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THE THEORY OF PERMUTABLE FUNCTIONS

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LECTURE I

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1. We shall begin with quite elementary and general notions.

First, let us recall the properties of a sum

$$a_1 + a_2 + \quad . \; . \; . \quad + a_n = \sum_{1}^{n} {}_i \, a_i \, .$$

This operation is both associative and commutative, that is,

$$(a+b) + c = a + (b+c)$$

and

$$a + b = b + a \, .$$

Now we can pass from a sum to an integral by a well-known limiting process. For the sake of simplicity, we shall make use of the definition of Riemann: Given a function $f(x)$ which is defined over an interval ab, we subdivide the interval ab into n parts h_1, h_2, h_3, h_4 . . . . . . h_n. Corresponding to every interval h_i we then take some value f_i of $f(x)$ lying between the upper and lower limits of $f(x)$ on h_i, and we form the sum

$$\sum_{1}^{n} {}_i \, f_i \, h_i \, .$$

Now suppose we allow $h_1, h_2, h_3, \ldots\ldots h_n$ to become indefinitely small. Then if a unique limit is approached by the sum regardless of the way in which the subdivision of *ab* is made, we have

$$lim \sum_1^n {}_i f_i h_i = \int_a^b f(x)\, dx \,.$$

Necessary and sufficient conditions for the existence of this limit are well known. In particular, if the function $f(x)$ is continuous over the interval *ab* or has at most a finite number of discontinuities, the limit and hence also the integral exists.

2. Now let us form the product

$$a \cdot b \cdot c \cdots\cdots\cdots \,.$$

This operation is associative and commutative, that is to say,

$$(ab)\, c = a\, (bc)$$

and

$$ab = ba \,.$$

It is not worth our while to consider the operation which could be obtained from a product by a limiting process such as the one employed in defining an integral. We should be led to logarithmic integration.

3. However, let us consider a limiting process which leads us to something more than these elementary operations.

Let us choose a set of numbers m_{is}, where $i, s = 1, 2, \ldots g$, which may be written in an array

$$\begin{matrix} m_{11} & m_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & m_{1g} \\ m_{21} & m_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & m_{2g} \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ m_{g1} & m_{g2} & \cdot & \cdot & \cdot & \cdot & \cdot & m_{gg} \end{matrix}$$

and numbers n_{is}, where $i, s = 1, 2, \ldots g$, that is,

$$\begin{matrix} n_{11} & n_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & n_{1g} \\ n_{21} & n_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & n_{2g} \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ n_{g1} & n_{g2} & \cdot & \cdot & \cdot & \cdot & \cdot & n_{gg} \,. \end{matrix}$$

We then consider the operation

$$(1) \quad \sum_{1}^{g}{}_{h} \, m_{ih} \, n_{hr}$$

which we shall call *composition of the second type*. This operation is associative, for if we also introduce a set of numbers p_{is}, where $i, s = 1, 2 \quad . \quad . \quad g$, and write the sum

$$\sum_{1}^{g}{}_h \sum_{1}^{g}{}_k \; m_{ih} \; n_{hk} \; p_{ks}$$

the expression which we thus obtain is equivalent to either of the forms

$$\sum_{1}^{g}{}_k \; (\sum_{1}^{g}{}_h \, m_{ih} \, n_{hk}) \; p_{ks}$$

$$\sum_{1}^{g}{}_h \; m_{ih} \; (\sum_{1}^{g}{}_k \, n_{hk} \, p_{ks})$$

which proves that the associative law is satisfied.

Making use of the notation

$$\sum_{1}^{g}{}_h \; m_{ih} \; n_{hr} = (m, n)_{ir}$$

we shall have

$$((m, n) \; p)_{is} = (m \; (n, p))_{is}$$

which may be written without the parenthesis, thus,

$$(m, n, p)_{is} \; .$$

The commutative law will in general not be satisfied. When it is, the quantities under consideration are called permutable, and we have

$$(m, n)_{ir} = (n, m)_{ir} .$$

We can at once give an example involving permutable quantities. All that is necessary is to consider

$$(m, m)_{ir} \text{ which may be written } (m^2)_{ir}$$
$$(m, m, m)_{ir} \text{ which may be written } (m^3)_{ir}$$

and so on. And it is clear that

$$(m^h, m^k)_{ir} = (m^k, m^h)_{ir} ,$$

since the associative law is satisfied.

4. We shall consider also another operation similar to the last, namely

$$(2) \quad \sum_{i+1}^{s-1}{}_h \, m_{ih} \, n_{hs}$$

which will be called *composition of the first type.* The sum (1) previously considered reduces to this one if we suppose that the numbers are zero unless the second subscript is greater than the first. In other words, we have in this case

$$\begin{matrix} 0 & m_{12} & m_{13} & . & . & . & m_{1g} \\ 0 & 0 & m_{23} & . & . & . & m_{2g} \\ & & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & m_{g-1,g} \\ 0 & 0 & 0 & . & . & . & 0 . \end{matrix}$$

Let us represent the sum (2) by

$$[m, n]_{is} \, .$$

This expression vanishes if s is less than or equal to $i + 1$. Moreover, if we write

$$[[m, n]\, p]_{is}$$

we shall have

$$[[m, n]\, p]_{is} = [m[n, p]]_{is} = [m, n, p]_{is}$$

which vanishes if s is less than or equal to $i + 2$, and so on.

In general, it is not true that

$$[m, n]_{is} = [n, m]_{is}$$

but when this condition is satisfied, the two quantities are called permutable. To distinguish this sort of permutability from that which we defined in section 3, we shall say that the new and the old are of types one and two respectively. In other words, if

$$(3) \quad \sum_{1}^{g}{}_h\, m_{ih}\, n_{hg} = \sum_{1}^{g}{}_h\, n_{ih}\, m_{hg}$$

we have permutability of the second type, whereas if

$$\sum_{i+1}^{s-1}{}_h\, m_{ih}\, n_{hs} = \sum_{i+1}^{s-1}{}_h\, n_{ih}\, m_{hs}$$

we have permutability of the first type.

Clearly, if we put

$$[m^h, m^k]_{is} = [m^k, m^h]_{is},$$

we at once obtain an example of permutability of the first type like the one mentioned in the last section. Nevertheless, we shall give another example which is of interest. Let us suppose that $n_{ih} = 1$. Then the condition for permutability becomes

$$\sum_{i+1}^{s-1}{}_h\, m_{ih} = \sum_{i+1}^{s-1}{}_h\, m_{hs} .$$

Putting $s = i + 2$, we have

$$m_{i,\, i+1} = m_{i+1,\, i+2} ,$$

and putting $s = i + 3$, we have

$$m_{i,\, i+1} + m_{i,\, i+2} = m_{i+1,\, i+3} + m_{i+2,\, i+3} ,$$

whence

$$m_{i,\, i+2} = m_{i+1,\, i+3}$$

and so on. Owing to the above, we have

$$m_{r,\, r+g} = m_{1,\, 1+g}$$

for all values of r, s, and g. From this it follows that the matrix of the m's is of the following type:

$$\begin{matrix} 0 & a_1 & a_2 & a_3 & . & . & . & a_{g-1} \\ 0 & 0 & a_1 & a_2 & . & . & . & a_{g-2} \\ 0 & 0 & 0 & a_1 & . & . & . & a_{g-3} \\ & & . & . & . & . & . & \\ & & . & . & . & . & . & \\ 0 & 0 & 0 & 0 & . & . & . & a_1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 . \end{matrix}$$

The law for this matrix may be expressed by

$$m_{is} = m_{s-i} ,$$

which puts into evidence the fact that the value of m_{is} depends upon the difference between the two subscripts s and i.

5. Now what do we find when we pass to the limit by a process analogous to the one employed in the integral calculus in going from a sum to an integral? We there passed from a set of quantities $f_1, f_2, f_3, \ldots\ldots f_n$ with single subscripts to a function $f(x)$ of one independent variable x, the variable taking the place of the subscripts. Here, we have instead a set of quantities m_{is} with double subscripts; hence, we must replace them by a function of two variables

$$f(x, y)$$

where the two variables take the place of the double subscripts. Moreover, we also have another set of quantities n_{is} which must be replaced by a different function of two variables $\phi(x,y)$. Finally, $\Sigma_h m_{ih} n_{hs}$ must be replaced by

$$\int f(x, \xi)\, \phi(\xi, y)\, d\xi .$$

We thus obtain two operations:

Composition of the first type:

$$\int_x^y f(x, \xi)\, \phi(\xi, y)\, d\xi ;$$

Composition of the second type:

$$\int_a^b f(x, \xi)\, \phi(\xi, y)\, d\xi .$$

The condition for permutability of the first type is

$$\int_x^y f(x, \xi)\, \phi(\xi, y)\, d\xi = \int_x^y \phi(x, \xi)\, f(\xi, y)\, d\xi ;$$

for permutability of the second type,

$$\int_a^b f(x, \xi)\, \phi(\xi, y)\, d\xi = \int_a^b \phi(x, \xi)\, f(\xi, y)\, d\xi .$$

The associative property is always satisfied.

