E. J. Senkler
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THE THEORY

OF

STATICS;

WITH NUMEROUS

PRACTICAL APPLICATIONS.

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PREFACE.

In a science that has existed from the time of Archimedes, and occupied the attention of the most celebrated Mathematicians of many centuries, much new matter cannot be expected. Little more now remains to be done than to collect, arrange, and simplify the results already obtained; this therefore is the task I proposed to myself, in undertaking the preparation of the following treatise on Statics.

That a treatise was necessary, which should present the subject under one continuous chain of reasoning, with a uniform system of notation, and having its several parts dependent on the same first principles and definitions, will not I think be denied. In what degree, however, the present will supply the want of such a treatise, must now be left to the public to determine.

The first chapter commences with some general notions of mechanical force. I have purposely avoided setting forth any abstract ideas of its nature; inasmuch as such considerations, so far from throwing light on the subject, do generally involve us in the greater darkness the further we pursue them.

It appeared to me, that while we are ignorant of motion, we must of necessity remain also ignorant of the *existence of force;* and as the idea of matter does not include in it the idea of motion, we infer that motion is not an essential, but a consequential state of matter; for it can scarcely be imagined how matter can pass into a certain state, which is not essential to its existence, without the agency of an independent cause. Wherefore, without further inquiry into the nature of this cause, I have denominated it *Force;* and as motion is the most simple, natural, and general evidence our senses can receive of the existence of force; and seeing, moreover, that we have originally (that is to say, independently of subsequent deductions of reason) no other test of its action, I have not scrupled to take motion as the measure of its *intensity* in Statics, as well as in Dynamics.

I am aware that the introduction of motion as a necessary fundamental idea in a treatise on Statics, (the science of Rest), does, in a certain degree, appear absurd, and has accordingly been reprobated by some writers; but, notwithstanding all that has been said upon the distinction between Statical and Dynamical force, it does not appear that any real difference in the *nature* of the two forces has ever been clearly made out; the only difference established having reference rather to the state of the body acted upon, or of that in which the force resides, than to the forces themselves. No one, I believe, will question the correctness of the two following fundamental principles:—

1st. We know nothing of force but what we learn from its effects.

2ndly. Force is that which produces, or tends to produce motion.

Now the latter virtually confesses that motion is the *only* natural and uninfluenced effect of force; and, therefore, we infer from the former, that motion is the only natural and uninfluenced means we can have of discovering the properties of force. Upon these grounds, as well as the considerations before mentioned,—that there is no difference in the nature of Dynamical and Statical force; I have (Art. 5) taken the direction in which a free particle begins to move when acted on by a force as the direction in which the force acts; and have used the quantity of motion produced in a given time as the measure of its intensity.

If this measure be found inconvenient in practice, a more convenient method of measuring forces must be *investigated,* but we cannot ***assume*** weight as such a measure, because such an assumption supposes us to be acquainted with the nature of the force of gravity, when, at the same time, we are supposed altogether ignorant of every force.

In Art. 24, the reader will find a new demonstration of the fundamental proposition of the science,—the composition of two equal forces acting at a point. It does not require a knowledge of Mathematics beyond the most simple elements of Algebra, and Plane Trigonometry; this simplicity of principle, it is hoped, will, when the importance of the proposition is considered, be deemed a sufficient apology for its length: for it does not appear, that brevity has ever yet been attained in it, without a violation of that degree of simplicity, which ought to be accounted superior to all other considerations, in establishing a proposition which is the basis of a science, and of the truth of which a clear perception is therefore absolutely necessary for securing a proper conviction of the truth of the numerous propositions which rest upon it.

Many writers on Statics commence with the consideration of the *lever,* and take certain experimental results* respecting its equilibrium as the fundamental principles of the science. This method, though allowable, on account of its great simplicity, when the science was in its infancy, ought not now to be adopted, when, as is agreed upon by all, the science has attained a very considerable degree of maturity. In such a system of Statics, the composition of forces acting at a point is usually deduced from the conditions of equilibrium of two forces acting on the bent lever, which have been previously deduced from the straight lever, by introducing the principle of the transmissibility of force. Now it is manifest, that when forces act at a point, no transmission can take place; and, consequently, the composition and resolution of forces would remain unaltered, if the principle of transmission were not true; wherefore, in establishing the composition of force by the lever, or by any other method involving the

* Among these, we find it stated as an axiom,—

That two equal weights hanging perpendicularly from the extremities of the equal arms of a straight lever, will be in equilibrium.

I am of opinion, however, that this ought to be deduced from the nature of force. It is, I suppose, considered as an axiom, that the equal weights will balance from the similarity of circumstances under which they exert their influence. But this is the kind of reasoning which in Roberval's Balance (page 274) would lead us into error, and therefore ought to be looked upon with suspicion. I believe it may be safely affirmed, that the action of force on no greater portion of matter than a single particle ought to be considered as axiomatic.

transmission of force, we introduce a superfluous principle, and thereby make it depend upon experiments foreign to the subject, when in fact it ought to follow from the definition of force, and be independent of all experiments, whose sole object is not limited to the illustration of the nature of force itself: and, therefore, every demonstration of it, which professes to be grounded on philosophic principles, and the rules of strict and logical reasoning, ought not to involve the transmission of force either directly or by implication.

In the second chapter the law of the transmission of force is proved by a simple and obvious experiment, and its effect in reducing two very extensive classes of forces to the formulæ of Chapter I, is pointed out in Arts. 44, 45. I have thought it necessary to insist upon the impossibility of establishing this principle by theory, because several authors have attempted to do so. Of attempts of this kind, those which depend upon removing successively different portions of the rigid body, until it is reduced to a straight line, are evidently not at all to the purpose, since they take for granted that these portions may be removed without prejudicing the effect, which is precisely the property to be proved. The demonstration given by Poinsot (Elémens de Statique, No. 13, page 16, 5ième édition,) also begs the question. His words are "Mais en considérant la force P (appliquée en A) et son égale et contraire —P′ appliquée en B, il est *manifeste que leur effet est aussi nul.* On peut donc les supprimer, ..." The property which is here said to be manifest is that which the demonstration was designed to prove.

It will not, I think, be said, that I have laid too great stress upon this point, when it is considered that there is scarcely a single property of forces acting on a rigid body which can be demonstrated without introducing the principle in question; and, indeed, it is of such importance, that were it not for the single exception of a very remarkable class of forces denominated *couples,* all forces whatever acting on a rigid body might at once be reduced by its means to others acting at a point. With respect to the last mentioned class of forces, considerable care has been taken clearly to point out their origin, and to shew that their separate treatment is not so much a matter of convenience as of necessity. In the enunciations and demonstrations of their properties I have, for the preservation of uniformity, frequently deviated considerably from their admirable inventor Poinsot.

The fourth chapter contains a completely new demonstration of the important principle of Virtual Velocities. The proof given by Lagrange (Mécanique Analytique, page 22, et seq.) in which the forces are replaced by systems of pullies, has of all others obtained the greatest celebrity; but is, I think, liable to two fatal objections.

First. In the sentence (page 24 ibid) "car le poids tendant toujours à descendre s'il y a un déplacement du système qui lui permette de descendre, il descendra nécessairement et produira ce déplacement dans le système," (which is the *point d'appui* of the demonstration, and which Lagrange seems, from the manner in which he states it, to think very evident) the principle to be proved is indirectly taken for granted.

Second. After the systems of pullies have been adjusted, and when a displacement of the body is being caused, it ought to be taken into account, that, as the intensities of the *original* forces depended upon the positions of the points at which they acted, they would be changed *simultaneously* with the position of the body, so that it is impossible for *any* displacement (no matter how *small*) to take place without causing an alteration of the original forces; and, consequently, as these forces are to be represented by the systems of pullies, a simultaneous change of systems of pullies ought also to be continually taking place, so long as the displacement of the body is being effected. If this objection be valid, no reasoning respecting either the ascent or descent of "le poids" can be instituted; and, therefore, the demonstration will become invalid. It will be no answer to this objection to say that the displacement is indefinitely small, for if there be a displacement at all, however small it may be, there will be a change of the forces, and consequently there must be a change of pullies.

It has been my aim, that each chapter should, as far as was possible, present one continuous chain of reasoning; and with the view of preserving this chain unbroken, I have thrown all the Problems into the last chapter; and have there also presented such other considerations and applications of theory, as could not well be introduced into their proper places without drawing the student's attention too far aside from the main object of the chapter to which they belong.

I have to apologize to the reader for using two systems of Differential Notation. I was induced to do so from a conviction that both have their peculiar advantages, and are equally good in principle. If it be recollected, that in functions of one variable, $\frac{dy}{dx}, \frac{ds}{dx}, \ldots\ldots$ in one system are represented by $d_{,}y$, $d_{,}s$ .... in the other, no great inconvenience can arise from it.

Though it was no part of my design, in compiling the present treatise, to enter into the lengthened details of practical Mechanics, yet I have been anxious so far to give it a practical bias as should make the student acquainted with the imperfections and deficiencies of theory, and the difficulties to be expected in applying it to practice; and in several instances (which may serve as precedents for other cases), I have shewn how, by proper experiments, such an extension of the theory is effected as enables us to overcome several of the most important of these impediments.

St. John's College, Cambridge,

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CHAPTER I.

PRELIMINARY NOTIONS, AND ON FORCES ACTING ON A SINGLE PARTICLE OF MATTER.

1. THE least conceivable portion of matter is called a *Particle*. The position of a particle can be determined by referring it to other bodies which are supposed to be fixed. If by this or any other method, the position be found to be different at successive instants of time, the particle is said to be in *Motion*: but so long as the position remains the same, the particle is at *Rest*.

2. The principal properties of matter which we assume in the present Science are *Inactivity, Mobility, Extension, Figure, and Impenetrability*.

3. We conceive of matter, that it *can* exist in a state of quiescence; or, in other words, that motion is not essential to its existence: and hence it would seem, that matter once at rest cannot possibly pass into a state of motion without the intervention of an external agent of some kind or other.

This agent, whatsoever its nature or essence may be, is known to us by its effects only, and is called *Force.*

4. A force may be considered as *sufficiently* determined when we know its *Intensity*, *Direction*, and *Point of application*,—that is, the point at which it immediately acts. For when these are known, the effects, with which alone we are concerned, can under any given circumstances be calculated; and to calculate these is the general object of the Science of MECHANICS. That part of Mechanics which relates to bodies at rest is called STATICS; the name DYNAMICS is given to the other part.

5. Since whatever we know of Force is gathered solely from its effects, the terms Intensity and Direction of a *force*, are to be interpreted solely of the unmodified *effect* which the force is known to produce upon an unimpeded particle of matter in free space. This principle is usually presented under the following form:

1st. A force produces motion in the direction in which it acts.

2ndly. The intensity, or, as it is generally called, the magnitude of a force is measured by the quantity of motion produced in a given time.

6. A force may be very conveniently represented in every respect by a straight line drawn from the point of application, in the direction of the force, and of a length proportional to the intensity.

7. The size or bulk of a body is called its *Volume*, and the quantity of matter it contains is called its *Mass*.

8. By experience we know, that the matter of which most bodies consist, can be compressed or squeezed into

less space; from which it appears that some bodies are composed of more compact matter than others: and, consequently, though two bodies may be of exactly the same size, yet they may contain very unequal quantities of matter.

9. A known body, composed of matter uniformly diffused through all its parts, is taken as a standard to which all others are referred. Its volume is called the unit of volume, and its quantity of matter, the unit of mass. If a body be V times the size, and contain M times the quantity of matter; V and M are taken as the *measures* of the volume and mass of the second body. Also, supposing the matter of the second body to be uniformly diffused through all its parts, if a portion of it, of the same size as the standard body, contain ρ times as much matter, ρ is called the density of the second body; and it is evident that

$$M = \rho V.$$

10. If several forces act simultaneously on a particle previously at rest, they will either put it in motion, or it will still remain at rest. In the latter case, the forces are said to be in equilibrium: but, in the former, the particle will begin to move in a certain direction; and as it might have been made to move exactly in the same manner by a single force only of a proper magnitude acting in that direction, we draw the important conclusion, that a single force of a proper magnitude, and applied in a proper direction, may, without producing any alteration in the effect, be substituted for any proposed set of forces. A single force, so applied, is called the *Resultant* of the other forces.

11. From the same considerations may also be deduced the converse property; viz.—that for any single force we may

substitute any set of forces of which it is the resultant; and the forces so substituted are called the *components* of the single force.

12. For any set of forces we may substitute any other set which has the same resultant.

For the resultant is the true measure of a set of forces, and this being the same for the two sets, it is indifferent which we use.

13. The words resultant and component are relative therefore respectively to the two mechanical properties just stated, and it is evident, from the very definition and nature of a resultant, that no system of forces can have more than one resultant: but the converse,—that no force can have different sets of components, is not true.

14. From the case of Art. 10, where the particle still remains at rest, after a set of forces has been applied, we draw a conclusion as important as the preceding, viz:

That a set of forces which are in equilibrium may altogether be removed or added, without altering the state of the particle, or the Statical effect of other forces acting on it, which are not in equilibrium.

This property is distinguished by the appellation of "The superposition of equilibrium."

15. Axiom. Two forces are equal, which in precisely similar circumstances would produce the same result.

Equal forces are usually defined to be such, as when applied at the opposite ends of a straight rod in contrary directions, coinciding with the direction of the rod, would keep each other in equilibrium. But this definition is not good, inasmuch as it involves the property of the transmissibility of force (see Chap. II.); and is, moreover, rather a demonstrable property, than an axiomatic definition, as will be shewn presently.

16. It follows at once from these properties, that when a set of forces is in equilibrium, they may all be increased or diminished in any proportion, without disturbing the equilibrium; and if they are not in equilibrium, and be all increased or diminished in any proportion, their resultant will be increased or diminished in the same proportion.

For doubling all the forces is the same as adding an equal set of forces, and the resultant of each will be equal, and coincide in direction; and the resultant of these resultants, or what is the same, that of all the forces, will be double the former; and similar reasoning may be used if they be altered in any other proportion.

17. Two equal forces acting on a particle in opposite directions are in equilibrium.

For if in consequence of these two forces motion ensue, we may refer its direction to one of the forces; let it for instance make an angle θ with the first force, then since the forces are equal and opposite, an equal motion will take place in a direction making an angle θ with the second force, and therefore opposite to the former direction: which is impossible, since the particle cannot move in opposite directions at the same time. Hence no motion will ensue, and the forces will be in equilibrium.

This property is proved at once by some writers in these words: "For no reason can be assigned why one should prevail rather than the other;" but as this appeals rather to our ignorance than to our knowledge, proof above given seems preferable.

18. If to a particle acted on by any forces not in equilibrium, we apply a force equal and opposite to their resultant, the whole will be in equilibrium.

For instead of the first forces, we may substitute their resultant (Art. 10), and then the whole is reduced to two equal and opposite forces acting on a point, which are in equilibrium by last article.

19. Conversely:—If two forces acting on a point are in equilibrium, they are necessarily equal and opposite.

For let two forces F_1, F_2 acting on the particle P (Fig. 1), in the directions PF_1, PF_2, be in equilibrium, and, if possible, let F_1PF_2 not be a straight line. Produce F_1P to f, and make the angle $f\,PF_2{}' =$ angle $f\,PF_2$. Then, since $PF_2{}'$ is in a similar situation to PF_2 with regard to PF_1, if F_2 were to act in the direction $PF_2{}'$, it would produce equilibrium with F_2; and, consequently, F_2 produces the same effect whether it acts in the direction PF_2 or $PF_2{}'$, which is impossible, and therefore PF_2 cannot be in a different direction from Pf. Hence, when two forces acting on a point are in equilibrium, they necessarily act in opposite directions; and they must also be equal, for if F_1 and F_2 (Fig. 2) acting on P in opposite directions are not equal, a force $F_2{}'$ equal to F_1 acting in place of F_2 would be in equilibrium with F_1 (Art. 17): and hence $F_2{}'$ and F_2 in the same situation produce equal effects, and are consequently equal (Art. 15). But F'_2 is equal to F_1, and consequently unequal to F_2, which is absurd. Hence F_1 and F_2 are equal, and they have been shewn to be opposite.

20. Hence when any forces acting on a particle are in equilibrium, any one of them is equal, and opposite to the resultant of the rest.

21. Since two forces which are in equilibrium must necessarily be equal and opposite, two forces such as F_1 and F_2 in Fig. 3, which are not opposite, must necessarily have a resultant, and as it is a matter of considerable importance, we shall proceed to determine its position.

1st. The resultant of two forces F_1 and F_2 is situated in the plane F_1PF_2.

For if it be not in that plane, it must be either above or below. But it cannot be above, for any reason which would assign it such a position might be used to assign it a similar position below; for these two positions are similarly situated with regard to the forces F_1 and F_2; there would consequently be two resultants, which (Art. 13) is impossible. The resultant then cannot be situated above the plane of the forces; and in a similar way we may shew that it cannot be situated below, and therefore it must be in the plane.

2ndly. It lies within the angle F_1PF_2.

For the tendency of F_1 is to draw the particle P in the direction PF_1, while that of F_2 is to draw it in the direction PF_2, and hence it is probable the real motion, which is the result of these united tendencies, will not be in the direction of either, but intermediate to both; and therefore within the angle F_1PF_2: consequently the resultant, which is a single force that would produce the same motion, must be situated within the angle F_1PF_2.

What is here stated with regard to this 2nd Case, can hardly be called a proof, but is rather a strong reason for presuming that the resultant is situated within the angle included by the forces. Perhaps, after what is said in the 1st Case, it may be regarded as an axiom.

22. Since F_1 and F_2 do in part hinder each other from producing their whole effects, it appears that their resultant must be less than their sum; for their resultant can only be equal to their sum when neither interferes with the other, which, as we have seen, is not the case; consequently

$$R < F_1 + F_2.$$

23. If the forces F_1 and F_2 are equal, their resultant R will bisect the angle F_1PF_2.

For if there be a reason why PR should lie nearer to PF_1 than to PF_2, there must be a similar reason why it should lie nearer to PF_2 than to PF_1, since the forces are equal; and hence there would be two resultants, which is impossible (Art. 13): consequently PR bisects the angle F_1PF_2.

24. Having thus determined the direction of the Resultant of two equal forces, we proceed to the more difficult problem of finding its magnitude.

Let F_1, f_1 (Fig. 4) be two equal forces acting on the particle P, and R their resultant bisecting the angle F_1Pf_1. Since R is less than the sum of the two forces F_1 and f_1 (Art. 22), it is clear that $\frac{R}{F_1+f_1}$, or its equal $\frac{R}{2F_1}$, is always less than 1; and, consequently, an angle θ may be found such that

$$\frac{R}{2F_1} = \cos\theta,$$

$$\text{or } R = 2F_1 \cos\theta.$$

The angle θ is perfectly unknown at present, but from Art. 16, we learn that so long as the angle F_1Pf_1 remains the same, θ continues unchanged; that is, if we have two sets of forces inclined at the same angle with each other respectively, we shall have $R = 2F_1 \cos\theta$, and $R' = 2F_1'$ $\cos\theta$, and therefore

$$R : R' :: F_1 : F_1' \; . \; . \; . \; . \; . \; . \; (A),$$

that is, the resultants are proportional to the components.

Let now F_2, f_2 be two other equal forces acting on P, whose resultant is also equal to R, the angles F_1PF_2, f_1Pf_2 being each equal to RPF_1 or RPf_1. Now at P apply four forces, each equal to x, two of them respectively in the directions PF_2, Pf_2, and the other two in the direction PR; and let them be of such magnitude, that F_1 may be the resultant of the one in the direction PF_2 and one in the direction PR. Then, since these two contain the same angle as F_1 and f_1, and F_1 is their resultant,

$$F_1 = 2x \cos\theta.$$

Also, if we substitute instead of F_1 and f_1, their components (Art. 10), we may consider R as the resultant of the forces x, x, x, and x; of which two act in the same direction as R; and, consequently, $R - 2x$ is the resultant of the two x, x, which act in the directions PF_2, Pf_2; and since, by hypothesis, R is the resultant of F_2 and f_2, which act in the same directions,

$$\therefore R : R - 2x :: F_2 : x, \text{ from (A)};$$

$$\therefore \frac{R}{F_2} = \frac{R}{x} - 2.$$

But $R = 2F_1 \cos\theta = 2 \, . \, 2x \cos\theta \, . \cos\theta = 4x \cos^2\theta$;

$$\therefore \frac{R}{F_2} = 4\cos^2\theta - 2$$

$$= 2\,(2\cos^2\theta - 1)$$

$$= 2\cos 2\theta;$$

$$\therefore R = 2F_2 \cos 2\theta.$$

It appears then, that in the formula

$$R = 2F_1 \cos\theta,$$

if we double the angle at which the forces are inclined, we must also double θ.

We will now suppose, that when the angle at which the forces act is a multiple n, or any inferior multiple of F_1Pf_1, it is true that in the same formula the corresponding equimultiple of θ is to be taken; so that

$$R = 2F_n \cos n\theta = 2F_{n-1} \cos(n-1)\theta = \ldots . = 2F_1 \cos\theta.$$

Apply (Fig. 5) at P, as before, four forces in the directions PF_{n+1}, PF_{n-1}, Pf_{n+1} and Pf_{n-1} respectively, each of such a magnitude x, that F_n may be the resultant of the two in the directions PF_{n+1}, PF_{n-1}, and f_n of the other two;

$$\therefore F_n = 2x\cos\theta.^*$$

But if, instead of the forces F_n, f_n, we substitute their four components, we may consider R as the resultant of the forces x, x, x, and x, of which two acting in the directions PF_{n-1}, Pf_{n-1} will have $2x\cos(n-1)\theta$ for their resultant in the direction PR, and consequently $R - 2x\cos(n-1)\theta$ is the resultant of the other two which act in the same directions as F_{n+1} and f_{n+1}; consequently, from (A),

$$R : R - 2x\cos(n-1)\theta :: F_{n+1} : x;$$

$$\therefore \frac{R}{F_{n+1}} = \frac{R}{x} - 2\cos(n-1)\theta,$$

$$= \frac{R}{x} - 2\cos\theta\cos n\theta - 2\sin\theta\sin n\theta.$$

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* For the $\angle F_{n+1}PF_{n-1} = \angle F_1Pf_1$, (Fig. 4).

But $R = 2F_n \cos n\theta = 4x \cos\theta \cos n\theta$,

$$\therefore \frac{R}{F_{n+1}} = 4 \cos\theta \cos n\theta - 2 \cos\theta \cos n\theta - 2 \sin\theta \sin n\theta,$$

$$= 2 (\cos\theta \cos n\theta - \sin\theta \sin n\theta),$$

$$= 2 \cos (n + 1) \theta;$$

$$\therefore R = 2F_{n+1} \cos (n + 1) \theta.$$

Hence the formula is true for a multiple $(n + 1)$ if it be true for n and all inferior multiples: but it has been shewn to be true for 2 and 1, and consequently it is true for multiples 3, 4, 5, 6, ... and generally, by induction, for any multiple whatever.

It appears then, that as we increase the angle at which two equal forces (F, F) act, we must increase the angle θ in the same proportion, and then, that the formula

$$R = 2F \cos\theta$$

still holds good. This, however, supposes the angle between the forces to be a multiple of $F_1 P f_1$ (Fig. 4), which may not happen to be the case; but by taking the original angle $F_1 P f_1$ exceedingly small, we may find a multiple of it which shall differ from FPF a proposed angle by less than any assignable quantity. It is evident then, that FPF and θ have an invariable ratio to each other, so that if FPR $= \phi$, then

$$\frac{\theta}{\phi} = \text{constant quantity} = c \text{ suppose};$$

$$\therefore R = 2F \cos c\phi.$$

To determine the value of c, we observe that (Art. 17) if F and F act at an angle π, or are opposite to each other, $\left(\text{in which case } \phi = \frac{\pi}{2}\right)$ they have no resultant,

$$\therefore 0 = 2F \cos \frac{c\pi}{2},$$

$$\therefore \cos \frac{c\pi}{2} = 0.$$

Now none but angles which are odd multiples of $\frac{\pi}{2}$ have their cosines $= 0$;

$$\therefore c = \text{an odd integer} = 1, \text{ as we shall shew.}$$

For if c is not $= 1$, let the angle FPF be such that $\phi = \frac{\pi}{2c}$, which is therefore less than a right angle, and then

$$R = 2F \cos c\phi = 2F \cos \frac{\pi}{2} = 0.$$

But since the angle FPF is, in this case, $= \frac{\pi}{c}$, and therefore less than π, the resultant cannot be $= 0$ (Art. 21), which is absurd, and consequently $c = 1$. We arrive therefore at the general result, that if F, F be two equal forces acting on a particle, and inclined to each other at the angle 2ϕ, their resultant R is inclined to each of them at the angle ϕ, and its magnitude is determined by the equation

$$R = 2F \cos \phi.$$

25. It will be immediately obvious, that since the forces F and F are perfectly equal and similarly situated with respect to PR, they contribute equally to the resultant R; and, consequently, the efficiency of each in the direction PR is equal to $\frac{1}{2}$ R, or F $\cos \phi$. This is the portion each contributes to make up R, and since R is less than their sum (Art. 22), that part of each force, which is lost, is spent in hindering the other from producing any effect, except in the direction PR. Hence, if one of them were by any impediment whatever hindered from producing effect, except in the direction PR, its efficiency in that direction would be F $\cos \phi$; for, in this case, the impediment merely supplies the place of that part of the other force which was employed in the same office.

We may therefore state, as general facts,

1st. That in compounding forces into one, force is lost; that is, the whole quantity of resultant force is not so great as the whole quantity of component force.

2ndly. That the efficiency (or, as it is often called, the resolved part) of a force F, in a direction making an angle ϕ with that in which it acts, is equal to F cos ϕ.

26. To determine the magnitude and direction of the resultant of any two forces acting on a particle.

Let F, f (Fig. 6) be the two forces acting on the particle P, R their resultant, perpendicular to which draw LPM; let α, β denote the angles FPR, fPR respectively, and ϕ the angle FPf between the forces. Then the efficiencies of F and f, in the direction PR, are respectively $F \cos \alpha, f \cos \beta$, the sum of which must be equal to R, since the efficiency of R is equivalent to the united efficiencies of F and f in any proposed direction, because R is their resultant.

$$\therefore R = F \cos \alpha + f \cos \beta \quad \ldots\ldots\ldots \quad (1).$$

Now the efficiency of R in the direction PL perpendicular to itself $= R \cos 90° = 0$; and the efficiency of F in the direction PL $= F \cos FPL$, and that of f in the same direction $= f \cos f PL$,

$$\therefore \; 0 = F \cos FPL + f \cos fPL,$$

$$\text{or } 0 = F \cos (90 - \alpha) + f \cos (90 + \beta),$$

$$\text{or } 0 = F \sin \alpha - f \sin \beta, \quad \ldots\ldots\ldots\ldots \quad (2);$$

and by squaring equations (1) and (2), we have

$$R^2 = F^2 \cos^2 \alpha + 2Ff \cos \alpha \cos \beta + f^2 \cos^2 \beta,$$

$$0 = F^2 \sin^2 \alpha - 2Ff \sin \alpha \sin \beta + f^2 \sin^2 \beta;$$

and adding these together,

$$R^2 = F^2 + 2Ff (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + f^2.$$

But because $\phi = \alpha + \beta$,

$\therefore \cos\phi = \cos\alpha \cos\beta - \sin\alpha \sin\beta$; and, consequently,

$$R^2 = F^2 + 2Ff\cos\phi + f^2.$$

27. The same result is sometimes thus stated:—If two sides of a triangle, taken in order, represent the magnitudes and directions of two forces acting on a point, the third side, taken in the opposite order, will represent the magnitude and direction of their resultant.

Let NP, PM (Fig. 7) be taken to represent the magnitudes and directions of two forces F, f, acting on the point P; complete the triangle PNM; and from N draw Nm in any direction whatever, and perpendicular to it draw Pp, Mm.

Then if we estimate the forces F and f in the direction Nm; Np being = NP cos PNm, and pm being = PM cos angle between PM and pm; Np, pm will respectively represent their efficiencies in that direction, and therefore Nm will represent their united efficiency in that direction. But we observe that Nm = NM cos MNm, and therefore NM represents the magnitude and direction of a *single* force which is equivalent to F and f in the direction Nm, and since this is any direction whatever, the single force represented by NM must be in every respect equivalent to the two F, f, and therefore it represents their resultant R; and it is manifest that NM, the direction of R, is in the contrary order to NP, PM, which are the respective directions of the given forces.

28. Hence it appears, that the three sides of a triangle, taken in order, will represent the magnitudes and directions of three forces which would keep a particle at rest (Art. 18).

Also, because the sides of a triangle are proportional to the sines of the opposite angles, it is manifest that when three forces acting on a particle are in equilibrium, any one of them is proportional to the sine of the angle contained between the directions of the other two.

29. The form under which the result of Art. 26 is best known, is the following, which is distinguished by the name of "The Parallelogram of Forces."

If two straight lines drawn from a point, representing the magnitudes and directions of two forces, be completed into a parallelogram, the diagonal drawn from the same point will represent the magnitude and direction of their resultant.

Let FP*f*R (Fig. 8) be a parallelogram so constructed; PF, P*f*, respectively representing the magnitudes and directions of the forces F, *f*; then PR will represent the magnitude and direction of their resultant. For, because FR is parallel and equal to P*f*, the two sides PF, FR of the triangle PFR represent the magnitudes and directions of F and *f*; and therefore PR, taken in opposite order (Art. 27), will represent their resultant.

30. The following proposition is somewhat more general than the preceding one, and is often called "The Polygon of Forces."

If all the sides of a polygon, except the last, taken in order, represent the magnitudes and directions of forces acting on a particle, the last side, taken in the opposite order, will represent the magnitude and direction of their resultant.

This is true whether the sides of the polygon be all situated in one plane or not.

Let PA, AB, BC, CD (Fig. 9), represent the magnitudes and directions of forces acting on a point P. Join PB, PC. Then the forces represented by PA, AB, are equivalent to that represented by PB (Art. 27); and, therefore, those represented by PA, AB, BC, are equivalent to those represented by PB, BC; and, consequently, to that represented by PC (Art. 27); and so on, till we come to the last side PD, which will represent the resultant of all the forces.

31. From this, and Art. 18, it appears, that the sides of any polygon represent the magnitudes and directions of forces which acting on a particle will be in equilibrium.

32. Hence we may resolve a force into as many component forces, acting in given directions, as we please.

For take a line to represent the magnitude and direction of the proposed force, and upon that line construct a polygon having its sides parallel to the directions in which it is proposed the components are to act, and they will represent the components both in magnitude and direction.

33. The most important case of the resolution of a force is, when the components are required to act parallel to three given lines which are at right angles to each other: and, for this reason, we shall enter upon this case more particularly than in last article.

Let Ox, Oy, Oz (Fig. 10) be the three given lines at right angles to each other. From O draw OF to represent the magnitude and direction of the force, which it is proposed to resolve into component forces parallel to Ox, Oy, Oz respectively. From F draw FM parallel to Oz, meeting the plane xOy, passing through Ox and Oy in M; from M draw MN perpendicular to Ox. Then ON, NM, MF represent the components of OF, and they are respectively parallel to the given lines. Let α, β, γ denote the angles

FOx, FOy, FOz, which the direction of F makes with the given lines; and X, Y, Z, the components, parallel to Ox, Oy, Oz. Then, because MF is parallel to Oz, and therefore perpendicular to OM, we have

$$\begin{aligned} MF &= OF \,.\, \cos OFM, \\ &= OF \,.\, \cos FOz, \\ &= OF \,.\, \cos \gamma. \end{aligned}$$

Now MF and OF represent Z and F respectively;

$$\therefore Z = F \cos\gamma \; . \; . \; . \; . \; (1).$$

In a similar manner,

$$Y = F \cos\beta \; . \; . \; . \; . \; (2),$$

$$\text{and } X = F \cos a \; . \; . \; . \; . \; (3).$$

34. Remark. The angles a, β, γ are not absolutely independent of each other.

For MF = OF cos γ; and so NM = OF cos β, and ON = OF cos a.

$$\begin{aligned} \text{But } OF^2 &= OM^2 + MF^2, \\ &= ON^2 + NM^2 + MF^2, \\ &= OF^2 \,.\, \cos^2 a + OF^2 \,.\, \cos^2\beta + OF^2 \,.\, \cos^2\gamma; \end{aligned}$$

$$\therefore 1 = \cos^2 a + \cos^2\beta + \cos^2\gamma.$$

From which equation, any two of the angles a, β, γ being given, the third may be found.

In employing the equations (1), (2) and (3) of last article, this equation must never be lost sight of.

35. The converse of Art. 33 may be easily effected: viz. Having giving three forces acting on a particle at right angles to each other, to find their resultant.

For let X, Y, Z be the three forces, F their resultant; and a, β, γ the angles which the direction of F makes with the directions of its three components X, Y, Z respectively.

$$\therefore X = F \cos \alpha,$$
$$Y = F \cos \beta,$$
$$Z = F \cos \gamma;$$

and, by adding together the squares of these equations,

$$X^2 + Y^2 + Z^2 = F^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma),$$
$$= F^2, \text{ by Art. 34};$$
$$\therefore F = \sqrt{X^2 + Y^2 + Z^2} \quad \ldots \text{(1)},$$

which determines the magnitude of F; and the equations

$$\cos \alpha = \frac{X}{F} = \frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \ldots \text{(2)},$$

$$\cos \beta = \frac{Y}{F} = \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \ldots \text{(3)},$$

$$\cos \gamma = \frac{Z}{F} = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \ldots \text{(4)},$$

determine its position.

36. Having given the magnitudes and directions of a set of forces acting on a particle, to determine their resultant.

Let F_1, F_2, F_3 . . . . . . be the forces, and α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3; . . . . . . the angles which their directions respectively make with three fixed lines Ox, Oy, Oz, at right angles to each other, as in Art. 33; and denote their respective components in the direction parallel to Ox by X_1, X_2, X_3, ... and those parallel to Oy and Oz by Y_1, Y_2, Y_3, ... and Z_1, Z_2, Z_3, ... which quantities are known from the equations

$$X_1 = F_1 \cos \alpha_1, \; X_2 = F_2 \cos \alpha_2, \; X_3 = F_3 \cos \alpha_3, \ldots$$
$$Y_1 = F_1 \cos \beta_1, \; Y_2 = F_2 \cos \beta_2, \; Y_3 = F_3 \cos \beta_3, \ldots$$
$$Z_1 = F_1 \cos \gamma_1, \; Z_2 = F_2 \cos \gamma_2, \; Z_3 = F_3 \cos \gamma_3, \ldots$$

The proposed forces are therefore equivalent to three forces, one in the direction of $Ox = X_1 + X_2 + X_3 + \dots$, one in the direction of $Oy = Y_1 + Y_2 + Y_3 + \dots$, and one in the direction of $Oz = Z_1 + Z_2 + Z_3 + \dots$, which denote respectively by ΣX, ΣY and ΣZ.

Now (Art. 12) the resultant of the proposed forces is the same as the resultant of these three; if, then, we denote it by R, and the angles which its direction makes with the lines Ox, Oy, Oz by α, β, γ, we have, by Art. 35,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

and $\cos\alpha = \dfrac{\Sigma X}{R}$, $\cos\beta = \dfrac{\Sigma Y}{R}$, and $\cos\gamma = \dfrac{\Sigma Z}{R}$; by which four equations every thing required is determined.

37. If Ox, Oy, Oz be taken as the co-ordinate axes, and a, b, c be the co-ordinates of the particle P, on which the forces act, and x, y, z be those of any point in the line of direction of the resultant; then the equations of this line are

$$\frac{x-a}{\Sigma X} = \frac{y-b}{\Sigma Y} = \frac{z-c}{\Sigma Z}.$$

If the particle on which the forces act be at the origin of co-ordinates, a, b, c are each equal to zero, and the equations of the line of direction of the resultant become

$$\frac{x}{\Sigma X} = \frac{y}{\Sigma Y} = \frac{z}{\Sigma Z}.$$

38. To find the equations of equilibrium of a set of forces acting on a particle.

If there be an equilibrium, the forces cannot have a resultant, and therefore, using the notation of Art. 36, R must be equal to zero;

$$\therefore 0 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2.$$

But the right hand number of this equation is composed of three quantities, which being squares, are essentially positive, and therefore their sum cannot be $= 0$, unless

$$\Sigma X = 0,\ \Sigma Y = 0,\ \text{and}\ \Sigma Z = 0\ ,\ \ldots\ldots\ (A),$$

these three equations are therefore equivalent to the one $R = 0$; and may be substituted for it. They are called the equations or conditions of equilibrium; for when they are satisfied there must be an equilibrium, because, in that case, $R = 0$; and unless they are all satisfied there cannot be an equilibrium, for then R would not to be equal to zero.

Being, then, both *necessary* to and *sufficient* for complete equilibrium, they are taken as the criteria of the equilibrium of forces acting on a point.

39. To find the efficiency, in a given direction, of a set of forces acting on a particle.

Using the notation of Art. 36, let ζ, η, θ, be the inclinations of the given direction to the three fixed lines Ox, Oy, Oz respectively; and ϕ the angle between this and the resultant R. Then

$$\cos\phi = \cos\alpha\cos\zeta + \cos\beta\cos\eta + \cos\gamma\cos\theta,$$

which determines ϕ.

And since the resultant R may be substituted for the proposed forces (Art. 10), and its efficiency in the proposed direction $= R\cos\phi$ (Art. 25), the efficiency of the set of forces in the given direction must be $R\cos\phi$, and consequently is equal to

$$\sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.(\cos\alpha\cos\zeta + \cos\beta\cos\eta + \cos\gamma\cos\theta).$$

40. In the case of equilibrium $R = 0$, and consequently $R\cos\phi = 0$, whatever be the value of ϕ; and

hence, when any set of forces acting on a point are in equilibrium, their efficiency in any direction whatever is equal to zero.

41. Remark. Since the directions of the lines Ox, Oy, Oz, may be taken ad libitum, and that the necessary conditions of an equilibrium are, that the efficiencies of the forces in these three directions shall be respectively equal to zero; it appears that if we can find *three* directions at right angles to each other, parallel to which the efficiencies of any set of forces are respectively equal to zero, then the efficiencies of the same set, in any other direction whatever, is equal to zero.

CHAPTER II.

ON FORCES ACTING ON A RIGID BODY.

42. A *rigid* body is one, the relative position of whose parts cannot be changed.

43. In treating of the action of force on rigid bodies, the first object is to ascertain what influence the form of the body has upon the direction in which a force is transmitted through it. For it is to be observed, that when a force acts upon a particle of a rigid body, its effect is not confined to that particle alone, but is distributed over the whole body. Now let M (Fig. 11) be a rigid body of any form, and F a force acting upon one of its particles at P, in the direction PF; produce FP through the body. It is found, by experiment, that the effect of the force F is distributed through the body in the same manner (or, in other words, that the effect which F produces is precisely the same) when it acts at any point Q in the line FP as when it acts at P. This is "The Law of the Transmissibility of Force;" from which it appears, that the effect of a force is not altered by changing the point of application to any other in the line of the direction in which the force acts.

When the point of application is changed according to this law, the force is said to be transmitted; and some writers have supposed the principle to be self-evident, and have, accordingly, dismissed the subject in few words; others have attempted a proof, but in every instance the proof will, upon examination, be found to take for granted the principle to be proved. The truth is, the principle is a most curious physical fact, which, in the present state of our knowledge, cannot be deduced from any reasoning, à priori, but must be established by experiment alone. A great variety of different experiments, by which the principle in question can be established, might be mentioned, but we shall select only one, which possesses the advantage of being very conclusive and easily made.

