and its more refined near relative, the Jolly balance, illustrate the linear principle in another way.

In short, every measuring instrument has some sort of *linear* scale, either straight or curved, on which some sort of indicator or pointer moves.

The reason for thus referring every kind of measurement to a simple one of length is mainly the one already referred to, that length is the only kind of magnitude that can be conveniently compared directly with its own unit. But there is another reason. The eye can estimate a length with far greater accuracy than the muscles can estimate a weight, the hand a temperature, or the consciousness an interval of time; and this process of estimation plays an all-important part, as will now be seen, in every kind of accurate measurement.

3. Estimation. — The degree of precision with which an observer can read a given linear scale depends upon two things, namely: (1) the definiteness or sharpness of the marks on the scale and of the pointer or indicator, and (2) the skill with which the observer can estimate fractional parts of one interval or scale-division.

The former item may be made clear by comparing the scale and indicator of an ordinary spring balance with those on a delicate ampere meter or aneroid barometer; or the graduations on a surveyor's leveling-rod with those on a silver-inlaid standard meter bar.

As to the second matter, it is of the utmost importance that the observer drill himself in this process of estimation. In no case does the accuracy of a single scale-reading end with the fineness of graduation of the scale, providing the scale lines and indicator are sharp and distinct; it can always be carried a step farther.

It is the custom of practical observers to make estimations of fractional units in *tenths*, not in halves, thirds, etc., and to record the readings decimally. No attempt is made to estimate the hundredths, unless it appears to the observer that the fraction is exactly one-fourth or three-fourths, when he would be likely to record .25 or .75; even this is a doubtful practice. The reading of any linear scale may be carried, in general, to an accuracy of one-tenth of the smallest scale-division by the estimation of the eye alone, or the eye aided by a magnifier if desirable.

In many instruments of precision, the linear scale is provided with some sort of *vernier*, which is a mechanical substitute for the estimation of fractional parts of scale divisions. Descriptions of the different kinds of verniers in use may be found in any elementary laboratory manual of physics, or in any encyclopedia. But even the use of the vernier requires the same sort of skill and judgment as estimation, namely, a correct idea of linear position and coincidence. And in the vast majority of measuring instruments, no vernier is provided, and the observer must be able to estimate tenths accurately and without hesitation.

4. The Impossibility of Exact Measurements. — Every scientist is familiar with the fact that there is no such thing as an absolutely exact measurement, for the simple reason that the quantity measured and the unit of measure are never commensurable.

If we weigh carefully a small piece of metal on a common balance, a typical result would be 3.9843 grams, and not a whole number, as four grams. This is, however, only an approximation to the true weight, even if correct to four decimal places, just as the number 3.1416 is only an approximation to the value of π. If a more sensitive balance is used, the result may be 3.984326 grams; but as the masses of the piece of metal and the gram weight are incommensurable, the true weight, even if it were possible to weigh without the inaccuracies that arise from imperfect apparatus and judgment in estimation, would be inexpressible in grams, and the result obtained could be true only to the degree of approximation represented by six places of decimals, that is, to the nearest millionth of a gram.

What is true of weighing is true of all measurement, and it will readily be seen that to obtain the *true* value of any actual concrete quantity is as hopeless as to obtain the true value of $\sqrt{2}$, or π, or $\log_{10} 17$.

5. Errors of Measurement. — Aside from the mere incommensurability of magnitudes, there is another and far more serious hindrance to the obtaining of correct values by measurement, and this is what is technically known as *error*.

Suppose the bit of metal, which was found on the more sensitive balance to weigh 3.984326 grams, be now weighed again, by the same person, in the same room, on the same balance and with the same weights. More likely than not, the result will turn out to be different from the former result by some millionths of a gram, perhaps thirty or forty millionths. This means simply that neither result is correct, even to the sixth decimal place.

Again, if we go out with a surveyor's transit of the finest construction and measure with the utmost care, to seconds even, each of the three angles of a triangle marked out by accurately centered stakes on level ground, and add the three results together, we shall probably find that their sum differs from 180° by several seconds one way or the other. We may repeat the operation with equal care and skill, and get a still different result, perhaps farther from 180° than the first. This illustration will be all the more striking, in that in this case the true value of the sum of the three angles is known from geometry, while in the case of the weights the true value is not and can never be known. Even here, the individual angles cannot be obtained exactly.

The causes of error in precise measurements are many and various. A single example will suffice to illustrate this. Suppose we wish to measure the distance from one stake to another with a surveyor's chain. Two men carry the chain. Each time they advance, one adjusts the following end to the rear marking-pin, the other sets a new pin at the leading end, and neither can do this work with absolute accuracy. They do not stretch the chain tight enough; they do not hold the chain horizontal in going up or down hill; they do not follow a straight line; they do not notice kinks in the chain, and they neglect the fact that the chain is wearing at the joints and getting longer. As a consequence of all these small items, and many others not mentioned, the measurement may in the end be several inches from the truth if the line to be measured be very long. This is only one instance showing how hundreds of little disturbances may combine and form one final resultant error which may be positive or negative, great or small, according to which kind of disturbances predominates (that is, whether they tend to make the result too large or too small), and to whether they happen to be about evenly balanced or not.

A systematic study of the occurrence of errors gives rise to a mathematical analysis, based essentially upon the principles of probability and known as the *Theory of Errors;* and our attempts to apply this theory to the results of measurements, with a view to getting the values that are probably nearest the truth, have resulted in the formulation of certain rules embraced in that part of the error theory known as the *Method of Least Squares.*

EXERCISES

6. The following exercises are intended for the use of students who have not done much laboratory work nor had the advantage of a course in laboratory measurements or field work. It will be seen that they are largely suggestive, and they may be modified as desired to suit the circumstances. For advanced students and research workers they may be omitted altogether.

1. Can you think of any kind of accurate measurement not ultimately employing some sort of linear scale?

2. Show wherein the following kinds of measurements are made to employ a linear scale: area of a piece of land; density of a solid; relative humidity of the atmosphere; index of refraction of a transparent substance; volume of liquid from a burette.

3. Determine the volume of a material sphere, cylinder or other geometrical solid in two ways: first by measuring its dimensions; and second by dropping it into a glass graduate partly filled with water and observing the displacement. Do the two results agree? Which do you consider the more precise method?

4. Measure a quantity of pure water in two ways: first by placing it in a glass graduate; and second by weighing it on a balance and computing the volume. The weighing may be done in the graduate, which has been weighed beforehand.

5. Lay off on a sheet of smooth paper, with a fine, hard pencil, a line of indefinite length, and mark two points on it at random somewhat less than 10 cm. apart. On the straight edge of a card, mark two points as nearly 10 cm. apart as possible. By means of direct comparison with this standard, estimate the length of the first line-segment in centimeters, writing down the result. Next compare the unknown line with a cardboard scale marked off in centimeters but not in millimeters, observing the number of centimeters and estimating the millimeters. Finally compare the same line with a millimeter scale, estimating the tenths of a millimeter. Notice how the three results agree, all being expressed in centimeters. Repeat this several times with different line-segments.

6. Devise and perform exercises, similar to Exercise 5, in the measurement of angles, using a large protractor and circular sectors of paper as measuring instruments.

7. Try measuring short intervals of time to tenths of a second by means of an ordinary watch. In order to test the results, let the period measured be the time of swing of a simple pendulum, and measure by the watch intervals of five, ten, fifteen, twenty and one hundred swings, finding the time of a single vibration from each measurement. Do the results agree? Have you any greater confidence in one than in another?

8. Familiarize yourself thoroughly with the use of as many different kinds of verniers as are available. *Before*

using the vernier in each case, *estimate* the fraction of a unit in tenths by the eye.

9. Weigh a small piece of iron by means of a Jolly balance, then on a trip scale, then on an equal arm balance. Compare the results. In which result have you the greatest confidence? Why?

10. Weigh a small object several times, with the highest degree of precision attainable, on a good balance. The pointer method should be used. Are the results all equal?

CHAPTER II

ON THE OCCURRENCE AND GENERAL PROPERTIES OF ERRORS

7. Errors and Residuals. — The term *error* has so far been used somewhat indefinitely, and it will be necessary, before going further, to explain its exact meaning, as well as that of another term closely connected with it.

We have seen that different measurements upon the same quantity generally give different results. These results evidently cannot all be correct, and it is very unlikely that any of them is correct, even to the degree of precision (that is, to the number of decimal places) attainable with the instruments and method used. The difference between the result of an observation and the true value of the quantity measured is called the *error* of the observation. In what follows we shall generally denote observations by the symbol s, the quantities upon which they are made by q, and the errors of the observations by x, the latter being defined, as just stated, as the difference

$$x = s - q, \tag{1}$$

which will be positive or negative according as the observation is too large or too small. The student should be careful to remember this definition, and to apply it to such illustrations as the following: If a line is exactly 437 feet long, and the result of a measurement upon it is 436.2 feet, then the error is $436.2 - 437 = -0.8$ ft.

While we cannot ordinarily obtain the *true* value of a measured quantity from one measurement, nor even by averaging many measurements, the method of least squares furnishes us, in the latter case, with a means of calculating what is called the *most probable value*, which is the closest approximation to the true value that the series of observations is capable of yielding. A familiar illustration is that of a series of direct observations upon a single quantity, in which case the most probable value is simply the arithmetical average of the several results.

Having obtained the most probable value from a series of observations in the manner hereafter to be explained, if we now subtract it from each measured result, we obtain a series of differences known as the *residuals* corresponding to the respective observations. The most probable value being denoted by m, and any observation by s, the residual corresponding to s is

$$\rho = s - m. \tag{2}$$

Thus, the *residual* bears the same relation to the *error* that the *most probable value* bears to the *true value*. If the number of observations be very large, and the observations be very precise, then the most probable value may be very, very close to, though *never equal to*, the true value; and in that case the residuals will be equally close to the corresponding true errors.

It is worthy of note that, since the true value of a quantity in terms of any arbitrarily selected unit is always an incommensurable number, while the most probable value is commensurable, it follows that the error of any observation is incommensurable, while the corresponding residual is commensurable. The true value and the errors are consequently forever unknown and figure only in theoretical discussions (with such exceptions as have been noted); and we deal in practice only with their close approximations, the most probable value and the residuals.

8. Classification of Errors. — It is now very important to point out that errors of observation may be divided naturally into two distinct classes, whose occurrence, and the methods of dealing with which, are entirely different.

First we may consider those errors which arise from causes that continue to operate in the same manner throughout the series of observations, and which may therefore be called *persistent* or *systematic* errors. In many cases, persistent errors not only occur in the same manner, but have the same value, throughout the investigation, and they may then be called *constant* errors.

The causes of persistent errors, which are often known to the observer and may in many cases be eliminated or avoided by methods presently to be explained, may be, for the most part, looked for under one or another of the following heads.

a. *Incorrect Instruments.* — The instruments or scales used may not be true. For example, if a 100-foot tape is actually only 99.99 feet in length, every measurement on a line made with that tape will tend to give a result one ten-thousandth too long, no matter how many times the observation is repeated; or if a clock used in scientific work gains one second a day, every measurement on an interval of time made with that clock is just the corresponding fraction too long. (In each of these cases, is the error positive or negative?)

b. *Imperfect Setting of Scale.* — Owing to carelessness or accident, the scale on a measuring instrument, though truly graduated, may be displaced from its proper position by a small amount. This is well illustrated by the mercurial barometer, on which the scale must be adjusted at each reading, to allow for the rise and fall of mercury in the reservoir; and a clock which, though running at the proper *rate*, has been set a little ahead or behind the true standard time, is an analogous case.

c. *Defective Mechanism.* — No instrument is absolutely perfect from a mechanical standpoint, and every instrument of precision must be frequently tested if we would rely upon the results of its use. The arms of a balance are never really equal, and, what is worse, they are continually changing their relative length, owing to changes of temperature. Nor has it been found possible to construct a clock that will run with absolutely constant rate, even at a constant temperature and in a vacuum.

d. *False Indicator Settings.* — In very delicate instruments, such as the balance or the aneroid barometer, the indicator frequently comes to rest, on account of friction, in a false position. In the case of the aneroid barometer, it often suffices to tap the dial gently, in order to make the indicator assume its true position. The same may be said of the magnetic compass-needle.

e. *Known External Disturbances.* — It is often the case that persistent errors are introduced by external causes whose nature is well understood, but which cannot be avoided. Thus, a heated body under experimental investigation always radiates some heat, in spite of the most elaborate precautions; and the length of a measuring rod or tape is certain to vary with changes of temperature.

f. *Personal Equation and Prejudice.* — Every observer exhibits peculiarities or habits of observation which cause him to have a tendency toward persistent error in the same direction. Thus, one observer may continually *overestimate* in the estimation of tenths, another will underestimate; a time observer requires a certain definite interval to respond to a stimulus, that is, to obey a signal of any sort. This unconscious, persistent error on the part of an observer is called his *personal equation.*

Somewhat analogous to personal equation is what may be called *prejudice.* After an observer has made one measurement of a quantity on a fixed scale, and made the estimation of tenths, there is a natural tendency for him to allow his first estimation to affect the subsequent ones. This difficulty is often met with in the use of the vernier, where it is necessary to judge as to which line coincides most nearly with its fellow on the scale.

The second class of errors referred to at the beginning of this article comprises those whose causes are temporary, existing through only one observation, and disappearing entirely upon a slight change of conditions. Such errors are not recognizable, and sometimes not even suspected, until their existence is demonstrated by the discrepancies between successive observations when all known disturbances have been eliminated. These are known as *accidental errors.*

Accidental errors may also be subdivided, as follows.

a. *Those Due to External Causes.* — Accidental errors may result from causes entirely foreign to the observer and of so complex a character as to be incapable of analysis. For example, in sighting a mark with a surveyor's transit, a sudden gust of wind may imperceptibly sway the instrument for a moment, or someone may, without the observer's knowledge, knock against the tripod and jar the telescope slightly out of place. In making delicate magnetic measurements, such rapid changes as often take place unexpectedly in the earth's magnetic field may momentarily affect the equilibrium of the needle. In sighting at a star with a telescope, currents of air in the upper atmosphere may cause it to waver and appear for a moment to one side of its mean apparent position. In using a balance, the zero of equilibrium may change slightly during the course of a single weighing, owing, perhaps, to an unsuspected fluctuation of temperature.

It will thus be seen that observations of all kinds are affected by multitudes of such causes, which are of greater or less importance, but which all tend to affect the accuracy of the results.

b. *Accidental Errors of Judgment.* — Aside from personal equation and prejudice, the observer himself is subject to *fluctuations* of judgment, both as to the adjustment of his instrument and as to the estimation of tenths. An attempt to analyze in detail the causes of these internal tendencies to err in judgment would belong to the realm of psychology; but we may mention as prominent among them the influences of imperfect vision, optical illusion, inattention and fatigue, the last mentioned cause probably affecting the others in a very large degree.

Some of the methods commonly employed in dealing with persistent errors are briefly mentioned in Art. 10. It is, however, the study of accidental errors, and of the laws which are found to govern their occurrence, that constitutes the special office of the method of least squares.

9. Mistakes. — Entirely distinct from *errors,* in the sense heretofore used, are those inaccuracies which are due purely to carelessness, and which should properly be called mistakes. They consist in such blunders as reading the wrong number on the scale, reading one number and putting another down in the notes, reading a vernier backward instead of forward, making a miscount in timing a pendulum, etc. Mistakes are usually easily detected, and there is no remedy except vigilance and careful checking. When measurements are made more than once the checking is a simple matter.

10. General Methods of Eliminating Persistent Errors. — In Art. 8 are enumerated several causes of persistent errors, with illustrations of each. Though their discussion does not properly belong to the general theory of errors, it may not be out of place to describe here some of the methods commonly employed in dealing with them, especially as the theory of errors is frequently applied in the processes of correction here referred to. The treatment of the several sources of persistent errors will be taken up in the same order as they are mentioned in Art. 8, and designated by the same letters.

a. *Incorrect Instruments. Adjustment and Standardization.* — As it is never certain that an instrument measures in true units, it is necessary to test it before relying upon the results of its use. (The tests may in some cases be made long after the measurements.) An instrument may sometimes be adjusted correctly, and remain so; more commonly it gets out of adjustment again, from wear or other causes. Actual adjustment may often be inconvenient or impossible. A more approved practice is *standardization,* which will apply to nearly every case. This consists in comparing the instrument with a standard and determining the true value of each of its scale divisions or units, and then, instead of trying to adjust the instrument, simply making the necessary corrections on the observations. (Where standardization extends over a whole scale, it is commonly called *calibration.*) Thus, the astronomer seldom corrects his clock; he simply determines its error from the stars at intervals, and thus deduces its error in rate, which is all the information needed at any time. Laboratory weights are seldom correct when purchased, and moreover they lose or gain weight by wear or corrosion; hence they should be compared from time to time with standards kept for the purpose. Numerous illustrations of the kind will occur to the reader.

b. *Imperfect Setting of Scale. Differential Method.* — The error due to imperfect setting of the scale may often be eliminated by the differential method, which consists in reading the position of the indicator when it should be at zero, then again when it is affected by the quantity to be measured, and taking the difference. This method applies only when the scale divisions are equal throughout the scale. The process is one very generally employed, as it has further advantages than the one here stated; very frequently it is the only method practicable. The use of a level and leveling rod in surveying illustrates the latter point, as does almost any kind of comparator; and when one wishes to weigh a portion of liquid, he must needs subtract the weight of the empty vessel from the weight of the vessel and contained liquid.

c. *Defective Mechanism. Compensation.* — Instrumental errors may often be made to react against themselves and automatically disappear. When this can be done, it is by far the best method of elimination. A simple example is the process of "double weighing," in which the effect of inequality in the arms of the balance is removed by weighing with the object first on one pan, then on the other, and taking the mean. (Strictly, the geometrical mean should be used.) If a spirit level, resting upon an imperfectly adjusted base, be simply reversed, end to end, the half-way point between the two positions of the bubble will indicate its true position as well as if it were in adjustment. The graduated circles used on surveying instruments, spectrometers, and the like, are usually provided with two diametrically opposite verniers, so that the error arising from the vernier system being out of center with the circle itself may disappear on taking the mean of the readings of the two verniers. In using a galvanometer it is well to reverse the current and read the deflection both ways on the scale. An interesting application of the method to the elimination of unknown external disturbance is the scheme devised by Rumford for neutralizing the effect of radiation in calorimetric measurements. A preliminary experiment is made to determine by what amount the temperature of the calorimeter will be raised; and then the initial temperature is so adjusted that it is about the same amount *below* the temperature of the surrounding air at the beginning of the experiment as it is *above* it at the close, so that practically the same amount of heat is absorbed during the first half of the operation as is radiated during the last half.

d. *False Indicator Settings. Oscillation.* — In cases where the indicator comes to rest in a false position, due to friction, the difficulty may often be removed by not allowing the indicator to come to rest at all, but reading it while still oscillating. This method has the further advantage of saving time in such instruments as the balance and undamped galvanometers or magnetometers. In order to compensate for diminishing amplitude, one more reading should be taken at one extreme of the swing than at the other, as in the following balance pointer readings and reduction:

Left		Right
7.8		13.1
8.0		13.0
8.1		2)26.1
3)23.9		13.05
7.97		
13.05		
2)21.02		
10.51	True reading.	

This result is much more quickly obtained and more accurate than one obtained by letting the pointer come to rest.

e. *Theoretical Corrections for Known External Disturbances.* — When the manner in which external disturbances operate is known, and their magnitude determined, the errors due to them are eliminated by simply applying the computed corrections. The temperature and stretch corrections applied to the steel tape in precise chaining, and the temperature corrections necessary with instruments, such as the barometer and pyknometer, depending upon the density of a liquid or the capacity of a hollow vessel, are familiar examples. Instead of employing Rumford's compensation in using the calorimeter, the amount of radiation per minute may be previously noted and allowed for in the reduction of the results. The refraction error in the observed altitude of a star, or in long range leveling, the vacuum correction in weighing, etc., are further familiar examples. It is for the purpose of obtaining data for such corrections that many investigations of the behavior of physical phenomena under varying conditions are carried on; indeed, this work constitutes a large part of quantitative scientific research.

f. *Corrections for Personal Equation and Prejudice.* — Personal equation may be eliminated, either by determining by means of specially devised experiments what the personal equation of the observer is for a given kind of measurement, or by arranging matters so as to make the personal error act in opposite directions in the two halves of the observation; or by a very different method, — that of employing a number of different observers on the same measurement, whose errors will tend to compensate in the long run, like accidental errors.

The effect of prejudice may often be avoided by altering the conditions. Thus, when repeatedly using the differential method, the whole measurement may be shifted each time to a different part of the scale. The oscillation method is not subject to prejudice, since, though the true reading may be the same in the successive observations, the oscillations approaching it will not be. An experienced observer will not allow prejudice to influence him to any great extent.

11. Exercises Leading to an Understanding of Error Distribution. — Before attempting any introduction to the methods of dealing with accidental errors in measurement, it is necessary that the student recognize the existence of a law governing their occurrence, and become to some extent familiar, through experience, with the operations of that law. To this end, it is deemed worth while to introduce at this point a number of laboratory exercises or experiments, in which the phenomenon to be studied is the distribution of errors as governed by the law of chance. The term "laboratory" refers to the method only; the exercises may be performed at one's study table without any special apparatus.

1. No better analogy to the behavior of accidental errors can be found than in the manner in which shots fired at a target are found to distribute themselves with respect to a point fired at. To illustrate this experimentally, take a sheet of ordinary foolscap or other ruled paper and with a black pencil make the ruled line nearest the middle of the sheet heavier than the others, so as to be distinctly visible a few feet away. Lay the paper on a board or smooth book, and place it, face upward, on the floor. Take a rather long pencil lightly between the extended finger-tips of both hands, and standing with the eye directly over the black line on the paper, hold the

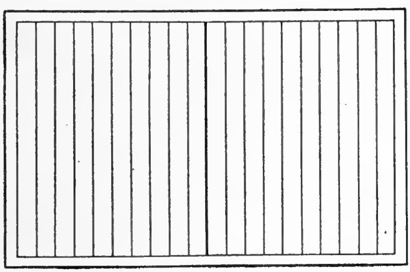

Fig. 1

pencil, point downward, over the line, and endeavor to drop it so as to strike the line with the descending pencil-point. In other words, make the central line a target; the shots will be self-recorded by the dots on the paper. Take at least a hundred shots in this manner, each time trying with all possible skill to hit the central line. Having done this, prepare another sheet of paper ruled off in a similar manner (ordinary coördinate paper will do) and plot on it a curve whose ordinates represent the relative number of shots found to have struck in each compartment of the ruled target and whose abscissas represent the distances of the respective compartments from the central line. In case a shot appears to have struck exactly upon one of the lines, assign it to the compartment on the side toward the center.

Can you think of any influence that might, in this experiment, be analogous to a persistent error in measurement? What effect would it have on the curve? *Keep the data for future use.*

2. On a sheet of smooth paper, draw a line with a hard, sharp pointed pencil and mark two points on it about a foot apart. The exercise is to measure this line with a metric scale to hundredths of a centimeter, estimating the hundredths as tenths of a millimeter. In order to avoid prejudice, it will be well to place a third point somewhere between the others, and measure the line in two segments, a and b. Now measure a and b alternately, using the differential method, until each has been measured, say, a hundred times. Add the corresponding pairs of values and record the sums as the measured lengths of the line. Find the mean of the hundred values to the nearest hundredth of a centimeter, and record the departure from it of each of the observations, plus or minus. These departures are the *residuals* of the observations (Art. 7). It will be noticed that a large number of residuals have the same value. Determine how many there are of each value, separating positive from negative, and plot a curve whose abscissas represent the values of the residuals and whose ordinates represent the numbers of residuals having those respective values. A convenient scale should be used: for example, on the abscissas, let 1 cm. represent 0.1 mm. of residual, and on the ordinates, let each residual be represented by a millimeter. *Keep the data for future use.*

What change would have to be made in the curve if the abscissas and ordinates were the values and numbers, respectively, of the *true errors* instead of the residuals, supposing that there is any means of knowing the former?

3. The preceding exercise may be varied by using, for the measured quantity, an angle of exactly 180°, measuring it in two segments with a protractor to tenths of a degree. In this case the true value, and hence the true errors, are known. *Keep the data.*

4. Do the curves obtained from the preceding exercises bear any resemblance to each other? Construct a smooth curve which seems to be typical of them. Does this curve resemble any familiar geometrical form? Plot the curve $y = 2^{-x^2}$, taking 10 cm. as the unit for both abscissas and ordinates and assigning to x the successive values 0, 0.1, 0.2, 0.3, etc., both positive and negative.

5. From the results of the foregoing exercises, does there appear to be any relation connecting the magnitude of an error with the frequency of its occurrence? Can you assign any reason for such a relation? Do positive errors appear to occur any more frequently, in the long run, than negative errors, or *vice versa?*

12. Remarks on the Distribution of Errors.—The curve to which the preceding exercises have introduced us is commonly called the *probability curve,* though a better name would be the *curve of departures,* as will appear later. Superficially it somewhat resembles the "witch," a typical case being shown in Fig. 2. The student must not expect that any curve plotted from the results of such experiments as the foregoing will be smooth and regular, like the curve here shown; actual curves are broken and irregular. But the greater the number of observations or data, the nearer will the actual departure curve assume the smooth, symmetrical form assigned by theory.

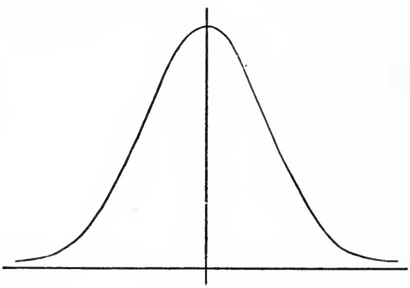

FIG. 2

The results of experiments, as we have seen, and theoretical considerations, as will appear, both point to the following facts regarding the distribution of accidental errors, all of which may be deduced from an examination of the curve.

1. *The frequency with which an accidental error of given magnitude occurs depends upon the magnitude of the error.*

2. *Large errors occur less frequently than small ones.*

3. *The error distribution is symmetrical; that is, positive and negative errors of the same magnitude occur with the same frequency.*

Though these laws do not of course apply absolutely in any one case, yet they express the general tendency of error, and, in fact, the general tendency of all accidental departures from the normal or mean, as, for example, the statures of individual people as compared with the average stature of the race. In theoretical discussions, the number of observations made, or of data considered, is regarded as infinite, and the curve as strictly symmetrical.

In the case of measurements, with which we are here concerned, if the results are affected by *persistent* error from any source, they will be found to cluster about the theoretical *most probable value* of the measured quantity instead of the true value, there being now an appreciable difference between the two. The whole curve of errors now becomes a curve of residuals, and is merely shifted a little to one side or the other according as the persistent error is positive or negative. If, for example, in the second exercise of Art. 11, the scale used had its spaces slightly too long, the whole curve would be shifted a little in the negative direction, simply because each observation tends to undervalue the line measured on account of the defect in the scale.