6. Let us begin by examining permuta-

bility of the first type. The most important facts are here summarized:

1. All of the functions which can be obtained by the composition of permutable functions are permutable with one another and also with the original functions.

2. All of the functions which can be obtained by the addition or the subtraction of permutable functions are permutable with one another and with the original functions.

Now, let us see how the following problem may be solved: To determine all the functions which are permutable with unity.

We can readily solve this problem if we recall a question which has already been answered. For before passing to the limit, we saw that if the functions m_{is} were permutable with unity, the condition

$$m_{is} = m_{s-i}$$

was satisfied. Now since, in the limit, the subscripts are replaced by the variables x and y, we are led to infer that

$$f(x, y) = f(y-x) .$$

This we can prove immediately. For if

$$\int_x^y f(x,\xi)\,d\xi = \int_x^y f(\xi,y)\,d\xi = \phi(x,y),$$

it must follow that

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \phi}{\partial x} = f(x,y).$$

Hence ϕ and f are of the forms $\phi(y-x)$ and $f(y-x)$ respectively.

Moreover, all of the functions of the type $f(y-x)$ are permutable with one another; for

$$\int_x^y f(y-\xi)\,\phi(\xi-x)\,d\xi = \int_x^y \phi(y-\xi)\,f(\xi-x)\,d\xi$$

as can be verified at once. The functions of type $f(y-x)$ form a group of permutable functions which is of especial interest. We have called it the *group of closed cycle*.* However, we shall not go into an examination of it here.

7. We have used several different notations representing the operation of composition. The simplest scheme where no confusion with multiplication is liable to arise, is merely to write

$$f\phi \quad \text{or} \quad f\phi(x,y)$$

*Leçons sur les equations intégrales et intégro-differéntielles. Paris: Gauthier-Villars. 1913. P. 150.

to represent the resultant of the composition of two function f and ϕ. But in a case where confusion might arise, we may place a small star over the letters, thus

$$\overset{*}{f}\,\overset{*}{\phi}\,.$$

We may also put the letters in square brackets

$$[f, \phi],$$

just as in the above we wrote $[m, n]_{hk}$ to represent the composition of the quantities m_{ih} and m_{ik}.

To indicate the composition of f with itself, the composition of the resultant thus obtained with f, and so on, we shall write

$$\overset{*}{f}{}^2,\ \overset{*}{f}{}^3,\ \ldots\ .$$

respectively, and if no confusion with multiplication is liable to arise, we may even omit the small stars and write

$$f^2,\ f^3,\ \ldots\ .$$

8. The notation in certain cases demands particular examination. Thus, to indicate the product of a constant a by f, we write af; and if b is also a constant and ϕ a function which is

permutable with f, then $b\phi$ is permutable with af and the composition of the two gives us

$$a\,b\,\overset{*}{f}\,\overset{*}{\phi}\,.$$

Moreover, if we have two polynomials made up of permutable functions with constant coefficients, these polynomials will also be premutable with one another, and to effect their composition all that is necessary is to apply the same rule which is used when polynomials are multiplied together.

But if a and b are constants, then $a + f$ and $b + \phi$ will not in general be permutable with one another. Nevertheless, we shall extend the definition so as to have in this case

$$(a+\overset{*}{f})\,(b+\overset{*}{\phi}) = a\,b + a\phi + bf + \overset{*}{f}\overset{*}{\phi}\,.$$

9. Before going further, let us consider what takes place in the case of composition of the second type.

All that we have said above concerning permutability of the first type can be established for permutability of the second type barring the remarks on permutability with unity. With this exception, all of the properties just mentioned may be extended to this case at once.

10. Let us pass in review some of the most interesting properties which can be derived from the operation of composition. We shall return once more to the finite case and consider the operation of composition for the numbers m_{is}. We saw above that if we composed the m's with the n's, the resultant thus obtained with the p's and so on, then after s-i-1 compositions, the resultant would be zero. In a like manner, all of the symbolic powers of m_{is} beginning with $[m^{s-i+1}]_{is}$ have a value zero.

Having seen this, let us consider any analytic function

$$\sum_0^\infty {}_n\, A_n z^n$$

which converges within a certain circle, and let us write

$$\sum_0^\infty {}_n\, A_n [m^n]_{is}\, z^n\,.$$

This new expression is evidently an integral rational function of z, that is, a polynomial.

Moreover, this result may be generalized. Consider the analytic function

$$\sum_0^\infty {}_{n_1} \sum_0^\infty {}_{n_2} \ldots \sum_0^\infty {}_{n_e}\, z_1^{n_1}\, z_2^{n_2} \ldots z_e^{n_e}\, A_{n_1 n_2 \ldots n_e}$$

of more than one variable, and write

$$\sum_0^\infty{}_{n_1} \sum_0^\infty{}_{n_2} \dots \sum_0^\infty{}_{n_e} A_{n_1 n_2 \dots n_e} \left[m^{n_1}, p^{n_2}, \dots r^{n_e} \right] z_1^{n_1} z_2^{n_2} \dots z_e^{n_e} .$$

We again obtain a polynomial.

11. Now, what does the above theorem become when we pass by a limiting process to the composition of functions? This we proceed to investigate.

Consider

$$\sum_0^\infty{}_n A_n \overset{*}{f}{}^n (x, y) z^n .$$

We shall prove that *if* f(x, y) *is finite, this expression is always an entire function of* z, *whatever may be the absolute value of* f(x, y).

To prove this theorem, we notice that

$$|A_n| < \frac{M}{R^n}$$

where M is some finite quantity and where R is less than the radius of convergence of the series. Moreover, let μ be a quantity which is larger than the absolute value of f. Then we shall have

$$|f(x,y)| < \mu, \quad |\overset{*}{f}{}^2(x,y)| < \mu^2 \int_x^y d\xi = \mu^2 (y-x),$$

$$|\overset{*}{f}{}^3(x,y)| < \mu^3 \int_x^y (y-\xi)\, d\xi = \mu^3 \int_0^{y-x} u du$$
$$= \frac{\mu^3 (y-x)^2}{2\,!}$$

$$|\overset{*}{f}{}^4(x,y)| < \frac{\mu^4 (y-x)^3}{3\,!} \quad . \; . \; . \; .$$

and so on, whence

$$|A_n \overset{*}{f}{}^n (x,y) z^n| < \frac{\mu^n (y-x)^{n-1}}{(n-1)\,!} \; \frac{M}{R^n} \; |z^n|,$$

which proves that the series is convergent for all values of z and is thus an entire function of z.

This theorem may also be generalized. Let us consider the series

$$\sum_0^\infty i_1 \; . \; . \; . \; \sum_0^\infty i_e \; A_{i_1} \; . \; . \; . \; {}_{i_e} \; z_1^{i_1} \; . \; . \; . \; z_e^{i_e}$$

which represents the expansion of a function $F(z_1, z_2, z_3, \ldots . z_e)$ about a point, and which converges if the absolute values of z_1, z_2, z_3, $\ldots . z_e$ do not exceed certain limits. Then the series

$$F = \sum_0^\infty i_1 \ldots \sum_0^\infty i_e \; A_{i_1} \ldots {}_{i_e} \overset{*}{f}{}_1^{i_1} \overset{*}{f}{}_2^{i_2} \ldots \overset{*}{f}{}_e^{i_e} \, z_1^{i_1} \ldots z_e^{i_e}$$

is always an entire function of $z_1, z_3, z_2, \ldots z_n$.

The proof is made in the previous case. Thus, if $f_1, f_2, f_3, \ldots f_e$ are each less than μ in absolute value, then

$$|\overset{*}{f}_1^{i_1} \ldots \overset{*}{f}_e^{i_e}| < \frac{\mu^{i_1+ \cdots +i_e} (y-x)^{i_1+ \cdots +i_{e-1}}}{(i_1+i_2+ \ldots +i_{e-1})!}$$

and hence the theorem may be verified immediately. We may also demonstrate another property besides the one just shown. Indeed, we have up to the present regarded the function $F(z_1, \ldots z_e, | x, y)$ as a function of z_1, z_2, z_3, z_4, ... z_e, but it is also a function of x and y. Regarded as a function of these two variables, the function is permutable with the functions f_1 .... f_e. This may be seen at once; for owing to the uniform convergence of the series, the operation of composition may be performed term by term, and since each term of the series is permutable with f_1, f_2, f_3, ... f_e, so also must be the sum.