Let M (Fig. 12) be a rigid body of any form, F a force acting upon a particle of it at the point P, and let Q (not in the line FP) be a fixed point about which the body is freely moveable. Draw QL parallel to PF. Then it will be found that the body will turn round the point Q until it comes into such a position that P is in the line QL, as in Fig. 13. The same result will be observed, if instead of having Q fixed, we apply there a force f in a direction opposite to F; for it is found that F acting at P, and f acting at Q, in opposite directions, are only in equilibrium when they are equal and situated in the line QPL, as in the Figure. This experiment shews most distinctly that force is transmitted in the line of its direction; for the force f will balance F at whatsoever point in the line FQ, F be applied; and therefore F produces the same effect at each point in that line; for otherwise it could not always be kept in equilibrium by f, which remains unaltered in every respect.

44. *Concurrent Forces* are those whose directions all pass through the same point, and since such forces, acting on a rigid body, may be transmitted to the point of con-

course, they are by this means at once reduced to forces acting on a point, and consequently fall under the preceding chapter, all the theorems and formulæ of which are applicable to them.

It may be objected, that when the point of concourse is situated without the body, the forces cannot be transmitted to it, for, by referring to the last article, it will be seen that transmission can only take place to points which are rigidly connected with and form part of the body. But the objection is removed, by considering that the forces do not in reality act at the point of concourse, but each at its own point; and this action is the same as it would be if the point of concourse were rigidly connected with the body, and the forces transmitted to it.

45. Two parallel forces act at different points of a rigid body, to determine their resultant.

Let F_1 and F_2 (Figs. 14, 15) be the two forces, A and B the points at which they act, or any other convenient points in the lines of the directions in which they act, to which they may be transmitted. At A and B apply two equal forces f_1 and f_2, of any convenient magnitude, in opposite directions. These being in equilibrium will not alter the effect or resultant of the other forces F_1 and F_2 (Art. 14). Now F_1 and f_1 will have a resultant m within the angle F_1Af_1 (Art. 21); and the resultant n of F_2 and f_2 will, in like manner, lie within the angle F_2Bf_2. Produce the directions of these two resultants to meet, *if possible*, in some point P, and transmit them to that point (Art. 43). The two parallel forces F_1 and F_2, acting at A and B, are consequently equivalent to the two m and n, acting at P, where we may again resolve the latter into their primitive components f_1, f_2, in opposite directions, parallel to AB; and F_1, F_2, parallel to the original forces.

The two f_1, f_2 being equal and opposite, may be removed (Art. 14); and hence, the two which remain are the original forces transposed parallel to themselves to the point P, and drawing PC parallel AF_1 or BF_2, we may now transmit them to any point in PC.

We see then, that there exists a certain line PC in the rigid body, to any point of which we may transpose the forces without altering their effect.

This line PC we call the *Diameter of Parallel Forces.*

We thus at once reduce the action of parallel forces on a rigid body, to the action of forces on a point.

Hence, when the forces act in the same direction, their resultant R is equal to the sum, but when they act in opposite directions it is equal to their difference, and in the direction of the greater.

46. To determine the position of the *diameter* PC, we observe that PC, CA, PA form a triangle, whose sides are in the directions of the forces F_1, f_1 and their resultant m, and are therefore proportional to them (Art. 27); hence,

$$F_1 : f_1 :: PC : AC,$$

$$\text{and similarly, } f_2 : F_2 :: BC : PC,$$

$$\therefore F_1 : F_2 :: BC : AC, \text{ because } f_1 = f_2.$$

The position of the diameter may therefore be thus determined:—take A, B, any two points in the lines in which the given forces act, and divide AB, when the forces act in the same direction, or AB produced on the side towards the greater, when they act in opposite directions, in the point C, so that the segments may be inversely proportional to the corresponding forces. The line drawn through C parallel to the given forces will be the diameter.

47. For the sake of simplifying our results, we shall consider forces which act in a certain direction as positive, and those which act in the contrary direction as negative. Hence, the signs + and −, when applied to forces, in like manner as to lines, merely denote contrariety of direction; and, as in Fig. 14,

$$R = F_1 + F_2;$$

this formula is equally applicable to Fig. 15, if we take care in using it to observe, that as F_2 acts in a direction opposite to that of F_1, it must be affected with the opposite sign − ; and thus, for that case,

$$R = F_1 - F_2.$$

Seeing, therefore, that we can deduce the results which belong to that case, from the results which belong to Fig. 14, by merely introducing the proper symbol of affectation of the forces, we shall use only the former formula, and state the general result under this form.

The resultant of two parallel forces is equal to their sum, and acts in their diameter.

The word *sum* being understood in the sense here explained, *i.e.* requiring each force to be affected with its proper symbol of direction.

48. We can also deduce another formula for the position of the diameter, which will be equally applicable to the two cases represented by Figs. 14, and 15.

In AB (Fig. 14) take any point O; then it has been shewn (Art. 46), that

$$F_1 : F_2 :: BC : AC,$$

$$:: OB - OC : OC - OA,$$

$$\therefore (F_1 + F_2) . OC = F_1 . OA + F_2 . OB,$$

$$\text{or } R . OC = F_1 . OA + F_2 . OB \; \ldots . (1).$$

But, in Fig. 15, $F_1 : F_2 :: BC : AC,$

$$:: OB - OC : OA - OC,$$

$$\text{and } \therefore (F_1 - F_2) . OC = F_1 . OA - F_2 . OB,$$

which we observe may be derived at once from (1) by merely affecting F_2 with the proper symbol of direction; we therefore take (1) as the general result, and state it thus:

If a line be drawn through a given point, cutting the lines of directions of two parallel forces and their diameter, the sum of the products of each force into the part intercepted between the given point and the line of its direction, is equal to the product of their resultant into the part intercepted between the point and the diameter.

49. For the sake of simplicity, the line is usually drawn at right angles to the direction of the forces, the intercepted parts then become perpendiculars from the given point upon the lines of direction of the forces, and upon their diameter.

Now the product of a force into a perpendicular, from a given point upon the line of the direction in which it acts, is called the moment of the force about that point; and hence we may state the general result of last article in more simple terms, thus:

The sum of the moments of two parallel forces about any point in their own plane, is equal to the moment of their resultant about the same point.

With respect to the perpendiculars drawn from the given point, we must observe that they are to be affected with their proper symbols of direction, so that if the point O lies between A and B, one of the perpendiculars must be affected with the sign $-$.

50. By means of these properties of parallel forces we can, without trouble, decompose any single force into two others parallel to it, and acting at given points; we shall, therefore, not dwell upon it, but proceed to observe that it is essential to the success of the demonstration of Art. 45, that the lines mA, nB should meet in a point P, which will always be the case, except when the given forces F_1, F_2 are equal, and act in contrary directions. If we construct a figure for this case, we shall find that mA, nB are parallel, and therefore never meet; the demonstration therefore fails on this hypothesis, and the results just obtained are inapplicable.

In fact, we can shew that it is impossible two such forces should have a resultant,* for any reasoning which would assign them a resultant, referred to one of the forces, would assign them an equal resultant in the contrary direction, upon referring it to the other force, for they are equal and act in opposite directions. Hence there does not exist any single force which can be substituted for them. This we may call the irreducible case of parallel forces, inasmuch as the given forces cannot be reduced to any thing simpler than themselves.

51. A system of two such forces (Fig. 16) is called a couple, and is usually written thus (F, — F). The perpendicular distance AB between them, is called the *arm* of the couple, and the product F . AB is called the moment of

* Some authors, however, considering this as an extreme case of the general proposition, have found, that the forces have a resultant O, acting at an infinite distance;—but as it does not seem to be very intelligible how a force O can act at all, so as to produce any effect, this method of treating the case of equal parallel forces has been thought too perplexing to be admitted into an elementary treatise.

the couple. This is agreeable to the definition in Art. 49. For let O be any point in the plane of the couple, then (Art. 49) the sum of the moments of F and $-$ F, that is, the moment of the couple

$$= F . OA - F . OB,$$
$$= F . (OA - OB),$$
$$= F . AB,$$

which is manifestly altogether independent of the position of the point O.

52. The *axis* of a couple is a straight line at right angles to the plane of the couple, passing through the middle point of its arm, and in length proportional to its moment; the *plane* of the couple is the plane which passes through the directions of its two forces.

We shall call all couples positive which tend to turn the body on which they act, in the direction of a right-handed screw; and those which tend to turn it in the direction of a left-handed screw, negative. For example, the couple exhibited in Fig. 16, is a positive couple.

53. All couples of the same kind (*i.e.* positive or negative) are equivalent, if their moments are equal and their planes parallel.

For let (F, $-$ F), and (f, $-f$) (Fig. 17), be two couples of the same kind, whose planes are parallel and moments equal; and let (F', $-F'$) be a parallel couple of equal moment and of an opposite kind; and,

1st. Suppose their arms AB, *ab*, A′B′, to be parallel. Join AA′, BB′, intersecting in C. Then, by similar triangles ACB, A′CB′,

$$AC : A'C :: AB : A'B'$$
$$:: F' : F,$$

because F.AB = F'.A'B', the moments of the couples being equal.

Hence (Art. 46) C is a point in the diameter of the two parallel forces F, F'. In a similar manner we may shew that C is a point in the diameter of the two parallel forces — F, — F'. We may therefore transpose them all to C (Art. 45); and, consequently, the two couples (F, — F), (F,' — F'), are equivalent to the four forces F, F', — F, — F', acting at C, and are therefore manifestly in equilibrium. In a similar manner (*f*, — *f*), and (F', — F') are in equilibrium, and consequently (F, — F), and (*f*, — *f*) are equivalent, since each of them balances (F', — F').

2ndly. Let (F, — F), and (F', — F'), Fig. 18, be two couples of opposite kinds, equal in every respect, and situated in the same plane, so as to have coincident axes passing through C. Let AF, A'F', intersect in D; and — FB, — F'B' in E; and let *m*C*n* pass through D, C, E; it bisects the angles FDF', F'EF. Now the forces F, F', may be transmitted to D, and then they will have a resultant *m*, in the direction D*m*, because they are equal (Art. 23). For a similar reason — F, — F', will have an equal resultant *n*, in the direction E*n*; and consequently the four forces of the two couples are equivalent to the two equal and opposite forces *m*, *n*, which, being transmitted to C, are in equilibrium (Art. 17). Hence the couple (F', — F') is balanced by (F, — F); and if this latter couple were still farther turned round its axis, it would still balance (F', — F'), and therefore, turning a couple round its axis, does not alter its effect.

Hence, reverting to Case 1, it appears, that if the arm of the couple (*f*, — *f*) be not parallel to the arm of the couple (F, — F), they will still be equivalent, for one of them may be turned round its axis, without altering its effect, until their arms become parallel.

54. By means of the last article, we can change any proposed couple into an equivalent one in a parallel plane, which shall either have an arm of a given length, or its forces of given magnitude.

For if (F, — F) and $(f, -f)$ be the couples, and AB, ab, their arms, they will be equivalent, if

$$F \cdot AB = f \cdot ab.$$

Hence, if the first couple be given, and the arm of the second, the magnitude of f will be known from the equation

$$f = \frac{AB}{ab} \cdot F.$$

Or if f be given, and its arm be required, it may be found from the equation

$$ab = \frac{F}{f} \cdot AB.$$

55. Hence, if we have any number of couples $(F_1, -F_1)$, $(F_2, -F_2)$, ... acting in parallel planes, whose arms are respectively equal to a_1, a_2, ... we can reduce them to couples with arms equal to b. In this case the new couples are,

$$\left(F_1\frac{a_1}{b}, -F_1\frac{a_1}{b}\right), \left(F_2\frac{a_2}{b}, -F_2\frac{a_2}{b}\right), \left(F_3\frac{a_3}{b}, -F_3\frac{a_3}{b}\right), \ldots .$$

and by removing them into one plane, so that their axes may coincide (Art. 53), and turning them round till their equal arms coincide, they will be reduced to a single couple, (S, — S) suppose, whose arm is b, and each of whose forces is equal to

$$F_1\frac{a_1}{b} + F_2\frac{a_2}{b} + F_3\frac{a_3}{b} + \ldots = S,$$

$$\therefore Sb = F_1a_1 + F_2a_2 + F_3a_3 + \ldots .$$

Hence, when couples act in parallel planes, they can be reduced to a single parallel resultant couple, whose moment is equal to the sum of all their respective moments.

56. Conversely:—We can decompose any proposed couple into as many other couples in parallel planes as we please, the only condition to be observed, being, that the sum of the moments of the component couples must be equal to the moment of the proposed couple.—The moment of a negative couple is, of course, to be accounted negative, and the word *sum* is used as explained in Art. 47.

57. Since two equal couples will evidently (Art. 16) produce double the effect of one of them; three, treble; and so on; it follows, that the efficiency of a couple may be properly measured by its moment. For a couple is equivalent to two or more couples when its moment is equal to the sum of their moments.

We shall, therefore, in what follows, designate a couple by its moment; and consider it as determined when its moment and the inclination of its plane are known.

58. Before proceeding further in the subject, we shall put the results of some of the preceding articles in a form better adapted to the future purposes of the chapter.

Let Ox, Oy, Oz (Fig. 19) be the three co-ordinate axes at right angles to each other, and suppose the two proposed forces, which we shall now denote by Z_1, Z_2, to act in a direction parallel to Oz, and therefore at right angles to the plane xOy. Transmit them to the points A, B, where their directions meet that plane; and let CR be their diameter. Draw Aa, Cc, Bb parallel to Oy, and let θ be the angle at which AB is inclined to Ox.

$$\therefore ac = AC \cos\theta,$$
$$bc = BC \cos\theta.$$

But if x_1, y_1, and x_2, y_2, are the co-ordinates of A and B, and x', y' those of C; that is, if $x_1 = Oa$, $x_2 = Ob$, $Oc = x'$, $y_1 = Aa$, $y_2 = Bb$, and $Cc = y'$; then

$$ac = x' - x_1,$$
$$\text{and } bc = x_2 - x',$$
$$\therefore x' - x_1 = AC \cos\theta,$$
$$\text{and } x_2 - x' = BC \cos\theta.$$

But, by Art. 46,

$$Z_1 : Z_2 :: BC : AC,$$
$$:: BC \cos\theta : AC \cos\theta,$$
$$:: x_2 - x' : x' - x_1;$$
$$\therefore Z_1 (x' - x_1) = Z_2 (x_2 - x'),$$
$$\therefore (Z_1 + Z_2) . x' = Z_1 x_1 + Z_2 x_2.$$

And, in a similar manner,

$$(Z_1 + Z_2) . y' = Z_1 y_1 + Z_2 y_2.$$

Or, if R_1 be the resultant of Z_1 and Z_2, it acts at C, in the direction CR, and is $= Z_1 + Z_2$,

$$\therefore \left. \begin{aligned} R_1 x' &= Z_1 x_1 + Z_2 x_2 \\ R_1 y' &= Z_1 y_1 + Z_2 y_2 \end{aligned} \right\} \ldots . (1).$$

59. Now according to the definition of Art. 49, $Z_1 y_1$, $Z_2 y_2$, $R_1 y'$, are the moments of the forces Z_1, Z_2, R_1 about the points a, b, c respectively, and as these points are situated in the line Ox, when we speak of these moments collectively, we call them the moments of Z_1, Z_2, R_1 about the line, or more frequently in this case the axis Ox. And, in a similar manner, by drawing perpendiculars from A, B, C upon Oy, we shall find that $Z_1 x_1$, $Z_2 x_2$, $R_1 x'$ are the respective moments of Z_1, Z_2, R_1 about the axis Oy.

From the equations (1) it appears,—that the sum of the moments of two parallel forces about any line, situated in a plane at right angles to the direction in which they act, is equal to the moment of their resultant about the same line.

The position of their diameter is determined by the equations

$$x' = \frac{Z_1 x_1 + Z_2 x_2}{Z_1 + Z_2},\ y' = \frac{Z_1 y_1 + Z_2 y_2}{Z_1 + Z_2}.$$

60. Suppose, now, that there is a third force Z_3 acting parallel to the former, at a point in the plane xOy, whose co-ordinates are x_3, y_3; and let R_2 be the resultant of the three, and x'', y'' the co-ordinates of the point where it may be supposed to act in the plane xOy.

Then, by (Art. 10) substituting R_1 instead of the forces Z_1 and Z_2, of which it is the resultant, we may consider R_2 as the resultant of R_1 and Z_3, and hence

$$\begin{aligned} R_2 &= R_1 + Z_3, \text{ by Art. 47,} \\ &= Z_1 + Z_2 + Z_3; \\ \text{and } R_2 x'' &= R_1 x' + Z_3 x_3, \\ &= Z_1 x_1 + Z_2 x_2 + Z_3 x_3; \\ \text{and, similarly, } R_2 y'' &= Z_1 y_1 + Z_2 y_2 + Z_3 y_3. \end{aligned}$$

By pursuing this method, it will appear that if instead of three we have any number (n) of parallel forces Z_1, Z_2, Z_3, ... Z_n; of which R is the resultant, and x, y the co-ordinates of the point in the plane xOy, at which it may be supposed to act, then

$$R = Z_1 + Z_2 + Z_3 + \ldots\ldots + Z_n = \Sigma Z \ \ldots . (1),$$

$$Rx = Z_1 x_1 + Z_2 x_2 + Z_3 x_3 + \ldots + Z_n x_n = \Sigma(Zx) \ \ldots (2),$$

$$\text{and } Ry = Z_1 y_1 + Z_2 y_2 + Z_3 y_3 + \ldots + Z_n y_n = \Sigma(Zy) \ \ldots (3).$$

61. It appears, from this demonstration, that there exists a certain line parallel to the direction of the forces, to any point of which they may all be transposed, without altering their effect; and thus the property and definition of Art. 45 become general. We observe also, that equations (2) and (3) extend the property enunciated for two forces, at the end of Art. 59, to any number of forces.

The position of the diameter is to be determined from the equations

$$\dot{x} = \frac{\Sigma(Zx)}{\Sigma Z}, \dot{y} = \frac{\Sigma(Zy)}{\Sigma Z}.$$

The symbol Σ is used as an abbreviation of the word *sum*, and in using these formulæ, care must be taken to affect each force and co-ordinate with their proper symbols of direction.

62. There is one case in which the demonstration of Art. 60 fails, it is when $Z_1 + Z_2 + Z_3 + \ldots + Z_n = 0$. But since some of the forces, in this instance, must act in a negative direction, we may find the resultant S of the positive forces, and the point A (Fig. 20) at which it acts; in the same way we may find $(-S)$ the resultant of all the negative forces, and the point B at which it acts. Draw Aa, Bb parallel to Oy. Then S, and $-$ S, form a couple; of which the moment about the axis Ox

$$= S \cdot Aa - S \cdot Bb,$$

$$= \left.\begin{array}{l}\text{moment of the positive forces} \\ \quad + \text{moment of the negative forces}\end{array}\right\} \text{about } Ox,$$

$$= \text{moment of all the forces about } Ox,$$

$$= \Sigma(Zy).$$

$$\text{But } S \cdot Aa - S \cdot Bb = S \cdot (Aa - Bb)$$
$$= - S \cdot BQ$$
$$= - S \cdot AB \cdot \sin \theta;$$

θ denoting the angle of inclination of AB, or the plane of the couple, to the line Ox; AQ also being parallel ab.

$$\therefore -\text{S}\,.\,\text{AB}.\sin\theta = \Sigma(\text{Z}y) \;.\;.\;.\;.\;.\; (1).$$

Again, the moment of the couple about Oy

$$\begin{aligned} &= \text{moment of S about O}y \\ &\quad + \text{moment of } (-\text{S}) \text{ about O}y, \\ &= \text{moment of the positive forces} \\ &\quad + \text{moment of the negative forces}, \\ &= \text{moment of all the forces about O}y, \\ &= -\Sigma(\text{Z}x).^* \end{aligned}$$

But the moment of S about Oy = − S . Oa,* and that of (− S) = S . Ob,

$$\begin{aligned} \therefore -\Sigma(\text{Z}x) &= -\text{S}\,.\,\text{O}a + \text{S}\,.\,\text{O}b, \\ &= \text{S}(\text{O}b - \text{O}a) \\ &= \text{S}\,.\,ab \\ &= \text{S}\,.\,\text{AB}\cos\theta\,; \end{aligned}$$

$$\therefore -\text{S}.\text{AB}.\cos\theta = \Sigma(\text{Z}x) \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (2).$$

Dividing (1) by (2), we have

$$\tan\theta = \frac{\Sigma(\text{Z}y)}{\Sigma(\text{Z}x)} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (3),$$

and adding their squares together,

$$(\text{S}\,.\,\text{AB})^2 = (\Sigma\text{Z}x)^2 + (\Sigma\text{Z}y)^2 \;.\;.\;.\;.\;.\; (4).$$

This last equation gives S . AB, the moment of the resultant couple; and (3) determines the position of its plane.

REMARK. It appears then, that when a couple is equivalent to a set of parallel forces, its moment round the axis of x is equal to the sum of their moments round the same

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* The sign − is used, because the positive forces tend to turn the body round Oy in a negative direction. See Art. 52.

axis; now the axis of x may be taken in any position whatever; and if the parallel forces be such as to form a set of couples, it will appear that the moment (round any axis) of a couple, which is equivalent to any set of couples, must be equal to the sum of the moments of all the couples about the same axis. Consequently, on the same principle as before (Art. 57), we may take the moment of a couple round an axis as the measure of its statical efficiency about that axis.

63. It appears, from the investigations of this chapter, that a set of parallel forces can be reduced to a single resultant force when ΣZ *is not* equal to zero, and to a single resultant couple when ΣZ is equal to zero.

64. We are now able to determine the conditions of equilibrium of a set of parallel forces acting on a rigid body.

As observed in last article, if there be not an equilibrium, there will be either a single resultant force, or a resultant couple; and, consequently, that there may not be a single resultant force, we must have

$$\Sigma Z = 0.$$

And that the resultant couple may vanish, we must have its arm $AB = 0$, for then it degenerates into two equal forces, acting at the same point in opposite directions, which, by Art. 17, are in equilibrium; hence

$$0 = S \,.\, AB,$$

$$\text{and } \therefore\ 0 = (\Sigma Zx)^2 + (\Sigma Zy)^2,$$

an equation which is equivalent to the two

$$\Sigma(Zx) = 0 \text{ and } \Sigma(Zy) = 0.$$

There are, therefore, three conditions of complete equilibrium of parallel forces acting on a rigid body, viz.—

$$\Sigma Z = 0; \text{ and } \Sigma(Zx) = 0,\ \Sigma(Zy) = 0;$$

which must be all satisfied, otherwise there cannot be an equilibrium; and if they are satisfied, there will necessarily be an equilibrium, for there will then be neither a single resultant force, nor a resultant couple.

65. We have found the resultant of forces, on the supposition that they are all parallel to one of the axes of co-ordinates, that being the form under which the results are most frequently wanted; but to render the subject complete, we shall suppose them to be parallel to each other, but not parallel to any one of the co-ordinate axes; and, besides this, we shall not transmit the forces, but shall refer each force to that particle on which it acts.

Let F_1, F_2, F_3, ... (Fig. 21) be the forces acting on the particles A, B, D, ... of a rigid body. Refer the body to the co-ordinate axes Ox, Oy, Oz. Join A, B; and let the diameter of the two forces F_1, F_2, pass through C. Draw Aa, Bb, Cc, ... parallel to Oz, and therefore perpendicular to the plane xOy.

Let x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3, ... be the respective co-ordinates of the points A, B, D, ... and x', y', z', of C; and let θ be the angle of inclination of AB to ab, then

$$z' - z_1 = Cc - Aa = AC \sin\theta,$$

$$\text{and } z_2 - z' = Bb - Cc = BC \sin\theta.$$

Now, by Art. 46, $F_1 : F_2 :: BC : AC,$

$$:: BC \,.\, \sin\theta : AC \,.\, \sin\theta,$$

$$:: z_2 - z' : z' - z_1;$$

$$\therefore F_1 (z' - z_1) = F_2 (z_2 - z'),$$

$$\therefore (F_1 + F_2) z' = F_1 z_1 + F_2 z_2.$$

Again, by transposing the forces F_1 and F_2 to C (Art. 45), or considering their resultant $F_1 + F_2$ (Art. 47) as acting there, we have, putting $x'' \; y'' \; z''$ for the co-ordinates of C′, the point through which passes the resultant of the three forces F_1, F_2, F_3, or which is the same, of the two $(F_1 + F_2)$ and F_3,

$$(F_1 + F_2 + F_3)\, z'' = (F_1 + F_2)\, z' + F_3 z_3,$$
$$= F_1 z_1 + F_2 z_2 + F_3 z_3.$$

In this manner, introducing successively a single force, until all are taken in, and putting $\dot{x} \; \dot{y} \; \dot{z}$ for the co-ordinates of the point at which the resultant of all the given forces acts, we shall at length obtain the final expression

$$(F_1 + F_2 + F_3 + \ldots + F_n)\dot{z} = F_1 z_1 + F_2 z_2 + F_3 z_3 + \ldots + F_n z_n;$$

or, more concisely,

$$\dot{z} \,.\, \Sigma F = \Sigma(Fz) \; . \; . \; . \; . \; . \; (1).$$

Very frequently the product of a force into the perpendicular distance of the particle on which it acts from a plane, is called the moment of the force with respect to that plane. Now it will be observed that z_1, z_2, z_3, ... are the perpendicular distances of the particles on which the forces F_1, F_2, F_3, ... respectively act from the plane xOy; and, consequently, as the plane xOy is not restricted as to position by the demonstration of the expression just given, we may enunciate equation (1) in general terms, thus:

The moment of any system of parallel forces, with respect to any plane, is equal to the moment of their resultant with respect to the same plane.

Hence for the planes xOz, and yOz, respectively, we have

$$\dot{y} \,.\, \Sigma F = \Sigma(Fy), \text{ and } \dot{x} \,.\, \Sigma F = \Sigma(Fx).$$

66. Since the expressions ΣF, $\Sigma(Fx)$, $\Sigma(Fy)$ and $\Sigma(Fz)$ do not at all depend upon the *direction* in which the system of parallel forces acts, but only on the magnitudes of the forces themselves, and on the positions of the particles on which they act, it follows that the position of the point, whose co-ordinates are $\dot{x}\dot{y}\dot{z}$, determined from the equations

$$\dot{x} = \frac{\Sigma(Fx)}{\Sigma F}, \dot{y} = \frac{\Sigma(Fy)}{\Sigma F}, \text{ and } \dot{z} = \frac{\Sigma(Fz)}{\Sigma F},$$

will be fixed in position, although the forces change their directions in any manner, so as still to continue parallel to each other.

There exists, then, a certain point in every rigid body, through which the diameter of a given system of parallel forces always passes, whatever be the position of the body with respect to the forces, providing the same force always acts on the same corresponding particle.

This point is of great importance in Mechanical Investigations, and has been called *The Centre of Parallel Forces.*

67. If the centre of parallel forces be in the origin O of co-ordinates, $\dot{x}\,\dot{y}\,\dot{z}$ are each $= 0$, and therefore

$$\Sigma(Fx) = 0, \;\; \Sigma(Fy) = 0, \;\; \Sigma(Fz) = 0.$$

Hence the moment of any system of parallel forces, with respect to any plane passing through the centre of parallel forces, is equal to zero.

The demonstration of Art. 65 fails when $\Sigma F = 0$, as it ought; for we know that, in that case, there is no *diameter* (Art. 63), and consequently no *centre* of parallel forces.

68. A force may be transposed parallel to itself, by the introduction of a couple in a plane parallel to it.

Let F be a force acting at the point A (Fig. 22), and suppose P the point to which F is to be transposed. At P apply two forces F_1 and $-$ F, each equal and parallel to F; they will not disturb the body (Art. 14). The single force F may consequently be considered as equivalent to the three F, $-$ F, and F_1, of which (F, $-$ F) form a couple, which may be transposed any where parallel to its own plane, and the latter F_1 is nothing else than the original force F transposed to P.

69. Since the moment of the introduced couple = F × perpendicular distance between AF and PF_1, and that this moment is positive or negative, according as P lies to the right or left of A, we can, by transposition of a force to a proper position, introduce either a positive or negative couple, of any proposed magnitude, by varying the perpendicular distance between PF and AF.

70. If the plane of a couple be parallel to the direction of any force, they may always be reduced to a single equivalent force.

Let F be the given force acting at *a* (Fig. 23), and (S, $-$ S) the given couple, whose arm is AB. Change it into another, whose forces $(f, -f)$ shall each be equal to F, and let its arm be equal to *ab*;

$$\therefore f \,.\, ab = S \,.\, AB, \text{ (Art. 54).}$$

Now turn the couple $(f, -f)$, round its axis (Art. 53), till its forces are both parallel to *a*F, which may be done, since the plane of the couple is parallel to *a*F; and then transpose the couple parallel to itself, to the position in the figure. Then, because F and $-f$ are equal and opposite, they may be removed (Art. 14), and there only remains the single force f, acting at b, which is, consequently, the resultant of the proposed couple and single force.

71. If from a point two straight lines be drawn parallel and equal to the axes of two couples, and completed into a parallelogram, the diagonal will be the axis of a couple equivalent to them both.

Let L and M be the moments of two couples; and in the line of intersection of their planes take AB (Fig. 24), of any convenient length, and reduce the couples to others (F, —F) and (f, $-f$) in the same planes respectively, having the common arm AB (Arts. 53, 54);

$$\therefore\ F \cdot AB = L, \text{ and } f \cdot AB = M.$$

Take AF, — FB, each equal to the axis of the couple (F, — F); and Af, $-f$B, each equal to that of the other couple; and complete the parallelograms, and draw the diagonals AR, BR, as in the figure.

Then, since AF, Af are equal to the axes of the two couples, they represent their moments;

$$\begin{aligned}\therefore\ AF : Af &:: L : M,\\ &:: F \cdot AB : f \cdot AB,\\ &:: F : f;\end{aligned}$$

that is, they are proportional to the forces F, f, and therefore (Art. 29) AR is proportional to their resultant (R), and it represents it in direction also. Similarly, BR is proportional to, and represents in direction, the resultant (— R) of — F and $-f$. From the equality of the parallelograms FAf, FBf, it is manifest that the angle FAR = angle FBR, and therefore AR is parallel to BR; consequently RABR is the plane of the resultant couple (R, — R). And since AB is at right angles to AF and Af, it is at right angles to the plane FAf, and therefore to AR; similarly, it is at right angles to BR; and therefore AB is the arm of the resultant couple. Its moment is R . AB; which denote by G.

Now from the triangle AFR we have

$$AF : Af : AR :: F : f : R,$$

$$:: F . AB : f . AB : R . AB,$$

$$:: L : M : G.$$

Consequently AR is equal to the axis of the resultant couple G, and RABR is its plane. But if lines be drawn, as in the enunciation of the proposition, they will be respectively equal and perpendicular to the lines AF, Af, AR; and will consequently form the two sides and diagonal of a parallelogram, the sides being the axes of the component couples, and the diagonal that of their resultant.

72. If ϕ be the angle between the planes of the two couples, we shall have

2LM cos ϕ + M²

$G^2 = L^2 +$ ~~$2LM + \cos\phi + M^2$.~~

For, in the triangle AFR,

$$AR^2 = AF^2 - 2AF . FR \cos AFR + FR^2,$$

$$= AF^2 - 2AF . Af \cos(\pi - FAf) + Af^2,$$

$$= AF^2 - 2AF . Af \cos(\pi - \phi) + Af^2,$$

$$= AF^2 + 2AF . Af \cos\phi + Af^2;$$

but it has been shewn, that AF, Af, AR, respectively represent L, M, G, and therefore

$$G^2 = L^2 + 2LM \cos\phi + M^2.$$

73. The most useful case of the composition and resolution of couples, is when the component couples are situated in planes at right angles to each other. Let θ be the angle between the planes of the couples L and G; then, on refering to Fig. 24, and supposing the angle FAf a right angle, we have

$$AF = AR\cos\theta, \text{ and } Af = AR\cos(90 - \theta) = AR\sin\theta,$$

$$\text{and therefore } L = G\cos\theta,$$

$$\text{and } M = G\sin\theta.$$

By these two formula we may resolve a proposed couple G into two other couples, situated in planes at right angles to each other. But if the components L and M are given, the resultant couple will be determined in magnitude and position from the two equations,

$$G^2 = G^2\cos^2\theta + G^2\sin^2\theta$$

$$= L^2 + M^2;$$

$$\tan\theta = \frac{G\sin\theta}{G\cos\theta} = \frac{M}{L}.$$

74. Hence any proposed couple G can be resolved into two other couples, the one situated in a plane parallel, and the other in a plane perpendicular, to a given line. And if θ be the inclination of the given line to the plane of the proposed couple, the moments of the component couples are respectively,

$$G\cos\theta, \text{ and } G\sin\theta.$$

75. If from a point three straight lines be drawn, respectively parallel and equal to the axes of three couples, and completed into a parallelopiped, the diagonal drawn from the same point will represent the axis of a couple which is equivalent to them all.

Let AB, AC, AD (Fig. 25) be respectively equal and parallel to the axes of three couples; complete them into a parallelopiped, and draw the diagonal AE. Join DE, AN. Then, by Art. 71, the couples whose axes are parallel to AC, AB, are equivalent to one whose axis is AN; and this and the couple whose axis is parallel to AD, are equivalent to one whose axis is AE; hence AE is parallel and equal to the axis of the resultant couple.

76. The most useful case of the last article is, when the planes of the three couples are mutually at right angles. The parallelopiped CEBD then becomes rectangular; and the lines AB, AC, AD, AE, being at right angles to the planes of the couples (Art. 52), will be mutually inclined to each other, in the same angles at which the planes of the couples and that of their resultant are inclined.

Let now L, M, N, denote respectively the moments of the couples, whose axes are parallel to AB, AC, AD; and G that of their resultant, whose axis is parallel to AE. λ, μ, ν the angles which the plane of this couple makes with the planes of its three components; which will, respectively, be equal to the angles at which their axes are inclined; viz.— the angles EAB, EAC, EAD. Then, as in Art. 33,

$$AB = AE \,.\, \cos EAB,$$

$$AC = AE \,.\, \cos EAC,$$

$$\text{and } AD = AE \,.\, \cos EAD.$$

But AB, AC, AD, AE, respectively represent the moments L, M, N, G (Art. 52), and therefore

$$L = G \cos \lambda,$$

$$M = G \cos \mu,$$

$$\text{and } N = G \cos \nu.$$

Which formulæ will enable us to resolve a given couple G into three other couples, acting parallel to three given rectangular planes.

77. If the three component couples L, M, and N, be given to find the magnitude and position of their resultant G, we must use the formulæ

$$G^2 = G^2 (\cos^2 \lambda + \cos^2 \mu + \cos^2 \nu) \; \ldots \; \text{(Art. 34)},$$
$$= (G \cos \lambda)^2 + (G \cos \mu)^2 + (G \cos \nu)^2,$$
$$= L^2 + M^2 + N^2;$$
$$\text{and } \cos \lambda = \frac{L}{G}, \; \cos \mu = \frac{M}{G}, \text{ and } \cos \nu = \frac{N}{G}.$$

78. We come now to determine the resultant of forces acting upon a rigid body in any directions whatever.

Let F_1, F_2, F_3, ... be the forces, and refering the body to a system of rectangular co-ordinate axes, let $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$, $x_3\, y_3\, z_3$, ... be the co-ordinates of the respective points on which they act; $\alpha_1\, \beta_1\, \gamma_1$, $\alpha_2\, \beta_2\, \gamma_2$, $\alpha_3\, \beta_3\, \gamma_3$, ... the respective inclinations of their directions to the co-ordinate axes of x, y, z; $X_1\, Y_1\, Z_1$, $X_2\, Y_2\, Z_2$, $X_3\, Y_3\, Z_3$, ... their respective rectangular components (Art. 33), parallel to the same axes.

$$\therefore\; X_1 = F_1 \cos \alpha_1, \quad Y_1 = F_1 \cos \beta_1, \quad Z_1 = F_1 \cos \gamma_1,$$
$$X_2 = F_2 \cos \alpha_2, \quad Y_2 = F_2 \cos \beta_2, \quad Z_2 = F_2 \cos \gamma_2,$$
$$X_3 = F_3 \cos \alpha_3, \quad Y_3 = F_3 \cos \beta_3, \quad Z_3 = F_3 \cos \gamma_3,$$
$$\ldots = \ldots \quad \ldots = \ldots \quad \ldots = \ldots$$

Take away the original forces F_1, F_2, ... and substitute, instead of them, their rectangular components thus determined; the whole system will then be reduced to three separate and distinct sets of parallel forces, viz.—

X_1, X_2, X_3, ... parallel to the axis of x;
Y_1, Y_2, Y_3, ... parallel to that of y,
and Z_1, Z_2, Z_3, ... parallel to that of z.

Let P (Fig. 26) be the point in the plane xOy, to which the force Z_1 may be transmitted, and at the origin O apply two forces Z_1 and $-Z_1$, each parallel and equal to Z_1; this force (Art. 69) is then reduced to an equal and parallel force Z_1 acting at the origin, and a couple $(Z_1, -Z_1)$ whose arm is OP; which, by Art. 73, we may resolve into two couples, the one in a plane parallel to xOz, whose moment $= -Z_1x_1$ (Art. 62), and the other in a plane parallel to yOz, whose moment $= Z_1y_1$. If we, in this manner, also resolve Z_2, Z_3, ... into corresponding equal forces acting at the origin O, and couples in planes parallel to xOz and yOz, it is manifest that Z_1, Z_2, Z_3 . . . . will be equal to a single force acting at the origin

$$= Z_1 + Z_2 + Z_3 + \dots\dots$$
$$= \Sigma(Z),$$

to a couple in a plane parallel to xOz (Art. 55), whose moment

$$= -Z_1x_1 - Z_2x_2 - Z_3x_3 - \dots\dots$$
$$= -\Sigma(Zx),$$

and to a second couple in a plane parallel to yOz, whose moment

$$= Z_1y_1 + Z_2y_2 + Z_3y_3 + \dots\dots$$
$$= \Sigma(Zy).$$

By a similar method we find that the forces Y_1, Y_2, Y_3 . . . . are equivalent to a single force ΣY, acting at the origin, and to two couples, the one in a plane parallel to zOy, whose moment $= -\Sigma(Yz)$, and the other in a plane parallel to xOy, whose moment $= \Sigma(Yx)$; and the forces X_1, X_2, X_3 . . . . are equivalent to a single force ΣX at the origin, and to two couples respectively parallel to the planes yOx, zOx, whose moments are $-\Sigma(Xy)$ and $\Sigma(Xz)$.

Upon the whole, then, it appears that the given forces F_1, F_2, F_3 . . . . are equivalent,—

1st. To three forces ΣX, ΣY, ΣZ, acting at the origin parallel to the co-ordinate axes; which may be reduced (Art. 36) to a single force R at the origin, in a direction making angles α, β, γ with the axes of co-ordinates, such that

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 \ldots\ldots\ldots (A),$$

$$\text{and } \cos\alpha = \frac{\Sigma X}{R},\ \cos\beta = \frac{\Sigma Y}{R},\ \cos\gamma = \frac{\Sigma Z}{R} \ldots\ldots (B).$$

2ndly. To two couples in planes parallel to zOy, whose moments are $\Sigma(Zy)$ and $-\Sigma(Yz)$; which, by Art. 55, are equivalent to a single parallel couple, whose moment

$$= \Sigma(Zy) - \Sigma(Yz), \text{ or } \Sigma(Zy - Yz);$$

to two other couples in planes parallel to xOz, whose moments are $\Sigma(Xz)$ and $-\Sigma(Zx)$, and which may be reduced to a single parallel couple, whose moment

$$= \Sigma(Xz - Zx).$$

And to two couples in planes parallel to yOx, whose moments are $\Sigma(Yx)$ and $-\Sigma(Xy)$, which, as before, are equivalent to a single parallel couple, whose moment

$$= \Sigma(Yx - Xy).$$

The six couples are thus reduced to three, acting respectively parallel to the planes zOy, xOz, and yOx, which are mutually at right angles. If, therefore, G be the moment of the couple, which is the resultant of these three; and λ, μ, ν the inclinations of its plane to their planes, that is, to the three co-ordinate planes; also, for brevity, writing

$$L \text{ for } \Sigma(Zy - Yz),$$

$$M \text{ for } \Sigma(Xz - Zx),$$

$$\text{and } N \text{ for } \Sigma(Yx - Xy),$$

we have, by Art. 77,

$$G^2 = L^2 + M^2 + N^2; \ldots\ldots\ldots \quad (C),$$

$$\cos\lambda = \frac{M}{G}, \cos\mu = \frac{L}{G}, \text{ and } \cos\nu = \frac{N}{G} \ldots\ldots (D).$$

It appears, then, that any proposed system of forces can be reduced to a single resultant force acting at the origin, its magnitude being determined by the equation (A), and its direction by equations (B); and to a single resultant couple, whose moment is determined by equation (C), and the position of its plane by equations (D).