From this consideration it is clear that when, as is really always the case, the true value of the measured quantity is not given by the measurements, a study of the curve of residuals will reveal nothing as to the presence or absence of *persistent* errors. The law of probability of error is concerned only with *accidental* errors, that is, those whose causes are of temporary duration, — the result, as we say, of pure chance.

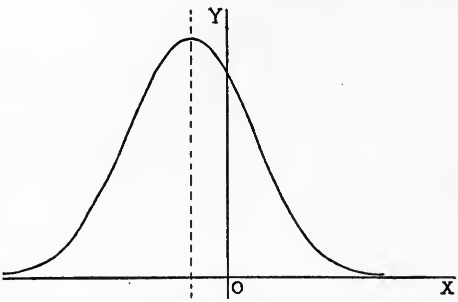

Fig. 3

13. Precision. — On comparison of the results of different sets of measurements, even upon the same quantity, it is found that the error curve is not of constant form. Every gradation is met with (Fig. 4), from low, flat curves to high, pointed ones. This peculiarity may be observed when we make several series of measurements upon the same quantity by different methods. The variation is easily interpreted.

Compare, for example, curves A and D. In the case of A there are nearly as many large errors as small ones.

For shots fired at a target, this would indicate poor marksmanship or long range; in measurement, it means random judgment, crude instruments, or circumstances which render the work difficult. In the case of *D*, on the other hand, the number of large errors is very small, the great body of results being crowded closely about the mean and indicating its position with considerable definiteness. From this it is clear that the form of the error or residual curve depends upon the precision with which the observations have been made.

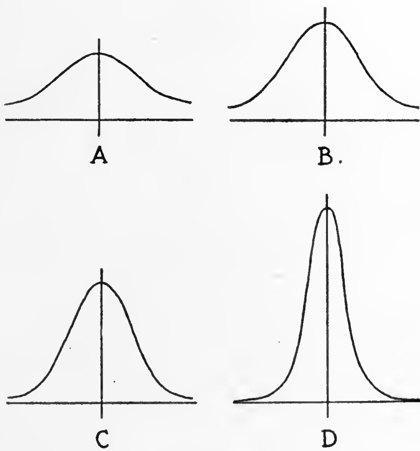

Fig. 4

To illustrate what is meant by *precision*, let two parties of observers each make a set of measurements on the distance between two stakes, the one with a ten-foot pole, the other with a steel tape. The most probable value deduced from one set may not differ much from that deduced from the other, but the residual curves plotted from the two sets of results will show considerable difference of precision, mainly on account of the larger number of times that the ten-foot pole must be laid down and its consequent greater liability to error.

The form of the residual curve may therefore be used as a test of the efficiency of an observer, an instrument or a method of measurement. It will be seen later that the same test can be applied by means of mathematical formulas, without the labor of plotting the curve (Chapter VIII).

14. Mathematical Expression of the Law of Error. — The evident existence of some law governing the distribution of errors leads us to inquire what that law is, and whether it is expressible by a simple mathematical relation. Some of the facts concerning the behavior of errors have already been deduced; but the theoretical expression of the law itself, and even the very language in which it is expressed, must be reserved until the student has reviewed some of the fundamental principles of the *theory of probabilities* and has been introduced to some of the special problems in probability upon which the theory of errors is found to depend. The following chapter is, therefore, devoted to this subject.

CHAPTER III

ON PROBABILITIES

15. Fundamental Principle. — It is a common remark that one thing is more likely to happen than another. In speaking thus, one concedes that either of the two events *may* happen, and attempts no prediction as to which *will* happen, if either; yet he recognizes a preponderance of the likelihood of one event over that of the other.

In the kind of magnitude here recognized, that is, likelihood or *probability*, there is, in the great majority of cases, no means of measuring or giving numerical expression to its relative degrees. It is said that corn growing on low ground is more likely to be caught by frost than that on high ground, but there is no means of telling *how many times* more likely it is.

It is possible, however, to give such a definite meaning to the term *probability* that the relative probabilities of some simpler events may be calculated and expressed. In framing such a definition, it is necessary to recognize an important principle in the operation of chance, governing the behavior of events whose causes are at least partly manifest, and lying at the foundation of the whole course of reasoning that gives rise to the idea of mathematical probability.

The principle is this. *If a number of different events are equally possible as regards constant conditions (that is, if there is no persistent reason why one should occur rather than another), and all are repeatedly given opportunity to occur, they will in the long run occur with equal average frequency.* The same principle may be expressed by saying that if we observe events occurring with equal frequency, we conclude that the constant conditions under which they occur are uniform.

The principle is well illustrated by the throwing of dice. If a die is exactly cubical, of homogeneous material (not "loaded") and the spots do not shift the center of gravity to one side, and if it be cast a great number of times absolutely at random, each face will come up, on the average, one throw out of six. (Of course these ideal conditions are not realized in practice.)

We are so accustomed to the operation of this *law of probability* in daily experience that it is taken as a matter of course, like the force of gravitation; yet its existence is really a mystery. We are here obliged to admit that there is a law controlling the operations of *chance*, — the one thing that would seem to obey no law.

16. Definition of Mathematical Probability. — Definite numerical significance may now be given to the probability of occurrence of certain classes of events.

If an event may occur in a equally possible ways, and at the same time b equally possible alternatives are presented in all (including the a ways in which the event may happen), then *the probability of the event in question is defined as the ratio*

$$p = \frac{a}{b}. \tag{3}$$

That is, there are a chances favoring the event out of a total of b possible chances; and according to the principle set forth above, if a great number of trials are made, the event does happen, on the average, a times out of b.

As an example, let us express the probability of drawing an ace from a deck of fifty-two playing cards, the drawing being done absolutely at random. Any one of the fifty-two cards may be drawn, so that the total number of alternatives is fifty-two. An ace may, however, be drawn in only four ways, viz., by drawing the ace of spades, the ace of clubs, the ace of hearts or the ace of diamonds. Here, then, $b = 52$, $a = 4$, and the probability of drawing an ace is $\frac{4}{52}$, or $\frac{1}{13}$.

What would be the probability of drawing a *red* ace? Of drawing the *ace of diamonds?*

All problems in probability may be solved by the application of the definition expressed in equation (3). But such direct application would be very difficult in the more complicated cases, and special rules and formulas are therefore to be devised which, when properly classified and applied, greatly simplify such problems.

From the definition, it follows that probability is a purely numerical ratio, and depends upon no unit of measure. Moreover, this ratio cannot exceed unity. The probability unity would denote *certainty*, since if an event may happen in n ways, and only n alternatives are possible, the event must happen. From this it follows that if the probability of an event is p, the probability of its failure to happen is

$$p' = 1 - p. \tag{4}$$

For, if the event can happen in a ways out of b, it can fail to happen in $b - a$ ways out of b, the probability of failure therefore being

$$\frac{b-a}{b} = 1 - \frac{a}{b} = 1 - p.$$

More generally, the sum of the probabilities of all possible alternatives is unity.

The probability zero, on the other hand, implies impossibility. It may be interpreted as meaning that there is no way for the event to happen, *i.e.*, $a = 0$; or in cases where the total number of alternatives is infinite, or at least extremely large, while the event in question may happen in only a very few ways, the zero or infinitely small probability denotes impossibility or at most only extremely remote possibility. But the distinction between absolute impossibility and the case in which the possibility is only remote is of some importance, as will be seen, in the theoretical discussion of the distribution of errors.

17. Permutations. — The solution of problems in probability involves the determination of the number of ways in which an event can occur, as well as the total number of possible alternatives. In very simple cases this may be done by inspection. For example, if one is expecting the arrival of three different persons, A, B, C, it is easy to determine the probability of their coming in the order named. There are obviously six different orders in which they may come; namely, ABC, ACB, BAC, BCA, CAB, CBA. The probability of their coming in the order ABC is therefore $\frac{1}{6}$. But let there be a hundred persons instead of three, and the number of orders becomes so enormous as to be unmanageable by inspection. We must then resort to the use of general formulas.

The *linear permutations* of a number of things are the different ways in which the things may be arranged in a row, or in which they may occur in order of time. There are, for example, six linear permutations of the letters A, B, C.

There is a general expression for the number of permutations of Q different things, derived easily by the following reasoning. Of one thing, there is evidently but one permutation. Of two things, since either may come first, there are two permutations. Of three things, any one may come first, and with a given one coming first, there are two arrangements of the two remaining; therefore the number of permutations of three things is $3 \times 2 = 6$. For four things, by the same reasoning, the number is $4 \times 3 \times 2 = 24$. And in general, the number of permutations of Q things is

$$P_Q = Q(Q-1)(Q-2) \cdots 3 \cdot 2 \cdot 1 = Q! \qquad (5)$$

We have here assumed that none of the Q things are duplicates. Let us now take a case where there are duplicates, as in the group of letters AABBBCCCC. If we distinguish between the different A's, etc., the case is, of course, the same as if the letters were all different. But if we consider one A, for example, the same as another, and permute without regard to which of them is being used in a particular place, the number of permutations is less. It will be easy for the student to show as an exercise that if there are, in a number of things, m of one kind, n of another, r of another, s of another, etc., the total number being $Q = m + n + r + s + \cdots$, the number of *distinguishable* permutations of the Q things is

$$P_Q^{(m \cdots s)} = \frac{Q\,!}{m\,!\, n\,!\, r\,!\, s\,! \cdots}. \tag{6}$$

Thus for the above set of nine letters, of which two are A's, three B's and four C's, the number is

$$P_9^{(2,\,3,\,4)} = \frac{9\,!}{2\,!\, 3\,!\, 4\,!} = 1260.$$

If there is only one thing of a kind in the group, so that $m, n, \ldots$ are each unity, (6) becomes equivalent to (5).

18. Combinations. — The different groups which can be formed from a number of things, taken so many at a time, are called *combinations*. The different combinations of the three letters A, B, C taken two at a time are AB, AC, BC.

If we further take into account the possible permutations of each combination, we have what may be called the *permuted combinations* of the series of things considered.

Thus the permuted combinations of A, B, C are AB, BA, AC, CA, BC, CB. It is easier to derive first the general formula for the number of permuted combinations.

Let the number of permuted combinations of Q things taken n at a time be designated by the symbol $PC_Q^{(n)}$. If they are taken two at a time, any one of the Q things may be taken as the first, and any one of the $Q - 1$ remaining things may be taken as the second, so that

$$PC_Q^{(2)} = Q(Q-1).$$

If taken by threes, any one of the $Q\,(Q - 1)$ permuted combinations of two each may constitute the first two, followed by any one of the $Q - 2$ remaining things as the third. Then

$$PC_Q^{(3)} = Q(Q-1)(Q-2).$$

By continuing the same reasoning until there are n things taken at a time, we readily deduce

$$PC_Q^{(n)} = Q(Q-1)(Q-2) \cdots \text{to } n \text{ factors.} \qquad (7)$$

If $n = Q$, this becomes identical with (5), since all the things are permuted at once.

To express now the number of combinations of Q things taken n at a time, without regard to their arrangement, it is necessary only to note that the $PC_Q^{(n)}$ permuted combinations include not merely those made up of different things, but all the permutations of each of the groups of n different things. Since n things are permuted in $n!$ different ways (5), there are only $\frac{1}{n!}$ as many combinations as permuted combinations. That is,

$$C_Q^{(n)} = \frac{Q(Q-1)\cdots \text{ to } n \text{ factors}}{n!}. \tag{8}$$

As an illustration of this problem, let us find how many different hands at whist, each made up of thirteen cards, could be drawn from a pack of fifty-two cards. Here $Q = 52$, $n = 13$, and the solution is

$$C_{52}^{(13)} = \frac{52 \cdot 51 \cdot 50 \cdots 40}{1 \cdot 2 \cdot 3 \cdots 13} = 635{,}013{,}559{,}600.$$

Then the probability of drawing any one specified hand is, by definition, the exceedingly small reciprocal of this number.

As a final problem in combinations, let there be s series of things, the number of things in the respective series being $Q_1, Q_2, \cdots, Q_s$; to determine how many different combinations can be formed by taking one thing from each series.

The number of combinations of two each which can thus be formed from the first two series is Q_1Q_2, since each of the Q_1 things in the first series can be successively combined with each of the Q_2 things in the second. Bringing in now the third series, each of the Q_1Q_2 combinations just considered may be combined with each of the Q_3 members of the third series, making $Q_1Q_2Q_3$ combinations; and so on. Clearly, then, the number of combinations that can be so formed from the s series is the product

$$_sC_{Q_1 \cdots Q_s} = Q_1Q_2Q_3 \cdots Q_s. \tag{9}$$

For example, let there be three series of letters:

$$\begin{array}{lll} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & \\ A_4 & & \end{array}$$

The number of combinations of the form ABC that can be selected from them is $4 \times 3 \times 2 = 24$. Let the student write these combinations.

19. Probability of Either of Two or More Events.—If the probability of an event A is p_a, that of an event B is p_b, that of an event C is p_c, etc., then it is easy to show that *the probability that one or another of these events will happen is* $p_a + p_b + p_c + \cdots$, it being understood that *only one* of these events can happen. For, suppose the event A may happen in a ways, the event B in b ways, etc., and that the total number of alternatives is T. (In general, T will be greater than the sum of a, b, etc.; that is, it is not necessary that any one of the events A, B, etc. shall happen.) Then by definition, the probabilities of the respective events are

$$p_a = \frac{a}{T}, \quad p_b = \frac{b}{T}, \text{ etc.}$$

If we designate by X the event of some one of the events A, B, etc. happening, without specifying which, then, since the number of ways in which X can occur is $a + b + \cdots$, the probability of X is

$$p_x = \frac{a + b + \cdots}{T} = p_a + p_b + \cdots. \tag{10}$$

As an example, let there be in a bag three balls of iron, two of glass, five of wood, seven of lead, six of rubber, one of ivory and four of copper, and let one be drawn at random.

The probability that a *metal* ball will be drawn is then $\frac{3}{28} + \frac{7}{28} + \frac{4}{28} = \frac{1}{2}$, since a metal ball is drawn if the result be an iron ball, a lead ball or a copper ball.

This principle of additive probabilities for alternative events is made use of in estimating premiums on so-called "joint" life insurance policies.

20. Probability of the Concurrence of Independent Events. — Quite a different problem is that of finding the probability that *all* of a specified set of independent events shall occur. As before, designate the respective events by A, B, C, etc., their respective separate probabilities by p_a, p_b, etc.; and designate the event of their *all* occurring by Z. Suppose the event A may occur independently in a ways out of α alternatives, B in b ways out of β alternatives, etc., so that

$$p_a = \frac{a}{\alpha}, \quad p_b = \frac{b}{\beta}, \text{ etc.}$$

It is of course understood that when all the events A, B, C, etc., are given opportunity to happen, some one of the α alternatives connected with A *will* happen, some one of the β alternatives connected with B *will* happen, etc., but that *only one* of each can happen. The total number of *possible* outcomes is therefore the number of combinations that can be formed by selecting one from each group of alternatives, namely, the product $\alpha\beta\gamma \cdots$ (9). Likewise, the number of different ways in which the events A, B, etc., can *all* occur is the product $abc \cdots$. It follows that the probability of *all* occurring, that is, the probability of the event Z, is

$$p_z = \frac{abc \cdots}{\alpha\beta\gamma \cdots} = \frac{a}{\alpha} \cdot \frac{b}{\beta} \cdot \frac{c}{\gamma} \cdots = p_a p_b p_c \cdots. \qquad (11)$$

That is to say, *the probability of the concurrence of two or more independent events is the product of the probabilities of the respective events considered separately.* This product is of course less than any one of its factors.

To make the meaning of this clear, suppose that it is known that a person A will spend five hours in a certain place between 6 A.M. and 6 P.M., and that another person B will spend three hours there during the same interval, but nothing is known as to when these hours will be. If we visit the place at any random moment, the probability of finding A there *at that moment* is $\frac{5}{12}$; the probability of finding B there *at that moment* is $\frac{3}{12}$. Then the probability of finding them *both* there *at that moment* is $\frac{5}{12} \times \frac{3}{12} = \frac{5}{48}$. But the probability of finding *either* A *or* B there is $\frac{5}{12} + \frac{3}{12} = \frac{2}{3}$. Let the student analyze this problem more closely, showing how the values stated for the probabilities can be deduced from the definition of probability.

21. The Coin Problem. — Suppose that the result of an experiment may be either one of two things, A and B, which are equally likely to occur, and that the result must be one or the other, but cannot be both. The probability of either result is then $\frac{1}{2}$. Let us determine what is the probability, if the experiment be performed Q times, that it will result n times one way and $Q - n$ times the other. A coin tossed at random illustrates the problem; for example, if it be tossed a hundred times, what is the probability that it will turn up heads thirty-eight times and tails sixty-two times? Here $Q = 100$, $n = 38$ (or 62). The required probability is a function of n; and, furthermore, it is evidently the same function of $Q - n$ that it is of n.

The first thing to determine is, in how many ways the result A may happen n times out of Q. In 100 throws of the coin, heads may come up 38 times and tails 62 times in a large variety of ways: for example, 1 H., 2 T., 37 H. and 60 T., in order, would fulfill the condition; or, equally well, 8 H., 5 T., 30 H., and 57 T. The number of ways in which A may happen n times and B, $Q - n$ times is readily seen to be equal to the number of *distinguishable* permutations of Q things, n being of one kind and $Q - n$ of the other (Art. 17), which is

$$P_Q^{(n,\, Q-n)} = \frac{Q\,!}{n\,!\ (Q-n)\,!}. \tag{12}$$

Or, it is equal to the number of combinations of Q things taken n at a time, since out of the totality of Q events, the n events A may be selected wherever desired. Hence another expression for the required number is equation (8), which the student may readily show to be equivalent to (12). We shall use equation (12).

Next we must determine the total number of possible alternatives. This may be done by adding together the values of the expression (12) obtained by giving n all integral values from 0 to Q. These are tabulated below.

n	$P_Q^{(n,\, Q-n)}$
0	1
1	Q
2	$\frac{Q(Q-1)}{2\,!}$
3	$\frac{Q(Q-1)(Q-2)}{3\,!}$
. .	. . . .
$Q-2$	$\frac{Q(Q-1)}{2\,!}$
$Q-1$	Q
Q	1

The expressions obtained for $P_Q^{(n,\, Q-n)}$ as n varies from 0 to Q are at once seen to be the successive coefficients of the expansion of a binomial with exponent Q, and their sum is therefore equal to 2^Q. That is,

$$\sum_{n=0}^{n=Q} \frac{Q\,!}{n\,!\;(Q-n)\,!} = 2^Q. \tag{13}$$

We now have the two elements of the solution of the coin problem, namely, the number of ways in which event A can happen n times and event B, $Q-n$ times, given by (12), and the total number of alternatives, given by (13). The required probability of the specified outcome is therefore

$$p_{n,\, Q-n} = \frac{Q\,!}{n\,!\;(Q-n)\,!\;2^Q}. \tag{14}$$

Thus the probability of the result 38 heads and 62 tails is

$$p_{38,\,62} = \frac{100\,!}{38\,!\;\;62\,!\;\;2^{100}}.$$

The reason for introducing the coin problem will appear later.

22. Important Exercise. — It is now very desirable, for the purpose in hand, that the student faithfully perform the following exercise. Suppose a coin tossed ten times. Find the probability of each of the following possible results:

n	$10-n$	n	$10-n$
10 heads and	0 tails	4 heads and	6 tails
9 heads and	1 tails	3 heads and	7 tails
8 heads and	2 tails	2 heads and	8 tails
7 heads and	3 tails	1 heads and	9 tails
6 heads and	4 tails	0 heads and	10 tails
5 heads and	5 tails		

Considerations of symmetry will shorten the work. Now plot a series of points, of which the abscissas shall represent the quantity $n - 5$ (n being the assumed number of heads) and the ordinates the computed probabilities of the respective results, using a convenient scale for each. Does the resulting curve resemble any other curve that has hitherto come to your notice? Test the theory by actual experiment with a coin, or better, a flat bone disk, recording the outcome of every ten throws. This exercise, if carefully performed and studied, will assist the student to a much better understanding of the behavior of error distribution than he could attain without it.

23. Empirical or Statistical Probability. — As has been before noted, in the majority of the events of life, the conditions are far too complicated to admit of any such analysis as has been applied to the problems concerning cards, balls, coins, etc. But it may happen that, when the conditions are sufficiently constant throughout a long series of observations, the probability of such a complex event may be deduced from the observed results. It is upon this principle that reliance is placed upon statistics. As a very important example, we cannot compute, by any theoretical formula, the probability that a person ten years old will live to be sixty. But if the statistics show that out of every 100,000 persons ten years of age, 58,000 do live to be sixty, we may conclude that the required probability is 0.58. In a similar manner it has been determined that the probability that a person sixty years old will live one more year is 0.97, since 97 per cent. of those attaining the age of sixty do live another year. The importance of such knowledge, and its bearing on the practical problems of the world, such as life insurance, are self-evident.

EXERCISES

24. 1. What is the probability of throwing a six in two throws of a single die? In a single cast of two dice?

2. How many possible arrangements are here of the letters in the word *travel?* How many distinguishable arrangements of the letters in the word *minimum?*

3. A hostess wishes to have as guests the same number of ladies as gentlemen. She has planned to have the guests find their partners by the matching of colored ribbons, each guest wearing two colors, there being but five different colors in all. How many guests may she invite? Would she be able to distinguish more couples by giving each guest three colors? Four colors?

4. How many different football elevens could be formed from a squad of fifteen players? What chance would any one player have of getting on a picked eleven if it were chosen by lot?

5. In a certain organization there are two candidates for an office and thirty voters. What is the probability that there will be a tie? That either candidate will receive a majority of exactly $\frac{2}{3}$?

6. By measuring a number of ordinates of the curve obtained from the target experiment (Art. 11), determine empirically the relative probabilities of the respective errors in aim.

7. Find the number of combinations of three things in eight; the number of permuted combinations.

8. A student council is to be made up of five members from each of four college classes, whose respective memberships are 150, 105, 75 and 56. In how many different ways may the council be made up?

9. Three things are selected at random from eight, then returned; and then another random selection of three is similarly made. What is the probability that the two selections will be exactly reverse permutations of the same three things?

10. A new janitor has a bunch of twenty-eight nearly similar keys, one for each door of the building. What is the probability of his being able to unlock the first three doors with only one trial each? Solve also on the supposition that he marks the keys as he discovers them.

11. What is the probability that a whole number of four figures, selected at random, will have two figures alike and the other two figures alike?

12. The following data are taken from the American Experience Mortality Tables used by life insurance companies in computing risks.

Out of 100,000 persons ten years of age,

100,000 live to be at least 10
92,637 live to be at least 20
89,032 live to be at least 25
85,441 live to be at least 30
81,822 live to be at least 35
78,106 live to be at least 40
74,173 live to be at least 45
69,804 live to be at least 50
64,563 live to be at least 55
57,917 live to be at least 60

49,341 live to be at least 65

38,569 live to be at least 70

Plot a curve representing these data.

13. Find your own chance of living to be at least seventy years old, using the curve in Ex. 12.

14. Three men are respectively 30, 27 and 22 years old. Find the probability that they will all live to be 60 or over.

15. Two brothers are respectively 25 and 35 years old, and their father is 60. The elder is to inherit the estate if living at the father's death, otherwise the younger will inherit it; and at the death of the elder son, the younger will, if living, inherit the estate from him. Find the probability that the elder son will own the estate five years hence; that the younger will own it ten years hence.

CHAPTER IV

THE ERROR EQUATION AND THE PRINCIPLE OF LEAST SQUARES

25. Analogy of Error Distribution to Coin Problem. — It was pointed out in Art. 5 that an error in measurement is the resultant of innumerable small disturbances of different kinds, the presence of many of which may not be even suspected. These disturbances operate, some in one way, some in the other; that is, some tend to produce positive error and some negative. The resultant error depends on the relation of the number of positive disturbances to the number of negative disturbances. If nearly all are positive, the error will be positive and large; if nearly all are negative, a large negative error will result; while if about the same number are positive as negative, the error will be small. This does not imply that the disturbances are all of the same magnitude. By way of illustration, suppose we select from a sand-heap, at random, a thousand grains of sand, and put eight hundred of them on the left pan of a balance and two hundred on the other. There is hardly a remote possibility that the former will not very largely overbalance the latter. But if we put five hundred on each pan, there will be little preponderance one way or the other. And this does not imply, by any means, that the grains are all equal in weight; some individual particles may be ten times heavier than others.

Now there is a remarkable and useful analogy between the theory of error distribution and the so-called coin problem (Art. 21), an analogy that the student has no doubt already observed. It is easily deduced that the most probable result of a number of throws of a coin is that they will be half heads and half tails. In general, this normal result will be departed from in greater or less degree, so that in one hundred throws we frequently obtain fifty-five heads and forty-five tails, or less frequently, sixty heads and forty tails, etc. This departure from the normal or most probable result may be looked upon as a sort of error. Like an error in measurement, it is complex in character, depending upon the result of each individual throw. Each throw, head or tail, affects the final outcome one way or the other, just as each small disturbance, positive or negative, affects the result of an observation in measurement. A little consideration of the two cases will bring out their analogy quite clearly.

We are therefore justified in assuming that the probability of the occurrence of an error is a function of the magnitude of the error in much the same manner as the probability of a departure from the half-and-half result in tossing the coin is a function of the extent of the departure. It is, in fact, upon this line of reasoning that Hagen's deduction of the error equation is based. The deduction is, however, rather cumbersome, and we shall follow instead the more elegant method due to Gauss.

It may be remarked here that the theory of departures is a very general one and finds application in a large variety of problems of common experience, such as the distribution of shots on a target and the distribution of given characteristics among the members of a biological group.

26. The Most Probable Value from a Series of Direct Measurements. The Arithmetical Mean. — If a series of measurements be made upon a single quantity under as nearly constant conditions as possible, the result is, in general, a series of different values, each approximating the true value of the measured quantity. No one of them *is* the true value, however, and it now becomes a matter of judgment to select, from all possible values, such a one as will make the actual distribution of the results appear most natural. An analogous case would be this: Suppose that after all the shots had been fired at the target in the first exercise of Art. 11, the central line aimed at were erased, and we were required, from the given distribution of the shots, to judge as to where the line had been; we could do no better than to select a position that, from the concentration of shots about it and their symmetry with respect to it, seems to be the *most probable* one. Likewise, in a series of measurements, we are aiming at a true value, the most probable location of which can only be estimated by an examination of the distributed results.

The symmetry of the distribution of errors in cases where the true value is known, as also in the analogous coin and target problems, leads at once to the common axiom of experience, that the best value to adopt in the case of a series of direct observations on a single quantity is the *arithmetical mean* or *average* of the observations. If the several measured results be designated by $s_1, s_2, \cdots, s_n$, and their mean be m, then the residuals (Art. 7) are respectively

$$\begin{aligned} \rho_1 &= s_1 - m, \\ \rho_2 &= s_2 - m, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho_n &= s_n - m. \end{aligned}$$

Adding these we obtain

$$\Sigma\rho = \Sigma s - nm = 0, \qquad (15)$$

which expresses the fact that the arithmetical mean of the results is the value *with respect to which they are symmetrically placed*, the algebraic sum of the differences being then equal to zero; and that therefore *this mean is the most probable value that can be assumed.*

27. Gauss's Deduction of the Error Equation. — Let q represent the unknown true value of a quantity and let a series of n measurements be made upon it, the number n being supposed very large. Let the errors arising from the respective measurements be $x_1, x_2, \cdots, x_n$. It has been seen that the probability of the occurrence of an error is some sort of inverse function of its magnitude. Designating the probabilities of these respective errors by $y_1, y_2, \cdots, y_n$, this fact may be expressed by the equations

$$\begin{aligned} y_1 &= f(x_1), \\ y_2 &= f(x_2), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= f(x_n). \end{aligned}$$

It is the form of this function $f(x)$ that we are seeking to determine.