To sum up, we have the following theorem: *If*

$$\Sigma \Sigma \ldots \Sigma A_{i_1 \ldots i_e} z_1^{i_1} \ldots z_e^{i_e}$$

is the expansion of an analytic function about a point, then

$$F(z_1 \ldots z_e \mid xy) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e} \overset{*}{f_1}{}^{i_1} \ldots \overset{*}{f_e}{}^{i_e} z_1^{i_1} \ldots z_e^{i_e}$$

where f_1, f_2, $f_3, \ldots . f_e$ *are permutable functions, is an entire function of* $z_1, z_2, z_3, \ldots . z_e$, *and as a function of* x *and* y *is permutable with the functions* $f_1, f_2, f_3 \ldots . f_e$.

Now if in $F(z_1, z_2, \ldots . z_e \mid x, y)$ we put $z_1 = z_2 = z_3 = \ldots . z_e = 1$, we obtain a series

$$\Sigma \ldots \Sigma A_{i_1 \ldots i_e} \overset{*}{f_1}{}^{i_1} \ldots \overset{*}{f_e}{}^{i_e}$$

which is convergent for all values of the *f's.*

12. The theorems which we have been deriving above suggest a method for investigating to a considerable extent the properties of permutable functions and for carrying out the operations of composition.

Thus, let us consider any analytic expression

$$F(z_1 \ldots z_e)$$

which can be expanded about the point $z_1 = z_2 = z_3 = z_4 = \ldots . z_e = 0$ in powers of z_1, $z_2, z_3, \ldots . z_e$. If we replace $z_1, z_2, z_3, \ldots . z_e$ in the series by $f_1, f_2, f_3, \ldots . f_e$ respectively, and write the operation of composition wherever we previously had multiplication, we shall always obtain a series which converges for all values of $f_1, f_2, f_3, \ldots f_e$, and which represents a function permutable with $f_1, f_2, f_3, \ldots f_e$. We may represent it by

$$F(\overset{*}{f}_1, \overset{*}{f}_2, \cdots \overset{*}{f}_e).$$

Thus, every algebraic expression takes on a new meaning for the operation of composition. For example,

$$\frac{z}{1+z} = z - z^2 + z^3 - \ldots$$

is a series which converges within the unit circle. But if we write

$$\frac{\overset{*}{f}}{1+\overset{*}{f}} = \overset{*}{f} - \overset{*}{f}{}^2 + \overset{*}{f}{}^3 - \ldots$$

we obtain a series which converges for all values of f and which is permutable with f. Consequently, a meaning has been ascribed to the expression on the left hand side of the equation.

Moreover, if we take two expressions

$$\frac{\overset{*}{f}}{1+\overset{*}{f}} = \overset{*}{f} - \overset{*}{f}{}^2 + \overset{*}{f}{}^3 \ldots$$

and

$$\frac{\overset{*}{f}}{1-\overset{*}{f}} = \overset{*}{f} + \overset{*}{f}{}^2 + \overset{*}{f}{}^3 + \ldots$$

then to make the composition of the two left-hand members, it is only necessary to apply the rules for finding their algebraic product and we shall have

$$\frac{\overset{*}{f}{}^2}{1-\overset{*}{f}{}^2} = \overset{*}{f}{}^2 + \overset{*}{f}{}^4 + \ldots$$

Hence, all the rules of ordinary algebra remain valid when we pass from the field of multiplication to the field of composition.

Some of the consequences which can be derived from this fact will be seen shortly.

13. Now let us see what takes place for the second type of composition.

Let

$$(4) \quad F(z) = \frac{\phi(z)}{1+\psi(z)}$$

be the ratio of two entire functions $\phi(z)$ and $1+\psi(z)$ which are such that $\phi(0) = \psi(0) = 0$.

Then we shall have an expansion

$$F(z) = A_1 z + A_2 z^2 + A_3 z^3 + \dots$$

which converges in general within a certain circle having the point $z = 0$ as center.

Now let us consider the expression

$$F_{is}(z) = A_1 m_{is} z + A_2 (m^2)_{is} z^2 + \dots$$

We shall prove that this new series is *the quotient of two entire functions.*

14. Let us first write

$$\phi(z) = B_1 z + B_2 z^2 + \dots$$

and determine a quantity

$$\phi_{is}(z) = B_1 m_{is} z + B_2 (m^2)_{is} z^2 + \dots$$

We say that $\phi_{is}(z)$ *is an entire function of* z. For let μ be greater in absolute value than m_{is}. We then have (§ 3)

$$|m_{is}| < \mu, \quad |(m^2)_{is}| < g\mu^2, \quad |(m^3)_{is}| < g^2 \mu^3, \dots$$

Moreover,

$$|B_1|\mu z + |B_2|\mu^2 g z^2 + |B_3|\mu^3 g^2 z^3 + \dots$$

converges for all values of z, and the theorem is proved.

For the same reason, if

$$\psi(z) = C_1 z + C_2 z^2 + \dots$$

then the series

$$\psi_{is}(z) = C_1 m_{is} z + C_2 (m^2)_{is} z^2 + \dots$$

is an entire function.

Bearing these facts in mind, let us consider the system of algebraic linear equations

$$X_{is} + \sum_{1}^{g} \psi_{ie} X_{es} = \phi_{is} \quad (i, s = 1, 2, \ldots g).$$

If we replace the unknowns X_{is} by F_{is} we can verify without trouble that these equations are identically satisfied. But if we solve for the unknowns X_{is} in the above system, the solution will be expressed as the quotients of rational entire functions of ψ_{is} and ϕ_{is} and hence the quantities X_{is} are quotients of entire functions of z.

It is clear that the determinant which constitutes the denominator of these quotients cannot vanish identically, and hence the theorem is proved.

It is not difficult to generalize this. Thus instead of the quotient (4), let us write

$$F(z_1, z_2 \ldots z_e) = \frac{\phi(z_1 \ldots . z_e)}{1 + \psi(z_1 \ldots . z_e)},$$

where ϕ and ψ are entire functions of the variables $z_1, z_2, z_3, \ldots . z_e$ which vanish for $z_1 = z_2 = z_3 = \ldots . z_e = 0$. If

$$F(z_1, \ldots z_e) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e} z_1^{i_1} \ldots z_e^{i_e}$$

is the expansion of F about the point for which

$z_1 = z_2 = z_3 = \ldots\ldots = z_e = 0$, and if we write $F_{is}(z_1, \ldots z_e) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e}(m^{i_1} n^{i_2} \ldots q^{i_e})\, z_1^{i_1} \ldots z_e^{i_e}$ where m, n, ... q are permutable, then the function $F(z_1, z_2, z_3, \ldots\ldots z_e \mid x, y)$ will be the quotient of two entire functions of z_1, z_2, z_3, ... z_e.

To make the proof in this case, it is only necessary to repeat the argument given above. We may add that F_{is} is permutable with m_{is}, n_{is}, ... q_{is}.

15. Let us now pass to the limit, that is, let us consider permutable functions of the second type*. All of the theorems remain valid. In other words, we have the theorem that

$$F(z_1, \ldots z_e \mid x, y) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e} f_1^{i_1} \cdots f_e^{i_e} z_1^{i_1} \ldots z_e^{i_e},$$

where f_1, f_2, f_3, ... f_e are permutable functions of the second type, is the quotient of two entire functions. Also,

$$F(z_1, \ldots z_e \mid x, y)$$

* To indicate composition of the second type, we shall make use of a double star thus:

$$\overset{**}{f_1}\ \overset{**}{f_2}$$

Bearing these facts in mind, let us consider the system of algebraic linear equations

$$X_{is} + \sum_1^g \psi_{ie} X_{es} = \phi_{is} \quad (i, s = 1, 2, \ldots g) .$$

If we replace the unknowns X_{is} by F_{is} we can verify without trouble that these equations are identically satisfied. But if we solve for the unknowns X_{is} in the above system, the solution will be expressed as the quotients of rational entire functions of ψ_{is} and ϕ_{is} and hence the quantities X_{is} are quotients of entire functions of z.

It is clear that the determinant which constitutes the denominator of these quotients cannot vanish identically, and hence the theorem is proved.

It is not difficult to generalize this. Thus instead of the quotient (4), let us write

$$F(z_1, z_2 \ldots z_e) = \frac{\phi(z_1 \ldots . z_e)}{1 + \psi(z_1 \ldots . z_e)} ,$$

where ϕ and ψ are entire functions of the variables z_1, z_2, z_3, ... z_e which vanish for $z_1 = z_2 = z_3 = \ldots . z_e = 0$. If

$$F(z_1, \ldots z_e) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e} z_1^{i_1} \ldots z_e^{i_e}$$

is the expansion of F about the point for which

$z_1 = z_2 = z_3 = \ldots\ldots = z_e = 0$, and if we write $F_{is}(z_1, \ldots z_e) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e}(m^{i_1} n^{i_2} \ldots q^{i_e}) z_1^{i_1} \ldots z_e^{i_e}$ where m , n , ... q are permutable, then the function $F(z_1, z_2, z_3, \ldots\ldots z_e \mid x, y)$ will be the quotient of two entire functions of z_1, z_2, z_3, ... z_e.

To make the proof in this case, it is only necessary to repeat the argument given above. We may add that F_{is} is permutable with m_{is}, n_{is}, ... q_{is}.