79. Hence any set of forces can either be reduced to a single resultant, or, at any rate, to two resultant forces.

For we have shewn that it can in general be reduced to a single force R, and a couple; transpose the couple, till the direction of one of its forces passes through the origin, transmit the force to that point, and compound it with R, which acts at the same point. If the transmitted force is equal and opposite to R, they may be removed (Arts. 17, 14), and the set is then reduced to the remaining force; but if they are not equal and opposite, their resultant and the remaining force of the couple are the two to which the system can be reduced.

80. If the resultant R should happen to be parallel to the plane of the couple G, the system may be still further reduced. For, as observed in Art. 70, such a system can be reduced to a single force, equal and parallel to R, and the the sole effect of this reduction will be to change the point at which R acts. By means of these properties we can investigate the condition, that a system of forces may have a single resultant, and find the point at which this resultant acts.

For the sine of the angle, at which the direction of R is inclined to the plane of G, is

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu;$$

and, in the case under consideration, this angle, and therefore its sine, is equal to zero;

$$\therefore 0 = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu,$$

$$= \frac{\Sigma X}{R}\cdot\frac{L}{G} + \frac{\Sigma Y}{R}\cdot\frac{M}{G} + \frac{\Sigma Z}{R}\cdot\frac{N}{G} \dots (1);$$

$$\therefore 0 = L\Sigma X + M\Sigma Y + N\Sigma Z \dots\dots\dots\dots (2)$$

is the condition required.

Let, now, O be the origin, and a, b, c be the co-ordinates of the point P, at which the resultant acts when the forces satisfy this condition. Then, since the method of investigation of Art. 78 reduces the forces to a resultant force acting at the origin and to a couple, if the origin had been at P instead of O, there would be no resultant couple. Now, by writing $x' + a$, $y' + b$, $z' + c$, instead of x, y, z in the equations of Art. 78, we shall obtain the same results as if we had begun with supposing P the origin; for this is nothing else than transposing the origin to P. These substitutions give us

$$L = \Sigma\,\{Z(y' + b) - Y(z' + c)\},$$

$$= \Sigma\,(Zy' - Yz') + b\Sigma Z - c\Sigma Y,$$

or denoting $\Sigma(Zy' - Yz')$ by L',

$$L = L' + b\Sigma Z - c\Sigma Y;$$

but, in a similar manner, denoting $\Sigma\,(Xz' - Zx')$, and $\Sigma\,(Yx' - Xy')$ by M' and N', and observing that the moment of the resultant couple must be equal to zero, there being in fact no couple at all, we have

$$0 = L'^2 + M'^2 + N'^2,$$

an equation which is equivalent to the three separate and independent ones,

$$L' = 0, \ M' = 0, \text{ and } N' = 0.$$

Hence

$$\left.\begin{aligned} L &= b\Sigma Z - c\Sigma Y; \\ \text{and, similarly, } M &= c\Sigma X - a\Sigma Z; \\ N &= a\Sigma Y - b\Sigma X. \end{aligned}\right\} \ldots (3).$$

If the first of these be multiplied by ΣX, the second by ΣY, and the results substituted in the equation of condition (2), we shall, after a little reduction, obtain the last equation. Hence it appears, that not more than two of the equations (3) are independent, and consequently they belong to a straight line, at any point of which the resultant of the forces may be supposed to act; and therefore (Art. 43) they are also the equations of the line in which the single resultant acts.

81. If a line be drawn parallel to the direction in which R acts, the origin of co-ordinates may be removed from one point to another of this line, without altering the moment and the inclination of the resultant couple.

For, remove the origin from O to a point whose co-ordinates are a, b, c;

$$\begin{aligned} \therefore L - L' &= b\Sigma Z - c\Sigma Y, \\ M - M' &= c\Sigma X - a\Sigma Z, \\ N - N' &= a\Sigma Y - b\Sigma X. \end{aligned}$$

Therefore L', M', N' will be constant for all values of a, b, c, which fulfil the conditions

$$\begin{aligned} b\Sigma Z - c\Sigma Y &= \text{constant}, \\ c\Sigma X - a\Sigma Z &= \text{constant}, \\ a\Sigma Y - b\Sigma X &= \text{constant}, \end{aligned}$$

which are the equations of a line parallel to the direction of R. Wherefore the origin may be changed from one point to another of this line, without altering the moments L′, M′, N′, and consequently without altering G′, the moment of the resultant couple, since

$$G'^2 = L'^2 + M'^2 + N'^2.$$

82. From Art. 37, it appears, that in all cases where the forces do not admit of a single resultant, the equations of the line in which the resultant R acts, when O is the origin, are

$$\frac{x}{\Sigma X} = \frac{y}{\Sigma Y} = \frac{z}{\Sigma Z} \quad \ldots \ldots (1).$$

Also, the equation of a plane through the origin O, parallel to the plane in which the resultant couple acts, is

$$Lx + My + Nz = 0 \quad \ldots \ldots (2);$$

which, taken in conjunction with the two equations

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$
$$G^2 = L^2 + M^2 + N^2,$$

completely determine the magnitudes and positions of the resultant force and couple to which a set of forces may be reduced; and may be used instead of the equations (A), (B), (C), (D) of Art. 78.

83. If ϕ be the angle between the direction of R and the plane of the couple G, then

$$\sin\phi = \cos a \cos\lambda + \cos\beta \cos\mu + \cos\gamma \cos\nu;$$

and, by Art. 73, we may resolve G into two couples; one in a plane parallel to R, and the other in a plane at right angles to R; the moments of these couples being

$G \cos\phi$, and $G \sin\phi$, respectively.

It must be observed, that the former of these two planes is not *any* plane passing through R, but only that one which is inclined to the plane of G, in the same angle ϕ as the line of the direction of R.

Now when we change the origin of co-ordinates, we introduce a couple (Art. 68), (k suppose) in a plane parallel to the direction of R, which we may compound with G cos ϕ, and their resultant will be a couple also situated in a plane parallel to the direction of R; and consequently G sin ϕ, which is perpendicular to R, is not at all altered by transposing the origin. It is also obvious, that we might transpose the origin to such a point Q (Art. 69) as to cause k, the introduced couple, to be equal and opposite to G cos ϕ; and, in this case, the only remaining couple is G sin ϕ, or

$$G (\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu),$$

$$\text{or } \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{R};$$

and it is evident, that this is the least possible resultant couple; call it K. Then, when Q is the origin, the forces F_1, F_2.... are reducible to a resultant R acting at Q, and to a resultant couple K in a plane at right angles to the direction of R. This is the most simple reduction of the proposed forces, and on this account Q is called the *Principal Origin:* and as this origin may be at any point in the line parallel to the direction of R passing through Q, such a line is called the *Central Axis;* and the couple K the *Principal Couple.*

84. To find the equation of the central axis.

Let a, b, c be the co-ordinates of Q any point in this line, and remove the origin to Q, by writing $x' + a$, $y' + b$, $z' + c$, for x, y, z, as in Art. 80;

$$\therefore L = L' + b\Sigma Z - c\Sigma Y,$$

$$M = M' + c\Sigma X - a\Sigma Z,$$

$$N = N' + a\Sigma Y - b\Sigma X.$$

But since K is the resultant in this case, and the angles which the plane of K makes with the co-ordinate planes are α, β, γ, for it is perpendicular to the direction of R (Art. 83),

$$\therefore L' = K \cos \alpha, \quad M' = K \cos \beta, \quad N' = K \cos \gamma,$$

$$\text{or } L' = \frac{K\Sigma X}{R}, \quad M' = \frac{K\Sigma Y}{R}, \quad N' = \frac{K\Sigma Z}{R};$$

consequently, the equations of the central axis are any two of the three

$$b\Sigma Z - c\Sigma Y = L - \frac{K\Sigma X}{R},$$

$$c\Sigma X - a\Sigma Z = M - \frac{K\Sigma Y}{R},$$

$$a\Sigma Y - b\Sigma X = N - \frac{K\Sigma Z}{R}.$$

85. If the original origin should happen to be the principal origin, then a, b, c are each equal to zero, and

$$L = \frac{K}{R} \cdot \Sigma X,$$

$$M = \frac{K}{R} \cdot \Sigma Y,$$

$$N = \frac{K}{R} \cdot \Sigma Z,$$

$$\text{and } \therefore \frac{L}{\Sigma X} = \frac{M}{\Sigma Y} = \frac{N}{\Sigma Z};$$

equations which are satisfied whenever the origin of co-ordinates happens to be a point in the central axis.

86. If it were required that the resultant should pass through a given point, we must transpose the origin of co-ordinates to that point, as in Art. 80.

87. To find locus of all the origins which give resultant couples of equal moments.

Take Q the principal origin for the origin of co-ordinates; and transpose the origin, as in Art. 68, from Q to any point P, at the distance p from the central axis. This will introduce a couple, whose moment is Rp (Art. 69), and whose plane is parallel to the direction of R, and, in this case, at right angles to the plane of K. Consequently, the moment of the couple which is compounded of Rp and K $= \sqrt{K^2 + (Rp)^2}$; which is therefore the resultant couple when P is the origin. Now we observe, that K and R are unchangeable, and consequently this moment is the same for all resultant couples corresponding to origins at the same distance p from the central axis; such a series of origins will evidently form a cylindrical surface, having the central axis for its axis.

88. In the general problem of Art. 78, if it should happen that R = 0, an equation which is equivalent to the three

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

equation (2) of Art. 80 is then satisfied, and it would thence appear, that the forces would admit of a single resultant, which is evidently not the case, since the resultant is a couple. This apparent contradiction, however, may be removed by observing, that the real condition to be satisfied in the Article referred to, is

$$\cos a \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu = 0,$$

$$\text{or } \frac{\Sigma X}{R} \cdot \frac{L}{G} + \frac{\Sigma Y}{R} \cdot \frac{M}{G} + \frac{\Sigma Z}{R} \cdot \frac{N}{G} = 0,$$

which is not satisfied, although the equation

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0$$

is satisfied.

There cannot be an equilibrium in this case, since, although R is evanescent, there yet remains a couple which will cause the body to have a tendency to rotatory motion.

Equation (2), therefore, in Art. 80, though it is in general the equation of condition that a force and a couple may admit of being reduced to a single force, does not hold good in the case under consideration; the proper equation being (1).

We come now to investigate the conditions of equilibrium of forces acting on a rigid body, and for the sake of distinctness we shall consider the rigid body;

1st. As having a fixed point.

2ndly. As having a fixed axis.

3rdly. As being perfectly free.

89. To determine the conditions of equilibrium of a rigid body, having a fixed point.

Since one point of the rigid body is fixed, we may determine the conditions of equilibrium by transposing the origin to that point.

Let a, b, c be the co-ordinates of the fixed point, transpose the origin, as in Art. 80; the effect of this will be to change the resultant couple into another G′, such that

$$G'^2 = L'^2 + M'^2 + N'^2.$$

But there cannot be an equilibrium unless this couple vanishes, for so long as it exists, it will produce a tendency to rotatory motion round the fixed point;

$$\therefore G' = 0,$$

an equation which is equivalent to the three

$$L' = 0, \quad M' = 0, \quad N' = 0.$$

Consequently, as in the article referred to, the three equations of condition are

$$\left.\begin{aligned} L &= b\Sigma Z - c\Sigma Y, \\ M &= c\Sigma X - a\Sigma Z, \\ N &= a\Sigma Y - b\Sigma X. \end{aligned}\right\} \quad . \; . \; . \; (A).$$

These equations are sufficient for equilibrium, for when they are satisfied

$$L' = 0, \quad M' = 0, \quad N' = 0,$$

and therefore the couple G′ vanishes, and the forces are reduced to the force R acting at the fixed point, which, of course, produces no effect upon the rigid body.

If the fixed point be at the origin of co-ordinates, a, b, c are each equal to zero, and the conditions of equilibrium are

$$L = 0, \quad M = 0, \quad N = 0.$$

90. To find the conditions of equilibrium of a rigid body, supposing it to have a fixed axis.

Let a, b, c be the co-ordinates of a known point P in the axis, O being the origin ; ϕ, χ, ψ the angles which it makes with the co-ordinate axis; then its equations are

$$\frac{x-a}{\cos\phi} = \frac{y-b}{\cos\chi} = \frac{z-c}{\cos\psi};$$

and remove the origin from O to P, by writing $x'+a$, $y'+b$, $z'+c$, for x, y, z, as in Art. 80 ; then the equations of the axis become

$$\frac{x'}{\cos\phi} = \frac{y'}{\cos\chi} = \frac{z'}{\cos\psi};$$

and the forces acting on the body are equivalent to a single force R acting at P, and to a couple G′ ; such that

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$
$$\text{and } G'^2 = L'^2 + M'^2 + N'^2;$$

and the angles λ', μ', ν', which the plane of this couple makes with the co-ordinate planes, are determined from the equations

$$\cos\lambda' = \frac{L'}{G'},\ \cos\mu' = \frac{M'}{G'},\ \cos\nu' = \frac{N'}{G'}.$$

Now the force R, acting at a point in the fixed axis, will produce no effect, being entirely resisted by it; and the couple G′ may be resolved (Art. 74) into two other couples, the one in a plane parallel to the fixed axis, whose moment is equal to $G'\cos\theta$, θ being the inclination of the fixed axis to the plane of G′; and the other, whose moment is equal to $G'\sin\theta$, or

$$G'(\cos\phi\cos\lambda' + \cos\chi\cos\mu' + \cos\psi\cos\nu'),$$

in a plane at right angles to the fixed axis. The former of these, being transposed into a plane passing through the fixed axis, will be wholly resisted by it, and the latter will be wholly effective in turning the body round the axis; there cannot therefore be an equilibrium, unless this couple vanish, and if it vanish, there will necessarily be an equilibrium. Hence the only condition of equilibrium of a rigid body having a fixed axis, is

$$0 = G'(\cos\phi\cos\lambda' + \cos\chi\cos\mu' + \cos\psi\cos\nu'),$$
$$= L'.\cos\phi + M'.\cos\chi + N'.\cos\psi,$$

or

$$0 = L\cos\phi + M\cos\chi + N\cos\psi + \left.\begin{array}{l}(c\Sigma Y - b\Sigma Z)\cos\phi \\ +(a\Sigma Z - c\Sigma X)\cos\chi \\ +(b\Sigma X - a\Sigma Y)\cos\psi\end{array}\right\} . \quad (B)$$

by substituting for L′, M′, N′ their respective values,

$$L + c\Sigma Y - b\Sigma Z,$$
$$M + a\Sigma Z - c\Sigma X,$$
$$N + b\Sigma X - a\Sigma Y.$$

Remark. The condition expressed by the equation (B) is, that the moment of all the forces round the fixed axis is equal to zero. The proof of this we shall reserve for the next chapter, (see Art. 100).

91. To find the condition of equilibrium of a body acted upon any forces, supposing it free.

In this case both R and G must be evanescent, which conditions are equivalent to the six independent conditions,

$$\left.\begin{aligned} \Sigma X &= 0, \\ \Sigma Y &= 0, \\ \Sigma Z &= 0, \\ L &= 0, \\ M &= 0, \\ N &= 0, \end{aligned}\right\} \ldots \text{(C)}.$$

And when these are satisfied, R and G are each evanescent; and there is no tendency to a motion, either of Translation or of Rotation; and, consequently, there is a complete equilibrium. And they are all necessary to equilibrium, for if they are not all satisfied, then either R or G will exist, and there will be a corresponding tendency to motion, either of Translation or of Rotation.

The six equations (C), since they are necessary and sufficient for equilibrium, (that is, there cannot be an equilibrium unless they are satisfied, and when they are satisfied there must be an equilibrium), are taken as the criteria of equilibrium of any forces acting on a free rigid body.

The first three provide, that there shall be no motion from place to place; and the other three, that there shall be no rotation or angular motion round any axis.

92. The six equations (C) of last article being applied to equation (B) of Art. 90, we find that the condition

expressed by it, is satisfied independently of the values of ϕ, χ, ψ; a, b, c; and, consequently, when the conditions

$$\begin{aligned} \Sigma X &= 0, \\ \Sigma Y &= 0, \\ \Sigma Z &= 0, \\ L &= 0, \\ M &= 0, \\ N &= 0, \end{aligned}$$

are satisfied, there is an equilibrium about any axis whatever. And, consequently, when a free body is in equilibrium, we may introduce any axis in any position, and the equation of condition of equilibrium about that axis will be satisfied; and, conversely, it must be satisfied, or there will not be an equilibrium. Consequently, if a rigid body be in equilibrium, the moment of the forces acting on it, about any axis whatever, must be equal to zero, a property which will be found extremely useful in a subsequent chapter, in the solution of Mechanical Problems of equilibrium.

CHAPTER III.

ON THE THEORY OF MOMENTS.

THE great importance of the moments of forces in the solutions of Statical Problems will have been already apparent, from its frequent occurrence in the preceding Theory. We shall therefore, in the present chapter, lay before the reader some of the most important theorems on the subject.

93. To find the moment of a force about a given axis.

Let F, the proposed force, act at the point C (Fig. 27) of a rigid body, which has a fixed axis AB. At P any point in AB, apply a force − F, equal and parallel to the force F, so as to form a couple (F, −F). Then the moment of the force − F about the axis AB is equal to zero, because it acts at a point in the axis; and hence the moment of F about AB

= moment of F about AB
+ moment of − F about AB,
= moment of the couple (F, − F) about AB.

Now the efficiency of a couple is measured by its moment (Art. 57).

Therefore the moment of F about AB measures the efficiency of the couple (F, − F) about AB. But, in order to discover the efficiency of the couple (F, − F) about AB, let the moment of (F, − F) be denoted by G, and the angle between the plane of G, and the axis AB by ϕ; and resolve the couple (F, − F) into two couples, the one in a plane parallel to the axis AB, whose moment, by Art. 74, will be G cos ϕ, and the other in a plane at right angles to AB, whose moment is G sin ϕ. The former of these couples, acting in a plane parallel to the axis, may be transposed parallel to itself, and then turned round till its arm coincides with the arm AB; and then its two forces, acting at two points in the fixed axis AB, can have no tendency to turn the body round AB; and, in fact, will produce no effect at all upon the body. Consequently the moment of F about AB, which measures the efficiency of (F, − F), and therefore also the efficiency of the components of (F, − F) measures that of the only efficient component, viz.—the one acting in a plane at right angles to AB; but this efficiency is also measured by G sin ϕ, because its plane is perpendicular to AB;

$$\therefore \text{ moment of } F = G \sin \phi.$$

It appears then, that to find the moment of a force about an axis, we must form it into a couple, by applying an equal and parallel force in the contrary direction at any point of the axis, and then the moment of that component of this couple, which acts in a plane at right angles to the given axis, will be equal to the moment of the proposed force.

94. To find the moment of any forces, acting on a rigid body about a fixed axis, passing through the origin of co-ordinates.

Apply at the origin for each force an equal one parallel to it and in the opposite direction; this process will transform the forces into couples, as in the last article, which will be identical with those introduced in Art. 78, by transposing the original forces to the origin. Then (using the notation of Art. 78) the moment of the resultant of these couples is G, and acts in a plane, making angles λ, μ, ν with the co-ordinate axes. Now denote the angles which the given fixed axis makes with the co-ordinate axes, by ϕ, χ, ψ; then the sine of the angle between this axis and the plane of G, is

$$\cos\lambda\cos\phi + \cos\mu\cos\chi + \cos\nu\cos\psi,$$

and consequently the moment of the forces about the given axis, which is equal to the moment of that component of G which is at right angles to the axis, is equal to

$$G(\cos\lambda\cos\phi + \cos\mu\cos\chi + \cos\nu\cos\psi),$$
$$= L\cos\phi + M\cos\chi + N\cos\psi.$$

95. The moment of the forces about the axis of x, is equal to that component of G, which is in a plane at right angles to the axis of $x = G\cos\lambda = L$. In like manner the moments about the axes of y and z, are M and N respectively.

96. $\text{Cos}\,\lambda\cos\phi + \cos\mu\cos\chi + \cos\nu\cos\psi$ is the cosine of the angle between the given axis and the axis of G. The product of a moment into the cosine of the angle between its axes and any given axis, is called the projection of the moment upon that axis. Hence L being $= G\cos\lambda$ is the projection of G upon the axis of x; and so M, N are its projections upon the axes of y and z respectively.

Also $L \cos \phi$, $M \cos \chi$, $N \cos \psi$ are the projections of L, M, N upon the given fixed axis, consequently the moment of a system of forces round an axis passing through the origin of co-ordinates, is equal to the sum of the projections of their moments about the co-ordinate axes upon that axis.

97. Of all axes passing through the origin to find that about which the moment is the greatest.

Since the moment round any axis is equal to the projection of G upon the axis, it will be the greatest possible when that projection is equal to G, or when the axis of G is parallel to the fixed axis. In this case the fixed axis makes angles λ, μ, ν with the co-ordinate axes, and is at right angles to the plane of G.

98. The moment of the forces about any axis through the origin, at right angles to the axis of greatest moment, is equal to zero.

For the moment about any such axis is equal to the projection of G upon it

$$= G \cos 90^\circ = 0.$$

99. The moment of the forces is the same round all axes through the origin, which are equally inclined to the axis of greatest moment, and which, therefore, form about it a conical surface, having its vertex in the origin of co-ordinates.

For the projection of G upon any axis of this kind is equal to G . cos of the angle, at which the axis in inclined to the axis of greatest moment, and is therefore constant.

100. To find the moment of the forces round any axis which does not pass through the origin of co-ordinates.

Use the notation of Art. 80, then the moment of the forces about the given axis is equal to

$$\begin{aligned} & L' \cos \phi + M' \cos \chi + N' \cos \psi \\ & = (L + c\Sigma Y - b\Sigma Z) \cos \phi \\ & + (M + a\Sigma Z - c\Sigma X) \cos \chi \\ & + (N + b\Sigma X - a\Sigma Y) \cos \psi, \\ & = L \cos \phi + M \cos \chi + N \cos \psi \\ & \quad + (c\Sigma Y - b\Sigma Z) \cos \phi \\ & \quad + (a\Sigma Z - c\Sigma X) \cos \chi \\ & \quad + (b\Sigma X - a\Sigma Y) \cos \psi. \end{aligned}$$

101. The moment of any forces about an axis is equal to the moment of their resultant force and couple about the same axis.

For, using the notation of last article, the moment of R, or, which is the same, the sum of the moments of ΣX, ΣY, ΣZ round the axis of x, is

$$c\Sigma Y - b\Sigma Z \;.\;.\;.\;.\;.\; (1),$$

the sum of their moments round the axis of y is

$$a\Sigma Z - c\Sigma X, \;.\;.\;.\;.\;.\; (2),$$

and round that of z is

$$b\Sigma X - a\Sigma Y \;.\;.\;.\;.\;.\; (3).$$

Hence the moment of R round the given axis, which is equal to the sum of the projections of the moments (1), (2), (3) (Art. 96) upon that axis, is

$$(c\Sigma Y - b\Sigma Z) \cos \phi + (a\Sigma Z - c\Sigma X) \cos \chi + (b\Sigma X - a\Sigma Y) \cos \psi.$$

The moment of the couple G, about the same axis, is found in like manner to be

$$L \cos \phi + M \cos \chi + N \cos \psi.$$

The sum of these two is equal to the moment of the forces about the given axis, as is evident by comparing the sum with the expression in last article for the moment of the forces.

102. In all cases where there is no resultant couple, the moment of the forces is equal to the moment of their resultant.

103. The reasoning of Art. 97 being true of any origin, it appears that there is an axis of maximum moment corresponding to every point that may be taken as the origin of co-ordinates; now it is proposed to find where the origin of co-ordinates must be situated, that the corresponding maximum moment may be less than the maximum moment corresponding to any other origin.

It appears, from the article just mentioned, that the maximum moment for any origin is equal to the moment of the resultant couple corresponding to it. Hence, then, the origin which gives the least resultant couple will be the one required. It is, in fact, the principal origin mentioned in Art. 83, and the axis required is the central axis, whose position is determined in Art. 84.

104. It will at once be evident, from the general method of reasoning on the subject of moments, and from the observation of Art. 87, that the maximum moment is the same for all origins situated in a cylindrical surface whose axis is the central axis.

CHAPTER IV.

ON THE PRINCIPLE OF VIRTUAL VELOCITIES.

105. We conceive of matter that it can move with any degree of quickness or slowness; and, in speaking of the precise degree of quickness or slowness, we use the term *velocity* as its measure. The velocity of a body when moving at a constant rate of motion, is the length of the path it describes in a given standard unit of time, the length being expressed in standard units of length. In common transactions the unit of time most generally employed, is an hour, and the unit of length a mile; thus we say, a coach travels at the rate of eight miles an hour. If these units of time and length were universally employed in speaking of motion, it would be sufficient, in the instance just mentioned, to say, the velocity of the coach is 8. As these units are not however in universal use, most English writers on the theory of Mechanics, have agreed to take a *foot* and a *second* as the unit of length and time;—an agreement which will be observed in this treatise.

106. There is one observation with respect to the velocity of a body, which (for the right understanding of the principle, which is the main object of consideration in

this chapter) it is of great importance should be made. When it is said that a coach travels at the *rate* of eight miles an hour, it is not necessary that the coach should actually travel so far as eight miles; the rate has nothing to do with the actual distance travelled; if the coach pass by us, and we see it only for an instant, we say that it is going at the rate of eight miles an hour, meaning, that if it were to continue travelling without altering its speed, it would have proceeded eight miles at the end of an hour. Hence, though the motion of a body continue but for an instant, it may have moved with the same velocity as if it had travelled for an hour. Sometimes the rate of a body's motion is continually changing, in which case we speak of its velocity at different points of its path; and the velocity at any proposed point is,—how many feet it would proceed in a second if the rate were not to be changed during that time. In this sense we speak, in the present chapter, of the velocity with which a body *begins* to move; but the body is not supposed actually to move through a finite space, but merely through a space so small that it can only be said to have *begun* to move.

107. Having given the velocity of a body in one direction, to find its velocity in any other direction.

Let AB (Fig. 28) represent the velocity and direction of the body's motion; EF the direction in which we are to estimate its velocity. Draw EG at right angles to EF; A*a*, B*b* parallel to EF, and AC parallel EG. Then every line perpendicular to EG in the plane FEG is parallel to, and therefore in the same direction as EF. Hence, to find the velocity of the body in the direction EF is the same as to find the velocity with which it recedes from the line EG; which is further evident, from the consideration that the body can only recede from the line EG, by increasing its distance from it, and every such increase of distance arises from motion parallel to, and therefore in the direction of EF.

Now at A the body's distance from EG is aA, and at the end of one second, when the body is at B, its distance is bB; consequently, in one second, the body has receded from EG, through the space $bB - aA$; that is, the velocity of the body in the direction EF is $bB - aA$.

Let v be the actual velocity of the body, the quantity which is represented by AB; and let θ be the angle at which AB is inclined to the direction EF, in which we are to estimate v; then the velocity in the direction EF

$$\begin{aligned} &= bB - aA, \\ &= bB - bC, \\ &= CB, \\ &= AB \cos ABC, \\ &= v \cos\theta. \end{aligned}$$

Hence we can estimate a body's velocity, in any proposed direction, by multiplying it by the cosine of the angle of inclination.

108. Hence the velocity in a direction at right angles to AB

$$= v \cos 90^\circ = 0.$$

109. If a rigid body move in any manner, the motion at any instant takes place about an imaginary fixed axis.

For let A, B (Fig. 29) be any two particles of a rigid body; Aa, Bb the paths they describe in the same instant; P, Q the centres of curvature of these paths; then the line joining PQ will be the axis about which the whole body turns during the instant that A takes to pass to a, and B to b.

For if the successive contemporaneous positions of A and B be joined, while passing from A to a, and from B to b, the joining lines will form a species of conical surface; and since, by reason of the rigidity of the body, they are all of the same length, the planes APa, BQb, in which lie the curves Aa, Bb, formed by their extremities, must be parallel. Now since A by turning round P describes the angle APa, in the same time that B by turning round Q describes the angle BQb, the figure APQB in the same time turns round PQ, and comes into the position aPQb, (for AP $=$ aP, and BQ $=$ bQ, because P and Q are the centres of curvature of Aa, Bb). Consequently every particle of the body situated in the line AB turns round the axis PQ.

Since, then, APa, BQb are the planes of motion of the particles A, B; and these motions have been shewn to take place about PQ; PQ must be at right angles to those planes. In like manner, if the motion of any other particle C takes place about the point R, PR must be at right angles to the planes of motion, APa, CRc; hence both PQ and PR are perpendicular to APa, which is impossible (Eucl. xi. 13) unless they coincide; in which case R is a point in PQ, and the motion of C takes place about PQ: and since C is any particle, therefore the motion of the whole body takes place round PQ. Wherefore, at the instant the body was in the first position ABC, its motion was about PQ.

The line PQ, whose existence is here determined, is that which, in Dynamics, is called the *axis of instantaneous rotation.*

110. When a body moves with the uniform velocity v during t seconds, the length (s) of the path described is tv.

For, by the definition of velocity (Art. 105), v is the length of the path described in one second; and since the motion is uniform, an equal length will be described in every succeeding second, therefore the whole space $s = tv$. The same equation holds if t be not an exact number of seconds, but contain a fraction of a second; for the motion being uniform, in half a second the space will be $\frac{1}{2}v$, in a quarter it will be $\frac{1}{4}v$, and so on.

111. When the motion of a body is not uniform, but varies continually, then $v = d_t s$; v being the velocity at the time t when the body has just described the space s.

For suppose the motion to continue for the additional time δt, at which moment let its velocity be $v + \delta v$, and the whole space described $s + \delta s$. Then the space δs has been described in the time δt, with the velocities varying *between* v and $v + \delta v$.

If the velocity had been uniform and equal to v during the time δt, the space described would have been $v\delta t$, by last article; but if it had been uniform and equal to $v + \delta v$, the other extreme, the space described would have been $(v + \delta v)\,\delta t$. Wherefore, v and $v + \delta v$ being the extremes of velocity, it is manifest that the extreme spaces which could be described in the time δt, are $v\delta t$, and $(v + \delta v)\,\delta t$; the one being greater and the other less than the actual space δs; seeing, then, that δs must always of necessity lie between the two quantities just mentioned, it follows, by dividing by δt, that $\frac{\delta s}{\delta t}$ *always* lies between v and $v + \delta v$. Now these latter quantities approach towards equality, the limit being v; and the quantity $\frac{\delta s}{\delta t}$ approaches to $d_t s$ as its limit; wherefore the three limits must be equal,

$$\therefore\ d_t s = v.$$

112. When a body turns uniformly round an axis, the angle through which it turns, in one second of time, is called its *angular* velocity. The velocity defined in Art. 105 is sometimes called *linear* velocity, to distinguish it from the velocity defined in this article.

113. The connection between the angular (ω) and linear velocity (v) of a particle, is expressed by the equation

$$v = \omega p;$$

p being the distance of the particle from the axis of rotation.

For the particle moves in a circle whose radius is p, and circumference $2p\pi$; and if t be the time (number of seconds) of turning once round, then

$$tv = 2p\pi;$$

for since v is the length of the path described in one second, tv must be the length described in t seconds.

Again, since ω is the angle described in one second, $t\omega$ must be the angle described in t seconds; but the angle described in t seconds is 2π,

$$\therefore 2\pi = t\omega,$$

$$\therefore tv = pt\omega,$$

$$\therefore v = p\omega.$$

It must be observed, that it is not necessary the body should actually move during one second, in order that this equation may hold; it will be equally true, if v and ω be the linear and angular velocities with which the particle *begins* to move. (See Art. 106).

114. When a body begins to turn round an axis, the direction in which any one of its particles begins to move is that of a tangent to the circle which it begins to describe.

For a tangent to a circle is a line which passes through two adjacent points of its circumference, and therefore a particle, in passing from the first of these points to the second, must move along the tangent.

115. The *virtual* velocity of a body acted on by a force, is its velocity estimated in the direction in which the force acts.

116. If a particle be restrained by a rod or a cord, so that it can only move in the circumference of a circle, of which the rod or the cord is the radius; the initial virtual velocity of the particle, with regard to the restraining force which the rod or cord exerts upon it, is equal to zero.

For the particle begins to move in the direction of a tangent (Art. 114) to the circle, and therefore in a direction at right angles to the radius, which is that in which the restraining force acts. Hence the virtual velocity $= 0$, Art. 108.

117. If a body rest against a smooth curve or surface, the restraining force of the curve or surface acts in the direction of a normal at the point against which it rests.

The demonstration of this property is very little more than an explanation of the word smooth. By a smooth curve or surface, we mean, one that can offer no impediment to the motion of a body along it. Now if the restraining force do not act at right angles to the curve or surface, let it act in a direction making an angle θ with it; and let F be the magnitude of the restraining force; then, by Art. 25, the resolved part of F in the direction of a tangent to the curve or surface

$$= F \cos \theta,$$

which can only be counteracted by an equal action of the curve or surface in the opposite direction.

Consequently the surface exerts a force F cos θ to impede the motion of a body along it, which is impossible, by the definition of the word smooth.

$$\therefore \text{F} \cos\theta = 0,$$

$$\therefore \cos\theta = 0,$$

$$\text{and } \theta = \frac{\pi}{2};$$

that is, the restraining force acts at right angels, and therefore in the direction of a normal, to the curve or surface.

118. Let a body rest against the point P (Fig. 30) of a curve AB, and let PQ be the radius of curvature at P; then when a body moves from P along the curve, it begins to move in the circumference of a circle whose centre is Q, and therefore its initial motion is in the direction of a tangent at P, and consequently at right angles to QP; consequently the virtual velocity (in the direction QP)

$= \text{(velocity of P along tangent at P)} . \cos\frac{\pi}{2}$ (Art. 107);

$= 0.$

119. If a system of rigid bodies, connected by hinges, inextensible rods or stretched cords, be in equilibrium, and the equilibrium be disturbed by any cause whatever, in a manner consistent with the connection of the parts of the system; that is, so that nothing be broken, that the cords continue stretched, and that such bodies as rested against

fixed points, curves or surfaces, still continue to rest against them, the equation

$$F_1v_1 + F_2v_2 + F_3v_3 + \ldots\ldots + F_nv_n = 0$$

is satisfied; F_1, F_2, F_3, ... F_n being the forces acting on the system, and v_1, v_2, v_3 .... v_n, the virtual velocities at the very beginning of the motion of the particles on which the forces respectively act. Those virtual velocities which take place in a direction opposite to that in which the corresponding forces act, are to be accounted negative.

This is the *Principle of Virtual Velocities.*

For the sake of clearness we shall call the bodies of the system A, B, C ...... L, and suppose that there are n of them.

Let F_1 (Fig. 31) be the force which acts on the particle P of the rigid body A; and when the disturbance takes place, let MN be the axis about which A begins to revolve. Draw qp perpendicular to PF_1 and MN; and through it draw a plane at right angles to MN, in which draw pf perpendicular to qp; pf will be a tangent to the circle which P describes, and therefore the direction (Art. 114) in which p begins to move. Let $\theta =$ the angle fpF_1; then, forming F_1 into a couple, according to Art. 93, by applying a force at q, its moment will be $F_1 . qp$, and the component of this, which is situated in the plane fpq (by Art. 74), is

$$F_1 . qp . \cos\theta,$$

which is therefore the moment of F_1 about MN. Now if ω be the initial angular velocity of the body about MN, the initial linear velocity of p will be equal to

$$qp . \omega \text{ (Art. 113)},$$

and will take place in the direction pf (Art. 114). Now, because P and p are situated in the line in which the force F_1 acts, the virtual velocity of P = that of p, otherwise the distance Pp would not be invariable, and the body would not be rigid. Hence

$$\begin{aligned} v_1 &= \text{the virtual velocity of P} \\ &= \text{velocity of } p \text{ in the direction } pF_1 \\ &= (\text{velocity of } p \text{ in the direction } pf) . \cos \angle fpF_1 \\ &= qp . \omega . \cos\theta, \end{aligned}$$

$$\therefore qp . \cos\theta = \frac{v_1}{\omega}.$$

Hence the moment of F_1 about MN $= F_1 \frac{v_1}{\omega}$.

Similarly, the moments of F_2, F_3 . . . . ., the other forces which act on A, are respectively $F_2 \frac{v_2}{\omega}$, $F_3 \frac{v_3}{\omega}$. . . . .

Amongst the forces F_1, F_2, F_3 . . . . here mentioned as acting on A, we include those which the other bodies exert on it through the medium of the connecting parts of the system, for these forces help to keep A in equilibrium. Therefore, by Art. 92, the sum of the moments of all the forces acting on A, is equal to zero; that is,

$$\frac{F_1 v_1}{\omega} + \frac{F_2 v_2}{\omega} + \frac{F_3 v_3}{\omega} + \ldots\ldots = 0.$$

$$\text{and } \therefore F_1 v_1 + F_2 v_2 + F_3 v_3 + \ldots\ldots = 0 \ldots\ldots (1).$$

In like manner, if F_r, F_{r+1} . . . . . . be the forces acting on B, then

$$F_r v_r + F_{r+1} v_{r+1} + \ldots\ldots = 0 \ldots\ldots (2).$$

And F_s, F_{s+1}, ... being those which act on C, we have

$$F_s v_s + F_{s+1} v_{s+1} + \ldots\ldots = 0 \ldots . (3);$$

and so on for the other bodies of the system. Adding together the n equations thus formed we have, including all the forces in the system,

$$F_1v_1 + F_2v_2 + F_3v_3 + \ldots + F_nv_n = 0;$$

$$\text{or, more briefly, } \Sigma(Fv) = 0 \ldots . (a).$$

Now, because there is an equilibrium before the disturbance, the forces which B, C, D . . . . L respectively exert on A, are exactly counterbalanced by, and therefore equal to those which A exerts on them; equation (1) therefore contains (n — 1) terms, belonging to the respective actions of B, C, D . . . . L on A; and, similarly, equations (2), (3), (4) . . . . (n) contain one term each for the action of A on B, C, D . . . . . . L, respectively; those in (1) are respectively equal to those which correspond to them in (2), (3), (4) . . . . (n) with contrary signs, so that in the addition made to obtain (a) they all disappear. The equality of the corresponding terms may be thus shewn. It has just been proved, that the force which A exerts on B is equal and opposite to that which B exerts on A; and, because that which connects A with B remains of unvariable length, if the virtual velocity of that point of A where B's action is communicated to it, be positive, the virtual velocity of that point of B, where A's action is communicated to it, must be equal to it and negative, and conversely. Hence, in adding (1) and (2) together, the actions of A and B on each other entirely disappear; in adding the other equations, the whole of A's action on the rest of the system disappears. In the same way it may be shewn, that the mutual actions of all the bodies disappear in the addition made to obtain (a). But if any bodies of the system be at all connected with *fixed points* by means of hinges, rods, or cords, or if they rest on points, curves, or surfaces, their action will not disappear by addition of the equation (1), (2) . . . . (n), but from another cause, viz.—because the virtual velocities corresponding to them are equal to zero (Art. 116). Hence, then, equation (a) contains no mutual forces, arising from the actions of the parts of the system against each other; and no actions of fixed points, curves, or surfaces, nor of rods, nor stretched cords, fastened so as to have one end unconnected with the system. In this state it constitutes what is called the principle of virtual velocities.