Now, as above noted, we do not know the true value q of the observed quantity, and therefore we do not know the true errors x. We may however assume various tentative values for q and study the resulting tentative systems of errors, particularly with a view to selecting that one which seems most naturally distributed, in accordance with the notions of error distribution that experience has taught us. In this sense, therefore, we may think of q and the errors x as *variables* subject to our control, and the probabilities y will then vary accordingly. With this understanding, then, we are seeking to find that system of values for the x's which, *as a whole,* has the greatest probability.

If the outcome of a series of measurements be the system of errors $x_1, x_2, \cdots, x_n$, this result may be looked upon as the concurrence of n independent events, each of which is the obtaining of one of the errors x. Then according to Art. 20, the probability of this outcome, designated by Y, is the product of the probabilities of the separate errors, namely

$$Y = y_1 y_2 \cdots y_n = f(x_1) \cdot f(x_2) \cdots f(x_n). \tag{16}$$

In order, therefore, that the system of x's shall have the greatest probability, as required, the value assumed for q

should be such that the expression Y is a maximum; which condition will be attained when

$$\frac{dY}{dq} = 0. \tag{17}$$

Differentiating (16)

$$\frac{dY}{dq} = \frac{Y}{f(x_1)} \cdot \frac{df(x_1)}{dq} + \frac{Y}{f(x_2)} \cdot \frac{df(x_2)}{dq} + \cdots$$

$$+ \frac{Y}{f(x_n)} \cdot \frac{df(x_n)}{dq} = 0,$$

or $$\frac{Y}{dq}\left[\frac{df(x_1)}{f(x_1)} + \frac{df(x_2)}{f(x_2)} + \cdots + \frac{df(x_n)}{f(x_n)}\right]$$

$$= \frac{Y}{dq}[d \log f(x_1) + d \log f(x_2) + \cdots + d \log f(x_n)] = 0.$$

Now let $d \log f(x) = \phi(x)dx$, where ϕ is another unknown function of x, thus simply related to f. Then canceling out the Y,

$$\phi(x_1)\frac{dx_1}{dq} + \phi(x_2)\frac{dx_2}{dq} + \cdots + \phi(x_n)\frac{dx_n}{dq} = 0. \tag{18}$$

If the results of the respective n measurements on q be designated by $s_1, s_2, \cdots, s_n$, having definite, fixed values, then the errors x are (Art. 7)

$$x_1 = s_1 - q,$$

$$x_2 = s_2 - q,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_n = s_n - q,$$

from which, at once,

$$\frac{dx_1}{dq} = \frac{dx_2}{dq} = \cdots = \frac{dx_n}{dq} = -1. \tag{19}$$

(18) then reduces to

$$\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n) = 0. \tag{20}$$

We already understand enough of the law of error distribution to know that *when the number of observations is very large,* the number of positive errors of given magnitude about equals the number of negative errors of the same magnitude, and that therefore the algebraic sum of the errors is approximately zero. Since in our theoretical discussion the number of observations is indefinitely large, we may write, therefore, as another condition fulfilled by the errors,

$$x_1 + x_2 + \cdots + x_n = 0. \tag{21}$$

It now remains to deduce from the two equations (20) and (21) the form of the function ϕ, from which the original function f may then be obtained. It is not difficult to see that the equations are satisfied if

$$\begin{aligned} \phi(x_1) &= Kx_1, \\ \phi(x_2) &= Kx_2, \\ &\cdots\cdots \\ \phi(x_n) &= Kx_n, \end{aligned}$$

where K is a constant. A mathematical proof that this is the *necessary* relation is given in Note *A*, Appendix, being omitted here to avoid distracting attention from the main problem. We may write, then,

$$\phi(x) = Kx; \tag{22}$$

or since $\phi(x)dx = d \log f(x) = d \log y,$

$$d \log y = Kxdx.$$

Integrating, $\log y = \frac{1}{2} Kx^2 + c',$

or

$$y = e^{\frac{1}{2}Kx^2 + c'}. \tag{23}$$

This is one form of the error equation.

The expression may, however, be so modified as to exhibit the relation to better advantage. We have seen that the larger the error, the less likely it is to occur: the larger x is, the smaller is y. Clearly, then, K must be a *negative* quantity. Replacing $\frac{1}{2} K$ by $-h^2$, and $e^{c'}$ by the constant c, the equation assumes the more usual and more useful form

$$y = ce^{-h^2x^2}. \tag{24}$$

This is the most important equation in the theory of errors, and should be committed to memory.

28. Discussion of the Error Equation. — It will be interesting to examine equation (24) to see how closely the law of error thereby expressed agrees with the conclusions already reached.

The bilateral symmetry of the function y is evident from the occurrence of x in the second degree only. This indicates the equal probability of positive and negative errors of the same magnitude. The function approaches zero as x increases in magnitude; which means that very great errors are extremely improbable. The derivatives of the function are

$$\frac{dy}{dx} = -2\,ch^2xe^{-h^2x^2}, \tag{25}$$

$$\frac{d^2y}{dx^2} = -2\,ch^2e^{-h^2x^2}[1 - 2\,h^2x^2]. \tag{26}$$

From these, since $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ when $x = 0$, there is a maximum value of y when $x = 0$; that is, the error zero has the greatest probability.

The curve shown in Fig. 5 represents the function, and has some interesting properties. Its symmetry, asymptotic character and central maximum merely illustrate what has just been deduced from the equation. The Y intercept, or maximum ordinate, is the quantity c, since $y = c$ when $x = 0$.

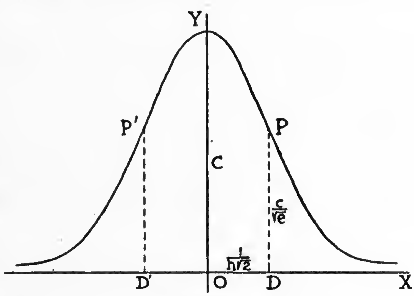

FIG. 5

If we put $\frac{d^2y}{dx^2}$ equal to zero, which is the condition for points of inflection, (26) gives

$$1 - 2\,h^2x_i^2 = 0,$$

$$x_i = \pm\frac{1}{h\sqrt{2}}. \tag{27}$$

This is the distance OD or OD', corresponding to the points of inflection P and P'. The ordinate of these points is, by substitution,

$$y_i = +\frac{c}{\sqrt{e}}, \tag{28}$$

and is therefore proportional to c.

The quantity c represents the probability of the error zero. Now the probability of any *given* error x is a function of both c and h, since it changes if we change either c or h. It would thus appear that c and h have something to do with the precision of the measurements, and that they are therefore connected with each other. We shall see later (Art. 54) that this is the case, and also that there is still another factor in the probability of a given error, depending upon the value of the smallest scale interval in terms of which the measurements are expressed.

29. The Principle of Least Squares in its Simplest Form. — We are now in position to make an introductory statement of the important principle which gives this branch of science its name, — the *principle of least squares*. Before we are through with the theory of errors, the principle will have been stated several times in successively more complicated forms, as the problems to which it is applied become more and more general. So far we have been considering only the simplest case, namely, that of observations of equal precision upon a single quantity; and while for this case the method of deducing the most probable value is clear without reference to the principle of least squares, still it will be interesting and instructive to observe how the assumption of the arithmetical mean as the most probable value may be shown to be in accordance with that principle in the simple form here stated.

The simple form of the principle referred to is as follows:

The most probable value of a measured quantity that can be deduced from a series of direct observations, made with equal care and skill, is that for which the sum of the squares of the residuals is a minimum.

The law governing the distribution of errors has already been deduced theoretically, and the experience of numberless experimenters testifies to its truth. We have therefore a right to expect that, when we have made a long series of measurements upon a single quantity, our observations will have grouped themselves around the true value in a manner approximately consistent with the error equation (24). Then it is logical for us to assume a value for the measured quantity, such that the results of the measurements will be so grouped with respect to it. This is the so-called *most probable value,* and it is the office of the principle of least squares, in any case, to point out the way of arriving at it.

Let the results of the n observations be s_1, s_2, $\cdots$, s_n. Then if we designate the most probable value sought by m, there will arise a corresponding series of residuals ρ_1, ρ_2, $\cdots$, ρ_n, each of which is found by subtracting m from the corresponding observation s (Art. 7). If m be properly chosen, the residuals derived from it will, like true errors, be found to be distributed in accordance with the exponential law of error probability (24), so that the probabilities of the respective residuals are

$$y_1 = ce^{-h^2\rho_1^2},$$
$$y_2 = ce^{-h^2\rho_2^2},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$y_n = ce^{-h^2\rho_n^2}.$$

The probability of the simultaneous occurrence of the assumed system of residuals is then (Art. 27)

$$Y = y_1 y_2 \cdots y_n = c^n e^{-h^2(\rho_1^2 + \rho_2^2 + \cdots + \rho_n^2)}. \qquad (29)$$

Now if m, and consequently the residuals ρ, are to be so chosen that the resulting distribution is the most probable one in accordance with the law of error, these quantities must be given such values that the probability Y is as great as possible. But this will be secured, evidently, by making $\rho_1^2 + \rho_2^2 + \cdots + \rho_n^2$ as small as possible, as will be seen at once from (29). That is to say, *m should be so chosen that $\Sigma\rho^2$ is a minimum,* which is the principle of least squares stated above.

In order to find what this required value of m is, we may write

$$\Sigma\rho^2 = (s_1 - m)^2 + (s_2 - m)^2 + \cdots + (s_n - m)^2$$
$$= \text{a minimum.}$$

Hence

$$\frac{d}{dm}\Sigma\rho^2 = -2[(s_1 - m) + (s_2 - m) + \cdots + (s_n - m)] = 0,$$

or reducing,
$$m = \frac{s_1 + s_2 + \cdots + s_n}{n}, \qquad (30)$$

which is simply the arithmetical mean of the observations s.

EXERCISES

30. 1. Show how, in the first experimental exercise of Art. 11, the errors of aim may be due to many minor causes, enumerating as many such possible causes as you can think of.

2. Find the algebraic sum of the errors of measurement in the third exercise of Art. 11; also the algebraic sum of the residuals.

3. Plot the curve $y = ce^{-h^2x^2}$, giving the value unity to each of the constants c and h. This may be done by use of logarithms ($e = 2.718 \cdots$). Let the unit abscissa be 10 squares and the unit ordinate 50 squares. Compare with the error curves obtained from Exercises 1, 2 and 3 of Art. 11, and with the coin problem curve obtained in Art. 22.

4. Draw a smooth, symmetrical curve which follows as closely as possible the irregular curve obtained in Ex. 3, Art. 11, making it conform to the known properties of the law of error as represented in Fig. 5. From this curve, determine the relative probabilities of the errors of magnitude 0°.1, 0°.2, etc., out to 5°. By locating the points of inflection, find an approximate numerical value for h.

5. Plot the curve represented by

$$y = (2 - x)^2 + (3 - x)^2 + (4 - x)^2 + (5 - x)^2 + (6 - x)^2.$$

Has it a minimum point? What does this illustrate?

6. The number of rays in the lower valve of a certain species of Atlantic mollusk was counted in 508 individual cases. Of these,

1 had 14 rays,
8 had 15 rays,
63 had 16 rays,
154 had 17 rays,
164 had 18 rays,
96 had 19 rays,
20 had 20 rays,
2 had 21 rays.

Plot a curve in which abscissas represent the number of rays and ordinates the corresponding number of individuals.

What is the probability that two of these mollusks, picked up at random, will each have exactly fifteen rays? (Data from Davenport, *Statistical Methods*.)

7. Tests were made on fifty schoolboys of equal age to ascertain strength of grip. The following data (Whipple, *Manual of Mental and Physical Tests*) are in hundreds of grams.

158	210	248	296	348
175	220	262	301	350
193	225	262	310	353
197	225	267	313	375
197	225	269	315	375
200	226	270	320	403
205	235	273	323	430
206	244	280	325	440
208	244	290	330	440
210	245	294	346	508

Arrange a suitable curve showing departures from the average or normal strength from these data.

8. About two hundred individuals were tested at the University of Iowa for accuracy of tone perception, the results being expressed by the number of vibrations in the departure from the true tone (international A, 435 per sec.) that the individual could distinguish. The data are expressed in per cent.

Departure, Vib.	Per Cent.
1	13.8
2	24.0
3	25.5
5	17.3
8	7.3
12	3.2
17	1.6
23	2.7
30 or over	4.6

Plot a curve representing this distribution, and discuss its form.

9. Out of a class of exactly 100 college freshmen, the age of

1 was 16,	2 was 22,
12 was 17,	1 was 23,
31 was 18,	0 was 24,
22 was 19,	0 was 25,
18 was 20,	1 was 26.
12 was 21,	

Plot curve and discuss its form.

10. Out of over 100,000 public school grades examined by Mr. L. L. Fishwild,

1 per cent. were 50,
1 per cent. were 55,
2 per cent. were 60,
2 per cent. were 65,
5 per cent. were 70,
6 per cent. were 75,
13 per cent. were 80,
13 per cent. were 85,
25 per cent. were 90,
23 per cent. were 95,
9 per cent. were 100.

Plot curve and discuss its form.

CHAPTER V

ON THE ADJUSTMENT OF INDIRECT OBSERVATIONS

31. Observations on Functions of a Single Quantity. — It has been pointed out that measurements are seldom made directly upon the quantities whose values are sought, but are usually made upon functions of them, or functions involving them with other unknown quantities. The former case being the simpler, we shall consider it first.

As a specific problem, let a number of measurements be made upon the diameter of a circle, with the object of determining its area. That is, the quantity really sought is the area, but the direct measurements are made upon the diameter, a function of the area. Supposing the observations to be all made in the same manner, the question arises, what is the most probable value of the area? Is it the arithmetical mean of the areas computed from the separate measurements on the diameter, or is it the area determined by taking, as the diameter, the mean of the measurements upon it? The two are of course not the same.

This question may be answered by the following general deduction. The quantity whose most probable value is sought being q, and the function of it, upon which the observations s are directly made, being $f(q)$, there arise the following approximate statements, known as *observation equations*, each of which represents one of the n measurements:

$$\left.\begin{aligned} f(q) &= s_1, \\ f(q) &= s_2, \\ &\cdot \quad \cdot \quad \cdot \\ f(q) &= s_n. \end{aligned}\right\} \tag{31}$$

$s_1, s_2, \cdots, s_n$ are the results of readings on some sort of scale or measuring instrument applied to the function directly measured.

The errors of the observations are represented by $s_1 - f(q)$, etc., but are not determinate. It is the *most probable value* of q that we are seeking, and if this be represented by m, the *residuals* of the n observations are

$$\left.\begin{aligned} \rho_1 &= s_1 - f(m), \\ \rho_2 &= s_2 - f(m), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho_n &= s_n - f(m). \end{aligned}\right\} \tag{32}$$

There is no reason why the principle of least squares should not apply to this case as well as to the case of direct measurements, since the law of error distribution, or the "law of departures," is universal in its scope. As relating to this sort of measurements, then, the principle of least squares takes the following form: *The most probable value of an unknown quantity that can be derived from a set of observations upon one of its functions is that for which* *the sum of the squares of the residuals arising from these observations is a minimum.*

The sum above referred to is expressed by

$$\Sigma\rho^2 = [s_1 - f(m)]^2 + [s_2 - f(m)]^2 + \cdots + [s_n - f(m)]^2, \quad (33)$$

in which m may be regarded as a variable whose value is to be so adjusted as to render $\Sigma\rho^2$ a minimum. This condition requires that

$$\frac{d}{dm}\Sigma\rho^2 = 0,$$

or, differentiating (33),

$$-2\,[\Sigma s - nf(m)]\,\frac{df}{dm} = 0,$$

$$f(m) = \frac{\Sigma s}{n}. \quad (34)$$

Therefore m, the most probable value of q, is that value whose f-function is the mean of the observations upon $f(q)$.

Thus, the most probable value of the area of a circle, as determined from measurements upon the diameter, is $\frac{\pi}{4}$ times the square of the arithmetical mean of the results of those measurements. A multitude of other illustrations of this principle will occur to any one familiar with such work.

32. Observation Equations for More Than One Unknown Quantity. — Very frequently, in an experimental research, occasion arises to determine, not merely one, but several, unknown quantities or constants which are so involved with each other and with the phenomena directly observed as to render their separate measurement impossible. The following illustrations will make this clear.

In the use of the zenith telescope for finding the latitude of a station, the quantities first sought are the zenith distances of two stars selected for the purpose. The *sum* of the zenith distances is equal to their difference in declination, as given in the star catalogues, and therefore depends upon the results of many very precise measurements made with other instruments at fixed observatories. The *difference* of the zenith distances is measured by means of the micrometer belonging to the zenith telescope, as the instrument is rotated from north to south about the vertical. In this way, neither zenith distance is separately determined, both being found by the simultaneous solution of the equations arising from the above observations.

Again, it is desired to find the relative proportions of sodium chloride ($NaCl$) and potassium chloride (KCl) in a mixture of the two salts. Or specifically, in a given specimen of the dry mixture, to find the number of grams, x, of sodium chloride and the number, y, of potassium chloride. First let the sample be weighed, with the result s_1. Then

$$x + y = s_1.$$

The sample is now dissolved and the chlorine precipitated with silver nitrate ($AgNO_3$), and the total amount of chlorine present calculated by weighing the precipitated silver chloride (AgCl). Denote the total chlorine by s_2. Now, sodium chloride is 0.6123 chlorine, and potassium chloride is 0.4754 chlorine. Hence in x grams of sodium chloride and y grams of potassium chloride, the total chlorine is

$$0.6123\,x + 0.4754\,y = s_2,$$

which furnishes the second observation equation necessary for obtaining x and y. This is another instance in which neither of the unknown quantities is measured separately.

Quite often only certain ones of the unknown quantities are really desired, the others being merely troublesome corrections or instrumental constants which must be determined or eliminated. The method of procedure, however, is the same in this as in any other case.

33. More Observations than Quantities. Normal Equations. — In the illustrations of the preceding article there were, in each case, two unknowns, and two independent observations were necessary to determine them. By *independent* observations are meant observations made on a different principle, or under such different conditions that the resulting observation equations will have different coefficients and not merely different absolute terms. To repeat the process of measuring the *sum* of two unknowns, without attempting to find some other relation between them (as, for example, their difference or their product), would give no information as to the separate values of the unknowns. And, in general, the determination of l unknown quantities requires a knowledge of l independent and consistent relations between them.

If measurements could be made without error, the solution of the l independent observation equations formed from such measurements would give us the values of the l unknowns exactly; more than l measurements would be superfluous. But, as in the simpler case of a single unknown, the existence of accidental errors makes it desirable to get as many observations as possible, and to devise some means of averaging them so as to find the *most probable* value of each of the unknowns. This problem is the most important that arises in least squares.

Let there be n observations upon functions of the l unknown connected quantities $q_1, q_2, \cdots, q_l$ $(n > l)$, and let the series of resulting observation equations be represented by

$$\left.\begin{array}{l} f_1(q_1, q_2, \cdots, q_l) = s_1, \\ f_2(q_1, q_2, \cdots, q_l) = s_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_n(q_1, q_2, \cdots, q_l) = s_n. \end{array}\right\} \tag{35}$$

Here, as in the simpler cases, there are errors and residuals obeying the same law of error distribution set forth in the error equation. We are seeking to obtain the most probable values, $m_1, m_2, \cdots, m_l$, of the unknown (and unknowable) quantities q that the observations will furnish, and when these are found, the n residuals will be given by

$$\left.\begin{array}{l} \rho_1 = s_1 - f_1(m_1, m_2, \cdots, m_l), \\ \rho_2 = s_2 - f_2(m_1, m_2, \cdots, m_l), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho_n = s_n - f_n(m_1, m_2, \cdots, m_l). \end{array}\right\} \tag{36}$$

The principle of least squares may now be slightly modified in wording to fit this case, thus: *The most probable values of unknown quantities connected by observation equations are those which will render the sum of the squares of the residuals arising from the observation equations a minimum.*

It is possible, through an application of this principle, to reduce the n residual equations (36) to a number l, equal to the number of unknowns, which can then be solved for the most probable values m. The process may be regarded as finding from the n observation equations (35) a set of l *most probable equations* whose solution will give the *most probable* values of the unknowns q.

From the principle of least squares, the sum $\rho_1^2 + \rho_2^2 + \cdots + \rho_n^2$ must be a minimum, and in order that $m_1, m_2, \cdots, m_l$ may be so selected that this will be the case, the first partial derivative of this $\Sigma\rho^2$ with respect to each of those quantities must be zero. (See any calculus.) That is,

$$\left.\begin{aligned}
&\frac{\partial}{\partial m_1}\sum_1^n [s - f(m_1, m_2, \cdots, m_l)]^2 = 0,\\
&\frac{\partial}{\partial m_2}\sum_1^n [s - f(m_1, m_2, \cdots, m_l)]^2 = 0,\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\frac{\partial}{\partial m_l}\sum_1^n [s - f(m_1, m_2, \cdots, m_l)]^2 = 0.
\end{aligned}\right\} \qquad (37)$$

The equations (37) resulting from these differentiations are the most probable or *normal equations* required, and being l in number, will yield, on solution, the most probable values $m_1, m_2, \cdots, m_l$, which are sought.

Equation (34) is a normal equation, containing only one unknown, m.

34. Reduction of Observation Equations of the First Degree. — In nearly all cases in which the method of least squares is used in the reduction of observations in accordance with the foregoing theory, the observation equations are either all of the first degree, or they may, by suitable substitutions, be replaced by equivalent observation equations which are of the first degree. The mathematical operations required in finding the normal equations are then comparatively simple, and can be performed without any knowledge of calculus.

Let the n first degree observation equations upon the l quantities q (corresponding to (35)) be as follows:

$$\left.\begin{array}{l} a_1q_1 + b_1q_2 + c_1q_3 + \cdots + r_1q_l = s_1, \\ a_2q_1 + b_2q_2 + c_2q_3 + \cdots + r_2q_l = s_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nq_1 + b_nq_2 + c_nq_3 + \cdots + r_nq_l = s_n. \end{array}\right\} \qquad (38)$$

The residuals will then be

$$\left.\begin{array}{l} \rho_1 = s_1 - (a_1m_1 + b_1m_2 + \cdots + r_1m_l), \\ \rho_2 = s_2 - (a_2m_1 + b_2m_2 + \cdots + r_2m_l), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho_n = s_n - (a_nm_1 + b_nm_2 + \cdots + r_nm_l). \end{array}\right\} \qquad (39)$$

Only one term in each of these expressions contains m_1; denote the balance of the expression in each case by a single letter, as B. Then

$$\rho_1 = -a_1m_1 + B_1, \text{ etc.},$$

and

$$\Sigma\rho^2 = (-a_1m_1 + B_1)^2 + \cdots + (-a_nm_1 + B_n)^2.$$

Differentiating with respect to m_1, as per equations (37),

$$\frac{\partial}{\partial m_1}\Sigma\rho^2 = -2a_1(-a_1m_1 + B_1) - \cdots$$
$$- 2a_n(-a_nm_1 + B_n) = 0.$$

Or dividing by 2 and remembering that $-am + B = \rho$ in each term,

$$-a_1\rho_1 - a_2\rho_2 - \cdots - a_n\rho_n = 0. \qquad (40)$$

This is the normal equation pertaining to m_1, and corresponds to the first of equations (37).

This result may be directly obtained by multiplying each of the residuals (39) by the coefficient of m_1 in the expression for that residual, adding the results and equating the sum to zero.

The remainder of the l normal equations required are determined with respect to $m_2, m_3, \cdots, m_l$ in the same manner.

The foregoing processes may be summed up in the following rule: *To adjust a set of observation equations of the first degree, write the expression for the residual corresponding to each observation equation, multiply it by the coefficient of the first unknown, in that expression, add the products and equate their sum to zero. The result is the normal equation pertaining to the said first unknown. Do likewise for each of the other unknowns. Then solve* *the l normal equations thus formed for the desired most probable values, $m_1, m_2, \cdots, m_l$.*

Let the student prove that taking the arithmetical mean of a number of direct observations upon a single quantity is merely a special application of this rule.

35. Illustrations from Physics. — It will be of material assistance to the student to have presented at this point a number of actual examples illustrating the application of least square adjustment in various departments of exact science. These examples are not "made up" for the purpose; they are drawn from actual experimental notes on research or field work.

1. *Bridge Wire.* — It was desired to measure the total resistance of a Wheatstone bridge wire and at the same time to calibrate it, by comparison with a standardized bridge of another type. The unknown (and unessential) resistance of the connections had also to be reckoned with and eliminated. The wire was 100 cm. long, and the measurement was conducted by observing the resistance of the first 10 cm., then of the first 20 cm., etc., and finally of the whole wire, the connections entering each time as a constant term in the observed resistance. The results follow.

No. Cm. Measured	Resist., Ohms	No. Cm. Measured	Resist., Ohms
10	0.116	60	0.595
20	0.205	70	0.675
30	0.295	80	0.760
40	0.388	90	0.850
50	0.503	100	0.926

Let x be the total resistance of the bridge wire, and c that of the connections. These are the two unknowns, the first of which is to be obtained with all possible precision, the second to be eliminated, as a mere correction. Mathematically they are equally important. The observation equations are

$$
\begin{aligned}
0.1\,x + c &= 0.116,\\
0.2\,x + c &= 0.205,\\
0.3\,x + c &= 0.295,\\
0.4\,x + c &= 0.388,\\
0.5\,x + c &= 0.503,\\
0.6\,x + c &= 0.595,\\
0.7\,x + c &= 0.675,\\
0.8\,x + c &= 0.760,\\
0.9\,x + c &= 0.850,\\
1.0\,x + c &= 0.926.
\end{aligned}
$$

In practice we need not take the trouble to change symbols in distinguishing between the *true* and *most probable* values of the unknown (" q " and " m "). If x and c now represent the most probable values sought, the first residual is $\rho_1 = 0.116 - (0.1\,x + c)$, etc. Let the student apply the rule developed in the preceding article to obtain the two normal equations, which he will find to be

$$
\begin{aligned}
3.85\,x + 5.5\,c &= 3.686,\\
5.5\,x + 10\,c &= 5.313,
\end{aligned}
$$

the solution of which gives

$$
\begin{aligned}
x &= 0.926 \text{ ohms},\\
c &= 0.022 \text{ ohms}.
\end{aligned}
$$

Let the student select *any two* of the observation equations and solve them for x and c, comparing the results with these *most probable* ones. The accompanying figure

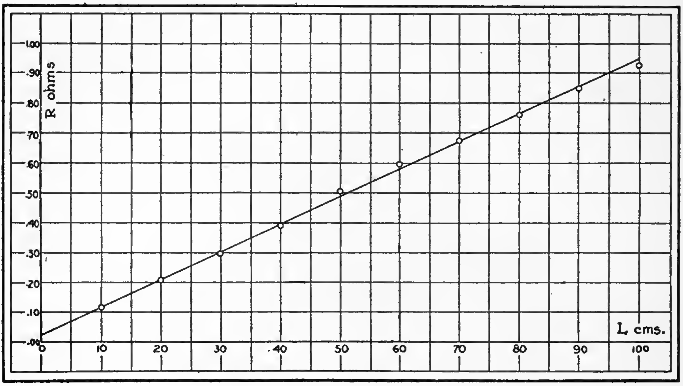

FIG. 6

shows the plotted observations, together with the straight line

$$0.926\frac{L}{100} + 0.022 = R,$$

upon which they all should lie were there no errors in the measurements nor irregularities in the wire itself. The departures of the plotted points from this most probable line represent the residuals of the ten observations.

