THE THEORY OF PROPORTION

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PREFACE

This little book is the outcome of the effort annually renewed over a long period to make clear to my students the principles on which the Theory of Proportion is based, with a view to its application to the study of the Properties of Similar Figures.

Its content formed recently the subject matter of a course of lectures to Teachers, delivered at University College, under an arrangement with the London County Council, and it is now being published in the hope of interesting a wider circle.

At the commencement of my career as a teacher I was accustomed, in accordance with the then established practice, to take for granted the definition of proportion as given by Euclid in the Fifth Definition of the Fifth Book of his *Elements** and to supply proofs of the other properties of proportion required in the Sixth Book which were valid only when the magnitudes considered were commensurable. Dissatisfied with the results of a method which could have no claim to be considered logical, after trying some other modes of exposition, I turned to the syllabus of the Fifth Book drawn up by the Association for the Improvement of Geometrical Teaching. But again I found this hard to explain, and it was evident that my students could not grasp the method as a whole, even when they were able to understand its steps singly.

After prolonged study I found that, in addition to the difficulty arising out of Euclid's notation, which is a matter of form and not of substance, and the difficulty that Euclid does not sufficiently define ratio, two reasons could be assigned for the great difficulty of his argument.

(1) Of the long array of definitions prefixed to the Fifth Book there are only two which effectively count. One of these, the Fifth, is the test for deciding when two ratios are equal; and the other, the Seventh, is the test for distinguishing

* The substance of the Fifth Book is usually attributed to Eudoxus.

between unequal ratios. They are intimately related, but when once stated they can be treated as independent.

Now it can be seen at once that if the test for deciding when two ratios are equal is a good and sound one, it should be possible to deduce from it all the properties of equal ratios, and in order to obtain these properties it should not be necessary to employ the test for distinguishing between unequal ratios.

But Euclid frequently employs this last-mentioned test, or propositions depending on it, to prove properties of equal ratios. In fact, it is not at all easy for any one trying to follow the course of his argument to see whether it leads naturally to the employment of the Fifth or of the Seventh Definition, or a proposition depending on the Seventh Definition. Euclid's proofs do not run on the same lines, and are so difficult and intricate that they have almost entirely fallen out of use. It will be shown in this book that *all the properties of equal ratios can be proved by the aid of the Fifth Definition, and that the Seventh Definition is not required.*

This is effected, without departing from the spirit or the rigour of Euclid's argument, by assimilating Euclid's proofs of those propositions in which the use of the Seventh Definition is directly or indirectly involved to his proofs of those propositions in which he employs the Fifth Definition only.

(2) I think it will appear to any one who reads this book that it is in a high degree probable that the two assumptions

(i) If $A = B$, then $(A:C) = (B:C)$,

and (ii) If $A > B$, then $(A:C) > (B:C)$

form the real bed-rock of Euclid's ideas, and that he deduced his Fifth and Seventh Definitions from these two fundamental assumptions as his starting-point, but that he finally re-arranged his argument so as to take the Fifth and Seventh Definitions as his starting-point and then deduced the above-mentioned assumptions as propositions.

An argument which does not follow the course of discovery is frequently very difficult to follow. De Morgan, in his *Theory of the Connexion of Number and Magnitude*, gives reasons for thinking that Euclid arrived at the conditions in the Fifth and Seventh Definitions from the consideration of a model representing a set of equidistant columns with a set of equidistant railings in front of them, and the relation between the model and the object it represented. However that may be it cannot, I think, be denied that these definitions appearing at the commencement of Euclid's argument without explanation present grave difficulties to the student. I hope to show that these difficulties can be removed and the whole argument presented in a simple form.

I have given a few geometrical illustrations in this book, some of which are not included in either of the two editions of my book entitled *The Contents of the Fifth and Sixth Books of Euclid's Elements,* published by the Cambridge University Press. I desire, however, to draw special attention to the very beautiful applications of Stolz's *Theorem* (Art. 40) to the proof of the proposition that the areas of circles are proportional to the squares on their radii (Euc. XII. 2), see Art. 61; and also to the proof of the same proposition on strictly Euclidean lines, for both of which I am indebted to my friend Mr. Rose-Innes (see Art. 61*a*). These proofs differ from Euclid's in a most important particular, viz. they do not assume the existence of the fourth proportional to three magnitudes of which the first and second are of the same kind. I think that any one who has tried to understand Euclid's argument will find the proofs here given much simpler and more direct. Euclid uses a *reductio ad absurdum.* As against methods other than Euclid's the infinitesimals are, by the aid of Euclid X. 1, handled in a manner which is far more convincing, at any rate to those who are commencing the study of infinitesimals.

I am aware that in bringing this subject forward, and in suggesting that a treatment of the Theory of Proportion, which is valid when the magnitudes concerned are incommensurable, should be included in the mathematical curriculum, I have immense prejudices to overcome.

On the one hand it is the outcome of all experience in teaching that Euclid's presentation of the subject is beyond the comprehension of most people whether old or young, a view with which I am in complete agreement. The matter is regarded as *res judicata*, and most teachers refuse to look at Euclid's work, or anything claiming kinship with it.

On the other hand, in suggesting any modification of

a 2

Euclid's argument, I have before me the dictum of that great Master of Logic, Augustus de Morgan, who said, "This same book (the Fifth Book of Euclid's *Elements*) and the logic of Aristotle are the two most unobjectionable and unassailable treatises which ever were written," and if that be so the usefulness of my work would be in dispute. What is presented here is a modification of Euclid's method, which requires for its understanding a knowledge of Elementary Algebra. I find no difficulty in explaining the first nine chapters, which form Part I., to students who are commencing the study of the properties of similar figures; and whose intellectual equipment in Geometry includes a knowledge of the subject matter of the first four books of Euclid's *Elements*. As I have ventured to make several criticisms on Euclid's argument, I hope it will not be supposed that I do not appreciate either the magnitude or the ingenuity of the work. Its ingenuity is in fact one of the obstacles, if not the greatest obstacle to its finding a place in the mathematical curriculum. What is claimed for the argument set out here is that an easier road to the same results has been found which is not deficient in rigour to that contained in the Euclidean text. Dedekind says in his *Essays on Number** that it was especially from the Fifth Definition of the Fifth Book that he drew the inspiration which led him to the theory of the "cut" or "section"† in the system of rational numbers, a theory which is fundamental in the Calculus. The propositions in this book furnish a number of easily understood examples of the "cut" and thus prepare the student for the study of irrational numbers in the Calculus. Its subject matter is thus very closely linked with modern ideas and well worthy of study.

The book is arranged in three parts. The first part, Chapters I.–IX., contains an elementary course, which can be explained to any one with average mathematical ability. The fourth, fifth, and sixth chapters should be carefully studied. Any difficulty that there may be in the first part will be found in these chapters. The table of contents gives a clear idea of their subject matter, and the main points that have to be borne in mind in the subsequent argument are summed up in Article

* Translated by Beman, p. 40. † Schnitt.

41. The frequent use of Archimedes' Axiom in this work is of great assistance to students when they enter upon the study of the Calculus.

The second part, Chapters X. and XI., is suitable for students preparing for an Honours Course and for Teachers. It is too difficult for an elementary course, and is not intended for those who are not really interested in mathematical study.

The third part, Chapter XII., is a commentary on the Fifth Book of Euclid's *Elements*, and contains remarks on matters which are of interest to those who are concerned with the history of the ideas involved.

This commentary is not intended to be a complete one, but deals only with some matters which have not been noticed in the earlier chapters. The reader who is interested in this part of the subject should consult Sir T. L. Heath's Edition of Euclid's *Elements.*

My acknowledgments are due to the Syndics of the Cambridge University Press for their courtesy in permitting me to use the methods employed in the two editions of my *Contents of the Fifth and Sixth Books of Euclid's Elements ;* and to the Editor of the *Mathematical Gazette* for permission to use a portion of the material of my Presidential Address to the London Branch of the Mathematical Association, published in the July and October numbers of the *Gazette* for 1912.

I am also under great obligation to De Morgan's *Treatise on the Connexion of Number and Magnitude*, and especially in connection with the matter of Chapter XII. to Sir T. L. Heath's great edition of Euclid's *Elements.*

Some further information will be found in my two papers on the Fifth Book of Euclid's *Elements* in the *Cambridge Philosophical Transactions,* Vol. XVI., Part IV., and Vol. XIX., Part II.

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CONTENTS

PART I

CHAPTER I

ARTICLES 1–3

Magnitudes OF THE SAME KIND.

CHAPTER II

ARTICLES 4–12

Propositions relating to Magnitudes and their Multiples.

CHAPTER III

Articles 13–18

The relations between Multiples of the same Magnitude. Commensurable Magnitudes.

CHAPTER IV

Articles 19–21

Magnitudes OF THE SAME KIND *which are* NOT *Multiples of the same magnitude. Incommensurable Magnitudes.*

CHAPTER V

ARTICLES 22–28

Extension of the Idea of Number.

CHAPTER VI

ARTICLES 29–41

On the Ratios of Magnitudes which have no Common Measure.

CHAPTER IX

Articles 54–57

Properties of Equal Ratios. Third Group of Propositions.

PART II

CHAPTER X

ARTICLES 58–67

Geometrical applications of Stolz's Theorem.

CHAPTER XI

ARTICLES 68–70

Further remarks on Irrational Numbers. The existence of the Fourth Proportional.

PART III

CHAPTER XII

ARTICLES 71–100

Commentary on the Fifth Book of Euclid's Elements.

THE THEORY OF PROPORTION

PART I

CHAPTER I

Articles 1–3

Magnitudes of the same kind.

Article 1

No attempt will be made to give a general definition of the term "Magnitude." It is sufficient to give a number of examples; e.g. lengths, areas, volumes, hours, minutes, seconds, weights, etc., are called magnitudes.

Article 2

It is, however, important to make precise the sense in which the term

"magnitudes *of the same kind*"

will be employed.

Some examples of what is meant will first be given.

All lengths are magnitudes *of the same kind.*

All areas are magnitudes *of the same kind.*

All volumes are magnitudes *of the same kind.*

All intervals of time are magnitudes *of the same kind.*

Article 3

Characteristics of Magnitudes of the same kind.

In the next place the characteristics of magnitudes *of the same kind* will be specified.*

* Stolz's account of the properties of absolute magnitudes in his *Allgemeine Arithmetik,* Erster Theil, page 69, is followed in essentials.

These will be readily admitted if we consider the magnitudes to be segments of lines, or areas, or volumes, or weights, etc.

A system of magnitudes is said to be *of the same kind* when the magnitudes possess the following characteristics :

(1) Any two magnitudes *of the same kind* may be regarded as equal or unequal.

In the latter case one of them is said to be the smaller, and the other the larger of the two.

(2) Two magnitudes *of the same kind* can be added together. The resulting magnitude is a magnitude *of the same kind* as the original magnitudes.

This property makes it possible to form multiples of a magnitude.

For denoting any magnitude by A, then $A+A$ is a magnitude *of the same kind* as A. It will be denoted by $2A$.

Then $2A+A$ is a magnitude *of the same kind* as A. It will be denoted by $3A$. And so on, if r denote any positive integer, $rA+A$ is a magnitude *of the same kind* as A and will be denoted by $(r+1)A$.

The Commutative and Associative Laws apply to the Addition of magnitudes *of the same kind*. So that

$A+B=B+A$. The Commutative Law.

$(A+B)+C=A+(B+C)$. The Associative Law.

These laws can be conveniently illustrated by taking the case in which A, B and C represent lengths.

(3) If A and B be two magnitudes *of the same kind*, and A be greater than B, then another magnitude X *of the same kind* as A and B exists such that

$$B+X=A.$$

This may also be written

$$X=A-B.$$

This can be illustrated by taking for A and B two lengths of which A is the longer. If, then, a length equal to B be cut off from A the remainder left is X.

(4) If A be any magnitude, and n any positive integer whatever, then a magnitude X *of the same kind* as A exists such that

$$nX=A.$$

This may also be written in either of the forms

$$X=\frac{1}{n}A$$

$$\text{or } X=\frac{A}{n}.$$

It can be illustrated by dividing a segment of a straight line into any number of equal parts.

It should be mentioned that if A represent an arc of a circle, although it is not in general possible by the aid of the ruler and compasses to divide A into n equal parts, yet it is assumed that an arc $X=\frac{A}{n}$ does exist.

(5) If A be greater than B, a multiple of B exists which is greater than A.

The fifth characteristic is known as the Axiom of Archimedes. It is not a consequence of the preceding four characteristics.

The following deduction from the above is specially useful in the Theory of Proportion:

If A and B are two magnitudes *of the same kind*, and any multiple whatever of A, say rA, is chosen, and any multiple whatever of B, say sB, is chosen, then one and only one of the alternatives

$$rA>sB,\ rA=sB,\ rA<sB$$

always exists, and it is assumed to be possible to determine which one of these alternatives exists.

CHAPTER II

ARTICLES 4–12

Propositions Relating to Magnitudes and their Multiples.

ARTICLE 4

(IN what follows, magnitudes are denoted by capital letters and positive integers by small letters.)

Prop. I. $n(A+B+C+\ldots)=nA+nB+nC+\ldots.$

Prop. II. $(a+b+c+\ldots)N=aN+bN+cN+\ldots.$

Prop. III. $(r(s))A=r(sA)=s(rA)=(s(r))A.$

Prop. IV. If $A>B$, then $r(A-B)=rA-rB.$

Prop. V. If $a>b$, then $(a-b)R=aR-bR.$

Prop. VI. If $A>B$, then $rA>rB.$
If $A=B$, then $rA=rB.$
If $A<B$, then $rA<rB.$

Conversely. If $rA>rB$, then $A>B.$
If $rA=rB$, then $A=B.$
If $rA<rB$, then $A<B.$

Prop. VII. If $a>b$, then $aR>bR.$
If $a=b$, then $aR=bR.$
If $a<b$, then $aR<bR.$

Conversely. If $aR>bR$, then $a>b.$
If $aR=bR$, then $a=b.$
If $aR<bR$, then $a<b.$

Prop. VIII. If X, Y, Z are magnitudes *of the same kind*, and if $X>Y+Z$, then an integer t exists such that $X>tZ>Y$.

Corollary. If A, B, C are magnitudes *of the same kind*, and $A>B$, then integers n, t exist such that $nA>tC>nB$.

ARTICLE 5

Prop. I. (Euc. V. 1.)

$$n(A+B+C+\ldots)=nA+nB+nC+\ldots.$$

The simplest case of this is

$$n(A+B)=nA+nB.$$

For a rigid deduction of this from the Associative and Commutative Laws I refer to my edition of *Euclid V and VI*, 2nd edition, pp. 125–6. It is tedious, and the beginner should not be stopped at this stage with it. It is sufficient to say that the effect of the Associative and Commutative Laws is this, that when any number of magnitudes are to be added together, they may be arranged in any order and grouped in any way, the magnitudes in each group may be first added together, and then finally the sum of the groups can be found, and that the result so obtained will always be the same.

Thus $n(A+B)$ is the sum of n groups, each of which is $A+B$.

The magnitude A occurs n times; and therefore taking these together, their sum is nA.

The magnitude B occurs n times; and taking these together, their sum is nB.

The sum of the two groups is $nA+nB$.

$$\therefore\ n(A+B)=nA+nB.$$

If on both sides B be replaced by C, and then A by $A+B$, it follows that

$$n((A+B)+C)=n(A+B)+nC.$$
$$\therefore n(A+B+C)=nA+nB+nC.$$

Proceeding in this way, it follows that

$$n(A+B+C+\ldots)=nA+nB+nC+\ldots.$$

Article 6

Prop. II. (Euc. V. 2.)

$$(a+b+c+\ldots)N=aN+bN+cN+\ldots$$

The simplest case of this is

$$(a+b)N=aN+bN.$$

Now $(a+b)N$ means that N is taken $a+b$ times.

Group the first a N's together. Their sum is aN.

Group the remaining b N's together. Their sum is bN.

$$\therefore\ (a+b)N=aN+bN.$$

If on both sides b be replaced by c, and then a by $(a+b)$ it follows that

$$((a+b)+c)N=(a+b)N+cN$$
$$\therefore\ (a+b+c)N=aN+bN+cN.$$

Proceeding in this way it follows that

$$(a+b+c+\ldots)N=aN+bN+cN+\ldots.$$

For a rigid deduction of the proposition from the Associative and Commutative Laws see my *Euclid V. and VI.*, 2nd edition, p. 127.

Article 7

Prop. III. $\qquad (r(s))A=r(sA)=s(rA)=(s(r))A.$

In Prop. I., suppose that each of the magnitudes $B, C, \ldots$ is equal to A, and that there are, including A, altogether s magnitudes.

Then $\qquad n(A+B+C+\ldots)$ is $n(sA)$,

and $\qquad nA+nB+nC+\ldots$ is $s(nA)$.

$$\therefore\ n(sA)=s(nA).$$

Or replacing n by r, $\quad r(sA)=s(rA)\ \ldots\ldots\ldots\ldots\ldots\ldots$(I).

Next in Prop. II. suppose that each of the integers $a, b, c, \ldots$ is equal to s; and that there are r such integers.

Then $\qquad (a+b+c+\ldots)N$ becomes $(r(s))N$,

and $\qquad aN+bN+cN+\ldots$ becomes $r(sN)$.

$$\therefore\ (r(s))\,N=r(sN),$$

or replacing N by A, $\qquad (r(s))\,A=r\,(sA)\ \ldots\ldots\ldots$(II).

Interchanging s and r, $\qquad (s(r))A=s(rA)\ldots\ldots\ldots$(III).

Then from (I), (II), (III) it follows that

$$(r(s))A=r(sA)=s(rA)=(s(r))A.$$

Corollary : $\qquad s[\{n(r)\}A]=r[\{n(s)\}A]$

To prove this, observe that

$$\{n(r)\}A=n(rA)=r(nA)$$
$$\begin{aligned}\therefore\ s[\{n(r)\}A]&=s[r(nA)]\\ &=r[s(nA)]\\ &=r[\{n(s)\}A]\end{aligned}$$

This Corollary is not required until Propositions XVIII. and XX. are reached (see Arts. 51, 53).

The effect of Prop. III. and the Corollary amount to this, that the factors of a product when they are all positive integers may be taken in any order and grouped in any way.

Beginners will find the Corollary a little difficult, and too much time ought not to be spent on it. It is enough to call attention to the effect of the Proposition and Corollary as just stated.

Article 8

Prop. IV. If $A > B$, then $r(A-B) = rA - rB$. (Euc. V. 5.)

Since $A > B$, then by Art. 3 (3) a magnitude C exists such that

$$A = B + C,$$
$$\therefore rA = rB + rC,$$
$$\therefore rC = rA - rB,$$
$$\text{but } C = A - B,$$
$$\therefore r(A-B) = rA - rB.$$

Article 9

Prop. V. If $a > b$, then $(a-b)R = aR - bR$. (Euc. V. 6.)

Since a, b are integers, and $a > b$, an integer c exists such that

$$a = b + c,$$
$$\therefore aR = (b+c)R = bR + cR \text{ (Prop. II.)}$$
$$\therefore cR = aR - bR,$$
$$\text{but since } a = b + c,$$
$$\therefore c = a - b,$$
$$\therefore (a-b)R = aR - bR.$$

Article 10

Prop. VI.
If $A > B$, then $rA > rB$,
If $A = B$, then $rA = rB$,
If $A < B$, then $rA < rB$.

Conversely,
If $rA > rB$, then $A > B$,
If $rA = rB$, then $A = B$,
If $rA < rB$, then $A < B$.

If $A > B$, then, as in Prop. IV.,

$$rA = rB + rC, \text{ where } A = B + C,$$
$$\therefore rA > rB.$$

If $A = B$, then rA means $(A + A + \ldots \text{ to } r \text{ terms})$

$$= (B + B + \ldots \text{ to } r \text{ terms})$$
$$= rB,$$
$$\therefore rA = rB.$$

If $A < B$, then $B > A$,

and $\therefore$ by the first case

$$rB > rA,$$
$$\therefore rA < rB.$$

The Converse Proposition follows as a logical consequence of the preceding.

Take for example the first part.

If $rA > rB$,

then since A and B are supposed to be *of the same kind*, one of the three alternatives must hold :

$$A > B, \text{ or } A = B, \text{ or } A < B.$$

If $A = B$, then $rA = rB$, by what has been shown already, which is contrary to the hypothesis that $rA > rB$.

Hence A is not equal to B.

If $A < B$, then $rA < rB$, by what has been shown already, which is contrary to the hypothesis that $rA > rB$.

Hence A is not less than B.

Consequently A is greater than B.

The remaining cases can be proved in like manner.

Article 11

Prop. VII. If $a > b$, then $aR > bR$,
If $a = b$, then $aR = bR$,
If $a < b$, then $aR < bR$.

Conversely, If $aR > bR$, then $a > b$,
If $aR = bR$, then $a = b$,
If $aR < bR$, then $a < b$.

If $a > b$, then, as in Prop. V.,

$$aR = bR + cR, \text{ where } a = b + c,$$
$$\therefore aR > bR.$$

If $a=b$, then aR means $(R+R+\ldots.$ to a terms)

$$=(R+R+\ldots. \text{ to } b \text{ terms})$$
$$=bR,$$
$$\therefore aR=bR.$$

If $a<b$, then $b>a$,

$$\therefore \text{ by the first case}$$
$$bR>aR,$$
$$\therefore aR<bR.$$

The Converse part of the Proposition follows as a logical consequence from the preceding, as in Prop. VI.

Article 12

Prop. VIII. If X, Y, Z are magnitudes *of the same kind*, and if $X>Y+Z$, then an integer t exists such that

$$X>tZ>Y.$$

Corollary. If A, B, C are magnitudes *of the same kind*, and if $A>B$, then integers n, t exist such that

$$nA>tC>nB.$$

$$\text{Since } X>Y+Z,$$
$$\therefore X>Z.$$

It may be that X is also greater than $2Z$ or $3Z$ or $4Z$, and so on.

Suppose that tZ is the greatest multiple of Z which is less than X.

Then $(t+1)Z$ must be either greater than X or equal to X.

If $(t+1)Z>X$, then since $X>Y+Z$, $\therefore (t+1)Z>Y+Z$, $\therefore tZ>Y$.	If $(t+1)Z=X$, then since $X>Y+Z$, $\therefore (t+1)Z>Y+Z$, $\therefore tZ>Y$.

Hence in both cases $tZ>Y$.

But also $tZ<X$.

Consequently $X>tZ>Y$.

This proposition is not an easy one for beginners to grasp. It may be illustrated graphically thus :

Suppose that X, Y, Z are lengths.

On one side of a straight line mark off a length $OB = Y$, and $BC = Z$.

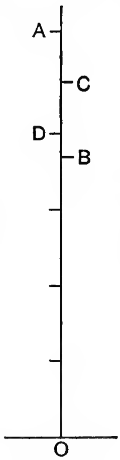

Fig. 1.