CHAPTER V.

ON THE CENTRE OF GRAVITY.

120. We have hitherto supposed matter to be devoid of tendencies of every kind, and ready to pay undivided obedience to any influence of the nature of force that might be impressed upon it. It is found, however, that all terrestrial bodies have a constant tendency towards the earth, and as, in practical Mechanics, we are chiefly concerned with bodies subject to this influence, it becomes absolutely necessary to take it into account. In order, however, that we may do this in the most convenient manner, it will be necessary to establish some properties of a certain point, called the Centre of Gravity, which exists in every rigid body; and to point out, and illustrate by examples, the various methods which have been employed for the determination of that point in bodies of different forms.

121. It is found by observation to be a general fact, that all matter has a natural tendency to come together, and to form one mass; and, consequently, that different portions of matter can only be kept asunder by some cause which is capable of counterbalancing this tendency. We are not

concerned to inquire into the nature and essence of the cause of this universal tendency in matter, it is sufficient for us to know, that since it produces motion in matter, it is a force.

The *force of Gravity* is that particular force which causes the tendency of every body towards the earth.

The *Weight* of a body is the tendency of it towards the earth, as compared with the tendency of another body taken as a standard; and is distinct from *heaviness* or *gravity*, which is its absolute tendency to the earth, no comparison being made with the tendency of other bodies. Weight is, in fact, the *measure* of heaviness.

122. With regard to the force of gravity, we observe that it penetrates the innermost parts of bodies, and acts equally upon every part. For we know, by experiment, that under the exhausted receiver of an air pump, bodies of unequal magnitudes, and altogether differing in form and kind, such as a ball of lead, a shilling, a feather, a shapeless piece of wood, or a particle of dust, fall from the top to the bottom of the receiver in the same time; whence we infer, that the molecules of a descending body fall in the same manner as if they were simply *contiguous* without being connected; and, consequently, that the force of gravity acts equally on every part.

We may hence conclude, that the comparative tendencies (at the same place) of different bodies to the earth's centre, or, in other words, their weights, are exactly proportional to the number of equal parts, or units of matter, which they respectively contain; the tendency of each body being, in fact, as we have just proved, equal to the sum of the equal tendencies of the equal units of which it consists.

Consequently, if the tendency of the standard unit of matter, mentioned in Art. 9, be represented by g, and W be the weight of a body whose mass is M and volume V, then

$$W = gM,$$
$$= g\rho V.$$

As a force is known only by its effects, the quantity g represents the force which produces the tendency of the standard unit to the earth; and (Art. 5) this force is measured by the velocity generated by it in a falling body in one second of time. In the latitude of London, it is found that a falling body acquires a velocity of 32.19 feet in a second, and therefore for that latitude

$$g = 32.19.$$

123. It is here to be observed, that the weight of the same body is different at different places on the earth's surface; it is the least at the equator, and increases as we advance towards the pole where it is the greatest. And even at the same distance from the equator it diminishes in the inverse duplicate ratio of its distance from the earth's centre as we ascend; and, as we descend into the earth, in the direct simple ratio of the distance from the centre. But in all bodies and systems of bodies, which are usually treated of in Statics, the difference of the distances of their respective parts from the earth's centre is so very small, in comparison of the radius of the earth (4000 miles), that we may generally, without sensible error, consider the force of gravity constant in most practical questions that occur.

124. The direction of gravity at any proposed place, is called the *Vertical* direction; it may be discovered by

suspending a heavy body by a thread, or by drawing a line perpendicular to the surface of still water. A plane at right angles to the vertical is called a *horizontal* plane, and it is evident, since the earth is spherical, that the horizontal plane which touches the surface of still water, changes its position in passing from place to place; and so the vertical, which is always at right angles to it, also changes its position; but since the mutual distances of the bodies of all systems, usually treated of in the present science, are exceedingly small, compared with the earth's radius, we may consider the surface of still water as a horizontal plane to a small extent, and consequently the verticals as parallel. We can, consequently, apply to all bodies acted on by gravity the theorems and principles of Chapter II.

125. From Art. 66, we therefore gather at once, that in every system of particles invariably connected together, there exists a certain point, through which the resultant of the forces which gravity exerts on the different parts always passes. It is there shewn, that for every system * of parallel forces, such a point exists; and that its situation depends, in general, upon the magnitude of the respective forces. That position of it which corresponds to the forces erected by gravity upon the system, is called the *Centre of Gravity*. Some writers have called this point also, the centre of parallel forces, but it is manifest that appellation is too general.

* There is an exception in the case of parallel forces, which are reducible only to a couple; but, in case of gravity, all the forces act in one direction, and are therefore reducible to a single resultant acting at the centre of parallel forces.

126. One property of the Centre of Gravity, particularly worthy of remark, is, that it does not depend at all upon the intensity of the force of gravity. For divide the whole system into very small equal molecules, the quantity of matter in each being m, and their number n, and denote the force exerted upon a unit of matter by g; then the force exerted on each molecule $= mg$ (Art. 122). And if x_1, y_1, z_1; $x_2, y_2, z_2, \ldots\ldots$ be the co-ordinates of the molecules, and $\dot{x}\ \dot{y}\ \dot{z}$ those of the centre of gravity, we have, by Art. 66,

$$\dot{x} = \frac{mg\,.\,x_1 + mgx_2 + mgx_3 + \ldots \text{ to } n \text{ terms}}{mg + mg + mg + \ldots \text{ to } n \text{ terms}},$$

$$= \frac{x_1 + x_2 + x_3 + \ldots\ldots}{n}.$$

$$\text{Similarly, } \dot{y} = \frac{y_1 + y_2 + y_3 + \ldots\ldots}{n},$$

$$\text{and } \dot{z} = \frac{z_1 + z_2 + z_3 + \ldots\ldots}{n}.$$

It appears then, that the co-ordinates of the centre of gravity are the means * of the co-ordinates of the molecules, and consequently its position is independent of the intensity of gravity. Hence the centre of gravity of any body is a certain point within it, the place of which depends only on the relative disposition of its equal molecules. The investigation of its place is therefore a matter purely geometrical, and may be applied to any body whatever; and for this reason we often speak of the centre of gravity of bodies far removed from the influence of the earth, and when, in fact, no reference is intended to be made either to the earth or to

* Hence the centre of gravity of two equal bodies is the middle point between them.

gravity; the point alluded to, being no other than the one determined from the geometrical principles just laid down, viz.—that its co-ordinates are the respective means of the co-ordinates of all the equal molecules of which the body is composed.

127. Since the resultant of the forces which act on the particles of a body passes through the centre of gravity, if that point be supported the body will be in equilibrium in every position. For instead of the forces themselves, we may substitute their resultant, which will be counteracted by the point of support, and this will be the case if the body be turned round that point into any position whatsoever.

128. And since the resultant may be applied at any point in the line of its direction (Art. 43), if the point of support be not in the centre of gravity, but in any point of a vertical passing through it, the body will be in equilibrium. And conversely, if a body be suspended from any point in it, it will not be at rest till the centre of gravity and the point of suspension, are situated in the same vertical.

This property may sometimes be employed in finding the centre of gravity. For if the body be successively suspended from two points in it, and the corresponding verticals be drawn upon or through the body, their common point of intersection will be the centre of gravity.

129. It follows at once, from Art. 127, that if all the particles which are situated in a line passing through the

centre of gravity be supported, the body will rest in equilibrium on that line in all positions. And the converse is true, viz.—that if a body rest in equilibrium, in all positions, on a fixed line, the centre of gravity must be in that line; for, unless the centre of gravity were in that line, a position might be found in which the vertical through the centre of gravity did not pass through a point of support, and consequently the body would not be in equilibrium in all positions, which is contrary to the hypothesis.

Hence, if we can find several lines on which a body will rest in all positions, the centre of gravity will be in their common point of intersection.

130. Many authors have defined the centre of gravity to be that point on which a body will rest in equilibrium, in all positions, when acted on by gravity; but it seems better to derive its definition from the general one of the centre of parallel forces. This method has the advantage of proving the existence of the point before we give it a name; and renders the explication of its various properties more simple and general.

131. Since the resultant of all the forces of gravity, which act on the particles of a body, may be supposed to act at the centre of gravity, and is equal to their sum (Art. 60), we may, in any investigation in which this resultant is required, suppose the whole mass united at the centre of gravity; and hence it becomes important to know the situation of this point in bodies of different figures.

132. It is not always convenient to divide a proposed body into equal molecules, as was done in Art. 126, it

therefore becomes necessary, in that case, to use other formulæ for the determination of the centre of gravity.

Let m_1, m_2, m_3, ... be very small masses into which the body may conveniently be supposed to be divided; $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$ . . . . their co-ordinates.

Then the forces which urge them are gm_1, gm_2, gm_3, ... respectively; and therefore, substituting in Art. 66, we obtain

$$\dot{x} = \frac{gm_1 . x_1 + gm_2 . x_2 + gm_3 . x_3 + \ldots .}{gm_1 + gm_2 + gm_3 + \ldots .},$$

$$= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots .}{m_1 + m_2 + m_3 + \ldots .},$$

$$= \frac{\Sigma(mx)}{\Sigma m}.$$

and, similarly,

$$\dot{y} = \frac{\Sigma(my)}{\Sigma m}, \quad \dot{z} = \frac{\Sigma(mz)}{\Sigma m}.$$

133. Since, whatever be the position of the plane yz, we have always

$$\dot{x} . \Sigma m = \Sigma(mx),$$

it appears that the moment, with respect to any plane, of the whole mass collected at its centre of gravity, is equal to the sum of the moments of all the molecules, with respect to the same plane.

134. If the origin of co-ordinates be in the centre of gravity, then $\Sigma(mx) = 0$, $\Sigma(my) = 0$, and $\Sigma(mz) = 0$; for $\dot{x}$, $\dot{y}$, and $\dot{z}$ are, in that case, each equal to zero.

135. Since the mass of a body of uniform density is measured by the product of its volume into its density

(Art. 9); if $\rho_1, \rho_2, \rho_3, \ldots$ be the densities, and $V_1, V_2, V_3, \ldots$ the volumes of the molecules $m_1, m_2, m_3 \ldots$ we shall have

$$m_1 = \rho_1 V_1,\ m_2 = \rho_2 V_2,\ m_3 = \rho_3 V_3, \ldots$$

the molecules being so small, that every part of each one may be considered of uniform density. Hence, by substitution in the formulæ of last article, we have

$$\dot{x} = \frac{\rho_1 V_1 . x_1 + \rho_2 V_2 . x_2 + \rho_3 V_3 . x_3 + \ldots}{\rho_1 V_1 + \rho_2 V_2 + \rho_3 V_3 + \ldots},$$

$$= \frac{\Sigma(\rho V x)}{\Sigma(\rho V)},$$

$$\text{and } \dot{y} = \frac{\Sigma(\rho V y)}{\Sigma(\rho V)},$$

$$\dot{z} = \frac{\Sigma(\rho V z)}{\Sigma(\rho V)}.$$

136. If the density of the whole system be the same, then $\rho_1 = \rho_2 = \rho_3 \ldots$ and these formulæ are simplified by dividing out ρ, thus,

$$\dot{x} = \frac{\Sigma(Vx)}{\Sigma V},\ \dot{y} = \frac{\Sigma(Vy)}{\Sigma V},\ \dot{z} = \frac{\Sigma(Vz)}{\Sigma V}.$$

But it is to be carefully observed, that these formulæ are only to be applied to such bodies as are of homogenous materials.

137. The general application of these formulæ depends on the Integral Calculus, but there are a few cases which can be made to depend upon the more simple principles of Art. 129, and with them we shall accordingly commence our series of examples on the subject of finding the position of the centre of gravity in bodies of various forms.

All bodies will be supposed homogenous, or of uniform density, unless the contrary is mentioned.

138. If through any figure a plane can be drawn, so that the figure shall be symmetrical with regard to it; that is, so that the two parts of the figure which are situated on opposite sides of that plane are perfectly similar and equal; the centre of gravity shall be in that plane.

For the moment of the volume on one side is exactly equal to the moment of that on the other side, with respect to that plane, and these moments will have contrary signs, and therefore their sum will be equal to zero. But this sum (Art. 133) is equal to the moment of the whole volume, collected at its centre of gravity, with respect to the same plane; which cannot be the case unless the centre of gravity be in that plane.

139. Hence, if we can find two such planes differently situated, the centre of gravity will be in the line of their intersection; and if we can find a third plane, the centre of gravity will be that point where it cuts the line of intersection of the other two; in other words, it will be the common point of intersection of any three planes, by which the figure can be symmetrically divided.

140. It follows, from these properties,—

1st. That the centre of gravity of a sphere, or of a spheroid, or of a cube, is its centre.

2ndly. That the centre of gravity of a parallelopiped is the middle point of one of its diagonals, and of a cylinder, the middle point of its axis.

3rdly. That the centre of gravity of any figure of revolution is some point in the axis.

141. When we speak of the centre of gravity of a line, or of a plane figure, it is to be understood that the line consists of material particles, and the plane figure of a single lamina of particles, or else, that the thickness is every where the same, and inconsiderable.

142. Hence the centre of gravity of a straight line is its middle point; of a circle, or ellipse, or square, its centre; and it will follow, from reasoning precisely similar to that of Art. 138, that if we can draw two straight lines in a plane, by each of which the figure is divided into two equal and symmetrical parts, the centre of gravity is the point of their intersection. This property will enable us to determine at once, by inspection, the centre of gravity of almost all regular plane figures.

143. To find the centre of gravity of a plane triangle.

Let ABC (Fig. 32) be the triangle, bisect one of the sides as BC in D, and join AD. Then we may suppose the triangle made up of material particles, arranged in lines parallel to BC; let *bc* be any one of them. Then, by the similar triangles BAD, *b*A*d*,

$$BD : DA :: bd : dA,$$

$$\text{and, similarly, } DA : DC :: dA : dc,$$

$$\therefore BD : DC :: bd : dc.$$

But BD = DC, therefore *bd* = *dc*; and, consequently, *d* is the centre of gravity of *bc*.

Similarly, the centre of gravity of every other line parallel to BC, of which the triangle consists, is somewhere in AD; consequently the whole triangle would rest in equilibrium on AD, consequently its centre of gravity is in AD (Art. 129). In the same manner it would appear that the centre of gravity of the whole triangle is in BE, which bisects AC, and hence G, the point of intersection of AD and BE, is the point required.

144. It may be observed that the line AD is divided in G, so that DG $= \frac{1}{3}$ AD.

For join DE, then because CA, CB are divided at E, D in the same proportion, viz.—each bisected, therefore DE is parallel to AB; and, therefore, the angle DEG is equal to the angle ABG, and angle EDG to the angle BAG, and consequently the triangles ABG, DEG are similar;

$$\therefore AG : DG :: AB : DE$$
$$:: AC : EC :: 2 : 1.$$

Hence $AG = 2DG$,

$$\text{and } \therefore AD = AG + DG = 3DG.$$
$$\therefore DG = \tfrac{1}{3} AD.$$

145. The following method, of finding the centre of gravity of a triangle, is very simple and elementary.

Let ABC (Fig. 33) be the triangle; DEF the triangle formed by joining the middle points of its sides. ABC is thus divided into four triangles AFE, FBD, EDC, DEF, perfectly equal and similar; each one of which we may suppose collected at its own centre of gravity (Art. 133), and then the distance of the centre of gravity of the whole triangle from BC, will be the mean of the distances of the centres of gravity of all the four equal ones, of which it is composed (Art. 126), from the same line BC.

Let h = the perpendicular from A upon BC, x = the distance of the centre of gravity of one of the small triangles from its base, which is, of course, the same for each of them.

Then the linear dimensions of the triangle ABC, being double those of any one of the similar small triangles, the distance of the centre of gravity of ABC from its base is $2x$;

$$\therefore 2x = \tfrac{1}{4}\{(\tfrac{1}{2}h + x) + (\tfrac{1}{2}h - x) + x + x\} = \frac{h}{4} + \frac{x}{2},$$

$$\therefore x = \frac{h}{6},$$

$$\text{and } 2x = \frac{h}{3};$$

which shews that the centre of gravity is distant from each side, exactly one third of the distance of the opposite angular point.

146. Before leaving the plane triangle we may remark, that if three equal bodies have their centres of gravity situated in the three angular points of the triangle ABC, the centre of gravity of these bodies will be the same as that of the triangle.

For to find the centre of gravity of the three bodies, we have only to observe, that BD being equal to DC, the two B, C will be in equilibrium on D, (see note, page 83); and therefore the three on a line passing through A, D; in the same manner they will be in equilibrium on BE, and therefore G is their common centre of gravity.

Hence (Art. 126) the distance of the centre of gravity of a triangle from any plane, is the mean of the distances of its angular points from the same plane.

147. To find the centre of gravity of a quadrilateral figure.

Let ABCD (Fig. 34) be the trapezium; AC, BD its diagonals intersecting in E; G its centre of gravity; GI, GK parallel to the diagonals. Then, supposing the trapezium to be made up of the two triangles ADC, ABC, we have (Art. 136),

(trapezium ABCD) . (perpendicular from G on AC)

$=(\triangle$ ABC).(perpendicular from its centre of gravity on AC)

$-(\triangle$ ADC).(perpendicular from its centre of gravity on AC),

$=\frac{1}{3}(\triangle$ ABC) . (perpendicular from B on AC)

$-\frac{1}{3}(\triangle$ ADC). (perpendicular from D upon AC).

Now the triangles ABC, ADC, having a common base AC, are proportional to the perpendiculars from B and D on AC, which are also proportional to BE, DE respectively; hence, in the above equation, instead of the triangles ABC, ADC, and the trapezium, which is their sum, write respectively the quantities BE, DE, and BE + DE, to which they are proportional; and, instead of the perpendiculars from B, D and G, or I, which is equal to it, write respectively BE, DE, and EI, which are proportional to them; and then we have

$$(BE + DE) . EI = \tfrac{1}{3} BE^2 - \tfrac{1}{3} DE^2$$
$$= \tfrac{1}{3} (BE + DE)(BE - DE),$$
$$\therefore EI = \tfrac{1}{3} (BE - DE).$$

And, similarly, $EK = \frac{1}{3}(AE - CE)$.

Hence, setting off EI equal one-third of the excess of EB above ED; and EK equal to one-third of the excess of AE above EC; and drawing IG, KG parallel to the diagonals of the trapezium, G will be the point required.

148. To find the centre of gravity of any other rectilinear figure we must divide it into triangles, and suppose each triangle collected at its own centre of gravity; we can then find the common centre of gravity of the whole by the formulæ of Art. 136.

149. To find the centre of gravity of a triangular pyramid.

Let A (Fig. 35) be the vertex, and BCD the base of the pyramid. E, H the centres of gravity of the base and the face ACD. Join AE, BH, BE, AH. Then, because E is the centre of gravity of the base, therefore BE produced, bisects CD. For a similar reason, AH produced, bisects CD; and therefore BE, AH intersect in F; consequently, AE, BH, which are in the plane ABF, intersect each other in some point G.

Now we may suppose the pyramid made up of triangular laminæ of particles, situated in planes parallel to the base; let *cbd* be one of them, cutting AF in *f*, and AE in *e*. This triangle is, of course, exactly similar to the base of the pyramid, and being parallel to it, *cd* must be parallel to CD; and therefore the triangles CAF, *cAf* are similar,

$$\therefore cf : Af :: CF : AF;$$

for a similar reason, $Af : df :: AF : DF$,

$$\therefore cf : df :: CF : DF;$$

but CF being equal to DF, *cf* must be equal to *df*, and consequently the centre of gravity of the triangle *cbd* must be in the line *bf*. Again, AFB being cut by parallel planes, *fb* must be parallel to FB, and the triangles FAE, *fAe* are similar,

$$\therefore fe : Ae :: FE : AE;$$

but, for a similar reason,

$$Ae : be :: AE : BE,$$

$$\therefore fe : be :: FE : BE.$$

But BE = 2FE, and therefore $be = 2fe$, consequently e is the centre of gravity of the lamina bcd. In the same manner, all the laminæ of which the pyramid consists, have their centres of gravity in AE, wherefore the pyramid would balance on AE in all positions; and, consequently, the centre of gravity is in that line. For like reasons, it is in the line BH, and therefore G, the point of intersection, is the centre of gravity of the pyramid.

Join HE. Then, because FE : FB :: 1 : 3 :: FH : FA, therefore HE is parallel to AB, consequently the triangles HEG, BAG are similar;

$$\therefore GE : AG :: EH : AB :: FE : FB :: 1 : 3,$$

$$\therefore AG = 3GE,$$

$$\therefore AE = AG + GE = 4GE,$$

$$\therefore GE = \tfrac{1}{4} . AE.$$

Hence, join the vertex and the centre of gravity of the base, and the centre of gravity of the solid will be at the distance of one-fourth of this line from the base.

150. It may be shewn, by a method very similar to the one in Art. 146, that if four equal bodies be placed in the four angular points of the pyramid, their common centre of gravity will coincide with the centre of gravity of the pyramid; and then the distance of the centre of gravity of any triangular pyramid, from any plane, is equal to the mean of the distances of its angular points from the same plane.

151. The line joining the centre of gravity of the base BCD, and that of any parallel section *bcd* of the pyramid being produced, passes through the vertex A.

152. If a plane be drawn through the centre of gravity of the pyramid parallel to the base, a fourth part of every line drawn from the vertex to any point in the base will be intercepted between this plane and the base.

For a fourth part of AE is intercepted, and therefore (Eucl. xi. 16) every line from the vertex to the base must be divided in the same proportion.

153. Hence, if a perpendicular be drawn from A upon the base, a fourth part of it will be intercepted between the base and a plane parallel to it through the centre of gravity of the pyramid. And, conversely, if we take a point in the perpendicular at the distance of one-fourth of its length from the base, a plane being drawn through that point parallel to the base will pass through the centre of gravity of the pyramid; consequently, all other triangular pyramids between the same parallel planes will have their centres of gravity situated in that plane.

154. To find the centre of gravity of any pyramid.

Let g (Fig. 36) be the centre of gravity of the base of the pyramid; join Ag. Then, by a method exactly similar to the one pursued in Art. 140, it may be shewn that the centres of gravity of all the plane laminæ, parallel to the base of which the pyramid may be supposed to be made up, are in Ag, and consequently the centre of gravity of the pyramid is in Ag.

But we can divide the base BCDEF into triangles, and suppose the pyramid made up of triangular pyramids, constituted upon these triangles as bases, and having the common vertex A. These, by the last article, will have their centres of gravity in a plane parallel to the base BCDEF, which divides Ag in G, so that G$g = \frac{1}{4}$ Ag; consequently the centre of gravity of the whole pyramid will be in that plane, and as it is also in Ag, it must be at G.

155. There is nothing in this demonstration to limit the number of sides of the base of the pyramid, and therefore in a cone, upon a curvilinear base of any form whatever, which we may suppose a polygon of an infinite number of sides, the centre of gravity will be found, by joining the vertex and the centre of gravity of the base, and taking a point in that line at the distance of one-fourth of its length from the base.

156. To find the centre of gravity of the frustum of a cone or pyramid cut off by a plane parallel to the base.

Let BCD (Fig. 37), *bcd* be the two ends of the frustum, which are, of course, similar figures; g, g' their centres of gravity; G the centre of gravity of the frustum, which will be in the line gg', because the centre of gravity of every lamina parallel to the base is in that line. Now, complete the frustum into a pyramid, its vertex A will be in gg' produced (Art. 151); and put a, b for the lengths of corresponding parts of the two ends of the frustum, and c for gg'.

Then Ag' and Ag being like dimensions of the top pyramid, and the whole pyramid as also b, and a; and, because the like demensions of similar figures are proportional,

$$\therefore a : b :: \text{A}g : \text{A}g',$$

$$\therefore a : a - b :: \text{Ag} : \text{Ag} - \text{Ag}' = gg' = c,$$

$$\therefore \text{Ag} = \frac{ac}{a-b}.$$

$$\text{Similarly, Ag}' = \frac{bc}{a-b}.$$

Now, measuring along gA, the distance of the centre of gravity of the whole pyramid from $g = \frac{1}{4} \cdot \frac{ac}{c-b}$; and the distance of the centre of gravity of the top pyramid from $g' = \frac{1}{4} \cdot \frac{bc}{a-b}$, and therefore, measuring from g, it $= c + \frac{1}{4} \cdot \frac{bc}{a-b}$; also, putting x for the distance of the centre of gravity of the frustum from g, measuring along gA, we have, by Art. 136,

$$(\text{whole pyramid}) . \tfrac{1}{4} . \frac{ac}{a-b}$$

$$= (\text{frustum}) . x + (\text{top pyramid}) . \left(c + \tfrac{1}{4} . \frac{bc}{a-b}\right).$$

But similar solid figures are as the cubes of their like dimensions, wherefore the whole pyramid, the top pyramid, and the frustum, which is the difference between them, are proportional to a^3, b^3, and $a^3 - b^3$ respectively; and substituting these in the last equation for the quantities to which they are proportional, we have

$$a^3 . \tfrac{1}{4} . \frac{ac}{a-b} = (a^3 - b^3)\, x + b^3 . \left(c + \tfrac{1}{4} . \frac{bc}{a-b}\right),$$

$$\therefore (a^3 - b^3)\, x = \frac{c}{4} \left\{ \frac{a^4 - b^4}{a-b} - 4b^3 \right\},$$

$$= \frac{c}{4} \left\{ a^3 + a^2 b + a b^2 + b^3 - 4b^3 \right\},$$

$$= \frac{c}{4} . \left\{ (a^3 - b^3) + (a^2 - b^2) b + (a-b) b^2 \right\};$$

therefore, by dividing the equation by $a - b$,

$$(a^2 + ab + b^2)\,x = \frac{c}{4}\cdot\left(a^2 + ab + b^2 + ab + b^2 + b^2\right),$$

$$= \frac{c}{4}\cdot\left(a^2 + 2ab + 3b^2\right);$$

$$\therefore\ x = \frac{c}{4}\cdot\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

General Properties of the Centre of Gravity.

I.

157. From Chap. I, and Art. 44, Chap. II, it appears that when forces F_1, F_2, F_3 . . . . acting in any direction converging to a point O (Fig. 38) are in equilibrium, their components in any proposed direction are such as would also produce equilibrium. Wherefore, if these forces be respectively represented by the lines OF_1, OF_2 . . . . the sum of their projections Of_1, Of_2, Of_3 . . . . on the axis OX ought to be equal to zero. But, if we draw through the point O a plane MN perpendicular to OX, these projections are equal to the respective distances of the points F_1, F_2, F_3 . . . . from that plane; and since their sum is equal to zero, their mean distance from it is also equal to zero. Consequently, when any concurrent forces acting on a rigid body are in equilibrium, the point to which the directions of the forces converge is the centre of gravity of a system of equal bodies or particles, placed at the extremities of lines drawn from that point of concourse and representing the forces in magnitudes and directions; and conversely,—

158. In a system of equal bodies or particles if we draw, from their respective centres of gravity, lines to the centre of gravity of the system, then any forces represented in magnitudes and directions by these lines will be in equilibrium.

For the mean distance of the extremities of these forces from any plane passing through the centre of gravity will be equal to zero; and the sum of the components of these forces in any proposed direction will be also equal to zero; and consequently the forces will in equilibrium.

159. It appears then, that if three forces tending to a point are in equilibrium, that point is the centre of gravity of the triangle formed by joining the extremities of the lines which represent the magnitudes and directions of the forces; for the centre of gravity of this triangle coincides with that of three equal bodies, placed at its angular points, (Art. 146).

And, in the same manner, if four forces tending to a point are in equilibrium, this point is the centre of gravity of the triangular pyramid, whose edges are the lines which join the extremities of the lines which represent the forces in magnitude and direction.

160. Conversely; there will be an equilibrium between forces which are represented in magnitude and direction, by the lines joining the centre of gravity of a triangle or pyramid with its angular points.

II.

161. If the mass of each particle of a system be multiplied by the square of its distance from a given point, the sum of the products will be the least possible when the given point is the centre of gravity of the system.

Let the centre of gravity of the system be taken for the origin of co-ordinates; and put a, b, c for the co-ordinates of the given point O (Fig. 39); $x_1\, y_1\, z_1,\, x_2\, y_2\, z_2, \ldots$ for those of the particles $m_1, m_2, \ldots$ of which the system consists.

Then

$$\begin{aligned}(\mathrm{O}m_1)^2 &= (x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2,\\ &= x_1^2 + y_1^2 + z_1^2 + a^2 + b^2 + c^2 - 2ax_1 - 2by_1 - 2cz_1,\\ &= (\mathrm{G}m_1)^2 + (\mathrm{GO})^2 - 2ax_1 - 2by_1 - 2cz_1;\end{aligned}$$

because $(\mathrm{G}m_1)^2 = x_1^2 + y_1^2 + z_1^2$, and $\mathrm{GO}^2 = a^2 + b^2 + c^2$.

Hence

$$m_1 \,.\, (\mathrm{O}m_1)^2 = m_1 \,.\, (\mathrm{G}m_1)^2 + m_1 \,.\, (\mathrm{GO})^2 - 2a \,.\, m_1x_1 - 2b \,.\, m_1y_1 - 2c \,.\, m_1z_1.$$

Similarly,

$$m_2 \,.\, (\mathrm{O}m_2)^2 = m_2 \,.\, (\mathrm{G}m_2)^2 + m_2 \,.\, (\mathrm{GO})^2 - 2a \,.\, m_2x_2 - 2b \,.\, m_2y_2 - 2c \,.\, m_2z_2,$$

$$m_3 \,.\, (\mathrm{O}m_3)^2 = m_3 \,.\, (\mathrm{G}m_3)^2 + m_3 \,.\, (\mathrm{GO})^2 - 2a \,.\, m_3x_3 - 2b \,.\, m_3y_3 - 2c \,.\, m_3z_3,$$

$$\ldots = \ldots\ldots\ldots\ldots$$

and, consequently, by adding the corresponding sides of the equations together,

$$\begin{aligned}&m_1 \,.\, (\mathrm{O}m_1)^2 + m_2 \,.\, (\mathrm{O}m_2)^2 + m_3 \,.\, (\mathrm{O}m_3)^2 + \ldots\\ &= m_1 \,.\, (\mathrm{G}m_1)^2 + m_3 \,.\, (\mathrm{G}m_2)^2 + m_3 \,.\, (\mathrm{G}m_3)^2 + \ldots\\ &+ (m_1 + m_2 + m_3 + \ldots\ldots) \,.\, (\mathrm{GO})^2\\ &- 2a \,.\, (m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots)\\ &- 2b \,.\, (m_1y_1 + m_2y_2 + m_3y_3 + \ldots\ldots)\\ &- 2c \,.\, (m_1z_1 + m_2z_2 + m_3z_3 + \ldots\ldots).\end{aligned}$$

But, because the centre of gravity of the system is in the origin of co-ordinates, we have, by Art. 134,

$$0 = m_1x_1 + m_2x_2 + m_3x_3 + \dots$$
$$0 = m_1y_1 + m_2y_2 + m_3y_3 + \dots$$
$$0 = m_1z_1 + m_2z_2 + m_3z_3 + \dots$$

Consequently,

$$m_1 . (Om_1)^2 + m_2 . (Om_2)^2 + m_3 . (Om_3)^2 + \dots$$
$$= m_1 . (Gm_1)^2 + m_2 . (Gm_2)^2 + m_3 . (Gm_3)^2 + \dots$$
$$+ (m_1 + m_2 + m_3 + \dots) . (GO)^2,$$

or, as it may be more conveniently written,

$$\Sigma \{m\,(Om)^2\} = \Sigma \{m\,(Gm)^2\} + \Sigma m . (GO)^2.$$

From this equation it appears, that the sum of the products of each particle into the square of its distance from the point O, is greater than $\Sigma\{m\,(Gm)^2\}$ by the quantity $\Sigma m . GO^2$; and since $\Sigma\{m\,(Gm)^2\}$ does not depend at all upon the position of the point O, the sum will be the least possible when $GO = 0$, that is, when the point O is in the centre of gravity of the system.

162. So long as the distance of O from G remains the same, the quantity $\Sigma\{m\,(Gm)^2\} + \Sigma m . GO^2$ retains the same value; if, therefore, O be fixed in space, and the body be made to turn round its centre of gravity, the sum of the products of each particle of the system into the square of its distance from O remains unaltered.

163. The two last articles are equally true if m_1, m_2, m_3, ... be large bodies instead of single particles, observing, in that case, that $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$, $x_3\, y_3\, z_3$, ... will be the co-ordinates of their respective centres of gravity.

164. Suppose the bodies all equal to m, and let their number be n, then

$$\Sigma\{m(\mathrm{O}m)^2\} = m_1 \,.\, (\mathrm{O}m_1)^2 + m_2 \,.\, (\mathrm{O}m_2)^2 + m_3 \,.\, (\mathrm{O}m_3)^2 + \ldots$$
$$= m\,\{(\mathrm{O}m_1)^2 + (\mathrm{O}m_2)^2 + (\mathrm{O}m_3)^2 + \ldots\}$$
$$= m \,.\, \Sigma(\mathrm{O}m)^2;$$

and, similarly, $\Sigma\{m(\mathrm{G}m)^2\} = m \,.\, \Sigma(\mathrm{G}m)^2$; also $\Sigma m = m_1 + m_2 + m_3 + \ldots = m + m + m + \ldots$ to n terms $= nm$; consequently, by substituting in the equation of Art. 161, we obtain

$$m \,.\, \Sigma(\mathrm{O}m)^2 = m \,.\, \Sigma \,.\, (\mathrm{G}m)^2 + nm \,.\, (\mathrm{GO})^2,$$
$$\therefore \Sigma(\mathrm{O}m)^2 = \Sigma \,.\, (\mathrm{G}m^2) + n \,.\, (\mathrm{GO})^2.$$

It appears then, that in a system of n equal bodies, the sum of the squares of the distances of their centres of gravity from a given point, is greater than the sum of the squares of the corresponding distances from the centre of gravity of the system, by n times the square of the distance of this latter from the given point.

Hence, if ABC be a triangle, G its centre of gravity, and O a point situated either in the plane of the triangle or not, we have

$$\mathrm{AO}^2 + \mathrm{BO}^2 + \mathrm{CO}^2 = \mathrm{AG}^2 + \mathrm{BG}^2 + \mathrm{CG}^2 + 3 \,.\, \mathrm{GO}^2.$$

And in a triangular pyramid whose angular points are A, B, C, D, and centre of gravity G,

$$\mathrm{AO}^2 + \mathrm{BO}^2 + \mathrm{CO}^2 + \mathrm{DO}^2$$
$$= \mathrm{AG}^2 + \mathrm{BG}^2 + \mathrm{CG}^2 + \mathrm{DG}^2 + 4 \,.\, \mathrm{GO}^2.$$

For, by Art. 146, the centre of gravity of the triangle coincides with that of three equal bodies placed at its angular points; and the centre of gravity of the pyramid with that of four equal bodies at its angular points, (Art. 150).

III.

165. If each particle of a system be multiplied, as in Art. 161, by the square of its distance from a given point, the sum of the products will be greater than it would be if the whole system were collected at its centre of gravity, by a quantity which is found by multiplying the products of the bodies taken two and two respectively, by the squares of their mutual distances, and dividing the sum of these products by the sum of all the bodies.

For let O (Fig. 40) be the given point, G the centre of gravity of the system of particles or bodies m_1, m_2, m_3, ... Take O for the origin, and let $\dot{x}$, $\dot{y}$, $\dot{z}$ be the co-ordinates of G; $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$, ... those of m_1, m_2, m_3 . . . .; also, let $(m_1 m_2)$, $(m_1 m_3)$, $(m_2 m_3)$ . . . . be used to denote the distances between m_1 and m_2, m_1 and m_3, m_2 and m_3, ...

Then, by Art. 132,

$$\dot{x} \, . \, \Sigma m = m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots .$$

$$\dot{y} \, . \, \Sigma m = m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots .$$

$$\dot{z} \, . \, \Sigma m = m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots .$$

squaring each of these equations, and adding the results, we obtain

$$\begin{aligned} OG^2.(\Sigma m)^2 = \; & m_1^2.(Om_1)^2 + m_2^2.(Om_2)^2 + m_3^2.(Om_3)^2 + \ldots . \\ & + 2m_1 m_2 . (x_1 x_2 + y_1 y_2 + z_1 z_2) + \ldots . \\ & + 2m_1 m_3 . (x_1 x_3 + y_1 y_3 + z_1 z_3) + \ldots . \\ & + 2m_2 m_3 . (x_2 x_3 + y_2 y_3 + z_2 z_3) + \ldots . \\ & + \ldots \ldots \ldots \ldots \end{aligned}$$

by writing OG^2, $(Om_1)^2$, $(Om_2)^2$, $(Om_3)^2$ . . . . for their equals $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, $x_1^2 + y_1^2 + z_1^2$, $x_2^2 + y_2^2 + z_2^2$, $x_3^2 + y_3^2 + z_3^2$, ... respectively.

But $(m_1 m_2)$ being the distance between two points, whose co-ordinates are $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$, we have

$$(m_1 m_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$
$$= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2$$
$$- 2(x_1 x_2 + y_1 y_2 + z_1 z_2),$$
$$= (Om_1)^2 + (Om_2)^2 - 2(x_1 x_2 + y_1 y_2 + z_1 z_2);$$
$$\therefore\ 2m_1 m_2 (x_1 x_2 + y_1 y_2 + z_1 z_2)$$
$$= m_1 m_2 \{(Om_1)^2 + (Om_2)^2 - (m_1 m_2)^2\}.$$

Similarly,

$$2m_1 m_3 (x_1 x_3 + y_1 y_3 + z_1 z_3)$$
$$= m_1 m_3 \{(Om_1)^2 + (Om_3)^2 - (m_1 m_3)^2\},$$
$$2m_2 m_3 (x_2 x_3 + y_2 y_3 + z_2 z_3)$$
$$= m_2 m_3 \{(Om_2)^3 + (Om_3)^2 - (m_2 m_3)^2\}.$$

Consequently, by substitution, $OG^2 (\Sigma m)^2$

$$= m_1^2.(Om_1)^2 + m_2^2.(Om_2)^2 + m_3^2.(Om_3)^2 + \dots$$
$$+ m_1 m_2 \{(Om_1)^2 + (Om_2)^2 - (m_1 m_2)^2\}$$
$$+ m_1 m_3 \{(Om_1)^2 + (Om_3)^2 - (m_1 m_3)^2\}$$
$$+ m_2 m_3 \{(Om_2)^2 + (Om_3)^2 - (m_2 m_3)^2\},$$
$$+ \dots\dots\dots$$
$$= (m_1 + m_2 + m_3 + \dots) m_1 (Om_1)^2$$
$$+ (m_1 + m_2 + m_3 + \dots) m_2 (Om_2)^2$$
$$+ (m_1 + m_2 + m_3 + \dots) m_3 (Om_3)^2 + \dots$$
$$- m_1 m_2.(m_1 m_2)^2 - m_1 m_3.(m_1 m_3)^2 - m_2 m_3.(m_2 m_3)^2 - \dots$$
$$= \Sigma m\,.\,\Sigma\{m\,(Om)^2\} - \Sigma\{m_1 m_2\,.\,(m_1 m_2)^2\};$$

the term $\Sigma\{m_1 m_2\,.\,(m_1 m_2)^2\}$ being understood to represent the sum of the products of the particles, taken two and two, into their mutual distances.

Hence dividing by Σm, and transposing,

$$\Sigma\{m\,(\mathrm{O}m)^2\} = (\Sigma m)\,.\,\mathrm{OG}^2 + \frac{\Sigma\{m_1 m_2\,.\,(m_1 m_2)^2\}}{\Sigma m},$$

which expresses the property to be proved.