15. Let us now pass to the limit, that is, let us consider permutable functions of the second type*. All of the theorems remain valid. In other words, we have the theorem that

$$F(z_1, \ldots z_e \mid x, y) = \Sigma \ldots \Sigma A_{i_1 \ldots i_e} f_1^{i_1} \cdots f_e^{i_e} z_1^{i_1} \ldots z_e^{i_e},$$

where f_1, f_2, f_3, ... f_e are permutable functions of the second type, is the quotient of two entire functions. Also,

$$F(z_1, \ldots z_e \mid x, y)$$

* To indicate composition of the second type, we shall make use of a double star thus:

$$\overset{**}{f_1} \; \overset{**}{f_2}$$

is a function which is permutable with f_1, f_2, f_3, ... f_e.

We shall study some applications of these fundamental theorems in the second lecture.

LECTURE II

LECTURE II

1. We shall begin by classifying integral equations into several categories. First, let us examine those which are linear. The simplest ones which we run across are the following:

$$(1) \quad f(y) + \int_0^y f(x)\, F(x, y)\, dx = \phi(y)\,,$$

known as *Volterra's equation of the second kind,* and

$$(1') \quad f(y) + \int_0^1 f(x)\, F(x, y)\, dx = \phi(y)\,,$$

Fredholm's equation of the second kind.
We shall also consider certain other kinds further on.

Let us look at equation (1). If we multiply both sides by $\Phi(y, z)$ and integrate with respect to y between the limits 0 and z, we obtain

$$\int_0^z \Phi(y,z) f(y)\, dy + \int_0^z f(x) dx \int_x^z F(x,y)\, \Phi(y,z)\, dy = \int_0^z \phi(y)\, \Phi(y, z)\, dy\,.$$

If now the function Φ be so chosen that

$$\text{(A)} \quad \int_x^z F(x,y)\,\Phi(y,z)\,dy = -\Phi(x,z) - F(x,z),$$

it will follow that

$$\int_0 \Phi(y,z)\,f(y)\,dy - \int_0^z \Phi(x,z)\,f(x)\,dx$$
$$- \int_0^z F(x,z)\,f(x)\,dx = \int_0^z \phi(y)\,\Phi(y,z)\,dy$$

which is reducible by (1) to

$$f(z) = \phi(z) + \int_0^z \phi(y)\,\Phi(y,z)\,dy,$$

so that the difficulty has been narrowed down to the solution of (A). In the symbols for composition of the first kind, this equation may be written as follows:

$$\text{(2)} \quad \Phi(x,y) + F(x,y) + \overset{*}{F}\overset{*}{\Phi}(x,y) = 0\,.$$

If we apply a similar argument to equation (1′) we find that the solution will be given in the form

$$f(z) = \phi(z) + \int_0^1 \phi(y)\,\Phi(y,z)\,dy$$

where

$$\text{(2′)} \quad \Phi(x,y) + F(x,y) + \overset{**}{F}\,\overset{**}{\Phi}(x,y) = 0\,.$$

2. Let us first see how equation (2) may be solved. If we write

$$z + z_1 + z z_1 = 0$$

we shall obtain

$$z_1 = \frac{-z}{1+z} = -z + z^2 - z^3 + \ldots$$

the solution being valid if $|z| < 1$.

But suppose we write

$$-\frac{\overset{*}{F}}{1+\overset{*}{F}} = -F + \overset{*}{F}{}^2 - \overset{*}{F}{}^3 + \ldots$$

Then, in this case, we have an expansion which converges for all values of F and which satisfies equation (2) by what we have proved. Hence we shall have

$$\Phi = -F + \overset{*}{F}{}^2 - \overset{*}{F}{}^3 + \ldots$$

and the integral equation (2) is solved.

If we replace $F(x, y)$ by $u\,F(x, y)$ in equation (2) we obtain

$$\Phi(x, y) + u\,F(x, y) + u\,\overset{*}{F}\overset{*}{\Phi}(x, y) = 0$$

$$\Phi = -u\,F + u^2 \overset{*}{F}{}^2 - u^3 \overset{*}{F}{}^3 + \ldots$$

which series is always an entire function of u.

3. Turning to equation (2′), let us replace F by uF as in the previous case. We shall then have

$$\Phi + uF + u\ddot{F}\ddot{\Phi} = 0\,,$$

and as a consequence of the last theorem of the first lecture, we have that Φ can be expressed as the quotient of two entire functions of u.

4. As soon as we have stated the fundamental problems in this form, it is easy to see that they are only special cases of other classes of problems of a much more general nature.

Indeed, let us consider any analytic function $F(z_1, z_2, z_3, \ldots . z_n)$ whatsoever and write the equation

$$(3) \qquad F(z_1, z_2, z_3, \ldots . z_n) = 0\,.$$

Furthermore, let us suppose that this equation is satisfied by $z_1 = z_2 = z_3 = \ldots . = z_n = 0$, and let us regard z_n as a function dependent upon $z_1, z_2, z_3, \ldots . z_{n-1}$. If the point $z_1 = z_2 = z_3 = \ldots . = z_n = 0$ is not a critical point, we may develop z_n as a power series in $z_1, z_2, \ldots . z_{n-1}$ and the expansion will be convergent within some region. We shall thus have

$$(4) \qquad z_n = \Sigma\,\Sigma \ldots \Sigma\, A_{i_1 \ldots i_{n-1}}\, z_1^{i_1} \ldots z_{n-1}^{i_{n-1}}\,,$$

$$A_{00\ldots 0} = 0\,.$$

Now suppose we replace $z_1, z_2, \ldots\ldots z_n$ in equation (3) by the permutable functions $f_1, f_2, \ldots\ldots f_n$ respectively and regard the operations as compositions of the first type. Then, in terms of our notation, we shall have

$$F(\overset{*}{f_1}, \overset{*}{f_2}, \ldots \overset{*}{f_n}) = 0 \ .$$

The equation which we have just found will no longer be algebraic or transcendental but will be an integral equation, since the operation of composition is an operation of integration. Nor will the equation in general be linear as was equation (2), but of any degree whatsoever. Nevertheless, if we regard f_n as the unknown function, we shall be able to find its solution by the same process which we used in solving equation (3). Indeed, it is only necessary to replace $z_1, z_2, z_3, \ldots\ldots z_n$ in the series (4) by $f_1, f_2, f_3, \ldots\ldots f_n$ respectively and to treat the operations as operations of composition. In this manner, we find

$$(5) \quad f_n = \Sigma\, \Sigma \ldots \Sigma\, A_{i_1 \ldots i_{n-1}}\, \overset{*}{f}{}_1^{i_1} \ldots \overset{*}{f}{}_{n-1}^{i_{n-1}} .$$

An interesting fact to be noticed is that whereas the expansion (4) is in general convergent over a limited region only, the solution (5) is valid for all values of $f_1, f_2, \ldots f_{n-1}$. Evidently problems of integration are of a more complicated nature than algebraic or transcendental problems, yet we have the surprising and interesting result that the solutions of the former are much more simple in the sense that the regions over which they are valid is infinite.

We may also replace $z_1, z_2, z_3, \ldots z_{n-1}$ in equation (3) by $z_1 f, z_2 f_2, \ldots z_{n-1} f_{n-1}$ respectively and write the equation

$$F(z_1 f_1, z_2 f_2, \ldots z_{n-1} f_{n-1}, f_n) = 0 .$$

Then the solution will be

$$f_n = \Sigma \ldots \Sigma A_{i_1 \ldots i_{n-1}} z_1^{i_1} \ldots z_{n-1}^{i_{n-1}} \overset{*}{f}{}_1^{i_1} \ldots \overset{*}{f}{}_{n-1}^{i_{n-1}}$$

which is of the form

$$f_n(z_1, \ldots z_{n-1} | x, y) .$$

The series will be an entire function of $z_1, z_2, z_3, \ldots z_{n-1}$, and with respect to x and y, $f_n(z_1, z_2, z_3, \ldots z_{n-1} | x, y)$ will be permutable with the functions $f_1, f_2, f_3, \ldots f_{n-1}$.