On the other side mark off $OA = X$.

$$\text{Then since} \quad X > Y + Z,$$
$$\therefore\ OA > OB + BC,$$
$$\therefore\ OA > OC.$$

If now, starting from O, successive lengths equal to BC be marked off, one of the markings must fall between B and A, because $BA > Z$. Let this marking be D. Let OD be equal to $t(BC)$, i.e. tZ.

$$\text{Then since} \quad OA > OD > OB,$$
$$\therefore\ X > tZ > Y.$$

To prove the Corollary, observe that since $A > B$, $\therefore\ A - B$ is a magnitude *of the same kind* as A and B, and therefore *of the same kind* as C.

Hence, by Archimedes' Axiom an integer n exists such that

$$n(A - B) > C,$$
$$\therefore\ nA - nB > C,$$
$$\therefore\ nA > nB + C.$$

Putting, in Prop. VIII.,

$$X = nA,\ Y = nB,\ Z = C,$$

it follows that an integer t exists such that

$$nA > tC > nB.$$

CHAPTER III

Articles 13–18

The Relations between Multiples of the same Magnitude. Commensurable Magnitudes.

Article 13

If two magnitudes are multiples of the same magnitude, they may be said to be measured by that magnitude. Thus lengths of 7 feet and 13 feet can be exactly measured by an *undivided* foot rule.

These two lengths are said to have a common measure, viz. 1 foot, and are called commensurable.

If a length of 2 feet and a length of 1 foot be taken, the first is said to be *twice* as great as the second, whilst the second is said to be *half* as great as the first.

Thus if these two lengths be considered, not separately, but in relation to one another, they determine two numbers, viz. 2 and $\frac{1}{2}$.

Note that in each case from the two lengths *and the order in which they are taken*, a number which is not a length has been determined, and that the unit in terms of which the lengths are measured does not appear in the result.

In this case it is said that

The ratio of 2 feet to 1 foot is 2.
,, ,, ,, 1 foot to 2 feet is $\frac{1}{2}$.

Similarly:

The ratio of 3 inches to 1 inch is 3.
,, ,, ,, 3 inches to 2 inches is $\frac{3}{2}$.
,, ,, ,, 2 inches to 3 inches is $\frac{2}{3}$.
,, ,, ,, 3 yards to 2 yards is $\frac{3}{2}$.
,, ,, ,, 2 yards to 3 yards is $\frac{2}{3}$.

The ratio of 3 yards to 2 feet
$=$the ratio of 9 feet to 2 feet$=\frac{9}{2}$.
The ratio of 2 yards to 3 feet
$=$the ratio of 6 feet to 3 feet$=\frac{6}{3}=2$.
The ratio of 5 miles to 7 miles$=\frac{5}{7}$.
The ratio of 5 miles to 7 furlongs
$=$the ratio of 40 furlongs to 7 furlongs$=\frac{40}{7}$.
The ratio of 7 minutes to 105 seconds
$=$the ratio of 420 seconds to 105 seconds$=\frac{420}{105}=4$.
The ratio of 13 hours to 2 days
$=$the ratio of 13 hours to 24 hours$=\frac{13}{24}$.

In all these cases the unit in terms of which the magnitudes are measured does not appear in the result. [When there are two units, those units are magnitudes of the same kind, e.g. an hour and a day, and the two magnitudes are expressible in terms of the same unit.]

Similarly it may be said that

The ratio of $3A$ to $2A=\frac{3}{2}$.
" " " $2A$ to $3A=\frac{2}{3}$.
" " " rA to $A=\frac{r}{1}=r$.
" " " nA to $rA=\frac{n}{r}$.
" " " rA to $nA=\frac{r}{n}$.

Thus if two multiples of the same magnitude are given, *and the order in which they are taken is fixed*, then these particulars determine a number.

This number is the quotient of one positive whole number by another. It is usually called a vulgar fraction. All positive and negative whole numbers or fractions, which are quotients of one whole number by another, are called rational numbers, but we shall only have to deal with those which are positive in this book.

The usual notation for the ratio of X to Y is

$$X : Y,$$

and consequently the ratio of Y to X is denoted by

$$Y : X.$$

It is advisable to write these in brackets, thus

$$(X : Y) \text{ and } (Y : X),$$

because beginners who have not grasped the idea that the whole symbol represents a single number not infrequently imagine that $X : Y$ still represents the two distinct things X and Y.

Geometrical Illustrations

Article 14

(i.) There are two parallelograms on bases 3 inches and 2 inches respectively.

The height of each parallelogram is 1 inch.

Prove that the ratio of the areas of the parallelograms is equal to the ratio of the lengths of the bases.

Let the parallelograms be $ABCD$ and $EFGH$.

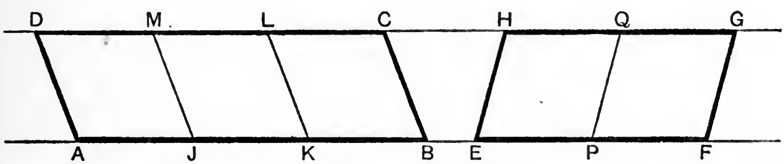

Fig. 2.

Since they have the same height they may be placed between the same parallels as in the figure.

Let the base AB represent 3 inches, and the base EF 2 inches.

Then $(AB : EF) = \frac{3}{2}$.

On AB mark off $AJ = JK = KB$ to represent 1 inch, and on EF mark off $EP = PF$ to represent 1 inch.

Draw JM, KL parallel to AD, and PQ parallel to EH.

Then the five parallelograms

$$AJMD,\ JKLM,\ KBCL,\ EPQH,\ PFGQ$$

are equal in area, for they stand on equal bases and are between the same parallels.

$$\begin{aligned} \text{Consequently}\quad ABCD &= 3(AJMD), \\ EFGH &= 2(AJMD), \\ \therefore\ (ABCD : EFGH) &= \tfrac{3}{2}, \\ \text{but}\quad (AB : EF) &= \tfrac{3}{2}, \\ \therefore\ (ABCD : EFGH) &= (AB : EF). \end{aligned}$$

Article 15

(ii.) There are two triangles on bases 4 inches and 3 inches respectively.

The height of each triangle is 2 inches.

It can be shown as in the last example that the ratio of the lengths of the bases of the triangles is $\frac{4}{3}$, and also that the ratio of the areas of the triangles is $\frac{4}{3}$.

Hence the ratio of the areas of the triangles is equal to the ratio of the lengths of their bases.

Article 16

(iii.) In two equal circles there are arcs whose lengths are $5A$ and $7A$ respectively, A representing the length of a certain arc.

Suppose that the arc A subtends an angle a at the centre of either circle; then the arc $5A$, being divisible into 5 equal parts, each of which subtends an angle a at the centre of its circle, will subtend an angle $5a$ at that centre.

Similarly the arc $7A$ subtends an angle $7a$ at the centre of its circle.

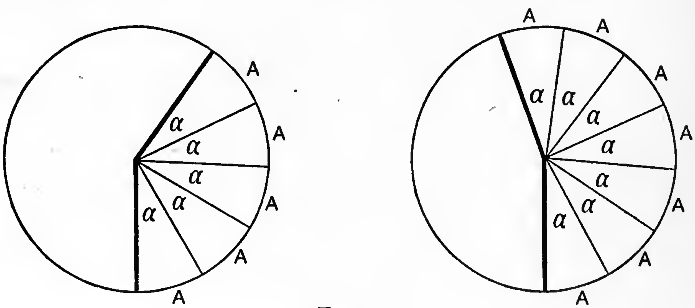

Fig. 3.

Now the ratio of the arcs$=(5A : 7A)=\frac{5}{7}$.

The ratio of the angles subtended by the arcs

$$=(5a : 7a)=\frac{5}{7}.$$

Hence the ratio of the arcs is equal to the ratio of the angles they subtend at the centre.

It will be noticed that in each of these examples the particular numbers which occur, viz. 3 and 2 in the first, 4 and

3 in the second, 5 and 7 in the third, do not appear in the final result, which is a general proposition having no apparent connection with the numbers that occur. And in fact each of these propositions can be generalised.

It will be sufficient to take the first.

Article 17

Two parallelograms, situated between the same parallels, have commensurable bases, to prove that the ratio of the area of the first parallelogram to the area of the second parallelogram is equal to the ratio of the length of the base of the first parallelogram to the length of the base of the second parallelogram.

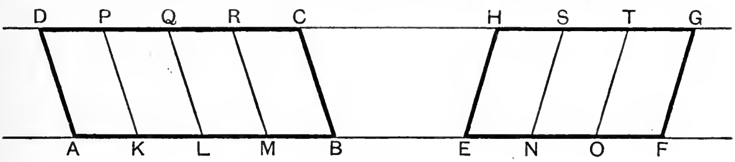

Fig. 4.

Let the parallelograms $ABCD$, $EFGH$ have their bases AB, EF commensurable.

Let AK be a common measure of AB, EF.

$$\text{Suppose that}\quad AB=r(AK),$$
$$EF=s(AK).$$

Let AB, EF be divided as in the figure into parts each equal to AK, and through the points of division of AB let straight lines be drawn parallel to AD; and through the points of division of EF let straight lines be drawn parallel to EH, so that each parallelogram is divided up into equal parallelograms.

Since the bases of all these parallelograms are equal, and they are situated between the same parallels, they are equal in area.

Since AB contains r lengths each equal to AK, therefore the parallelogram $ABCD$ contains r parallelograms each equal to $AKPD$.

$$\therefore\ ABCD=r(AKPD).$$

$$\text{Also } EF=s(EN)=s(AK).$$

Thus EF contains s lengths each equal to AK,

$\therefore$ the parallelogram $EFGH$ contains s parallelograms each equal to $AKPD$,

$$\therefore EFGH = s(AKPD).$$

$$\text{Since } AB = r(AK),$$

$$\text{and } EF = s(AK);$$

$$\therefore (AB : EF) = \frac{r}{s}.$$

$$\text{Since } ABCD = r(AKPD),$$

$$\text{and } EFGH = s(AKPD);$$

$$\therefore (ABCD : EFGH) = \frac{r}{s}.$$

$$\therefore (ABCD : EFGH) = (AB : EF).$$

Another step may now be taken.

Article 18

Suppose that there are three magnitudes A, B, C which are all multiples of the same magnitude G.

Let $A = aG$, $B = bG$, $C = cG$, where a, b, c are some positive integers.

Then the ratio of A to C,

i.e. of aG to cG,

is by definition $\frac{a}{c}$.

Similarly the ratio of B to C is $\frac{b}{c}$.

Now if $A > B$,	If $A = B$,	If $A < B$,
then $aG > bG$,	then $aG = bG$,	then $aG < bG$,
$\therefore a > b$,	$\therefore a = b$,	$\therefore a < b$ (Prop. VII).
$\therefore \frac{a}{c} > \frac{b}{c}$,	$\therefore \frac{a}{c} = \frac{b}{c}$,	$\therefore \frac{a}{c} < \frac{b}{c}$.
$\therefore (A : C) > (B : C)$.	$\therefore (A : C) = (B : C)$.	$\therefore (A : C) < (B : C)$.

Hence if A, B, C are multiples of the same magnitude,

Then, if $A > B$,	If $A = B$,	If $A < B$,
$(A : C) > (B : C)$.	then $(A : C) = (B : C)$.	then $(A : C) < (B : C)$.

These results have been obtained on the hypothesis that A, B, C are multiples of the same magnitude.

It will be noticed that the fact that A, B, C are multiples of the same magnitude does not appear *plainly* in the statement of the results. It appears *indirectly* because only in this case has the meaning of the symbols $(A : C)$ and $(B : C)$ been defined.

Let it be supposed that A, B, C represent any lengths, then, if $A = B$, the magnitude of A compared with that of C is the same as that of B compared with that of C. This idea is fundamental.

It will be noticed that it does not involve the condition that A, B, C have a common measure. The result is expressed by saying that if $A = B$, then the ratio of A to C is equal to that of B to C, or more shortly $(A : C) = (B : C)$, and it will be proved later that the statement has a meaning when A, B, C have no common measure.

If, next, A be greater than B the magnitude of A compared with that of C is greater than that of B compared with that of C. This idea, too, is fundamental. It will be noticed that it does not involve the condition that A, B, C have a common measure. The result is expressed by saying that if A be greater than B, then the ratio of A to C is greater than that of B to C, or more shortly $(A : C) > (B : C)$, and it will be proved later that the statement has a meaning when A, B, C have no common measure.

I think it must be evident to any one who compares these two fundamental ideas with the Fifth and Seventh Definitions of Euclid's Fifth Book that they are of a far simpler nature than those definitions, and that they must have formed the starting-point from which the book was built up.

CHAPTER IV

ARTICLES 19–21

Magnitudes OF THE SAME KIND *which are* NOT *Multiples of the same Magnitude. Incommensurable Magnitudes.*

ARTICLE 19

IT will now be shown that if two magnitudes *of the same kind* be chosen at random they *may* not have a common measure.

To prove this all that is necessary is to show that it is possible to choose some two magnitudes *of the same kind*, out of the infinite number that exist, which have no common measure.

ARTICLE 20

Take the diagonal and side of a square.

If possible let them have a common measure, viz. a length L. Let the side of the square be p times L, and let the diagonal of the square be q times L, where p, q are some positive whole numbers.

Now the square on the diagonal has twice the area of the square on the side.

$$\therefore q^2 = 2p^2.$$

If p, q have a common factor, let their greatest common factor be g.

Let $\frac{q}{g} = r$, $\frac{p}{g} = s$.

Then r, s have no common factor and $r^2 = 2s^2$.

As r, s have no common factor they cannot both be even, and therefore the following cases only need be considered :

(i.) r odd, s odd.

(ii.) r odd, s even.

(iii.) r even, s odd.

In the first and second cases r^2 is odd, and cannot therefore be equal to $2s^2$, which is even. Hence the first and second alternatives cannot hold.

In the third case put $r=2m$.

This gives $2m^2=s^2$.

But $2m^2$ is even, and s^2 is odd.

Hence the third alternative cannot hold.

Hence the equation $r^2=2s^2$ cannot hold.

Consequently the side and diagonal of a square have no common measure.

It has thus been proved that magnitudes *of the same kind* exist which have no common measure.

ARTICLE 21

Having reached this result the question arises:

If two magnitudes have no common measure, can one of them have a ratio to the other, and if so how is it to be measured?

Let us return to the study of the case of the diagonal and side of a square and see where it leads us.

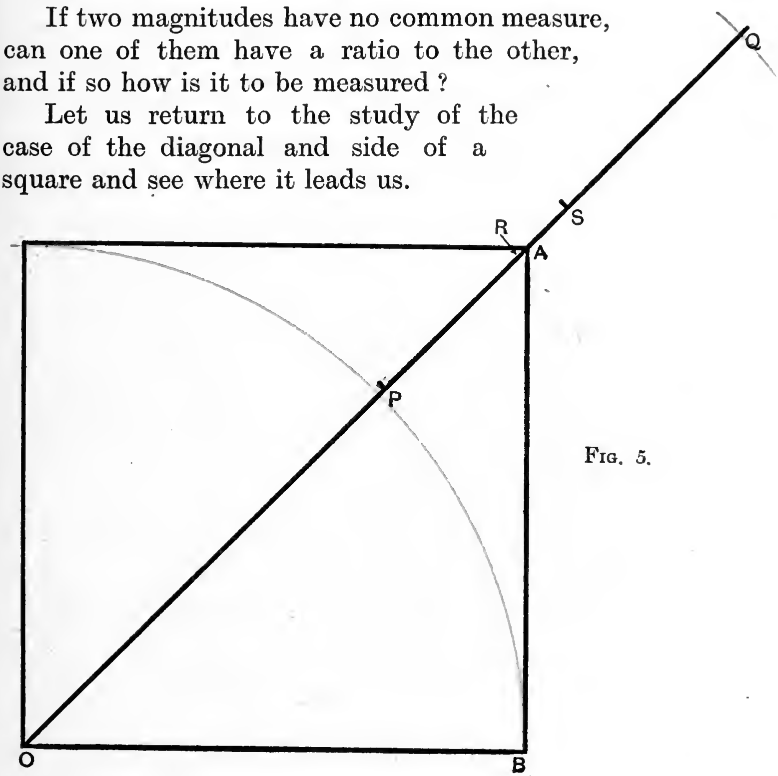

FIG. 5.

Suppose that OA is the diagonal and OB the side of the square.

On OA measure off $OP=OB$, and $OQ=2OB$.

It will be found that

$$OP<OA<OQ.$$

$$\text{Now}\quad (OP:OB)=(OB:OB)=1,$$
$$(OQ:OB)=(2OB:OB)=2.$$

Next divide OB into tenths, and set off tenths of OB along the diagonal OA.

If $OR=14\left(\frac{OB}{10}\right)$ and $OS=15\left(\frac{OB}{10}\right)$ it will be found that

$$OR<OA<OS.$$

$$\text{Now } (OR:OB)=\left(14\left(\frac{OB}{10}\right):10\left(\frac{OB}{10}\right)\right)=\frac{14}{10},$$
$$(OS:OB)=\left(15\left(\frac{OB}{10}\right):10\left(\frac{OB}{10}\right)\right)=\frac{15}{10}.$$

Next divide OB into hundredths and set off hundredths of OB along OA.

If $OT=141\left(\frac{OB}{100}\right)$ and $OU=142\left(\frac{OB}{100}\right)$, it will be found that

$$OT<OA<OU.$$

$$\text{Now } (OT:OB)=\left(141\left(\frac{OB}{100}\right):100\left(\frac{OB}{100}\right)\right)=\frac{141}{100},$$
$$(OU:OB)=\left(142\left(\frac{OB}{100}\right):100\left(\frac{OB}{100}\right)\right)=\frac{142}{100}.$$

This process may be continued indefinitely.

It is possible to construct a series of steadily increasing lengths

$$OP, OR, OT, \ldots .$$

approaching closer and closer in length to the diagonal OA, but never actually reaching it, each of them having a ratio to OB; and also a series of steadily decreasing lengths

$$OQ, OS, OU, \ldots .\text{*}$$

approaching closer and closer in length to the diagonal OA,

* The scale on which the figure is drawn is too small for the insertion of the points T, U.

but never actually reaching it, and each of them having a ratio to OB.

This process never comes to an end, because however much OB may be subdivided, whether into tenths, hundredths, etc., or into *any* number of equal parts, no part of OB can ever be found of which OA is a multiple, as has just been proved in Art. 20.

It would be incorrect to conclude from these facts that OA has not a ratio to OB.

All that they justify is that if OA has a ratio to OB it is not a rational number. If, then, OA has a ratio to OB, and this ratio is a number of some kind, then there must be numbers which are not rational.

CHAPTER V

Articles 22–28

Extension of the Idea of Number.

Article 22

We are thus led to enquire whether it is possible to widen the idea of number so as to include numbers which are not rational. Such numbers will be shown to exist, and when they have been defined it will be possible to construct a theory of ratio which is applicable when A and B are two magnitudes *of the same kind* which have no common measure, and then it will appear that the ratio of A to B is a number of this kind. Such numbers are called irrational numbers.

The subject will be made clearer by going back to some earlier stages in the extension of the idea of number.

Commencing with the series of positive whole numbers, it is seen that if any two positive whole numbers are *added* together, their sum is also a positive whole number, and no new kind of number is required to express the result of the operation of *addition*.

If, however, any two positive numbers are taken at random, and one is *subtracted* from the other, then the result is not always a positive whole number. In order that it may be possible to express the result of the *subtraction* as a number in all cases, it is necessary to widen the idea of number by introducing the idea of the negative whole number.

If, next, any two positive whole numbers be taken and *multiplied* together the result is always a positive whole number, and no new kind of number is required to express the result of the *multiplication*.

If, however, any two positive numbers be taken and one of them *divided* by the other, the result of the *division* cannot always be expressed by a positive whole number. It cannot in general be expressed as a number at all until the idea of number is widened by introducing the idea of the vulgar fraction.

To express the result of subtracting any positive vulgar fraction from any other it is necessary to introduce the idea of the negative vulgar fraction.

In all these cases the idea of number has been widened by endeavouring to express as numbers the results of certain operations.

Thus, starting from the idea of the positive whole number, the idea of number has gradually been widened so as to include positive vulgar fractions, negative whole numbers and negative vulgar fractions.

Positive whole numbers and positive vulgar fractions may be regarded as magnitudes *of the same kind* in the technical sense explained in Article 3.

Similarly negative whole numbers and negative vulgar fractions may be regarded as magnitudes *of the same kind.*

All these numbers together are said to form

THE SYSTEM OF RATIONAL NUMBERS.

ARTICLE 23

Every number in this system has a definite place WITH REGARD TO *the other numbers,* and provided that all fractions are supposed to be reduced to their lowest terms, each place is occupied by one and only one number.

If $\frac{a}{b}$ and $\frac{c}{d}$ be any two positive numbers in the system, then the rule for determining their order is as follows:

$$\text{Since } \frac{a}{b}=\frac{ad}{bd},$$

$$\text{and since } \frac{c}{d}=\frac{bc}{bd},$$

$\frac{a}{b}$ will be said to precede or be less than $\frac{c}{d}$ if $ad<bc$;

$\frac{a}{b}$ will be said to follow or be greater than $\frac{c}{d}$ if $ad>bc$.

The case in which $ad=bc$ can only occur when either

$\frac{a}{b}$ or $\frac{c}{d}$ or both of them have not been reduced to their lowest terms. In this case they are said to be equal.

Suppose that when reduced to their lowest terms the result is $\frac{e}{f}$. Then both are replaced by the single number $\frac{e}{f}$.*

Article 24†

Let us now put to ourselves the question :

Does anything exist, which is not a RATIONAL *number, which is nevertheless entitled to be ranked as a number?*

If so we may agree that it must be in the technical sense of the words, a magnitude *of the same kind* as the rational numbers.

Now the first of the characteristics of magnitudes *of the same kind*, enumerated in Chapter I (Article 3 (1)) is this :

"Any two magnitudes *of the same kind* may be regarded as equal or unequal.

"In the latter case one of them is said to be the smaller and the other the larger of the two."

Suppose that $\frac{r}{s}$ is any rational number whatever, and that $\bar{i}$ is a magnitude *of the same kind* in the technical sense as the rational numbers, but yet is not a rational number.

Thus $\bar{i}$ and $\frac{r}{s}$ are in the technical sense magnitudes *of the same kind.*

Hence either $\bar{i}$ is equal to $\frac{r}{s}$, or $\bar{i}$ is not equal to $\frac{r}{s}$.

Now $\bar{i}$ cannot be equal to $\frac{r}{s}$, for then $\bar{i}$ would be a rational number contrary to the hypothesis.

$$\text{Hence } \bar{i} \text{ is not equal to } \frac{r}{s},$$

$$\text{and } \therefore \text{ either } \bar{i} > \frac{r}{s} \text{ or } \bar{i} < \frac{r}{s}.$$

* The negative numbers precede the positive numbers, and are arranged according to the following rule :

$$\text{If } \frac{a}{b} \text{ precede } \frac{c}{d},$$
$$\text{then } -\frac{c}{d} \text{ precedes } -\frac{a}{b}.$$
$$\text{These may be written } \frac{a}{b} < \frac{c}{d},$$
$$\text{and } -\frac{c}{d} < -\frac{a}{b}.$$

For the purposes of this book the negative numbers will not be required.