166. From Art. 161 we have

$$\Sigma\{m\,(\mathrm{O}m)^2\} = \Sigma\{m\,(\mathrm{G}m)^2\} + \Sigma m\,(\mathrm{GO})^2;$$

which, substituted in the equation above obtained, gives

$$\Sigma\{m\,(\mathrm{G}m)^2\} = \frac{\Sigma\{m_1 m_2\,.\,(m_1 m_2)^2\}}{\Sigma m}.$$

A result which might have been obtained at once, without the aid of Art. 161, by supposing O to coincide with G.

167. If now, as in Art. 165, we suppose all the bodies equal, and n in number, the last equation becomes

$$m\,.\,\Sigma(\mathrm{G}m)^2 = \frac{m}{n}\,.\,\Sigma(m_1 m_2)^2;$$

$$\therefore\ \Sigma(m_1 m_2)^2 = n\,.\,\Sigma(\mathrm{G}m)^2.$$

Hence, in any system of n equal bodies, the sum of the squares of the lines joining their centres of gravity, two and two, is equal to n times the sum of the squares of the distances of those points from the centre of gravity of the system.

Consequently, in the case of the triangle (Art. 146),

$$\mathrm{BC}^2 + \mathrm{AC}^2 + \mathrm{AB}^2 = 3\,.\,(\mathrm{AG}^2 + \mathrm{BG}^2 + \mathrm{CG}^2).$$

Hence the sum of the squares of the sides of a triangle is equal to three times the sum of the squares of the distances of its angular points from its centre of gravity.

168. In the case of the triangular pyramid we have

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2$$
$$= 4\,(AG^2 + BG^2 + CG^2 + DG^2).$$

Hence the sum of the squares of the six edges of a pyramid is equal to four times the sum of the squares of the distances of its angular points from its centre of gravity.

IV.

Guldin's Properties of the Centre of Gravity.

169. 1st. Let BC (Fig. 41) be a plane curve line, made up of equal material particles, adjacent to each other; and suppose this material curve to revolve about a fixed axis AD, situated in its own plane, through any angle θ; it will trace out a surface of revolution, which we may also suppose to consist of equal material particles. Draw BA, Pp, CD perpendicular to the axis AD. Then any particle as P will trace out a portion of the circumference of a circle, whose radius is Pp, comprehended between two radii inclined to each other, at an angle θ; and since this angle is the same for every part of the generating curve, the portion of the surface generated by any particle will be proportional to its distance from the axis; and, consequently, the whole surface generated will be the same as if all the particles of the generating curve had been placed in a line parallel to the axis AD, at their mean distance; that is, since the particles are equal (Art. 126) at the distance of the centre of gravity of the curve BC from AD. Now this latter surface will be a

portion of a cylindrical surface, whose length is equal to that of BC, and the radius of whose base is equal to the distance of the centre of gravity of BC from AD. The surface of such a portion of the cylinder is found by multiplying its length by the circumference of the base, which latter being equal to the path described by the centre of gravity of BC, it appears

That the surface generated by a curve line, revolving about a fixed axis, is equal to the product of the length of the curve, by the length of the path described by its centre of gravity. This is the first of Guldin's properties.

170. The second may be thus stated:—*The volume generated by a plane area, revolving about a fixed axis in its own plane, is equal to the product of the area into the length of the path described by its centre of gravity.*

For let BC (Fig. 42) be the given plane area, which we may suppose to consist of equal adjacent particles. Then, as before, the portion of the whole volume generated by any particle P, is proportional to its distance Pp from the axis AD; and, consequently, the volume generated by BC, is the same as would be generated if they were all placed at their mean distance from AD, that is, at the distance of the centre of gravity G, of the given area BC from AD. Let now a be the portion of the area BC occupied by a single particle; then the volume which this would generate when placed at the same distance as G from AD, is equal to $a \times$ length of its path $= a \times$ length of the path of G. And if all the particles were placed at this distance, the volumes generated by each would be equal, and therefore the whole volume proportional to the number of generating particles. Consequently, if there be n particles in BC, the volume which would be generated by them all, when placed at the same distance as G from AD $= n \times a \times$ length of G's path. But since a is the area occupied by one particle, na will be the area occupied by all the particles, that is, $na =$ area BC; and since the volume actually generated by BC is the same as would be generated if all its particles were removed to the same distance as G from AD, it must be equal to

(area of BC) . (length of G's path),

which agrees with the enunciation.

V.

171. When a system of connected bodies, acted on by gravity, is in equilibrium, the altitude of the common centre of gravity is either a maximum or a minimum; the meaning of the terms maximum and minimum, as here used, being that the centre of gravity would necessarily descend in the one case, and ascend in the other, respectively, in passing out of a position of equilibrium.

Let $m_1, m_2, m_3 \ldots$ be the particles of the system in equilibrium; $z_1, z_2, z_3 \ldots$ their respective altitudes above a given horizontal plane; z the altitude of their common centre of gravity above the same plane; g the force exerted by gravity on a unit of matter. Then the forces acting on the particles $m_1, m_2, m_3 \ldots$ are $m_1 g$, $m_2 g, m_3 g, \ldots$ respectively, as in Art. 126, and therefore, by the principle of virtual velocities,

$$m_1 g \cdot v_1 + m_2 g \cdot v_2 + m_3 g \cdot v_3 + \ldots = 0,$$

$$\therefore m_1 v_1 + m_2 v_2 + m_3 v_3 + \ldots = 0.$$

Now $v_1 = d_t z_1$, $v_2 = d_t z_2$, $v_3 = d_t z_3$, ...

$$\therefore\ m_1 d_t z_1 + m_2 d_t z_2 + m_3 d_t z_3 + \dots . = 0.$$

But, by Art. 132, $m_1 z_1 + m_2 z_2 + m_3 z_3 + \dots . = \dot{z}\Sigma m$; therefore, by differentiation,

$$m_1 d_t z_1 + m_2 d_t z_2 + m_3 d_t z_3 + \dots . = d_t\dot{z}.\Sigma m;$$

$$\therefore\ d_t\dot{z}\,.\,\Sigma m = 0;$$

$$\therefore\ d_t\dot{z} = 0.$$

Since, then, the differential coefficient of $\dot{z}$ is equal to zero, $\dot{z}$ will in general be either a maximum or a minimum, though it may happen to be neither, according to circumstances, which we shall proceed to investigate.

Let the system be disturbed from its position of equilibrium at the time t, and let the motion continue during a time h, then the altitude of the centre of gravity at the time $t + h$ will, by Taylor's theorem, be

$$\dot{z} + d_t\dot{z}\,.\,h + \tfrac{1}{2}\,d_t^2\dot{z}\,.\,h^2 + \tfrac{1}{6}\,d_t^3\dot{z}\,.\,h^3 + \dots .$$

$$= \dot{z} + \tfrac{1}{2}\,d_t^2\dot{z}\,.\,h^2 + \tfrac{1}{6}\,d_t^3\dot{z}\,.\,h^3 + \dots .$$

Wherefore it is evident that the centre of gravity will have ascended or descended, in consequence of the disturbance, according as the quantity

$$\tfrac{1}{2}d_t^2\dot{z}\,.\,h^2 + \tfrac{1}{6}d_t^3\dot{z}\,.\,h^3 + \dots .$$

is positive or negative, which is the remaining condition, that $\dot{z}$ may be a minimum or a maximum respectively.

If the centre of gravity have neither ascended nor descended during the disturbance, then

$$0 = \tfrac{1}{2}d_t^2\dot{z}\,.\,h^2 + \tfrac{1}{6}d_t^3\dot{z}\,.\,h^3 + \dots .$$

for all the values h successively takes during the time of the disturbance, wherefore

$$0 = d_t^2\dot{z},\quad 0 = d_t^3\dot{z},\quad 0 = d_t^4\dot{z},\quad \dots .$$

and therefore $\dot{z}$ is neither a maximum nor a minimum, being, in fact, a constant quantity.

172. By Art. 131, it appears that if the centre of gravity of the system be rigidly connected with the bodies of the system, we may transpose all the forces m_1g, m_2g, m_3g . . . . to that point. Let this be done; then,—

1st. If the altitude of the centre of gravity be a maximum when in equilibrium, as soon as a disturbance takes place it will begin to descend, and the forces which act upon it will cause it to descend still farther from the position of equilibrium. This is called a position of *unstable* equilibrium.

2ndly. But if the altitude be a minimum, and a disturbance takes place, it will ascend, and the forces which act upon it will endeavour to destroy its motion, and ultimately bring it back again into the position of equilibrium. This is called a position of *stable* equilibrium.

3rdly. If the centre of gravity neither ascend nor descend, the forces which are transposed to it can neither carry it forward nor backward, for it still continues in a position of equilibrium. This is called a position of *neuter* equilibrium.

VI.

173. If a body be placed with its base upon a plane, it will stand or fall, according as a vertical through its centre of gravity falls within or without its base.

Let the vertical Gg, through the centre of gravity G of a body of any form, fall within the base AB in Fig. 43, and without it in Fig. 44. And let Ag be perpendicular on Gg.

Then, if m_1, m_2, m_3 . . . be the particles of the body, $m_1 g$, $m_2 g$, $m_3 g$ . . . will be the forces impressed upon them, which we may transpose (Art. 131) to G; the whole force acting there will be

$$\begin{aligned} & m_1 g + m_2 g + m_3 g + \ldots\ldots \\ & \quad = g\,(m_1 + m_2 + m_3 + \ldots), \\ & \quad = g\Sigma m, \\ & \quad = \text{W}, \end{aligned}$$

W denoting the weight of the body.

Now it is manifest the body cannot fall over without turning round some extreme point of its base; let this point be A; then the effort of gravity to turn the body about A

$$= \text{W} \,.\, \text{A}g.$$

In Fig. 43, this effort is exerted to turn the body in a positive direction, which is evidently impossible, because the particles in contact with the base AB cannot be moved in that direction, wherefore the body cannot fall over about A; in the same way it may be shewn that it cannot fall over about any other point in the base, and therefore it will stand.

But, in Fig. 44, the effort $W . Ag$ is exerted to turn the body in a negative direction, and as there is nothing to hinder the particles of the base, which are in contact with the plane, from moving in that direction, the body will fall over.

174. The body, in Fig. 43, cannot be overturned about A, except by a force which tends to turn it in a negative direction; and the only force to oppose it is W, whose moment is

$$W . Ag;$$

consequently, such a body cannot be overturned by any force whose moment is less than

$$W . Ag.$$

Remark. The reasoning of the last two articles is equally good if the plane on which the body is placed be not horizontal but inclined, providing the body be prevented from sliding, by the roughness of the plane, or any equivalent cause.

175. Since a force whose moment is just greater than $W . Ag$ can overturn the body, if the base and therefore Ag be extremely small, a very small force will be sufficient to overturn it; hence it must be extremely difficult to balance a body upon a point, as the slightest agitation of the air would overturn it.

176. Although when the vertical through the centre of gravity falls within the base, bodies will stand firm, yet they will have different degrees of firmness or stability. If the body be overthrown about A, its centre of gravity will describe the arc of a circle whose centre is A and radius AG, (Fig. 45); and as soon as G gets beyond G′,

the highest point of this arc, the vertical from G will fall to the left of A, and then the body will fall over of itself. In order then, that a force may overthrow the body, it must be capable of elevating its centre of gravity to G′; the higher therefore G′ is above G, that is, the greater is

$$AG' - Gg, \text{ or } AG - Gg,$$

the more stable will the body be. We may therefore take

$$W(AG - Gg),$$

$$\text{or } \frac{W(AG^2 - Gg^2)}{AG + Gg},$$

$$\text{or } \frac{W \cdot Ag^2}{2Gg}, \text{ very nearly,}$$

as the measure of the stability of a body standing upon a horizontal plane. From this measure of stability, it appears,—

1st. That if two bodies be perfectly similar and equal, the heavier will stand firmer than the lighter.

2ndly. That if two bodies of equal weights stand on equal bases, the one whose centre of gravity is highest will be most easily overturned.

Hence a waggon loaded with hay is more easily overturned then if loaded with an equal weight of a heavier substance; and a coach will not be so liable to be overthrown when it has inside passengers, as when it has none.

Remark. If a body rest upon points instead of a flat base, it will be stable or not, according as a vertical through its centre of gravity falls within the polygon formed, by passing a thread round the points on which it rests.

Hence no animal can stand unless the vertical through its centre of gravity falls within the polygon, whose angular points are its feet; and this being very large for all four-footed animals, they find it very difficult to stand on two feet.

177. To find the centre of gravity of a plane curve.

Let AB (Fig. 46) be the curve line; Ox, Oy rectangular axes in its plane to which it is referred by its equation. P any point in AB, and Q very near to P. Let OM $= x$, MP $= y$, MN $= \delta x$, AP $= s$, PQ $= \delta s$, $u =$ the moment of AP about Oy, $\delta u =$ that of PQ about Oy.

Now the moment of PQ about Oy is greater than it would be if PQ were all collected at P, that is,

$$\delta u > x \, . \, \delta s;$$

and it is less than if PQ were all collected at Q, that is,

$$\delta u < (x + \delta x) \, . \, \delta s.$$

Hence $\frac{\delta u}{\delta x}$ is always greater than $x \, . \frac{\delta s}{\delta x}$,

and always less than $x \frac{\delta s}{\delta x} + \delta s$.

Now this being always the case; and seeing that the two latter quantities are ultimately (that is, have their limits) equal; it is evident, that all three are ultimately equal. The limit of $\frac{\delta u}{\delta x}$ is $d_x u$, and that of $x \frac{\delta s}{\delta x}$ and $x \frac{\delta s}{\delta x} + \delta s$ is $x d_x s$, wherefore

$$d_x u = x d_x s,$$

$$\therefore u = \int_x (x d_x s).$$

But, if $\dot{x}$, $\dot{y}$ are the co-ordinates of the centre of gravity of AP, we have, by Art. 136,

$$\dot{x} s = u = \int_x (x d_x s).$$

And, similarly, we shall find

$$\dot{y}s = u = \int_y (yd_ys),$$
$$= \int_s (yd_s s),$$

by changing the independent variable.

From these two equations $\dot{x}$ and $\dot{y}$ are to be determined.

Remark. The same result may be deduced from Guldin's first theorem.

For let the curve AP revolve about the axis Ox, then the surface generated by it is known, by the Differential Calculus, to be

$$2\pi \int_s (yd_s s).$$

But, by the theorem of Guldin, this surface is equal to the product of the arc AP into the length of the path of its centre of gravity.

Now the distance of the centre of gravity of AP from Ox is $\dot{y}$, and its path is the circumference of a circle whose radius is $\dot{y}$, its length is therefore equal to $2\pi\dot{y}$.

$$\therefore 2\pi\dot{y} . s = 2\pi \int_s (yd_s s).$$
$$\therefore \dot{y}s = \int_s (yd_s s) = \int_y (yd_y s).$$

And, similarly,

$$\dot{x} . s = \int_y (xd_y s) = \int_s (xd_s s).$$

178. These results may be adapted to polar co-ordinates, by the usual method.

For join OP, and put OP $= r$, xOP $= \theta$,

$$\text{then } x = r\cos\theta,\ y = r\sin\theta,$$

and therefore

$$d_s s = \sqrt{(d_s r)^2 + r^2 d_s \theta^2} = \sqrt{(d_\theta r)^2 + r^2} . d_s\theta,$$
$$\therefore s = \int_s \{\sqrt{(d_\theta r)^2 + r^2} . d_s\theta\},$$
$$= \int_\theta \sqrt{(d_\theta r)^2 + r^2}.$$

$$\text{And } \int_s (xd_s s) = \int_\theta (xd_\theta s),$$

$$= \int_\theta (r\cos\theta\sqrt{(d_\theta r)^2 + r^2});$$

$$\therefore \dot{x}\int_\theta \sqrt{(d_\theta r)^2 + r^2} = \int_\theta (r\cos\theta\sqrt{(d_\theta r)^2 + r^2});$$

and, similarly,

$$\dot{y}\int_\theta \sqrt{(d_\theta r)^2 + r^2} = \int_\theta (r\sin\theta\sqrt{(d_\theta r)^2 + r^2}).$$

Ex. 1. To find the centre of gravity of an arc of a circle.

Let AB (Fig. 47) be the arc, G its centre of gravity, C its middle point, O the centre of the circle. Join OC, this will pass through G, because it divides AB symmetrically (Art. 138).

Let O be taken for the origin of co-ordinates; and OC for the axis of x; x, y the co-ordinates of any point in AB. Draw AM perpendicular to OC.

$$\text{Then } d_x s = \frac{a}{\sqrt{a^2 - x^2}};$$

$$\therefore \int_x (xd_x s) = \int_x \frac{ax}{\sqrt{a^2 - x^2}} = -a\sqrt{a^2 - x^2} + C = -ay + C.$$

The integral taken from $x = \text{OM}$ to $x = \text{OC}$, or from $y = \text{AM}$ to $y = 0$, will give the moment of AC, the double of which will be $2a$. AM, the moment of AB.

$$\therefore \dot{x}\,.\,(\text{arc}) = 2a\,.\,\text{AM} = (\text{radius})\,.\,(\text{chord of AB}),$$

$$\therefore \dot{x} = \frac{(\text{rad.})\,.\,(\text{chord})}{\text{arc}}.$$

Remark. Suppose the arc AB to revolve about the axis OY, through an angle θ, the length of the path described by its centre of gravity $= \dot{x}\theta$, and therefore the surface generated $=$ (arc AB) . $\dot{x}\theta$, (Art. 169); which

= (chord of AB). OC . θ. Now OC . θ is the length of the path of the point C, and therefore (chord of AB).OC.θ is equal to the surface that would be generated by a tangent at C, equal in length to the chord. Hence the surface of any zone of a sphere is equal to the curve surface of its circumscribing cylinder.

Ex. 2. To find the centre of gravity of the arc of a semicycloid.

Let BC (Fig. 48) be the base, and AB the axis of the simicycloid; P any point in the arc AC; PM perpendicular to AB. AM $= x$, MP $= y$, $s =$ AP, AB $= 2a$. Then the equation of the cycloidal arc AC is

$$y = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a};$$

$$\therefore d_x y = \frac{a - x}{\sqrt{2ax - x^2}} + \frac{a}{\sqrt{2ax - x^2}}, \text{ by differentiation,}$$

$$= \frac{2a - x}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a}{x} - 1};$$

$$\therefore d_x s = \sqrt{1 + (d_x y)^2} = \sqrt{\frac{2a}{x}};$$

$$\therefore s = \int_x \sqrt{\frac{2a}{x}} = 2\sqrt{2ax}.$$

Also,

$$\int_x (x d_x s) = \int_x \left(x \sqrt{\frac{2a}{x}}\right) = \int_x \sqrt{2ax} = \frac{2x}{3} \cdot \sqrt{2ax} = \frac{xs}{3};$$

$$\therefore \bar{x}s = \frac{xs}{3};$$

$$\therefore \bar{x} = \frac{x}{3} = \tfrac{1}{3} . \text{AM}.$$

Again,

$$\int_x (yd_x s) = ys - \int_x (sd_x y), \text{ by integrating by parts,}$$

$$= ys - \int_x \left(2\sqrt{2ax} . \sqrt{\frac{2a}{x} - 1} \right),$$

$$= ys - 2\sqrt{2a} . \int_x \sqrt{2a - x},$$

$$= ys + 2\sqrt{2a} \cdot \frac{2}{3} (2a - x)^{\frac{3}{2}} + \text{C}.$$

Now this integral ought to vanish when $x = 0$,

$$\text{and } \therefore \text{C} = -\frac{16a^2}{3};$$

$$\therefore \int_x (yd_x s) = ys + \frac{4}{3}\sqrt{2a} . (2a - x)^{\frac{3}{2}} - \frac{16}{3} . a^2.$$

$$\therefore \bar{y}s = ys + \frac{4}{3}\sqrt{2a} (2a - x)^{\frac{3}{2}} - \frac{16}{3} a^2,$$

which gives $\bar{y}$. Hence, $\bar{x}\,\bar{y}$ the co-ordinates of the centre of gravity of the arc AP are known, and by writing $2a$ for x, we shall find

$$\tfrac{1}{3} . \text{AB}, \text{ and BC} - \tfrac{2}{3} . \text{AB},$$

to be the co-ordinates of the centre of gravity of the whole arc AC.

Ex. 3. As an example of the application of the formulæ for spirals, let us find the centre of gravity of the semiarc of one node of the Lemniscate.

Let P (Fig. 49) be any point in the semiarc APB, put $r = \text{AP}$, $\theta = x\text{AP}$, $\text{AB} = a$. Then the equation of the Lemniscate is

$$r^2 = a^2 \cos 2\theta;$$

$$\therefore rd_\theta r = - a^2 \sin 2\theta, \text{ by differentiating;}$$

$\therefore r^2 (d_\theta r)^2 + r^4 = a^4$, by adding the squares of these two equations;

$$\therefore \sqrt{(d_\theta r)^2 + r^2} = \frac{a^2}{r},$$

$$\therefore \int_\theta \sqrt{(d_\theta r)^2 + r^2} = \int_\theta \frac{a^2}{r} = \int_\theta \frac{a}{\sqrt{\cos 2\theta}},$$

$$= \int_\theta \frac{a}{\sqrt{1 - 2\sin^2\theta}}.$$

The integral of this, taken from $\theta = 0$, to $\theta = 45°$,* gives the length of the arc APB.

Again,

$$\int_\theta \left(r \cos\theta \sqrt{(d_\theta r)^2 + r^2}\right) = \int_\theta \left(r \cos\theta . \frac{a^2}{r}\right) = a^2 \sin\theta;$$

which taken between the same limits as before, gives $\frac{a^2}{\sqrt{2}}$.

$$\therefore \dot{x} . \text{arc APB} = \frac{a^2}{\sqrt{2}}, \text{ which gives } \dot{x}.$$

Also,

$$\int_\theta \left(r \sin\theta \sqrt{(d_\theta r)^2 + r^2}\right) = \int_\theta \left(r \sin\theta . \frac{a^2}{r}\right) = -a^2 \cos\theta + C,$$

which, between the same limits as before, gives

$$a^2 \left(1 - \frac{1}{\sqrt{2}}\right);$$

$$\therefore \dot{y} . \text{arc APB} = a^2 \left(1 - \frac{1}{\sqrt{2}}\right), \text{ which gives } \dot{y}.$$

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* The operation may be simplified by assuming an angle ϕ, such that $\sin\phi = \sqrt{2} . \sin\theta$; $\therefore \sqrt{1 - 2\sin^2\theta} = \cos\phi$, and

$$\int_\theta \frac{a}{\sqrt{1-2\sin^2\theta}} = \int_\phi \frac{a d_\phi \theta}{\cos\phi} = \int_\phi . \frac{a}{\sqrt{2} . \cos\theta} = \frac{a}{\sqrt{2}} . \int_\phi \frac{1}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}.$$

This integral is to be taken from $\phi = 0$, to $\phi = \frac{\pi}{2}$.

The following examples are added for practice.

Ex. 4. The equation of a catenary being $\frac{x}{a} = \frac{1}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right)$, the co-ordinates of the centre of gravity of the semiarc, may be found from the equations

$$2s\dot{x} = sx + ay,$$
$$s\dot{y} = sy - ax.$$

Ex. 5. The equation of a parabola being $y^2 = 4ax$, the co-ordinates of the centre of gravity of the arc, cut off by the latus rectum, are

$$\dot{x} = \frac{a}{4} \cdot \frac{3\sqrt{2} - \log_e(1+\sqrt{2})}{\sqrt{2} + \log_e(1+\sqrt{2})}, \text{ and } \dot{y} = 0.$$

179. To find the centre of gravity of an area.

Use the same construction and notation as in Art. 177, except that $\dot{x}\ \dot{y}$ now denote the co-ordinates of the centre of gravity of the area PAM. Let A equal the area PAM, and δA equal the area of the small increment PMNQ; u equal the moment of A about OY; then δu = moment of δA about Oy will be greater than if δA were collected into one mass in PM, and less than if collected into one mass in QN; consequently,

$$\delta u > x \,.\, \delta A,$$
$$\text{and } \delta u < (x + \delta x) \,.\, \delta A$$
$$< x \,.\, \delta A + \delta x \,.\, \delta A;$$
$$\therefore \frac{\delta u}{\delta x} > x \,.\, \frac{\delta A}{\delta x},$$
$$\text{and } \frac{\delta u}{\delta x} < x \,.\, \frac{\delta A}{\delta x} + \delta A.$$

Seeing, therefore, that $\frac{\delta u}{\delta x}$ always lies between these two quantities $x\,\frac{\delta \mathrm{A}}{\delta x}$ and $x\,\frac{\delta \mathrm{A}}{\delta x} + \delta \mathrm{A}$, both of which, as δx is diminished, tend to $x d_x \mathrm{A}$ as their limit, and that $\frac{\delta u}{\delta x}$ has $d_x u$ for its limit, these limits must be equal; wherefore

$$d_x u = x d_x \mathrm{A}.$$

But, by the Differential Calculus, $d_x \mathrm{A} = y$,

$$\therefore\ d_x u = xy,$$

$$\therefore\ u = \int_x (xy).$$

But, by Art. 136,

$$u = \dot{x}\mathrm{A} = \dot{x} \,.\, \int_x y,$$

$$\therefore\ \dot{x} \int_x y = \int_x (xy) \ \ldots\ldots\ (1).$$

Again, let now u denote the moment of A about ox, and therefore δu the moment of $\delta \mathrm{A}$ about ox.

Since PMNQ is very nearly a parallelogram, the distance of its centre of gravity from ox will be greater than $\frac{1}{2}$ PM, and less than $\frac{1}{2}$ QN, and, consequently,

$$\delta u > \tfrac{1}{2} y \,.\, \delta \mathrm{A} \text{ and } < \tfrac{1}{2}(y + \delta y) \,.\, \delta \mathrm{A};$$

$$\therefore\ \frac{\delta u}{\delta y} > \tfrac{1}{2} y \,.\, \frac{\delta \mathrm{A}}{\delta y} \text{ and } < \tfrac{1}{2} y \,.\, \frac{\delta \mathrm{A}}{\delta y} + \tfrac{1}{2} \delta \mathrm{A};$$

and from these, by similar reasoning to that employed above, we obtain

$$d_y u = \tfrac{1}{2} y d_y \mathrm{A},$$

or, by changing the independent variable,

$$d_x u = \tfrac{1}{2} y d_x \mathrm{A} = \tfrac{1}{2} y^2,$$

$$\therefore\ u = \tfrac{1}{2} \int_x (y^2).$$

But $u = \dot{y}\mathrm{A} = \dot{y} \,.\, \int_x y$,

$$\therefore\ \dot{y} \cdot \int y = \tfrac{1}{2} \int_x (y^2) \ \ldots\ldots\ (2).$$

Equations (1) and (2) determine the position of the centre of gravity of the area AM.

Ex. 6. To find the centre of gravity of the common parabola, whose equation is $y^2 = 4ax$.

Let PAM (Fig. 50) be the parabola, $\text{AM} = x$, $\text{MP} = y$. Then,

$$\int_x y = \int_x \sqrt{4ax} = \sqrt{4a} \, . \int_x x^{\frac{1}{2}},$$
$$= \frac{2}{3} \sqrt{4a} \, . \, x^{\frac{3}{2}},$$
$$= \frac{2}{3} \, . \, xy.$$

$$\text{And} \int_x (xy) = \int_x x \sqrt{4ax} = \sqrt{4a} \, . \int_x x^{\frac{3}{2}},$$
$$= \frac{2}{5} \sqrt{4a} \, . \, x^{\frac{5}{2}},$$
$$= \frac{2}{5} \, x^2 y.$$

$$\text{Also} \int_x (y^2) = \int_x (4ax) = 4a \, . \int_x x,$$
$$= 2a \, . \, x^2,$$
$$= \tfrac{1}{2} \, xy^2.$$

Consequently,

$$\dot{x} = \frac{\frac{2}{5} x^2 y}{\frac{2}{3} xy} = \frac{3}{5} x = \frac{3}{5} \, . \, \text{AM},$$

$$\text{and } \dot{y} = \frac{\frac{1}{4} xy^2}{\frac{2}{3} xy} = \frac{3}{8} y = \frac{3}{8} \, . \, \text{PM}.$$

Ex. 7. To find the centre of gravity of the area of a semicycloid.

Using the notation and figure of Ex. 2, we have

$\int_x y = xy - \int_x (x d_x y)$, by integrating by parts;

$= xy - \int_x \left(x \sqrt{\frac{2a}{x} - 1} \right)$, by substituting for $d_x y$;

$= xy - \int_x \sqrt{2ax - x^2}$.

But

$$\int_x \sqrt{2ax-x^2} = x\sqrt{2ax-x^2} - \int_x \frac{ax-x^2}{\sqrt{2ax-x^2}},$$ by integrating by parts;

$$= \ldots\ldots\ldots - \int_x \frac{2ax-x^2}{\sqrt{2ax-x^2}} + \int_x \frac{ax}{\sqrt{2ax-x^2}},$$

$$= \ldots\ldots\ldots - \int_x \sqrt{2ax-x^2} - a\int_x \frac{a-x}{\sqrt{2ax-x^2}}$$

$$+ a^2 \int_x \frac{1}{\sqrt{2ax-x^2}};$$

therefore, by transposing and dividing by 2,

$$\int_x \sqrt{2ax-x^2} = \tfrac{1}{2}x\sqrt{2ax-x^2} - \tfrac{1}{2}a\int_x \frac{a-x}{\sqrt{2ax-x^2}}$$

$$+ \tfrac{1}{2}a^2 \int_x \frac{1}{\sqrt{2ax-x^2}},$$

$$= \tfrac{1}{2}(x-a)\sqrt{2ax-x^2} + \tfrac{1}{2}a^2 \operatorname{vers}^{-1}\frac{x}{a};$$

$$\therefore \int_x y = xy - \tfrac{1}{2}(x-a)\sqrt{2ax-x^2} - \tfrac{1}{2}a^2 \operatorname{vers}^{-1}\frac{x}{a},$$

$$= \tfrac{1}{2}(a+x)\sqrt{2ax-x^2} + \tfrac{1}{2}a(2x-a)\operatorname{vers}^{-1}\frac{x}{a};$$

and, by writing $2a$ for x, this becomes $\frac{3}{2}a^2\pi$, the area of the semicycloid.

Again,

$\int_x (xy) = \frac{1}{2}x^2y - \frac{1}{2}\int_x (x^2 d_x y)$, by integrating by parts;

$= \frac{1}{2}x^2y - \frac{1}{2}\int_x x\sqrt{2ax-x^2}$, by substituting for $d_x y$;

$$= \tfrac{1}{2}x^2y + \tfrac{1}{2}\int_x (a-x)\sqrt{2ax-x^2} - \tfrac{1}{2}a\int_x \sqrt{2ax-x^2},$$

$$= \tfrac{1}{2}x^2y + \tfrac{1}{6}(2ax-x^2)^{\frac{3}{2}} - \tfrac{1}{4}a(x-a)\sqrt{2ax-x^2}$$

$$- \tfrac{1}{4}a^3 \operatorname{vers}^{-1}\frac{x}{a},$$

$$= \tfrac{1}{12}(4x^2+ax+3a^2)\sqrt{2ax-x^2}$$

$+ \frac{1}{4}a(2x^2-a^2)\operatorname{vers}^{-1}\frac{x}{a}$, by substituting for y;

and, by writing $2a$ for x, this gives $\frac{3}{4}a^3\pi$ for the moment of the semicycloid about the axis of y.

Also,

$$\int_x(y^2) = y^2x - 2\int_x(xy\,d_xy), \text{ by integrating by parts;}$$

$$= y^2x - 2\int_x(y\sqrt{2ax-x^2}), \text{ by substituting for } d_xy.$$

Now,

$$2\int_x(y\sqrt{2ax-x^2}) = 2\int_x\left(2ax-x^2+a.\sqrt{2ax-x^2},\ \text{vers}^{-1}\frac{x}{a}\right),$$

by substituting for y;

$$= 2ax^2 - \tfrac{2}{3}x^3 + 2a\int_x\left(\sqrt{2ax-x^2}.\text{vers}^{-1}\frac{x}{a}\right).$$

Integrating by parts,

$$2\int_x\left(\sqrt{2ax-x^2}.\text{vers}^{-1}\frac{x}{a}\right) =$$

$$\left((x-a).\sqrt{2ax-x^2} + a^2\,\text{vers}^{-1}\frac{x}{a}\right).\text{vers}^{-1}\frac{x}{a}$$

$$-\int_x\left(x-a+a^2\,\text{vers}^{-1}\frac{x}{a}.\,d_x\,\text{vers}^{-1}\frac{x}{a}\right);$$

$$\left(\text{because } d_x\,\text{vers}^{-1}\frac{x}{a} = \frac{1}{\sqrt{2ax-x^2}}\right),$$

$$=\left\{(x-a)\sqrt{2ax-x^2} + a^2\,\text{vers}^{-1}\frac{x}{a}\right\}.\text{vers}^{-1}\frac{x}{a}$$

$$-\left\{\tfrac{1}{2}x^2 - ax + \tfrac{1}{2}a^2\left(\text{vers}^{-1}\frac{x}{a}\right)^2\right\},$$

$$= \tfrac{1}{2}(2ax-x^2)+(x-a)\sqrt{2ax-x^2}.\text{vers}^{-1}\frac{x}{a} + \tfrac{1}{2}a^2\left(\text{vers}^{-1}\frac{x}{a}\right)^2.$$

Whence we have,

$$\int_x y^2 = y^2x - 2ax^2 + \tfrac{2}{3}x^3 - \tfrac{1}{2}a(2ax-x^2)$$

$$- a(x-a)\sqrt{2ax-x^2}.\text{vers}^{-1}\frac{x}{a} - \tfrac{1}{2}a^3\left(\text{vers}^{-1}\frac{x}{a}\right)^2,$$

$$= -\tfrac{1}{2}a(2ax-x^2) - \tfrac{1}{3}x^3 + a(a+x)\sqrt{2ax-x^2}.\text{vers}^{-1}\frac{x}{a}$$

$$+ \tfrac{1}{2}a^2(2x-a)\left(\text{vers}^{-1}\frac{x}{a}\right)^2.$$

And, by writing $2a$ for x, this becomes $-\frac{8}{3}a^3 + \frac{3\pi^2}{2} \cdot a^3$,

the half of which is $\frac{1}{2}a^3\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$, the moment of the semi-cycloid about the axis of x.

Consequently,

$$\dot{x} \cdot \tfrac{3}{2}a^2\pi = \tfrac{1}{4}a^3\pi,$$

$$\text{and } \dot{y} \cdot \tfrac{3}{2}a^2\pi = \tfrac{1}{2}a^3\left(\frac{3\pi^2}{2} - \frac{8}{3}\right),$$

which give

$$\dot{x} = \tfrac{1}{6}a, \text{ and } \dot{y} = a\left(\frac{\pi}{2} - \frac{8}{9\pi}\right).$$

180. To find the centre of gravity of a curve, considered as a spiral.

Let P (Fig. 51) be any point in the curve, Q very near to P, $r = \text{OP}$, $\theta = \angle x\text{OP}$, $\delta\theta = \angle \text{POQ}$. Then, by the Differential Calculus, the area of the triangle POQ is $\frac{1}{2}r^2\delta\theta$, and if g be its centre of gravity, the ultimate position of g is in OP, at a distance from O $= \frac{2}{3}r$ (Art. 144); and we may ultimately suppose the whole triangle collected at the ultimate position of g, and therefore its moment about $\text{O}x = \frac{1}{2}r^2\delta\theta \cdot gm = \frac{1}{2}r^2\delta\theta \cdot \frac{2}{3}r\sin\theta$. Hence, if u be the moment of the area AOP about Ox,

$$\delta u = \tfrac{1}{3}r^3\sin\theta \cdot \delta\theta.$$

Wherefore, dividing by $\delta\theta$ and taking the limits,

$$d_\theta u = \tfrac{1}{3}r^3\sin\theta;$$

$$\therefore u = \tfrac{1}{3}\textstyle\int_\theta(r^3\sin\theta).$$

But $u = \dot{y} \cdot \text{area AOP} = \dot{y} \cdot \frac{1}{2}\int_\theta(r^2)$,

$$\therefore \dot{y} \cdot \tfrac{1}{2}\textstyle\int_\theta(r^2) = \tfrac{1}{3} \cdot \int_\theta(r^3\sin\theta),$$

$$\therefore \dot{y} = \tfrac{2}{3} \cdot \frac{\int_\theta(r^3\sin\theta)}{\int_\theta(r^2)}.$$

In a similar manner, we find

$$\dot{x} = \tfrac{2}{3} \cdot \frac{\int_\theta (r^3 \cos\theta)}{\int_\theta (r^2)}.$$

Ex. 8. To find the centre of gravity of a portion of a parabola cut off by a chord through its focus.