5. We might also start with a system of equations, as

$$(6)\quad \begin{cases} F_1(z_1, \ldots z_n, u_1, \ldots u_p) = 0 \\ F_2(z_1, \ldots z_n, u_1, \ldots u_p) = 0 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F_p(z_1, \ldots z_n, u_1, \ldots u_p) = 0 \end{cases}$$

which are satisfied when $z_1 = u_1 = z_2 = u_2 = \ldots . = z_n = 0$. Now let us suppose that we can define

$$(7)\quad \begin{cases} u_1 = \Sigma\,\Sigma \ldots \Sigma\, A^{(1)}_{i_1 \ldots i_n}\, z_1^{i_1} \ldots z_n^{i_n} \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_p = \Sigma\,\Sigma \ldots \Sigma\, A^{(p)}_{i_1 \ldots i_n}\, z_1^{i_1} \ldots z_n^{i_n} \end{cases}$$

as implicit functions of $z_1, z_2, z_3, \ldots . z_n$ which have no critical point at $z_1 = z_2 = \ldots . z_n = 0$. Then if we write the integral equations

$$F_1(z_1 \overset{*}{f}_1, \ldots z_n \overset{*}{f}_n, \overset{*}{\phi}_1, \ldots \overset{*}{\phi}_p) = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$F_p(z_1 \overset{*}{f}_1, \ldots z_n \overset{*}{f}_n, \overset{*}{\phi}_1, \ldots \overset{*}{\phi}_p) = 0$$

where $f_1, f_2, f_3, \ldots . f_n$ are permutable functions, the solution of the system will be

$$\phi_1 = \Sigma \ldots \Sigma\, A^{(1)}_{i_1 \ldots i_n}\, z_1^{i_1} \ldots z_n^{i_n}\, \overset{*}{f}_1^{i_1} \ldots \overset{*}{f}_n^{i_n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\phi_p = \Sigma \ldots \Sigma\, A^{(p)}_{i_1 \ldots i_n}\, z_1^{i_1} \ldots z_n^{i_n}\, \overset{*}{f}_1^{i_1} \ldots \overset{*}{f}_n^{i_n}$$

and the functions thus obtained will be entire functions of $z_1, z_2, z_3, \ldots . z_n$ for all values of $f_1, f_2, f_3, \ldots . f_n$. Moreover, these solutions will be permutable with the given functions.

All of the equations which we have been considering involve only integrals for which the limits of integration are x and y; that is to say, they are equations with variable limits. Let us see what the situation is when the limits are constant. Returning to the set of equations (6), we shall suppose that the solutions (7) are quotients of entire functions. Then let us examine the integral equations

$$F_1(z_1 \overset{**}{f_1}, \ldots z_n \overset{**}{f_n}, \overset{**}{\phi_1}, \ldots \overset{**}{\phi_p}) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_p(z_1 \overset{**}{f_1}, \ldots z_n \overset{**}{f_n}, \overset{**}{\phi_1}, \ldots \overset{**}{\phi_p}) = 0$$

where $f_1, f_2, f_3, \ldots . f_n$ are permutable functions of the second type. In these equations the limits of integration are constant, since we are concerned with composition of the second type.

If now we put

$$\phi_1 = \Sigma \ldots \Sigma A^{(1)}_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n} \overset{**}{f}_1^{i_1} \ldots \overset{**}{f}_n^{i_n}$$

. . . .

. . . .

$$\phi_p = \Sigma \ldots \Sigma A^{(p)}_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n} \overset{**}{f}_1^{i_1} \ldots \overset{**}{f}_n^{i_n}$$

the functions thus obtained

1) satisfy the preceding system of equations,

2) are quotients of entire functions, and

3) have permutability of the second type with the original functions $f_1, f_2, \ldots . f_n$.

Thus we see that linear equations are only a very special type of integral equations and that we can pass from their study to that of a more general class.

6. Let us prove certain important properties about functions which may be found by a process like the one above outlined. More precisely, let us show what certain algebraic properties become when we pass from multiplication to composition. We shall begin by giving an example:

We consider the exponential function

$$e^z = 1 + z + \frac{z^2}{2\,!} + \frac{z^3}{3\,!} + \dots$$

Associated with it is an addition theorem

$$e^{(z+z_1)} = e^z\, e^{z_1}\,.$$

Suppose we put

$$f(z) = e^z - 1\,.$$

We then have

$$(8) \quad f(z+z_1) = f(z)\, f(z_1) + f(z) + f(z_1)\,.$$

Keeping the above in mind, let us write the function

$$V(z\,|\,x,y) = zf + \frac{z^2 \overset{*}{f}{}^2}{2\,!} + \frac{z^3 \overset{*}{f}{}^3}{3\,!} + \dots$$

We can see at once what the relation (8) becomes. Indeed, we have only to replace multiplication by composition. We shall therefore have

$$V(z+z_1\,|\,x,y) = V(z\,|\,x,y) + V(z_1\,|\,x,y) + \overset{*}{V}(z\,|\,x,y)\, \overset{*}{V}(z_1\,|\,x,y)\,,$$

that is

$$V(z+z_1\,|\,x,y) = V(z\,|\,x,y) + V(z_1\,|\,x,y) + \int_x^y V(z\,|\,x,\xi)\, V(z_1\,|\,\xi,y)\, d\xi\,.$$

In other words, the theorem of algebraic addition for the exponential function becomes for this new function a theorem of integral addition as we have called it.*

7. To go from the particular case to the general involves no difficulty. Consequently, we may state the theorem: *To every theorem of algebraic addition, there corresponds a theorem of integral addition.*

Thus, for example, if we consider elliptic functions, we can pass from these to entire functions by the process of § 11 of the preceding lecture. To the addition theorems for elliptic functions, there correspond new addition theorems for new functions. In a like manner, let us consider the σ function of Weierstrass. Suppose we examine for a moment the expansion of this function and replace u in the expression by $uf(x, y)$ where the powers of f represent operations of composition. The three-term equation for σ leads us to a three-term equation for the new function

* Evans has studied in a systematic manner an Algebra of permutable functions. (Memorie Lincei, S.V. Vol. VIII, 1911; also Rendiconti del Circolo di Palermo, Vol. XXXIV, 1912.)

which is of the integral type since we have replaced products by compositions.

8. What we have said about compositions of the first type may be repeated for compositions of the second. Returning once more to the example involving the exponential function, let us put

$$W(z\,|\,x,y) = zf + \frac{z^2 \overset{\bullet\bullet}{f}{}^2}{2\,!} + \frac{z^3 \overset{\bullet\bullet}{f}{}^3}{3\,!} + \ldots$$

This function is also an entire function and we shall have

$$W(z+z_1\,|\,x,y) = W(z\,|\,x,y) + W(z_1\,|\,x,y) + \overset{\bullet\bullet}{W}(z\,|\,x,y)\ \overset{\bullet\bullet}{W}(z_1\,|\,x,y)\,,$$

or in other words,

$$W(z+z_1\,|\,x,y) = W(z\,|\,x,y) + W(z_1\,|\,x,y) + \int_a^b W(z\,|\,x,\xi)\ W(z\,|\,\xi,y)\,d\xi\ .$$

It is hardly necessary to prove that the above is true in the general case.

9. A similar theorem may also be stated for the case of more than one variable; hence, all the theorems about Abelian functions may be carried over to the domain of integration by a process like the one which we have indicated.

10. Let us now return to linear integral equations. As indicated, equations (1) and (1′) are of the second kind. Those of the first kind of the Volterra and Fredholm types respectively may be written

$$(9) \quad \int_0^y f(x)\, F(x,y)\, dx = \phi(y),$$

$$(9') \quad \int_0^1 f(x)\, F(x,y)\, dx = \phi(y).$$

Leaving out of consideration equations (9′) which can only be attacked by methods of a different sort, let us consider equations (9). The latter may be reduced to equations of the second kind. For we can differentiate and obtain

$$F(y,y)\, f(y) + \int_0^y f(x) \frac{\partial F(x,y)}{\partial y}\, dx = \frac{d\phi}{dy}.$$

If $F(y, y)$ does not vanish, we can divide by $F(y, y)$ and get an equation of the second kind.

If $F(y, y)$ vanishes identically, the last equation is still of the first kind. But if

$$\left(\frac{\partial F(x,y)}{\partial y}\right)_{x=y}$$

is not zero, then by a second differentiation, we shall get an equation of the second kind, and so on.

If $F(x, y)$ is such that $F(x, x) \lessgtr 0$, we shall call it a function of the first order. If $F(x, x) = 0$ and $\left(\frac{\partial F}{\partial y}\right)_{x=y} \gtrless 0$, we shall call it a function of the second order, and so on. Hence, if the order of the function $F(x, y)$ in equation (9) is determinate, the equation can always be reduced to one of the second kind by a finite number of differentiations, and hence can be solved by the method which we have indicated.

But the order of $F(x, y)$ may not be determinate. A case in point is where $F(x, x)$ is in general different from zero but vanishes for certain values of x. Then the nature of the problem changes, and to solve it, new methods must be used. To develope these would lead us too far afield. The solution of this question has been the goal of numerous enquiries. We were the first to take up the matter and since then Lalesco and others have studied it.*

Instead of considering equation (9) which

* See: *Lalesco, Introduction á la théorie des équations intègrales.* Paris: Hermann, 1912. Troisième partie I.

is of the first kind, we may consider the equation

$$\int_x^y F(x,\xi)\ \Phi(\xi,y)\ d\xi = \psi(x, y),$$

where we can regard F and ψ as the known functions and Φ as unknown. For we have only to suppose that x is a constant, when the equation reduces at once to equation (9).