† For a more complete treatment of this subject see Chapter XI., Arts. 68–69

In order that this result may be of any use, it is necessary to have the means of deciding whether $\bar{i}$ is greater or less than any rational number whatever.

Article 25

As a particular case take the number known as the square root of 2. If it be defined as that number whose square is 2, then it is plainly not a rational number $\frac{r}{s}$, for it has been shown that no integers r and s exist such that $r^2=2s^2$, and therefore there is no rational number $\frac{r}{s}$ such that

$$\left(\frac{r}{s}\right)^2=2.$$

Let us now consider the process for finding approximately the square root of 2. In essence it is a process for obtaining rapidly the results described below.

Commencing with the natural numbers 1, 2, ... we find that 2 lies between 1^2 and 2^2; we say, therefore, that $\sqrt{2}$ lies between 1 and 2. Take next the numbers between 1 and 2 which have a figure in the first place of decimals, and include 1 and 2 as well.

These are 1, 1·1, 1·2, 1·3, 1·4, 1·5, 1·6, 1·7, 1·8, 1·9, 2. Squaring each of these it will be found that 2 lies between $(1{\cdot}4)^2$ and $(1{\cdot}5)^2$.

We say therefore that $\sqrt{2}$ lies between 1·4 and 1·5.

Taking next between 1·4 and 1·5 inclusive the numbers:

1·4, 1·41, 1·42, 1·43, 1·44, 1·45, 1·46, 1·47, 1·48, 1·49, 1·5

it will be found that

2 lies between $(1{\cdot}41)^2$ and $(1{\cdot}42)^2$.

We say, therefore, that

$\sqrt{2}$ lies between 1·41 and 1·42.

This is a process that can be continued indefinitely. It gives us an infinite series of numbers,

1, 1·4, 1·41, 1·414, 1·4142, ...

in ascending order of magnitude whose squares are all less than 2; and we say that $\sqrt{2}$ is greater than each member of this series. It gives us also an infinite series of numbers,

2, 1·5, 1·42, 1·415, 1·4143, ...

in descending order of magnitude whose squares are all greater than 2; and we say that $\sqrt{2}$ is less than each member of this series.

Now suppose $\frac{r}{s}$ is any rational number; then it is known that either

$$r^2<2s^2 \text{ or else } r^2>2s^2,$$

$$\text{and } \therefore \text{ either } \left(\frac{r}{s}\right)^2<2 \text{ or else } \left(\frac{r}{s}\right)^2>2.$$

Then, as in the preceding cases, we shall say if $\left(\frac{r}{s}\right)^2<2$, then $\left(\frac{r}{s}\right)$ is less than the square root of 2; but if $\left(\frac{r}{s}\right)^2>2$, then $\frac{r}{s}$ is greater than the square root of 2.

On this understanding *the square root of* 2 *has a definite place* WITH REGARD TO *the system of rational numbers.* It is not itself a rational number, but whenever any rational number is assigned, it is possible to say whether it is greater or less than the square root of 2. The square root of 2 fills the gap between those rational numbers whose squares are greater than 2 and those whose squares are less than 2. We proceed to generalise the idea we have reached.

ARTICLE 26

An irrational number $\bar{i}$ will be regarded as known, whenever any rule has been given which will make it possible to distinguish those rational numbers which are greater than $\bar{i}$ from those which are less than $\bar{i}$, because this knowledge makes it possible to determine all the properties of $\bar{i}$. The effect of the adoption of such a rule is that

Any irrational number has a definite place with regard to the system of rational numbers. It fills the gap between those rational numbers which are greater than it and those rational numbers which are less than it.

Mode of Distinguishing between Unequal Irrational Numbers.

ARTICLE 27

Suppose next that $\bar{i}$ and $\bar{j}$ are two irrational numbers.

If they are not equal to one another, they occupy different places with regard to the system of rational numbers, and therefore some rational number $\frac{r}{s}$ must fall between them.

$$\text{Hence either } \bar{i} < \frac{r}{s} < \bar{j},$$

$$\text{and then } \bar{i} \text{ is said to be less than } \bar{j};$$

$$\text{or else } \bar{i} > \frac{r}{s} > \bar{j},$$

and then $\bar{i}$ is said to be greater than $\bar{j}$.

Conditions for Equality of Irrational Numbers.

Article 28

If, however, $\bar{i}$ and $\bar{j}$ are the same irrational number, then they have the same place with regard to the system of rational numbers, hence no rational number can lie between them.

$$\text{If } \therefore\ \bar{i} = \bar{j},$$

and if $\frac{r}{s}$ represent any rational number whatever, then

$$\text{if } \bar{i} > \frac{r}{s}, \text{ it is necessary that } \bar{j} > \frac{r}{s};$$

$$\text{but if } \bar{i} < \frac{r}{s}, \text{ it is necessary that } \bar{j} < \frac{r}{s}.$$

Conversely, if $\frac{r}{s}$ represent *any* rational number *whatever*, and if it be known

$$\text{that whenever } \bar{i} > \frac{r}{s}, \text{ then } \bar{j} > \frac{r}{s};$$

$$\text{and whenever } \bar{i} < \frac{r}{s}, \text{ then } \bar{j} < \frac{r}{s},$$

$$\text{then will } \bar{i} = \bar{j}.$$

For since $\frac{r}{s}$ represents *any* rational *number* whatever, the data amount to the statement that no rational number whatever can lie between $\bar{i}$ and $\bar{j}$, and $\therefore\ \bar{i} = \bar{j}$.

CHAPTER VI

ARTICLES 29–41

ON THE RATIOS OF MAGNITUDES WHICH HAVE NO COMMON MEASURE.

Principles on which the Theory of the Ratio of Magnitudes which have no Common Measure is based.

ARTICLE 29

HAVING now explained how irrational numbers are defined the next step is to lay down certain principles upon which a theory of the ratio of magnitudes which have no common measure can be constructed.

It was seen in the third chapter that when A, B, C were magnitudes *of the same kind* which had a common measure, then

(1) if $A > B$, then $(A : C) > (B : C)$; | (2) if $A = B$, then $(A : C) = (B : C)$;

(3)* if $A < B$, then $(A : C) < (B : C)$.

The new theory of ratio is constructed so as to satisfy the above conditions (1), (2), and (3) in all cases, whether A, B, C have or have not a common measure.

Since these conditions are exactly those which held good when A, B, C have a common measure, there will be no contradiction between this and what precedes.

This will have prepared the way for

ARTICLE 30

Prop. IX. Let it be assumed to be true that, when A, B, C

* It is to be observed that condition (3) is not distinct from (1).

are magnitudes *of the same kind, whether they are multiples of the same magnitude or not*,*

(1) if $A > B$, then $(A : C) > (B : C)$; | (2) if $A = B$, then $(A : C) = (B : C)$;

(3) if $A < B$, then $(A : C) < (B : C)$;

then it will be proved that it follows as a logical consequence that

(4) if $(A : C) > (B : C)$, then $A > B$; | (5) if $(A : C) = (B : C)$, then $A = B$;

(6) if $(A : C) < (B : C)$, then $A < B$.

It will be sufficient to take case (4).

In this case $(A : C) > (B : C)$.

Now since A and B are magnitudes *of the same kind*,

$$\therefore A > B \text{ or } A = B \text{ or } A < B.$$

Now if $A = B$, then, by (2), $(A : C) = (B : C)$, which is contrary to the hypothesis that $(A : C) > (B : C)$.

Hence A is not equal to B.

And if $A < B$, then, by (3), $(A : C) < (B : C)$, which is contrary to the hypothesis that $(A : C) > (B : C)$.

Hence A is not less than B, and it was shown that A was not equal to B.

$\therefore A$ must be greater than B.

The other two cases can be proved in like manner.

We shall refer to the assumptions

(1) if $A > B$, then $(A : C) > (B : C)$; | (2) if $A = B$, then $(A : C) = (B : C)$;

(3) if $A < B$, then $(A : C) < (B : C)$,

in what follows as the fundamental assumptions or principles on which the theory of ratio is constructed.

* It will be proved later on that in this case the symbols $(A : C)$ and $(B : C)$ have a meaning.

Since the third is included in the first, they are equivalent to only two assumptions. It will be found that they make it possible to construct a theory of the ratio of magnitudes *of the same kind*, whether they have or have not a common measure, which is consistent with and includes the former theory.

We shall require the following Proposition:

Article 31

Prop. X.

(i.) If $rA > sB$, then is $(A : B) > \frac{s}{r}$;

(ii.) If $rA = sB$, then is $(A : B) = \frac{s}{r}$;

(iii.) If $rA < sB$, then is $(A : B) < \frac{s}{r}$.

Conversely

(iv.) If $(A : B) > \frac{s}{r}$, then is $rA > sB$;

(v.) If $(A : B) = \frac{s}{r}$, then is $rA = sB$;

(vi.) If $(A : B) < \frac{s}{r}$, then is $rA < sB$.

To prove (i.):

By Art. 3 (4)

a magnitude X exists such that

$$B = rX,$$

$\therefore$ if $rA > sB$,

then $rA > s(rX)$,

$\therefore rA > r(sX)$ . . . . Prop. III.

$\therefore A > sX$ . . . . Prop. VI.

$\therefore$ by the fundamental assumption (Art. 29)

$$(A : B) > (sX : B),$$

$$\therefore (A : B) > (sX : rX),$$

$$\therefore (A : B) > \frac{s}{r}.$$

To prove (ii.):

In this case $rA = sB$,

but $B = rX$,

$\therefore rA = s(rX)$,

$\therefore rA = r(sX)$ . . . . Prop. III.

$\therefore A = sX$ . . . . Prop. VI.

$\therefore$ by the fundamental assumption (Art. 29)

$$(A : B) = (sX : rX) = \frac{s}{r}.$$

(It is to be specially noted that if any relation of the form $rA = sB$ exist, then A and B have a common measure. This result will be useful afterwards.)

To prove (iii.):

In this case $rA < sB$,

$B = rX$,

$\therefore rA < s(rX)$,

$\therefore rA < r(sX)$, ... Prop. III.

$\therefore A < sX$. .... Prop. VI.

$\therefore$ by the fundamental assumption (Art. 29)

$(A : B) < (sX : B)$,

$\therefore (A : B) < (sX : rX)$,

$\therefore (A : B) < \frac{s}{r}$.

To prove (iv.):

In this case $(A : B) > \frac{s}{r}$.

Now $B = rX$,

and $\frac{s}{r} = (sX : rX)$

$= (sX : B)$,

$\therefore (A : B) > (sX : B)$,

$\therefore A > sX$, ... Prop. IX.

$\therefore rA > r(sX)$,

$\therefore rA > s(rX)$. .... Prop. III.

$\therefore rA > sB$.

To prove (v.):

In this case $(A : B) = \frac{s}{r}$.

Now $B = rX$,

$\frac{s}{r} = (sX : rX)$

$= (sX : B)$,

$\therefore (A : B) = (sX : B)$,

$\therefore A = sX$, ... Prop. IX.

$\therefore rA = r(sX)$

$= s(rX)$. .... Prop. III.

$\therefore rA = sB$.

To prove (vi.):

In this case $(A : B) < \frac{s}{r}$.

$$\text{Now } B = rX$$
$$\frac{s}{r} = (sX : rX)$$
$$= (sX : B),$$
$$\therefore (A : B) < (sX : B),$$
$$\therefore A < sX, \quad \ldots\ldots\ldots\ldots \quad \text{Prop. IX.}$$
$$\therefore rA < r(sX),$$
$$\therefore rA < s(rX), \quad \ldots\ldots\ldots\ldots \quad \text{Prop. III.}$$
$$\therefore rA < sB.$$

The Ratio of Two Magnitudes OF THE SAME KIND *is a Number Rational or Irrational.*

ARTICLE 32

It is now possible to show that if A and B be two magnitudes *of the same kind*, then the ratio of A to B is a number rational or irrational.

Let any rational number $\frac{t}{u}$ be taken, it will be shown that it is possible to determine whether $(A : B)$ is greater than $\frac{t}{u}$, or equal to $\frac{t}{u}$, or less than $\frac{t}{u}$. For since A and B are magnitudes *of the same kind*, uA and tB are magnitudes *of the same kind*, and hence one of the following alternatives must hold:

(1) $uA > tB$; (2) $uA = tB$; (3) $uA < tB$;

and hence by Prop. X.

$$\text{If } uA > tB, \text{ then } (A : B) > \frac{t}{u}.$$
$$\text{If } uA = tB, \text{ then } (A : B) = \frac{t}{u}.$$
$$\text{If } uA < tB, \text{ then } (A : B) < \frac{t}{u}.$$

Hence it is possible to determine the position of $(A : B)$ with regard to the system of rational numbers.

Hence $(A : B)$ is a number rational or irrational.

Note 1.—In the case in which some integers t, u exist such that $(A : B) = \frac{t}{u}$, then $(A : B)$ is a rational number.

In this case A and B have a common measure (see Prop. X.).

Note 2.—If A and B have no common measure, then no relation of the form

$$uA=tB \text{ or } (A:B)=\frac{t}{u}$$

can exist.

The preceding theory can now be applied to find tests for determining :

(*a*) When two ratios are equal ;
(*b*) When two ratios are unequal.

Equal Ratios

Article 33

If two ratios are equal they occupy the same place with regard to the system of rational numbers.

Hence no rational number can lie between them.

We give the following definition of Equal Ratios :

*Two ratios are said to be equal when no rational number lies between them.**

This may be expressed in greater detail as follows :

Suppose that $(A:B)=(C:D)$.

Let $\frac{r}{s}$ represent *any* rational number *whatever*.

If now $(A:B)$ be compared with $\frac{r}{s}$, one of three alternatives must hold.

(1) $(A:B)>\frac{r}{s}$; (2) $(A:B)=\frac{r}{s}$; (3) $(A:B)<\frac{r}{s}$.

(1) If $(A:B)>\frac{r}{s}$, then in order that $\frac{r}{s}$ may not lie between $(A:B)$ and $(C:D)$ it is necessary that $(C:D)>\frac{r}{s}$.

If $\therefore$ $(A:B)>\frac{r}{s}$, then $(C:D)>\frac{r}{s}$.

(2) If $(A:B)=\frac{r}{s}$, then in order that no rational number may lie between $(A:B)$ and $(C:D)$, it is necessary that $(C:D)=\frac{r}{s}$.

If $\therefore$ $(A:B)=\frac{r}{s}$, then $(C:D)=\frac{}{s}$;

(3) If $(A:B)<\frac{r}{s}$, then in order that $\frac{r}{s}$ may not lie between $(A:B)$ and $(C:D)$, it is necessary that $(C:D)<\frac{r}{s}$.

Hence if $(A:B)<\frac{r}{s}$, then $(C:D)<\frac{r}{s}$.

* This definition does not conflict with what has been said before about equal ratios, see Note in Art. 34.

NOTE ON ARTICLE 33

Article 34

In the preceding work some cases of equal ratios have been considered.

(1) Ratios which were equal to the same rational number were said to be equal.

(2) When $A = B$ it was laid down as a fundamental principle that

$$(A : C) = (B : C).$$

In the first case it is obvious that no rational number can lie between the equal ratios, and it will now be proved that, when $A = B$, no rational number can lie between

$$(A : C) \text{ and } (B : C).$$

For if possible let some rational number $\frac{p}{q}$ lie between $(A : C)$ and $(B : C)$. Let $(A : C)$ be greater than $(B : C)$.

$$\text{Then } (A : C) > \frac{p}{q} > (B : C).$$

$$\text{Since } (A : C) > \frac{p}{q},$$

$$\therefore qA > pC.$$

$$\text{Since } (B : C) < \frac{p}{q},$$

$$\therefore qB < pC,$$

$$\therefore qA > pC > qB,$$

$$\therefore qA > qB,$$

$$\therefore A > B,$$

which is contrary to the hypothesis that $A = B$.

Hence if $A = B$ no rational number can fall between $(A : C)$ and $(B : C)$.

The Test for Equal Ratios

Article 35

Conversely, if *whatever* the integers r and s may be, then

if $(A : B) > \frac{r}{s}$, it is also true that $(C : D) > \frac{r}{s}$;

but if $(A : B) = \frac{r}{s}$, it is also true that $(C : D) = \frac{r}{s}$;

and if $(A : B) < \frac{r}{s}$, it is also true that $(C : D) < \frac{r}{s}$,

then will $(A : B) = (C : D)$,

for these conditions simply express the fact that no rational number lies between $(A : B)$ and $(C : D)$,

$$\text{and } \therefore (A : B) = (C : D).$$

Euclid V., Definition 5

ARTICLE 36

Derivation from the preceding article of the conditions of the Fifth Definition of the Fifth Book of Euclid's *Elements*. This follows immediately by the aid of Prop. X.

If $sA > rB$,

then $(A : B) > \frac{r}{s}$, ...Prop. X.

$\therefore (C : D) > \frac{r}{s}$, ...Art. 35.

$\therefore sC > rD$. ....Prop. X.

If $sA = rB$,

then $(A : B) = \frac{r}{s}$, ...Prop. X.

$\therefore (C : D) = \frac{r}{s}$, ...Art. 35.

$\therefore sC = rD$. ....Prop. X.

If $sA < rB$,

then $(A : B) < \frac{r}{s}$, ...Prop. X.

$\therefore (C : D) < \frac{r}{s}$, ...Art. 35.

$\therefore sC < rD$. ....Prop. X.

Hence, if whatever the integers r, s may be, it is true that

when $sA > rB$, then $sC > rD$; (i.)
but when $sA = rB$, then $sC = rD$; (ii.)
and when $sA < rB$, then $sC < rD$; (iii.)
then is $(A : B) = (C : D)$.

And these are Euclid's conditions.

It will appear in Prop. XI., Art. 40 (Stolz's Theorem) that the set of conditions marked (ii.) is superfluous.

Unequal Ratios

ARTICLE 37

If two ratios are unequal they occupy different places with regard to the system of rational numbers.

Hence some rational number falls between them.

We give the following definition :

*Two ratios are said to be unequal when some rational number falls between them.**

Let $(A : B)$ and $(C : D)$ be two unequal ratios. Then some rational number $\frac{u}{v}$ falls between them, and

$$\text{either } (A : B) > \frac{u}{v} > (C : D),$$
$$\text{or else } (A : B) < \frac{u}{v} < (C : D).$$

In the first case $(A : B)$ is said to be greater than $(C : D)$; in the second case $(A : B)$ is said to be less than $(C : D)$.

In the first case,

$$\text{since } (A : B) > \frac{u}{v},$$
$$\therefore vA > uB;$$
$$\text{and since } (C : D) < \frac{u}{v},$$
$$\therefore vC < uD.$$

This may be put thus :

The Test for distinguishing between Unequal Ratios

ARTICLE 38

If $(A : B) > (C : D)$, then integers u, v exist such that

$$vA > uB, \text{ but } vC < uD.$$

It will be proved at a later stage that these conditions are equivalent to, though they are not the same in form as, the conditions laid down in the Seventh Definition of the Fifth Book of Euclid's *Elements*. For the present these conditions will not be required. They have been mentioned here only on account of their close connection in thought with the preceding work.

In the case where $(A : B) < (C : D)$, some rational number $\frac{u}{v}$ exists such that

$$(A : B) < \frac{u}{v} < (C : D).$$

Hence integers u, v exist such that

$$vA < uB, \text{ but } vC > uD.$$

* This definition does not conflict with what was said before about unequal ratios, see Note in Art. 39.

NOTE ON ARTICLES 37–38

Article 39

In the preceding work a case of unequal ratios was considered. It was laid down as a fundamental principle that

$$\text{if } A > B,$$
$$\text{then } (A : C) > (B : C).$$

In this case it will be proved that a rational number falls between $(A : C)$ and $(B : C)$.

It was shown in the Corollary to Prop. VIII. that if $A > B$, then integers n, t exist such that

$$nA > tC > nB.$$
$$\text{Since } nA > tC,$$
$$\therefore (A : C) > \frac{t}{n}.$$
$$\text{Since } nB < tC,$$
$$\therefore (B : C) < \frac{t}{n},$$
$$\therefore (A : C) > \frac{t}{n} > (B : C).$$

Hence the rational number $\frac{t}{n}$ falls between $(A : C)$ and $(B : C)$.

Simplification of the Test for Equal Ratios (*Stolz's Theorem*)

Article 40

I proceed now to show that the second of the three sets of conditions in the Test for Equal Ratios (Art. 35) is superfluous.

Prop. XI. If all values of r, s which make

$sA > rB$ also make $sC > rD$. . . . . (I.),

and if all values of r, s which make

$sA < rB$ also make $sC < rD$. . . . .(II.),

and if any values of r, s exist, say $r = r_1$, $s = s_1$ which make $s_1A = r_1B$, then must also $s_1C = r_1D$.

By hypothesis $s_1A = r_1B$.
Suppose if possible s_1C is not equal to r_1D.
Hence either

(i.) $s_1C > r_1D$.

$\therefore s_1C - r_1D$ is a magnitude *of the same kind* as D.

$$\therefore \text{ An integer } n \text{ exists such that}$$
$$n(s_1C - r_1D) > D,$$
$$\therefore ns_1C > (nr_1+1)D,$$
$$\text{but } s_1A = r_1B,$$
$$\therefore ns_1A = nr_1B < (nr_1+1)B.$$
$$\text{Hence } ns_1A < (nr_1+1)B,$$
$$\text{but } ns_1C > (nr_1+1)D,$$

and $\therefore$ putting $s = ns_1$, $r = nr_1 + 1$, it is seen that for these values of r, s the hypothesis (II.) is not satisfied.

$$\therefore s_1C \text{ is not greater than } r_1D.$$

Or

(ii.) $s_1C < r_1D$.

$\therefore$ $r_1D - s_1C$ is a magnitude *of the same kind* as D.

$$\therefore \text{ An integer } n \text{ exists such that}$$
$$n(r_1D - s_1C) > D,$$
$$\therefore ns_1C < (nr_1-1)D,$$
$$\text{but } s_1A = r_1B,$$
$$\therefore ns_1A = nr_1B > (nr_1-1)B.$$
$$\text{Hence } ns_1A > (nr_1-1)B,$$
$$\text{but } ns_1C < (nr_1-1)D,$$

and $\therefore$ putting $s = ns_1$, $r = nr_1 - 1$, it is seen that for these values of r, s the hypothesis (I.) is not satisfied.

$$\therefore s_1C \text{ is not less than } r_1D.$$

It has now been shown that s_1C is neither greater nor less than r_1D.

$$\therefore s_1C = r_1D.$$

Hence the second set of conditions in Euclid's Definition is involved in the first and third sets of conditions. It is therefore superfluous.

$$\text{Hence } (A : B) = (C : D),$$

if all values of r, s which make $sA > rB$ also make $sC > rD$, and if all values of r, s which make $sA < rB$ also make $sC < rD$.