Let A (Fig. 52) be the vertex, S the focus, and BC the focal chord; P any point in the parabola, SP $= r$, ASP $= \theta$, AS $= a$; then the polar equation of the curve, is

$$r = a \sec^2 \frac{\theta}{2};$$

$$\therefore \int_\theta (r^2) = a^2 \int_\theta \sec^4 \frac{\theta}{2};$$

$$= 2a^2 \int_\theta \left(\sec^2 \frac{\theta}{2} . d_\theta \tan \frac{\theta}{2}\right);$$

$$\left(\text{because } d_\theta \tan \frac{\theta}{2} = \tfrac{1}{2} \sec^2 \frac{\theta}{2}\right),$$

$$= 2a^2 \int_\theta \left(d_\theta \tan \frac{\theta}{2} + \tan^2 \frac{\theta}{2} . d_\theta \tan \frac{\theta}{2}\right),$$

$$\left(\text{because } \sec^2 \frac{\theta}{2} = 1 + \tan^2 \frac{\theta}{2}\right),$$

$$= 2a^2 \left(\tan \frac{\theta}{2} + \tfrac{1}{3} \tan^3 \frac{\theta}{2}\right).$$

If $a = \angle$ BSA; by writing a for θ in this expression, the value of $\int_\theta (r^2)$ for ASB, is

$$2a^2 \left(\tan \frac{a}{2} + \tfrac{1}{3} \tan^3 \frac{a}{2}\right).$$

And since $\angle$ CSA $= 180 - a$, by writing this for θ in the same expression, the value of $\int_\theta (r^2)$ for ASC, is

$$2a^2 \left(\cot \frac{a}{2} + \tfrac{1}{3} \cot^3 \frac{a}{2}\right).$$

Where the whole value of $\int_\theta(r^2)$ is

$$\tfrac{2}{3}\,a^2\left(3\cot\frac{a}{2}+\cot^3\frac{a}{2}+3\tan\frac{a}{2}+\tan^3\frac{a}{2}\right)$$

$$=\tfrac{2}{3}\,a^2\left(\cot\frac{a}{2}+\tan\frac{a}{2}\right)^3,$$

$$=\tfrac{2}{3}\cdot\frac{a^2}{\sin^3\frac{a}{2}\cos^3\frac{a}{2}},$$

by writing $\dfrac{\cos\frac{a}{2}}{\sin\frac{a}{2}}$ and $\dfrac{\sin\frac{a}{2}}{\cos\frac{a}{2}}$ for $\cot\frac{a}{2}$ and $\tan\frac{a}{2}$;

$$=\tfrac{16}{3}\cdot\frac{a^2}{\sin^3 a},\ \text{because}\ \sin a=2\sin\frac{a}{2}\cos\frac{a}{2}.$$

Again, $\int_\theta(r^3\cos\theta)=a^3\int_\theta\left(\sec^6\frac{\theta}{2}\cos\theta\right)$,

$$=2a^3.\int_\theta\left(\sec^2\frac{\theta}{2}\cdot\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}.d_\theta\tan\frac{\theta}{2}\right),$$

$$=2a^3\int_\theta\left\{\left(1-\tan^4\frac{\theta}{2}\right)d_\theta\tan\frac{\theta}{2}\right\},$$

$$=2a^3\int_\theta\left(d_\theta\tan\frac{\theta}{2}-\tan^4\frac{\theta}{2}\,d_\theta\tan\frac{\theta}{2}\right),$$

$$=2a^3\left(\tan\frac{\theta}{2}-\tfrac{1}{5}\tan^5\frac{\theta}{2}\right).$$

The value of which, corresponding to the whole area, is

$$\tfrac{2}{5}\,a^3\left(5\cot\frac{a}{2}-\cot^5\frac{a}{2}+5\tan\frac{a}{2}-\tan^5\frac{a}{2}\right),$$

$$=\tfrac{2}{5}\,a^3\left(\cot\frac{a}{2}+\tan\frac{a}{2}\right)^3\left\{5-\left(\cot\frac{a}{2}+\tan\frac{a}{2}\right)^2\right\},$$

$$=\tfrac{16}{5}\cdot\frac{a^3}{\sin^3 a}\cdot\left(5-\frac{4}{\sin^2 a}\right);$$

$$\therefore \dot{x} \cdot \tfrac{16}{3} \cdot \frac{a^2}{\sin^3 \alpha} = \tfrac{2}{3} \cdot \tfrac{16}{5} \cdot \frac{a^3}{\sin^3 \alpha} \cdot \left(5 - \frac{4}{\sin^2 \alpha}\right);$$

$$\therefore \dot{x} = \tfrac{2}{5} a\,(5 - 4 \operatorname{cosec}^2 \alpha),$$

$$= \tfrac{2}{5} a\,(1 - 4 \cot^2 \alpha).$$

Also,

$$\int_\theta (r^3 \sin\theta) = 2a^3 \int_\theta \left(\sec^6 \frac{\theta}{2} \cdot \sin\frac{\theta}{2} \cos\frac{\theta}{2}\right),$$

$$= 2a^3 \int_\theta \left(\sec^4 \frac{\theta}{2} \cdot \tan\frac{\theta}{2}\right),$$

$$= 4a^3 \int_\theta \left(\sec^2 \frac{\theta}{2} \cdot \tan\frac{\theta}{2} \cdot d_\theta \tan\frac{\theta}{2}\right),$$

$$= 4a^3 \int_\theta \left(\tan\frac{\theta}{2}\, d_\theta \tan\frac{\theta}{2} + \tan^3\frac{\theta}{2}\, d_\theta \tan\frac{\theta}{2}\right),$$

$$= 4a^3 \left(\tfrac{1}{2} \tan^2 \frac{\theta}{2} + \tfrac{1}{4} \tan^4 \frac{\theta}{2}\right).$$

The value of this corresponding to the whole area, observing that the moments of BSA and of CSA about AS have contrary signs, is

$$a^3 \left(2 \tan^2 \frac{\alpha}{2} + \tan^4 \frac{\alpha}{2} - 2 \cot^2 \frac{\alpha}{2} - \cot^4 \frac{\alpha}{2}\right),$$

$$= a^3 \left(\tan\frac{\alpha}{2} + \cot\frac{\alpha}{2}\right)^3 \cdot \left(\tan\frac{\alpha}{2} - \cot\frac{\alpha}{2}\right),$$

$$= -\frac{8a^3}{\sin^3 \alpha} \cdot 2 \cot \alpha;$$

$$\therefore \dot{y} \cdot \tfrac{16}{3} \cdot \frac{a^2}{\sin^3 \alpha} = -\tfrac{2}{3} \cdot \frac{16 a^3}{\sin^3 \alpha} \cdot \cot \alpha,$$

$$\therefore \dot{y} = -\,2a \cot \alpha.$$

Ex. 9. To find the centre of gravity of a quadrant of a circle, whose equation is $x^2 + y^2 = a^2$.

$$\dot{x} = \tfrac{4}{3} \cdot \frac{a}{\pi}, \quad \dot{y} = \tfrac{4}{3} \cdot \frac{a}{\pi}.$$

Ex. 10. To find the centre of gravity of the sector of a circle.

The centre being the origin, and the line bisecting the sector being the axis of x, it will be found that

$$\dot{x} = \tfrac{2}{3} \cdot \frac{(\text{radius}) \cdot (\text{chord})}{\text{arc}}, \text{ and } \dot{y} = 0.$$

Ex. 11. To find the centre of gravity of a node of the Lemniscate.

$$\dot{x} = \frac{\pi \cdot a}{4\sqrt{2}}, \quad \dot{y} = 0.$$

181. To find the centre of gravity of a solid of revolution.

Let AB (Fig. 46) be the curve, by the revolution of which round Ox the given solid is generated. Make the same construction and notation as before. Let V denote the volume of the solid generated by the revolution of AMP, and δV that generated by PMNQ; $u =$ the moment of V round Oy, and δu that of δV about Oy.

The moment of δV about Oy is greater than it would be if δV were all collected in the circular plane generated by PM, that is,

$$\delta u > x \cdot \delta V;$$

and it is less than it would be if δV were all collected in the circular plane generated by QN, that is,

$$\delta u < (x + \delta x) \cdot \delta V.$$

Hence $\dfrac{\delta u}{\delta x}$ is always greater than $x \cdot \dfrac{\delta V}{\delta x}$,

and always less than $x \cdot \dfrac{\delta V}{\delta x} + \delta V$.

Whence, as in Art. 177,

$$d_x u = x d_x \mathrm{V};$$

$$\therefore u = \textstyle\int_x (x d_x \mathrm{V}).$$

But $\dot{x}$, $\dot{y}$ being the co-ordinates of the centre of gravity of V,

$$\dot{x} . \mathrm{V} = u = \textstyle\int_x (x d_x \mathrm{V}),$$

Now $d_x \mathrm{V} = \pi y^2$, by the Differential Calculus; and, therefore, $\mathrm{V} = \pi \int_x y^2$; consequently

$$\dot{x} . \pi \textstyle\int_x (y^2) = \pi \int_x (x y^2);$$

$$\text{and } \therefore \dot{x} . \textstyle\int_x (y^2) = \int_x (x y^2).$$

From Art. 140, it is manifest that

$$\dot{y} = 0.$$

Ex. 12. To find the centre of gravity of a hemisphere.

A hemisphere is generated by the revolution of a quadrant, whose equation is

$$y^2 = 2ax - x^2.$$

$$\therefore \textstyle\int_x (y^2) = ax^2 - \frac{1}{3} x^3,$$

which gives, for the whole hemisphere, by writing a for x, the quantity $\frac{2}{3} a^3$.

Again,

$$\textstyle\int_x (x y^2) = \int_x (2ax^2 - x^3),$$

$$= \tfrac{2}{3} a x^3 - \tfrac{1}{4} x^4;$$

which, by writing a for x, becomes $\frac{5}{12} a^4$,

$$\therefore \dot{x} = \frac{\frac{5}{12} a^4}{\frac{2}{3} a^3},$$

$$= \tfrac{5}{8} a = \tfrac{5}{8} \text{ of the radius.}$$

Ex. 13. Given the altitude and the radii of the ends of a parabolic frustum, to find its centre of gravity.

Let PM (Fig. 53) and QN be (a, b) the radii of the ends, and MN (c) the altitude of the frustum; G its centre of gravity; also let $y^2 = 4mx$ be the equation of the generating parabola APQ. Then

$$\int_x (y^2) = 4m \int_x x = 2mx^2 + C.$$

This taken from $x = AM$, to $x = AN$, gives for the frustum the expression

$$2m\,(AN)^2 - 2m\,.\,(AM)^2,$$

which is equal to $2m\,(x_2^2 - x_1^2)$, denoting AN, AM by x_2, x_1 respectively,

$$= 2m\,(x_2 + x_1)(x_2 - x_1),$$

$$= 2m\,(x_2 + x_1)\,c.$$

Again,

$$\int_x (xy^2) = 4m \int_x (x^2),$$

$$= \tfrac{4}{3} mx^3 + C;$$

which, taken between the same limits as before, becomes

$$\tfrac{4}{3}\, m\,(x_2^3 - x_1^3),$$

$$= \tfrac{4}{3}\, m\,(x_2^2 + x_2x_1 + x_1^2)(x_2 - x_1),$$

$$= \tfrac{4}{3}\, m\,(x_2^2 + x_2x_1 + x_1^2)\,c.$$

$$\text{Hence AG} = \frac{\tfrac{4}{3}\, mc\,(x_2^2 + x_2x_1 + x_1^2)}{2mc\,(x_2 + x_1)},$$

$$= \tfrac{2}{3}\cdot\frac{x_2^2 + x_2x_1 + x_1^2}{x_2 + x_1};$$

$$\therefore \text{MG} = \text{AG} - x_1,$$

$$= \tfrac{2}{3}\,\frac{x_2^2 + x_2x_1 + x_1^2}{x_2 + x_1} - x_1,$$

$$= \tfrac{1}{3}\cdot\frac{2x_2^2 - x_2x_1 - x_1^2}{x_2 + x_1},$$

$$= \tfrac{1}{3}\cdot\frac{(x_2 - x_1)(2x_2 + x_1)}{x_2 + x_1}.$$

$$= \frac{c}{3} \cdot \frac{8mx_2 + 4mx_1}{4mx_2 + 4mx_1},$$

$$= \frac{c}{3} \cdot \frac{2b^2 + a^2}{b^2 + a^2},$$

because $4mx_1 = a^2$, and $4mx_2 = b^2$.

This formula gives the distance of the centre of gravity from the smaller end of the frustum.

Ex. 14. In a cone, generated by the revolution of a right angled triangle about one of it sides.

$$\dot{x} = \tfrac{3}{4} \text{ . of that side.}$$

Ex. 15. In the solid formed by the revolution of a semicycloid about its axis.

$$\dot{x} = \frac{a}{3} \cdot \frac{9\pi^2 - 32}{9\pi^2 - 16}.$$

$\dot{x}$ being from the base along the axis.

Ex. 16. In the paraboloid, formed by the revolution of the parabola, whose equation is $y^{m+n} = a^m x^n$,

$$\dot{x} = \frac{m + 3n}{m + 2n} \cdot \frac{x}{2}.$$

182. To find the centre of gravity of a solid of any form.

Let Ox, Oy, Oz (Fig. 54) be the rectangular co-ordinates, to which the solid is referred by its equation. Let ABPC be a portion of the surface of the solid, comprehended between the co-ordinate planes xOz, yOz, and the planes $PpNC$, $PpMB$ respectively parallel to them. Through the point S very near to P draw planes $Ssnc$, $Ssmb$ parallel to the former. Let x, y, z, be the co-ordinates of P, and $x + \delta x$, $y + \delta y$, $z + \delta z$ those of S. Then, denoting the volume of the parallelopiped Ps by A, its moment about the axis Ox is greater than if it were all collected in the plane Pq, and less than if collected in the plane Rs; that is, the moment of A is

$$\text{greater than } y\text{A},$$
$$\text{and less than } (y + \delta y)\,\text{A}.$$

But now if u be the moment of the solid PO about Ox, the moment of SOBmPn about Ox will be (by Taylor's theorem applied to two variables x, y),

$$\begin{aligned} &d_x u\,.\,\delta x + \tfrac{1}{2} d_x^2 u\,.\,(\delta x)^2 + \,\ldots\,. \\ &d_y u\,.\,\delta y + d_x d_y u\,.\,\delta x \delta y + \,\ldots\,. \\ &\qquad\qquad + \tfrac{1}{2} d_y^2 u\,.\,(\delta y)^2 + \,\ldots\,. \\ &\qquad\qquad\qquad\qquad + \,\ldots\,. \end{aligned}$$

and by the same theorem, applied to the variable x, the moment of the solid BmP about Ox, is

$$d_x u\,.\,\delta x + \tfrac{1}{2} d_x^2 u\,.\,(\delta x)^2 + \,\ldots\,.$$

and, similarly, the moment of the solid CnP,

$$d_y u\,.\,\delta y + \tfrac{1}{2} d_y^2 u\,.\,(\delta y)^2 + \,\ldots\,.$$

Subtracting both these from the former, we find the moment of the parallelopiped Ps to be equal to $d_x d_y u\,.\,\delta x \delta y + \,\ldots\,.$; consequently, this quantity always lies between yA and $(y + \delta y)$ A; and, therefore, $d_x d_y u + \,\ldots\ldots$ always lies between

$$y\,.\,\frac{\text{A}}{\delta x \delta y} \text{ and } y\,.\,\frac{\text{A}}{\delta x \delta y} + \delta y\,.\,\frac{\text{A}}{\delta x \delta y}.$$

Now $\dfrac{\text{A}}{\delta x \delta y}$ tends to z as its limit, and consequently the two quantities $y \cdot \frac{A}{\delta x \, \delta y}$ and $y \cdot \frac{A}{\delta x \, \delta y} + \delta y \cdot \frac{A}{\delta x \, \delta y}$ tend to equality with yz; and $d_x d_y u + \ldots$ which always lies between them, tends to $d_x d_y u$ as its limit; the three limits are therefore equal; consequently,

$$d_x d_y u = yz;$$
$$\therefore u = \int_x \int_y (yz).$$

Now the volume of PO is equal to $\int_x \int_y z$, and its moment about Ox is

$$\dot{y} \cdot \int_x \int_y z;$$

wherefore, by Art. 136,

$$\dot{y} \cdot \int_x \int_y z = \int_x \int_y (yz) \quad \ldots\ldots \quad (2).$$

By a similar investigation, we should find

$$\dot{x} \cdot \int_x \int_y z = \int_x \int_y (xz) \quad \ldots\ldots \quad (1).$$

And observing that the centre of gravity of the parallelopiped A is ultimately in its middle point (Art. 140), we should find

$$\dot{z} \cdot \int_x \int_y z = \tfrac{1}{2} \int_x \int_y (z^2) \quad \ldots . \quad (3).$$

Remark. It is evident, that by taking an elementary parallelopiped A, at right angles to the plane yOz, we might also obtain

$$\dot{x} \cdot \int_x \int_z y = \int_x \int_z (xy),$$
$$\dot{y} \cdot \int_x \int_z y = \tfrac{1}{2} \int_x \int_z (y^2),$$
$$\dot{z} \cdot \int_x \int_z y = \int_x \int_z (xy);$$

and if the elementary parallelopiped were at right angles to the plane yOz, we should find

$$\dot{x} \cdot \int_y \int_z x = \tfrac{1}{2} \int_y \int_z (x^2),$$
$$\dot{y} \cdot \int_y \int_z y = \int_y \int_z (xy),$$
$$\dot{z} \cdot \int_y \int_z z = \int_y \int_z (xz).$$

These formulæ are, in fact, often more convenient than those first given; and the most convenient is to be determined by the form of the body and its situation with respect to the co-ordinate axes; the choice must, however, be left to the skill of the reader, as no general rules can be laid down. In every case, the greatest care is requisite in taking the integrals between proper limits.

All the three sets of formulæ are comprehended in the following,—

$$\bar{x} \cdot \textstyle\int_x \int_y \int_z 1 = \int_x \int_y \int_z x,$$
$$\bar{y} \cdot \textstyle\int_x \int_y \int_z 1 = \int_x \int_y \int_z y,$$
$$\bar{z} \cdot \textstyle\int_x \int_y \int_z 1 = \int_x \int_y \int_z z.$$

Ex. 17. To find the centre of gravity of a portion of a cone, comprehended between two planes passing through the axis at right angles to each other.

Let AOBC (Fig. 55) be the portion of the cone, O the centre of the base, being the origin of co-ordinates. Let $a = \text{OC}$, $b = \text{OA}$. Then the equation of the surface of the cone is

$$z = a - \frac{a}{b}\sqrt{x^2 + y^2}.$$

The volume of the whole cone is $\frac{\pi}{3}ab^2$, and consequently the volume of AOBC is $\frac{\pi}{12}ab^2$.

Also,

$$\textstyle\int_x (xz) = \int_x \left(ax - \frac{a}{b}x\sqrt{x^2 + y^2}\right),$$
$$= \tfrac{1}{2}ax^2 - \frac{a}{3b}(x^2 + y^2)^{\frac{3}{2}} + \text{C}.$$

This integral is to be taken between the limits $x = 0$ and $z = 0$, or $x = 0$ and $x = \sqrt{b^2 - y^2}$, consequently between the proper limits,

$$\int_x (xz) = \tfrac{1}{2} a (b^2 - y^2) - \tfrac{1}{3} ab^2 + \frac{ay^3}{3b},$$

$$= \tfrac{1}{6} ab^2 - \tfrac{1}{2} ay^2 + \frac{ay^3}{3b};$$

therefore,

$$\int_x \cdot \int_y (xz) = \int_y \cdot \int_x (xz) = \tfrac{1}{6} ab^2 y - \tfrac{1}{6} ay^3 + \frac{ay^4}{12b} + \text{C}.$$

This being taken from $y = 0$, to $y = \text{OB} = b$, gives

$$\int_x \int_y (xz) = \tfrac{1}{6} ab^3 - \tfrac{1}{6} ab^3 + \tfrac{1}{12} \cdot ab^3,$$

$$= \frac{ab^3}{12};$$

$$\therefore \dot{x} \cdot \frac{\pi}{12} ab^2 = \frac{ab^3}{12}, \text{ from equation (1)};$$

$$\therefore \dot{x} = \frac{b}{\pi}.$$

From the symmetrical form of the solid, we know that $\dot{y} = \dot{x}$; and, from Art. 155, $\dot{z} = \tfrac{1}{4} a$.

Ex. 18. To find the centre of gravity of the eighth part of an ellipsoid.

Let ABC (Fig. 56) be the eighth part of an ellipsoid, its centre coinciding with O the origin of co-ordinates. Then OA, OB, OC being represented by a, b, c respectively, the equation of the surface of the ellipsoid will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\therefore \int_y z = c \int_y \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}},$$

$$= \frac{c}{b} \cdot \int_y \left\{\left(b^2 - \frac{b^2 x^2}{a^2}\right) - y^2\right\}^{\frac{1}{2}},$$

$$= \tfrac{1}{2} \cdot \frac{c}{b} \cdot y \left(b^2 - \frac{b^2 x^2}{a^2} - y^2\right)^{\frac{1}{2}}$$

$$+ \tfrac{1}{2} \cdot \frac{c}{b} \cdot \left(b^2 - \frac{b^2 x^2}{a^2}\right) \sin^{-1}. \frac{y}{\left(b^2 - \frac{b^2 x^2}{a^2}\right)^{\frac{1}{2}}} + C,$$

$$= \tfrac{1}{2} c y \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$+ \frac{bc}{2a} (a^2 - x^2) \sin^{-1} \frac{ay}{b\sqrt{a^2 - x^2}} + C.$$

This integral is to be taken from $y = 0$, to $z = 0$; or from $y = 0$, to $y = \frac{b}{a} \sqrt{a^2 - x^2}$; and, between these limits,

$$\int_y z = \frac{bc\pi}{4a^2} (a^2 - x^2);$$

$$\therefore \int_x \int_y z = \frac{bc\pi}{4a^2} \int_x (a^2 - x^2),$$

$$= \frac{bc\pi}{4a^2} \cdot (a^2 x - \tfrac{1}{3} x^3 + C).$$

This integral is to be taken from $x = 0$, to $x = a$; and between these limits we have

$$\int_x \int_y z = \frac{bc\pi}{4a^2} \cdot \frac{2a^3}{3} = \frac{\pi}{6} \cdot abc.$$

Again, to find the value of $\int_x \int_y (xz)$ we observe that

$$\int_y (xz) = x \int_y z,$$

$$= x \cdot \frac{bc\pi}{4a^2} (a^2 - x^2),$$

$$= \frac{bc\pi}{4a^2} (a^2 x - x^3);$$

$$\therefore \int_x \int_y (xz) = \frac{bc\pi}{4a^2} \cdot \int_x (a^2x - x^3),$$

$$= \frac{bc\pi}{4a^2} (\tfrac{1}{2}a^2x^2 - \tfrac{1}{4}x^4) + C;$$

which, taken between the same limits as before, viz.— $x = 0$, and $x = a$, gives

$$\int_x \int_y (xz) = \frac{\pi a^2 bc}{16}.$$

$$\text{Hence } \dot{x} \cdot \frac{\pi abc}{6} = \frac{\pi a^2 bc}{16};$$

$$\therefore \dot{x} = \tfrac{3}{8}a.$$

$$\text{Similarly, } \dot{y} = \tfrac{3}{8}b;$$

$$\text{and } \dot{z} = \tfrac{3}{8}c.$$

Ex. 19. To find the centre of gravity of a portion of a paraboloid, comprehended between two planes passing through the axis at right angles to each other.

If a be its length, and b the radius of its base, the co-ordinates of its centre of gravity will be

$$\dot{x} = \tfrac{2}{3}a, \quad \dot{y} = \dot{z} = \frac{16b}{15\pi}.$$

183. To find the centre of gravity of a surface of revolution.

Employing the notation and figure of Art. 117, let u be the moment of the surface generated by the arc AP, and therefore δu the moment of that generated by PQ; let S denote the former, and δS the latter of these surfaces so generated. Then the moment of δS about Oy is greater than if it were all collected in the circumference of the circle described by P, and less than if collected in the circumference of that described by Q, that is,

$$\delta u \text{ is greater than } x\,.\,\delta S,$$
$$\text{and less than } (x + \delta x)\,.\,\delta S;$$
$$\therefore \frac{\delta u}{\delta x} \text{ is greater than } x\frac{\delta S}{\delta x},$$
$$\text{and less than } x\frac{\delta S}{\delta x} + \delta S.$$

Equating the limits, as before, we have

$$d_s u = x d_s S,$$
$$= 2\pi x y d_s s, \text{ since } d_s S = 2\pi y d_s s;$$
$$\therefore u = 2\pi \int_s (x y d_s s).$$

But

$$u = \text{the moment of S about } Oy = \dot{x}S = \dot{x}\,.\,2\pi\int_s (y d_s s),$$
$$\therefore \dot{x}\,.\,2\pi\int_s (x d_s s) = 2\pi\int_s (x y d_s s),$$
$$\therefore \dot{x}\int_s (x d_s s) = \int_s (x y d_s s).$$

And it is evident, from the symmetrical form of the surface, that $\dot{y} = 0$.

184. We may adapt this formula to polar co-ordinates, by writing $r\cos\theta$, $r\sin\theta$, for x and y respectively.

Hence

$$\dot{x}\,.\int_\theta \{r\sqrt{r^2 + (d_\theta r)^2}\,.\sin\theta\} = \int_\theta \{r^2\sqrt{r^2 + (d_\theta r)^2}\,.\sin\theta\cos\theta\}.$$

Ex. 20. To find the centre of gravity of the surface of a cone.

Here the generating curve is a straight line OB; and if a equal the altitude OA (Fig. 57), and b equal the radius AB of the base of the cone, the equation of this line will be

$$y = \frac{bx}{a};$$

$$\therefore d_x s = \sqrt{1 + (d_x y)^2},$$

$$= \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}};$$

$$\therefore \int_x (y d_x s) = \int_x \frac{bx}{a} \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}},$$

$$= \tfrac{1}{2} \cdot \frac{bx^2}{a} \cdot \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}} + \text{C};$$

which, taken between the limits $x = 0$, and $x = \text{OA} = a$, gives

$$\int_x (y d_x s) = \tfrac{1}{2} b \sqrt{a^2 + b^2}.$$

Also,

$$\int_x (x y d_x s) = \int_x \frac{bx^2}{a} \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}},$$

$$= \tfrac{1}{3} \frac{bx^3}{a} \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}} + \text{C};$$

which, between the same limits, gives

$$\int_x (x y d_x s) = \tfrac{1}{3} ab \sqrt{a^2 + b^2};$$

$$\therefore \dot{x} \cdot \tfrac{1}{2} b \sqrt{a^2 + b^2} = \tfrac{1}{3} ab \sqrt{a^2 + b^2};$$

$$\therefore \dot{x} = \tfrac{2}{3} a.$$

Ex. 21. To find the centre of any position of a sphere.

This surface is generated by the revolution of an arc OB, (Fig. 58) whose equation is

$$y = \sqrt{2ax - x^2},$$

x being $=$ OA, and $y =$ AB;

$$\therefore d_x s = \sqrt{1 + (d_x y)^2},$$

$$= \frac{a}{\sqrt{2ax - x^2}} = \frac{a}{y};$$

$$\therefore \int_x(yd_x s) = \int_x a = ax,$$

$$\text{and } \int_x(xyd_x s) = \int_x ax = \tfrac{1}{2}\, ax^2;$$

$$\therefore \bar{x} \,.\, ax = \tfrac{1}{2}\, ax^2;$$

$$\therefore \bar{x} = \tfrac{1}{2}\, x.$$

Ex. 22. To find the centre of gravity of the surface generated by the revolution of a semicycloid about its axis.

$$\bar{x} = \frac{2a}{3} \cdot \frac{\pi - \frac{8}{15}}{\pi - \frac{4}{3}}.$$

Ex. 23. To find the centre of gravity of the surface of a paraboloid.

Taking the focus as origin of co-ordinates, we find the distance of the centre of gravity from the directrix

$$= \tfrac{3}{5}\, a \,.\, \frac{\sec^5 \dfrac{\theta}{2} - 1}{\sec^3 \dfrac{\theta}{2} - 1}.$$

Ex. 24. To find the centre of gravity of the surface generated by the revolution of a node of the Lemniscate.

$$\bar{x} = \frac{a}{6} \cdot \frac{1 - \cos^{\frac{3}{2}} 2\theta}{1 - \cos\theta}.$$

185. To find the centre of gravity of a surface of any form.

If, in Art. 182, we use A to denote the elementary surface PQ instead of the prism P*s*, we shall have

$$\text{the limit of } \frac{A}{\delta x \delta y} = \sqrt{1 + (d_x z)^2 + (d_y z)^2};$$

and by proceeding exactly as in that article, we shall find

$$\dot{x}\,.\int_x\int_y\sqrt{1+(d_xz)^2+(d_yz)^2}=\int_x\int_y\{x\sqrt{1+(d_xz)^2+(d_yz)^2}\},$$

$$\dot{y}\,.\int_x\int_y\sqrt{1+(d_xz)^2+(d_yz)^2}=\int_x\int_y\{y\sqrt{1+(d_xz)^2+(d_yz)^2}\},$$

$$\dot{z}\,.\int_x\int_y\sqrt{1+(d_xz)^2+(d_yz)^2}=\int_x\int_y\{z\sqrt{1+(d_xz)^2+(d_yz)^2}\}.$$

186. To find the centre of gravity of a curve of double curvature.

If we use S for the length of the curve line, and δS for the length of a very small portion of it, we shall have the limit of $\frac{\delta S}{\delta x} = d_xS = \sqrt{1 + (d_xy)^2 + (d_xz)^2}$, and it will be found that

$$\dot{x}S = \int_x x\sqrt{1 + (d_xy)^2 + (d_xz)^2},$$

$$\dot{y}S = \int_x y\sqrt{1 + (d_xy)^2 + (d_xz)^2},$$

$$\dot{z}S = \int_x z\sqrt{1 + (d_xy)^2 + (d_xz)^2}.$$

187. We shall now add a few examples of finding the centre of gravity when the density is variable. Questions of this kind depend upon the formulæ of Art. 135, viz.—

$$\dot{x} = \frac{\Sigma(\rho Vx)}{\Sigma(\rho V)};\ \dot{y} = \frac{\Sigma(\rho Vy)}{\Sigma(\rho V)};\ \dot{z} = \frac{\Sigma(\rho Vz)}{\Sigma(\rho V)}.$$

188. To find the centre of gravity of a physical line, the density of which, at any point, varies as the n^{th} power of its distance from a given point in the line produced.

Ex. 25. Let AB (Fig. 59) be the given line, and C the given point; μ = the density at a point in AB, whose distance from C = 1; a = CA, b = CB, x = CP, δx = PQ. Since a physical line is of uniform thickness throughout, we may take the length of any portion of it as the measure of the volume of that portion; hence $\delta x =$ the volume of PQ, and as the density varies as (distance from C)^n,

$$\therefore\ 1^n : x^n :: \mu : \mu x^n.$$

Wherefore the density at P is μx^n, and PQ is ultimately of uniform density, therefore the value of ρV for PQ, is

$$= \mu x^n \delta x\,;$$

$$\therefore\ \Sigma(\rho \mathrm{V}) = \Sigma(\mu x^n \delta x),$$

$$= \mu \Sigma(x^n \delta x),$$

$$= \mu \textstyle\int_x x^n,$$

$$= \mu . \frac{x^{n+1}}{n+1} + \mathrm{C},$$

$$= \mu . \frac{b^{n+1} - a^{n+1}}{n+1},$$

between the limits $x = a$ and $x = b$.

Again,

$$\Sigma(\rho \mathrm{V} x) = \Sigma(\mu x^{n+1} \delta x),$$

$$= \mu \textstyle\int_x x^{n+1},$$

$$= \mu . \frac{x^{n+2}}{n+2} + \mathrm{C},$$

$$= \mu . \frac{b^{n+2} - a^{n+2}}{n+2},$$

between the same limits as before.

Wherefore $\dot{x}$ being the distance of the centre of gravity of the line from C, we have

$$\dot{x} = \frac{\Sigma(\rho \mathrm{V} x)}{\Sigma(\rho \mathrm{V})},$$

$$= \frac{n+1}{n+2} . \frac{b^{n+2} - a^{n+2}}{b^{n+1} - a^{n+1}}.$$

REMARK. When $n = -1$,

$$\Sigma(\rho V) = \mu \int_x x^n,$$

$$= \mu \int_x \frac{1}{x},$$

$$= \mu \,.\, \log_e x + C,$$

$$= \mu \log_e \frac{b}{a}.$$

And $\Sigma(\rho V x) = \mu (b - a)$;

$$\therefore \dot{x} = \frac{b-a}{\log_e \left(\frac{b}{a}\right)}.$$

Again, when $n = -2$,

$$\Sigma(\rho V) = \mu \left(\frac{1}{a} - \frac{1}{b}\right),$$

$$= \frac{\mu (b-a)}{ab};$$

and $\Sigma(\rho V x) = \mu \int_x x^{n+1}$,

$$= \mu \int_x \frac{1}{x},$$

$$= \mu \log_e \frac{b}{a};$$

$$\therefore \dot{x} = \frac{ab}{a-b} \,.\, \log_e \frac{b}{a}.$$

Ex. 26. To find the centre of gravity of a triangular plate, of uniform thickness, the density of which at any point varies as the n^{th} power of its distance from a line through the vertex parallel to the base.

Let ABC be the triangle, CD a line through its vertex parallel to its base; μ the density at a point in the triangle at the distance from CD is 1; P, Q two points in AC very near each other, through which draw Pp, Qq parallel to the base; $b = \text{AC}$, $c = \text{AB}$, $x = \text{CP}$, $\delta x = \text{PQ}$, $\theta = \angle\ \text{CAB} = \text{ACD}$ (Fig. 60).

Then the density at every point in the line Pp = $\mu (x \sin\theta)^n$, which may be ultimately taken as the density at every point of the element Pq. We may regard Pq as a parallelogram, whose base

$$= \text{P}p,$$

$$= \frac{xc}{b}, \text{ by similar triangles ACB, PC}p;$$

and whose altitude is PQ $\sin\theta = \delta x . \sin\theta$; its area, which we may take as the measure of its volume, is therefore

$$= \frac{xc}{b} . \delta x . \sin\theta;$$

and its mass

$$= \mu (x \sin\theta)^n . \frac{c}{b} \cdot x\delta x . \sin\theta,$$

$$= \frac{\mu c}{b} (x \sin\theta)^{n+1}\, \delta x;$$

$$\therefore \Sigma(\rho \text{V}) = \Sigma \left\{\frac{\mu c}{b} (x \sin\theta)^{n+1}\, \delta x\right\},$$

$$= \frac{\mu c}{b} \cdot (\sin\theta)^{n+1} \textstyle\int_x x^{n+1},$$

$$= \frac{\mu c}{b} \cdot (\sin\theta)^{n+1} . \frac{x^{n+2}}{n+2} + \text{C},$$

$$= \frac{\mu c}{b} \cdot (\sin\theta)^{n+1} . \frac{b^{n+2}}{n+2},$$

$$= \frac{\mu c}{n+2} \cdot (b \sin\theta)^{n+1}.$$

And the moment of the element Pq about CD

$$= \frac{\mu c}{b} (x \sin \theta)^{n+1} \delta x \,.\, x \sin \theta,$$

$$= \frac{\mu c}{b} (x \sin \theta)^{n+2} \delta x.$$

Therefore the moment of the triangle about CD

$$= \frac{\mu c}{b} \cdot \int_x (x \sin \theta)^{n+2},$$

$$= \frac{\mu c}{b} (\sin \theta)^{n+2} \cdot \frac{x^{n+3}}{n+3} + C,$$

$$= \frac{\mu c}{b} (\sin \theta)^{n+2} \cdot \frac{b^{n+3}}{n+3},$$

$$= \frac{\mu c}{n+3} \cdot (b \sin \theta)^{n+2}.$$

Wherefore, if a line passing through the centre of gravity of the triangle, parallel to the base, cut AC at a distance $\dot{x}$ from C, the distance of the centre of gravity from CD will be $\dot{x} \sin \theta$, and

$$\therefore \dot{x} \sin \theta = \frac{\frac{\mu c}{n+3} \cdot (b \sin \theta)^{n+2}}{\frac{\mu c}{n+2} \cdot (b \sin \theta)^{n+1}},$$

$$= \frac{n+2}{n+3} \cdot b \sin \theta;$$

$$\therefore \dot{x} = \frac{n+2}{n+3} \cdot AC.$$

And if CE be drawn bisecting the base, the centre of gravity must be in that line; hence we have two lines passing through the centre of gravity, and consequently it is the point of their intersection.

Ex. 27. To find the centre of gravity of a quadrant of a circle, the density at any point of which varies as the n^{th} power of its distance from the centre.

Let ACB (Fig. 61) be the quadrant; CD, Cd two radii making angles with CA respectively equal to θ, $\theta + \delta\theta$; $AC = a$, $CP = Cp = r$, $PQ = pq = \delta r$; μ = density at the distance 1 from the centre; therefore the density at P or $p = \mu r^n$. Now we may ultimately consider Pq as a parallelogram, whose sides are PQ and Pp, or δr and $r\delta\theta$, and its area $= r\delta r \,.\, \delta\theta$, which may be taken as the measure of its volume;

$$\therefore \Sigma(\rho V) = \Sigma(\mu r^n \,.\, r\delta r \,.\, \delta\theta),$$
$$= \textstyle\int_\theta \int_r (\mu r^{n+1}).$$

$$\text{Now} \textstyle\int_r (\mu r^{n+1}) = \frac{\mu}{n+2} \cdot r^{n+2} + C,$$

$$= \frac{\mu}{n+2} \cdot a^{n+2} \text{ from } r = 0, \text{ to } r = a.$$

$$\therefore \Sigma(\rho V) = \frac{\mu}{n+2} \cdot \textstyle\int_\theta a^{n+2},$$

$$= \frac{\mu}{n+2} \cdot a^{n+2}\, \theta + C,$$

$$= \frac{\mu}{n+2} \cdot \frac{\pi}{2} \cdot a^{n+2},$$

from $\theta = 0$, to $\theta = \frac{\pi}{2}$.

Again, $x = r \cos \theta$, and

$$\Sigma(\rho V x) = \Sigma(\mu r^n \,.\, r\delta r \,.\, \delta\theta \,.\, r \cos \theta),$$
$$= \textstyle\int_\theta \int_r (\mu r^{n+2} \cos \theta).$$

$$\text{But} \textstyle\int_r (\mu r^{n+2} \cos \theta) = \frac{\mu}{n+3} \cdot r^{n+3} \cos \theta + C,$$

$$= \frac{\mu}{n+3} \cdot a^{n+3} \cos \theta;$$

$$\therefore \Sigma(\rho V x) = \frac{\mu}{n+3} \cdot \int_\theta a^{n+3} \cos\theta,$$

$$= \frac{\mu}{n+3} \cdot a^{n+3} \sin\theta + C,$$

$$= \frac{\mu}{n+3} \cdot a^{n+3},$$

between the same limits as before.

$$\therefore \dot{x} = \frac{\Sigma(\rho V x)}{\Sigma(\rho V)},$$

$$= \frac{\frac{\mu}{n+3} \cdot a^{n+3}}{\frac{\mu}{n+2} \cdot \frac{\pi}{2} \cdot a^{n+2}},$$

$$= \frac{n+2}{n+3} \cdot \frac{2a}{\pi}.$$

And it is manifest, from the symmetrical form of the figure, with regard to CA and CB, that $\dot{y} = \dot{x}$.

Ex. 28. A sector of a circle ACB (Fig. 62) revolves round one of its radii AC through a given angle (β), and generates a solid, the density at any point of which varies as the $(n)^{th}$ power of its distance from the centre C; to find the centre of gravity of the solid.

Since the solid is perfectly symmetrical with regard to a plane passing through AC, and bisecting the angle β, the centre of gravity must be in that plane. Let CA be the axis of x, and a line in the plane BCA at right angles to AC, the axis of y; the axis of z being at right angles to both.

$$\therefore \dot{z} = \dot{y} \tan\frac{\beta}{2}.$$

Let $a =$ AC, $\alpha = \angle$ BCA, $\theta =$ ECA, $\delta\theta =$ FCE, CP $=$ Cp $= r$, PQ $= pq = \delta r$, $\mu =$ the density at the distance 1 from C. Then the area of the parallelogram Qp

$$= r\delta\theta \,.\, \delta r;$$

and when the sector revolves about AC, this parallelogram generates a volume

$$= r \sin\theta \,.\, \beta \,.\, r\delta\theta \,.\, \delta r, \text{ (Art. 170)},$$
$$= \beta r^2 \delta r \,.\, \sin\theta\,\delta\theta;$$

for P's distance from AC is $r \sin\theta$, and in revolving through the angle β, the length of its path is $r \sin\theta \,.\, \beta$. The density of this volume

$$= \mu r^n,$$

and therefore

$$\Sigma(\rho \mathrm{V}) = \Sigma(\mu r^n \,.\, \beta r^2 \delta r \,.\, \sin\theta \,.\, \delta\theta),$$
$$= \mu\beta \textstyle\int_\theta \int_r r^{n+2} \sin\theta.$$

$$\text{But } \textstyle\int_r r^{n+2} \sin\theta = \dfrac{r^{n+3}}{n+3} \cdot \sin\theta + \mathrm{C},$$

$$= \frac{a^{n+3}}{n+3} \cdot \sin\theta,$$

from $r = 0$, to $r = a$.

$$\therefore\ \Sigma(\rho\mathrm{V}) = \frac{\mu\beta}{n+3} \cdot a^{n+3} \textstyle\int_\theta \sin\theta,$$

$$= -\frac{\mu\beta}{n+3} \cdot {}^{n+3} \cos\theta + \mathrm{C},$$

$$= \frac{\mu\beta}{n+3} \cdot a^{n+3} (1 - \cos\alpha),$$

$$= \frac{2\mu\beta}{n+3} \cdot a^{n+3} \sin^2 \frac{\alpha}{2},$$

from $\theta = 0$, to $\theta = \alpha$.

$$\text{Again, } \Sigma(\rho V x) = \Sigma(\mu\beta r^{n+2}\sin\theta . \delta r . \delta\theta . r\cos\theta),$$
$$= \mu\beta \textstyle\int_\theta\int_r (r^{n+3}\sin\theta\cos\theta),$$
$$= \frac{\mu\beta}{n+4} . a^{n+4} \textstyle\int_\theta (\sin\theta\cos\theta),$$
$$= \tfrac{1}{2} . \frac{\mu\beta}{n+4} . a^{n+4}\sin^2\theta + C,$$
$$= \tfrac{1}{2} . \frac{\mu\beta}{n+4} . a^{n+4}\sin^2 a;$$
$$\therefore \dot{x} = \frac{\Sigma(\rho V x)}{\Sigma(\rho V)},$$
$$= \tfrac{1}{4} . \frac{n+3}{n+4} . a . \frac{\sin^2 a}{\sin^2 \frac{a}{2}},$$
$$= \frac{n+3}{n+4} . a\cos^2\frac{a}{2}.$$

In order to find $\dot{z}$, we must divide the volume generated by the revolution of the parallelogram Pq into elements; to this end, let there be two planes passing through AC and inclined to the plane BCA, at the angles ϕ and $\phi + \delta\phi$ respectively; then the portion comprehended between them will be equal to the volume generated by Pq, in revolving through an angle $\delta\phi$, and therefore is

$$= r\sin\theta . \delta\phi . r\delta\theta . \delta r \text{ (Art. 170)},$$
$$= r^2\delta r . \sin\theta\,\delta\theta . \delta\phi.$$

And the density of this element is μr^n, and therefore its mass is

$$\mu r^{n+2}\,\delta r . \sin\theta\,\delta\theta . \delta\phi,$$

and its distance from the plane ABC is $r\sin\theta . \sin\phi$, as is evident from the construction; and therefore its moment

$$= \mu r^{n+3}\,\delta r . \sin^2\theta\,\delta\theta . \sin\phi\,\delta\phi;$$

$$\therefore \Sigma(\rho V z) = \mu \textstyle\int_r \int_\theta \int_\phi (r^{n+3} \sin^2\theta \sin\phi),$$

$$= \frac{\mu a^{n+4}}{n+4} \cdot \textstyle\int_\theta \int_\phi (\sin^2\theta \sin\phi),$$

$$= \frac{\mu a^{n+4}}{n+4} \textstyle\int_\theta (-\sin^2\theta \cos\phi + C),$$

$$= \frac{\mu a^{n+4}}{n+4} \textstyle\int_\theta (1 - \cos\beta) \sin^2\theta,$$

taken from $\phi = 0$, to $\phi = \beta$.

$$\text{Now} \textstyle\int_\theta \sin^2\theta = \frac{1}{2} \int_\theta (1 - \cos 2\theta),$$

$$= \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta) + C,$$

$$= \frac{a}{2} - \frac{\sin 2a}{4},$$

taken from $\theta = 0$, to $\theta = a$.

$$\therefore \Sigma(\rho V z) = \frac{\mu a^{n+4}}{n+4} (1 - \cos\beta) \tfrac{1}{2}(a - \tfrac{1}{2}\sin 2a),$$

$$= \frac{\mu a^{n+4}}{n+4} \sin^2\frac{\beta}{2} (a - \sin a \cos a);$$

$$\therefore \dot{z} = \frac{\Sigma(\rho V z)}{\Sigma(\rho V)},$$

$$= \frac{n+3}{n+4} \cdot \frac{a}{2} \cdot \left\{ \frac{\sin\frac{\beta}{2}}{\sin\frac{a}{2}} \right\}^2 \frac{a - \sin a \cos a}{\beta};$$

and therefore $\dot{y} = \dot{z} \cot\frac{\beta}{2}$,

$$= \frac{n+3}{n+4} \cdot \frac{a}{2} \cdot \frac{\cos\frac{\beta}{2} \sin\frac{\beta}{2}}{\sin^2\frac{a}{2}} \cdot \frac{a - \sin a \cos a}{\beta}.$$

Ex. 29. Find the centre of gravity of a cone, the density at every point of which varies as the square of its distance from a plane through the vertex parallel to the base.