If we take the equation of the first kind in this form, we may also write it

$$\overset{*}{F}\ \overset{*}{\Phi} = \psi\ ;$$

that is to say, the problem is of the following nature: Given a function ψ which is the resultant of the composition of F and Φ, and given one of the factors F of the composition, to find the other factor Φ. If for the moment we were to replace the operation of composition by that of multiplication, the problem would reduce to that of finding the inverse operation; that is, we are dealing with a problem which is analogous to the problem of division.

Now it is necessary to observe that certain conditions must be satisfied if the problem is to have finite solutions. *The order of* ψ *must be greater than the order of* F *by at least unity.* For when $x=y$, ψ vanishes to a higher order than F. If F is of order m and ψ is of order n, then Φ must be of order m-n. Moreover, two cases may arise according as the functions F and ψ are or are not permutable with one another. Clearly in the latter case, Φ cannot be permutable with F, otherwise the resultant of the composition of the two would be permutable with either. But if F and ψ are permutable, will Φ be permutable with F and ψ?

We shall prove that this property is actually realized. In fact, we have

$$\overset{*}{F}\,\overset{*}{\Phi}\,\overset{*}{F} = \overset{*}{\psi}\,\overset{*}{F}\,, \qquad \overset{*}{F}\,\overset{*}{F}\,\overset{*}{\Phi} = \overset{*}{F}\,\overset{*}{\psi}\,.$$

Hence

$$F(\Phi F) = F(F\Phi)\,,$$

and since this integral equation has but one solution,

$$\Phi F = F\Phi\,,$$

and the theorem is proved.

11. Furthermore, when the problem of

linear integral equations of the first kind has been put in the form

$$\overset{*}{F}\,\overset{*}{\Phi} = \psi$$

other problems suggest themselves at once. Thus, if F, Φ and ψ are known functions, we may set the problem of determining a quantity such that

$$(10)\quad \overset{*}{F}\,\overset{*}{X} + \overset{*}{X}\,\overset{*}{\Phi} = \psi;$$

or again the following problem: given the known quantities F_1, F_2, F_3, F_4, and ψ to calculate a quantity Φ such that

$$(11)\quad \overset{*}{F}_1\,\overset{*}{\Phi} + \overset{*}{\Phi}\,\overset{*}{F}_2 + \overset{*}{F}_3\,\overset{*}{\Phi}\,\overset{*}{F}_4 = \psi\,.$$

The above are new equations which up to the present have never been studied and with which we shall now concern ourselves.

First, let us consider equation (10) which we can write

$$\int_x^y F(x,\xi)\,X(\xi,y)\,d\xi + \int_x^y X(x,\xi)\,\Phi(\xi,y)\,d\xi = \psi(x,y).$$

Let us put

$$S(x,y) = \int_x^y F(x,\xi)\;X(\xi,y)\;d\xi = \psi(x,y) - \int_x^y X(x,\xi)\;\Phi(\xi,y)\;d\xi\,.$$

Then we shall have

$$\frac{\partial S}{\partial x} = -F(x,x)\, X(x,y) + \int_x^y F_1(x,\xi)\, X(\xi,y) d\xi\,,$$

$$\frac{\partial S}{\partial y} = -\Phi(y,y)\, X(x,y) - \int_x^y X(x,\xi)\, \Phi_2(\xi,y)\, d\xi + \frac{\partial \psi}{\partial y}\,,$$

where we have put

$$F_1 = \frac{\partial F}{\partial x}\,, \qquad \Phi_2 = \frac{\partial \Phi}{\partial y}\,.$$

From the first equation, we derive

$$X(x,y) = -\frac{1}{F(x,x)} \frac{\partial S(x,y)}{\partial x} + \int_x^y f(x,\xi) \frac{\partial S(\xi,y)}{\partial \xi}\, d\xi\,,$$

and from the second

$$X(x,y) = -\frac{1}{\Phi(y,y)} \frac{\partial S(x,y)}{\partial y}$$
$$+ \int_x^y \phi(\xi,y)\, \frac{\partial S(x,\xi)}{\partial \xi}\, d\xi + H(x,y).$$

where f and ϕ are two known functions which one can obtain from F and Φ and where

$$H = -\frac{1}{\Phi(y,y)} \frac{\partial \psi(x,y)}{\partial y} + \int_x^y \phi(\xi,y)\, \frac{\partial \psi(x,\xi)}{\partial \xi}\, d\xi$$

is also a known function. Then by subtracting the second equation from the first, we have at once

$$\frac{1}{\Phi(y,y)}\frac{\partial S(x,y)}{\partial y}-\frac{1}{F(x,x)}\frac{\partial S(x,y)}{\partial x}-H(x,y)$$
$$+\int_x^y f(x,\xi)\frac{\partial S(\xi,y)}{\partial \xi}\,d\xi$$
$$-\int_x^y \phi(\xi,y)\frac{\partial S(x,\xi)}{\partial \xi}\,d\xi,$$

and integration by parts gives

$$(12)\quad \frac{1}{\Phi(y,y)}\frac{\partial S(x,y)}{\partial y}-\frac{1}{F(x,x)}\frac{\partial S(x,y)}{\partial x}-S(x,y)f(x,x)$$
$$-\int_x^y \frac{\partial f(x,\xi)}{\partial \xi}\,S(\xi,y)\,d\xi-S(x,y)\,\phi(y,y)$$
$$+\int_x^y \frac{\partial \phi(\xi,y)}{\partial \xi}\,S(x,\xi)\,d\xi-H(x,y)=0\,.$$

Thus we are led to the following result: To solve the integral equation (10) we must solve the problem which presents itself in the shape of the last equation. This problem is nothing more than the integration of an integro-differential equation. Indeed, equation (12) is both of the type of an integral equation and of a differential equation.

The above problem admits of a solution, but we shall not go into details of the solution. The interesting point to notice is that integro-differential equations arise in a great variety of problems. We have examined these equations in a number of forms and have made a particular study of the integro-differential equations of the second order and of the elliptic or hyperbolic types which arise in connection with certain problems of mathematical physics.*

The problem we were considering is of a different type. It is of the first order, and since two dependent variables appear, it corresponds to problems involving partial derivatives. The case of equation (11) may be handled in a similar manner.

12. We wish to demonstrate certain interesting results which are closely connected with the problems we have been discussing. Let us go back to equation (9). In certain cases, this equation has a finite number of solutions, while in others the number of solutions is infinite and the solutions involve an arbitrary function.

To see this, we need only to consider the equation

$$F\chi + \chi\Phi = 0$$

* Leçons sur les fonctions de lignes. Paris: Gauthier-Villars. 1913.

and to determine under what conditions $\chi = 0$ is the only solution and under what conditions solutions other than 0 exist.

To simplify matters, we shall assume that the functions F and Φ are of the first order and shall determine under what circumstances the equation has a solution of the first order.

Suppose we write our equation in the form

$$\text{(B)} \quad \int_x^y F(x,\xi)\chi(\xi,y)\,d\xi + \int_x^y \chi(x,\xi)\,\Phi(\xi,y)\,d\xi = 0.$$

Then by differentiation with respect to y, we have

$$F(x,y)\chi(y,y) + \int_x^y F(x,\xi)\,\chi_2(\xi,y)\,d\xi + \chi(x,y)\,\Phi(y,y) + \int_x^y \chi(x,\xi)\,\Phi_2(\xi,x)\,d\xi = 0$$

and when $x = y$,

$$F(y,y) + \Phi(y,y) = 0\,,$$

which gives us a necessary condition.

Moreover, by suitable transformations of a simple sort, we are always led to the case where

$$\text{(12)} \quad F(x,x) = -\Phi(x,x) = 1\,,$$

$$\text{(12}'\text{)} \quad F_1(x,x) = F_2(x,x) = \Phi_1(x,x) = \Phi_2(x,x) = 0\,,$$

where the subscripts 1 and 2 denote partial differentiation with respect to x and y respectively; that is

$$F_1(x,y)=\frac{\partial F(x,y)}{\partial x}\,,\quad F_2(x,y)=\frac{\partial F(x,y)}{\partial y}\,,$$

$$\Phi_1(x,y)=\frac{\partial \Phi(x,y)}{\partial x}\,,\quad \Phi_2(x,y)=\frac{\partial \Phi(x,y)}{\partial y}\,.$$

For we can first write

$$x=f(x_1)\,,\qquad y=f(y_1)\,,$$

$$F'(x_1,y_1)=\pm\sqrt{f'(x_1)\,f'(y_1)}\;F(x,y)\,,$$

$$\Phi'(x_1,y_1)=\pm\sqrt{f'(x_1)\,f'(y_1)}\;\Phi(x,y)\,,$$

$$\chi'(x_1,y_1)=\pm\sqrt{f'(x_1)\,f'(y_1)}\;\chi(x,y)\,,$$

Then if we take

$$f'(x_1)=\frac{\pm 1}{F(x,x)}\,,$$

we clearly see that the equation (B) becomes

$$\int_{x_1}^{y_1}F'(x_1,\xi_1)\,\chi'(\xi_1,y_1)\,d\xi_1 + \int_{x_1}^{y_1}\chi'(y_1,\xi_1)\Phi'(\xi_1,y_1)\,d\xi_1=0\,,$$

where

$$F'(x_1,x_1)=-\Phi'(y_1,y_1)=1\,.$$

Hence we can suppose at the outset that conditions (12) are satisfied.