2. *Balance Constants.*—The general theory of the equal-arm balance is somewhat complicated, but in the equation used to express the sensibility in terms of the load, the various instrumental constants may all be involved in two quantities a and b, the equation being

$$a + bw = \frac{1}{s}.$$

Here w is the load on either pan (grams) and s the gram sensibility, or one thousand times the deflection produced by a milligram weight laid on one pan. The constants a and b are to be estimated from the following observations.

w Grams	s Scale Div.	w Grams	s Scale Div.
0	2212	50	2389
10	2265	75	2449
20	2320	100	2563
30	2343	125	2590
40	2316		

The observation equations are then

$$a + 0\,b = 1 \div 2212,$$
$$a + 10\,b = 1 \div 2265,$$
$$a + 20\,b = 1 \div 2320,$$
$$a + 30\,b = 1 \div 2343,$$
$$a + 40\,b = 1 \div 2316,$$
$$a + 50\,b = 1 \div 2389,$$
$$a + 75\,b = 1 \div 2449,$$
$$a + 100\,b = 1 \div 2563,$$
$$a + 125\,b = 1 \div 2590.$$

The adjustment of these by the foregoing method gives as the most probable values sought,

$$a = +\,0.0004466,$$
$$b = -\,0.000000518.$$

Let the student perform this reduction.

36. Illustrations from Chemistry.

1. *Volumetric Solutions.* — It is desired to test certain acid and alkaline solutions to be used in volumetric chemical analysis, in order to ascertain their exact strengths. Two common reagents, in the form of tenth-normal solutions, may be tested first, then others may be compared to these. If the two reagents chosen be hydrochloric acid and potassium hydroxide, the following procedure may be employed.

A quantity of each solution is placed in an accurately graduated burette, the two burettes being supported side by side. A small amount (say about 0.2 g.) of finely pulverized pure calcium carbonate (chalk, $CaCO_3$) is carefully weighed, placed in a white porcelain dish and treated with an excess (say about 50 cc.) of the HCl solution from the burette, the amount being accurately observed. The chalk dissolves and neutralizes part of the acid, the CO_2 gas escaping. The porcelain dish is now set under the KOH burette, and just enough of the alkaline solution allowed to flow into it to render it exactly neutral, this point being determined by a drop or two of methyl orange or other sensitive indicator previously added to the mixture in the dish. The amount of KOH solution thus used is also carefully noted. Part of the acid is neutralized by the $CaCO_3$ and the remainder by the KOH. The chemical equations representing the two reactions are as follows:

$$\text{(I)}\quad \overset{72.36}{2\,HCl} + \overset{99.32}{CaCO_3} = CaCl_2 + CO_2 + H_2O,$$

$$\text{(II)} \quad \overset{36.18}{HCl} + \overset{55.70}{KOH} = KCl + H_2O.$$

The small numbers above the symbols are obtained from the molecular weights, and represent the *relative* weights of the substances engaging in the reaction.

$$\left.\begin{aligned} \text{Let } q_1 &= \text{wt. HCl in 1 cc. HCl sol.} \\ q_2 &= \text{wt. KOH in 1 cc. KOH sol.} \end{aligned}\right\} \text{Unknown.}$$

$$\begin{aligned} a &= \text{total volume HCl sol. used.} \\ \alpha &= \text{vol. HCl sol. neutralized by } CaCO_3. \\ a - \alpha &= \text{vol. HCl sol. neutralized by KOH.} \\ b &= \text{vol. KOH sol. used in neutralization.} \\ c &= \text{wt. } CaCO_3 \text{ powder used.} \end{aligned}$$

$$\begin{aligned} \text{Then } \alpha q_1 &= \text{wt. HCl neutralized by } CaCO_3. \\ (a - \alpha)q_1 &= \text{wt. HCl neutralized by KOH.} \\ bq_2 &= \text{wt. KOH used.} \end{aligned}$$

From (I)

$$\alpha q_1 : c = 72.36 : 99.32 = 0.73,$$

or

$$\alpha q_1 = 0.73\ c.$$

From (II)

$$(a - \alpha)q_1 : bq_2 = 36.18 : 55.70 = 0.65,$$

or

$$aq_1 - \alpha q_1 = 0.65\ bq_2.$$

From these two equations α is eliminated by addition, giving finally

$$aq_1 - 0.65\ bq_2 = 0.73\ c.$$

This is an observation equation, the quantities a, b, c having been measured, and q_1, q_2 being the two unknowns; and a series of such experiments (at least two) will yield the most probable values required. In some of the experiments the $CaCO_3$ powder may be omitted entirely, giving $c = 0$; but not in all of them. (Why?)

a Vol. HCl Sol. used	b Vol. KOH Sol. used	c Wt. $CaCO_3$ Powder used
cc.	cc.	g.
50.00	10.33	0.1779
50.00	7.88	0.1936
11.23	9.98	none
11.25	10.00	none
11.25	10.00	none
11.34	10.10	none

The above data yield the following observation equations:

$$
\begin{aligned}
50\, q_1 - 0.65 \times 10.33\, q_2 &= 0.73 \times 0.1779, \\
50\, q_1 - 0.65 \times 7.88\, q_2 &= 0.73 \times 0.1936, \\
11.23\, q_1 - 0.65 \times 9.98\, q_2 &= 0, \\
11.25\, q_1 - 0.65 \times 10.00\, q_2 &= 0, \\
11.25\, q_1 - 0.65 \times 10.00\, q_2 &= 0, \\
11.34\, q_1 - 0.65 \times 10.10\, q_2 &= 0.
\end{aligned}
$$

Let the student reduce these to normal equations and solve for the most probable values of q_1 and q_2.

2. *Pyknometer Constants.* — The expansion of a pyknometer (specific gravity bottle), like any solid, is in approximate accordance with the linear law

$$V = V_0 + Kt,$$

V being the capacity at temperature t, V_0 the capacity at zero and K a constant involving the coefficient of expansion of the glass. The two constants V_0 and K must be experimentally determined from time to time for any pyknometer that is used in accurate measurements of density. This may be done by finding the capacity at several different temperatures over the required range.* The following is a tabulation of eight such determinations, using distilled water and corrected for buoyancy of the air.

t	V	t	V
19°.20	25.2628 cc.	35°.50	25.2687 cc.
19.75	.2634	39.30	.2691
25.61	.2664	39.75	.2692
30.92	.2681	46.45	.2734

Let the student form the eight observation equations and the two normal equations, and reduce for the most probable values of V_0 and K. (The approximate answers are, $V_0 = 25.2509$, $K = 0.0005244$.)

37. Illustrations from Surveying.

1. *Locating a Distant Station.* — Some of the best writers on surveying strongly recommend the use of rec-

* If the range be large, K will vary somewhat. The range may be subdivided, say into 10-degree intervals, and the constants found for each; or better, a quadratic relation assumed, with three constants. See Art. 45.

tangular coördinates in surveying and mapping; certainly their use reduces many calculations to a more scientific basis. The problem in hand is as follows: Given, the

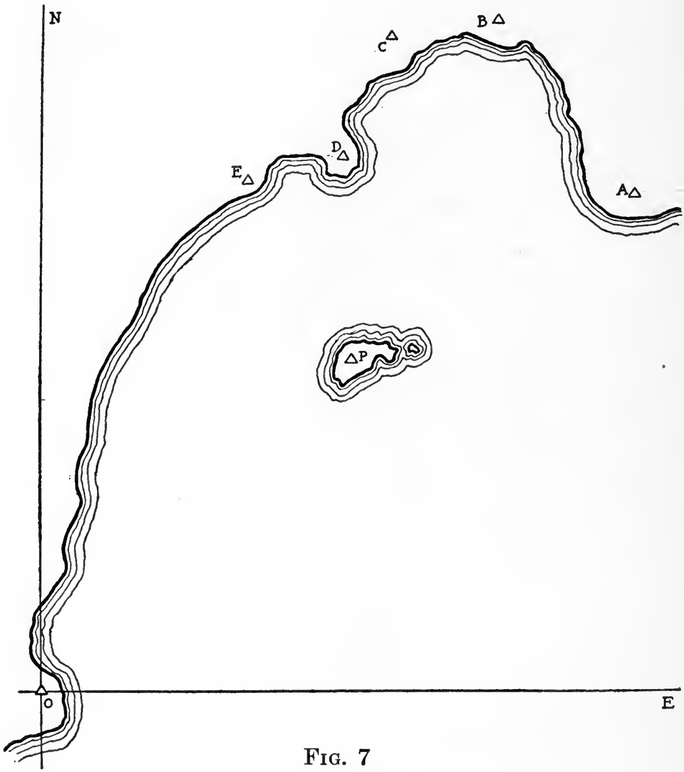

Fig. 7

coördinates of a number of stations A, B, C, etc., with reference to an origin O, and the bearing of an unknown station P from each of these stations; to find the most probable coördinates of P. For instance, the unknown station P is a reef near the coast along which the points A, B, etc., are located. The numerical data are as follows, for five stations.

Station	Coördinates (Ft.) E.	N.	Bearing of P	Vector	Angle θ
A	1785	1501	S. 58° 57′ W.	PA	31° 3′
B	1372	2020	S. 22 5 W.	PB	67 55
C	1052	1971	S. 5 29 W.	PC	84 31
D	909	1609	S. 4 43 E.	PD	94 43
E	620	1533	S. 32 43 E.	PE	122 43

The vectorial angles in the last column are the angles made by the vectors PA, PB, etc., with the line drawn eastward through P, calculated from the given bearings. Using coördinates x and y to locate P, and x_a, y_a, etc., for A, etc., we can write

$$\frac{y_a - y}{x_a - x} = \tan \theta_a,$$

or

$$x \tan \theta_a - y = x_a \tan \theta_a - y_a,$$

that is,

$$x \tan 31° 3' - y = 1785 \tan 31° 3' - 1501,$$

etc., as the observation equations, there being as many of these as there are known stations. These equations, being of the first degree in x and y, may be adjusted in the usual manner. Let the student do this. (The results should be, approximately, $x = 930$, $y = 1000$; that is, $P = 930$ E., 1000 N.)

2. *Relative Levels of Stations.* — The next illustration is taken from Merriman's *Least Squares*, and is typical of many kinds of measurements in which the quantity sought is measured by parts or segments. The same method is, for example, applied to a number of angles at one station. Given, a number of determinations on the relative altitudes of several stations, obtained by precise leveling, to find the most probable values of their altitudes above one of them taken as a datum. Following are the results of the levelings.

A above *O* 573.08 ft.
B above *A* 2.60 ft.
B above *O* 575.27 ft.
C above *B* 167.33 ft.
D above *C* 3.80 ft.
D above *B* 170.28 ft.
D above *E* 425.00 ft.
E above *O* 319.91 ft. (one way)
E above *O* 319.75 ft. (another way)

Representing by a, b, etc., the elevations of the respective stations above O as a datum, the following simple observation equations at once result.

$$a = 573.08$$
$$b - a = 2.60$$
$$b = 575.27$$
$$c - b = 167.33$$
$$d - c = 3.80$$

$$d - b = 170.28$$
$$d - e = 425.00$$
$$e = 319.91$$
$$e = 319.75$$

The student can readily adjust these in the usual manner. It will be interesting in this case to compare the adjusted values of a, b and e with their values as directly measured.

38. Illustrations from Astronomy.

1. *Errors of the Transit and Clock.*—Astronomical time is ascertained, at any observatory, by observations upon the stars. To this end an instrument not unlike a surveyor's transit is used. It is larger, however, and fixed on a solid pier, and is incapable of rotating horizontally, being swung in the vertical plane of the meridian. This instrument is the *astronomical transit* or the *meridian circle.*

When used for time observations, the telescope is set at the proper angle of altitude for some star to traverse its field as it crosses the meridian. The exact sidereal time of meridian passage, or *transit,* is known as the *right ascension* * of the star, and is given in the star catalogues. In order to correct the clock, therefore, it is necessary only to note at what time, by the clock, the star is actually observed to cross the meridian.

* *Right ascension* on the celestial sphere, as shown by the star maps, is closely analogous to longitude on the earth, only it is usually expressed in hours, minutes and seconds, reading toward the east. *Declination* corresponds to terrestrial latitude.

On account of the extreme accuracy demanded in astronomical work, this apparently simple procedure requires the elimination of certain recognized instrumental errors. These are: (1) the *level error,* arising from the non-horizontality of the bearings or trunnions on which the telescope turns, so that its revolution does not exactly coincide with the meridian plane; (2) the *azimuth error,* or failure of this axis of rotation to coincide with the east and west line, which has a similar effect on the plane of rotation; (3) the *collimation error,* due to the fact that the cross-wires in the telescope, which determine its line of sight, are not exactly in the optic axis, being a little to one side of the center of the field. In addition to these, there is the error of the clock, which is the quantity really wanted. The level error is ascertained by a direct application of the stride level resting on the trunnions and having a very sensitive graduated spirit-bubble. The other errors must be found simultaneously from several observations on different stars, the level error reading being simply a part of the determination.

Without entering into the applications of spherical astronomy required, it may be simply stated that the observation equations involved are of the first degree. If

q_1 = the true clock error (clock minus true time),
q_2 = the azimuth error,
q_3 = the collimation error,
l = the level error,

all being expressed in seconds of time, then the form of the observation equation is

$$q_1 + aq_2 + cq_3 = d - bl,$$

d being the apparent clock error, or the time indicated by the clock at apparent transit minus the true time of transit, or right ascension, of the star. The quantities *a*, *b* and *c* are known as *Meyer's coefficients*, and are calculated from the following formulas:

$$a = \frac{\sin(\lambda - \delta)}{\cos \delta},$$

$$b = \frac{\cos(\lambda - \delta)}{\cos \delta},$$

$$c = \sec \delta,$$

in which λ is the latitude of the observatory and δ the declination of the star used. Tables of these coefficients are at hand in every observatory.

Of course three observations on different stars, at least, are required to determine q_1 and eliminate q_2 and q_3. If more are made, least-square reduction may be applied to their adjustment. Following is a typical set of data of this sort, based on the observed transits of six stars.

IOWA CITY, IOWA, Lat. 41° 40′ November 16, 1896

STAR	DECLINATION	RIGHT ASCENSION	OBSERVED TIME OF TRANSIT	*l*	*a*	*b*	*c*
		h. m. s.	h. m. s.	s.			
κ Draconis . .	109° 38′	0 29 4.28	0 29 39.14	0.47	2.76	−1.12	−2.97
β Ceti . . .	−18 33	0 38 26.50	0 39 34.07	0.44	0.91	0.52	1.05
γ Cassiopeiæ .	60 9	0 50 30.72	0 52 4.77	0.38	−0.64	1.91	2.01
σ Ursæ Majoris	112 27	21 1 21.85	21 2 1.41	0.59	2.47	−0.86	−2.61
ζ Cygni . . .	29 48	21 8 32.81	21 9 52.14	0.59	0.24	1.13	1.15
α Cephei . .	62 9	21 16 6.35	21 17 42.64	0.59	−0.75	2.01	2.14

No allowance has here been made for any error in the clock's rate during the progress of the observations.

2. *Parallax and Proper Motion.*—Stellar parallax is the apparent change in the position of a star, during the year, caused by the earth's motion in its orbit. In addition to this, there is the actual, or "proper," motion of the star itself through space. These two are superposed and produce one resultant effect upon the star's apparent position at any time. Their separation into distinguishable components is the problem here presented.

Modern astronomical measurements are conducted very largely by photography. The star in question is photographed on the same plate with others so immensely farther away that they have no perceptible parallax or proper motion, and then the positions of the images are measured at leisure on very accurate measuring machines.

Let π = the parallax in a given direction,
μ = the proper motion in that direction,
s = the measured displacement of the star in that direction, with reference to its apparent position at some previous date T days past.

Then the observation equation is shown in practical astronomy to be

$$P\pi + T\mu + c = s.$$

P is the parallax factor, easily calculated from the direction of the star and the position of the earth in its orbit. c is an unknown constant, depending on the peculiarities of the measuring machine, and to be eliminated. The three unknowns are, then, π, μ and c. The coefficients P and T and the quantity s are varied by making observations on many different dates, and from the resulting series of observation equations, the most probable values of the proper motion and the parallax are obtained. The latter gives the most probable distance of the star. The details of the process being somewhat technical, no numerical example is here given.

39. Observation Equations Not of First Degree. — If the observation equations are not of the first degree, recourse may be had to the general method explained in Art. 33, that is, to the application of the principle of least squares through the general equations (37). This would often lead, however, to normal equations that would be exceedingly inconvenient to solve.

In many such cases, the difficulty may be at once avoided by a suitable application of logarithms. A standard measurement in the physical laboratory, for example, is the simultaneous determination of the magnetic field of the earth H and the magnetic moment M of the bar magnet used for the purpose. One experiment gives the product,

$$MH = s_1, \tag{41}$$

and another the quotient

$$\frac{M}{H} = s_2, \tag{42}$$

of the unknown quantities. These observation equations may be made linear by using instead of M and H, as unknowns, their common logarithms:

$$\left.\begin{aligned}\log M + \log H &= \log s_1,\\ \log M - \log H &= \log s_2,\end{aligned}\right\} \tag{43}$$

the most probable values of log M and log H being then found in the usual manner.

Again, the solubility of a chemical salt is given by the theoretical formula *

$$s = s_0 e^{\frac{ct}{273+t}}, \tag{44}$$

in terms of the centigrade temperature t. s_0 is the solubility of the salt at 0° C. and c is a constant depending on its heat of solution. s_0 and c are unknowns, to be determined for each substance by means of several measurements on s at different temperatures. For this purpose the observation equation may be written

$$\log s_0 + \frac{t}{273+t} \log e \cdot c = \log s, \tag{45}$$

the most probable values of c and $\log s_0$ being the values directly found. The following data pertain to the solubility of potassium chlorate ($KClO_3$) in water.

t	s (obs.)	s (calc.)
0°		0.0247
5	0.0299	.0317
10	.0406	.0402
15	.0512	.0507
20	.0672	.0634
25	.0774	.0787
30	.1027	.0970
35	.1145	.1187
40	.1405	.1444

* See Arrhenius, *Electrochemistry*.

The eight observations on t and s furnish eight observation equations of the above form, which when adjusted give as most probable values $\log s_0 = -1.6073$, whence $s_0 = 0.0247$; and $c = 13.82$. The solubility formula for this substance may now be written in its original form, or more conveniently retained in the logarithmic form:

$$\log s = 6.0014 \frac{t}{273 + t} - 1.6073,$$

from which the values of s given in the third column are calculated. The student will find it instructive to plot the observations on t and s, and also the smooth curve corresponding to the calculated values of s. It would be difficult to imagine a more typical application of least-square adjustment than the one just given.

Another method of procedure when the observation equations are not of the first degree, somewhat analogous to Horner's method of approximation for higher algebraic equations, is explained in Note *B*, Appendix.

40. Observations upon Quantities Subject to Rigorous Conditions. — It often happens that unknown quantities involved in observation equations are further connected by known mathematical conditions, which the final adjusted values must rigorously satisfy. For example, the most probable values of the angles of a triangle could not be a set of angles whose sum is other than exactly 180°; the sum of all the percentages in a chemical analysis must be 100; etc. Observations upon such quantities are known as *conditioned observations.*

Suppose that the results of measurements upon the three angles of a triangle are

$$\left.\begin{aligned} q_1 &= s_1, \\ q_2 &= s_2, \\ q_3 &= s_3. \end{aligned}\right\} \tag{46}$$

These are the observation equations. To this list there must be added a fourth equation, namely:

$$q_1 + q_2 + q_3 = 180°, \tag{47}$$

which is called an *equation of condition.* It differs from the others in that it is known to be exactly true, while the others are not. The three most probable values, when deduced, must satisfy this equation exactly; the others must be satisfied as nearly as may be. This equation of condition cannot, therefore, be classed as an observation equation and treated like the others.

In general, we may have n observations involving l unknowns, which are further subject to m rigorous conditions, expressed as equations of condition. m must be less than l; for if equal to it, the unknowns would be absolutely determined by the given conditions, and the measurements would be superfluous; and if greater, no set of quantities could, in general, be found to satisfy all the conditions.

There being fewer conditions than unknowns, there is an unlimited number of sets of values of the unknowns which might satisfy the conditions, and we have to determine from the n observations which of these sets is the most probable.

Though the m conditions do not give the values of the l unknowns, they enable us to express m of the unknowns rigorously *in terms of the remaining ones;* and if we now substitute these expressions for the m unknowns in the observation equations, the latter may then be adjusted for the most probable values of the $l - m$ quantities remaining. The most probable values of the m replaced quantities may now also be calculated so that the conditions are exactly satisfied.

Applying this to the case of the angles of a triangle, subject to one condition (47), one of the angles, say q_3, may be expressed by means of it in terms of the other two:

$$q_3 = 180° - q_1 - q_2. \tag{48}$$

The three observation equations then appear:

$$\left.\begin{aligned} q_1 &= s_1, \\ q_2 &= s_2, \\ 180° - q_1 - q_2 &= s_3. \end{aligned}\right\} \tag{49}$$

Let the student adjust these and show that the most probable values sought are

$$\left.\begin{aligned} q_1 &= s_1 + \tfrac{1}{3}[180° - (s_1 + s_2 + s_3)], \\ q_2 &= s_2 + \tfrac{1}{3}[180° - (s_1 + s_2 + s_3)], \\ q_3 &= s_3 + \tfrac{1}{3}[180° - (s_1 + s_2 + s_3)], \end{aligned}\right\} \tag{50}$$

the third result following from the other two through substitution in (48); which shows that the results sought can be obtained by adding to each measured angle one-third the discrepancy between the sum of the measured angles and 180°, so as to make the sum correct. This is on the assumption that all three of the measurements are equally trustworthy. (See Chap. VII.) The same proceeding is to be followed in every case where the observation equations represent the separately measured values of the unknowns, while the one equation of condition rigorously gives their sum. Cases like this are of common occurrence.

If the sides of the triangle are measured, as well as the angles, there will be six observation equations (at least), and three equations of condition. Of these latter, one will be the same as (47), the other two arising from the requirements of trigonometry as to sides and angles.

A case of special importance to the surveyor is the adjustment of the sides and angles of a polygon of land. In addition to whatever measurements are made upon the lengths and bearings of the sides, there are two rigorous conditions to be fulfilled, namely: that the algebraic sum of the projections of the sides on an east-and-west line is zero, and the algebraic sum of their projections on a north-and-south line is zero. This adjustment will be found explained in detail in the more advanced works on plane surveying.

EXERCISES

41. 1. Draw a large triangle on paper with a fine pencil, and measure with a protractor each of the angles. Form the observation equations and the equation of condition, and from them deduce the most probable values of the angles.

2. Lay off on a straight line four points, A, B, C, D. Measure AB, BC, CD, AC, BD, AD. From these measurements form observation equations and compute the most probable values of AB, BC, CD. These segments may be conveniently lettered x, y, z.

3. The following measurements were made upon a rectangular metallic tank to determine its dimensions:

Length (inside) 27.31 cm.
Width (inside) 16.08 cm.
Depth (inside) 9.67 cm.
Capacity by standard graduates, 4.3217 liters.

Find the most probable dimensions.

4. Draw five lines radiating accurately from a common point O, the further extremities being A, B, C, D, E. Measure with a protractor, by the differential method, and turning the protractor at each measurement, each of the angles AOB, AOC, AOD, AOE, BOC, BOD, BOE, COD, COE, DOE. Determine the most probable values of the angles AOB, BOC, COD, DOE.

5. The following are the results of an analysis of a certain medicinal compound:

Salts of calcium 1.26 per cent.
Salts of sodium 2.53 per cent.
Salts of iron 0.23 per cent.
Salts of manganese 0.14 per cent.
Salts of quinine 0.07 per cent.

Salts of strychnine 0.02 per cent.
Water 95.67 per cent.

Find the most probable values of the several percentages.

6. Six points, supposed to lie on the arc of a circle, have the following measured coördinates:

x	y	x	y
3.15	2.49	1.07	5.33
2.67	3.72	− 0.20	5.98
1.80	4.69	− 1.84	6.25

Find the most probable coördinates of the center and most probable radius.

7. A steel tape was measured under different conditions of stretch and temperature, as follows:

Centigrade Temp.	Tension, Lb.	Obs. Length, Ft.
0°	0	100.031
20	10	.064
25	8	.068
18	12	.063
21	15	.069
15	15	.062

Using the approximate formula

$$l = l_0 + at + bf,$$

in which t = temperature and f = tension, adjust for the most probable values of l_0, a, b.

8. (Adapted from Wright's *Adjustment of Observations.*) Let D be the difference in length of two standard meter bars at 62° F. and Δ the difference in their coefficients of expansion. Then the difference d in length at any temperature t is

$$d = D + (t - 62)\,\Delta.$$

Observations were made as follows:

t	d
24°.7	0.00791 inch
37.1	811 inch
61.7	833 inch
49.3	820 inch
66.8	847 inch
71.5	849 inch

Adjust for the most probable values of D and Δ.

9. Van der Waal's equation for pressure and volume of a gas at absolute temperature T may be put in the form

$$v^2TR - va + pv^2b + ab = pv^3.$$

The measurements of Amagat on air at moderate pressures and at 16° C. (289°.1 absolute) were published as follows:

p in cm. Mercury	pv
76	1.0000
2000	0.9930
2500	.9919
3000	.9908
3500	.9899
4000	.9896

Form the observation equations by the method of Note *B*, Appendix, and adjust for *a*, *b*, *R*.

10. The electrical conductivity of selenium is found to vary with the intensity of light falling on it according to the equation

$$C = \sqrt{a\sqrt{I} + b}.$$

The following data were furnished by Dr. F. C. Brown.

Intensity I	Conductivity C	Intensity I	Conductivity C
0	83	33	319
3	188	44	348
11	250	50	361
17	285	100	446
25	303		

Adjust for the most probable values of *a* and *b*. (Note. — In working the above problem, it will be found necessary, as is sometimes the case, to use caution in dropping decimal places, as the normal equations happen to be quite "sensitive" to slight changes in the coefficients.)