Ex. 30. Find the centre of gravity of the eighth part of a sphere, the density at any point whose distance from the centre is s, being proportional to

$$\frac{a}{s} \sin \frac{\pi s}{2a},$$

where a denotes the radius of the sphere.

There is reason to believe the law of density stated in this question does approximately represent that of the earth.

CHAPTER VI.

ON THE MECHANICAL POWERS.

189. In laying down the theory of Statics we have considered force in a general point of view, and our reasonings are accordingly applicable to every cause which can produce motion. We have mentioned in the last chapter, that the earth possesses the property of producing motion towards its centre in all matter abandoned to its influence. This influence is exerted incessantly, and gives weight and motion to many solids and fluids; which, in their turn, often become the causes of motion in other matter. All living animals also have the property of exerting force in a limited degree; this capability is called *strength;* by the exercise of it they move from place to place and communicate motion to other bodies at will. It is to be observed, however, that every animal employed in moving bodies spends of necessity a certain portion of its strength in moving itself at the same time; the quantity of strength so spent depends upon the speed with which the animal is required to move, and there is, in fact, a certain velocity for each particular animal with which it can only just move itself without any load at all. For a like reason there is a limit to the velocity with which a man can move his arm,

or his hand, or his finger; and therefore it would be as useless for him to attempt to perform an operation with any of these members, requiring a greater velocity than it can move with, as it would be for him to attempt to lift a weight beyond his strength.

For the purpose of rendering his strength available in such instances as these, and to enable him to call in the assistance of animals, stronger or fleeter than himself, and to make use of those natural forces mentioned above, MACHINES have been invented; and, as it appears that we are hindered from accomplishing certain objects by three causes, viz.—

1. Not being able to apply our strength in a proper manner;

2. Not being able to move with sufficient velocity;

3. Not having sufficient strength;

so it will be found, that notwithstanding an almost infinite variety of machines have been invented, their effects may nevertheless be reduced to three corresponding classes; viz.—

1. To change the direction of the force employed.

2. To render velocity produced greater than that with which the agent employed can move.

3. To enable the force employed to overcome a much larger force.

Every machine, however complicated its construction, is found to be reducible to a set of simple ones, called the *Mechanical Powers*. These, though authors differ considerably on the subject, are generally said to be six in number; viz.—

1. The Lever;
2. The Pulley;
3. The Wheel and Axle;
4. The Inclined Plane;
5. The Screw;
6. The Wedge.

These are not the most simple machines; for, *rods* used in pushing, and *cords* used in pulling, are much more simple; in fact, every machine will be found to be a combination of levers, cords, and inclined planes, and these might consequently be called the simple Mechanical Powers, with much greater propriety than the six before mentioned. As, however, these are not very complicated in construction and application, and as levers, cords, and inclined planes do always, in actual practice, present themselves in machinery, in one or more of these six combinations, it will very much facilitate our enquiries into any proposed machine, to be acquainted with their forms and the advantages to be expected from their use.

In speaking of any machine, the force which is applied to work it is called the *working power*, or, simply, the *Power;* the weight to be raised, or resistance to be overcome, is called the *Weight;* the point where the machine is applied to produce its effect is called the *working point;* and the fraction

$$\frac{\text{Weight}}{\text{Power}}$$

is called the *Mechanical Advantage* (by some authors the *Power*, but this creates confusion by confounding it with the former definition of power) of the machine.

190. Every machine is useless until put in motion, and therefore its parts ought to be so arranged and adapted that the given power may be able to overcome the proposed weight, and move it with the requisite degree of celerity; but, in discussing the theory of the Mechanical Powers, it will be sufficient to determine the ratio of the weight to the power when they balance each other, for then the slightest addition made to the power will cause it to preponderate and put the machine in motion.

191. It is very important to remark, that when a power is employed in working a machine, a very considerable portion of it is found not to reach the working point, being spent in overcoming the stiffness of the cords and the roughness of surfaces which rub against each other. Much power is also lost through the imperfection of workmanship, the bending of rods, beams and other materials, which are intended to be rigid, the resistance of the air, &c.; but the introduction of the consideration of these things, though very important in a practical point of view, would only tend to embarrass the student by rendering our investigations tedious and perplexing. We shall therefore at first suppose cords to be perfectly flexible, surfaces quite smooth, workmanship geometrically exact, rods and beams perfectly rigid, the air to offer no resistance; &c.

"It is scarcely necessary to state, that, all these suppositions being false, none of the consequences deduced from them can be true. Nevertheless, as it is the business of Art to bring machines as near to this state of ideal perfection as possible, the conclusions which are thus obtained, though false in a strict sense, yet deviate from the truth in but a small degree. Like the first outline of a picture, they resemble in their general features that truth, to which, after many subsequent corrections, they must finally approximate.

"After a first approximation has been made on the several suppositions which have been mentioned, various effects, which have been previously neglected, are successively taken into account. Roughness, rigidity, imperfect flexibility, the resistance of air and other fluids, the effects of the weight and inertia of the machine, are severally examined, and their laws and properties detected. The modifications and corrections thus suggested, as necessary to be introduced into our former conclusions, are applied, and a second approximation, but still *only* an approximation to truth is made. For, in investigating the laws which regulate the several effects just mentioned, we are compelled to proceed upon a new group of false suppositions. To determine the laws which regulate the friction of surfaces, it is necessary to assume that every part of the surfaces of contact are uniformly rough; that the solid parts which are imperfectly rigid, and the cords which are imperfectly flexible, are constituted throughout their entire dimensions of a uniform material; so that the imperfection does not prevail more in one part than another. Thus all irregularity is left out of account, and a general average of the effects taken. It is obvious therefore, that by these means we have still failed in obtaining a result exactly conformable to the real state of things; but it is equally obvious, that we have obtained one much more conformable to that state than had been previously accomplished, and sufficiently near it for most practical purposes.

"This apparent imperfection in our instruments and powers of investigation, is not peculiar to Mechanics; it pervades all departments of natural science. In Astronomy, the motions of the celestial bodies, and their various changes and appearances, as developed by theory, assisted by observation and experience, are only approximations to the real motions and appearances which take place in nature. It is true that these approximations are susceptible of almost unlimited accuracy; but still they are, and ever will continue to be, only approximations. Optics, and all other branches of natural science, are liable to the same observations."*

192. We shall introduce the theory of the Mechanical Powers with the following general proposition, which is applicable to any machine whatever.

If the power P balance the weight W on any machine; and if when the machine is put in motion the point at which P acts begins to move with a virtual velocity v, and the working point moves with a velocity w estimated in the direction opposite to that in which W acts, then

$$Pv = Ww.$$

For since the virtual velocities of P and W are respectively v and $-w$, and since the mutual actions of the parts of the machine against each other may be neglected, the principle of virtual velocities (Art. 119) gives

$$Pv + W(-w) = 0;$$

$$\therefore Pv = Ww.$$

The quantity w will always be positive, since W is a force to be *overcome*, and therefore, in working the machine, the working point will move in a direction opposed to the action of W.

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* Captain Kater's Treatise on Machines.

Remark. From the equation above given, we have

$$\frac{W}{P} = \frac{v}{w}.$$

Hence w is less than v in the same proportion as P is less than W; and therefore, whatever advantage we may gain by a machine in moving large weights with small powers, we lose again in time, for W moves slower than P in the proportion $w : v$ or P : W. In like manner, what we lose in power we gain in time. We shall illustrate this principle by an example.

Let it be required to raise a weight of 600*lbs.* with a force of 150*lbs.* through 10 feet, with any machine whatever.

Without employing any machine, the power 150*lbs.* can raise 600*lbs.* in four separate and equal portions through 10 feet each, and the labour exercised, or power spent, is the same as in raising one portion through four times 10 feet, that is, through 40 feet. But if a machine be employed, the power's velocity is to the weight's velocity :: W : P :: 4 : 1, and therefore the power must descend through 40 feet to raise W through 10 feet, the same as before. The only real advantage therefore gained by machinery, in this case, is that it enables us to do that at *once,* which we should otherwise be obliged to do at *four separate* times.

193. We may further observe, that if the construction of a machine be such that when the point at which P acts moves uniformly, the working point also moves uniformly; or, which is the same thing, if the velocity of P always has to that of W a constant ratio (which is the case with a great number of machines), then

$$Ps = Ws'.$$

s, s' being the spaces described by P and W in any equal time.

For, in this case, s and s' are proportional to the actual velocities of P and W, and these again are proportional to the virtual velocities v and w.

194. It is a peculiar property of machines of this class, that P and W are in a state of *neuter equilibrium;* and therefore balance in all positions (Art. 172), and consequently the centre of gravity of P and W neither ascends nor descends.

I. *On the Lever.*

195. Def. A *Lever* is a rigid rod, moveable in a certain plane about one of its points, which is fixed and called its fulcrum.

196. In a lever the power is to the weight inversely as the perpendiculars from the fulcrum upon the directions in which they act.

(Both the power and weight are supposed to act in the plane in which the lever is moveable, which is technically called the plane of the lever).

Let AB (Figs. 63, 65) or AC (Fig. 64) or BC (Fig. 66) be a lever whose fulcrum is C; A, B the points at which the power P and weight W act; CY, CZ perpendiculars from C upon their directions. Then the equilibrium will not be disturbed by applying at C two forces P', $-P'$ parallel and equal to P, and two others W', $-W'$ parallel and equal to W.

We have thus, six forces acting on the lever, of which (P, – P′) and (W, – W′) form two couples, and the two remaining forces P′, W′ being counterbalanced by the reaction of the fulcrum, may be removed. Hence the couple (P, – P′) whose arm is CY, balances the couple (W, – W′) whose arm is CZ, consequently their moments must be equal;

$$\therefore P . CY = W . CZ.$$

To find the pressure on the fulcrum C.

We have shewn that P and W are equivalent to two forces P′, W′ acting at C, and two equal couples (P, – P′), (W, – W′); these couples may be removed, because they are equal and opposite, and therefore balance each other. It appears then, that P and W are equivalent to P′ and W′ acting at C. Consequently the pressure on the fulcrum is the same as if the power and weight were both transposed to it parallel to themselves.

197. We have considered the weight of the lever inconsiderable when compared with P and W, but if this should not be the case, let w be its weight, G its centre of gravity. Then we may suppose the whole force w, which gravity exerts upon the lever, to be applied at G (Art. 131); this force may be converted into a couple whose moment is $w . CG$, and as there is an equilibrium between the three couples, the sum of the moments of the two which act in one direction (*i.e.* positive or negative) must be equal to that of the third;

$$\therefore P . CY + w . CG = W . CZ,$$

the equation of equilibrium in this case.

Remark. Examples of levers of the same kind as the one in Fig. 63, are the common balance, steelyards, pokers, &c.;

and scissors, pincers, &c. are instances of two such levers having a common fulcrum.

Examples of levers of the same kind as those in Figs. 64, 66, are the oars and rudders of boats, cutting-knives moveable about one end, &c.; and tongs, sheep-shears, &c. are instances of the combination of two such levers with a common fulcrum.

Examples of the bent lever, in Fig. 65, are gavelocks, jemmies, bones of all animals, &c.

198. We have defined a lever to be a rigid rod, but we may consider any rigid body having a fixed axis as a lever, whose fulcrum is the axis; and if powers P_1, P_2, $P_3 \ldots . P_n$, act upon this lever, and balance the weights W_1, W_2, $W_3 \ldots . W_m$, then

$$P_1 p_1 + P_2 p_2 + \ldots . + P_n p_n$$
$$= W_1 w_1 + W_2 w_2 + \ldots . + W_m w_m,$$
$$\text{or } \Sigma(Pp) = \Sigma(Ww).$$

The powers and weights being supposed to act in planes at right angles to the axis, and p_1, $p_2 \ldots . p_n$; w_1, $w_2 \ldots w_m$ being the respective perpendiculars from the axis upon the directions in which the powers and weights act.

This may be proved as before, by converting the powers and weights into couples, and then transposing them into one plane; and it will also appear, that the pressure on the axis or fulcrum is the same as it would be if all the forces were transposed in their own planes parallel to themselves to the axis.

II. *On the Pulley.*

199. Def. *A Pulley* is a wheel of wood or metal, turning on an axis through its centre at right angles to its plane, and usually enclosed in a frame or case, called its *block,* which admits a rope to pass freely over the circumference of the pulley, in which there is usually a groove to receive it, and prevent its slipping out. The pulley is said to be *fixed* or *moveable,* according as its axis is stationary or not. An assemblage of several pullies is called a *system of pulleys* or a *mouffle.*

200. It will be necessary before investigating the properties of the pulley to premise, that if a cord be stretched by two equal forces applied at its extremities in contrary directions, there will be a tendency to break; the force which the rope, in consequence of the cohesion of its particles, exerts to resist this tendency, must be equal and opposite to that which causes the tendency; it is called the *tension* of the rope. Hence tension is a force which is exerted equally in every part, tends from the extremities of a cord towards the middle, and is always equal to either of the equal forces, by which the cord is stretched. If one end of the cord, instead of being acted on by a force, be fastened to a fixed point, the tension will not be altered; for the fixed point will, by its reaction, exactly supply the place of the force.

201. In the single fixed pulley the power and weight are equal.

Let ABK (Fig. 67) be the pulley, C its centre, CN its block; P and W the power and weight acting at the extremities of the cord passing over the pulley, and having the part AB in contact with it. Then we may consider ABK as a lever whose fulcrum is C, and therefore drawing the radii CA, CB to the points A and B, we have

$$P \,.\, CA = W \,.\, CB, \text{ (Art. 196)};$$
$$\therefore P = W.$$

Hence it appears that no mechanical *advantage* is gained by the use of this pulley; the only purpose for which it is used is to change the direction in which a force is transmitted.

To determine the pressure on the fulcrum C, transpose the forces P and W to that point (Art. 196), and put θ for the angle at which AP and BW are inclined to each other, and let R be the pressure, which is, of course, the resultant of these transposed forces.

$$\begin{aligned} \therefore R^2 &= P^2 + 2PW \cos\theta + W^2 \text{ (Art. 26)}, \\ &= P^2 + 2P^2 \cos\theta + P^2, \\ &= 2P^2 (1 + \cos\theta), \\ &= 4P^2 \,.\, \cos^2\frac{\theta}{2}; \end{aligned}$$

$$\therefore R = 2P \cos\frac{\theta}{2}.$$

This pressure bisects the angle P′CW′, and is transmitted to N, the fixed point to which the block is attached.

202. In the single moveable pulley, the power is to the weight :: 1 : 2 × cosine of half the angle between the strings.

Let the power P act at the extremity P of the cord PABD (Fig. 68), which passes under the pulley; has the part AB in contact with it; and its other extremity fastened at D.

The weight W hangs from the block at N.

Exactly as in the last case, we find the pressure on the centre C to be

$$2P \cos \frac{\theta}{2},$$

θ being the angle between the strings AP, BD; this force is transmitted through the block in the direction CN, bisecting the angle θ, wherefore the action of W must be equal to it and in the opposite direction, otherwise there cannot be an equilibrium;

$$\therefore W = 2P \cos \frac{\theta}{2},$$

$$\text{and consequently } P : W :: 1 : 2 \cos \frac{\theta}{2}.$$

203. No mechanical *advantage* can be gained by the use of this pulley, unless

$$2 \cos \frac{\theta}{2} > 1,$$

$$\text{and } \therefore \cos \frac{\theta}{2} > \tfrac{1}{2} > \cos 60^\circ;$$

$$\therefore \theta < 120^\circ;$$

that is, unless the strings are inclined to each other at a less angle than 120°.

The greatest possible *advantage* will be gained when the strings are parallel, for then $\theta = 0$, and $\cos \frac{\theta}{2} = 1$,

$$\text{and therefore } W = 2P.$$

204. If the weight of the pulley and its block be considerable, it must be considered as an additional weight, and added to W in the above expressions.

205. To find the conditions of equilibrium in a system of pullies, where each pulley hangs by a separate string, the strings being all parallel.

Let A_1, A_2, A_3 . . . (Fig. 69) be the pullies; M_1, M_2, M_3 . . . the points where the strings are fastened. Then P is equal to the tension of the string passing under A_1, and W equal that of the string passing under A_2. The two strings A_1P, A_1M have to support the tension of N_1A_2; N_1A_2 and M_2A_2 support that of N_2A_3, and so on; therefore, by (Art. 203),

$$\begin{aligned} (P =) \text{ tension of } A_1P : \text{tension of } N_1A_2 &:: 1 : 2, \\ \text{tension of } N_1A_2 : \text{tension of } N_2A_3 &:: 1 : 2, \\ \cdots\cdots\cdots\cdots \\ \text{tension of } N_2A_3 : \text{tension of } N_3W\ (= W) &:: 1 : 2 : \end{aligned}$$

$$\therefore P : W :: 1 \times 1 \times 1 \times \ldots : 2 \times 2 \times 2 \ldots$$

If n be the number of moveable pullies, then

$$P : W :: 1^n : 2^n;$$
$$\therefore W = 2^n P.$$

206. If the weights of the pullies and blocks are considerable, let A_1, A_2, A_3 . . . . represent the weights of the pullies and blocks denoted by those letters in the figure; and let T_1, T_2 . . . . be the tensions of the strings N_1A_2, N_2A_3, ... Then, as before, the weights of the pullies must be added to the tensions of the respective cords which they support;

$$\therefore P : T_1 + A_1 :: 1 : 2;$$
$$\therefore T_1 = 2P - A_1.$$

$$\begin{aligned} \text{Similarly, } T_2 &= 2T_1 - A_2, \\ &= 2^2P - 2A_1 - A_2, \\ T_3 &= 2T_2 - A_3, \\ &= 2^3P - 2^2A_1 - 2A_2 - A_3, \end{aligned}$$

and so on, the law being manifest; then, since the tension of the last string = W, we have

$$W = 2^n P - 2^{n-1} A_1 - 2^{n-2} A_2 - 2^{n-3} A_3 - \ldots\ldots - A_n.$$

It appears from this expression, that the weights of the pullies diminish the advantage of this system.

207. If all the pullies are equal, then

$$W = 2^n P - A_1 (2^{n-1} + 2^{n-2} + \ldots\ldots + 1),$$
$$= 2^n P - A_1 \frac{2^n - 1}{2 - 1},$$
$$= 2^n P - (2^n - 1) A_1,$$
$$= 2^n (P - A_1) + A_1;$$
$$\therefore W - A_1 = 2^n (P - A_1).$$

Hence, if we suppose both the power and weight diminished by the weight of a pulley, we may then neglect the consideration of the heaviness of the pullies.

208. In the system (Fig. 70) where each string is attached to the weight, let T_1, T_2, . . . be the tensions of the first, second, . . . strings; then if the weights of the pullies are inconsiderable, we have

$$T_1 = P,$$
$$T_2 = 2T_1 = 2P \text{ (Art. 203)},$$
$$T_3 = 2T_2 = 2^2 P,$$
$$T_4 = 2T_3 = 2^3 P;$$

and if there be n separate strings,

$$T_n = 2^{n-1} P.$$

Now W is supported by the tensions of the n strings fastened to the block B, and

$$\therefore W = T_1 + T_2 + \ldots\ldots + T_n,$$
$$= P(1 + 2 + 2^2 + \ldots 2^{n-1}),$$
$$= P.\frac{2^n - 1}{2 - 1},$$
$$= P(2^n - 1).$$

209. In the system (Fig. 71), let T_1, T_2, ... be the tensions of the first, second .... strings; then T_2 has to support three tensions equal to P;

$$\therefore T_1 = P,$$
$$T_2 = 3T_1 = 3P,$$
$$T_3 = 3T_2 = 3^2P,$$
$$T_4 = 3T_3 = 3^3P;$$

and if there be (n) different strings, the tension of the last is

$$T_n = 3^{n-1}P.$$

Now the weight W is supported by two strings whose tensions are each equal to T_1, two of which the tensions are equal to T_2, &c.

$$\therefore W = 2T_1 + 2T_2 + \ldots + 2T_n,$$
$$= 2P(1 + 3 + 3^2 + \ldots\ldots 3^{n-1}),$$
$$= 2P.\frac{3^n - 1}{3 - 1},$$
$$= P.(3^n - 1).$$

Remark. If the weights of the pullies and blocks are not inconsiderable, they may be taken into account, in this and every other system, by adding each to the tension of that string which supports it, as in Art. 206.

210. In the system, Fig. 72, the weight W is supported by the tensions of all the strings at the lower block, and as it is the same string which passes round all the pullies, the tension of every part = P; wherefore, if there be n pullies in the lower block, there are $2n$ strings supporting the weight, and therefore

$$W = 2nP.$$

III. *On the Wheel and Axle.*

211. The wheel and axle consists of a cylinder and a wheel firmly attached to each other, and being moveable about a fixed axis coinciding with the axis of the cylinder, and passing through the centre of the wheel at right angles to its plane, as in Fig. 73.

The power P acts by means of a cord wrapped round the circumference of the wheel C, and the weight W is fastened to a cord which is wound upon the cylinder AB as P turns the machine round its axis; and thus W is raised.

212. To find the condition of equilibrium on the wheel and axle.

We may consider P and W as forces acting upon a rigid body with a fixed axis, and therefore their moments about that axis must be equal;

$\therefore$ P $\times$ (perpendicular upon its direction from the axis),
= W . (perpendicular upon its direction from the axis).

Now these perpendiculars are respectively the radii of the wheel and of the clylinder;

$\therefore$ P . (radius of the wheel) = W . (radius of the axle).

213. If the thickness of the rope be considerable, it must be taken into account.

We may suppose the actions of P and W to be transmitted along the middle or axis of the rope, and then the perpendiculars upon the directions of P and W will be respectively equal to

radius of wheel + radius of rope,

and radius of axle + radius of rope,

and the condition of equilibrium is

$$P.(\text{rad. wheel} + \text{rad. of rope}) = W\,(\text{rad. axle} + \text{rad. of rope})$$

This diminishes the *advantage* of the machine.

214. The pressure on the axis of this machine may be found by transposing P and W in their own planes parallel to themselves to the axis.

IV. *On the Inclined Plane.*

215. This machine is nothing more than a plane inclined to the horizon. The condition of equilibrium may be thus found.

Let AB (Fig. 74) be the plane; AC parallel and BC perpendicular to the horizon; W the weight, P the power. Draw WR perpendicular to the plane, WG perpendicular to the horizon. P is supposed to act in the plane RWB. The weight W is kept at rest by three forces, viz. P in the direction WP; gravity (= W) in the direction WG, and reaction R of the plane in the direction WR (Art. 117).

Denote the angle PWB by θ, and the inclination BAC of the plane to the horizon by A; and resolve the three forces, acting on the point W, in a direction parallel to the plane, the sum will be

$$P \cos PWB - W \cos AWG + R \cos RWB$$
$$= P \cos\theta - W . \sin A.$$

But since there is an equilibrium, this sum must be equal to zero, (Art. 40.);

$$\therefore P \cos\theta = W \sin A.$$

216. If P's direction should happen to be parallel to the base, $\theta = 0$ and $\cos\theta = 1$;

$$\therefore P = W \sin A.$$

But if P's direction should happen to be parallel to the horizon, $\theta = - A$ and $\cos(-A) = \cos A$;

$$\therefore P \cos A = W \sin A;$$
$$\therefore P = W \tan A.$$

217. To find the reaction (which is equal to the pressure on the plane) resolve the forces in a direction at right angles to that in which P acts;

$$\therefore R \cos RWP + W \cos GWP = 0, \text{ (Art. 40)};$$
$$\text{or } R \sin\theta + W \cos(90 + A + \theta) = 0;$$
$$\therefore R = W . \frac{\sin(A + \theta)}{\sin\theta}.$$

V. *On the Screw.*

218. This mechanical power is a combination of the lever and inclined plane; it may be conceived to be thus generated.

Let ABCD (Fig. 75) be a cylinder; BEFC a rectangle whose base BE is equal to the circumference of the cylinder.

Divide this rectangle into any convenient number of equal rectangles GE, IK, CK; and draw their diagonals BH, GK, IF. Then, if this rectangle CE be wrapped upon the cylinder, so that BE coincides with the circumference of the base, E, H, K, F will respectively fall upon the points B, G, I, C of the cylinder, and the lines BH, GK, IF will trace out upon its surface a continuous spiral thread BLGMINC winding uniformly up the cylinder. The cylinder is usually made protuberant where the spiral line BLGMINC falls upon it so that the thread becomes a winding inclined plane, projecting from the cylinder as in Fig. 76, and differing from the inclined plane BH * in nothing but its winding course. This is the external screw. The internal screw is formed by applying the parallelogram BEFC to a hollow cylinder, equal to the former, and making a groove where the thread falls to fit the protuberant thread of the external screw. This internal screw is often called a *nut*, and the other the *screw*. When the two screws

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* The following illustration renders this very clear:—

"When a road directly ascends the side of a hill, it is to be considered as an inclined plane; but it will not lose this mechanical character, if, instead of directly ascending towards the top of the hill, it winds successively round it, and gradually ascends so as after several revolutions to reach the top. In the same manner a path may be conceived to surround a pillar by which the ascent may be facilitated upon the principle of the inclined plane. Winding stairs constructed in the interior of great columns partake of this character; for although the ascent be produced by successive steps, yet if a floor could be made sufficiently rough to prevent the feet from slipping, the ascent would be accomplished with equal facility. In such a case the winding path would be equivalent to an inclined plane, bent into such a form as to accommodate it to the peculiar circumstances in which it would be required to be used. It will not be difficult to trace the resemblance between such an adaptation of the inclined plane and the appearances presented by the thread of the *screw;* and it may hence be easily understood that a screw is nothing more than an inclined plane, constructed upon the surface of a cylinder."—Captain Kater's *Machines.*

are thus adapted to each other, the external or the internal screw, as the case requires, may be moved by means of a lever about their common axis, as in Figs. 77, 78. The force being applied to the lever at right angles to it, in a plane parallel to the base of the cylinder.

The screw and nut thus applied to each other, resemble two inclined planes, such as BHG and HBE, one of which is laid upon, and slides down the other; and as the planes wind round the cylinder a rotatory motion ensues. When the machine is worked, the weight is laid upon the nut, and thus causes its inclined plane to press upon that of the screw in the direction of gravity. The consequence would be, that the nut and weight with it would begin to slide down the thread of the screw and descend, but this is prevented by confining the nut so that it cannot have a rotatory motion, but only one of ascent or descent. The screw is then turned round by means of a lever passing through its head, and thus its inclined thread sliding under that of the nut, forces the nut and the weight upon it to ascend, just as by pushing the inclined plane EBH in the direction EB, the plane GBH would be made to ascend. One turn of the screw raises the weight through an altitude equal to the distance between two threads. Sometimes, however, the nut is firmly fixed so as to admit of no motion whatever, (as in Figs. 77, 78); and then the thread of the screw, in sliding under that of the nut, forces the screw to descend and press violently against any obstacle which may be opposed to it. In some cases the weight is not applied to the nut, but to the screw; but as the two inclined planes are perfectly equal and similar, it will require the same force to support a weight on one as on the other, and for this reason one investigation will serve for both.

As before observed, the screw is worked by applying a power P at the end of a lever; and the moment of P to turn the screw round

$$= P \times \text{length of the lever},$$

and therefore P is equivalent to a force

$$\frac{P \times \text{length of the lever}}{\text{rad. cylinder}}$$

acting immediately at the thread of the screw in a horizontal direction parallel to that in which P acts. Now the inclined plane on which W rests, by means of the nut, is only BH wrapped round the cylinder; its inclination to the horizon or base of the cylinder is therefore HBE.

Hence, by Art. 216, we have

$$P \times \frac{\text{length of lever}}{\text{rad. of cylinder}} = W . \tan HBE,$$

$$= W . \frac{HE}{BE},$$

$$= W . \frac{\text{distance between two threads}}{\text{circumf. of cylinder}}.$$

But the radii of circles are proportional to their circumferences;

$$\therefore \frac{\text{length of lever}}{\text{rad. of cylinder}} = \frac{\text{circumf. described by power}}{\text{circumf. of cylinder}};$$

$$\therefore P . \frac{\text{circumf. by power}}{\text{circumf. of cylinder}} = W . \frac{\text{dist. between two threads}}{\text{circumf. of cylinder}};$$

$$\therefore P = W . \frac{\text{dist. between two threads}}{\text{circumf. described by power}}.$$

As the distance between two successive threads can be made very small, and the circumference described by the power as large as we please, the advantage of this machine is very great; and it is remarkable, that it does not depend upon the thickness of the screw.

Remark. When the nut is fixed, the end of the screw moves through the distance between two threads, while the end of the lever at which the power acts, describes one whole circumference. Now suppose that the length of the lever is $3\frac{1}{2}$ inches, then the circumference described by its end is 11 inches nearly; and if there be 50 threads in one inch of length of the screw, the lever must turn round 50 times, and therefore its end will move through

$$50 \times 11 = 550 \text{ inches}$$

while the end of the screw descends through one inch. And if the end of the lever describe one inch, the end of the screw will describe the $(\frac{1}{550})^{th}$ part of an inch. The very slow motion of the screw, which may be thus communicated by a very considerable motion of the end of the lever, renders it a very convenient instrument for the measurement of small spaces. When the lever is made to move over a graduated circle, the screw is called a Micrometer screw, and is very useful in Astronomy when very minute portions of the divisions of a graduated circle are to be ascertained.

VI. *On the Wedge.*

219. A wedge is the solid figure defined by Euclid (book xii. def. iv.) as a triangular prism. Its two ends are equal and similar triangles, and its three sides rectangular parallelograms, (see Fig. 79). It is principally used in splitting timber, and separating bodies which are very strongly united, and in raising very heavy weights through a small altitude, for the purpose of introducing a lever, or some other more convenient machine. AB is called its *edge*, CDEF its *head*, CABD and FABE its *faces*.

When used, its edge is introduced into a small cleft prepared to receive it, and then by violent blows with a hammer on its head its body is driven between the substances, which are thus separated by an interval equal to the breadth of the head. After this, a larger wedge may be introduced, if necessary, and treated as before, until the requisite degree of separation is effected.

As the wedge is driven in by violent blows, if its sides were perfectly smooth it would start back by the pressure of the obstacles upon them in the interval between the strokes; and thus we should fail in effecting and maintaining the requisite degree of separation, and the machine would be rendered useless. In practice, however, the friction in this machine is always so great as to prevent any recoil, and forms, in fact, the principal resistance to be overcome in driving the wedge. The mode of working this machine will at once present itself to the reader as being *totally different in principle* from that of all the other machines we have described. These are made to work by the *constant* and *steady* exertion of a power, uniformly pressing upon that point of the machine at which it is applied, and *gradually* producing motion in the weight; but in this machine motion is accumulated in a hammer, by suffering it to descend from an altitude, and is *suddenly* by an *impulse* transferred to the wedge. In this case it must be evidently a useless labour to attempt to calculate the ratio of P to W, when they act by pressures, as in the other mechanical powers, and are in equilibrium. It is true, when we know this ratio, a slight increase* of P will gradually produce a motion in W, and

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* This, however, supposes the sides to be perfectly smooth, for otherwise the friction itself, without the assistance of any power at all, would preserve the equilibrium.

thus separate the obstacles; but this mode of working the machine is so widely different from that actually practised, that it would be a waste of time and labour to attempt an explication on Statical principles. The slightest stroke with a hammer is found to be far more effective than several tons of pressure. The only theoretical property of the wedge which agrees with practice is that its *advantage* is increased by diminishing its angle DBE.

All cutting instruments, such as knives, swords, hatchets, chisels, planes used by carpenters, nails, pins, needles, &c. are modifications of the wedge. Of these, knives, planes, pins and needles, are usually worked by pressure, but swords, hatchets, chisels, nails, &c. are worked by percussion.

CHAPTER VII.

ON FRICTION, AND THE RIGIDITY OF CORDS.

220. We have defined force to be that which can produce motion; this, however, does not embrace the forces to be considered in this chapter. And hence it is necessary to observe, that there are two kinds of forces, which have been denominated by some authors *active* and *passive*. The former appellation belongs to all those which fall under our definition of force; but the latter is applied to such forces, as having no energy in themselves to put a body in motion, do nevertheless constantly diminish existing motion, and oppose its production in a body at rest. That this opposition is of the nature of force, in those effects which it can produce, (and we know nothing of force but by its effects) is very manifest; for the same diminution of existing motion, and the same opposition to the production of new motion, which are the effects of *passive* forces, might be effected by employing *active* forces in directions opposite the existing motion in the one case, and to the direction in which there was a previous tendency to a production of motion in the other case. Since then, we might theoretically employ *active* for *passive* forces, the latter must be of the same *essential nature* as the former, so far as they have effects, but the former having a more extensive nature, embrace within their effects those

belonging to the latter. Care must therefore be taken, if it be found necessary to substitute theoretically *active* for *passive* forces in any investigation, that we suppose them divested of such effects as passive forces do not possess in common with them. Among the most familiar instances of passive forces we might enumerate, the resistance offered by the air, water, and all fluids, whether elastic or not, to the motion of bodies within them; and the resistance to rotatory and progressive motion in bodies which rub against surfaces with which they are in contact; this latter resistance is called friction, and is distinguishable into two kinds.

1st. *Statical* friction, or resistance to the *production* of motion in a quiescent body.

2ndly. *Dynamical* friction, or the resistance which diminishes *existing* motion.

Of these two kinds, since all machines are designed to work, the latter is of more importance in practical Mechanics; and it would accordingly not be improper to postpone the consideration of friction, till the student has made himself acquainted with some of the most elementary parts of Dynamics. Under this supposition, we shall not hesitate to introduce some simple considerations depending upon that branch of Mechanics.

221. Friction, as before observed, is the resistance of one surface to the motion of another upon it, and as there are three ways in which one surface can move upon another, it will be convenient to subdivide both Statical and Dynamical friction into three corresponding heads.

1st. When the surfaces in contact are two planes.

2ndly. When the surfaces in contact are a solid and a hollow cylinder.

3rdly. When a cylinder rolls (without rubbing) upon a plane.

The laws which govern the action of friction cannot be deduced from theoretical considerations, though these will render us great assistance in our researches by pointing out the experiments which are most likely to lead us to the discovery of them, as well as shewing the inconclusiveness of other experiments, on which we might otherwise be induced to rely. It is to be regretted, however, that the experiments which have been made upon the subject by different philosophers are frequently at variance; and, consequently, the theory cannot be said to have arrived at that state of perfection which is desirable.

222. The statical friction of plane surfaces is, under like circumstances, proportional to the pressure.

For let AB, *ab* (Fig. 80) be two planes in contact, placed in a horizontal position, the lower one AB being firmly fixed, but the upper one *ab* free to slide upon it. To *ab* attach a horizontal string *b*D passing over a pulley D, and having a dish C suspended from it. Load *ab* with a weight *w*, and denote the whole pressure of the plane *ab* on AB by W. Pour fine sand into the dish C until it begins to move, and then the weight of the dish and sand is the measure of the statical friction of the planes corresponding to the pressure W. If *ab* be loaded with more weights until the pressure is 2W, the friction is found to be double of what it was before; when the pressure is 3W, the friction is trebled; and so on. Wherefore the statical friction of plane surfaces is proportional to the pressure.

This result was confirmed by Coulomb and Ximines for very considerable pressures; in extreme cases, where the pressures were very large indeed, the friction was observed to be rather less in proportion than for small pressures; the deviation from the above law was however so small, even for extreme cases, that we shall not fall into any very considerable error, in supposing the law to be uninversally true.

The following method of establishing the property of the proportionality of the friction to the pressure, is very convenient for experiments.

Let the body W (Fig. 81) be placed upon an inclined plane AB, and then let the altitude BC be slowly increased until the plane has acquired such an elevation that W begins to slide down it; at this moment the friction just balances the weight W, and since it acts parallel to the plane in the direction AB, we may consider W as kept in equilibrium by a power in that direction,

$$\left.\begin{aligned}\frac{\text{friction}}{\text{W}} &= \sin A,\\ \frac{\text{W}}{\text{pressure}} &= \frac{1}{\cos A},\end{aligned}\right\}\ \text{(Art. 216)},$$

$$\therefore \frac{\text{friction}}{\text{pressure}} = \frac{\sin A}{\cos A} = \tan A;$$

$$\therefore \text{friction} = (\text{pressure}) \,.\, \tan A.$$

223. The fraction $\frac{\text{friction}}{\text{pressure}}$, is usually called the coefficient of friction, and is taken as its measure. It appears then, that in the last experiment the coefficient of friction is equal to the tangent of the inclination of the plane.

224. It being granted that the friction is proportional to the pressure when the surfaces are given, then, whatever be the magnitude of the surfaces in contact, the friction will remain the same, so long as the pressure is the same.