The above having been established, equation

(B) may be written

$$\int_x^y \frac{\alpha(x)}{\alpha(\xi)} F(x, \xi)\, \alpha(\xi)\, \chi(\xi, y)\, \beta(y)\, d\xi$$
$$+ \int_x^y \alpha(x)\, \chi(x, \xi)\, \beta(\xi)\, \Phi(\xi, y) \frac{\beta(y)}{\beta(\xi)}\, d\xi = 0\ .$$

If we put

$$\frac{\alpha(x)}{\alpha(y)} F(x, y) = F'(x, y)\,,$$
$$\frac{\beta(y)}{\beta(x)} \Phi(x, y) = \Phi'(x, y)\,,$$
$$\alpha(x)\, \chi(x, y)\, \beta(y) = \chi'(x, y)\ ,$$

we shall have

$$\int_x^y F'(x,\xi)\, \chi'(\xi, y)\, d\xi + \int_x^y \chi'(x, \xi)\, \Phi'(\xi, y)\, d\xi = 0.$$

But we can make use of the arbitrariness of α and β to choose

$$F'_1(x, x) = F'_2(x, x) = \Phi'_1(x, x) = \Phi'_2(x, x) = 0\ ,$$

which shows that we can always assume that condition (12′) is satisfied.

Now let us write

$$\psi = -\int_x^y F(x, \xi)\, \chi(\xi, y)\, d\xi = \int_x^y \chi(x, \xi)\, \Phi(\xi, y)\, d\xi\, .$$

We shall have

$$\frac{\partial \psi}{\partial x} = \chi(x,y) - \int_x^y F_1(x,\xi)\,\chi(\xi,y)\,d\xi\ ,$$

$$\frac{\partial \psi}{\partial y} = \chi(x,y) + \int_x^y \chi(x,\xi)\,\Phi_2(\xi,y)\,d\xi\ ,$$

whence we derive

$$\chi(x,y) = \frac{\partial \psi}{\partial x} + \int_x^y f_1(x,\xi)\frac{\partial \psi}{\partial \xi}\ ,$$

$$\chi(x,y) = \frac{\partial \psi}{\partial y} + \int_x^y \frac{\partial \psi}{\partial \xi}\phi_2(\xi,y)\,d\xi,$$

where

$$f_1(x,y) = -F_1(x_1,y) + \overset{*}{F}{}_1^2(x,y) + \overset{*}{F}{}_1^3(x,y) + \ldots,$$

$$\phi_2(x,y) = -\Phi_2(x,y) + \overset{*}{\Phi}{}_2^2(x,y) + \overset{*}{\Phi}{}_2^3(x,y) + \ldots,$$

and therefore (see Lecture II, § 1)

$$f_1(x,x) = \phi_2(x,x) = 0\ .$$

Hence, integrating by parts, we have

$$\chi = \frac{\partial \psi}{\partial x} + \int_x^y \lambda(x,\xi)\,\psi(\xi,y)\,d\xi\ ,$$

$$\chi = \frac{\partial \psi}{\partial y} + \int_x^y \psi(x,\xi)\,\mu(\xi,y)\,d\xi\ ,$$

where

$$\lambda(x,\xi) = \frac{\partial f_1(x,\xi)}{\partial \xi}\ , \qquad \mu(\xi,y) = \frac{\partial \phi_2(\xi,x)}{\partial \xi}$$

and therefore

$$\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} + \int_x^y [\lambda(x,\xi)\,\psi(\xi,y) - \psi(x,\xi)\,\mu(\xi,y)]\,d\xi = 0.$$

This integro-differential equation may be integrated.

If we write

$$G(x, y) = \int_x^y [\lambda(x, \xi)\, \psi(\xi, y) - \psi(x, \xi)\, \mu(\xi, y)]\, d\xi\,,$$

we have

$$(16) \quad \psi(x, y) = \theta(v) - \int_v^u G(\xi + v, v - \xi)\, d\xi$$

where θ is an arbitrary function and

$$v = \frac{x + y}{2}\,, \quad u = \frac{x - y}{2}\,.$$

The solution of the equation (16) is obtained by the method of successive approximations.

Applications of the above will be brought out in the next lecture.

LECTURE III

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1. We shall begin with some applications of the work developed in the last lecture.

We have solved the problem of finding the function $\chi\ (x, y)$ which satisfies the equation

$$\int_x^y F(x,\xi)\,\chi(\xi,y)\,d\xi+\int_x^y \chi(x,\xi)\,\Phi(\xi,y)\,d\xi=0$$

on the hypothesis that F and Φ are of the first order. Now suppose we put

$$\Phi(x,y) = -F(x,y)\ .$$

Then the condition

$$\Phi(x,x) + F(x,x) = 0$$

is clearly satisfied, and hence we shall be able to calculate all of the functions $\chi\,(x, y)$ which satisfy the relation

$$\int_x^y F(x,\xi)\,\chi(\xi,y)\,d\xi=\int_x^y \chi(x,\xi)\,F(\xi,y)\,d\xi$$

in other words, all of the functions which have permutability of type one with a given function. However, in the last lecture (§ 12) this problem was solved only in the special case where the given function is of the first order. If the function is of higher order, the method breaks down.

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in other words, all of the functions which have permutability of type one with a given function. However, in the last lecture (§ 12) this problem was solved only in the special case where the given function is of the first order. If the function is of higher order, the method breaks down.

We have seen that the problem may be reduced to the solution of an integro-differential equation of the first order. If the given function is of the second order, the integro-differential equation which we must solve is of the second order and admits of a solution. An arbitrary function always enters in.

As we increase the order of the given function, the problem becomes more and more complicated, hence we shall not go into details on this question as we should be led too far afield. In the general case where the functions are analytic the question has been answered by M. Pérès.*

2. We wish to present some of the properties of permutable functions. The very method which enables us to calculate all of the functions that are permutable with a given function also leads us to the result that if F and Φ are permutable and if F is of the first order, then we must have

$$\frac{\Phi(x, x)}{F(x, x)} = \text{const.}$$

We shall give a rigorous proof of this fact.

* Rendiconti déi Lincei. 1913-14.

We write

$$\int_x^y F(x,\xi)\,\Phi(\xi,y)\,d\xi = \int_x^y \Phi(x,\xi)\,F(\xi,y)\,d\xi\ .$$

Differentiating with respect to y, we have

$$F(x,y)\,\Phi(y,y) + \int_x^y F(x,\xi)\,\Phi_2(\xi,y)\,d\xi$$

$$= \Phi(x,y)\,F(y,y) + \int_x^y \Phi(x,\xi)\,F_2(\xi,y)\,d\xi\ ,$$

and differentiating this last expression with respect to x,

$$F_1(x,y)\,\Phi(y,y) + F(x,x)\,\Phi_2(x,y)$$

$$+ \int_x^y F_1(x,\xi)\,\Phi_2(\xi,y)\,d\xi = \Phi_1(x,y)\,F(y,y)$$

$$- \Phi(x,x)\,F_2(x,y) + \int_x^y \Phi_1(x,\xi)\,F_2(\xi,y)\,d\xi\ .$$

Suppose we put $x = y$. We shall then have

$$F_1(y,y)\,\Phi(y,y) - F(y,y)\,\Phi_2(y,y)$$

$$= \Phi_1(y,y)\,F(y,y) - \Phi(y,y)\,F_2(y,y)\ ,$$

that is

$$[F_1(y,y) + F_2(y,y)]\,\Phi(y,y) =$$

$$F(y,y)\,[\Phi_1(y,y) + \Phi_2(y,y)]\ .$$

Moreover, if we put

$$F(y,y) = f(y)\ , \qquad \Phi(y,y) = \phi(y)\ ,$$

we have

$$F_1(y,y) + F_2(y,y) = f'(y)\ ,$$

$$\Phi_1(y,y) + \Phi_2(y,y) = \phi'(y)\ ,$$

and consequently

$$f'(y)\phi(y) = f(y)\phi'(y) ,$$

whence the theorem.

3. It is a simple matter to find the expansion of any function ψ which is permutable with a function F of the first order.

For by the last theorem

$$\frac{\psi(x, x)}{F(x, x)} = c_1$$

where c_1 is a constant. The expression

$$\psi(x, y) - c_1 F(x, y)$$

will be of higher order than F and permutable with F.