11. The E.M.F. of a thermo-couple for a given temperature difference *t* between junctions may be represented by the equation

$$e = at + bt^2.$$

The following values for a copper-tellurium couple with one junction at 0°, in which *e* is in volts, were furnished by Mr. W. E. Tisdale.

t	$\frac{e}{t}$	t	$\frac{e}{t}$
50	0.000243	150	0.000254
82	242	162	262
92	245	180	265
100	248	186	267
113	249	190	266
117	250	195	267
128	252	200	268
140	251		

Calculate the most probable values of the constants a and b. Plot the curve.

12. In a sine intensity magnetometer, let the pole strength of the bar magnet be P, the distance between its poles l, and the distance from its center to the needle pivot a. δ is the needle deflection and H the horizontal intensity of the earth's magnetism. The equation connecting these quantities is

$$\frac{2a^2}{H} Pl + \frac{Pl^3}{H} = a^5 \sin \delta.$$

The following data were obtained at a station where $H = 0.1884$ (c.g.s.).

a (CM).	δ	a (CM).	δ
20	24° 17′	40	1° 49′
25	12 46	45	1 35
30	6 48	50	1 27
35	3 26		

Find the most probable values of P and l. Use the method of Note B, Appendix.

13. The specific volume of a certain liquid was measured at different temperatures by a quick secondary method which was known to have certain small persistent errors. At three of the temperatures, known as "tie points," the specific volume was also measured by a more laborious absolute method, free from the said sources of error. The results follow:

Temp., C.	Sp. Vol., Secondary	Sp. Vol., Absolute
23°.0	0.952750	
23.5	2879	
24.0	3003	
24.5	3177	0.953322
25.0	3339	3488
25.5	3505	
26.0	3678	
26.5	3840	4059
27.0	4012	
27.5	4187	
28.0	4366	
28.5	4526	
29.0	4701	
29.5	4872	
30.0	5075	
31.0	5426	

In order to correct all the data in the second column, use was made of the equation

$$Y = AX + B,$$

in which X is the specific volume by the secondary method and Y the corresponding corrected value, A and B being assumed constant. By using the values of X and Y at the three "tie points," find the most probable values of A and B and correct all the secondary data accordingly.

14. A glass sinker, used for precision measurements of liquid density by the Archimedes buoyancy method, was calibrated for expansion, the data being as follows:

Temp. C.	Vol. Sinker (cc.)
24°.0	34.03894
24.5	3907
25.0	3984
25.5	4085
26.0	4102
26.5	4134
27.0	4191
27.5	4203
28.0	4231
28.5	4240
29.0	4290
30.0	4393

Find the most probable zero volume and coefficient of expansion of the sinker, assuming a linear relation.

15. Guthe and Worthing's formula for the vapor pressure of water at temperature $t°$ C. is

$$\log_{10} p = 7.39992 - \frac{a}{(t+273)^b}.$$

From the following data, find the most probable values of a and b.

t	p (mm. of Mercury)
10°	9
20	17
30	32
40	55
50	92
60	149
80	355
100	760

16. The angles and the sides of a triangle ABC were measured, with the following results.

Angle	A	51° 9′
	B	95 4
	C	33 51
Side	BC	1721.3 ft.
	AC	2207.5
	AB	1233.0

Introduce the necessary geometric conditions and adjust for the most probable values of sides and angles.

17. Draw accurately a large quadrilateral $ABCD$ and its two diagonals AC and BD. Measure with a millimeter scale the four sides AB, BC, CD, DA, and with a protractor the angles DAB, DAC, CAB, ABC, ABD, DBC, BCD, BCA, ACD, CDA, CDB, BDA. Introduce the eight necessary geometric conditions and adjust for the most probable values of the sides and angles of the quadrilateral.

If the diagonals had also been measured, how many conditions would be introduced?

18. Adjust the transit observations given in Art. 38.

19. (Adapted from Crandall's *Geodesy and Least Squares*.) Adjust the following transit observation equations for x, y, z.

$$\begin{aligned}
-0.07\,x + 1.41\,y + z &= -0.65,\\
+0.68\,x + 1.00\,y + z &= +0.18,\\
+0.52\,x + 1.02\,y + z &= +0.13,\\
+2.51\,x - 2.67\,y + z &= +3.96,\\
-0.73\,x + 2.13\,y + z &= -1.88,\\
+0.75\,x + 1.01\,y + z &= +0.02,\\
+0.53\,x + 1.02\,y + z &= +0.13,\\
+0.68\,x + 1.00\,y + z &= +0.44,\\
+0.81\,x + 1.02\,y + z &= +0.29,\\
+0.09\,x + 1.27\,y + z &= -0.76.
\end{aligned}$$

20. In the following observation equations the unknowns n and K are constants of metallic reflection.

$$n\sqrt{1+K^2}\left[1 - \tfrac{1}{2}\sin^2\phi\,\frac{1-K^2}{n^2(1+K^2)^2}\right] = \sin\phi\tan\phi,$$

$$K\left[1 + \frac{\sin^2\phi}{n^2(1+K^2)^2}\right] = \tan 2\,\psi.$$

Assuming that approximate values of n and K are known, transform these into observation equations of the first degree by the method of Note B, Appendix.

CHAPTER VI

EMPIRICAL FORMULAS

42. Classification of Formulas. — If we examine into the many formulas employed to represent natural or physical laws, it is found that they fall into two fairly distinct classes, which may be called, respectively, *rational* and *empirical* formulas.

To the former class belong those which have been deduced through processes of mathematical reasoning from the elementary and established laws of the science to which they pertain; hence the term *rational.* Such, for example, are the equations for the motion of falling bodies, the expressions for electric or gravitational force at a point, the equations of the balance, the error equations for the astronomical transit, etc. In these there appear certain *constants* or *coefficients,* the determination of which is often a matter of great scientific importance.

Empirical formulas, on the other hand, are those whose form is inferred wholly from the results of experiment or observation, and which have not been deduced theoretically. Some of the best examples of these are to be found in engineering, such as the formulas for the flow of water in pipes and channels, or for steam pressure as a function of temperature. Empirical formulas also contain constants, which are determined in exactly the same manner as if the formulas were rational, and whose determination depends upon experiment and measurement.

A closer examination into the subject reveals, however, the fact that the boundary between these two classes is by no means a sharp one, for the reason that a very large proportion of the rational formulas purporting to represent natural laws have been deduced upon more or less empirical and approximate assumptions, which have been adopted for the sake of simplicity of form, or for want of better information. In fact, it may well be doubted whether there exist any absolutely rational formulas pertaining to material magnitudes. Even Newton's great law of gravitation has its experimental basis; and it is possible that some future investigation in astronomy may demonstrate it to be inaccurate.

43. Uses and Limitations of Empirical Formulas. — Empirical formulas owe their existence to the fact that in many cases no rational formula can be deduced to represent the law of behavior of a phenomenon, but that, nevertheless, experiment shows *some law* is being obeyed which appears to be simple in character and is therefore presumably expressible, at least approximately, in mathematical symbols. Not being able to trace the mechanism operating between cause and effect, on account of its complexity or for other reasons, the experimenter must seek more or less blindly for a functional relation that will satisfactorily connect them. It may happen that the finding of such a relation as accords perfectly with the observations will throw much light on the nature of the mechanism itself, and lead to a theory relative to it, which can be tested by more intelligently directed later experiments. Stefan's fourth-power law of cooling, which, though wholly empirical as far as Stefan was concerned, has led to the important modern theory of radiation, is an excellent example of this sort.

But the great majority of empirical formulas are confessedly artificial, and reveal nothing of the real nature of the connection between the phenomena involved. Many do not even pretend to consistency in the matter of dimensions; the writer has estimated railroad culvert openings, for example, on the crude working rule that the area of opening, in square feet, should be equal to the square root of the drainage area, in acres — an area equal to a length.

Nevertheless these formulas are capable of the utmost practical usefulness; for by means of them, depending upon the principle of continuity, we may accurately interpolate the values of the unknown function between points actually observed, and even, in a limited way, extrapolate beyond the experimental region into conditions unattainable in practice.

There is still another class of empirical formulas, more or less in the nature of scientific curiosities, which represent, in the experimental region only, a relation between variables that have no conceivable connection with each other. It is thus possible to construct an artificial formula which will follow, with fair accuracy, the increase in population of the United States, or of a city, with time, or even the fluctuations of the stock market over a given interval of time. Such formulas are, however, of little value, as they are merely a sort of cast of a series of statistics which are themselves available; and since the variable represented may not even be continuous, interpolation and extrapolation with any certainty are impossible.

It must also be pointed out that empirical formulas cannot be allowed to enter into theoretical developments on the same basis as rational ones, unless their physical nature is first carefully looked into and the region in which they are assumed to apply is properly circumscribed. Where the true functional relation (supposing one to exist) can be dealt with mathematically with safety, an artificial one closely approximating it may lead, if so used, to altogether erroneous conclusions.

44. Illustrations of Empirical Formulas.

1. *Reduction of Pendulum to Zero Arc.* — The Kater's reversible pendulum is familiar to nearly every physical laboratory student as a means of obtaining the acceleration of a falling body, g, or the value of "gravity." When so adjusted that the time of swing is the same from both supports, *i.e.*, when the knife edges are at conjugate points, the pendulum swings in a period given by the ideal simple pendulum formula

$$T = \pi\sqrt{\frac{l}{g}},$$

in which l is the distance between the knife edges. The determination of g is therefore a matter of measuring the period of oscillation and the distance l.

This equation affords, however, an excellent example of the class referred to in the last paragraph of Art. 42. For in its deduction it is assumed that the pendulum swings without any kind of friction from a perfectly rigid support, and that the amplitude of vibration is infinitely small, none of which conditions is attainable. The writer has attacked these difficulties in the following manner, with good results.

Apparatus is arranged to release the pendulum so as to swing with any desired initial angle of amplitude, and the time accurately observed for each of several small amplitudes. The following results, obtained by one of my students, are typical. ϕ is the half amplitude in degrees, T the period in seconds.

ϕ	T	ϕ	T
1°	0.878489	6°	0.878807
2	8543	7	8874
3	8622	8	8938
4	8679	9	8975
5	8740	10	9029

The steady increase of period with amplitude includes all factors: the true, theoretical increase that would exist under ideal conditions, and the influences of air friction, pivot friction and bending of supports. The results are plotted in Fig. 8, which shows an unmistakable curvature with downward concavity. The relation is therefore not linear, but may be approximately quadratic. The empirical formula

$$T = a + b\phi + c\phi^2 \tag{51}$$

is now assumed to represent the variable T as a function of ϕ. This is treated as a form of observation equation in

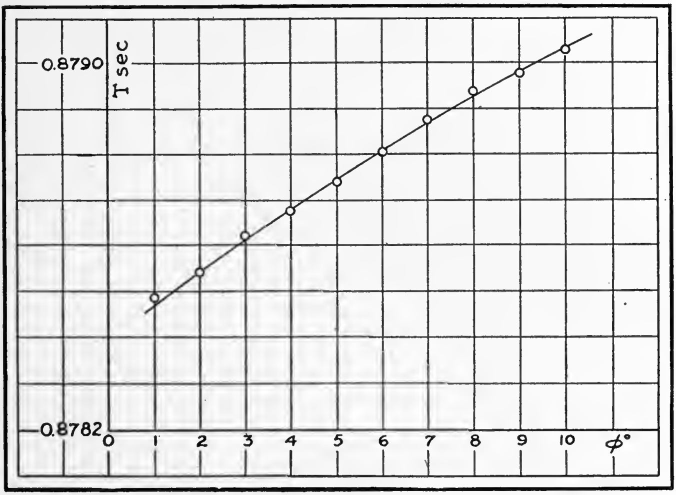

FIG. 8

which the coefficients a, b, c are the unknowns. The observation equations and residuals are written out as usual, the normal equations deduced and solved, with the approximate results

$$a = +0.878400,$$
$$b = +0.000073,$$
$$c = -0.000001,$$

which give

$$T = 0.878400 + 0.000073\,\phi - 0.000001\,\phi^2 \qquad (52)$$

as the relation desired. In reality only a is wanted, for it is the value of T for zero amplitude that we are seeking; that is, the limit approached by T as ϕ approaches zero. This is 0.878400 sec. By this slight extrapolation, therefore, it is possible to extend the experiments into a region unattainable otherwise. The value of g may now be calculated from this result and the measured distance between knife edges.

2. *Solubility Formula.* — Previous to the theoretical calculation of a rational formula for solubility in terms of temperature (Art. 39), the relation was represented by an empirical formula of simple power terms:

$$s = a + bt + ct^2 + dt^3. \qquad (53)$$

The data given in Art. 39 will suffice for the determination of the constants a, b, c, d, a result being obtained which will fit the observations nearly, if not quite, as well as the rational expression. This exercise is left to the student.

3. *Gordon's Formula for Rectangular Columns.* — The ultimate strength of a rectangular column under compression is found to depend fundamentally upon how slender it is; specifically, upon the ratio of its length to its shorter transverse dimension. For long, slender columns, the relation is found to be expressed satisfactorily by the following formula, in which U is the ultimate compressive strength per square inch of cross section, and R the ratio of length to least width.

$$U = \frac{a}{1 + bR^2}. \tag{54}$$

a and b are the empirical constants to be determined. The following data refer to white-oak timber columns or posts, U being expressed in pounds per square inch, and will serve as a further exercise for the student.

R	*U*	*R*	*U*
10	845	25	585
12	820	28	540
15	770	30	510
18	715	35	435
20	675	38	400
22	640	40	375

a and b should be about 925 and 0.00091, respectively.

45. Choice of Mathematical Expression. — The reader will now wish to know by what process the form to be used, as representing the unknown relation between variables, may be arrived at. There is no general rule covering this matter. The empirical form once being settled upon, the calculation of the empirical constants is a direct process; but the selection of a mathematical expression which can be made, by the use of proper constants, to fit the facts with sufficient accuracy, is often a problem calling for the exercise of the highest degree of ingenuity, especially where there is more than one independent variable.

The first step will probably always be to plot the results of the observations, or the data to be represented, to a suitable scale on coördinate paper. The result will be some sort of curve, which, if at all regular, will give an idea as to the nature of the variation, and will often suggest an equation through its resemblance to some well-known locus, such as the straight line, parabola, etc. A few general forms have been found especially adaptable.

The equation

$$y = a + bx + cx^2 + dx^3 + \cdots, \tag{55}$$

continued so far as may be necessary, may be used for curves which are not periodic, nor asymptotic, nor very irregular. The number of terms to be used will be limited by the fact that the coefficients of powers that may be omitted turn out to be negligibly small. This form was used in two of the three examples in the preceding article, and might probably have been used with some success in the third. It was remarked in connection with the second example of Art. 36 that the volume coefficient of expansion really varies when carried over a considerable range. This might have been allowed for by adding a term involving the square of t, with a third unknown constant coefficient, to those used. When such a form as (55) is used, it will be well to apply to it the values of x and y belonging to five or six of the observed points that seem to lie most accurately on the curve, as a preliminary calculation, and determine from them approximate values of an equal number of the coefficients without least squares, in order to ascertain where the series may safely be stopped. It may be found, in this way, that only two or three terms are necessary, the coefficients beyond these being negligible.

The following equations are also quite adaptable to many physical phenomena, particularly those involving variables which approach a limit, or in which maxima and minima do not appear.

$$y = a + b \log (x + K). \qquad (56)$$

$$x = a + b \log (y + K). \qquad (57)$$

$$ax + by + c = xy. \qquad (58)$$

$$\log y = a + b \log x. \qquad (59)$$

(56) is asymptotic in the y direction, (57) in the x direction; (58) is an equilateral hyperbola, asymptotic in both directions. The logarithmic formulas may easily be put into exponential form if desired. The constant K may sometimes be theoretically assigned.

The equation

$$ax + by + c = x^n y \qquad (60)$$

is more general and includes (58). (54) will also be seen to be a special case of it. The curves represented by (60) may have maxima or minima and points of inflection. n may be given a small integral value, as 1, 2, 3, and a, b, c will be the empirical constants to be determined. The student will do well to plot these equations, using assumed values of the constants.

For functions that are apparently periodic, or which have many "ups and downs" in the course of the variation, there may be used a limited number of terms of the trigonometric series

$$\begin{aligned} y = a &+ b \sin nx + c \cos nx \\ &+ d \sin 2\,nx + c \cos 2\,nx \\ &+ f \sin 3\,nx + g \cos 3\,nx + \cdots . \end{aligned} \tag{61}$$

This is a Fourier's series, and can be made to fit any curve with any desired degree of approximation by carrying it to a sufficient number of terms. The calculation of the constants may become extremely laborious, and Prof. A. A. Michelson devised, some years ago, a mechanism known as the *harmonic analyzer*, which will give their approximate values. By the aid of this machine it is possible to analyze very complicated phenomena, such as the tides or the variations in terrestrial magnetism, into harmonic components, and often to reveal their component causes. But it is also possible, by this means, to express in an altogether artificial manner such phenomena as are referred to toward the close of Art. 43. To empirical formulas of this class applies, particularly, the caution against treating them on the same basis as rational formulas in mathematical analysis.

Very often the problem of selecting the proper form will be facilitated by giving attention to obvious limiting conditions, such as the fact that effect is zero where cause is zero, etc. This amounts to making the selection partly rational, and only emphasizes the statement that there is no sharp distinction between rational and empirical expressions. After all has been said, however, the student will still find true the remark, previously made, that this matter calls for skill and ingenuity of a high order.

EXERCISES

46. 1. Experiments were made upon the index of refraction of a solution of varying concentration and density, sodium light being used. The results follow:

Density x	Index y	Density x	Index y
1.200	1.378	1.146	1.365
1.187	1.374	1.132	1.361
1.178	1.371	1.123	1.359
1.167	1.369	1.115	1.356
1.156	1.367	1.098	1.352

Express the variation by a suitable empirical formula, deducing the constants. Would it be safe to infer from this formula the index for pure water?

2. A galvanometer attached to a thermo-electric couple gave the following readings y, for the corresponding differences of temperature x:

x	y	x	y
0°	0	45°	5.50
20	2.50	50	6.15
25	3.10	60	7.60
30	3.70	70	8.65
35	4.30	80	9.90
40	4.90	90	10.90

Prepare suitable empirical formula, deducing constants.

3. The following are observed positions of points on a curve:

x	y	x	y
0	0	5	15.0
1	0.5	6	23.0
2	2.5	7	31.0
3	6.0	8	40.5
4	10.5	9	51.5

Obtain an equation whose graph will fit these points as nearly as possible, and plot it.

4. The temperature of a heated body, cooling in the air, was taken each minute for ten minutes, the results being here tabulated:

Time t	Temp. θ	Time t	Temp. θ
0	84°.9	6	61°.9
1	79.9	7	59.9
2	75.0	8	57.6
3	70.7	9	55.6
4	67.2	10	53.4
5	64.3		

The temperature of the air was 20°. Deduce an equation expressing θ in terms of t.

5. Measurements were made upon the radioactivity of a deposit of pure thorium at intervals after its formation, as follows:

Time	Activity	Time	Activity
10 min.	100.0	5 hr.	107.0
20	104.3	6	101.1
40	110.8	8	89.1
60	115.8	10	78.3
80	118.2	12	68.7
100	119.6	15	56.6
120	119.8	18	46.2
3 hr.	117.9	20	40.7
4	113.0		

Express the relation as an empirical formula. (Note that activity will die out with time.)

6. The quantity of discharge Q, in cubic feet per minute, of a 10-inch sewer pipe was found to vary with the slope (percentage grade) s as per the following data:

s	Q	s	Q
0.1 %	64	2.0 %	146
0.2	75	3.0	170
0.4	88	4.0	190
0.6	95	5.0	208
0.8	108	10.0	279
1.0	116		

Work out an empirical formula and plot it.

7. The means of many observations upon a certain variable star of short period gave the following variations of magnitude:

Time (Days)	Mag.	Time (Days)	Mag.
0	4.65	8	4.20
1	4.10	9	3.57
2	3.50	10	3.70
3	3.80	11	3.93
4	4.00	12	4.07
5	4.10	13	4.35
6	4.40	14	4.64
7	4.65	15	4.40

Represent this variation by as simple a formula as possible.

8. The atmospheric refraction R for a star above the horizon at various altitudes α is given approximately by the following table, corresponding to temperature 50° F. and normal pressure:

α	R	α	R
0°	34′ 50″	10°	5′ 16″
2	18 6	20	2 37
4	11 37	40	1 9
6	8 23	60	0 33
8	6 29	90	0 0

Represent these as nearly as possible by means of an empirical formula.

9. Amagat's experiments on air at very high pressures gave the following results:

Press., Atmos.	Vol.	Press., Atmos.	Vol.
1	1.000000	2000	0.001566
750	0.002200	2500	0.001469
1000	0.001974	3000	0.001401
1500	0.001709		

Represent these by an empirical formula.

10. The current through the field coils of a certain dynamo was varied and the voltage generated by the machine simultaneously measured, as follows:

Field Current, Amps.	Armature Volts	Field Current, Amps.	Armature Volts
0.000	0.0	1.416	21.0
0.472	8.5	1.650	23.2
0.709	12.1	1.888	25.5
0.943	15.4	2.125	27.3
1.180	18.3	2.360	29.0

Represent these by an empirical formula.

11. The specific gravity of dilute sulphuric acid at different concentrations is given in the following table:

Conc. (Per Cent.)	Sp. Grav.	Conc. (Per Cent.)	Sp. Grav.
5	1.033	30	1.218
10	1.068	35	1.257
15	1.101	40	1.300
20	1.139	45	1.345
25	1.178	50	1.389

Represent these by an empirical formula.

12. A pyknometer being tested for evaporation was allowed to stand in a desiccator and weighed at intervals, as follows:

Sept. 30	3:15 P.M.	44.4226 grams
	4:00 P.M.	.4223 grams
Oct. 2	11:00 A.M.	.3855 grams
	3:30 P.M.	.3821 grams
Oct. 3	8:00 A.M.	.3695 grams
	4:00 P.M.	.3622 grams

Find the most probable weight at noon October 1.

13. Simultaneous observations were made upon two connected variables x and y with the following results:

x	y	x	y
26.5	0.002442	31.5	0.005315
27.0	2571	32.0	5607
27.5	2582	32.5	6039
28.0	2885	33.0	6407
28.5	3165	33.5	6947
29.0	3500	34.0	7238
29.5	3738	34.5	7703
30.0	4311	35.0	8092
30.5	4548	35.5	8438
31.0	4991	36.0	8870

Represent these by an empirical formula.

14. Following are vapor pressures, in mm. of mercury, of methyl alcohol at various temperatures:

t	p	t	p
0°	30	35°	204
5	40	40	259
10	54	45	327
15	71	50	409
20	94	55	508
25	123	60	624
30	159	65	761

Represent these by an empirical formula.

15. Assuming the form

$$\log n = \log N - \log \frac{104 - s}{15} - a \log^2 \frac{104 - s}{15},$$

in which n is per cent. and s is grade, deduce N and a from the data of Ex. 10, Art. 30. Plot the curve.

16. The following average heights and weights for men 35 to 40 years of age were compiled by the medical director of the Connecticut Mutual Life Insurance Co.

Height	Weight	Height	Weight
5 ft. 0 in.	131	5 ft. 8 in.	157
1	131	9	162
2	133	10	167
3	136	11	173
4	140	6 0	179
5	143	1	185
6	147	2	192
7	152	3	200

Represent these by an empirical formula.

17. The Society for the Promotion of Engineering Education reports its growth in membership as follows:

Year	No. Members	Year	No. Members
1894	156	1905	400
1895	188	1906	415
1896	203	1907	503
1897	226	1908	675
1898	244	1909	747
1899	251	1910	938
1900	266	1911	1040
1901	261	1912	1166
1902	275	1913	1291
1903	326	1914	1358
1904	379		

Try to calculate the most probable membership in 1915 from these data.

18. Try to represent the data plotted in Ex. 8, Art. 30, by means of an empirical formula.

19. The following measurements give the average length of the head in schoolboys at different ages (West, *Science*, Vol. 21, 1893):

Age	Length (mm.)	Age	Length (mm.)
5	176	14	187
6	177	15	188
7	179	16	191
8	180	17	189
9	181	18	192
10	182	19	192
11	183	20	195
12	183	21	192
13	184		

Represent these by an empirical formula.

20. Records of the magnetic declination (departure of compass from the true north) at 25° N. lat., 110° W. long. over a series of years are as follows (U. S. Mag. Tables for 1905):

1840	9° 28′ E.	1875	10° 24′ E.
1845	38	1880	25
1850	49	1885	25
1855	10 00	1890	26
1860	09	1895	30
1865	16	1900	36
1870	21	1905	48

Represent these by an empirical formula.

CHAPTER VII

WEIGHTED OBSERVATIONS

47. Relative Reliability of Observations. Weights. — We have hitherto regarded each one of a set of several observations as having been made with equal mechanical refinement, care and skill, and the results as meriting, therefore, the same degree of confidence. This assumption is often, however, far from the truth. The position of a star, for example, as measured with an engineer's transit, is less reliable than it would be if measured with a large meridian circle; and the results of a series of difficult observations made by a tired research worker in a cold, drafty laboratory are not worth as much as a similar series made by the same person when rested and under favorable conditions. Again, the mean of a long series of careful observations upon a quantity is certainly of more value than the result of a single measurement upon the same quantity.

It is therefore evident that, in practical work, it is necessary to employ some means whereby differences in reliability may be taken into account. This can be done by using a method of adjustment in which the more trustworthy results are allowed to have more influence upon the final most probable values than the less reliable ones, thus giving each result a degree of prominence proportional to its reliability.

To accomplish this, it is the practice of observers to assign to different observations, numbers, which are supposed to represent their relative degrees of reliability, and which are called *weights*. Thus an observation to which the weight 3 has been assigned is considered to merit only half as much attention in the adjustment as one with the weight 6; etc.

In order to have some basis of estimation, we may regard an observation of given reliability as being equivalent to the mean of a certain number of observations considered as having standard or unit weight, and this number is the weight of the observation in question. The assignment of the weight 10 to an observation means that in the opinion of the observer the result is as trustworthy as the average of ten observations of unit weight. Any standard of trustworthiness may be taken as a unit, but it should be such as to render the weights of all the observations referred to it simple, whole numbers. It is to be remembered that weights are purely relative quantities.

The assignment of weights to the several observations of a set is a task demanding the exercise of skill and careful judgment. If each observation is actually the mean of several elementary observations and all are of the same kind, the matter is comparatively simple, since there is in this case a numerical basis of estimate. Otherwise, and especially when the observations are of different kinds, the assignment is not so easy. The problem presents many analogies to that of giving numerical grades to pupils.

Like other processes of the sort, the weighting of observations cannot be covered by any set of definite rules. It may be suggested that the observer should note and record in detail the peculiar circumstances, if any, attending each observation or set of observations which is to enter into the final adjustment, and allow no source of unusual disturbance to go unnoticed. Often it is well to assign weights at the time of the observation, while all the circumstances are fresh in the mind, but this should not take the place of recording the circumstances. It sometimes happens that some one else examines the original notes and prefers to assign weights for himself. I recall a case of this sort, in which the weighting depended solely upon the records which the observer had kept of the weather conditions prevailing at the time of each experiment. This was because wind and fluctuations of temperature were causes of marked disturbance in this particular work.