Hence if whenever $(A : B) > \frac{r}{s}$, then also $(C : D) > \frac{r}{s}$,
and if whenever $(A : B) < \frac{r}{s}$, then also $(C : D) < \frac{r}{s}$,

whatever integers r, s may be, then will $(A : B)=(C : D)$, and it is not necessary to show also that

$$\text{if } (A : B)=\frac{r}{s}, \text{ then } (C : D)=\frac{r}{s}.$$

Magnitudes in Proportion

If $(A : B)=(C : D)$, the magnitudes A, B, C, D are said to be proportionals, or in proportion.

The proportion is usually written thus:

$$A : B :: C : D,$$

and is read "the ratio of A to B is the same as the ratio of C to D."

A and D are called the extremes, B and C the means of the proportion. D is called the fourth proportional to A, B and C.

If $C=B$ so that $(A : B)=(B : D)$, then the three magnitudes A, B, D are said to be in proportion, B is said to be a mean proportional between A and D, and D is said to be a third proportional to A and B.

Recapitulation of the Chief Points of the Preceding Theory

Article 41

Before illustrating the preceding theory it is well to recapitulate the chief points which have to be borne in mind in what follows.

(1) Numbers exist which are not rational numbers. They are called irrational numbers. They are in the technical sense of the words magnitudes *of the same kind* as the rational numbers (Art. 24).

(2) An irrational number is determined when a rule is given which makes it possible to decide whether the irrational number is greater or less than *any* rational number *whatever*. *An irrational number has therefore a definite place amongst the rational numbers* (Art. 26).

(3) If A and B are any two magnitudes *of the same kind*, then the ratio of A to B is a number rational or irrational (Art. 32).

(4) Two ratios are equal when no rational number lies between them (Art. 33).

CHAPTER VII

Articles 42–49

Properties of Equal Ratios. First Group of Propositions.

Article 42

The proofs of the following propositions may be conducted on the same lines. They are independent of one another and may be taken in any order.

Prop. XII. If $(A : B) = (C : D)$,
then $(rA : sB) = (rC : sD)$. ....Euc. V. 4.

Prop. XIII. If $(A : B) = (C : D)$,
then $(B : A) = (D : C)$.
Euc. V. Corollary to 4.

Prop. XIV. If $(A : B) = (C : D) = (E : F)$, and if all the magnitudes are *of the same kind*, then $(A : B) = (A + C + E : B + D + F)$.
Euc. V. 12.

Prop. XV. $(A : B) = (nA : nB)$. ....Euc. V. 15.

Prop. XVI. If $(A : B) = (X : Y)$,
then $(A + B : B) = (X + Y : Y)$. ..Euc. V. 18.

Prop. XVII. If $(A + B : B) = (X + Y : Y)$,
then $(A : B) = (X : Y)$. ......Euc. V. 17.

Article 43

Prop. XII. If $(A : B) = (C : D)$, to show that
$(rA : sB) = (rC : sD)$. ....Euc. V. 4.

Let $\frac{p}{q}$ denote *any* rational number *whatever*, then it follows from Art. 40 that it is sufficient to consider the following alternatives :

Either $(rA : sB) > \frac{p}{q}$,	Or $(rA : sB) < \frac{p}{q}$,
$\therefore q(rA) > p(sB)$,	$\therefore q(rA) < p(sB)$,
$\therefore (q(r))A > (p(s))B$,	$\therefore (q(r))A < (p(s))B$,
$\therefore (A : B) > \frac{p(s)}{q(r)}$.	$\therefore (A : B) < \frac{p(s)}{q(r)}$.
But $(A : B) = (C : D)$,	But $(A : B) = (C : D)$,
$\therefore (C : D) > \frac{p(s)}{q(r)}$,	$\therefore (C : D) < \frac{p(s)}{q(r)}$,
$\therefore (q(r))C > (p(s))D$,	$\therefore (q(r))C < (p(s))D$,
$\therefore q(rC) > p(sD)$,	$\therefore q(rC) < p(sD)$,
$\therefore (rC : sD) > \frac{p}{q}$.	$\therefore (rC : sD) < \frac{p}{q}$.
Hence if $(rA : sB) > \frac{p}{q}$,	Hence if $(rA : sB) < \frac{p}{q}$,
then $(rC : sD) > \frac{p}{q}$.	then $(rC : sD) < \frac{p}{q}$.

Hence no rational number can lie between $(rA : sB)$ and $(rC : sD)$.

$$\therefore (rA : sB) = (rC : sD).$$

Article 44

Prop. XIII. If $(A : B) = (C : D)$,
then $(B : A) = (D : C)$. ......Euc. V. Cor. to 4.

It has to be shown that no rational number falls between $(B : A)$ and $(D : C)$.

Let $\frac{s}{t}$ be *any* rational number *whatever*. Then it is sufficient to consider the alternatives :

If $(B : A) > \frac{s}{t}$,	If $(B : A) < \frac{s}{t}$,
then $tB > sA$,	then $tB < sA$,
$\therefore sA < tB$,	$\therefore sA > tB$,
$\therefore (A : B) < \frac{t}{s}$.	$\therefore (A : B) > \frac{t}{s}$.
But $(A : B) = (C : D)$,	But $(A : B) = (C : D)$,
$\therefore (C : D) < \frac{t}{s}$,	$\therefore (C : D) > \frac{t}{s}$,
$\therefore sC < tD$,	$\therefore sC > tD$,
$\therefore tD > sC$,	$\therefore tD < sC$,
$\therefore (D : C) > \frac{s}{t}$.	$\therefore (D : C) < \frac{s}{t}$.

Hence if $(B : A) > \frac{s}{t}$, then $(D : C) > \frac{s}{t}$.	Hence if $(B : A) < \frac{s}{t}$, then $(D : C) < \frac{s}{t}$.

Hence no rational number lies between $(B : A)$ and $(D : C)$.

$$\therefore (B : A) = (D : C).$$

Article 45

Prop. XIV. If $(A : B) = (C : D) = (E : F)$, and if all the magnitudes are *of the same kind*,

$$\text{then } (A : B) = (A + C + E : B + D + F).$$

Euc. V. 12.

It has to be shown that no rational number lies between

$$(A : B) \text{ and } (A + C + E : B + D + F).$$

Let $\frac{p}{q}$ be *any* rational number *whatever*. Then it is sufficient to consider the alternatives :

$(A : B) > \frac{p}{q}$.	$(A : B) < \frac{p}{q}$.
Then since $(A : B) = (C : D) = (E : F)$,	Then since $(A : B) = (C : D) = (E : F)$,
it follows that $(C : D) > \frac{p}{q}$, $(E : F) > \frac{p}{q}$.	it follows that $(C : D) < \frac{p}{q}$, $(E : F) < \frac{p}{q}$.
Hence $qA > pB$, $qC > pD$, $qE > pF$,	Hence $qA < pB$, $qC < pD$, $qE < pF$,
$\therefore q(A + C + E) > p(B + D + F)$,	$\therefore q(A + C + E) < p(B + D + F)$,
$\therefore (A + C + E : B + D + F) > \frac{p}{q}$.	$\therefore (A + C + E : B + D + F) < \frac{p}{q}$.
Hence if $(A : B) > \frac{p}{q}$, then $(A + C + E : B + D + F) > \frac{p}{q}$.	Hence if $(A : B) < \frac{p}{q}$, then $(A + C + E : B + D + F) < \frac{p}{q}$.

Hence no rational number falls between

$$(A : B) \text{ and } (A + C + E : B + D + F),$$

$$\therefore (A : B) = (A + C + E : B + D + F).$$

In like manner it can be proved that

$$\text{if } (A_1 : B_1) = (A_2 : B_2) = \ldots = (A_n : B_n),$$

and if all the magnitudes are *of the same kind*,

$$\text{then } (A_1 : B_1) = (A_1 + A_2 + \ldots + A_n : B_1 + B_2 + \ldots + B_n).$$

Article 46

Prop. XV. To prove that $(A : B)=(nA : nB)$. . .Euc. V. 15. It has to be shown that no rational number falls between

$$(A : B) \text{ and } (nA : nB).$$

Let $\frac{t}{v}$ be *any* rational number *whatever*. Then it is enough to consider the alternatives :

$(A : B) > \frac{t}{v}$,	$(A : B) < \frac{t}{v}$,
$\therefore vA > tB$,	$vA < tB$,
$\therefore n(vA) > n(tB)$,	$n(vA) < n(tB)$,
$\therefore v(nA) > t(nB)$,	$\therefore v(nA) < t(nB)$,
$\therefore (nA : nB) > \frac{t}{v}$.	$\therefore (nA : nB) < \frac{t}{v}$.
Hence if $(A : B) > \frac{t}{v}$,	Hence if $(A : B) < \frac{t}{v}$,
then $(nA : nB) > \frac{t}{v}$.	then $(nA : nB) < \frac{t}{v}$.

Hence no rational number lies between

$$(A : B) \text{ and } (nA : nB),$$
$$\therefore (A : B)=(nA : nB).$$

Note.—This is a particular case of the preceding proposition, viz. that in which $A_1=A_2=\ldots=A_n=A$,
and $B_1=B_2=\ldots=B_n=B$.

Article 47

Prop. XVI. If $(A : B)=(X : Y)$,
then $(A+B : B)=(X+Y : Y)$. . .Euc. V. 18.

Compare $(A+B : B)$ with *any* rational number *whatever*, $\frac{r}{s}$. It is sufficient to consider the alternatives :

Either $(A+B : B) > \frac{r}{s}$,	Or $(A+B : B) < \frac{r}{s}$,
$\therefore s(A+B) > rB$.	$\therefore s(A+B) < rB$.
It is necessary to take separately the cases $s<r, s=r, s>r$.	In this case s must be $<r$,
(i.) $s<r$, $sA > (r-s)B$,	$\therefore sA < (r-s)B$,
$(A : B) > \frac{r-s}{s}$,	$(A : B) < \frac{r-s}{s}$,
but $(A : B)=(X : Y)$,	but $(A : B)=(X : Y)$,

$\therefore (X : Y) > \frac{r-s}{s}$,
$\therefore sX > (r-s)Y$,
$\therefore s(X+Y) > rY$,
$\therefore (X+Y : Y) > \frac{r}{s}$.

(ii.) $s = r$, $sY = rY$,
$\therefore s(X+Y) > rY$,
$\therefore (X+Y : Y) > \frac{r}{s}$.

(iii.) $s > r$, $sY > rY$,
$\therefore s(X+Y) > rY$,
$\therefore (X+Y : Y) > \frac{r}{s}$.

Hence if $(A+B : B) > \frac{r}{s}$,
then $(X+Y : Y) > \frac{r}{s}$.

$\therefore (X : Y) < \frac{r-s}{s}$,
$\therefore sX < (r-s)Y$,
$\therefore s(X+Y) < rY$,
$\therefore (X+Y : Y) < \frac{r}{s}$.

Hence if $(A+B : B) < \frac{r}{s}$,
then $(X+Y : Y) < \frac{r}{s}$.

Hence no rational number lies between $(A+B : B)$ and $(X+Y : Y)$.

$$\therefore (A+B : B) = (X+Y : Y).$$

ARTICLE 48

Prop. XVII. If $(A+B : B) = (X+Y : Y)$ it is required to prove that $(A : B) = (X : Y)$.........Euc. V. 17.

Compare $(A : B)$ with *any* rational number *whatever*, $\frac{r}{s}$.
Then it is sufficient to consider the alternatives:

(i.) $(A : B) > \frac{r}{s}$.
In this case $sA > rB$,
$\therefore s(A+B) > (r+s)B$,
$\therefore (A+B : B) > \frac{r+s}{s}$,
but $(A+B : B) = (X+Y : Y)$,
$\therefore (X+Y : Y) > \frac{r+s}{s}$,
$\therefore s(X+Y) > (r+s)Y$,
$\therefore sX > rY$,
$\therefore (X : Y) > \frac{r}{s}$.
Hence if $(A : B) > \frac{r}{s}$,
then $(X : Y) > \frac{r}{s}$.

(ii.) $(A : B) < \frac{r}{s}$.
In this case $sA < rB$,
$\therefore s(A+B) < (r+s)B$,
$\therefore (A+B : B) < \frac{r+s}{s}$,
but $(A+B : B) = (X+Y : Y)$,
$\therefore (X+Y : Y) < \frac{r+s}{s}$,
$\therefore s(X+Y) < (r+s)Y$,
$\therefore sX < rY$,
$\therefore (X : Y) < \frac{r}{s}$.
Hence if $(A : B) < \frac{r}{s}$,
then $(X : Y) < \frac{r}{s}$.

Hence no rational number can lie between

$$(A : B) \text{ and } (X : Y),$$
$$\therefore (A : B) = (X : Y).$$

Article 49

As a geometrical illustration take the proposition that the ratio of the areas of two triangles of equal altitudes is equal to the ratio of the lengths of their bases (Euc. VI. 1).

The proof depends on the following:

(1) If two triangles of equal altitudes have equal bases their areas are equal.

(2) If two triangles of equal altitudes have unequal bases the one with the larger base has the larger area.

(3) If two triangles having the same altitude are such that the base of one is equal to r times the base of the other, then the area of the first is equal to r times the area of the second (it being understood that r is some positive whole number).

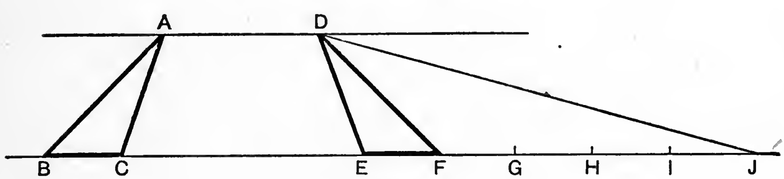

Fig. 6.

Let the triangles ABC, DEJ have the same altitude.

Suppose that $EJ = r(BC)$.

It is required to prove that

$$\triangle DEJ = r(\triangle ABC).$$

EJ can be divided into r pieces each equal to BC.

Suppose that EF, FG, GH, HI, IJ are these r pieces.

Then the triangles DEF, DFG, DGH, DHI, DIJ are each equal to the triangle ABC.

Now EJ contains r parts each equal to BC.

Hence the triangle DEJ is divided into r triangles each equal to ABC.

$$\therefore\ \triangle DEJ = r(\triangle ABC).$$

Proceeding now to the proposition.

Let ABC and DEF be any two triangles having equal altitudes, and let BC, EF be their bases. It is required to prove that

$$(BC : EF) = (\triangle ABC : \triangle DEF).$$

It has to be shown that no rational number can fall between the two ratios.

Take *any* rational number *whatever*, $\frac{s}{r}$. If it be compared with $(BC : EF)$, then one of the following alternatives must occur:

(i.) $(BC : EF) > \frac{s}{r}$. (ii.) $(BC : EF) = \frac{s}{r}$. (iii.) $(BC : EF) < \frac{s}{r}$.

It is known by Prop. XI. (Stolz's Theorem), Art. 40, that we need not consider the second alternative.*

Take the first alternative:

$$(BC : EF) > \frac{s}{r},$$
$$\therefore\ r(BC) > s(EF).$$
Now set off $BR = r(BC)$,
and $ES = s(EF)$.

Since the triangles have the same altitude they may be placed between the same parallels.

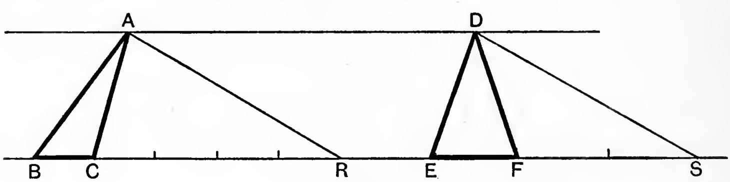

Fig. 7.

Join AR and DS.

Then since $BR = r(BC)$,
$\therefore$ by (3) above $\triangle ABR = r(\triangle ABC)$,
and since $ES = s(EF)$,
$\therefore$ by (3) above $\triangle DES = s(\triangle DEF)$.

* There is no difficulty in considering it in this case, as there is in some of the propositions which follow.

$$\text{Now } r(BC) > s(EF),$$
$$\text{i.e. } BR > ES,$$
$$\therefore \text{ by (2) above } \triangle ABR > \triangle DES,$$
$$\text{i.e. } r(\triangle ABC) > s(\triangle DEF),$$
$$\therefore (\triangle ABC : \triangle DEF) > \frac{s}{r}.$$

$$\left.\begin{array}{r}\text{Hence if } (BC : EF) > \frac{s}{r}, \\ \text{then } (\triangle ABC : \triangle DEF) > \frac{s}{r}.\end{array}\right\}\text{(I.)}$$

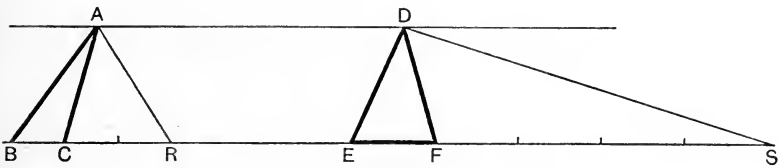

Fig. 8.

Take now the alternative:

$$(BC : EF) < \frac{s}{r},$$
$$\therefore r(BC) < s(EF).$$
$$\text{As before set off, } BR = r(BC)$$
$$\text{and } ES = s(EF).$$

Join AR, DS.

$$\text{As before, } \triangle ABR = r(\triangle ABC)$$
$$\text{and } \triangle DES = s(\triangle DEF),$$
$$\text{Since } r(BC) < s(EF),$$
$$\therefore BR < ES.$$

Hence by (2) above $\triangle ABR < \triangle DES$,

$$\text{i.e. } r(\triangle ABC) < s(\triangle DEF),$$
$$\therefore (\triangle ABC : \triangle DEF) < \frac{s}{r}.$$

$$\left.\begin{array}{r}\text{Hence if } (BC : EF) < \frac{s}{r}, \\ \text{then } (\triangle ABC : \triangle DEF) < \frac{s}{r}.\end{array}\right\}\text{(II.)}$$

It follows from (I.) and (II.) that no rational number falls between

$$(BC : EF) \text{ and } (\triangle ABC : \triangle DEF),$$
$$\therefore (BC : EF) = (\triangle ABC : \triangle DEF).$$

CHAPTER VIII

ARTICLES 50–53

Properties of Equal Ratios. Second Group of Propositions.

ARTICLE 50

Prop. XVIII. If A, B, C, D be four magnitudes *of the same kind*, and if

$$(A : B) = (C : D),$$

then $(A : C) = (B : D)$. . . . . Euc. V. 16.

Corollary :

If also $A > C$, then $B > D$;
but if $A = C$, then $B = D$;
and if $A < C$, then $B < D$. . . . . Euc. V. 14.

Prop. XIX. If $(A : B) = (T : U)$,
and if $(B : C) = (U : V)$,
then $(A : C) = (T : V)$. . . . . Euc. V. 22.

From this follows as a corollary :

If also $A > C$, then $T > V$;
but if $A = C$, then $T = V$.
and if $A < C$, then $T < V$. . . . . Euc. V. 20.

Prop. XX. If $(A : B) = (U : V)$,
and if $(B : C) = (T : U)$,
then $(A : C) = (T : V)$. . . . . Euc. V. 23.

From this follows a corollary (Euc. V. 21) the statement of which is identical with that of the Corollary to Prop. XIX.

These propositions are independent of one another, and they may therefore be proved in any order.

The proofs differ from those of the preceding propositions in that they depend on Prop. VIII. or its Corollary.

The proofs of Props. XVIII. and XX. require in addition the Corollary to Prop. III.

Prop. III. and its Corollary amount to this, that the factors of a product can be taken in any order.

In explaining these propositions to beginners the teacher would, I think, be well advised to take this result for granted, as I have done in the proofs of Props. XVIII. and XIX.

On the other hand, the proof of Prop. XX. is set out in a perfectly formal way with reference to the Corollary to Prop. III.

Article 51

Prop. XVIII. If A, B, C, D be four magnitudes *of the same kind*, and if

$$(A : B) = (C : D),$$

to prove that $(A : C) = (B : D)$. . . . . Euc. V. 16.

It has to be shown that *no* rational fraction *whatever* can lie between $(A : C)$ and $(B : D)$.

Let $\frac{s}{r}$ represent *any* rational number *whatever*. Comparing it with $(A : C)$, it is necessary to consider only the alternatives:

(1) $(A : C) > \frac{s}{r}$,

$$\therefore rA > sC.$$

Hence $rA - sC$ is a magnitude *of the same kind* as A, B, C, D.

Compare it with either B or D, say with B.

Then by Archimedes' Axiom an integer n exists such that

$$n(rA - sC) > B,$$

$$\therefore nrA > nsC + B,$$

$\therefore$ an integer t exists such that

$$nrA > tB > nsC$$

(Prop. VIII.).

Since $nrA > tB$,

$$\therefore (A : B) > \frac{t}{nr}.$$

(2) $(A : C) < \frac{s}{r}$,

$$\therefore rA < sC.$$

Hence $sC - rA$ is a magnitude *of the same kind* as A, B, C, D.

Compare it with either B or D, say with D.

Then by Archimedes' Axiom an integer p exists such that

$$p(sC - rA) > D,$$

$$psC > prA + D,$$

$\therefore$ an integer u exists such that

$$psC > uD > prA$$

(Prop. VIII.).

Since $psC > uD$,

$$\therefore (C : D) > \frac{u}{ps}.$$

But $(A:B)=(C:D)$,	But $(A:B)=(C:D)$,
$\therefore (C:D)>\frac{t}{nr}$,	$\therefore (A:B)>\frac{u}{ps}$,
$\therefore nrC>tD$.	$\therefore psA>uB$.
But $tB>nsC$,	But $uD>prA$,
$\therefore rtB>rnsC>stD$,	$\therefore ruB<rpsA<suD$,
$rtB>stD$,	$\therefore ruB<suD$,
$rB>sD$,	$\therefore rB<sD$,
$\therefore (B:D)>\frac{s}{r}$.	$\therefore (B:D)<\frac{s}{r}$.

Hence if $(A:C)>\frac{s}{r}$, then $(B:D)>\frac{s}{r}$; but if $(A:C)<\frac{s}{r}$, then $(B:D)<\frac{s}{r}$.

Hence no rational number lies between $(A:C)$ and $(B:D)$.

$$\therefore (A:C)=(B:D).$$

Corollary:

To prove that with the data in the proposition

If $A>C$, then $B>D$.
If $A=C$, then $B=D$.
If $A<C$, then $B<D$. .... Euc. V. 14.

Since $(A:C)=(B:D)$,
let us compare $(A:C)$ with the rational number 1.

Then one of the following alternatives must hold:

(1) $(A:C)>1$,	(2) $(A:C)=1$,	(3) $(A:C)<1$.
$\therefore (B:D)>1$.	$\therefore (B:D)=1$.	$\therefore (B:D)<1$,
Hence if $A>C$,	Hence if $A=C$,	Hence if $A<C$,
then $B>D$.	then $B=D$.	then $B<D$.