Now by one of the theorems which we proved in the last lecture, we may write

$$\psi(x,y) - c_1 F(x,y) = \overset{*}{F}\overset{*}{\Phi}_1$$

where Φ_1 will be permutable with F. Then there will be a constant c_2 such that

$$\Phi_1 - c_2 F = \overset{*}{F}\overset{*}{\Phi}_2$$

and consequently we shall have

$$\psi = c_1 F + c_2 \overset{*}{F}{}^2 + \dots$$

If this process can be carried on indefinitely, we shall have under certain conditions an expansion of ψ in terms of F, $\overset{*}{F}{}^2$, ...

4. We shall give a short survey of the results which can be obtained by the introduction of a new symbol. If we put

$$\overset{*}{F}\overset{*}{\Phi} = \psi ,$$

we can write

$$F = \overset{*}{\Phi}{}^{-1}\, \overset{*}{\psi} , \qquad \Phi = \overset{*}{\psi}\, \overset{*}{F}{}^{-1} ,$$

where F^{-1} and Φ^{-1} are merely symbols which do not represent functions but which may be treated as if they did. If the functions are permutable, we can write

$$F = \overset{*}{\psi}\, \overset{*}{\Phi}{}^{-1} = \overset{*}{\Phi}{}^{-1}\, \overset{*}{\psi} ,$$

$$\Phi = \overset{*}{\psi}\, \overset{*}{F}{}^{-1} = \overset{*}{F}{}^{-1}\overset{*}{\psi} ,$$

and if

$$\overset{*}{F}\overset{*}{\Phi}\overset{*}{\Theta} = \chi ,$$

$$\Theta = \overset{*}{F}{}^{-1}\overset{*}{\Phi}{}^{-1}\, \overset{*}{\chi} = \overset{*}{\Phi}{}^{-1}\overset{*}{F}{}^{-1}\, \overset{*}{\chi} ,$$

hence the symbols Φ^{-1} and F^{-1} are themselves permutable.

Let us assume that we have permutability. We wish to determine

$$\overset{*}{\Phi}{}^{-1} + \overset{*}{F}{}^{-1} .$$

In other words, let

$$\overset{*}{\Phi}\overset{*}{\Theta}_1 = \psi\,, \qquad \overset{*}{F}\overset{*}{\Theta}_2 = \psi\,.$$

We shall then have

$$\overset{*}{F}\overset{*}{\Phi}\overset{*}{\Theta}_1 = \overset{*}{F}\overset{*}{\psi}\,, \qquad \overset{*}{\Phi}\overset{*}{F}\overset{*}{\Theta}_2 = \overset{*}{\Phi}\overset{*}{\psi}\,,$$

and owing to the property of permutability,

$$\overset{*}{F}\overset{*}{\Phi}(\Theta_1 \overset{*}{+} \Theta_2) = (F \overset{*}{+} \Phi)\,\overset{*}{\psi}\,,$$

whence

$$\begin{aligned}\Theta_1 + \Theta_2 &= (\overset{*}{F}\phi)^{-1}\,(F \overset{*}{+} \Phi)\,\overset{*}{\psi} \\ &= (F \overset{*}{+} \Phi)\,(\overset{*}{F}\Phi)^{-1}\,\overset{*}{\psi}\,,\end{aligned}$$

and we may write

$$\overset{*}{\Phi}^{-1} + \overset{*}{F}^{-1} = (F \overset{*}{+} \Phi)\,(F\overset{*}{\Phi})^{-1}\,,$$

that is, *the rule for the sum of two fractions may be applied.*

Thus we see that we can develop as it were an arithmetic for the symbol $\overset{*}{F}^{-1}$ quite analogous to the theory of fractions.

5. We have seen (§ 1) that if Φ and ψ are of the first type and if

$$\Phi(x, x) = \psi(x, x)\,,$$

then a function $\psi\,(x, y)$ may always be found such that

$$\overset{*}{\Phi}\overset{*}{\chi} = \overset{*}{\chi}\overset{*}{\psi}\,.$$

Hence we can write

$$(1) \qquad \chi = \overset{*}{\Phi}{}^{-1}\overset{*}{\chi}\overset{*}{\psi} = \overset{*}{\Phi}\overset{*}{\chi}\overset{*}{\psi}{}^{-1},$$

$$\Phi = \overset{*}{\chi}\overset{*}{\psi}\overset{*}{\chi}{}^{-1}.$$

And by solving the equation

$$\overset{*}{\chi}{}^{1}\overset{*}{\Phi} = \overset{*}{\psi}\overset{*}{\chi}{}^{1},$$

we shall have that

$$(2) \qquad \Phi = \overset{*}{\chi}{}'^{-1}\overset{*}{\chi}\overset{*}{\psi}{}',$$

Therefore the two functions Φ and ψ can always be obtained the one from the other by a transformation through the functions χ or χ'.

In particular, if

$$\psi(x, x) = 1,$$

we shall always be able to find

$$(3) \qquad \overset{*}{\chi}\overset{*}{\psi}\overset{*}{\chi}{}^{-1} = 1.$$

The relations (1) and (2) may be obtained even if Φ and ψ are permutable. In this case, χ and χ' do not belong to the group of functions which are permutable with the given ones. In particular, equation (3) may hold even if ψ is permutable with unity.

6. We shall bring these lectures on permutable functions to a close by extending some

of the results which were obtained in the first lecture. (§ 11.)

A function which depends upon all the values of a certain function $f(x)$ between the limits a and b admits of an expansion

$$(4)\quad A_0 + \int_a^b f(x_1)\, F_1(x_1)\, dx_1 + \int_a^b \int_a^b f(x_1)\, f(x_2)\, F_2(x_1, x_2)\, dx_1\, dx_2 + \ldots,$$

provided certain conditions are satisfied; where $F_2(x_1, x_2)$ and $F_3(x_1, x_2, x_3)$, etc., are symmetric functions. The expansion in question corresponds to Taylor's expansion (or to a power series) in ordinary analysis.*

With these facts before us, let $f(x, y \mid \alpha)$ be a set of permutable functions of type one, that is of such a sort that if α be given any two values α_1 and α_2, the two functions thereby obtained will be permutable with one another.

As an example, we give

$$f(x-y \mid \alpha)$$

which has the above properties.

* See: *Leçons sur les équations intégrales et intégro-différentielles.* Paris: Gauthier-Villars. 1913. Chap. I, § VIII. *Leçons sur les fonctions de lignes.* Paris: Gauthier-Villars. 1913. Chap. II. *Lectures delivered at Clark University,* Worcester, Mass., 1912. Third lecture, § IV.

Now, let us write

$$\int_x^y f(x, \xi \,|\, \alpha)\, f(\xi, y \,|\, \beta)\, d\xi = f(x, y \,|\, \alpha, \beta)\,,$$

$$\int_x^y f(x, \xi \,|\, \beta)\, f(\xi, y \,|\, \alpha)\, d\xi = f(x, y \,|\, \alpha, \beta)\,.$$

The function $f(x, y) \,|\, \alpha, \beta)$ is permutable with the original ones.

Again let us write

$$\int_x^y f(x,\xi \,|\, \gamma) f(\xi,y \,|\, \alpha,\beta)\, d\xi = \int_x^y f(x,\xi \,|\, \alpha,\beta)\, f(\xi,y \,|\, \gamma)\, d\xi$$

$$= f(x, y \,|\, \alpha, \beta, \gamma)$$

and so on, and let us suppose that the series (4) is convergent when $|\, f(x) \,|$ is less than a certain quantity. Then if we write the series

$$A_0 + \int_a^b f(x, y \,|\, x_1)\, F_1(x_1)\, dx_1$$
$$+ \int_a^b \int_a^b f(x,y \,|\, x_1,x_2)\, F(x_1,x_2)\, dx_1\, dx_2 + \ldots,$$

it will converge no matter what the absolute value of $f(x, y \,|\, \alpha)$ may be.

Moreover, let us consider the series

$$(5) \quad \Phi(\xi) = f(\xi) + \int_a^b f(x_1)\, F_1(x_1, \xi)\, dx_1$$
$$+ \int_a^b \int_a^b f(x_1)\, f(x_2)\, F_2(x_1, x_2 \,|\, \xi)\, dx_1 dx_2 + \ldots$$

If the determinant of the linear integral equation which we obtain by taking into consideration only the first two terms does not vanish, we can derive $f(\xi)$ as a function of $\Phi(\xi)$ from equation (5), provided $|\Phi(\xi)|$ does not exceed a certain value.*

But let us examine the series

$$\Phi(x,y\,|\,\xi)=f(x,y\,|\,\xi)+\int_a^b f(x,y\,|\,x_1)\,F_1(x,\xi)\,dx_1+\dots$$

Then if Φ is known, we can derive $f(x, y\,|\,\xi)$ in the form of a series which converges no matter what the absolute value of $\Phi(x, y\,|\,\xi)$ may be.

This is the latest theorem which we have derived in the field of research we have been developing.

* *Leçons sur les équations intégrales et intégro-différentielles.* Paris: Gauthier-Villars. 1913. Chap. III, § XVI.