48. Adjustment of Observations of Unequal Weight. — In adjusting a set of observations to which different weights have been assigned, we have but to remember that the weight w signifies that the observation in question is the equivalent in importance of w observations of unit weight. It is therefore necessary only to repeat the corresponding observation equation w times, and then proceed as usual with the reduction to normal equations. That is, if the first observation has weight 2, the second 5, the third 3, etc., then simply write the first observation equation twice, the second five times, the third three times, etc. The number of observation equations is now Σw, the sum of the weights.

A simple illustration of this is the case of n direct observations on a single quantity q. If the results are s_1, s_2, $\cdots$, s_n with weights $w_1, w_2, \cdots, w_n$, the most probable value as deduced on the above principle is

$$m_w = \frac{w_1 s_1 + w_2 s_2 + \cdots + w_n s_n}{w_1 + w_2 + \cdots + w_n},$$

or

$$m_w = \frac{\Sigma(ws)}{\Sigma w} \tag{62}$$

This is called the *weighted mean.* If all the weights are equal, it becomes simply the mean.

With observation equations *of the first degree* involving several unknowns, the process can be effected by first multiplying the expression for each residual by the coefficient of the unknown contained therein (as in the rule at the close of Art. 34), then multiplying by the weight of the corresponding observation, adding the results and equating the sum to zero, to form the normal equation. In this way each residual is represented in each normal equation a number of times equal to its weight. The same thing may be attained by first multiplying each of the original observation equations by the *square root* of its weight and then proceeding with the reduction as usual. These square roots need only be indicated, by means of radical signs, as they will disappear on reduction. (Let the student show why the *square roots* of the weights should be thus used, and not the weights themselves.)

In the reduction of observations upon quantities limited by conditions (Art. 40), it is evident that the equations of condition are not to be weighted, but only the observation equations. In the process of adjustment, the weighting should be introduced after the conditions have been involved in the observation equations, but before the reduction of the latter to normal equations. Some of the following examples will illustrate this.

EXERCISES

49. 1. Measurements were made upon the segments of a line AB, formed by points C, D upon it, as follows:

Mean of 2 observations on AC = 45.10 ft.
Mean of 3 observations on AD = 77.96 ft.
Mean of 2 observations on CD = 32.95 ft.
Mean of 3 observations on CB = 98.36 ft.
Mean of 2 observations on DB = 65.55 ft.
Mean of 4 observations on AB = 143.55 ft.

Find the most probable values of AC, CD, DB.

2. In one time-observation with a transit instrument, only five of the nineteen lines of the reticle were used, viz., Nos. 2, 5, 10, 15, 18. A second observation employed all the lines. What can be said as to the relative weights of the two observations, the method of observing being the same in both cases? (Fig. 9.)

3. In determining the constants of a balance, it was borne in mind that the instrument was to be used repeatedly for the weighing of an object varying slightly in weight but always in the neighborhood of 43 to 45 grams. Hence the sensibility was measured twenty-five times with a load of 45 grams, giving a mean of 2.402 scale divisions per milligram, and only four times with zero load, giving a mean of 2.767 scale divisions. Determine the most probable values of the balance constants (Art. 35, Ex. 2).

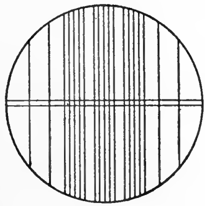

FIG. 9

4. Draw a triangle and measure its angles with a protractor, one angle being measured but once, the second three times, the third eight times (or some other set of unequal numbers), all the measurements being made differentially. Introduce the necessary condition, assign the proper weights and deduce the most probable values of the angles.

5. The following pointings were made at three stations in the triangulation of California, using a 50-cm. direction theodolite (U. S. Coast Survey Report, 1904):

K

Station	Pointing on	Circle Reading	Wt.
San Pedro	Wilson Peak	73° 11′ 40″.97	6.1
	San Juan	118 57 57 .51	6.1
San Juan	San Pedro	16 54 50 .29	7.6
	Wilson Peak	84 26 21 .03	7.6
Wilson Peak	San Juan	241 39 01 .29	6.7
	San Pedro	308 21 21 .51	6.7

Adjust for the most probable angles.

6. The range of magnitude of the variable spectroscopic binary star α *Geminorum* was measured by a selenium photometer on different nights as follows (Stebbins, *Astrophysical Journal*):

Range	Wt.	Range	Wt.	Range	Wt.
0.237	5	0.235	3	0.218	4
.217	4	.197	5	.233	4
.233	5	.217	5	.209	3
.231	5	.210	5	.224	5
.217	5	.222	5	.227	5
.205	5	.213	5	.189	3
.207	5	.223	5	.220	5
.227	5	.250	5	.211	4
.231	5	.219	5		

Find the weighted mean of these observations.

7. Following are results from precise leveling in Texas (U. S. Coast Survey Report, 1911). The weights assigned are inversely proportional to the squares of the distances between the stations.

	Meters	Weight
Lavernia above Serita	+ 57.47	4.8
Thomas above Serita	+ 45.73	1.0
Serita above Stockdale	+ 10.56	2.9
Serita above Ruckman	+ 33.14	0.6
Stockdale above Ruckman	+ 23.62	1.1
Stockdale above Karnes	+ 30.89	0.6
Ruckman above Karnes	+ 6.42	1.8
Ruckman above Bryde	− 11.83	0.9
Karnes above Bryde	− 18.66	4.8
Ruckman above Choate.	+ 20.17	0.9
Bryde above Choate	+ 32.65	3.6
Bryde above Pettus	+ 23.34	4.8
Choate above Pettus	− 9.36	10.9
Bryde above Barroum	+ 11.51	5.1
Pettus above Barroum	− 11.71	7.9
Pettus above Wiess	+ 26.19	7.5
Choate above Wiess	+ 17.40	3.3

Adjust for the most probable elevations above the lowest station in the list.

8. Experiments were made for the purpose of rating a Price current meter, used in measuring the velocity of streams. The data are the velocity V of the current in feet per second and the number R of revolutions per second of the meter (Raymond, *Plane Surveying*).

V	R	Wt.	V	R	Wt.
3.774	1.886	2	1.036	0.466	1
4.544	2.295	1	1.105	0.503	1
4.878	2.464	1	7.142	3.678	1
1.613	0.774	1	2.740	1.342	1
1.316	0.618	1	6.896	3.552	1

Assume a linear relation and deduce the two constants.

9. Four points, A, B, C, D, lie consecutively in a straight line. The following distances are measured with a steel tape.

AD	2871.2 (Ave. of 2)	BC	1392.2
AB	1042.0	BD	1828.6
AC	2433.5	CD	437.5

Apply the principle that, in chaining, the weights of similarly measured lines are inversely proportional to the squares of their measured lengths, and adjust the above values accordingly.

10. Zenith telescope observations were made at Roslyn Station, Virginia, upon the latitude of that station with various pairs of stars, as follows (Chauvenet, *Practical Astronomy*). The weights were assigned from the number of observations involved and the precision with which the declinations of the stars employed had been measured.

Observed Lat.	Wt.	Observed Lat.	Wt.
37° 14′ 24″.78	0.44	37° 14′ 25″.15	0.59
25 .05	.67	25 .22	.67
24 .83	.82	24 .84	.67
26 .20	.59	25 .36	.67
25 .91	.43	26 .02	.62
22 .73	.00	25 .42	.44
25 .93	.70	26 .08	.44
25 .18	.65	25 .72	.67
25 .89	1.09	25 .70	1.33
25 .79	1.33	25 .93	1.20
24 .53	0.29		

Find the most probable latitude.

11. (Adapted from Chauvenet, *Practical Astronomy.*) At a station O of the U. S. Coast Survey, angles were read on each of four other stations, A, B, C, D, as follows:

Angle				Wt.	Angle				Wt.
AOB	65°	11′	52″.5	3	*COD*	87°	2′	24″.7	3
BOC	66	24	15 .6	3	*DOA*	141	21	21 .8	1

Adjust for the most probable angles.

12. Spectrographic radial velocity measurements were made upon the Orion nebula, using different spectrum lines on different dates, as follows (Lick Observatory Bulletin No. 19):

Date	Line	Velocity (Km.)	Wt.
Dec. 8, 1901	H_γ	+ 17.1	3
16	H_β	16.1	2
17	H_β	17.0	2
18	H_γ	14.8	3

Find the most probable radial velocity.

13. A certain critical coefficient of expansion was measured several times with different apparatus.

Observed Val.	Wt.	Observed Val.	Wt.
0.0045	3	0.0036	2
39	2	26	2
34	5	27	1
30	4	43	3

Find the most probable value from these data.

14. The following data are right ascension corrections to the Berlin *Jahrbuch* made by the photographic transit at Georgetown Observatory for the star ζ *Ophiuchi* on different dates.

Cor.	Wt.	Cor.	Wt.	Cor.	Wt.	Cor.	Wt.
− 0.03 s.	2	+ 0.02 s.	2	− 0.01	3	+ 0.02	3
− .03	3	.00	2	− .04	2	+ .02	3
− .01	1	+ .04	1	+ .03	2	.00	2
− .02	0	− .04	1	− .02	3	− .04	3
− .03	1	− .05	1	− .06	2	− .06	3

Find the weighted mean.

15. (Adapted from Wright's *Adjustment of Observations.*) The following trigonometric levelings were made between two terminal stations *A* and *B*, as follows:

Stations	Meters	Wt.	Stations	Meters	Wt.
A above 12	914.96	23	3 above 9	216.46	1
A above 10	1287.75	17	5 above 9	899.87	1
A above 11	1299.27	2	5 above 8	1075.77	1
A above 9	1553.09	5	3 above 8	391.74	1
12 above 10	372.73	5	7 above 8	901.78	1
12 above 11	384.41	2	5 above 7	174.45	7
12 above 9	638.30	3	4 above 3	296.69	60
12 above 8	814.35	1	7 above 3	509.49	4
10 above 11	11.60	3	*B* above 3	1376.19	14
10 above 9	265.48	6	5 above 4	387.24	20
10 above 8	441.10	2	7 above 4	212.75	7
11 above 9	253.87	1	*B* above 4	1079.50	30
11 above 8	429.55	10	*B* above 5	692.35	15
9 above 8	175.37	1			

By precise spirit leveling, A was found to be 39.05 meters above B, which may be taken as correct. Adjust the heights of the other stations above B accordingly.

50. Wild or Doubtful Observations. — It sometimes happens that, in the course of a series of measurements, results occur which are so doubtful that the observer is tempted to reject them altogether. In technical language, their weight is so small as to be seemingly negligible, and it is a question whether their retention may not do more harm than good.

The doubt may arise from the existence of unusual or disturbing conditions, known to the observer. On one occasion I was making a quantitative analysis to determine the exact concentration of a solution, and during the process of drying, accidentally spilled a few drops of hydrant water into the residue. My final result was to be an average from the analyses of several specimens, and the accident would unquestionably vitiate the result of this observation; but the specimens were obtained with difficulty and I could ill afford to spare any of the data. Was the result to be rejected or not?

Again, suspicion may be due to a marked difference between the result in question and all the others of the set. This does not refer to mistakes (Art. 9), which may usually be easily rectified. To the observer's best knowledge, the doubtful observation deserves as much weight as the others, having been made with the same care; but he dislikes to retain it, as it is so far out of agreement.

The former class of doubtful observations should, in the opinion of the writer, be rejected unless some idea of the extent of the disturbance can be obtained and due correction made for it if necessary. What I did in the case cited was to test the hydrant water and ascertain that the amount of solids contained in a few drops would not be sufficient to affect the result at all seriously; but I gave only half as much weight to this observation as to the others.

With the latter class the case is more doubtful. Just because a result differs from the others is no proof that it is any farther from the truth, especially when the number of observations is small. In casting out such a result, one may be throwing away his most valuable observation. Certain criteria have been proposed for deciding whether to retain or reject a "wild" observation, based upon the law of error distribution. Probably the best decision will be based upon the observer's judgment, it being borne in mind that results of observations should not be tampered with unthinkingly. Where wide deviations occur, it will be well, if possible, to continue the observations until a sufficient number are accumulated to show the law of distribution with some distinctness and symmetry.

51. The Precision Index *h*.—It was pointed out in Art. 28 that the quantity h in the error equation has to do with the precision of the observations (Art. 13), and that the greater the value of h, the greater is the precision indicated. h may thus be termed the "precision index" or "measure of precision." We are here naturally led to inquire what connection exists between the precision index and the weight of an observation. For, if we have two sets of measurements, one of which is more precise than the other, the value of h belonging to the error distribution in one set will be larger than that belonging to the other; while at the same time the weight of one observation from the first set is greater than that of one from the second set.

Let h_1 and c_1 be the constants in the equation of error distribution corresponding to the first set, and let w_1 be the weight of an observation from that set, supposing them all to have equal weight; and let h_2, c_2, w_2 be the corresponding quantities relating to the second set. The probability of an error x occurring in the first set is

$$y_1 = c_1 e^{-h_1^2 x^2}. \tag{63}$$

Let the value of the precision index corresponding to a set in which the observations are of unit weight be h. This may be called a "standard index," though no absolute value can as yet be assigned to it. An observation from the first set is equivalent in worth to w_1 observations from the standard set, in each of which the probability of an error x is

$$y = ce^{-h^2 x^2}.$$

Therefore the probability y_1 of the error x occurring in the first set is that of its occurring w_1 times in the standard set, which is y^{w_1}, giving

$$y_1 = y^{w_1} = c^{w_1} e^{-w_1 h^2 x^2}. \tag{64}$$

The error x being supposed the same in (63) and (64), and these equations holding for all values of x, comparison gives at once

$$h_1^2 = w_1 h^2.$$

Likewise

$$h_2^2 = w_2 h^2,$$

referring to the observations of the second set, having weight w_2. That is,

$$h_1^2 : h_2^2 : h_3^2 : \cdots = w_1 : w_2 : w_3 : \cdots, \tag{65}$$

or *the weights of observations are in proportion to the squares of their precision indices.*

In order to illustrate this principle, let the error distribution of the first set be represented by A, Fig. 10, and

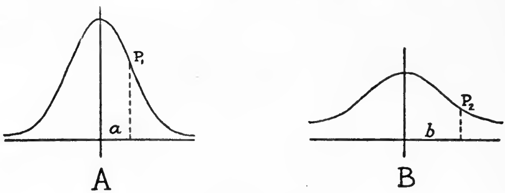

FIG. 10

that of the second by B. A represents the more precise set of measurements. Let the points of inflection be distant a and b from the y-axis in the two curves respectively. From (27), Art. 28,

$$a : b = \frac{1}{h_1\sqrt{2}} : \frac{1}{h_2\sqrt{2}},$$

or

$$h_1 : h_2 = b : a. \tag{66}$$

Then from (65),

$$w_1 : w_2 = b^2 : a^2. \tag{67}$$

It is thus possible, by means of a study of residual curves, to estimate the relative weights of observations made by different processes, or with different instruments, or by different observers. In the next chapter (Art. 59) will be presented a mathematical means of obtaining the same information, without plotting the curves. This does not, of course, refer to the weighting of individual observations of the same set, which must depend upon the judgment of the observer as to the conditions existing at the time.

52. General Statement of the Principle of Least Squares. — The principle of least squares, already enunciated in three ways adapted to increasingly complicated cases of adjustment (Arts. 29, 31, 33), may now be deduced in its general form, which includes all the others as special cases.

Let a series of n observations be made, whose weights are respectively $w_1, w_2, \cdots, w_n$, and let the residuals be $\rho_1, \rho_2, \cdots, \rho_n$. The probabilities of these residuals are

(64)
$$\begin{aligned} y_1 &= c^{w_1} e^{-w_1 h^2 \rho_1^2}, \\ y_2 &= c^{w_2} e^{-w_2 h^2 \rho_2^2}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= c^{w_n} c^{-w_n h^2 \rho_n^2}, \end{aligned}$$

in which c and h are the precision constants corresponding to an observation of unit weight. The probability of the occurrence of all of this particular set of residuals is

$$Y = y_1 y_2 \cdots y_n = c^{w_1 + w_2 + \cdots + w_n} e^{-h^2(w_1\rho_1^2 + w_2\rho_2^2 + \cdots + w_n\rho_n^2)}. \quad (68)$$

The most probable set of residuals, and hence those determined by the most probable values of the unknown quantities involved in the observations, are those for which Y is a maximum, and hence those for which $\Sigma(w\rho^2)$ is a minimum.

The general statement follows: *The most probable values of unknown quantities connected by observation equations to which weights have been assigned are those which will render the sum of the weighted squares of the residuals a minimum.* The meaning of the term "weighted squares" is obvious from the above.

The rules of Art. 48 for the adjustment of weighted observations might have been deduced from the principle as above stated, in the same manner as the deduction was made for the simpler case of equal precision (Arts. 33, 34).

CHAPTER VIII

PRECISION AND THE PROBABLE ERROR

53. Discontinuity of the Error Variable. — There is one point in the foregoing discussions of the law of error that has not been emphasized. In all of the mathematical work, we have treated the error as if it were a true continuous variable x, which might have any value whatever from $-\infty$ to $+\infty$. But to assume this would be to assume an infinitely minute graduation of our measuring scale. To illustrate the fact, let us suppose that the measured quantity is an angle. If the error were a continuous variable, successive measured values of the angle need not differ by so much as a billionth of a second, yet might be different; and the probability of any particular error out of the infinity of possible ones would be infinitesimally small. It is thus seen that the variable error x, instead of varying by infinitesimal increments dx, really has equal finite discontinuities Δ, which represent the smallest fraction of a unit in which the measured results are expressed. On a surveyor's transit, for example, Δ is usually one minute for single angle-readings; while with the micrometers used on large equatorial telescopes, angular measurements are made which may be expressed in hundredths of a second.

The error curve may then be represented as a sort of stairway with equal treads and unequal risers, and the errors considered as falling into compartments corresponding to the several Δ's, just as do the shots in the target experiment of Art. 11.

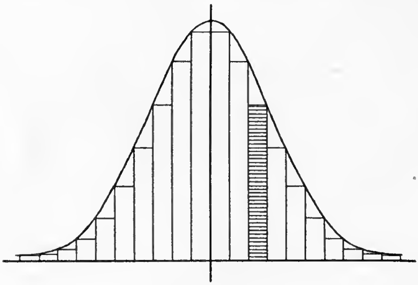

FIG. 11

The width of one of these error "compartments" being Δ and its height y, it may be looked upon as a narrow strip of finite area $y\Delta$. y is the probability that an error will fall into the compartment to which the ordinate y corresponds. Let us imagine all the strips placed end to end, the total length being $\Sigma y = 1$, since this is the sum of the probabilities of all possible errors (Art. 16). But the area of this long strip is the area of the curve:

$$A = c \sum_{x=-\infty}^{x=+\infty} e^{-h^2x^2} \cdot \Delta$$

$$= 2\,c \sum_{x=0}^{x=\infty} e^{-h^2x^2} \cdot \Delta;$$

and

$$\Sigma y = \frac{A}{\Delta} = \frac{2\,c}{\Delta} \sum_{x=0}^{x=\infty} e^{-h^2x^2} \cdot \Delta = 1. \qquad (69)$$

Δ being small, the summation Σ in (69) is for practical purposes represented by the definite integral

$$I = \int_0^\infty e^{-h^2x^2} dx. \tag{70}$$

Whence, from (69),

$$c = \frac{\Delta}{2I}. \tag{71}$$

The integral I is a function of h, and when we have determined *what* function of h, we shall have found the relation between the two constants c and h of the error equation, which was referred to in Art. 28.

54. Value of the Integral I, and the Relation between c and h. — The evaluation of the definite integral I of the preceding article is worked out in Note C of the Appendix, this being a problem belonging to the theory of definite integrals and an interesting example of a method often employed in such cases. The result is

$$I = \int_0^\infty e^{-h^2x^2} dx = \frac{\sqrt{\pi}}{2h}. \tag{72}$$

The student who does not care to follow out the proof may verify the result by plotting the function for two or three chosen values of h and integrating the curve with a planimeter. However, the note referred to is not difficult to read, and students are advised to do so.

Substituting the value of I here obtained in (71), the expression for c is found to be

$$c = \frac{h\Delta}{\sqrt{\pi}}, \tag{73}$$

and is therefore proportional to h.

We may now write the error equation in a final and more satisfactory form:

$$y = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2x^2}, \tag{74}$$

in which the law of error distribution is made to depend upon the scale-interval Δ, which is readily obtained from the recorded observations, and upon a constant h which we have come to refer to as the *precision index* (Art. 51) and which, as will be seen later, can also be calculated from the results of the observations.

55. Probability of an Error Lying between Given Limits. The Probability Integral. — An important problem in the theory of errors is to find the probability that an error will lie between two given limits X_1 and X_2. This may be obtained in terms of the precision index h. For, the result sought is merely the sum of the probabilities of all errors between X_1 and X_2, which is, by (74)

$$Y_{1,2} = \frac{h}{\sqrt{\pi}} \sum_{x=X_1}^{x=X_2} e^{-h^2x^2} \cdot \Delta,$$

or replacing Δ by dx as in equations (69), (70), in order to convert the summation approximately into an integral,

$$Y_{1,2} = \frac{h}{\sqrt{\pi}} \int_{X_1}^{X_2} e^{-h^2x^2} dx$$

$$= \frac{h}{\sqrt{\pi}} \int_0^{X_2} e^{-h^2x^2} dx - \frac{h}{\sqrt{\pi}} \int_0^{X_1} e^{-h^2x^2} dx. \tag{75}$$

The definite integral occurring in (75), viz.,

$$Y = \frac{h}{\sqrt{\pi}} \int_0^X e^{-h^2x^2} dx,$$

is simplified by a substitution. Let $hx = z$, $h^2x^2 = z^2$, $hdx = dz$; when $x = X$, $z = hX$. We then have

$$Y = \frac{1}{\sqrt{\pi}} \int_0^{hX} e^{-z^2} dz = \frac{1}{\sqrt{\pi}} \int_0^{hX} e^{-x^2} dx. \qquad (76)$$

This expression Y is commonly called the *probability integral,* and is evidently a function of the upper limit hX. It expresses the probability that an error will lie between 0 and $+X$. As applied to the question of precision, the value of Y itself is less useful than that of $2Y$, which is the probability that an error will lie between $+X$ and $-X$, that is, that a measured result will be within X of the true value.

Our problem now requires that we be able to calculate Y from the given value of hX. Y cannot, however, be expressed directly as a function of hX, but must be evaluated through the use of infinite series. This mathematical work is given in Note D of the Appendix, to which the student is referred. Tables of the values of the integral, thus calculated, are standard, and given in every book on the theory of errors. In the accompanying table the values of $2 \cdot Y$ are given, corresponding to the argument hX.

hX	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0113	0.0226	0.0338	0.0451	0.0564	0.0676	0.0789	0.0901	0.1013
0.1	.1125	.1236	.1348	.1459	.1569	.1680	.1790	.1900	.2009	.2118
0.2	.2227	.2335	.2443	.2550	.2657	.2763	.2869	.2974	.3079	.3183
0.3	.3286	.3389	.3491	.3593	.3694	.3794	.3893	.3992	.4090	.4187
0.4	.4284	.4380	.4475	.4569	.4662	.4755	.4847	.4937	.5027	.5117
0.5	0.5205	0.5292	0.5379	0.5465	0.5549	0.5633	0.5716	0.5798	0.5879	0.5959
0.6	.6039	.6117	.6194	.6270	.6346	.6420	.6494	.6566	.6638	.6708
0.7	.6778	.6847	.6914	.6981	.7047	.7112	.7175	.7238	.7300	.7361
0.8	.7421	.7480	.7538	.7595	.7651	.7707	.7761	.7814	.7867	.7918
0.9	.7969	.8019	.8068	.8116	.8163	.8209	.8254	.8299	.8342	.8385
1.0	0.8427	0.8468	0.8508	0.8548	0.8586	0.8624	0.8661	0.8698	0.8733	0.8768
1.1	.8802	.8835	.8868	.8900	.8931	.8961	.8991	.9020	.9048	.9076
1.2	.9103	.9130	.9155	.9181	.9205	.9229	.9252	.9275	.9297	.9319
1.3	.9340	.9361	.9381	.9400	.9419	.9438	.9456	.9473	.9490	.9507
1.4	.9523	.9539	.9554	.9569	.9583	.9597	.9611	.9624	.9637	.9649
1.5	0.9661	0.9673	0.9684	0.9695	0.9706	0.9716	0.9726	0.9736	0.9745	0.9755
1.6	.9763	.9772	.9780	.9788	.9796	.9804	.9811	.9818	.9825	.9832
1.7	.9838	.9844	.9850	.9856	.9861	.9867	.9872	.9877	.9882	.9886
1.8	.9891	.9895	.9899	.9903	.9907	.9911	.9915	.9918	.9922	.9925
1.9	.9928	.9931	.9934	.9937	.9939	.9942	.9944	.9947	.9949	.9951
2.0	0.9953	0.9955	0.9957	0.9959	0.9961	0.9963	0.9964	0.9966	0.9967	0.9969
2.1	.9970	.9972	.9973	.9974	.9975	.9976	.9977	.9979	.9980	.9980
2.2	.9981	.9982	.9983	.9984	.9985	.9985	.9986	.9987	.9987	.9988
2.3	.9989	.9989	.9990	.9990	.9991	.9991	.9992	.9992	.9992	.9993
2.4	.9993	.9993	.9994	.9994	.9994	.9995	.9995	.9995	.9995	.9996
2.	0.9953	0.9970	0 9981	0.9989	0.9993	0.9996	0.9998	0.9999	0.9999	0.9999
∞	1.0000									

56. Calculation of the Precision Index from the Residuals. — The table in the preceding article enables us to find the value of $2\,Y$ corresponding to any given value of X, providing the precision index h is known. That is, if we have h, we can find from the table the probability that an error will not exceed a given value X in numerical value.

Reciprocally, if the value of $2\,Y$ corresponding to a given limiting error X can be determined by any means, the same table will give the value of hX and hence of h, just as a number can be found from its logarithm, or an angle from its tangent, by interpolation. This may be accomplished from a study of the residuals if the number of observations is large enough.

For, the probability $2Y$ that a residual will lie between $+X$ and $-X$ may be obtained by finding what proportion of them *do* lie between these limits. By choosing several different values of X, as many values for hX may be found, which may be combined like observation equations for the unknown quantity h.