Article 52

Prop. XIX. If A, B, C are three magnitudes *of the same kind*, and if T, U, V are three magnitudes *of the same kind*, and if

$$(A:B)=(T:U),$$
and if $(B:C)=(U:V)$,
to prove that $(A:C)=(T:V)$. .... Euc. V. 22.

It has to be shown that no rational number falls between

$$(A:C) \text{ and } (T:V).$$

Let $\frac{k}{l}$ denote *any* rational number *whatever*. Then comparing it with $(A:C)$, it is necessary to consider only the alternatives:

(1) $(A:C)>\frac{k}{l}$,

$\therefore lA>kC$,

$\therefore lA-kC$ is a magnitude *of the same kind* as A, B, C.

Comparing it with B, it follows by Archimedes' Axiom that an integer v exists such that

$$v(lA-kC)>B,$$
$$\therefore vlA>vkC+B,$$

$\therefore$ an integer w exists such that

$$vlA>wB>vkC$$

(Prop. VIII.).

Since $vlA>wB$,

$$\therefore (A:B)>\frac{w}{vl}.$$

But $(A:B)=(T:U)$,

$$\therefore (T:U)>\frac{w}{vl},$$
$$\therefore vlT>wU.$$

But $wB>vkC$,

$$\therefore (B:C)>\frac{vk}{w}.$$

Now $(B:C)=(U:V)$,

$$\therefore (U:V)>\frac{vk}{w},$$
$$\therefore wU>vkV,$$
$$\therefore vlT>wU>vkV,$$
$$\therefore vlT>vkV,$$
$$lT>kV,$$
$$(T:V)>\frac{k}{l}.$$

(2) $(A:C)<\frac{k}{l}$,

$\therefore lA<kC$,

$\therefore kC-lA$ is a magnitude *of the same kind* as A, B, C.

Comparing it with B, it follows by Archimedes' Axiom that an integer n exists such that

$$n(kC-lA)>B,$$
$$nkC>nlA+B,$$

$\therefore$ an integer r exists such that

$$nlA<rB<nkC$$

(Prop. VIII.).

Since $nlA<rB$,

$$\therefore (A:B)<\frac{r}{nl}.$$

But $(A:B)=(T:U)$,

$$\therefore (T:U)<\frac{r}{nl},$$
$$\therefore nlT<rU.$$

But $rB<nkC$,

$$\therefore (B:C)<\frac{nk}{r}.$$

Now $(B:C)=(U:V)$,

$$\therefore (U:V)<\frac{nk}{r},$$
$$\therefore rU<nkV,$$
$$\therefore nlT<rU<nkV,$$
$$\therefore nlT<nkV,$$
$$\therefore lT<kV,$$
$$\therefore (T:V)<\frac{k}{l}.$$

Hence if $(A:C)>\frac{k}{l}$, then $(T:V)>\frac{k}{l}$; but if $(A:C)<\frac{k}{l}$, then $(T:V)<\frac{k}{l}$.

Hence no rational number falls between

$$(A : C) \text{ and } (T : V),$$
$$\therefore (A : C) = (T : V).$$

Corollary (Euc. V. 20) :

To prove that with the data in the proposition

$$\text{If } A > C, \text{ then } T > V.$$
$$\text{If } A = C, \text{ then } T = V.$$
$$\text{If } A < C, \text{ then } T < V.$$

The proof can be derived from that of the Corollary to Prop. XVIII. by changing therein B into T, and D into V.

Article 53

Prop. XX. If $(A : B) = (U : V)$, and if $(B : C) = (T : U)$, then $(A : C) = (T : V)$. (Euc. V. 23).

Compare $A : C$ with *any* rational number *whatever*, $\frac{r}{s}$.

It is necessary to consider the two following alternatives only :

Either $(A : C) > \frac{r}{s}$,

$$\therefore sA > rC.$$

Now B is a magnitude *of the same kind* as sA, rC.

Hence by the Corollary to Prop. VIII. integers n, t exist such that

$$n(sA) > tB > n(rC).$$

Now $(n(s))A = n(sA) > tB$,

$$\therefore (A : B) > \frac{t}{n(s)},$$

But $(U : V) = (A : B) > \frac{t}{n(s)}$,

$$\therefore (n(s))U > tV \quad \ldots .(\text{I.})$$

Also $tB > n(rC)$,

$$\therefore tB > (n(r))C,$$
$$\therefore (B : C) > \frac{n(r)}{t}.$$

Or $(A : C) < \frac{r}{s}$,

$$\therefore sA < rC.$$

Now B is a magnitude *of the same kind* as sA, rC.

Hence by the Corollary to Prop. VIII. integers n, t exist such that

$$n(sA) < tB < n(rC).$$

Now $(n(s))A = n(sA) < tB$,

$$\therefore (A : B) < \frac{t}{n(s)}.$$

But $(U : V) = (A : B) < \frac{t}{n(s)}$,

$$\therefore (n(s))U < tV \ldots (\text{III.})$$

Also $tB < n(rC)$,

$$\therefore tB < (n(r))C,$$
$$\therefore (B : C) < \frac{n(r)}{t}.$$

Now $(T:U)=(B:C)>\frac{n(r)}{t}$,

$\therefore tT>(n(r))U$. (II.)

We have to eliminate U between (I.) and (II.),

$s(tT)>s[(n(r))U]$ from (II.),
$s[(n(r))U]=r[(n(s))U]$,
by Cor. to Prop. III.

Also $r[(n(s))U]>r(tV)$ from (I.),

$$\therefore s(tT)>r(tV),$$
$$\therefore t(sT)>t(rV),$$
$$\therefore sT>rV,$$
$$\therefore (T:V)>\frac{r}{s}.$$

If $\therefore (A:C)>\frac{r}{s}$,

then $(T:V)>\frac{r}{s}$.

Now $(T:U)=(B:C)<\frac{n(r)}{t}$,

$\therefore tT<(n(r))U$ ... (IV.)

We have to eliminate U between (III.) and (IV.),

$s(tT)<s[(n(r))U]$ from (IV.),
$s[(n(r))U]=r[(n(s))U]$,
by Cor. to Prop. III.

Also $r[(n(s))U]<r(tV)$ from (III.),

$$\therefore s(tT)<r(tV),$$
$$\therefore t(sT)<t(rV),$$
$$\therefore sT<rV,$$
$$\therefore (T:V)<\frac{r}{s}.$$

If $\therefore (A:C)<\frac{r}{s}$,

then $(T:V)<\frac{r}{s}$.

Hence no rational number lies between $(A:C)$ and $(T:V)$.

$$\therefore (A:C)=(T:V).$$

Corollary (Euc. V. 21):

To prove that with the data in the proposition

If $A>C$, then $T>V$.
If $A=C$, then $T=V$.
If $A<C$, then $T<V$.

The proof can be derived from that of the Corollary to Prop. XVIII. by changing therein B into T, and D into V.

CHAPTER IX

ARTICLES 54–57

Properties of Equal Ratios. Third Group of Propositions.

ARTICLE 54

Prop. XXI. If $(A+C : B+D)=(C : D)$,
then $(A : B)=(C : D)$. .....Euc. V. 19.

Prop. XXII. If $(A : C)=(X : Z)$, and if $(B : C)=(Y : Z)$,
then $(A+B : C)=(X+Y : Z)$.
Euc. V. 24.

Prop. XXIII. If A, B, C, D are four magnitudes *of the same kind*, if A be the greatest of them,

and if $(A : B)=(C : D)$,
then $A+D>B+C$...........Euc. V. 25.

As in the Fifth Book, the proofs of these propositions are made to depend on the proofs of the preceding propositions. They could be proved in a manner having some resemblance to those of the earlier propositions, but the proofs are complicated and difficult; and altogether unsuited for an elementary course of instruction.

It will be seen that the proofs of these propositions are not automatic like those which have gone before. They involve a considerable strain on the memory, but on the whole they are very much simpler than any other proofs known to me.

ARTICLE 55

Prop. XXI. *If* $(A+C : B+D)=(C : D)$,
to prove that $(A : B)=(C : D)$.....Euc. V. 19.

Since $(A+C : B+D)=(C : D)$,

$\therefore$ by Prop. XVIII.

$$(A+C : C)=(B+D : D),$$

$\therefore$ by Prop. XVII.

$$(A : C)=(B : D),$$

$\therefore$ by Prop. XVIII.

$$(A : B)=(C : D).$$

Article 56

Prop. XXII. (Euc. V. 24).

If $(A : C)=(X : Z)$,

and if $(B : C)=(Y : Z)$,

to prove that $(A+B : C)=(X+Y : Z)$.

Since $(B : C)=(Y : Z)$,

$\therefore$ by Prop. XIII.

$$(C : B)=(Z : Y).$$

Now $(A : C)=(X : Z)$,

and $(C : B)=(Z : Y)$,

$\therefore$ by Prop. XIX.

$$(A : B)=(X : Y),$$

$\therefore$ by Prop. XVI.

$$(A+B : B)=(X+Y : Y).$$

But $(B : C)=(Y : Z)$,

$\therefore$ by Prop. XIX.

$$(A+B : C)=(X+Y : Z).$$

Article 57

Prop. XXIII. If A, B, C, D are four magnitudes *of the same kind*, if A be the greatest of them,

and if $(A : B)=(C : D)$,

then $(A+D)>(B+C)$.......Euc. V. 25.

Since $A>B$,

$\therefore (A : B)>1$,

$\therefore (C : D)>1$,

$\therefore C>D$.

Since $(A : B)=(C : D)$,

$\therefore$ by Prop. XVIII.

$$(A : C)=(B : D).$$

But $A>C$,

$\therefore (A : C)>1$,

$\therefore (B : D)>1$,

$\therefore B>D$.

Hence D is the smallest of the four magnitudes.

$$\text{Since } (A : B) = (C : D),$$

$\therefore$ by Prop. XVII.

$$(A - B : B) = (C - D : D),$$

$\therefore$ by Prop. XVIII.

$$(A - B : C - D) = (B : D).$$

$$\text{But } B > D,$$

$$\therefore (B : D) > 1,$$

$$\therefore (A - B : C - D) > 1,$$

$$\therefore A - B > C - D,$$

$$\therefore A + D > B + C.$$

PART II

CHAPTER X

Articles 58–67

Geometrical Applications of Stolz's Theorem (Art. 40).

Article 58

All the properties of equal ratios that can be put into an elementary course have now been given.

I will now give a remarkable application of Stolz's Theorem, due to my friend, Mr. Rose-Innes, to prove that the areas of circles are proportional to the squares described on their radii.

Some preliminary propositions from the Tenth and Twelfth Books of Euclid's *Elements* are required. They are set out here in order to make the argument complete in itself.

Euclid X. 1

If A and B be two magnitudes OF THE SAME KIND, *of which A is the larger, and if from A more than its half be taken away, leaving a remainder R_1; and if from R_1 more than its half be taken away, leaving a remainder R_2; and so on, then if this process be continued long enough, the remainder left will be less than B.*

This is deduced by repeated applications of the following:

If X and Y be two magnitudes *of the same kind*, and if X be greater than Y, then if from X less than its half be taken away, and if from Y more than its half be taken away, then the remainder of X left is greater than the remainder of Y.

If from X less than $\frac{1}{2}X$ is taken, more than $\frac{1}{2}X$ is left.

If from Y more than $\frac{1}{2}Y$ is taken, less than $\frac{1}{2}Y$ is left.

$$\text{But since } X > Y,$$
$$\tfrac{1}{2}X > \tfrac{1}{2}Y,$$
$$\therefore (\text{more than } \tfrac{1}{2}X) > (\text{less than } \tfrac{1}{2}Y).$$

To apply this to Euc. X. 1.

Since $A > B$ it follows by Archimedes' Axiom that an integer n exists such that

$$nB > A.$$

From the greater magnitude nB take away B, which is less than $\frac{1}{2}nB$. The remainder is $(n-1)B$.

From the smaller magnitude A take away more than its half. Let the remainder be R_1,

$$\therefore (n-1)B > R_1 \text{ by what has just been proved.}$$

From the greater magnitude $(n-1)B$, take away B, which is less than $\frac{1}{2}(n-1)B$. The remainder is $(n-2)B$.

From the smaller magnitude R_1 take away more than its half. Let the remainder be R_2,

$$\therefore (n-2)B > R_2.$$

Proceeding thus we get after s applications

$$(n-s)B > R_s,$$

and $\therefore$ after $(n-2)$ applications

$$2B > R_{n-2}.$$

Now take from R_{n-2} more than its half.

Let the remainder be R_{n-1}.

$$\text{Then } \tfrac{1}{2}R_{n-2} > R_{n-1}.$$
$$\text{But } B > \tfrac{1}{2}R_{n-2},$$
$$\therefore B > R_{n-1}.$$

So that after $(n-1)$ operations on A, the remainder of A left, viz. R_{n-1}, is less than B.

Article 59

As a geometrical application of this result take the following from the Second Proposition of the Twelfth Book of Euclid's *Elements* :

If a regular polygon of 2^n sides be inscribed in a circle, then the part of the circular area outside the polygon can be made as small as we please by making n *large enough.*

Take a segment of a circle which is a semicircle or *less* than a semicircle.

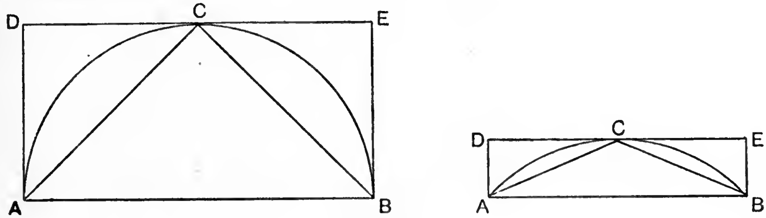

Fig. 9.

Let its chord be AB, and the middle point of its arc C.

Through C draw a tangent to the circle. This is parallel to AB.

Let the perpendiculars to AB, through A and B, cut the tangent at C at D, E.

Then the triangle ABC is equal to the sum of the triangles ACD, BCE.

Hence the triangle ABC is greater than the sum of the segments cut off by AC, BC.

Hence the triangle ABC is greater than half the area of the segment ABC; and therefore if the triangular area ABC be removed from the segment, the remainder left is less than half the original segment.

Suppose now that a square $ABCD$ is inscribed in a circle, then if the square be cut out from the circular area less than half the circular area is left. The part cut out is shaded.

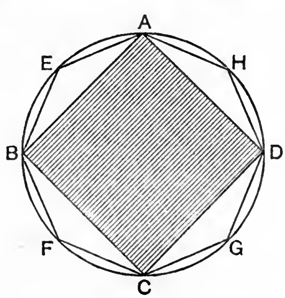

Fig. 10.

Let each of the arcs AB, BC, CD, DA be bisected at E, F, G, H respectively.

Then from the unshaded area remove the triangles

$$AEB, BFC, CGD, DHA.$$

Then by what has been proved the triangles

$$AEB, BFC, CGD, DHA$$

are greater than half the segments

AEB, BFC, CGD, DHA respectively.

And therefore, if the triangles are removed, more than half of the portion of the circle outside the square will have been removed.

We have left, then, only the unshaded areas shown in the next figure.

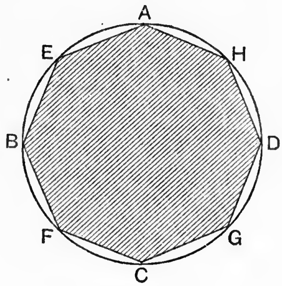

FIG. 11.

This process of bisecting the arcs and removing the triangular areas can be continued indefinitely.

At each step more than half the remaining area is removed, and therefore, if the process be carried on long enough, there will at length remain an area less than any area, say D, which may be fixed in advance.

Euclid XII. 1

ARTICLE 60

The areas of similar polygons inscribed in two circles are proportional to the areas of the squares described on the radii of the circles.

The areas of any two similar polygons are proportional to the squares described on corresponding sides.

If the similar polygons are inscribed in two circles, then corresponding sides are proportional to the radii of the circles, and therefore the squares on corresponding sides are proportional to the squares described on the radii of the circles.

Hence the areas of similar polygons inscribed in two circles are proportional to the areas of the squares described on the radii of the circles.

Euclid XII. 2

Article 61

The areas of circles are proportional to the squares described on their radii.

Let C_1 be the area of a circle whose radius is r_1.

Let C_2 be the area of a circle whose radius is r_2.

*If the ratio $(C_1 : C_2)$ be compared with any rational number $\frac{t}{u}$, then it is sufficient to consider the alternatives $(C_1 : C_2) > \frac{t}{u}$ and $(C_1 : C_2) < \frac{t}{u}$.

$$\text{Suppose } (C_1 : C_2) > \frac{t}{u},$$
$$\therefore\ uC_1 > Ct_2,$$
$$\therefore\ uC_1 - tC_2 = \text{some area } C.$$

Inside C_1 describe the polygon P_1 as explained in Art. 59,

$$\text{so that } C_1 - P_1 < \frac{C}{u},$$
$$\therefore\ uC_1 - uP_1 < C,$$
$$\text{but } uC_1 - tC_2 = C,$$
$$\therefore\ uC_1 - uP_1 < uC_1 - tC_2.$$
$$\therefore\ tC_2 < uP_1.$$

Inside C_2 describe a polygon P_2 similar to P_1.

$$\text{Then } C_2 > P_2,$$
$$\therefore\ tC_2 > tP_2,$$
$$\text{but } uP_1 > tC_2,$$
$$\therefore\ uP_1 > tP_2,$$
$$\therefore\ (P_1 : P_2) > \frac{t}{u}.$$

* From this point onwards the argument is applicable to many other propositions than the one under consideration.

$$\text{Hence if } (C_1 : C_2 > \frac{t}{u},$$
$$\text{then } (P_1 : P_2) > \frac{t}{u}.$$

Now let S_1 be the square described on r_1,
and S_2 be the square described on r_2.

$$\text{Then } (P_1 : P_2) = (S_1 : S_2).$$
$$\text{But } (P_1 : P_2) > \frac{t}{u},$$
$$\therefore (S_1 : S_2) > \frac{t}{u}.$$

$$\left.\begin{array}{r}\text{Hence if } (C_1 : C_2) > \dfrac{t}{u},\\ \text{then } (S_1 : S_2) > \dfrac{t}{u}.\end{array}\right\}\text{(I.)}$$

Suppose next

$$(C_1 : C_2) < \frac{t}{u},$$
$$\therefore uC_1 < tC_2,$$
$$\therefore tC_2 - uC_1 = \text{some area } D.$$

Inside C_2 inscribe a polygon Q_2 as explained in Art. 59,

$$\text{so that } C_2 - Q_2 < \frac{D}{t},$$
$$\therefore tC_2 - tQ_2 < D,$$
$$\therefore tC_2 - tQ_2 < tC_2 - uC_1,$$
$$\therefore tQ_2 > uC_1.$$

Inside C_1 inscribe a polygon Q_1 similar to Q_2.

$$\text{Then } C_1 > Q_1,$$
$$\therefore uC_1 > uQ_1,$$
$$\therefore tQ_2 > uC_1 > uQ_1,$$
$$\therefore uQ_1 < tQ_2,$$
$$\therefore (Q_1 : Q_2) < \frac{t}{u}.$$
$$\text{But } (Q_1 : Q_2) = (S_1 : S_2).$$
$$\text{Hence } (S_1 : S_2) < \frac{t}{u}.$$

$$\left.\begin{array}{r}\text{Hence if } (C_1 : C_2) < \dfrac{t}{u},\\ \text{then } (S_1 : S_2) < \dfrac{t}{u}.\end{array}\right\}\text{(II.)}$$

It follows from (I.) and (II.) that no rational number falls between $(C_1 : C_2)$ and $(S_1 : S_2)$,

$$\therefore (C_1 : C_2) = (S_1 : S_2).$$

ARTICLE 61 (*a*)

This result has been arrived at by the aid of Stolz's Theorem (Art. 40), which is involved.

It is of interest to see how Mr. Rose-Innes completes the proof of the proposition on strictly Euclidean lines, without the aid of Stolz's Theorem.

It has been shown that

$$\text{if } uC_1 > tC_2, \text{ then } uS_1 > tS_2 \quad \text{(I.)}$$
$$\text{and if } uC_1 < tC_2, \text{ then } uS_1 < tS_2 \quad \text{(II.)}$$

From these results it follows that

$$\text{if } uS_1 = tS_2, \text{ then must } uC_1 = tC_2.$$

For if $uC_1 > tC_2$, then by (I.) $uS_1 > tS_2$, which is contrary to the hypothesis.

And if $uC_1 < tC_2$, then by (II.) $uS_1 < tS_2$, which is contrary to the hypothesis.

Hence if $uS_1 = tS_2$, then $uC_1 = tC_2$. .... (III.)

Suppose now that $uC_1 = tC_2$, construct a rectangle R with one side equal to t times the side of S_2, and the other side equal to $\frac{1}{u}$ of the side of S_2, and make a square S equal to the rectangle R.

Then $uS = tS_2$.

Let C be the area of the circle, whose radius is equal to the side of S.

Then from (III.) it follows that

$$\text{since } uS = tS_2,$$
$$\therefore uC = tC_2.$$
$$\text{But } uC_1 = tC_2,$$
$$\therefore uC = uC_1,$$
$$\therefore C = C_1.$$

Hence the radii of these circles are equal,

$$\therefore S = S_1.$$
$$\text{But } uS = tS_2,$$
$$\therefore uS_1 = tS_2.$$

Hence it has now been shown that

$$\text{if } uC_1 = tC_2, \text{ then } uS_1 = tS_2 \quad \text{(IV.)}$$

Using (I.), (II.), and (IV.) and Euclid's Fifth Definition, it follows that

$$(C_1 : C_2) = (S_1 : S_2).$$

Article 61 (*b*)

There is still another way, due also to Mr. Rose-Innes, of obtaining the result.

Having proved the results marked (I.) and (II.) above, suppose next that

$$uS_1 > tS_2.$$

Let I_1, I_2 be the squares inscribed in the circles C_1, C_2 respectively.

$$\text{Then } I_1 = 2S_1,\ I_2 = 2S_2,$$
$$\therefore\ uI_1 > tI_2.$$

Now let us go on doubling the number of sides of the polygons inscribed in the circle C_2 until we reach a polygon P_2 such that

$$C_2 - P_2 < \frac{1}{t}(uI_1 - tI_2).$$
$$\text{Now } (I_1 : I_2) = (S_1 : S_2),$$
$$(P_1 : P_2) = (S_1 : S_2);$$
$$\therefore\ (I_1 : I_2) = (P_1 : P_2),$$
$$\therefore\ (uI_1 : tI_2) = (uP_1 : tP_2).$$
$$\text{Now } uI_1 > tI_2,$$
$$\therefore\ uP_1 > tP_2.$$
$$\text{Also } uI_1 - tI_2 : tI_2 = uP_1 - tP_2 : tP_2.$$
$$\text{But } tI_2 < tP_2,$$
$$\therefore\ uI_1 - tI_2 < uP_1 - tP_2,$$
$$\therefore\ C_2 - P_2 < \frac{1}{t}(uP_1 - tP_2),$$
$$\therefore\ tC_2 - tP_2 < uP_1 - tP_2,$$
$$\therefore\ tC_2 < uP_1.$$
$$\text{But } P_1 < C_1,$$
$$\therefore\ tC_2 < uC_1.$$

Hence if $uS_1 > tS_2$, then $uC_1 > tC_2$.

In a similar way it can be shown that

$$\text{if } uS_1 < tS_2, \text{ then } uC_1 < tC_2.$$

We have now the four results:

if $uC_1 > tC_2$, then $uS_1 > tS_2$; (i.)
if $uC_1 < tC_2$, then $uS_1 < tS_2$; (ii.)
if $uS_1 > tS_2$, then $uC_1 > tC_2$; (iii.)
if $uS_1 < tS_2$, then $uC_1 < tC_2$; (iv.)

Suppose now that $uC_1 = tC_2$, then if we compare uS_1 with tS_2 the logical alternatives are

$$uS_1 > tS_2 \text{ or } uS_1 < tS_2 \text{ or } uS_1 = tS_2.$$

But if $uS_1 > tS_2$, then $uC_1 > tC_2$ by (iii.), which is contrary to the hypothesis that $uC_1 = tC_2$.

And if $uS_1 < tS_2$, then $uC_1 < tC_2$ by (iv.), which is contrary to the hypothesis that $uC_1 = tC_2$.

Hence if $uC_1 = tC_2$, we must have $uS_1 = tS_2$ (v.).

From (i.), (ii.), and (v.) it follows that

$$(C_1 : C_2) = (S_1 : S_2).$$

[In the *first* edition of my Euclid, p. 18, Art. 37, I employed the four sets of conditions (i.), (ii.), (iii.), and (iv.), and did not use Stolz's Theorem in the form in which he himself gave it, which will be found in the *second* edition of my Euclid, V and VI, p. 29, and has been given above (Art. 40.)]

Article 62

If we now go over the steps of the proposition proved in Art. 61 and try to distinguish what is incidental to the particular proposition from what is necessary to obtain the final result, we get the following:

Let C_1, C_2 represent the contents of any two lengths, or of any two areas, or of any two volumes of such a nature that it is possible to inscribe in the first an infinite series of figures, which may be denoted by P_1, and in the second a series of related figures P_2 (not necessarily similar to P_1), but such that the ratio of the contents of two related figures $(P_1 : P_2)$ always has a FIXED *value, say $(S_1 : S_2)$, and if further by increasing the number of sides of P_1 it is possible to make the difference between C_1 and P_1 as small as we please, and if in like manner it is possible to make the difference between C_2 and P_2 as small as we please, then will $(C_1 : C_2) = (S_1 : S_2)$,* for the argument of Art. 61 from the point specially noted therein applies.

Article 63

As a first illustration I will prove that *the lengths of the circumferences of circles are proportional to their radii.*

Without going too deeply into the matter, let us assume that the length of the circumference is the limit to which the length of the perimeter of an inscribed regular polygon of 2^n sides approaches as n tends to infinity.

Now let C_1, C_2 denote the lengths of the circumferences of two circles whose radii are r_1, r_2.

Inscribe in each circle a regular polygon of 2^n sides.

The two inscribed polygons are similar.

Let their perimeters be P_1, P_2.

Then $(P_1 : P_2)=(r_1 : r_2)$ and is therefore fixed.

Hence the fixed ratio $(S_1 : S_2)$ of Art. 62 is in this case $=(r_1 : r_2)$.

Now by the assumption we have made as to the meaning of the length of the circumference of a circle, we know that C_1-P_1 and C_2-P_2 can be made as small as we please by increasing n sufficiently.

$\therefore$ the argument of Arts. 61, 62 applies,

$\therefore (C_1 : C_2)=(S_1 : S_2)$, which is here $(r_1 : r_2)$,

$\therefore (C_1 : C_2)=(r_1 : r_2)$.

Hence the circumferences of circles are proportional to their radii.

Article 64

As a second illustration it will be shown that *the area of the radian sector of a circle is equal to half the area of the square described on its radius.*

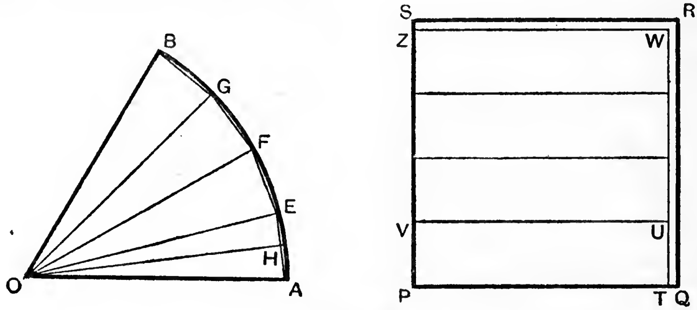

Fig. 12.

Let AB be an arc of a circle whose centre is O, such that the length of the arc AB is equal to the radius OA, it is required to prove that the area of the sector OAB is equal to half the area of the square described on OA as side.

I will call the sector OAB the radian sector.

When we say that the length of the arc AB is equal to the radius we may agree to mean that if we divide the arc into 2^n equal parts, and join the points of division by straight lines, then the sum of the lengths of the joining lines increases as n increases up to a limit which is equal to the length of the radius.

In the figure the arc AB is divided into equal parts AE, EF, FG, GB.

OH is drawn perpendicular to AE.

$PQRS$ is a square on $PQ=OA$.

In it we take $PT=OH$,

$PV=AE$,

$PZ=AE+EF+FG+GB$.

Then the rectangle $PTUV$ is double the triangle OAE, and the rectangle $PTWZ$ is double the figure $OAEFGB$ inscribed in the radian sector.

Moreover, by increasing the number of points of division of the arc AB the difference between the radian sector and the figure $OAEFGB$ can be made as small as we please. When this is done OH and $\therefore$ PT tends to the radius as its limiting value, and $\therefore$ the point T tends to Q.

Also $AE+EF+FG+GB$ tends to the radius as its limiting value, and $\therefore$ PZ tends to PS, and $\therefore$ Z to S.

Hence the rectangle $PTWZ$ tends to coincide with $PQRS$, and can be made to differ from it by as little as we please.

In this case the argument of Arts. 61, 62 applies if we make C_1 the area of the radian sector, C_2 the area of the square described on the radius, P_1 the figures represented by $OAEFGB$, and P_2 the figures represented by $PTWZ$.

And here the fixed ratio $(S_1 : S_2)$ is now the ratio $(1 : 2)$.

Hence (the area of the radian sector : the area of the square on the radius)$=(1 : 2)$.

$\therefore$ area of radian sector$=\frac{1}{2}$(the square on the radius).

Article 65

The area of a circle whose radius is r is πr^2.

Let us now compare the area of the whole circle with that of the radian sector.

Since the areas of sectors of the same circle are proportional to their arcs,

$\therefore$ (area of whole circle : area of radian sector)
$=$(length of circumference : length of radius),
$\therefore$ (area of whole circle : $\frac{1}{2}r^2$)$=$($2\pi r : r$),
$\therefore$ area of whole circle$=\pi r^2$.

Article 66

As a third illustration it will be proved that *the volumes of tetrahedra standing on the same base are proportional to their altitudes.*

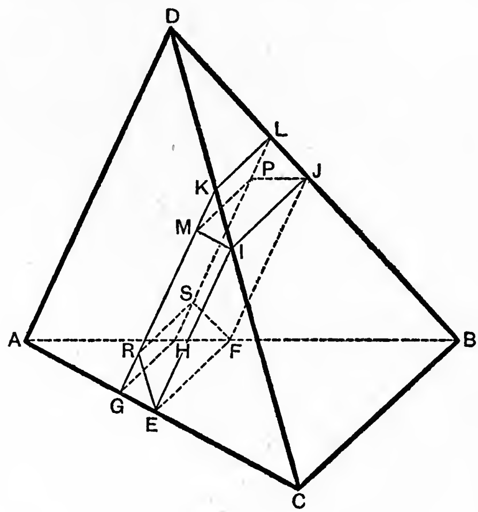

Fig. 13.

Only one tetrahedron $ABCD$ is drawn in the figure.
Let the side AC be divided into n equal parts.
Let E, G be two consecutive points of division.

Draw EF, GH parallel to BC.
,, EI, GK ,, AD.
,, FJ, HL ,, AD.

Then it can be shown that

KL, IJ are parallel to CB.
Also if IM be drawn parallel to CA,
and JP be drawn parallel to BA,

then the figure standing on $EFHG$ as base, and having the equal and parallel edges EI, FJ, HP, GM is a prism whose volume is

base $EFHG \times$ perpendicular from I on the plane ABC.

If, through all the points of division of AC, parallels be drawn to BC such as EF, GH, and if on each of the areas such as $EFHG$ there be erected a prism corresponding to the one just described, then the aggregate of all these prisms will be a polyhedron inscribed in $ABCD$, which, using the notation of Art. 62, may be called P_1. The volume of the tetrahedron $ABCD$ may, with the notation of Art. 62, be denoted by C_1.

The volume of P_1 differs from that of C_1 by the sum of the pieces such as $MIJPLK$, which is a frustum of the prism whose parallel edges IJ, MP, KL are cut by the non-parallel planes IMK, JPL.

$$\text{Now } MI = GE = \frac{1}{n}(AC).$$
$$\text{Also } (KM : MI) = (DA : AC),$$
$$\therefore\ KM = \frac{1}{n}(DA).$$

Also $MIJP$ is congruent with $GEFH$.

Hence if the piece $MIJPLK$ be detached from its position, and made to slide until $MIJP$ falls upon $GEFH$, K will fall at a point R, such that $GR = \frac{1}{n}(DA)$, and L will fall at a point S such that $HS = \frac{1}{n}(DA)$. $MIJPLK$ will coincide with $GEFHSR$.

If all the pieces like $MIJPLK$ are treated in the same way, their bases will together cover up the base ABC and the lines such as RS will all fall on a plane parallel to ABC, cutting AD at a distance from $A = \frac{1}{n}(AD)$.

Their total volume will therefore be less than that of a prism whose base is ABC and whose height is $\frac{1}{n}$ of the perpendicular from D on ABC.

Hence their total volume is less than

$\frac{1}{n}$(base ABC) (perpendicular from D on ABC).

Hence by increasing n the difference between the volume of the tetrahedron and the aggregate of the prisms such as $EFHGMIJP$ can be made as small as we please.

Call the figure inscribed in the second tetrahedron, which corresponds to P_1 in the first, P_2.

It will be proved that

$$(P_1 : P_2) = (h_1 : h_2),$$

where h_1, h_2 are the heights of the tetrahedrons.

Suppose that accented letters denote the points in the second tetrahedron which correspond to unaccented letters in the first.

The ratio of the volumes of corresponding prisms,

$$\text{i.e. } (EFHGMIJP : E'F'H'G'M'I'J'P')$$

$=(EFHG \times \text{perp. from } I \text{ on } ABC) : (E'F'H'G' \times \text{perp. from } I' \text{ on } A'B'C')$

$=(\text{perp. from } I \text{ on } ABC) : (\text{perp. from } I' \text{ on } A'B'C')$, because the bases ABC and $A'B'C'$ are the same, and so $EFHG$ and $E'F'H'G'$ coincide.

Further

$$(\text{perp. from } I \text{ on } ABC) : (\text{perp. from } D \text{ on } ABC)$$
$$=(IC : DC) = (EC : AC).$$

Also

$$(\text{perp. from } I' \text{ on } A'B'C') : (\text{perp. from } D' \text{ on } A'B'C')$$
$$=(I'C' : D'C') = (E'C' : A'C') = (EC : AC)$$

because the bases $A'B'C'$, ABC are the same; and the points A, E, C coincide with A', E', C' respectively.

$$\therefore\ (\text{perp. from } I \text{ on } ABC) : (\text{perp. from } D \text{ on } ABC)$$
$$=(\text{perp. from } I' \text{ on } A'B'C') : (\text{perp. from } D' \text{ on } A'B'C'),$$
$$\therefore\ (\text{perp. from } I \text{ on } ABC) : (\text{perp. from } I' \text{ on } A'B'C')$$
$$=(\text{perp. from } D \text{ on } ABC) : (\text{perp. from } D' \text{ on } A'B'C')$$
$$=(h_1 : h_2),$$
$$\therefore\ (EFHGMIJP : E'F'H'G'M'I'J'P') = (h_1 : h_2),$$

i.e. the ratio of the volumes of corresponding prisms$=(h_1 : h_2)$.

Hence by the aid of Prop. XIV. it can be shown that $(P_1 : P_2) = (h_1 : h_2)$.

We are now in a position to apply the argument of Arts. 61, 62.

Here C_1, C_2 are the volumes of the two tetrahedra. P_1, P_2 are the volumes of the inscribed polyhedra made up of the prisms such as $EFHGMIJP$ and $E'F'H'G'M'I'J'P'$.

Both $C_1 - P_1$ and $C_2 - P_2$ can be made as small as we please by sufficiently increasing n.

Also $(P_1 : P_2)$ has the fixed value $(h_1 : h_2)$.

So here the $(S_1 : S_2)$ of Art. 62 is $(h_1 : h_2)$.

Hence $(C_1 : C_2) = (S_1 : S_2)$, which is $(h_1 : h_2)$.

$$\therefore (C_1 : C_2) = (h_1 : h_2).$$

Article 67

The volumes of tetrahedra are proportional to their bases and altitudes jointly.

It has been proved (Art. 66) that the volumes of any two tetrahedra standing on *identical* bases are proportional to their altitudes.

The next step is to show how to take account of an alteration in the bases of the tetrahedra.

Let $DABC$ and $D'A'B'C'$ be any two tetrahedra.

On AB take a length $AK = A'B'$.

In the plane ABC make $K\hat{A}L = B'\hat{A}'C'$,

and take $AL = A'C'$.

Join KL. Let AC meet KL at R. Join DR, CK.

Then the triangles AKL, $A'B'C'$ are congruent, and it is possible to move the tetrahedron $D'A'B'C'$ until its base $A'B'C'$ coincides with the triangle AKL.

Then by what has been shown as to the ratio of the volumes of tetrahedra standing on the same base,

vol. of $DAKL$: vol. of $D'A'B'C'$
= perp. from D on AKL : perp. from D' on $A'B'C'$
= perp. from D on ABC : perp. from D' on $A'B'C'$.

The next step is to find the ratio of the volumes of the tetrahedra $DABC$, $DAKL$. This is obtained by comparing

(1) $DABC$ with $DACK$,
then (2) $DACK$ with $DARK$,
and then (3) $DARK$ with $DAKL$.

If $DABC$ be compared with $DACK$, they may be regarded as standing on the same base DAC, and their heights are then the perpendiculars from B and K on the plane DAC.

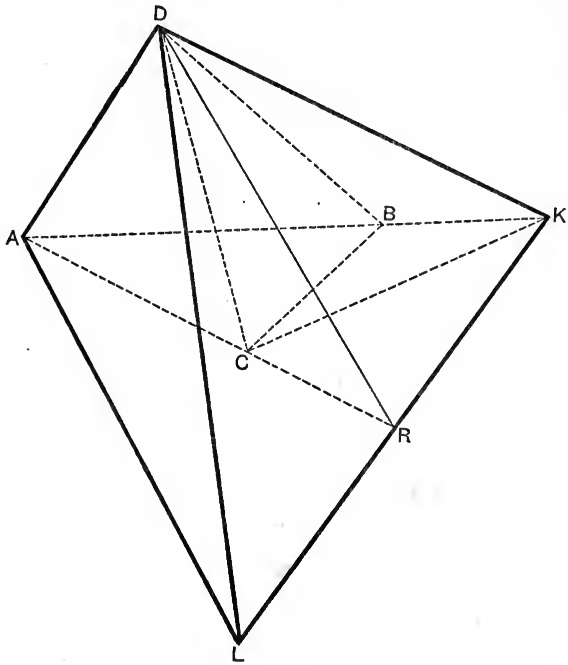

Fig. 14.

Now these heights are proportional to AB and AK.

$$\therefore\ DABC : DACK = AB : AK = \triangle ABC : \triangle AKC,$$
$$\therefore\ DABC : DACK = \triangle ABC : \triangle AKC \text{ ....(1).}$$

Next compare $DACK$ with $DARK$.

They may be considered as standing on the base DAK, and their heights to be the perpendiculars from C and R on the plane DAK. These heights are proportional to AC and AR.

$$\therefore\ DACK : DARK = AC : AR = \triangle ACK : \triangle ARK,$$
$$\therefore\ DACK : DARK = \triangle AKC : \triangle ARK \ldots\ldots\ldots\ldots(2).$$

Lastly, consider the tetrahedra $DARK$, $DAKL$.

They may be considered as standing on the base DAK, and their heights are the perpendiculars from R and L on the plane DAK.

These heights are proportional to KR and KL.

$$\therefore\ DARK : DAKL = KR : KL = \triangle ARK : \triangle ALK,$$
$$\therefore\ DARK : DAKL = \triangle ARK : \triangle A'B'C' \ldots\ldots\ldots(3).$$

From (1) and (2) by Prop. XIX. it follows that

$$DABC : DARK = \triangle ABC : \triangle ARK \ldots\ldots\ldots\ldots(4).$$

From (4) and (3) by Prop. XIX. it follows that

$$DABC : DAKL = \triangle ABC : \triangle A'B'C' \ldots\ldots\ldots(5).$$

Now since the bases AKL and $A'B'C'$ are identical,

$$\therefore\ DAKL : D'A'B'C'$$
$$= \text{perp. from } D \text{ on } AKL : \text{perp. from } D' \text{ on } A'B'C',$$
$$\therefore\ DAKL : D'A'B'C'$$
$$= \text{perp. from } D \text{ on } ABC : \text{perp. from } D' \text{ on } A'B'C' \ldots(6).$$

From (5) and (6) together it follows that the volumes of tetrahedra are proportional to their bases and altitudes jointly.

If now $DABC$ and $D'A'B'C'$ be two tetrahedra on equal bases and having equal altitudes, it follows from (5) and (6) that

$$DABC = D'A'B'C'.$$

From this it follows in the well-known way that the volume of a tetrahedron is equal to one-third of the base multiplied by the height.

CHAPTER XI

Articles 68–70

Further remarks on the Irrational Number.
The existence of the Fourth Proportional.

Article 68

Before proceeding to discuss the existence of the Fourth Proportional, it is necessary to say a little more about the Irrational Number. The subject is dealt with more fully in the Note on Irrational Numbers appended to the Second Edition of my book on the Contents of the Fifth and Sixth Books of Euclid's *Elements*, pp. 147–162.

The Irrational Number

Let some rule be given by means of which the system of all the rational numbers can be separated into two classes such that every number in one class (called the lower class) is less than every number in the other class (called the upper class); then the following cases have to be distinguished.

(i.) The lower class has no greatest number, and the upper class has no least number. Then between the two classes there is a gap. This gap is filled by the creation of a number. It cannot be a *rational* number, because every *rational* number falls by hypothesis into one of the two classes. Consequently it is called an *irrational* number; and it separates the whole system of *rational* numbers into two classes, such that every number of the lower class is less than every number of the upper class.

It is regarded as known because all its properties can be inferred from a knowledge of all the rational numbers which are less than it, and a knowledge of all the rational numbers which are greater than it.

(ii.) The case in which the lower class contains a greatest number. This greatest number is a rational number, because it belongs to the lower class. Calling it $\frac{r}{s}$, the upper class contains all the rational numbers greater than $\frac{r}{s}$, and therefore can contain no least number.

In this case the number $\frac{r}{s}$ separates the whole system of rational numbers into two classes such that each number in the lower class is less than each number in the upper class.

The lower class contains $\frac{r}{s}$ and all rational numbers less than $\frac{r}{s}$; the upper class contains all the rational numbers greater than $\frac{r}{s}$.

(iii.) The case in which the upper class contains a least number.

This least number is a rational number because it belongs to the upper class. Calling it $\frac{r}{s}$, the lower class contains all the numbers less than $\frac{r}{s}$ and therefore can contain no greatest number.

In this case the number $\frac{r}{s}$ separates the whole system of rational numbers into two classes such that each number in the lower class is less than each number in the upper class.

The upper class contains $\frac{r}{s}$ and all numbers greater than $\frac{r}{s}$; the lower class contains all the numbers less than $\frac{r}{s}$.

Cases (ii.) and (iii.) are regarded as not essentially distinct. In both the separation of the whole system of rational numbers into two classes can be regarded as being effected by the number $\frac{r}{s}$. In any of the cases (i.), (ii.), (iii.) the separation of the system of all the rational numbers into two classes is such that every number in the lower class is less than every number in the upper class.

Article 69

The following is a geometrical analogy to the preceding.

If P be a point in any straight line, then all the points in the straight line may be separated into two classes P_1, P_2 as follows:

The first-class P_1 contains all the points that lie on one side, say the left, of P.

The second class P_2 contains all the points that lie on the right of P.

The point P itself may be put into either class, it does not matter which.

If P be put into the class P_1, then the class P_1 has a point, viz. P, which is the farthest to the right, but the class P_2 has no point which is farthest to the left.

If P be put into the class P_2, then the class P_2 has a point, viz. P, which is the farthest to the left, but the class P_1 has no point which is farthest to the right.

In either case the separation of all the points on the straight line into the two classes P_1, P_2 is of such a nature that every point of the first class P_1 is on the left of every point of the second class P_2.

The converse of the above statement cannot be proved. The assumption of its truth is known as the Cantor-Dedekind Axiom. It explains what is meant by ascribing continuity to the straight line.

The Cantor-Dedekind Axiom

If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this separation of all points of the straight line into two classes.

Now it has been seen that the system of rational numbers can be separated into two classes, such that every number in the lower class is less than every number of the upper class, and that this separation can be produced by a number which is not a rational number. From this it follows that the system of rational numbers is not continuous.

The existence of the Fourth Proportional

ARTICLE 70

I come now to a very important proposition, which is far too difficult to be included in an elementary course, except in the special case in which the magnitudes concerned are segments of straight lines or cases readily reducible thereto.

Euclid assumed it in V. 18, and XII. 2, 5, 11, 12, and 18. The proposition, as will be seen, rests for its validity on an axiom corresponding to the Cantor-Dedekind Axiom. It is fundamental in the Theory of Ratio. It is as follows:

Prop. XXIV. If A and B be magnitudes *of the same kind*, and if C be any third magnitude, then there exists a fourth magnitude Z *of the same kind* as C such that

$$(A:B)=(C:Z).$$

$$\text{Now if } (A:B)=(C:Z),$$
$$\text{then } (Z:C)=(B:A).$$

(1) Suppose that A and B have a common measure, and

$$\therefore \text{ integers } r, s \text{ exist such that } (B:A)=\frac{r}{s}.$$
$$\text{Then } Z=\frac{rC}{s}.$$
$$\text{For } \frac{rC}{s}:C=(rC:sC)=\frac{r}{s}=(B:A).$$

(2) Suppose B and A have no common measure.

Let $B:A$ be equal to the irrational number ρ.