While this is rather too laborious a method for practical purposes, it will be found a very useful means of getting clearly in mind the relation of h to the precision of the measurements. We shall therefore apply it, by way of illustration, to the following results of 144 measurements upon the length of a line. Of these:

		Residuals			Residuals
1 gave	2178.1 feet	−1.0	18 gave	2179.2 feet	+0.1
1 gave	.2 feet	−0.9	18 gave	.3 feet	+0.2
2 gave	.4 feet	−0.7	10 gave	.4 feet	+0.3
3 gave	.5 feet	−0.6	7 gave	.5 feet	+0.4
4 gave	.6 feet	−0.5	5 gave	.6 feet	+0.5
5 gave	.7 feet	−0.4	2 gave	.7 feet	+0.6
11 gave	.8 feet	−0.3	4 gave	.8 feet	+0.7
16 gave	.9 feet	−0.2	144	Mean	$\Sigma\rho^2$
21 gave	2179.0 feet	−0.1		=2179.1	=14.36
16 gave	.1 feet	0.0			

The interval Δ between successive residuals is 0.1 foot. An examination of the residuals gives the following facts:

N	$\pm X$	$2Y$
55 are numerically not greater than	0.1	0.3819
89 are numerically not greater than	0.2	0.6166
110 are numerically not greater than	0.3	0.7638
122 are numerically not greater than	0.4	0.8472
131 are numerically not greater than	0.5	0.9098
136 are numerically not greater than	0.6	0.9444
142 are numerically not greater than	0.7	0.9861
142 are numerically not greater than	0.8	0.9861
143 are numerically not greater than	0.9	0.9930

The numbers in the column headed $2Y$ are simply the values of the fraction $\frac{N}{144}$; *e.g.*, $\frac{55}{144} = 0.3819$, etc. The values of hX corresponding to these values of $2Y$, obtained from the probability integral table, are as follows:

X	hX	X	hX
0.1	0.353	0.6	1.353
0.2	0.616	0.7	1.740
0.3	0.838	0.8	1.740
0.4	1.011	0.9	1.907
0.5	1.198		

any one of which will give an approximate value of h. By using the above values, we may write simple observation equations for h, thus:

$$0.1\, h = 0.353,$$
$$0.2\, h = 0.616,$$
$$\text{etc.},$$

which when reduced in the usual manner (Art. 34) give as the most probable value

$$h = 2.33.$$

Using this as the value of h and 0.1 as that of Δ, and substituting them in (74), gives as the equation of error distribution for this case

$$y = 0.1314\, e^{-5.43 x^2}.$$

Let the student plot this curve and the actual distribution of residuals together on the same sheet for the purpose of comparison. The ordinates had better be laid off on five or ten times as great a scale as the abscissas, for convenience.

57. Approximate Formulas for the Precision Index. — The foregoing is doubtless as accurate a method of obtaining the precision index h as could be desired where the number of observations is large, and where it can therefore be assumed that the residuals distribute themselves in accordance with the error law. It is however too laborious for practical purposes, and can be replaced by shorter methods. The first one here presented depends, in fact, upon the same principle as that used in the foregoing calculation, being simply more direct.

Let there be n observations made upon the unknown quantity or upon functions of it, the errors being x_1, $x_2, \cdots, x_n$. From (74), the probabilities of these errors are, respectively,

$$y_1 = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2 x_1^2},$$

$$y_2 = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2 x_2^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_n = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2 x_n^2},$$

and the probability of the given set of errors is

$$Y = \left(\frac{\Delta}{\sqrt{\pi}}\right)^n h^n e^{-h^2 \Sigma x^2}. \tag{77}$$

The most probable value of h that can be afforded by these observations is the one giving rise to the most probable distribution of errors, a condition which is equivalent to the statement that Y is to be a maximum. Hence, regarding h as a variable and obtaining the condition for Y a maximum by differentiation of (77), we have

$$\frac{\partial Y}{\partial h} = \left(\frac{\Delta}{\sqrt{\pi}}\right)^n e^{-h^2 \Sigma x^2} [n h^{n-1} - 2\, h^{n+1} \Sigma x^2] = 0,$$

whence

$$h = \sqrt{\frac{n}{2\, \Sigma x^2}}. \tag{78}$$

This would be adequate if the true errors x were known, and does very well in any case, where there are many observations, if we simply use the residuals ρ instead of the true errors. The discrepancy between (78) and the value obtained by using the residuals is discussed by many writers at some length, and it is a question whether it deserves such attention, inasmuch as only an approximate value of h is usually required. It may be easily seen that Σx^2 is greater than $\Sigma \rho^2$, since by the principle of least squares $\Sigma \rho^2$ is to be a minimum. Hence if $\Sigma \rho^2$ is to be used instead of Σx^2 in (78), something must be done to reduce the numerator as well as the denominator. The general practice is to make it $n - 1$ instead of n, a procedure which has some theoretical support. The formula for h now becomes

$$h = \sqrt{\frac{n-1}{2\,\Sigma \rho^2}}. \tag{79}$$

This formula is of the greatest importance in the calculation of what is known as the *probable error* (Art. 58). Its use is somewhat laborious, owing to the necessity of squaring all the residuals. Another formula for h, first used by Peters, can be derived upon the following reasoning.

The total number of errors, both $+$ and $-$, is n. Then if n_x be the number of errors having the particular value x, their probability is

$$y = \frac{n_x}{n} = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2x^2}. \tag{80}$$

Let us consider only $+$ errors, the average value of which is the same as that of all the errors, $+$ and $-$, taken without signs. The sum of all the $+$ errors is, from (80),

$$\sum_{x=0}^{x=\infty} (n_x x) = \frac{nh}{\sqrt{\pi}} \sum_{x=0}^{x=\infty} (x e^{-h^2x^2})\cdot\Delta,$$

or approximately (see Art. 53),

$$\sum_{x=0}^{x=\infty}(n_x x) = \frac{nh}{\sqrt{\pi}}\int_0^\infty xe^{-h^2x^2}dx = \frac{n}{2\,h\sqrt{\pi}}.$$

The average of the + errors, and hence of all the errors (disregarding sign), is therefore

$$\frac{\Sigma x}{n} = \frac{\sum_{x=0}^{x=\infty}(n_x x)}{\frac{1}{2}\,n} = \frac{1}{h\sqrt{\pi}};$$

whence

$$h = \frac{n}{\sqrt{\pi}\Sigma x}. \tag{81}$$

This formula, like (78), is in terms of the errors x. In order to reduce (78) to the expression (79) for h in terms of the residuals ρ, the numerator was reduced in the ratio

$$\sqrt{n-1} : \sqrt{n}.$$

If we apply the same process to (81), at the same time replacing the x's by the ρ's, we obtain as the analogue of (79),

$$h = \frac{\sqrt{n(n-1)}}{\sqrt{\pi}\Sigma\rho}, \tag{82}$$

which will be referred to as *Peters' formula* for h, whereas (79) will be called the *standard formula* for h.

It is to be noted that the above reasoning applies only to observations of equal weight. The question of weight, as related to precision, will be introduced in Art. 59.

58. The Probable Error of an Observation. — We have heretofore treated the precision of observations in a more or less abstract and relative way, and the need is apparent for a more concrete and tangible expression for it. In short, we desire something that will convey to the mind an idea of the accuracy attained, in terms of the units of measurement used. This has been secured in several ways.

One of the simplest quantities of this sort is the *average residual,* taken without reference to sign. Its relation to the precision index is obtainable from (82), which gives as the average residual

$$\frac{\Sigma \rho}{n} = \frac{1}{h}\sqrt{\frac{n-1}{\pi n}}, \tag{83}$$

or if n is very large, approximately,

$$\frac{\Sigma \rho}{n} = \frac{1}{h\sqrt{\pi}}, \tag{84}$$

which is equivalent to (81).

Again, there is the *virtual* or *radical mean square* (R.M.S.) residual, which from (79) is

$$\sqrt{\frac{\Sigma \rho^2}{n}} = \frac{1}{h}\sqrt{\frac{n-1}{2\,n}}, \tag{85}$$

or if n is large, approximately,

$$\sqrt{\frac{\Sigma \rho^2}{n}} = \frac{1}{h\sqrt{2}}. \tag{86}$$

The concrete significance of this quantity lies in the fact that it represents the abscissa of the point of inflection on the error curve (Art. 28, Eq. 27).

The most approved expression used for this purpose is, however, the *probable error*. In Art. 55 it is shown how to calculate the probability that an error will not exceed a given limit X, providing the precision index h is known, the table of the probability integral being used in the calculation. We may now give the following definition.

Designated by ϵ, *the probable error of an observation is such that the probability that the given observation differs from the truth by an amount numerically less than* ϵ *is equal to the probability that it differs by an amount numerically greater than* ϵ.

More briefly, any error is just as likely to be less than the *probable error* ϵ as it is to be greater; or in other words, the probability that an error lies between $+\epsilon$ and $-\epsilon$ is $\frac{1}{2}$. In the long run, half the errors will lie within ϵ, and half will exceed it.

Therefore, ϵ is that value of the limit X, appearing in the argument hX, which corresponds to $2Y = \frac{1}{2} = 0.5000$ in the table of the probability integral. Interpolation in this table gives as the value of the argument hX for which $2Y = 0.5000$,

$$h\epsilon = 0.4769,$$

whence the probable error is

$$\epsilon = \frac{0.4769}{h}. \tag{87}$$

Using the standard formula (79) for h, this gives

$$\epsilon = 0.6745\sqrt{\frac{\Sigma\rho^2}{n-1}}, \tag{88}$$

in terms of the sum of the squares of the residuals; or using Peters' formula (82), we obtain Peters' formula for the probable error,

$$\epsilon = 0.8453 \frac{\Sigma \rho}{\sqrt{n(n-1)}}, \tag{89}$$

in terms of the sum of the residuals without sign. From Peters' formula it will be seen that when n is large, approximately,

$$\epsilon = 0.85 \frac{\Sigma \rho}{n}; \tag{90}$$

or *the probable error of an observation is approximately equal to* 85 *per cent. of the average residual,* taken without sign. This simple rule is sufficiently accurate for most practical purposes, in the case of a long series of observations of equal weight.

The notation by which probable errors are expressed uses the double sign. For example, if the mass of an object, obtained by weighing, is stated as 24.830726 ± 0.000014 grams, this means that the probable error of the weighing is 0.000014 gram. This quantity would be obtained, as explained above, by taking a series of weighings on the same balance under the same conditions, finding the residuals, and applying (88), (89) or (90).

59. Relation between Probable Error and Weight. — When the several observations of the same series are assigned different weights, the probable error of a single observation has no significance without further qualification, since the precision index h, and hence the probable error, is supposed different for the different observations. We may express both, however, with reference to observations of some selected precision, as those which have been assigned unit weight. We shall designate by h the precision index and by ϵ the probable error of observations of unit weight, and refer to them also as observations of *standard* precision. Other observations whose weights are $w_1, w_2, \cdots$, have precision indices $h_1, h_2, \cdots$, and probable errors $\epsilon_1, \epsilon_2, \cdots$.

We have already deduced the relation between h and w (Art. 51, Eq. 65). It is

$$h_1 : h_2 : h_3 : \cdots = \sqrt{w_1} : \sqrt{w_2} : \sqrt{w_3} : \cdots.$$

From (87), $$h_1 : h_2 : h_3 : \cdots = \frac{1}{\epsilon_1} : \frac{1}{\epsilon_2} : \frac{1}{\epsilon_3} : \cdots.$$

Combining these proportions,

$$\epsilon_1 : \epsilon_2 : \epsilon_3 : \cdots = \frac{1}{\sqrt{w_1}} : \frac{1}{\sqrt{w_2}} : \frac{1}{\sqrt{w_3}} : \cdots, \qquad (91)$$

which expresses the very important principle that *the probable errors of different observations in the same series are inversely proportional to the square roots of their weights;* or reciprocally, *the weights of observations of the same kind are inversely proportional to the squares of their probable errors.*

To illustrate this, if the probable error of a measured quantity obtained by one method is found to be only one-half as great as that obtained by a less precise method, then the weight assigned to the former in combining them should be four times that assigned to the latter. In other words, one observation by the former method is worth four made by the latter.

If ϵ be the probable error of an observation of unit weight, found from (88), (89) or (90), then by the foregoing principle, the probable error of an observation of weight w is given by

$$\epsilon_w = \frac{\epsilon}{\sqrt{w}}. \tag{92}$$

This will shortly be seen to have an important application to the finding of probable errors of adjusted or most probable values of unknown quantities.

If the probable error of an observation of unit weight is to be calculated from a series of weighted observations, we may generalize the reasoning of Art. 57 as follows. The weights are $w_1, w_2, \cdots$, and the corresponding precision indices $h_1, h_2, \cdots$. h being the standard index,

$$h_1 = h\sqrt{w_1},$$
$$h_2 = h\sqrt{w_2},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$h_n = h\sqrt{w_n}.$$

The probabilities of the respective errors are now given by

$$y_1 = \frac{h_1\Delta}{\sqrt{\pi}}e^{-h_1^2x_1^2} = \frac{\Delta}{\sqrt{\pi}}h\sqrt{w_1}\,e^{-h^2w_1x_1^2},$$

$$y_2 = \frac{\Delta}{\sqrt{\pi}}h\sqrt{w_2}\,e^{-h^2w_2x_2^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_n = \frac{\Delta}{\sqrt{\pi}}h\sqrt{w_n}\,e^{-h^2w_nx_n^2}.$$

The joint probability, corresponding to (77), is now

$$Y = \left(\frac{\Delta}{\sqrt{\pi}}\right)^n \sqrt{w_1 w_2 \cdots w_n}\, h^n e^{-h^2\Sigma(wx^2)};$$

from which, by the same reasoning as that leading to (78) and (79), the standard index of precision is

$$h = \sqrt{\frac{n-1}{2\,\Sigma(w\rho^2)}}. \tag{93}$$

Instead of (88) we now have, by substitution of this new expression for h in (87),

$$\epsilon = 0.6745\sqrt{\frac{\Sigma(w\rho^2)}{n-1}}, \tag{94}$$

the important standard formula for the probable error of an observation of unit weight, as obtained from a series of weighted observations. In this formula, before summing the squares of the residuals, each square is multiplied by the corresponding weight; or, otherwise, each residual is multiplied by the square root of the corresponding weight. (See Art. 52.)

The same modification may be made in the Peters' formula (89) to adapt it to weighted observations, giving

$$\epsilon = 0.8453\frac{\Sigma(\sqrt{w}\rho)}{\sqrt{n(n-1)}}, \tag{95}$$

or if n is large, approximately,

$$\epsilon = 0.85\,\frac{\Sigma(\sqrt{w}\rho)}{n}, \tag{96}$$

which corresponds to (90).

EXERCISES

60. **1.** Two specific gravity bottles, one of which, No. 7701 *a*, was of the ordinary type, and the other, No. 7701 *c*, of a special improved design, were each filled with water five times at the same temperature, the following being the results of the weighings, which were made on the same balance in the same manner:

No. 7701 *a*	No. 7701 *c*
42.602818	45.345518
42.604108	45.345852
42.603512	45.345597
42.602062	45.346437
42.602947	45.346219

Find the probable error of a single filling and weighing with each of the two bottles, and the relative weights of a single observation in the two cases.

2. Eighteen measures of a horizontal angle were made by means of a large Coast Survey theodolite, as follows, the observations being of equal weight:

13° 31′ 17″.6	13° 31′ 20″.4
21 .5	20 .9
19 .0	23 .5
21 .5	18 .4
26 .2	14 .2
17 .1	21 .0
22 .1	21 .8
20 .1	22 .4
17 .9	17 .6

Find the probable error of a single observation of this series by means of each of the formulas (88), (89) and (90).

Regarding the mean as an observation of weight 18, find the probable error of the mean.

3. Find the probable error of one shot in your own target experiment of Art. 11, Ex. 1.

4. Find the probable error of one observation in the series of measurements which you made upon a line in Art. 11, Ex. 2. Also, find the probable error of the mean.

5. Six separate researches, by different observers, upon the velocity of light gave the following mean results, with their probable errors, in kilometers per second:

$$298000 \pm 1000$$
$$298500 \pm 1000$$
$$299930 \pm 100$$
$$299990 \pm 200$$
$$300100 \pm 1000$$
$$299944 \pm 50$$

Assign the proper relative weights and find the probable error of an observation of unit weight.

Also, regarding the weighted mean as an observation of weight Σw, find its probable error.

Explain why the answer to the first part of the problem is not 1000, supposing the first observation to be assigned unit weight. From the answer to the second part, do the less precise observations add to the value of the whole? Give reason for your conclusion.

6. The constant of a Babinet compensator is determined by measuring the distance between two successive dark bands as seen through the analyzer. Micrometer readings were taken as follows:

1st Band	2d Band	1st Band	2d Band
267	225	267	225
269	224	265	227
268	226	268	223
267	227	267	227
264	226	264	226
266	226	266	225
266	227	264	227
268	225	267	226
268	224	266	224
264	225	267	226

Find the probable error of one measurement of the difference in readings; of the mean.

7. Ten measurements were made upon the magnitude of a certain bright star, with the following results:

0.600	0.470
.460	.483
.477	.475
.500	.490
.467	.475

Find the probable error of one measurement and of the mean.

8. Syntheses of carbonic acid gas made from different kinds of carbon by Dumas and Stas gave the following

results (Freund, *Chemical Composition*). The numbers represent the percentage of carbon in the gas.

Natural Graphite	Artificial Graphite	Diamond
27.241	27.237	27.251
.268	.253	.276
.270	.281	.301
.258	.307	.263
.248		.275

Find the probable error of one determination and of the mean.

9. In a series of base line measurements made with both steel and invar tapes, the following probable errors were found (U. S. Coast Survey Report, 1907):

Base Line	Steel	Invar
	one part in	one part in
Point Isabel	1 300 000	2 310 000
Willamette	1 730 000	3 340 000
Tacoma	1 630 000	2 980 000
Stephen	1 120 000	2 040 000
Brown Valley	1 420 000	3 110 000
Royalton	2 260 000	2 460 000

Averaging these, find the relative weights of base line measurements made with these two tapes.

10. Apply Peters' formula (95) to find the probable error of an observation of unit weight for the data of Ex. 14, Art. 49.

61. Probable Errors of Functions of Observed Quantities. — An important phase of the subject of precision is what may be termed the "propagation of error" and illustrated by an example: The probable error of the diameter of a circle, obtained by measurement, is ϵ; what is the probable error E of the area calculated therefrom? Or generally, given the probable error of a measured value of a quantity, to find the corresponding probable error of any function of that quantity.

Let the measured quantity be q, and the function

$$Q = f(q).$$

Let an observation be made upon q with error x, and let the corresponding error affecting the function Q, as a result of this, be X. Then if x be small, we have approximately

$$X : x = dQ : dq,$$

or

$$X = \frac{dQ}{dq} x.$$

It may now readily be seen that if ϵ and E are the probable errors of the measured q and of Q, respectively, then

$$E = \frac{dQ}{dq} \epsilon. \tag{97}$$

This may, however, be shown as follows.

If $x_1, x_2, \cdots, x_n$ are a series of errors committed in measurements upon q, and $X_1, X_2, \cdots, X_n$ are the resulting errors in Q, then as above,

$$X_1 = \frac{dQ}{dq} x_1,$$

$$X_2 = \frac{dQ}{dq} x_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_n = \frac{dQ}{dq} x_n;$$

or squaring and adding,

$$\Sigma X^2 = \left(\frac{dQ}{dq}\right)^2 \Sigma x^2. \tag{98}$$

Now from (87), substituting the value of h given in (78), since the x's and the X's are true errors (not residuals), the probable error of q is

$$\epsilon = 0.6745 \sqrt{\frac{\Sigma x^2}{n}},$$

and that of Q,

$$E = 0.6745 \sqrt{\frac{\Sigma X^2}{n}};$$

from which

$$\Sigma x^2 = \frac{\epsilon^2 n}{0.6745^2},$$

$$\Sigma X^2 = \frac{E^2 n}{0.6745^2}.$$

The substitution of these in (98) with subsequent reduction gives (97).

That is, *the probable error of a function of a single measured quantity is equal to the derivative of the function times the probable error of the measured quantity.*

For example, if the measured radius of a circle be $q = 9.67 \pm 0.02$ cm., the computed area is $Q = \pi q^2 =$

293.7663 sq. cm., and its probable error is $E = \pm 2\,\pi q \times 0.02 = \pm 1.215$ sq. cm.

In general, Q is a function of several (l) measured quantities:

$$Q = f(q_1, q_2, \cdots, q_l). \tag{99}$$

Then if $x_1, x_2, \cdots, x_l$ are the errors of the respective values of $q_1, q_2, \cdots, q_l$ simultaneously substituted in (99), the resulting error of Q is given approximately by

$$X = \frac{\partial Q}{\partial q_1} x_1 + \frac{\partial Q}{\partial q_2} x_2 + \cdots + \frac{\partial Q}{\partial q_l} x_l. \tag{100}$$

Let there be a number (n) of series of observations upon the q's, each giving rise to an error X as represented in (100), viz., $X_1, X_2, \cdots, X_n$. Then approximately,

$$\Sigma X^2 = \left(\frac{\partial Q}{\partial q_1}\right)^2 \Sigma x_1^2 + \left(\frac{\partial Q}{\partial q_2}\right)^2 \Sigma x_2^2 + \cdots + \left(\frac{\partial Q}{\partial q_l}\right)^2 \Sigma x_l^2, \tag{101}$$

which is obtained from the X's upon squaring (100) and omitting the product terms of the expansion. This omission is justified by the fact that there will be in the long run as many + products as −, and they will be distributed approximately in accordance with the error law, and will hence practically cancel each other; whereas, the square terms are all +, and must be retained.

By the same reasoning as that employed in the simpler case following (98), we now readily obtain

$$E = \sqrt{\left(\frac{\partial Q}{\partial q_1}\right)^2 \epsilon_1^2 + \left(\frac{\partial Q}{\partial q_2}\right)^2 \epsilon_2^2 + \cdots + \left(\frac{\partial Q}{\partial q_l}\right)^2 \epsilon_l^2}, \tag{102}$$

of which (97) may be regarded as a special case. The quantities $\epsilon_1, \epsilon_2, \cdots, \epsilon_l$ are the probable errors of measured values of $q_1, q_2, \cdots, q_l$, and E the probable error of Q resulting from substituting these values in the function (99).

As special cases of importance, we may take the following:

(*a*) If $$Q = K_1q_1 + K_2q_2 + \cdots + K_lq_l,$$

then $$E = \sqrt{K_1^2\epsilon_1^2 + K_2^2\epsilon_2^2 + \cdots + K_l^2\epsilon_l^2}. \quad (103)$$

(*b*) If $$Q = Kq_1^a q_2^b \cdots q_l^r,$$

then $$E = \sqrt{\left(\frac{aQ}{q_1}\right)^2 \epsilon_1^2 + \left(\frac{bQ}{q_2}\right)^2 \epsilon_2^2 + \cdots + \left(\frac{rQ}{q_l}\right)^2 \epsilon_l^2}. \quad (104)$$

Let the student deduce these results.

62. Probable Errors of Adjusted Values. — The discussions of the probable error heretofore have been confined to the results of single measurements. The values finally taken as the most probable, for the unknown quantities, from a series of measurements may, however, be more trustworthy than that of any single measurement, and the manner of their calculation from the observations enables us, by applying the laws developed in the preceding article, to calculate the probable errors of these adjusted values, regarding them as functions of the observations.

As the simplest illustration, we shall first take the case of direct observations of equal weight upon a single quantity q. The observation equations are

$$q = s_1,$$
$$q = s_2,$$
$$\cdot \quad \cdot \quad \cdot$$
$$q = s_n,$$

there being n observations. The most probable value m may be given in the form

$$m = \frac{1}{n}s_1 + \frac{1}{n}s_2 + \cdots + \frac{1}{n}s_n.$$

This is a function of the form (a) of the preceding article, and the probable error is given by (103). Each observation s has the same probable error, designated by (88),

$$\epsilon = 0.6745\sqrt{\frac{\Sigma\rho^2}{n-1}}.$$

Then by (103) the probable error of the arithmetical mean m is

$$\epsilon_m = \sqrt{\frac{\epsilon^2}{n^2} + \frac{\epsilon^2}{n^2} + \cdots + \frac{\epsilon^2}{n^2}} = \frac{\epsilon}{\sqrt{n}} = 0.6745\sqrt{\frac{\Sigma\rho^2}{n(n-1)}}, \quad (105)$$

which is the formula ordinarily applied. It may be obtained at once from (88) and (92) by regarding the arithmetical mean of n observations of unit weight as an observation of weight n, as suggested in certain of the problems of Art. 60.

By a similar course of reasoning, if there are n observations upon a single quantity having weights $w_1, w_2, \cdots, w_n$ assigned, the probable error of the weighted mean (Art. 48) is

$$\epsilon_{mw} = 0.6745\sqrt{\frac{\Sigma(w\rho^2)}{(n-1)\Sigma w}}. \quad (106)$$

The general case, in which there are n observations of different weight upon functions of l unknowns $q_1, q_2, \cdots, q_l$, is somewhat more complicated.* We shall deal only

* See article by the author, *Popular Astronomy*, Vol. XIX, p 239.

with the usual problem of first-degree observation equations, represented by (38), Art. 34. The residuals are then given by (39), from which their numerical values must first be calculated. The probable error of an observation of unit weight is now calculated from these residuals in the usual manner, or by another formula

$$\epsilon = 0.6745\sqrt{\frac{\Sigma(w\rho^2)}{n-l}}, \tag{107}$$

which is commonly taken as being more satisfactory, and which certainly differs little from (94) when n is, as it should be, large compared with l. (94) may be regarded as a special case of (107), in which $l = 1$.

In many kinds of work, the probable error of an observation of unit weight is known to the observer through long experience with his instruments, and need not be calculated with reference to each series adjusted. At any rate, we shall suppose the probable errors of $s_1, s_2, \cdots, s_n$ in equations (38) to be known, and designate them by $e_1, e_2, \cdots, e_n$. It is now required to find the probable errors of $m_1, m_2, \cdots, m_l$, which may be designated by $\epsilon_1, \epsilon_2, \cdots, \epsilon_l$.

In the process of adjustment of such a set of observations as here referred to, there arises a set of l most probable or *normal* equations, which, from the mode of arriving at them (Arts. 34 and 48), may be symbolized as follows:

$$\Sigma(aws) - (A_1m_1 + B_1m_2 + \cdots + R_1m_l) = 0,$$

etc., or more conveniently,

$$\left.\begin{aligned}\Sigma(aws) &= A_1m_1 + B_1m_2 + \cdots + R_1m_l,\\ \Sigma(bws) &= A_2m_1 + B_2m_2 + \cdots + R_2m_l,\\ &\cdots\cdots\cdots\cdots\\ \Sigma(rws) &= A_lm_1 + B_lm_2 + \cdots + R_lm_l.\end{aligned}\right\} \quad (108)$$

If we represent by $E_1, E_2, \cdots, E_l$ the probable errors of the members of these respective normal equations, then these E's may each be expressed in two ways. In the first place, since the first member of the first normal equation is

$$\Sigma(aws) = a_1w_1s_1 + a_2w_2s_2 + \cdots + a_nw_ns_n,$$

its probable error E_1, as a function of the s's, is given by

$$E_1^2 = a_1^2w_1^2e_1^2 + a_2^2w_2^2e_2^2 + \cdots + a_n^2w_n^2e_n^2 = \Sigma(a^2w^2e^2), \quad (109)$$

with similar relations for the other normal equations. Again, the probable error E_1 of the second member of the first normal equation (108), as a function of the m's, is given by

$$E_1^2 = A_1^2\epsilon_1^2 + B_1^2\epsilon_2^2 + \cdots + R_1^2\epsilon_l^2, \quad (110)$$

with similar relations for $E_2, E_3, \cdots, E_l$. Then equating (109) and (110), and the other similar pairs, we obtain the system

$$\left.\begin{aligned}A_1^2\epsilon_1^2 + B_1^2\epsilon_2^2 + \cdots + R_1^2\epsilon_l^2 &= \Sigma(a^2w^2e^2),\\ A_2^2\epsilon_1^2 + B_2^2\epsilon_2^2 + \cdots + R_2^2\epsilon_l^2 &= \Sigma(b^2w^2e^2),\\ &\cdots\cdots\cdots\cdots\\ A_l^2\epsilon_1^2 + B_l^2\epsilon_2^2 + \cdots + R_l^2\epsilon_l^2 &= \Sigma(r^2w^2e^2).\end{aligned}\right\} \quad (111)$$

These equations are of the first degree in $\epsilon_1^2, \epsilon_2^2, \cdots, \epsilon_l^2$ and may be readily solved for these values, the required probable errors themselves being then obtained by extracting the square roots.