Let $p_1, p_1', p_1'', \ldots$ represent rational numbers in the lower class determined by ρ in *ascending* order of magnitude. These contain no greatest number.

Let $p_2, p_2', p_2'', \ldots$ represent rational numbers in the upper class determined by ρ in *descending* order of magnitude. These contain no least number.

Now if p_1 represent any rational number $\frac{u}{v}$, p_1C denotes $\frac{u}{v}C$, i.e. $\frac{uC}{v}$, which means the vth part of the magnitude uC, so that the magnitude p_1C can be constructed.

Construct the magnitudes $p_1C, p_1'C, p_1''C, \ldots$, and call them magnitudes of the form p_1C.

Construct the magnitudes $p_2C, p_2'C, p_2''C, \ldots$, and call them magnitudes of the form p_2C.

Then $p_1C<p_1'C<p_1''C<\ldots<p_2''C<p_2'C<p_2C$.

Let us separate all the magnitudes *of the same kind* as C into two classes by the following rules:

(i.) Put all those into the upper class which are greater than every one of the magnitudes of the form p_1C.

Call any one of these a magnitude Y.

(ii.) Into the lower class put all those magnitudes which are not greater than every one of the magnitudes of the form p_1C.

Call any one of these magnitudes a magnitude X.

It will be proved in the *first* place that

Every magnitude Y is greater than every magnitude X.

From the definition of the magnitudes X, it appears that any X, say X_1, does not exceed every magnitude of the form p_1C.

$$\text{Suppose } X_1 \leqq p_1'C.$$

But every Y exceeds every magnitude of the form p_1C.

$$\therefore\ Y > p_1'C,$$
$$\therefore\ Y > X_1,$$
$$\therefore \text{ every } Y \text{ exceeds every } X.$$

It will be proved in the *second* place that

The set of magnitudes X includes no greatest magnitude.

The characteristic of the magnitudes X is that they are not greater than every magnitude of the form p_1C.

Suppose X' is one of the magnitudes X.

$$\text{Let } X' \lesseqgtr p_1'C.$$

Now there is no greatest p_1.

$$\text{Suppose } p_1' < p_1'',$$
$$\therefore\ p_1'C < p_1''C.$$

It is then permissible to take $X'' = p_1''C$,

$$\therefore\ X' < X'',$$

and so on; other magnitudes X can be found continually increasing in order of magnitude.

Therefore the set of magnitudes X includes no greatest magnitude.

It will be proved in the *third* place that on the assumption of the truth of the axiom referred to above

The set of magnitudes Y includes a least magnitude.

Now the magnitudes X and Y together include all the magnitudes *of the same kind* as C. In regard to these magnitudes we assume an axiom corresponding to the Cantor-Dedekind Axiom for the straight line, as follows:

If all the magnitudes *of the same kind* as C be separated into two classes such that every magnitude of the one class is less than every magnitude of the other class, then there is one and only one magnitude *of the same kind* as C which produces this separation, and it is either the greatest magnitude of the one class, or the least magnitude of the other class.

Now it has been proved that the magnitudes X include no greatest magnitude.

Therefore the magnitudes Y include a least magnitude.

Call this least magnitude Z.

It will be proved in the *fourth* place

$$\textit{That } (Z : C) > \textit{every } p_1.$$

Since Z is a magnitude Y, it is greater than every magnitude of the form p_1C. Write this thus:

$$Z > \text{every } p_1C,$$
$$\therefore (Z : C) > \text{every } p_1.$$

It will be proved in the *fifth* place

$$\textit{That } (Z : C) < \textit{every } p_2,$$
$$\text{i.e. } Z < \text{every } p_2C.$$
$$\text{Suppose if possible } Z \geqq \text{some } p_2C,$$
$$\text{say } Z \geqq p_2''C.$$

Then since the rational numbers p_2 include no least rational number, take

$$p_2''' < p_2''.$$
$$\text{Then } p_2'''C < p_2''C,$$
$$\therefore p_2'''C < Z.$$
$$\text{Now every } p_2 > \text{every } p_1,$$
$$\therefore p_2''' > \text{every } p_1,$$
$$\therefore p_2'''C > \text{every } p_1C,$$
$$\therefore p_2'''C \text{ is a magnitude } Y.$$
$$\text{But } p_2'''C < Z.$$

Hence there is a magnitude Y which is less than Z.

But this is contrary to the definition of Z, viz. that it was the least of the magnitudes Y.

$$\text{Hence } Z < \text{every } p_2C,$$
$$\therefore (Z : C) < \text{every } p_2.$$

It has now been proved both that

$$(Z:C) > \text{every } p_1,$$
$$\text{and } (Z:C) < \text{every } p_2.$$

Hence $Z:C$ determines the same separation of the whole system of rational numbers as the irrational number ρ, i.e. the same separation as $B:A$,

$$\therefore (Z:C)=(B:A),$$
$$\therefore (A:B)=(C:Z).$$

PART III

CHAPTER XII

Articles 71–100

Commentary on the Fifth Book of Euclid's Elements.

I. Definitions

Article 71

The most important definitions are the 3rd, the 4th, the 5th, and the 7th.

(*a*) The *third* definition is translated thus by De Morgan in his Treatise on the Connexion of Number and Magnitude.

" Ratio is a certain mutual habitude* of two magnitudes of the same kind depending upon their quantuplicity."†

He says (*l.c.*, p. 63, line 4): " Ratio is relative magnitude."

If we say that the ratio of A to B is the measure of the relative magnitude of A as compared with B, I think that this will give us all that can be extracted from Euclid's third definition.

(*b*) The *fourth* definition is translated thus by De Morgan:

" Magnitudes are said to have a ratio to each other which can, being multiplied, exceed ' one the other.' "

I have given De Morgan's translation above. It is sometimes rendered, " Two magnitudes are said to have a ratio when the less can be multiplied so as to exceed the greater."

But De Morgan contends that this does not represent the meaning of the Greek original.

Whichever form be accepted, the Axiom of Archimedes is

* σχέσις, method of holding or having, mode or kind of existence.

† πηλικότης, for which there is no English word; it means relative greatness, and is the substantive which refers to the number of times or parts of times one is in the other.

assumed, viz.: If A and B are two magnitudes *of the same kind*, it is always possible to find a multiple of the less which will exceed the greater.

The third and fourth definitions may be regarded as doing two things:

In the first place they call attention to important properties of two magnitudes *of the same kind.*

In the second place, the fourth definition taken together with the third says that if A and B are two magnitudes *of the same kind,* then they also determine something else which Euclid calls a ratio, and which we regard as a number. It is denoted by $A : B$ or $B : A$, according to the order in which the magnitudes are taken.

The definition does not say what a ratio is, or how it is to be determined, *and this omission is one of the great difficulties* which present themselves to those who try to understand the argument.

The Fifth Definition

(The test for determining if two ratios are equal.)

Article 72

If A, B, C, D are four magnitudes, then A has the same ratio to B as C has to D, if when *any* integers *whatever*, r, s have been chosen, the following sets of conditions are satisfied:

(1) If the integers r, s are such that $rA > sB$, it is necessary that $rC > sD$.

(2) If the integers r, s are such that $rA = sB$, it is necessary that $rC = sD$.

(3) If the integers r, s are such that $rA < sB$, it is necessary that $rC < sD$.

Article 73

The first thing to observe about this definition is that it is not necessary to bring in the idea of ratio at all. It merely states that the four quantities A, B, C, D are such that the multiples of A are distributed amongst the multiples of B in the same way as the multiples of C are distributed amongst the multiples of D.

For suppose that pA lies between qB and $(q+1)B$,

$$\begin{aligned} \text{then } & pA > qB, \\ \therefore\ & pC > qD. \\ \text{Also } & pA < (q+1)B, \\ \therefore\ & pC < (q+1)D. \end{aligned}$$

Hence pC lies between qD and $(q+1)D$.

If, however, $pA = tB$, then $pC = tD$.

To fix the ideas take a particular example. Let A represent a straight line 3 inches long, and let B represent a straight line 4 inches long.

Then if we arrange the multiples of A and B in ascending order of magnitude we get the following diagram :

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 9A \\ 8A,\ 6B \\ 7A \\ 5B \\ 6A \\ 4B \\ 5A \\ 4A,\ 3B \\ 3A \\ 2B \\ 2A \\ 1B \\ 1A \end{array}$$

which, of course, may be continued indefinitely upwards.

This was what De Morgan called the relative multiple scale of A, B.

Its properties may be seen more clearly by placing A and B at the bottom of the column, and arranging the digits above them in accordance with the following rules :

If $rA > sB$, then r above A is to be placed higher than s above B.

If $rA = sB$, then r above A is to be placed on the same level as s above B.

If $rA < sB$, then r above A is to be placed lower than s above B.

If we do this in the above case we get the following diagram; for the relative multiple scale of 3 inches and 4 inches:

9	
8	6
7	
	5
6	
	4
5	
4	3
3	
	2
2	
	1
1	
A=3	B=4

FIG. 15.

If we call the part of the diagram, above the line on which A and B are written, supposed continued indefinitely upwards, the relative multiple scale of A, B; then the conditions set out in Euclid's Fifth Definition simply amount to the statement that the ratio of A to B is the same as that of C to D when the relative multiple scale of A, B is the same as that of C, D.

Now all the propositions in the Fifth Book which deal with equal ratios can be expressed as propositions dealing with the sameness of relative multiple scales.

For example, instead of saying that the ratio of A to B is the same as that of nA to nB, we might say that the relative multiple scale of A, B is the same as the relative multiple scale of nA, nB; or if we do not use the technical term "relative multiple scale" we might say that the multiples of A are distributed amongst the multiples of B in the same way as the multiples of nA are distributed amongst the multiples of nB.

In this way the introduction of the idea of ratio might be avoided, and the difficulty arising from the fact that Euclid does not define ratio would not then force itself on any one trying to understand his argument.

It is true that when one passes on to consider unequal ratios the difficulty of avoiding the use of the term "ratio" becomes greater.

On the other hand, as has been explained in the preceding chapters, all the properties of Equal Ratios proved in the Fifth Book can be proved without using the Seventh Definition.

Article 74

Let us now make some further study of the three sets of conditions which appear in the Fifth Definition.

It will be proved in the *first* place that if the sets of conditions marked (1), (2), and (3) in Art. 72 in the Fifth Definition hold good, then conversely,

(4) if $rC > sD$, it will follow that $rA > sB$;
(5) if $rC = sD$, it will follow that $rA = sB$;
(6) if $rC < sD$, it will follow that $rA < sB$.

Suppose if possible that (1), (2), and (3) hold, but that when $r = r_1$, $s = s_1$ we have

$r_1C > s_1D$, but r_1A not greater than s_1B;
so that either (i.) $r_1C > s_1D$ and $r_1A = s_1B$,
or (ii.) $r_1C > s_1D$ and $r_1A < s_1B$.

(i.) If $r_1C > s_1D$ and $r_1A = s_1B$,

then since $r_1A = s_1B$, it follows by (2) that $r_1C = s_1D$, which is contrary to the hypothesis that $r_1C > s_1D$.

Hence r_1A is not equal to s_1B.

(ii.) If next $r_1C > s_1D$ and $r_1A < s_1B$,

then since $r_1A < s_1B$, it follows by (3) that $r_1C < s_1D$, which is contrary to the hypothesis that $r_1C > s_1D$.

Hence r_1A is not less than s_1B.

Hence if $r_1C > s_1D$, then is $r_1A > s_1B$.

Hence (4) holds.

In like manner it can be shown that (5) and (6) hold.

Article 75

In the *second* place it may be observed that it has already been shown in the discussion of Stolz's Theorem, Prop. XI., Art. 40, that if (1) and (3) hold, then (2) must hold.

Similarly if (4) and (6) hold, then (5) must hold.

Article 76

It will be proved in the *third* place that it is sufficient for (1) and (4) to hold.

Suppose that all values of r, s which make

$$rA > sB \text{ also make } rC > sD, \ldots\ldots\ldots\ldots \quad (1)$$

and that all values of r, s which make

$$rC > sD \text{ also make } rA > sB, \ldots\ldots\ldots\ldots \quad (4)$$

it will be shown that (3) holds.

Suppose then that

$$r_1A < s_1B,$$

and it is required to prove that $r_1C < s_1D$.

If not, then either (i.) $r_1C > s_1D$ or (ii.) $r_1C = s_1D$.

(i.) Consider first the alternative

$$r_1A < s_1B, \text{ but } r_1C > s_1D.$$

This is impossible, for by (4),

$$\text{if } r_1C > s_1D,$$
$$\text{then } r_1A > s_1B,$$

which is contrary to the hypothesis that

$$r_1A < s_1B.$$

Hence r_1C is not greater than s_1D.

(ii.) Consider next the alternative

$$r_1A < s_1B, \text{ but } r_1C = s_1D.$$

Then by Archimedes' Axiom an integer n exists such that

$$n(s_1B-r_1A)>A,$$
$$\therefore\ (nr_1+1)A<ns_1B,$$
$$\text{but } nr_1C=ns_1D,$$
$$\therefore\ (nr_1+1)C>ns_1D,$$

i.e. $(nr_1+1)C>(ns_1)D$, but $(nr_1+1)A<(ns_1)B$,
but by (4),

$$\text{putting } r=nr_1+1,$$
$$s=ns_1,$$

it is necessary, when $(nr_1+1)C>(ns_1)D$, that

$$(nr_1+1)A>ns_1B,$$
$$\text{and not } (nr_1+1)A<ns_1B.$$

Hence r_1C is not equal to s_1D.

Consequently it is necessary that

$$\text{when } r_1A<s_1B,$$
$$\text{then } r_1C<s_1D.$$

Hence (3) holds.

But when (1) and (3) hold, then by Stolz's Theorem

$$(A:B)=(C:D).$$

Hence (1) and (4) are sufficient.

Similarly (3) and (6) are sufficient.

Article 77

Suppose in the *fourth* place that (2) holds good for a *single* value of r, say r_1, and a *single* value of s, say s_1,

$$\text{i.e. } r_1A=s_1B,\ r_1C=s_1D.$$

Now suppose that for any pair of values of r, s, say $r=r_2$, $s=s_2$,

$$r_2A>s_2B,$$
$$\text{to prove that } r_2C>s_2D,$$
$$\because\ r_1A=s_1B,$$
$$\therefore\ r_1s_2A=s_1s_2B,$$
$$\because\ r_2A>s_2B,$$
$$\therefore\ s_1r_2A>s_1s_2B,$$
$$\therefore\ s_1r_2A>r_1s_2A,$$

$$\therefore s_1 r_2 > r_1 s_2,$$
$$\therefore s_1 r_2 C > r_1 s_2 C,$$
$$\therefore s_1 r_2 C > s_2 (r_1 C),$$
$$\therefore s_1 r_2 C > s_2 s_1 D,$$
$$\therefore r_2 C > s_2 D.$$

Similarly it can be proved that if

$$r_3 A < s_3 B,$$
$$\text{then } r_3 C < s_3 D.$$

So that if (2) hold for a *single* value of r and a single *value* of s, then (1) or (3) will hold for *any* values of r, s whatever.

The like conclusion follows in regard to (5).

The Seventh Definition

Article 78

The *seventh* definition is the test for distinguishing the greater of two *Unequal* Ratios from the smaller.

The ratio of A to B is greater than that of C to D if a single pair of integers r s can be found such that

$$\text{if } rA > sB, \text{ then either } rC = sD \text{ or } rC < sD.$$

$$\text{Hence } (A : B) > (C : D)$$

if a single pair of integers r, s exist such that either

$$rA > sB, \text{ but } rC < sD, \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$\text{or } rA > sB, \text{ but } rC = sD. \quad \ldots\ldots\ldots\ldots\ldots (2)$$

With these might have been included also the possibility that

$$rA = sB, \text{ but } rC < sD. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

It will now be shown that in the cases (2) and (3) other integers r', s' exist such that

$$r'A > s'B, \text{ but } r'C < s'D,$$

which is of the form (1).

Suppose that when $r = r_1$, $s = s_1$, the form (2) holds.

$$\therefore r_1 A > s_1 B, \text{ but } r_1 C = s_1 D.$$

By Archimedes' Axiom an integer n exists such that

$$n(r_1A-s_1B)>B,$$
$$\therefore\ nr_1A>(ns_1+1)B.$$
$$\text{Since } r_1C=s_1D,$$
$$\therefore\ nr_1C=ns_1D,$$
$$\therefore\ nr_1C<(ns_1+1)D.$$

Hence putting $nr_1=r'$, $ns_1+1=s'$ it follows that integers r', s' exist such that

$$r'A>s'B, \text{ but } r'C<s'D,$$

which is of the form (1).

Article 79

Suppose next that when $r=r_2$, $s=s_2$ the form (3) holds.

$$\therefore\ r_2A=s_2B, \text{ but } r_2C<s_2D.$$

Then by Archimedes' Axiom an integer n exists such that

$$n(s_2D-r_2C)>C,$$
$$\therefore\ (nr_2+1)C<ns_2D,$$
$$\text{but } r_2A=s_2B,$$
$$\therefore\ nr_2A=ns_2B,$$
$$\therefore\ (nr_2+1)A>ns_2B.$$

Hence putting $nr_2+1=r'$, $ns_2=s'$ integers r', s' exist such that

$$r'A>s'B, \text{ but } r'C<s'D,$$

which is of the form (1).

Consequently it is possible to leave out of consideration the forms (2) and (3), since values of r, s can always be found for which the form (1) holds when either (2) or (3) holds.

Article 80

In connection with the Seventh Definition an important point arises, which is not considered by Euclid.*

In order to simplify the explanation I will take the condition that $(A:B)$ may be greater than $(C:D)$ in the form that some integers r, s must exist such that

$$rA>sB, \text{ but } rC<sD. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

* Heath's Euclid, Vol. II, p. 130.

Therefore the condition that $(A : B)$ may be less than $(C : D)$ will be that some integers r', s' must exist such that

$$r'A < s'B, \text{ but } r'C > s'D. \qquad \dots\dots\dots\dots\dots (4)$$

Now inasmuch as Euclid has not defined a ratio as a magnitude, it is essential *from his point of view* to show that no integers r, s, r', s' can exist which satisfy simultaneously (1) and (4).

To prove this :

It follows from (1) that

$$s'rA > s'sB, \; s'rC < s'sD,$$

and from (4)

$$sr'A < ss'B, \; sr'C > ss'D.$$

$$\text{Since } s'sB = ss'B,$$

$$\text{it follows that } s'rA > sr'A,$$

$$\therefore\; s'r > sr'. \qquad \dots\dots\dots\dots\dots (5)$$

$$\text{Since } s'sD = ss'D,$$

$$\text{it follows that } s'rC < sr'C,$$

$$\therefore\; s'r < sr'. \qquad \dots\dots\dots\dots\dots (6)$$

But (5) and (6) cannot co-exist.

Hence no integers r, s, r', s' can exist which satisfy (1) and (4) simultaneously.*

That the conditions (1) and (4) can never hold simultaneously is seen at once when the mode of treating the subject adopted in the preceding chapters is followed, in which $(A : B)$ and $(C : D)$ are numbers.

For (1) is equivalent to

$$(A : B) > \frac{s}{r} > (C : D),$$

and therefore makes the number $(A : B)$ greater than the number $(C : D)$.

Whilst (4) is equivalent to

$$(A : B) < \frac{s'}{r'} < (C : D),$$

and therefore makes the number $(A : B)$ less than the number $(C : D)$.

But the number $(A : B)$ cannot be at the same time both greater and less than the number $(C : D)$. Consequently (1) and (4) can never be satisfied simultaneously.

* See Heath's Euclid, Book V. Def. 7.

The Fifth and the Seventh Definitions constitute together the basis on which the structure of the Fifth Book of Euclid's *Elements* is reared. They alone of all the definitions prefixed to the Fifth Book effectively count.

The Third and the Fourth Definitions are not sufficiently definite in form to enter in reality into the argument.

Article 81

Euclid's Fifth Definition does not define ratio, but it makes it possible to decide the extremely important question whether two ratios are equal, whether they be ratios of commensurable or incommensurable magnitudes. Let us consider for a little the question, What did Euclid understand by a ratio? In particular, did he or did he not regard it as a number? To this question there is in his work no clear or unambiguous answer. I can only set forth the evidence on both sides.

On the one hand

(*a*) If he regarded a ratio as a magnitude, why does he give a demonstration of Prop. 11, viz.:

$$\begin{aligned} \text{If } (A:B)&=(C:D),\\ \text{and if } (E:F)&=(C:D),\\ \text{then } (A:B)&=(E:F). \end{aligned}$$

Simson says, " The words greater, the same or equal, lesser have a quite different meaning when applied to magnitudes and ratios, as is plain from the Fifth and Seventh Definitions of Book V."

" That those things which are equal to the same are equal to one another is a most evident axiom when understood of magnitudes, yet Euclid does not make use of it or infer that those ratios which are the same to the same ratio, are the same to one another: but explicitly demonstrates this in Prop. 11 of Book V."

(*b*) If he regarded a ratio as a magnitude, why does he give a demonstration of Prop. 13; viz:

$$\begin{aligned} \text{If } (A:B)&=(C:D),\\ \text{and if } (C:D)&>(E:F),\\ \text{then } (A:B)&>E:F), \end{aligned}$$

which on the hypothesis that ratios are magnitudes amounts only to this:

If each of the symbols X, Y, Z represents a ratio, and if $X=Y$, and $Y>Z$, then is $X>Z$.

On the other hand, if he did not regard a ratio as a magnitude,

(*c*) Why does he speak of one ratio being greater than another in the Seventh Definition? The term greater, if used in the ordinary sense, can refer only to magnitudes.

(*d*) Why does he not supply a demonstration in connection with the Seventh Definition showing that if the four magnitudes A, B, C, D are such that integers r, s exist, such that

$$rA>sB, \text{ but } rC<sD, \quad \ldots\ldots\ldots\ldots\ldots (\text{I.})$$

which are the conditions that $A:B>C:D$; then no integers r', s' can exist, such that

$$r'A<s'B, \text{ but } r'C>s'D, \quad \ldots\ldots\ldots\ldots (\text{II.})$$

which are the conditions that $A:B<C:D$?

Such a demonstration is unnecessary if a ratio is a magnitude, for $(A:B)>(C:D)$ and $(A:B)<(C:D)$ are inconsistent if $(A:B)$ and $(C:D)$ are magnitudes. On the other hand, if it is only a question of a comparison of the distribution of the multiples of A amongst those of B with the distribution of the multiples of C amongst those of D, then a demonstration of the incompatibility of the conditions (I.) with the conditions (II.) is essential (see Art. 80).

(*e*) Why should he say in the proof of Prop. 10, viz.:

$$\begin{aligned} &\text{If } (A:C)>(B:C), \\ &\quad \text{then } A>B. \\ &\text{If } (C:A)>(C:B), \\ &\quad \text{then } A<B, \end{aligned}$$

that the statements

$$(A:C)=(B:C) \text{ and } (A:C)<(B:C)$$

are inconsistent with

$$(A:C)>(B:C)\ ?$$

For if ratios are magnitudes, this needs no proof; but if they are not so regarded, then the proof given by Euclid of this proposition does not follow from his definitions (see Art. 88).