The actual process is often not as complicated as might be supposed, especially when the number of unknowns is not large. The coefficients $A, B, \cdots, R$ appearing in (111) are already known from the normal equations. The weights w are simple numbers, frequently all unity. The work is greatly facilitated by the use of tables of squares and square roots, and the slide rule. And it may finally be remarked that, since no great precision is required in determining probable errors, superfluous decimal places may be dispensed with in the several stages of their calculation.

Another method of calculating the probable errors of adjusted values from observation equations of the first degree is given, without proof, in the Appendix, Note E.

63. Probable Errors of Conditioned Observations. — The laws developed in Art. 61 make possible the extension of the foregoing processes to the case of observations upon quantities limited by rigorous conditions (Art. 40). It will be remembered that the m equations of condition are first used to obtain the value of m of the unknowns in terms of the others, those values being substituted for them in the observation equations before adjustment. The probable errors of the quantities still involved in the observation equations may now be found as explained in the preceding article. This being done, the probable errors of the m eliminated quantities may be found as functions of the others, by means of (102).

EXERCISES

64. 1. (The formula for double weighing on a balance is $W = p + \frac{r_1 - r_2}{2s}$, in which p is the sum of the weights used, r_1 and r_2 are the pointer readings when object is on *left* and *right* pans, respectively, and s is the sensibility of the balance.) For ten weighings of the same object, the values of $r_1 - r_2$ were as follows:

0.96	0.93
1.08	0.95
0.99	1.12
1.02	1.05
0.92	1.10

The factor $\frac{1}{2s}$ for the load used was 0.0002753. Find the probable error of one weighing, and of the mean of the ten weighings. (Does p need to be given for this purpose?)

2. Given, the probable errors of two measured quantities, to find the probable error of their calculated sum or difference.

3. All of the weighings, the data for which are given in Ex. 1, Art. 60, were made on the same balance and by the same method as those giving rise to the data of Ex. 1 of this article. The latter data refer to ten weighings of the empty bottle No. 7701 *a*, for which $p = 17.423$ g. The former data, referring to bottle No. 7701 *a*, are for fillings with pure water at 21° C., at which temperature the specific volume of water is 1.001957 cc. per gram. Find the most probable capacity of the bottle at this temperature, and the probable error of the result.

Find the probable error of a single determination of the capacity, the above data being regarded as five determinations.

4. Given, the probable errors of the measured legs of a right triangle, to find the probable error of the calculated hypothenuse.

5. Given, the probable error of a measured angle, to find the probable error of its sine, derived therefrom; of its tangent.

6. Find the probable errors of the constants V_0 and K calculated from the data of illustration 2, Art. 36 (p. 80).

7. Find the probable errors of the constants a, b, c calculated from the data of illustration 1, Art. 44 (p. 107).

8. (From J. P. Bartlett, *Least Squares*.) The following weighted observations were made upon the differences of longitude of four American observatories:

Cambridge — Washington	23 m. 41.041 s.,	wt. 30
Cambridge — Cleveland	42 m. 14.875 s.,	wt. 7
Cambridge — Columbus	47 m. 27.713 s.,	wt. 8
Washington — Columbus	23 m. 46.816 s.,	wt. 7
Cleveland — Columbus	5 m. 12.929 s.,	wt. 5

The longitude of Washington being taken as 5 h. 8 m. 15.78 s., find the most probable longitude of each of the other stations, with their respective probable errors.

9. The probable error of a single observation upon an angle with a surveyor's transit is known to be $\pm 1' 4''$. If the angle A of a triangle is measured three times, B five times and C six times with this instrument, calculate the probable errors of the most probable values of A, B, C, obtained by adjusting these measurements.

10. Adapt Peters' formulas to the probable errors of adjusted values, (105), (106); also to (107).

11. If the current in a galvanometer corresponding to deflection δ is $c = K \tan \delta$, find the probable error of a current determined from a deflection reading whose probable error is $4'$.

12. The following table occurs in a certain Coast Survey report, referring to the probable errors of the various sections of a base line, in millimeters.

Sec.	P. E. Due to Uncertainty in Lengths of Tapes	Due to Uncertainty in Coefficients of Exp.	Due to Accidental Errors of Meas.	Combined P. E. of Each Section
1	±0.21	±0.05	±1.75	±
2	.21	.06	0.71	
3	.21	.01	0.54	
4	.21	.02	0.24	
5	.22	.09	1.72	
6	.22	.09	0.54	
7	.22	.04	2.06	
8	.22	.13	2.53	
9	.21	.15	0.30	
10	.21	.11	1.08	
11	.21	.09	1.75	
12	.21	.08	0.20	
13	.21	.12	1.92	
14	.03	.02	0.07	

Fill out the last column and obtain the probable error of the whole base line.

13. The most probable length of the Stanton Base is 13191.3417 ± 0.0052 meters. Find the most probable value and probable error of its logarithm. (Omit unless at least a seven-place table is available.)

14. The weights of three measured angles, *BAC*, *CAD*, *DAE*, are 2, 1, 5, respectively. Find the corresponding weight of the angle *BAE* obtained by adding the measurements.

15. The probable error of a circle reading on a transit is 0′.2, and of a pointing at a signal, 0′.1. What is the probable error of a single differential angle measurement?

16. The probable error of a setting on a mark being ϵ_1, and of a circle reading, ϵ_2, find the probable error of an angle measurement by the cumulative method, using n turns of the circle.

17. The probable error of a scale reading on a cathetometer is 0.07 mm.; of a setting of the telescope on a mark, 0′.1; and of an adjustment of the level, 0′.07. Find the probable error of the mean of ten readings on a mark 2 meters from the instrument. If you had to use a cathetometer, would you analyze the probable error in this way? How would you find it?

18. Following are the results of three series of measurements on the combining weight of lithium, made by different chemists (Freund, *Chemical Composition*):

Diehl	59.417 ± 0.0060
Troost . . .	59.456 ± 0.0200
Dittmar . . .	59.638 ± 0.0173

Weight these results and obtain the weighted mean and its probable error.

19. Five independent series of determinations of the atomic weight of silver gave the following results (Freund, *Chemical Composition*):

107.9401 ± 0.0058
107.9406 ± 0.0049
107.9233 ± 0.0140
107.9371 ± 0.0045
107.9270 ± 0.0090

Assign weights and obtain the weighted mean and its probable error.

What would be the effect of a persistent error entering one of such a series? Might the probable error of a weighted mean ever be greater than that of any one of the observations entering into it?

20. Apply the appropriate Peters' formula to finding the probable error of the weighted mean of the observations of Ex. 14, Art. 49. (See Ex. 10, Art. 60.)

21. The probable error of the mean of fifty observations is found to be 0.1 per cent. How many more observations would be necessary to reduce it to 0.01 per cent.?

APPENDIX

SUPPLEMENTARY NOTES

A. **Proof of the Necessary Functional Relation Assumed in Deriving the Error Law. (Supplementary to Art. 27.)** — To deduce the form of the function ϕ, such that any set of values of $x_1, x_2, \ldots, x_n$ that will render

$$X = x_1 + x_2 + \cdots + x_n = 0 \tag{a}$$

will simultaneously render

$$\Phi = \phi(x_1) + \phi(x_2) + \cdots + \phi(x_n) = 0. \tag{b}$$

Let us add a small finite quantity ϵ to *any* one of the x's, say x_r, and subtract it from *any* other, say x_s, making the new values of these quantities $x_r' = x_r + \epsilon$, $x_s' = x_s - \epsilon$. This will not alter the condition $X = 0$, and hence will not alter the condition $\Phi = 0$, since, by the hypothesis, these conditions are to be simultaneous. This necessitates that

$$\phi(x_r) + \phi(x_s) = \phi(x_r + \epsilon) + \phi(x_s - \epsilon),$$

or that

$$[\phi(x_r + \epsilon) - \phi(x_r)] + [\phi(x_s - \epsilon) - \phi(x_s)] = 0,$$

whatever the values of x_r, x_s and ϵ.

Dividing through by ϵ, this may be written

$$\frac{\phi(x_r + \epsilon) - \phi(x_r)}{\epsilon} - \frac{\phi(x_s - \epsilon) - \phi(x_s)}{-\epsilon} = 0.$$

Allowing ϵ to approach zero, this becomes at the limit

$$\frac{d}{dx_r}\phi(x_r) - \frac{d}{dx_s}(\phi x_s) = 0;$$

or since x_r and x_s represent *any two* of the x's, in general,

$$\frac{d}{dx_1}\phi(x_1) = \frac{d}{dx_2}\phi(x_2) = \cdots = \frac{d}{dx_n}\phi(x_n).$$

It follows at once that, since the x's may be varied in any manner among themselves, only so condition (a) holds, $\frac{d}{dx}\phi(x)$ is a constant, say K. Therefore, integrating,

$$\phi(x) = Kx + c.$$

That $c = 0$ follows from (a) and (b) jointly, since substitution in (b) gives

$$\Phi = K(x_1 + x_2 + \cdots + x_n) + nc = 0,$$

the first term of which vanishes by (a).

Hence, necessarily, $\phi(x) = Kx$,
which is Eq. (22), Art. 27.

***B*. Approximation Method for Observation Equations Not of the First Degree. (Supplementary to Art. 39.)** — This method requires that the values of the unknowns be very approximately known beforehand, as by choosing such of the observation equations as, when solved simultaneously, will yield values for all of them. Attention is then given to the unknown *small corrections* that must be applied to these approximate values; a procedure somewhat resembling Horner's method of approximation for algebraic equations.

The approximate values, however obtained, being designated by $a_1, a_2, \cdots, a_l$, and the corrections required by $q'_1, q'_2, \cdots, q'_l$, the true values of the unknowns are

$$\left.\begin{aligned} q_1 &= a_1 + q'_1, \\ q_2 &= a_2 + q'_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ q_l &= a_l + q'_l. \end{aligned}\right\} \qquad (c)$$

Let the non-linear observation equations to be dealt with be typified by

$$f(q_1, q_2, \cdots, q_l) = s. \qquad (d)$$

The substitution of the values (*c*) in (*d*) gives

$$f(a_1 + q'_1, a_2 + q'_2, \cdots, a_l + q'_l) = s, \qquad (e)$$

an observation equation in which the unknowns are the *small* corrections $q'_1, q'_2, \cdots, q'_l$. Expanding the first member of this by the general Taylor's theorem,

$$f(a_1 + q'_1, a_2 + q'_2, \cdots, a_l + q'_l) = f(a_1, a_2, \cdots, a_l)$$

$$+ q'_1 \frac{\partial}{\partial a_1} f(a_1, a_2, \cdots, a_l) + q'_2 \frac{\partial}{\partial a_2} f(a_1, a_2, \cdots, a_l) + \cdots$$

$$+ q'_l \frac{\partial}{\partial a_l} f(a_1, a_2, \cdots, a_l) + R,$$

R being the remainder of the series, which involves higher powers of the very small corrections q'_1, etc., and which *may therefore be neglected without serious inaccuracy.*

Denoting $f(a_1, a_2, \cdots, a_l)$ by F,

$$\frac{\partial F}{\partial a_1} \quad \text{by } a,$$

$$\frac{\partial F}{\partial a_2} \quad \text{by } b,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial F}{\partial a_l} \quad \text{by } r,$$

we may therefore replace (e) by

$$F + aq'_1 + bq'_2 + \cdots + rq'_l = s,$$

or

$$aq'_1 + bq'_2 + \cdots + rq'_l = s', \qquad (f)$$

in which s' denotes $s - F$.

This is an observation equation of the first degree, and may be used as such, in combination with other observations similarly obtained from their respective originals, for finding the most probable values of the corrections. This being done, the most probable values of the unknown quantities themselves are found by adding the most probable corrections to the approximate values a.

C. Evaluation of the Integral $\int_0^\infty e^{-h^2x^2}dx$. (Supplementary to Art. 54.) — Equation (70) is

$$I = \int_0^\infty e^{-h^2x^2}dx.$$

This may be transformed into

$$I = \frac{1}{h}\int_0^\infty e^{-h^2x^2}hdx = \frac{1}{h}I'. \qquad (g)$$

The new integral I' is independent of h. For, let $hx = z$, then $hdx = dz$, and

$$I' = \int_0^\infty e^{-z^2}dz = \int_0^\infty e^{-x^2}dx. \qquad (h)$$

Returning, however, to the original form of I' in (g), multiply it by $e^{-h^2}dh$:

$$I'e^{-h^2}dh = \int_0^\infty [e^{-(1+x^2)h^2} \cdot hdh]dx.$$

I' and x being independent of h, we may integrate both members of this equation *with respect to h as a parameter*, assigning the limits 0 and ∞ to this integration also, thus:

$$I'\int_0^\infty e^{-h^2}dh = \int_0^\infty \left[\int_0^\infty e^{-(1+x^2)h^2} \cdot hdh\right]dx. \qquad (i)$$

Now the integral within the brackets is readily determined:

$$\int_0^\infty e^{-(1+x^2)h^2} \cdot hdh = \frac{1}{2(1+x^2)}\int_0^\infty e^{-(1+x^2)h^2} \cdot 2(1+x^2)hdh$$

$$= \frac{1}{2(1+x^2)}$$

Substituting this in (i) gives

$$I'\int_0^\infty e^{-h^2}dh = \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

But by Eq. (h) the integral $\int_0^\infty e^{-h^2}dh$ in the first member is equal to I'. Hence $I'^2 = \frac{\pi}{4}$, and from (g)

$$I = \frac{\sqrt{\pi}}{2\,h},$$

which is equation (72).

***D.* Evaluation of the Probability Integral. (Supplementary to Art. 55.)** — The value of the integral

$$Y = \frac{1}{\sqrt{\pi}}\int_0^{hX} e^{-x^2}dx \tag{76}$$

may be found when $hX < 1$ by developing e^{-x^2} into a series and integrating the terms separately. By Maclaurin's theorem,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots,$$

whence

$$\int_0^{hX} e^{-x^2}dx = hX - \frac{(hX)^3}{3} + \frac{1}{2!}\frac{(hX)^5}{5} - \frac{1}{3!}\frac{(hX)^7}{7} + \cdots. \tag{j}$$

This series converges rapidly for values of hX less than or equal to unity, and may therefore be employed in the calculation of Y in this case. When $hX > 1$, however, it is divergent. We may write

$$\int e^{-x^2}dx = \int\left[-\frac{1}{2x}\right][-2xe^{-x^2}dx] = \int\left[-\frac{1}{2x}\right]d(e^{-x^2}).$$

Integrating successively by parts,

$$\int e^{-x^2}dx = -\frac{1}{2x}e^{-x^2} - \frac{1}{2}\int\frac{e^{-x^2}}{x^2}dx$$

$$= -\frac{e^{-x^2}}{2x} + \frac{e^{-x^2}}{4x^3} + \frac{3}{4}\int\frac{e^{-x^2}}{x^4}dx$$

$$= -\frac{e^{-x^2}}{2x} + \frac{e^{-x^2}}{4x^3} - \frac{3e^{-x^2}}{8x^5} - \frac{3\cdot 5}{8}\int\frac{e^{-x^2}}{x^6}dx = \cdots$$

$$= e^{-x^2}\left[-\frac{1}{2\,x}+\frac{1}{4\,x^3}-\frac{3}{8\,x^5}+\frac{3\cdot 5}{16\,x^7}\right.$$

$$\left.-\frac{3\cdot 5\cdot 7}{32\,x^9}+\frac{3\cdot 5\cdot 7\cdot 9}{64\,x^{11}}-\cdots\right]. \qquad (k)$$

Now $$\int_0^{hX} e^{-x^2}dx = \int_0^{\infty} e^{-x^2}dx - \int_{hX}^{\infty} e^{-x^2}dx$$

$$= \frac{\sqrt{\pi}}{2} - \int_{hX}^{\infty} e^{-x^2}dx. \qquad (l)$$

(See Note C.) The value of the integral in this last expression can be found by applying the limits hX and ∞ to the successive terms of (k), giving

$$\int_{hX}^{\infty} e^{-x^2}dx = e^{-(hX)^2}\left[\frac{1}{2\,hX}-\frac{1}{4\,(hX)^3}\right.$$

$$\left.+\frac{3}{8}\frac{1}{(hX)^5}-\frac{15}{16}\frac{1}{(hX)^7}+\cdots\right], \qquad (m)$$

which converges rapidly when $hX > 1$. Equation (l) will now give values for the integral appearing in Y for this case.

Therefore, for $hX \leqq 1$, use series (j); for $hX > 1$, use series (m) substituted in (l). (76) will then yield the values of the probability integral desired. Let the student verify these calculations for, say, $hX = \frac{1}{2}$ and $hX = 2$.

E. Outline of Another Method for Probable Errors of Adjusted Values. (Supplementary to Art. 62.) — The method referred to at the end of Art. 62 is given here without proof. (See Merriman, *Method of Least Squares*, Art. 74.)

Let the normal equations found from the observation equations be

$$A_1m_1 + B_1m_2 + \cdots + R_1m_l = K_1,$$
$$A_2m_1 + B_2m_2 + \cdots + R_2m_l = K_2,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$A_lm_1 + B_lm_2 + \cdots + R_lm_l = K_l,$$

in which the quantities K take the place of $\Sigma(aws)$, etc., in equations (108). Let the *literal* form of these quantities K be preserved throughout the solution of the normal equations, which will then yield

$$\left.\begin{aligned} m_1 &= \alpha_1K_1 + \beta_1K_2 + \cdots + \lambda_1K_l, \\ m_2 &= \alpha_2K_1 + \beta_2K_2 + \cdots + \lambda_2K_l, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ m_l &= \alpha_lK_1 + \beta_lK_2 + \cdots + \lambda_lK_l, \end{aligned}\right\} \qquad (n)$$

the quantities α, β, $\cdots$, λ being numerical coefficients of the literal quantities K.

It may then be shown that the weight of m_1 is $\frac{1}{\alpha_1}$, that of m_2 is $\frac{1}{\beta_2}$, that of m_3 is $\frac{1}{\gamma_3}$, etc. That is, the weight of any most probable value m_p is the reciprocal of the coefficient of the absolute term K_p of the normal equation corresponding to m_p as K_p appears in the solution (n) for m_p.

The weights of all the most probable values m_1, m_2, $\cdots$, m_l are thus calculated. The probable error of an observation of unit weight is given by (107), or, as remarked in Art. 62, may be already known from experience. The probable error of each m may therefore now be found by dividing this standard probable error by the square root of the weight of m (92), as above determined.

F. Collection of Important Definitions, Theorems, Rules and Formulas for Convenient Reference.

DEFINITIONS

Error. — The result of a measurement minus the true value of the quantity measured. (Art. 7.)

Residual. — The result of a measurement minus the most probable value of the quantity, as derived from a series of measurements. (Art. 7.)

Most Probable Value. — A calculated value of an unknown quantity, based upon the results of measurements, such that the residuals arising therefrom will be most nearly in accord with the normal error distribution. (Arts. 7, 29.)

Adjustment. — The process of obtaining from the results of measurements the most probable values of the unknown quantities sought. (Chap. V.)

Observation Equation. — An equation, in general only approximately true, connecting one or more unknown quantities, or functions of them, with the result of a measurement. (Art. 31.)

Normal Equation. — An equation, in general one of a set of simultaneous equations, whose solution gives the most probable values of the unknowns involved in the observation equations. (Art. 33.)

Equation of Condition. — An equation expressing a theoretical condition which must be exactly satisfied by the calculated most probable values of the unknowns. (Art. 40.)

Empirical Formula. — A formula expressing a relation between variables, whose mathematical form is inferred from the results of experience or experiment, and which is not deduced theoretically. (Art. 42.)

Weights. — Numbers assigned to observations, or to the adjusted values of unknowns, representing the relative degrees of confidence which the respective observations or values are supposed to merit. (Art. 47.)

Weighted Mean. — The most probable value of a single unknown quantity obtained by multiplying each observation upon that quantity by its weight, adding the products, and dividing by the sum of the weights. (Art. 48.)

Probable Error. — A theoretical quantity ϵ, so related to the precision of a system of observations, that the probability of the error of any observation or adjusted value being numerically less than ϵ is equal to the probability of its being numerically greater. (Art. 58.)

RULES AND THEOREMS

1. *Principle of Least Squares.* — (*a*) The most probable value of a measured quantity that can be deduced from a series of direct observations, made with equal care and skill, is that for which the sum of the squares of the residuals is a minimum. (Art. 29.)

(*b*) The most probable value of an unknown quantity that can be deduced from a set of observations upon one of its functions is that for which the sum of the squares of the residuals is a minimum. (Art. 31.)

(*c*) The most probable values of unknown quantities connected by observation equations are those for which the sum of the squares of the residuals of those equations is a minimum. (Art. 33.)

(*d*) The most probable values of unknown quantities connected by weighted observation equations are those for which the sum of the weighted squares of the residuals is a minimum. (Art. 52.)

2. *Rules for Adjusting Observation Equations of the First Degree.* — (*a*) Write the expression for the residual corresponding to each observation equation, multiply it by the coefficient of the first unknown, in that expression, add the products, and equate their sum to zero. The result is the normal equation pertaining to the said first unknown. Do likewise for each of the other unknowns. Then solve the normal equations thus formed for the desired most probable values of the unknowns. (Art. 34.)

(*b*) In the case of weighted observation equations, after multiplying the residual by the coefficient of the unknown, multiply again by the weight of the corresponding observation; then add and proceed as above stated. (Art. 48.)

3. *Weight and Precision Index.* — The weights of observations are directly proportional to the squares of their precision indices. (Art. 51.)

4. *Weight and Probable Error.* — The weights of observations are inversely proportional to the squares of their probable errors. (Art. 59.)

FORMULAS

1. *The Error Equation.* (Art. 54.)

$$y = \frac{h\Delta}{\sqrt{\pi}} e^{-h^2x^2}. \tag{74}$$

2. *Formulas for the Precision Index.* (Arts. 57, 59.)

(*a*) For observations of equal precision, standard formula,

$$h = \sqrt{\frac{n-1}{2\,\Sigma\rho^2}}. \tag{79}$$

(*b*) For weighted observations, standard formula,

$$h = \sqrt{\frac{n-1}{2\,\Sigma(w\rho^2)}}. \tag{93}$$

(*c*) Peters' formula, disregarding signs of residuals, observations not weighted,

$$h = \frac{\sqrt{n(n-1)}}{\sqrt{\pi}\Sigma\rho}. \tag{82}$$

3. *Formulas for the Probable Errors of Observations in Terms of Residuals.* (Arts. 58, 59.)

(*a*) Probable error of single observation, no weights, standard formula,

$$\epsilon = 0.6745\sqrt{\frac{\Sigma\rho^2}{n-1}}. \tag{88}$$

(*b*) With weights assigned, probable error of single observation of unit weight, standard,

$$\epsilon = 0.6745\sqrt{\frac{\Sigma(w\rho^2)}{n-1}}. \tag{94}$$

(*c*) For an observation of unit weight, there being l unknown quantities, standard,

$$\epsilon = 0.6745\sqrt{\frac{\Sigma(w\rho^2)}{n-l}}. \tag{107}$$

(*d*) Peters' formulas corresponding to the above (*a*), (*b*) and (*c*), disregarding signs of residuals,

$$\epsilon = 0.8453\frac{\Sigma\rho}{\sqrt{n(n-1)}}. \tag{89}$$

$$\epsilon = 0.8453\frac{\Sigma(\sqrt{w}\rho)}{\sqrt{n(n-1)}}. \tag{95}$$

$$\epsilon = 0.8453\frac{\Sigma(\sqrt{w}\rho)}{\sqrt{n(n-l)}}.$$

(*e*) Simplified Peters' formulas corresponding to the above (*a*) and (*b*), adapted to approximate calculation when n is large, disregarding signs of residuals,

$$\epsilon = 0.85\frac{\Sigma\rho}{n}. \tag{90}$$

$$\epsilon = 0.85\frac{\Sigma(\sqrt{w}\rho)}{n}. \tag{96}$$

4. *Formulas for Probable Errors of Functions of Quantities, in Terms of Probable Errors of Quantities Themselves.* (Art. 61.)

(*a*) Function Q of a single quantity q,

$$E = \frac{dQ}{dq}\epsilon. \tag{97}$$

(*b*) Function Q of several quantities, $q_1, q_2, \cdots, q_l$,

$$E = \sqrt{\left(\frac{\partial Q}{\partial q_1}\right)^2 \epsilon_1{}^2 + \left(\frac{\partial Q}{\partial q_2}\right)^2 \epsilon_2{}^2 + \cdots + \left(\frac{\partial Q}{\partial q_l}\right)^2 \epsilon_l{}^2}. \quad (102)$$

(*c*) Function $Q = K_1 q_1 + K_2 q_2 + \cdots + K_l q_l$,

$$E = \sqrt{K_1{}^2 \epsilon_1{}^2 + K_2{}^2 \epsilon_2{}^2 + \cdots + K_l{}^2 \epsilon_l{}^2}. \quad (103)$$

(*d*) Function $Q = K q_1{}^a q_2{}^b \cdots q_l{}^r$,

$$E = \sqrt{\left(\frac{aQ}{q_1}\right)^2 \epsilon_1{}^2 + \left(\frac{bQ}{q_2}\right)^2 \epsilon_2{}^2 + \cdots + \left(\frac{rQ}{q_l}\right)^2 \epsilon_l{}^2}. \quad (104)$$

5. *Formulas for Probable Errors of Adjusted Values.* (Art. 62.)

(*a*) For the arithmetical mean, standard,

$$\epsilon_m = 0.6745 \sqrt{\frac{\Sigma \rho^2}{n(n-1)}}. \quad (105)$$

(*b*) For the weighted mean, standard,

$$\epsilon_{mw} = 0.6745 \sqrt{\frac{\Sigma (w\rho^2)}{(n-1)\Sigma w}}. \quad (106)$$

(*c*) Peters' formulas corresponding to the above (*a*) and (*b*) (Ex. 10, Art. 64),

$$\epsilon_m = 0.8453 \frac{\Sigma \rho}{n\sqrt{n-1}}.$$

$$\epsilon_{mw} = 0.8453 \frac{\Sigma (\sqrt{w}\rho)}{\sqrt{n(n-1)\Sigma w}}.$$

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