Article 82

As to this matter, modern writers are equally in conflict. On the one hand, Stolz, in his *Vorlesungen über allgemeine Arithmetik* (Erster Theil, p. 94), says, "In den auf uns gekommenen geometrischen Schriften des Alterthumes findet sich keine deutliche Spur der Ansicht, dass das Verhältniss zweier incommensurabelen Grössen eine Zahl sei."

Whilst, on the other hand, Max Simon (*Euklid und die sechs planimetrischen Bücher*), so far from agreeing in the usual view that the Greeks saw in the irrational no number, thinks it clear from Euclid, Book V., that they possessed a notion of number in all its generality.

The Arithmeticians and Algebraists of the Middle Ages called the ratios of incommensurable magnitudes "Numeri ficti" or "Numeri surdi," and regarded them as a necessary evil which had to be endured. Michael Stifel, in his *Arithmetica Integra*, published in 1544, treated them as real numbers. His words amount to an assertion that each irrational number as well as each rational number has a single definite place in the ordered number series.*

II. Propositions (First Group). Nos. 1, 2, 3, 5, 6.

Article 83

Denoting positive integers by small letters and magnitudes by large letters these propositions are :

1. $r(A+B+C+\ldots)=rA+rB+rC+\ldots.$
2. $(a+b+c+\ldots)R=aR+bR+cR+\ldots.$
3. $r(sA)$ is the same multiple of A as $r(sB)$ is of B.
5. If $A>B$, then $r(A-B)=rA-rB$.
6. If $a>b$, then $(a-b)R=aR-bR$.

With regard to the first group of propositions I merely call attention to the fact that I have in the preceding chapters replaced No. 3 by another proposition which is more useful, viz. :

$$(r(s))A=r(sA)=s(rA)=(s(r))A.$$

* *Encyklopädie der mathematischen Wissenschaften*, Vol. I. A. 3, p. 51.

Euclid would have found very great difficulty in expressing this clearly in his notation.

III. Propositions (Second Group). Nos. 4, 7, 11, 12, 15 and 17

Article 84

These propositions express properties of Equal Ratios, and are deduced by Euclid *directly* from the Fifth Definition.

(The sign of equality will be used in place of the sign : : for "is the same as.")

4. If $(A : B)=(C : D)$, then $(rA : sB)=(rC : sD)$.
7. If $A=B$, then $(A : C)=(B : C)$ and $(C : A)=(C : B)$.
11. If $(A : B)=(C : D)$, and if $(E : F)=(C : D)$, then $(A : B)=(E : F)$.
12. If $(A : B)=(C : D)=(E : F)$, and if all the magnitudes are of the same kind, then
$$(A : B)=(A+C+E : B+D+F).$$
15. $(A : B)=(nA : nB)$.
17. If $(A+B : B)=(C+D : D)$, then $(A : B)=(C : D)$.

No. 15 is a particular case of No. 12 (Art. 46).

Article 85

From No. 17 a very important deduction can be made, bearing on Euclid X. 6, viz.:

$$\textit{If } (a : b)=(X : Y),$$
$$\textit{then } X=aG,\ Y=bG.$$

Suppose that X is divided into a equal parts and we call each part G.

$$\text{Then } (a : b)=(aG : bG).$$
$$\text{But } X=aG,$$
$$\therefore (a : b)=(X : bG),$$
$$\text{but } (a : b)=(X : Y),$$
$$\therefore (X : bG)=(X : Y),$$
$$\therefore Y=bG.$$

Hence X, Y have a common measure G.

It will be noticed that the proof of this depends on the assumption made by Euclid in this and one other place only that the magnitude X can be divided into any number a of equal parts (see Art. 3 (4)).

I am indebted to my friend Mr. Rose-Innes for the following proof of this proposition which does not make this assumption.

It follows from Euc. V. 17, viz.:

$$\text{If } (A+B:B)=(C+D:D),$$
$$\text{then } (A:B)=(C:D),$$

that

$$\text{If } (A:B)=(C:D)$$
$$\text{and } A>B,$$
$$\text{then } (A-B:B)=(C-D:D).$$

We start from

$$(a:b)=(X:Y).$$

We may suppose $a>b$, for if not we could begin with

$$(b:a)=(Y:X).$$

Suppose then

$$a>b,$$
$$(a:b)=(X:Y),$$
$$\therefore (a-b:b)=(X-Y:Y)\ldots\ldots \text{ Euc. V. 17.}$$

If $a-b>b$, this can be repeated

$$(a-2b:b)=(X-2Y:Y).$$

Suppose this can be done q_1 times,

$$\therefore (a-q_1b:b)=(X-q_1Y:Y).$$

Let $a-q_1b=r_1$, $X-q_1Y=R_1$,

$$\therefore (r_1:b)=(R_1:Y).$$
$$\text{Then } (b:r_1)=(Y:R_1).$$

Apply Euc. V. 17 again as many times as possible.

Suppose we get

$$(b-q_2r_1:r_1)=(Y-q_2R_1:R_1).$$

Suppose $b-q_2r_1=r_2$, $Y-q_2R_1=R_2$

$$\text{we get } (r_2:r_1)=(R_2:R_1).$$

Going on thus, it is plain from the relations

$$a-q_1b=r_1,$$
$$b-q_2r_1=r_2,$$
$$\ldots\ldots\ldots\ldots$$

that we are in fact applying the process for finding the greatest common measure of the integers a, b.

Now the integers a, b have certainly unity for a common measure. Hence the process will ultimately come to an end after a *finite* number of steps.

Suppose we have

$$a - q_1 b = r_1,$$
$$b - q_2 r_1 = r_2,$$
$$\cdots\cdots\cdots\cdots$$
$$r_{n-3} - q_{n-1} r_{n-2} = r_{n-1},$$
$$r_{n-2} - q_n r_{n-1} = 0.$$

Then we have at the same time

$$X - q_1 Y = R_1,$$
$$Y - q_2 R_1 = R_2,$$
$$\cdots\cdots\cdots\cdots$$
$$R_{n-3} - q_{n-1} R_{n-2} = R_{n-1},$$
$$R_{n-2} - q_n R_{n-1} = 0,$$
$$\therefore R_{n-1} \text{ measures } R_{n-2},$$
$$\therefore R_{n-1} \text{ measures } R_{n-3},$$

and so on.

$$R_{n-1} \text{ measures } Y \text{ and } X.$$

Thus X and Y have a common measure R_{n-1}.

X contains R_{n-1} as many times as a contains r_{n-1}.

Y contains R_{n-1} as many times as b contains r_{n-1}.

It follows that Euclid might have avoided making the assumption that a magnitude can be divided into any number of equal parts.

In the mode of treatment adopted in this book the assumption was required for the demonstration of Prop. X. (Art. 31). Without it ratios of magnitudes of the same kind could not have been compared with rational numbers.

IV. Propositions (Third Group). Nos. 8, 10, 13.

Article 86

8. (i.) If $A > B$, then $(A : C) > (B : C)$.
   (ii.) If $A < B$, then $(C : A) > (C : B)$.

10. (i.) If $(A:C)>(B:C)$, then $A>B$.
(ii.) If $(C:A)>(C:B)$, then $A<B$.
13. If $(A:B)=(C:D)$, and if $(C:D)>(E:F)$,
then $(A:B)>(E:F)$.

These propositions deal with properties of Unequal Ratios. The effective part of them is contained in the 8th Proposition, which consists almost wholly in proving that if A, B, C be three magnitudes *of the same kind*, and if A be greater than B, then integers n, t exist such that

$$nA>tB>nC.$$

(This is the Corollary to Prop. VIII., Art. 12, in this book.)

In the proof of this the idea of ratio is not involved.

Euclid's proofs of the propositions in this group depend on his Seventh Definition, but he employs them to prove properties of Equal Ratios and for this purpose only.

Proposition 8

Article 87

The first part of this proposition, viz.:

If $A>B$, then $(A:C)>(B:C)$,

is deduced by Euclid from his Seventh Definition.

In the arrangement of the subject presented in this book the fact expressed by the proposition is regarded as one of the fundamental principles from which the conditions of both the Fifth and Seventh Definitions are deduced.

Proposition 10

Article 88

Euclid assumes in his proof that the statements

$$(A:C)=(B:C) \text{ and } (A:C)<(B:C)$$

are inconsistent with

$$(A:C)>(B:C).$$

Of course, if ratios are magnitudes, this needs no proof; but Euclid's Fifth and Seventh Definitions do not entitle him to regard them as such.

As Simson pointed out, the proof on Euclid's lines should be as follows:

If $(A:C)>(B:C)$, then by Definition 7 a pair of integers r, s exist such that

rA is greater than sC,
but rB is not greater than sC.
Consequently $rA>rB$,
$\therefore A>B$.
Hence if $(A:C)>(B:C)$,
then $A>B$.

This proof is not open to criticism.

V. Propositions (Fourth Group). Nos. 9, 14, 16, and 18–25.

Article 89

9. (i.) If $(A:C)=(B:C)$, then $A=B$.
(ii.) If $(C:A)=(C:B)$, then $A=B$.

14. If A, B, C, D are magnitudes of the same kind, and if $(A:B)=(C:D)$, then $B \gtreqless D$ according as $A \gtreqless C$.

16. If A, B, C, D are magnitudes of the same kind, and if $(A:B)=(C:D)$, then $(A:C)=(B:D)$.

18. If $(A:B)=(C:D)$, then $(A+B:B)=(C+D:D)$.

19. If $(A+C:B+D)=(C:D)$, then $(A:B)=(A+C:B+D)$.

20. If $(A:B)=(T:U)$, and if $(B:C)=(U:V)$, then $T \gtreqless V$ according as $A \gtreqless C$.

21. If $(A:B)=(U:V)$, and if $(B:C)=(T:U)$, then $T \gtreqless V$ according as $A \gtreqless C$.

22. If $(A:B)=(T:U)$, and if $(B:C)=(U:V)$, then $(A:C)=(T:V)$.

23. If $(A:B)=(U:V)$, and if $(B:C)=(T:U)$, then $(A:C)=(T:V)$.

24. If $(A:C)=(X:Z)$, and if $(B:C)=(Y:Z)$, then $(A+B:C)=(X+Y:Z)$.

25. If A, B, C, D are four magnitudes of the same kind, and if $(A:B)=(C:D)$, and if A be the greatest of them, then $A+D>B+C$.

All these propositions deal with properties of *equal* ratios, but Euclid's proofs depend directly or indirectly on Props. 8, 10, 13, and therefore ultimately on the Seventh Definition, so that the proofs depend on properties of *unequal* ratios.

Their proofs can, however, be obtained from the Fifth Definition, without using the Seventh Definition, and then 14 can be deduced immediately from 16, 20 from 22, and 21 from 23.

The proof of this last statement is given in the preceding chapters.

Article 90

There is one point as to the connection between the Fifth and Seventh Definitions in regard to which a misconception may arise. In obtaining the conditions for the equality of ratios in the preceding chapters, use was made of such inequalities as

$$(A : B) > \frac{n}{r} \text{ and } (A : B) < \frac{n}{r},$$

and also of the fundamental assumption that

$$\text{if } A > B, \text{ then } (A : C) > (B : C),$$

but the Seventh Definition itself was not used.

Thus the Fifth and Seventh Definitions are independent of one another.

Hence arguing a priori *it might be expected that it would prove to be possible to obtain all the properties of Equal Ratios by means of the Fifth Definition only, if that definition is a full and complete one ; as has in fact been shown.*

I proceed now to make remarks on some propositions in this group.

Propositions 14 *and* 16

Article 91

In the preceding chapters Prop. 16 was proved first, and Prop. 14 was deduced as a Corollary.

Euclid proves Prop. 14 first, and derives Prop. 16 from it.

If Euclid's proofs given below be compared with the proof of Prop. 16 (see Chapter VIII, Prop. XVIII., Art. 51) it will be seen how many steps there are in Euclid's work which do not suggest themselves to any one trying to follow his argument ; whilst the proof given in this book is much more direct, the successive steps arising naturally from one another.

Euclid's Proof of Props. V. 14, 16.

Prop. 14. If $(A:B)=(C:D)$,
and if $A>C$,
$\therefore (A:B)>(C:B)$, ... Euc. V. 8.
$\therefore (C:D)>(C:B)$,
$\therefore D<B$, ... Euc. V. 10.
$\therefore B>D$.
If $A=C$,
$(A:B)=(C:B)$, ... Euc. V. 7.
$\therefore (C:D)=(C:B)$,
$\therefore D=B$, ... Euc. V. 9.
$\therefore B=D$.
If $A<C$,
$(A:B)<(C:B)$, ... Euc. V. 8.
$(C:D)<(C:B)$,
$\therefore D>B$, ... Euc. V. 10.
$\therefore B<D$.

Prop. 16. If $(A:B)=(C:D)$,
then $(A:B)=(rA:rB)$,....... Euc. V. 15.
$(C:D)=(sC:sD)$,....... Euc. V. 15.
$\therefore (rA:rB)=(sC:sD)$........ Euc. V. 11.
If $\therefore rA>sC$, then $rB>sD$,..... Euc. V. 14.
$rA=sC$ .... $rB=sD$,..... Euc. V. 14.
$rA<sC$ .... $rB<sD$..... Euc. V. 14.
$\therefore (A:C)=(B:D)$.

The use of V. 10 in the proof of V. 14 is not an easy matter for the beginner, and there is nothing to suggest the introduction of V. 15 in the proof of V. 16 in the two places where it is employed.

Proposition 18

Article 92

A very important point is raised by the form of Euclid's proof of Prop. 18. He assumes that if A and B be two magnitudes *of the same kind*, and C be another magnitude, then a magnitude D *of the same kind* as C exists, such that $(A : B) = (C : D)$.

This, as was first pointed out by Saccheri, does not follow from Euclid's definitions. Simson showed how the proof could be made to depend on the definitions,* but the proposition assumed is one of great importance. De Morgan's attempt to prove it (*l.c.*, pp. 60–61, and Heath, Vol. II., p. 171) is insufficient, and in fact the theorem cannot be proved, except in the special case where A, B, C are segments of straight lines, or in cases easily reducible to this case. The result assumed depends on the Axiom by which continuity is ascribed to the system of magnitudes *of the same kind* as C. This Axiom corresponds to the Cantor-Dedekind Axiom, by which the gaps in the system of rational numbers are filled up by the creation of irrational numbers (see Chapter XI., Prop. XXIV., Art. 70).

Euclid makes the same assumption as to the existence of the Fourth Proportional in the Twelfth Book in Propositions 2, 5, 11, 12, and 18. It has been seen in Arts. 61, 61 (*a*), 61 (*b*) how by the aid of Stolz's Theorem or without it the assumption can be avoided in Euc. XII. 2, and the same method can be employed to avoid the assumption in Euc. XII. 5, 11, 12, and 18.

Propositions 20 *and* 22

Article 93

In the preceding chapters Prop. 22 was proved first and Prop. 20 deduced as a Corollary.

Euclid proves Prop. 20 first and uses it to prove Prop. 22.

Propositions 21 *and* 23

Article 94

In the preceding chapters Prop. 23 was proved first and Prop. 21 deduced as a Corollary.

Euclid proves Prop. 21 first and uses it to prove Prop. 23.

Remarks similar to those made on Props. 14 and 16 apply to Props. 20 and 22, and to Props. 21 and 23.

* See Prop. XVI. Art. 47

Propositions 22 *and* 23

ARTICLE 95

The 22nd and 23rd Propositions bear on what Euclid calls the Compounding of Ratios, but what we should call the Multiplication of Ratios.

Prop. 23 shows that the order of multiplication has no effect on the result.

It is proved that if $(A : B)=(U : V)$, and if $(B : C)=(T : U)$, then $(A : C)=(T : V)$.

Now, if A and B were commensurable, it would be possible to write $A=\lambda B$, and then λ would be a rational fraction. But if A and B are *not* commensurable, *and if we still put* $A=\lambda B$, *we must regard* λ *as the symbol of some kind of operation performed on* B, *which gives as its result the magnitude* A.

Let us further regard the relation $(A : B)=(U : V)$ *as justifying the statement that since* $A=\lambda B$, $\therefore$ $U=\lambda V$; i.e. we consider that U can be obtained by performing on V an operation which, if performed on B, would give A.

In like manner, if $(B : C)=(T : U)$, and if $B=\mu C$, then $T=\mu U$, where μ is the symbol of some other operation.

$$\therefore\ A=\lambda B, \text{ and } B=\mu C; \quad \therefore\ A=\lambda(\mu C).$$
$$\text{Also } T=\mu U, \text{ and } U=\lambda V; \quad \therefore\ T=\mu(\lambda V).$$

But by Prop. 23, if $(A : B)=(U : V)$, and if $(B : C)=(T : U)$, then $(A : C)=(T : V)$.

If therefore we put $A=\nu C$, then we must have $T=\nu V$, where ν is the symbol of another operation.

Equating the two values of A, we have

$$\lambda(\mu C)=\nu C;$$
$$\therefore\ \lambda(\mu \text{ and } \nu \text{ are equivalent operations.}$$

Equating the two values of T, we have

$$\mu(\lambda V)=\nu V;$$
$$\therefore\ \mu(\lambda \text{ and } \nu \text{ are equivalent operations.}$$

Hence the operation denoted by ν is the same as those denoted by $\lambda($$\mu$ and $\mu($$\lambda$ respectively;

$$\therefore\ \lambda(\mu \text{ and } \mu(\lambda \text{ are equivalent operations.}$$

Hence the same result is obtained if the operation μ is first performed and then the operation λ, as is obtained when the operation λ is first performed and then the operation μ.

Proposition 24

Article 96

The 24th Proposition corresponds to what we should call the Addition of Ratios. If we attempt to make use of it, or of the 22nd or 23rd Proposition, the question of the existence of the fourth proportional to three magnitudes, of which the first and second are *of the same kind*, is immediately raised, a matter to which I have already alluded in Art. 92.

Proposition 25

Article 97

"If four magnitudes of the same kind are in proportion, then the sum of the greatest and least exceeds the sum of the other two"—is an important one.

My interest in it was first aroused by the fact that De Morgan (*l.c.*, p. 72) points out its bearing upon the notions on which the first theory of logarithms was founded.

Euclid does not make use of this proposition in his *Elements*. Sir T. L. Heath tells me that it appears once only in Greek Geometry, viz. in the 69th Theorem of the 7th Book of the *Collectiones Mathematicae* of Pappus, where it is used to show that the shortest intercept on all straight lines drawn through the middle point of the base of an isosceles triangle is the base itself, and that as the straight line turns round the middle point of the base, the intercept made by the sides continually increases until the line becomes parallel to one of the equal sides of the triangle.

It is difficult, if not impossible, to ascertain for what purpose Euclid recorded it. I imagine that most people who look at it think it a comparatively useless result, except in the special case where the second and third terms of the proportion are equal, when it expresses the fact that the arithmetic mean of two magnitudes exceeds their geometric mean.

Article 98

This special case is, however, very important. It can be used to prove two limits of the utmost value in the Calculus.

(i.) If n be a positive integer, and $a>1$, then $\underset{n\to\infty}{\mathrm{L}}\, a^n=+\infty$.

To see this, let $V, V_1, V_2 \ldots V_n$ be in continued proportion ;

$$\therefore V : V_1 = V_1 : V_2 = \ldots = V_{n-2} : V_{n-1} = V_{n-1} : V_n.$$

Suppose $V<V_1$, then from $V : V_1 = V_1 : V_2$, it follows by Euc. V. 25 that $V+V_2>2V_1$;

$$\therefore V_2-V_1>V_1-V.$$

Similarly,

$$V_3-V_2>V_2-V_1>V_1-V$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$V_n-V_{n-1}>V_{n-1}-V_{n-2}>V_1-V.$$

From these it follows that $V_n-V_1>(n-1)(V_1-V)$; and

$$\therefore V_n-V>n(V_1-V) \text{ and } \therefore V_n>n(V_1-V) ;$$

$$\therefore \frac{V_n}{V}>n\left(\frac{V_1}{V}-1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $V<V_1$, if we put $\frac{V_1}{V}=a$, then $a>1$; and from the continued proportion it follows that $\frac{V_n}{V}=a^n$; $\therefore$ (1) gives $a^n>n(a-1)$.

From this, by the aid of Archimedes' Axiom, it follows that, if $a>1$, and n be a positive integer, $\underset{n\to\infty}{\mathrm{L}}\, a^n=+\infty$. (Cf. De Morgan, *l.c.*, p. 72.)

Article 99

(ii.) If n be a positive integer, and $a<1$, then $\underset{n\to\infty}{\mathrm{L}}\, a^n=+0$.

If in the same continued proportion as in the last case we put $\frac{V}{V_1}=a$, then $a<1$, and $\frac{V_1}{V_n}=a^{n-1}$.

Now, from $V_n>n(V_1-V)$ it follows that $\frac{V_n}{V_1}>n\left(1-\frac{V}{V_1}\right)$,

$$\text{i.e. } \frac{1}{a^{n-1}}>n(1-a) ;$$

$$\therefore \frac{1}{n(1-a)} > a^{n-1} > a^n; \quad \because a<1, \quad \therefore \frac{1}{n(1-a)} > a^n.*$$

From this, by the aid of Archimedes' Axiom, it follows that $\underset{n\to\infty}{\mathrm{L}}\, a^n = +0$, where $a<1$, and n is a positive integer.

Upon this last result depends the proof of the convergency of an infinite geometric progression, when the common ratio is less than unity.

Article 100

The proof which Euclid gives of X. 1 covers the proposition that

$$\underset{n\to\infty}{\mathrm{L}}\, a^n = +0, \text{ if } a<\tfrac{1}{2}.$$

It may be that Euclid included the 25th Proposition in the Fifth Book merely to establish the fact that the arithmetic mean of two unequal magnitudes was greater than their geometric mean, though if that were the case one would think he would have said so explicitly, or one may perhaps be tempted to believe that he had by means of Euc. V. 25 obtained a proof of the result $\underset{n\to\infty}{\mathrm{L}}\, a^n = +0$, not merely for the range $0<a<\frac{1}{2}$, but also for the wider range $0<a<1$; but that he put it aside because it was not required for the geometrical applications for which Euc. X. 1 was sufficient, and also because the statement of its proof would have been somewhat intricate in his notation. But this of course is mere speculation.

* There is a simpler way of obtaining this result, due, I think, to Professor Hudson, and published some time ago in the *Mathematical Gazette*, as follows:

In the identity

$$\frac{1-a^n}{1-a} = 1 + a + a^2 + \ldots + a^{n-1}$$

suppose $0 < a < 1$,

then each term on the right is greater than a^n, and therefore the right-hand side is greater than na^n.

The left-hand side is less than $\frac{1}{1-a}$,

$$\therefore \frac{1}{1-a} > na^n; \qquad \therefore \frac{1}{n(1-a)} > a^n;$$

if $0<a<1$ and n be a positive integer.

INDEX

(The references are to the articles.)

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