Let the body W (Fig. 81) have faces; whose areas are C and D square inches; then when the first face is in contact with the plane, the whole pressure is supported on C square inches, and therefore the pressure on each square inch, is equal to

$$\frac{\text{pressure}}{C};$$

and therefore the friction upon each square inch of surface

$$= \frac{\text{pressure}}{C} \cdot \tan A.$$

Consequently the friction upon the whole surface

$$= \frac{\text{pressure}}{C} \cdot \tan A \times \text{number of square inches},$$

$$= \frac{\text{pressure}}{C} \cdot \tan A \times C,$$

$$= (\text{pressure}) \cdot \tan A.$$

In the same way it may be shewn that the friction upon the second surface

$$= (\text{pressure}) \cdot \tan A,$$

and therefore the friction of a body is the same whether the surface on which it rests be large or small. When the surface is very small, the pressure on each square inch becomes very large, and then the friction, as observed in Art. 222, becomes somewhat less in proportion to the pressure; and therefore the friction is less, in a slight degree, when the body rests upon a small surface than a larger.

225. The Dynamical friction of plane surfaces is a uniformly retarding force; which diminishes as the pressure increases.

Let W (Fig. 82) be the body whose friction is to be determined; AB the plane on which the body presses; and let W be drawn along the plane by the weight P; and let F denote the friction. Then if F be supposed a uniform force, we shall derive the formula for the motion of W or P from the common principles of Dynamics.

Let g = the force of gravity; then the moving force which P exerts on W = P, and that which friction exerts is F;

$$\therefore \text{ the whole moving force on W} = \text{P} - \text{F},$$

and the mass moved is $\frac{\text{W} + \text{P}}{g}$, (Art. 122, Statics);

therefore the accelerating force on W or P = $\frac{\text{P} - \text{F}}{\text{W} + \text{P}} \cdot g$ (Art. 16, *Dynamics*); and therefore, if s be the space ascended by W or descended by P in the time t, we have, by Art. 53, *Dynamics*,

$$s = \tfrac{1}{2} \frac{\text{P} - \text{F}}{\text{W} + \text{P}} \cdot gt^2;$$

$$\therefore s \propto \frac{\text{P} - \text{F}}{\text{P} + \text{W}} \cdot t^2.$$

It appears then, that if the space varies as the square of the time, then the friction must be uniform; and if when P and W are increased or diminished in the same proportion, the space still bears the same proportion to the square of the time, and in that case

$$\frac{\text{P} - \text{F}}{\text{P} + \text{W}} = \text{constant};$$

$$\therefore \text{P} + \text{W} \propto \text{P} - \text{F};$$

$$\therefore \frac{\text{P}}{\text{W}} + 1 \propto \frac{\text{P}}{\text{W}} - \frac{\text{F}}{\text{W}};$$

but $\frac{\text{P}}{\text{W}}$ is constant in this case, and therefore

$$\frac{\text{F}}{\text{W}} \text{ is constant};$$

$$\therefore \text{F} \propto \text{W},$$

and therefore the Dynamical friction will be proportional to the pressure.

The late Professor Vince, having first premised, what we have just proved, that if the space described varies as the square of the time, then friction is a uniform force; and if, further, the space and time continue unchanged when P and W are increased or diminished in the same proportion, then the friction is proportional to the pressure, but otherwise not; instituted some experiments, from which we select the following, for the purpose of establishing the proposition at the head of this article. He adjusted a plane parallel to the horizon and so placed a pulley that it could be elevated or depressed, (see Fig. 82) in order to keep the string parallel to the plane. An accurately divided scale was placed near the pulley perpendicular to the horizon, by the side of which P descended. Upon this scale was a moveable stage, which the Professor adjusted to the space through which the moving force descended in any given time; the time was measured by a well regulated pendulum clock beating seconds.

1st. Experiment.* A body was placed upon the horizontal plane, and a moving force P applied, which from repeated trials was found to descend $52\frac{1}{2}$ inches in 4 seconds; (by the beat of the clock and the sound which P made when it struck against the stage, the space could by moving the stage be very accurately adjusted to the time). The stage was then removed to that point to which P would descend in 3 seconds, upon the supposition that the spaces described by P were proportional to the squares of the times, and this space was found to agree very accurately with the time. The stage was then removed to that point to which P ought to descend in 2 seconds on the same supposition as before; and this was also found to agree very accurately with the time. Lastly, the space was adjusted to that space which

* Gregory's Mechanics.

ought to be described in 1 second, and this was also found to agree very accurately with the time. Now in order to find whether a difference in the time of descent could be observed by removing the stage a little above and below the positions which corresponded to the above times, the experiment was tried, and the descent was always found too soon in the former case, and too late in the latter case; by which the Professor was assured that the spaces first mentioned corresponded exactly to the times. For the greater certainty, each descent was repeated eight or ten times. Other experiments were made with different bodies moving with different velocities, and results were always obtained confirmatory of the one just detailed; by which it was clearly established,—

That Friction is a uniformly retarding force.

This is however only true when the surfaces in contact are hard; for from experiments made with bodies covered with cloth, woollen, &c. the friction was found to increase with the velocity.

2nd. Experiment. When W was 10 ounces and P 4 ounces, then P descended through 51 inches in 2 seconds; but when W and P were loaded, so as to become 20 and 8 ounces, then P descended through 56 inches in 2 seconds; and, again, when W and P were increased to 30 and 12 ounces, P descended through 63 inches in 2 seconds; and therefore the friction was diminished, with respect to the pressure, as the pressure increased.

To determine the friction accurately in these cases, we must revert to the formula

$$s = \frac{g}{2} \frac{P - F}{P + W} \cdot t^2,$$

where $g = 32 . 19$ feet $= 386$ inches nearly.

Let f_1, f_2, f_3 be the coefficients of friction in the three cases respectively,

$$\therefore \text{ in the first, } 51 = \frac{g}{2} \cdot \frac{4 - 10f_1}{4 + 10} \cdot 4;$$

$$\therefore f_1 = .307.$$

$$\text{In the second, } 56 = \frac{g}{2} \cdot \frac{8 - 20f_2}{8 + 20} \cdot 4;$$

$$\therefore f_2 = .298.$$

$$\text{In the third, } 63 = \frac{g}{2} \cdot \frac{12 - 30f_3}{12 + 30} \cdot 4;$$

$$\therefore f_3 = .285.$$

From which it appears, that the coefficient of friction decreases as the pressure increases; and this result was confirmed by a great number of experiments with different pressures and velocities.

226. From this result, precisely as in Art. 224, it follows that a body will be less retarded when it slides with a small face in contact with a plane, than when it moves with a larger face in contact; but as this inference was directly at variance with the received opinions, Professor Vince deemed it necessary to confirm it by actual experiments.

3rd. Experiment. A body was taken whose flat side was to its edge as 22 : 9; and it described 33½ inches in 2 seconds on its flat side, and 47 inches in 2 seconds on its edge; P being the same in both cases.

4th. Experiment. Another body was taken, and one of its faces being covered with fine rough paper, it described on that face 25 inches in 2 seconds; but the paper being taken off from the middle of the face, so as only to leave a very small slip $\frac{3}{8}$ of an inch in breadth at the two ends, it described 40 inches in 2 seconds.

These and many other experiments, which it is not necessary to detail here, agreed in confirming the inference drawn from the preceding experiments.

227. In the same body Statical friction is greater than Dynamical friction; that is, it requires a greater force to put a body at rest in motion, than is requisite to preserve the motion undiminished when once it is produced.

This was thought by Professor Vince to arise from the cohesion of the body to the plane when it is at rest, and which does not happen when the body is in motion.

5th. Experiment. A body whose weight was 16 ounces was laid upon the plane (Fig. 82), and it was found that *after* the body had been put in motion, a power P of 4 ounces would continue the motion without acceleration; in this case, therefore, the friction must have just been equal to the accelerating force of P. But when the body was stopped, P could not put it in motion again until it had been increased by 2 ounces.

Other experiments of a similar kind were made, and they all agreed in shewing that the Statical friction is considerably greater than the Dynamical. This difference not having been sufficiently attended to by some philosophers, great discrepancies are found by comparing the results of their experiments with those of Professor Vince.

Remark. Notwithstanding, however, that these experiments appear to have been conducted with all possible care to guard against deception, yet Coulomb, who was supported in almost all his results by Ximines, has arrived at very different conclusions, from experiments apparently quite as carefully conducted. For fear of extending this chapter to too great a length, we shall merely give a few of the most important of his results, referring the reader for further satisfaction to Coulomb's original paper, in the 10th volume of the Memoires des Savans étrangers; and to the Terria e Pentica delle Resist. de sol ne' loro Attr. of Ximines.

228. (1). Both Statical and Dynamical friction are proportional to the pressures.*

(2). When a body of wood is first laid upon another, the Statical friction increases for a few minutes, when it attains its maximum, and no further alteration takes place. In making experiments, therefore, it is necessary to wait some time before the body is put in motion.

(3). Friction between substances of the same kind is greater than when they are of different kinds.

(4). The velocity has very little, if any influence, except when one body is composed of wood and the other of metal, in which case the resistance increases with the velocity.

It is also found that friction is diminished by oiling and polishing the surfaces in contact. There is a limit however to the latter, for if they be very highly polished, the resistance increases.

* There is a reason for preferring the result obtained by Professor Vince to this; for if a body be placed upon an inclined plane (as in Art. 222), not quite sufficiently elevated to overcome the friction; upon being loaded with a small additional weight, it will begin to descend without further elevation of the plane, which contradicts at once the principle of Coulomb, and confirms that of Vince.

229. The friction of cylinders rolling on planes, is proportional to their pressures directly and their radii inversely.

Let AB, CD (Fig. 83) be two planes, so placed that when a cylinder EF is laid upon them, it coincides with both. To the extremities of a string which is wrapped once or twice round the cylinder to prevent it from sliding, two equal scales G, H are attached. Into these equal weights are placed till the requisite degree of pressure of the cylinder upon the planes is produced. Fine sand is then to be poured into one of them until motion commences, its weight is then equal to the friction. We may also pour the sand into the other scale, and take a mean of the results, as a more accurate result. In this way Coulomb found, from numerous experiments,—

1st. That with the same cylinder and different pressures, the friction is proportional to the pressure.

2ndly. That with the same pressure and cylinders of different diameters, the friction is inversely proportional to their radii.

Wherefore, when both the pressures and radii are different, the friction varies as

$$\frac{\text{pressure}}{\text{radius}}.$$

230. It is remarkable, that friction of this kind, unlike that between two planes, is not diminished by greasing or oiling the surfaces of the planes and cylinder. Also the Statical friction is somewhat greater than the Dynamical friction. When a cylinder of mahogany, about 3 inches in diameter, was rolled upon a plane of oak, the coefficient of friction was .056; but when it was rolled upon a plane

of elm, the coefficient was not more than .01. This kind of friction, therefore, is much less than that between planes.

231. To determine the friction of solid cylinders revolving in hollow cylinders.

An apparatus for this purpose may be used, such as that exhibited in Fig. 84; any requisite degree of pressure may be produced by loading the scales with equal weights, and then the friction will be determined by pouring fine sand into one of the scales until motion ensues. The experiment must be tried with both scales, and the mean of the results taken. Let Fig. 85, be a projection of the apparatus on a plane at right angles to the cylinder. A, B the centres of the two cylinders; C the point of contact; pAq perpendicular to the strings pP, Qq; and AD parallel to them. Then the motion will commence by the point C of the solid cylinder sliding down the arc of the hollow cylinder. Let f be the coefficient of friction. To find the pressure at C, let w be equal to the weight of the cylinder, r equal to its radius; P and Q the weights in the scales, θ equal the angle CAD. Resolve P, Q and w in the direction AC, their sum is

$$(P + Q + w) \cos \theta,$$

which must be equal to the pressure at C.

And because there is an equilibrium, the sum of the moments of P, Q, and w about C, is equal to zero (Art. 92);

$$\therefore 0 = P(r - r \sin \theta) - wr \sin \theta - Q(r + r \sin \theta);$$

$$\therefore \sin \theta = \frac{P - Q}{P + Q + w} = \frac{\text{weight of the sand}}{P + Q + w}.$$

Hence the pressure $(P + Q + w) \cos \theta$ is known. The friction at $C = f(P + Q + w) \cos \theta$, and takes place in the direction of a tangent at C perpendicular to BC. Now resolving P, Q, and w in this direction, their sum is $(P + Q + w) \sin\theta$, which must be equal to the friction;

$$\therefore f(P + Q + w) \cos\theta = (P + Q + w) \sin\theta;$$

$$\therefore f = \tan\theta,$$

which is known because $\sin\theta$ is known.

By means of numerous experiments made with the apparatus of Fig. 84, the fraction

$$\frac{\text{weight of sand}}{P + Q + w}$$

is found to be nearly constant, except for large pressures, when it diminishes; wherefore f, which is equal to the tangent of that angle, of which this fraction is the sine, is nearly constant, but diminishes with large pressures. This is the Statical friction; but the Dynamical friction, which is the most useful in practice, is to be determined in a manner somewhat different, as follows. Suppose the friction to be uniform, then because the point A remains stationary during the motion, the angular motion of the cylinder takes place about A. Now $\frac{w}{g} \cdot \frac{r^2}{2}$ is the moment of inertia of the cylinder; and $\frac{P + Q}{g} \cdot r^2$ is the moment of inertia of P and Q; also the moment of the forces affecting rotation about A, is

$$Pr - Qr - Fr,$$

F denoting the friction; wherefore the accelerating force on P's descent

$$= \frac{(P - Q - F)\, r^2}{\frac{P + Q}{g} \cdot r^2 + \frac{w}{g} \cdot \frac{r^2}{2}} = \frac{P - Q - F}{P + Q + \frac{1}{2}w} \cdot g,$$

which being constant, we shall have

$$s = \frac{1}{2} \cdot \frac{P - Q - F}{P + Q + \frac{1}{2}w} \cdot gt^2$$

for the relation between the space and time; wherefore if the friction is constant, the space descended by P varies as the square of the time.

Now, by experiment, it is found that the space descended by P does vary very nearly as the square of the time, and therefore the friction is nearly uniform. Coulomb, in his experiments, made the hollow cylinder revolve; his apparatus was similar to that of Fig. 86. This is the case of the wheels of carriages, which revolve round the axles. He also deduced, from his experiments, that the friction is considerably less when the two cylinders are of different substances, than when of the same. Hence the propriety of having the axles of wheels in clocks, watches, and other machines, which are of steel, to run in brass.

We may remark, with respect to the mechanical powers;

1st. That in the way the lever is commonly used, it is not subject to friction.

2ndly. That in the pulley the Dynamical friction is very great, on account of the rubbing of the sides of the pulley against its block.

3rdly. That in the wheel and axle, the Dynamical friction is very small, and is nearly proportional to the pressure × radius of the axis × angular velocity.

4thly. That in the inclined plane, the Dynamical friction is not very great, being generally intermediate to the two last mentioned; and it is of two kinds, according as the body slides or rolls; the former being very much greater than the latter.

5thly. That in the screw the Dynamical friction is large, being frequently sufficient to prevent the recoil of the weight after the power ceases to act; there is, however, less friction with a square than a triangular thread.

6thly. That in the wedge, the friction, as already explained, is by far the greatest force to be overcome; it however diminishes with the angle of the wedge.

Remark. Besides the various kinds of friction here treated of, Coulomb has made a great number of experiments on the friction of pivots, but as the subject is rather long, and does not seem to admit of compression, we must beg leave to refer to the original paper, in the Memoirs of of the French Academy, for the year 1790.

232. To explain the effect of the rigidity of cordage upon pullies, wheel and axles, &c.

We have hitherto, for reasons before mentioned, considered ropes and cords as perfectly flexible, and this supposition will not be very erroneous, when the work to be done by a machine is so light, that very fine silken threads may be used with safety. But upon a more enlarged scale, the cords have a considerable degree of rigidity; and, consequently, it becomes necessary to take into consideration the effect of this rigidity upon pullies, capstans and other machines, in which ropes are used.

The theory of rigidity, which has been invented for this case, is fortunately found to be much more satisfactory than any of the theories of friction which have hitherto been discovered.

Let P and W (Fig. 87) be the power and weight, suspended at the ends of a cord passing over the pulley AB, whose centre is C. Draw the horizontal diameter ACB, then when P descends, drawing up W, A and B will still be the extreme points of contact of the rope with the pulley, which will now take the form exhibited in Fig. 88, in consequence of its rigidity. Draw aCb a horizontal diameter in this position, and Pp, Ww perpendicular to it. We may transmit the forces P and W, which act at P and W, to the points p and w; and, consequently, the effect of rigidity is to increase the arm at which the weight acts by the quantity bw, and diminish that at which the power acts by the quantity ap. Both these lessen the mechanical advantage of the machine; but by numerous experiments, it is found that ap is so very small, that its effect may be neglected; we have, consequently, only to determine bw, by which the arm at which the weight acts is lengthened.

To this end it is necessary to observe, that the magnitude of bw depends upon three things,—

1st. The tension of the rope.

2ndly. Its materials, and the manner in which it has been twisted in the making of it.

3rdly. Its diameter.

Now denote bw by x, the radius of the rope by r, and the radius of the cylinder over which the rope is bent by R.

Then (Fig. 88) $P \,.\, aC = W \,.\, Cw$,

$$\text{or } P \,.\, R = W \,.\, (R + x),$$

$$P - W = W \,.\, \frac{x}{R}.$$

Now $P - W$ is the force necessary to overcome the rigidity of the rope, and put the apparatus in motion; but this force is rendered necessary, in consequence of the three causes above mentioned; of which the portion arising from the first (tension), varies as W, and may be represented by bW; that arising from the second, is evidently constant for the same rope, and may be represented by a; and that from the third, depends on r, and may be proportional to some power of it (r^n); upon the whole we may represent this force by

$$\frac{r^n}{R}(a + bW),$$

which must therefore be equal to $W \cdot \frac{x}{R}$;

$$\therefore r^n (a + bW) = Wx;$$

$$\therefore x = \frac{r^n}{W}(a + bW).$$

a, b, n being constant quantities for the same rope, to be determined by experiment. This formula, though empirical is found to agree very accurately with the results deduced from experiments.

In his experiments on this subject, Coulomb employed an apparatus (Fig. 89) similar in principle to that of Fig. 83. Having first estimated the friction of the cylinder by the method there pointed out, and changed the thread to which the scales are suspended for the rope; he added small weights to one of the scales, uutil an extremely slow motion commenced.

Let W_1, W_2, W_3 be the values of W in three experiments with the same rope, passing over cylinders, whose radii are R_1, R_2, R_3 respectively; and let Q_1, Q_2, Q_3 be the corresponding additional weights added before motion commences;

$$\therefore Q_1 = \frac{r^n}{R_1}(a + bW_1),$$

$$Q_2 = \frac{r^n}{R_2}(a + bW_2),$$

$$Q_3 = \frac{r^n}{R_3}(a + bW_3).$$

These three equations will enable us to determine a, b and n in terms of known quantities; and then x is known from the equation

$$x = \frac{r^n}{W}(a + bW),$$

for any value of W whatever. If we make a fourth experiment, we shall have

$$Q_4 = \frac{r^n}{R_4}(a + bW_4),$$

which will serve as a criterion of the accuracy of the empirical form, which is made the foundation of the investigation.

The value of n was found to be very nearly equal to $\sqrt{3}$, but it gradually diminished as the cord was worn, until it reached $\sqrt{2}$; and the value of x, or the effect of the rigidity, was then nearly constant for all velocities; and, consequently, there is always a constant part of the power spent in surmounting the friction and stiffness of the cords employed.

CHAPTER VIII.

MISCELLANEOUS PROBLEMS.

(1). Three forces acting on a point are found to be in equilibrium, when their directions make angles 105°, 120° and 135° with each other. Find the proportion of the forces to each other.

In Art. 28, it was shewn that any one of the forces is proportional to the sine of the angle between the directions of the other two; if therefore F_1, F_2, F_3 are the forces, and 105°, 120°, 135° the angles to which their directions are opposite, we shall have

$$F_1 : F_2 : F_3 :: \sin 105^\circ : \sin 120^\circ : \sin 135^\circ,$$

$$:: \cos 15^\circ : \cos 30^\circ : \cos 45^\circ,$$

$$:: \frac{\sqrt{3}+1}{2} : \frac{\sqrt{3}}{2} : \frac{\sqrt{2}}{2},$$

$$:: \sqrt{3}+1 : \sqrt{3} : \sqrt{2};$$

$$\therefore \frac{F_1}{\sqrt{3}+1} = \frac{F_2}{\sqrt{3}} = \frac{F_3}{\sqrt{2}};$$

from which, when the magnitude of any one of the forces is given, the magnitudes of the others will be known.

(2). A cord PAQ (Fig. 90) is knotted to a fixed point A, and drawn in different directions by forces P and Q, such that the pressure on A is an arithmetic mean between the forces. Required the angle PAQ.

The pressure on A must be equal and opposite (Art. 20) to the resultant P and Q, otherwise there would not be an equilibrium; and as it is, by the question, equal to $\frac{1}{2}(P+Q)$, we have, by Art. 26,

$$\tfrac{1}{4}(P+Q)^2 = P^2 + 2PQ \,.\, \cos PAQ + Q^2;$$

$$\therefore \tfrac{3}{4}(P^2+Q^2) = -\tfrac{3}{2}PQ \cos PAQ = \tfrac{3}{2}PQ . \cos(\pi - PAQ);$$

$$\therefore \cos(\pi - PAQ) = \tfrac{1}{2}\left(\frac{P}{Q} + \frac{Q}{P}\right);$$

from which equation the angle PAQ will be known.

(3). A weight W is sustained upon an inclined plane by three forces, each equal to $\frac{1}{3}$W, one acting vertically upwards, another parallel to the plane, and the third horizontally; required the inclination of the plane.

Let F_1, F_2, F_3 (Fig. 91) be the forces, θ equal the inclination of the plane to the horizon. The weight W is kept in equilibrium by five forces F_1, F_2, F_3, the force of gravity W, and the reaction R of the plane.

Now by Art. 40, the sum of the resolved parts of these forces in any direction is equal to zero; and since they are all known, except the reaction, which is perpendicular to the plane, resolve them in a direction at right angles to R; that is, parallel to the plane, in order that the resolved part of the unknown force R may disappear in the resulting equation.

Wherefore, resolving F_1, F_2, F_3, W, R parallel to the plane, the resolved parts are respectively,

$F_1\cos(90^\circ - \theta)$, $F_2\cos 0$, $F_3\cos\theta$, $-W\cos(90^\circ - \theta)$, $R\cos 90^\circ$,
or $F_1 \sin\theta$, F_2 , $F_3\cos\theta$, $-W\sin\theta$, 0.

Wherefore,

$$F_1 \sin\theta + F_2 + F_3 \cos\theta - W \sin\theta = 0;$$

$$\therefore \tfrac{1}{3} W \sin\theta + \tfrac{1}{3} W + \tfrac{1}{3} W \cos\theta - W \sin\theta = 0;$$

$$\therefore 1 + \cos\theta = 2\sin\theta;$$

$$\therefore 2\cos^2\frac{\theta}{2} = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2};$$

$$\therefore \tan\frac{\theta}{2} = \tfrac{1}{2}, \text{ by dividing by } 2\cos^2\frac{\theta}{2};$$

$$\therefore \theta = 2\tan^{-1}\tfrac{1}{2},$$
$$= 2 \times 26^\circ\, 33'\, 54'',$$
$$= 53^\circ\, 7'\, 48'',$$

the inclination required.

(4). A given sphere rests upon two inclined planes; to find the pressure upon each.

Let AB, AD (Fig. 92) be the planes; θ, ϕ their respective inclinations; C the centre of the sphere, W its weight; B, D the points against which it presses.

Now the sphere may be supposed collected at its centre of gravity C (Art. 131); and since the reactions of the planes at B, D, take place in directions at right angles to the planes (Art. 117), and therefore passing through C, we may transmit all the forces to C. Wherefore, denoting the pressures at B and D, by R_1, R_2, we shall have, by Art. 28,

$$R_1 : R_2 : W :: \sin R_2CW : \sin R_1CW : \sin R_1CR_2.$$

Now $\angle R_2CW = 90^\circ + \theta$, as will be evident by drawing a line from C parallel to the horizon;

also $\angle R_1CW = 90^\circ + \phi$,

and $\angle R_1CR_2 = DCB = 180^\circ - DAB = \phi + \theta$;

$$\therefore R_1 : R_2 : W :: \sin(90^\circ + \theta) : \sin(90^\circ + \phi) : \sin(\phi + \theta),$$
$$:: \cos\theta : \cos\phi : \sin(\phi + \theta);$$

$$\therefore R_1 = \frac{W\cos\theta}{\sin(\phi + \theta)},$$

$$\text{and } R_2 = \frac{W\cos\phi}{\sin(\phi + \theta)}.$$

(5). Two weights support each other, upon two given inclined planes, having a common vertex, by means of a string passing over a pulley at the common intersection of the planes; required the proportion of the weights; the parts of the strings being parallel to the planes (Fig. 93).

Let a, β be the inclinations of the two planes; P, Q the weights resting upon them. Then it is evident that each of the weights is hindered from falling down the plane on which it rests by the tension of the same string; if therefore T be this tension, since it acts parallel to the plane in each case, we have, by Art. 216,

$$T = P \sin a,$$

$$\text{and } T = Q \sin \beta;$$

$$\therefore P \sin a = Q \sin \beta;$$

$$\therefore \frac{P}{Q} = \frac{\sin \beta}{\sin a} = \frac{AC}{BC};$$

that is, each weight is proportional to the length of the plane on which it rests.

(6). Three equal forces act upon a particle, so that their directions include angles 105°, 120°, and 135°; to find the magnitude and position of their resultant.

Let F_1, F_2, F_3 (Fig. 94) be the forces acting at the point O; take OF_1 for the axis of x, and OY perpendicular to it for that of y; and let the resultant R make an angle a with Ox.

Then the resolved part of the forces in the direction Ox

$$= F_1 \cos 0^\circ + F_2 \cos F_2Ox + F_3 \cos F_3Ox,$$

$$= F_1 + F_2 \cos 120^\circ + F_3 \cos 135^\circ,$$

$$= F_1 - F_2 \sin 30^\circ - F_3 \sin 45^\circ,$$

$$= F_1\left(1 - \tfrac{1}{2} - \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2} - 1}{2} . F_1,$$

which must be equal to R cos a.

And in a similar manner, resolving the forces in the direction Oy, we obtain

$$R \sin a = F_1 \sin 0^\circ + F_2 \sin F_2Ox - F_3 \sin F_3Ox,$$
$$= F_2 \sin 120^\circ - F_3 \sin 135^\circ,$$
$$= F_2 \cos 30^\circ - F_3 \cos 45^\circ,$$
$$= F_1 \cdot \frac{\sqrt{3}}{2} - F_1 \cdot \frac{\sqrt{2}}{2},$$
$$= F_1 \cdot \frac{\sqrt{3} - \sqrt{2}}{2};$$
$$\therefore \tan a = \frac{R \sin a}{R \cos a} = -\frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1};$$
$$\therefore \tan(180^\circ - a) = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2,$$
$$= .7673269$$
$$= \tan 37^\circ\, 30';$$
$$\therefore 180^\circ - a = 37^\circ\, 30';$$
$$\therefore a = 142^\circ\, 30',$$

which determines the direction of the resultant.

$$\text{Again,} \quad R^2 = R^2 \cos^2 a + R^2 \sin^2 a,$$
$$= \frac{3 - 2\sqrt{2}}{4} \cdot F_1^2 + \frac{5 - 2\sqrt{6}}{4} \cdot F_1^2,$$
$$= F_1^2 \cdot \left\{2 - \frac{\sqrt{6} + \sqrt{2}}{2}\right\},$$
$$= F_1^2 \times .0681484;$$
$$\therefore R = F_1 \times .2610525,$$

which determines the magnitude of the resultant.

(7). Two weights P, Q are connected by a string passing over two pullies A, B, situated in a horizontal line; and support a weight W which hangs from a ring C, which slides upon the string AB; to determine the position of equilibrium (Fig. 95).

Since both P and Q are supported by the tension of the same string, we must have

$$P = \text{tension},$$

$$\text{and } Q = \text{tension};$$

$$\therefore P = Q.$$

Also, we may consider the point C as kept at rest by the tensions of BC and AC, and the weight W; and, therefore, any one of these forces is proportional to the sine of the angle between the directions of the other two; wherefore the angle ACW = angle BCW, and consequently the string from which W hangs being produced bisects the angle ACB; wherefore, if ACB = θ, then

$$BCW = \pi - \frac{\theta}{2}, \text{ and}$$

$$P : W :: \sin BCW : \sin ACB,$$

$$:: \sin\left(\pi - \frac{\theta}{2}\right) : \sin\theta,$$

$$:: \sin\frac{\theta}{2} : 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2},$$

$$:: 1 : 2\cos\frac{\theta}{2};$$

$$\therefore P = \tfrac{1}{2} W \sec\frac{\theta}{2}.$$

Hence if it were possible for ACB to be a straight line, we should have $\frac{\theta}{2} = 90°$, and $\therefore \sec\frac{\theta}{2} = \infty$; and, consequently, P is infinitely greater than W. Whence we infer that it is impossible for any weights, however great they may be, to stretch the cord ACB until it becomes straight.

We may also remark, that since $\sec\frac{\theta}{2}$ is necessarily greater than 1,

$$P > \tfrac{1}{2} W.$$

And therefore an equilibrium will be impossible, unless P and Q are together greater than W.

(8). Two weights P and Q of 3 and 4 lbs. respectively, are suspended from a bent lever ACB (Fig. 96), whose fulcrum is C, and arms AC, BC are 15 and 12 inches; to find the position of equilibrium; the angle ACB at which the arms are inclined being 120°.

From C draw Ca, Cb perpendicular to the directions of the strings AP, BQ; therefore the condition of equilibrium is

$$P . Ca = Q . Cb, \text{ (Art. 196).}$$

Now to express this equation in terms of known quantities, let ACa = $\alpha + \theta$, and BCb = $\alpha - \theta$, and therefore ACB $= 180° - (\alpha + \theta) - (\alpha - \theta) = 180° - 2\alpha = 120°$;

$$\therefore 2\alpha = 180° - 120° = 60°;$$

$$\therefore \alpha = 30°.$$

Also, let CA = a, CB = b, and consequently Ca = $a \cos(\alpha + \theta)$, and Cb = $b \cos(\alpha - \theta)$;

$$\therefore P . a \cos(\alpha + \theta) = Q . b \cos(\alpha - \theta);$$

$$\therefore Pa(\cos\alpha \cos\theta - \sin\alpha \sin\theta) = Qb.(\cos\alpha \cos\theta + \sin\alpha \sin\theta);$$

$$\therefore Pa(1 - \tan\alpha \tan\theta) = Qb(1 + \tan\alpha \tan\theta),$$

by dividing by $\cos\alpha \cos\theta$;

$$\therefore \tan\theta = \frac{Pa - Qb}{Pa + Qb} \cdot \cot\alpha,$$

$$= \frac{3 . 15 - 4 . 12}{3 . 15 + 4 . 12} \cdot \cot 30°,$$

$$= -\frac{3}{93} \cdot \sqrt{3},$$

$$= -\frac{\sqrt{3}}{31};$$

$$\therefore \tan(-\theta) = \frac{\sqrt{3}}{31} = .0558726,$$

$$= \tan 3^\circ\ 11'\ 53'';$$

$$\therefore \theta = -3^\circ\ 11'\ 53'';$$

$$\therefore ACa = a + \theta = 26^\circ\ 48'\ \ 7'',$$

$$\text{and } BCb = a - \theta = 33^\circ\ 11'\ 53'',$$

which are the inclinations of the arms of the lever to the horizon.

(9). AE is a straight lever weighing $3\frac{1}{2}$ lbs., at the points A, B, C, D, E hang weights equal to 3, 7, 1, 5 and 2 lbs. respectively; required the point O, on which the whole will rest in equilibrium; AB, BC, CD, DE being equal to 8, 6, 2, 10 inches respectively (Fig. 97).

Let G be the centre of gravity of the lever, W its weight, $F_1, F_2 \ldots F_5$ the weights suspended from it. Then we may suppose the matter of the lever suspended at G (Art. 131); and, therefore, by Art. 198,

$$F_1 . OA + F_2 . OB = F_3 . OC + F_4 . OD + F_5 . OE + W . OG,$$

$$\text{or } F_1 . OA + F_2 . (AO - AB) = F_3 . (AC - AO)$$
$$+ F_4 (AD - AO) + F_5 (AE - AO) + W (AG - AO);$$

$$\therefore AO = \frac{F_2 . AB + F_3 . AC + F_4 . AD + F_5 . AE + W . AG}{F_1 + F_2 + F_3 + F_4 + F_5 + W}.$$

Now $AG = \frac{1}{2} AE = 13$ inches;

$$\therefore AO = \frac{7.8 + 1.14 + 5.16 + 2.26 + 3\frac{1}{2}.13}{3 + 7 + 1 + 5 + 2 + 3\frac{1}{2}},$$

$$= \frac{495}{43} = 11\frac{22}{43} \text{ inches.}$$

(10). LM is a sphere whose radius is 6 inches and weight $3\frac{1}{2}$ lbs., upon the plane AM, inclined to the horizon at an angle 60°; AB is a beam whose weight is 100 lbs. and length

6 feet, moveable about a hinge at A, and by its pressure on the sphere preventing it from rolling down the plane. Determine the position of the beam and sphere (Fig. 98).

Let W = weight of the beam which we may suppose collected at its centre of gravity G. w = that of the ball; $AB = 2a$, and r = radius of the ball; α = the inclination of the plane to the horizon, and $2\theta = BAM$; C the centre of the sphere; L, M the points of contact.

The beam AB is kept in equilibrium about the fulcrum A by its gravity (= W) acting at G, and the reaction R at L. Now $\frac{LC}{AL} = \tan\theta$; $\therefore AL = r\cot\theta$; and, therefore, the moment of R about A $= R.r\cot\theta$; and that of $W = W.AG.\cos(\alpha + 2\theta) = W.a\cos(\alpha + 2\theta)$;

$$\therefore Rr\cot\theta = Wa\cos(\alpha + 2\theta);$$

$$\therefore R = \frac{Wa}{r}.\tan\theta\cos(\alpha + 2\theta).$$

Again, the sphere is kept at rest by the reactions at L and M and its own weight w, all of whose directions pass through C, they may therefore be transmitted to that point; wherefore resolving these forces parallel to the plane AM, in order that the resolved part of the unknown reaction at M may disappear, we have

$$R\sin 2\theta - w\sin\alpha = 0, \text{ Art. } 40;$$

$$\therefore \frac{Wa}{r}.\sin 2\theta\tan\theta\cos(\alpha + 2\theta) = w\sin\alpha;$$

$$\therefore 2\sin^2\theta\cos(\alpha + 2\theta) = \frac{wr}{Wa}.\sin\alpha;$$

$$\therefore 2\log\sin\theta + \log\cos(\alpha + 2\theta) = \log\left(\frac{wr}{2Wa}\right) + \log\sin\alpha + 20.$$

Now $\cos(\alpha + 2\theta) = \cos(60^\circ + 2\theta) = \sin(30^\circ - 2\theta)$;

therefore

$$2 \log \sin \theta + \log \sin (30^\circ - 2\theta) = \log \frac{7}{2400} + \log \sin 60^\circ + 20,$$

$$= 27.4024174.$$

By approximation we find that the value of θ, which fulfils this equation, is

$$\theta = 4^\circ\ 54'\ 30'' \text{ nearly},$$

$$\text{and therefore } \alpha + 2\theta = 69^\circ\ 49',$$

which is the inclination of the beam to the horizon.

The position of the sphere is known from the equation

$$\text{AM} = \text{AL} = r \cot \theta,$$

$$= \tfrac{1}{2} \cot 4^\circ\ 54'\ 30'',$$

$$= 5.822314 \text{ feet}.$$

(11.) A uniform beam rests with its ends upon two planes inclined to the horizon at angles 45° and 30°; to determine the position of equilibrium, (Fig. 99).

Let AB be the beam, G its centre of gravity; α, β the inclinations of the two planes, and θ that of the beam to the horizon; R, R′ the reactions at A and B. Then we may consider AB as a rigid body kept at rest by the forces R, R′ and its own weight at G. Resolve these forces parallel to the horizon,

$$\therefore \text{R} \sin \alpha - \text{R}' \sin \beta = 0;$$

$$\therefore \frac{\text{R}}{\text{R}'} = \frac{\sin \beta}{\sin \alpha};$$

also, the moments about an axis through G must be equal to zero, (Art. 92).

$$\text{Now the moment of R} = \text{R} . \text{AG} \sin \text{GAR},$$

$$= \text{R} . \text{AG} . \cos \text{GAO},$$

$$= \text{R} . \text{AG} \cos (\alpha + \theta),$$

for GAO $= \alpha + \theta$, as will be found by drawing from A a line parallel to the horizon.

Also, the moment of R′ = − R′ . BG cos (β − θ):

$$\therefore \text{R.AG} \cos (\alpha + \theta) - \text{R}' . \text{BG} \cos (\beta - \theta) = 0;$$

$$\therefore \frac{\text{BG}}{\text{AG}} \cdot \frac{\cos(\beta - \theta)}{\cos(\alpha + \theta)} = \frac{\text{R}}{\text{R}'} = \frac{\sin\beta}{\sin\alpha};$$

$$\therefore \text{BG}.\frac{\cos(\beta - \theta)}{\sin\beta\cos\theta} = \text{AG}.\frac{\cos(\alpha + \theta)}{\sin\alpha\cos\theta},$$

or $\text{BG} (\cot\beta + \tan\theta) = \text{AG} (\cot\alpha - \tan\theta)$;

$$\begin{aligned} \therefore \tan\theta &= \frac{\text{AG}}{\text{AB}}.\cot\alpha - \frac{\text{BG}}{\text{AB}}.\cot\beta, \\ &= \tfrac{1}{3}\cot 30^\circ - \tfrac{1}{2}\cot 45^\circ, \\ &= \frac{\sqrt{3} - 1}{2}, \\ &= .3660254, \\ &= \tan 20^\circ\, 6'\, 14''; \\ \therefore \theta &= 20^\circ\, 6'\, 14'', \end{aligned}$$

the inclination of the beam to the horizon.

(12.) A weight W (Fig. 100) is suspended from one extremity of a string, which passes through a ring C at its other extremity; to find the position of equilibrium; the string passing over two pullies A and B.

Since the tension of the string is the same in every part, we may consider the point C as being acted upon by three forces, each equal to the tension W; wherefore the angles between the strings at C must be all equal, (Art. 28), and therefore each equal to 120°. To determine the position of C draw A*a*, B*b* perpendicular to the horizon, and make the angles CA*a* and CB*b* each = 60°, and the point C in which AC and BC intersect will be the point required.

(13.) Let AC (Fig. 101) be a curve in a vertical plane; P, Q weights attached to a string passing over a pulley B in its axis BA*x*; to determine the position of equilibrium.

Draw QM perpendicular to Bx, and let QG be a normal at Q. $x = \text{BM}$, $y = \text{QM}$. Then Q is kept at rest by the action of three forces; viz. the tension (= P) of the string QB, the gravity of Q (= Q) in a direction parallel to BG, and the reaction (= R) in the direction GQ; these three directions are parallel to the sides of the triangle GBQ; its sides therefore represent the magnitudes of the forces, (Art. 28);

$$\therefore \frac{\text{P}}{\text{Q}} = \frac{\text{BQ}}{\text{BG}} = \frac{\sqrt{x^2 + y^2}}{x + yd_x y},$$

for $\text{BG} = \text{BM} + \text{MG} = x + yd_x y$. This formula will be simplified by representing BQ by r, for then

$$r = \sqrt{x^2 + y^2},$$

$$\text{and } rd_x r = x + yd_x y;$$

$$\therefore \frac{\text{P}}{\text{Q}} = \frac{r}{rd_x r} = \frac{dx}{dr},$$

$$\therefore \text{P}dr = \text{Q}dx.$$

REMARK. We might have deduced this result immediately from the principle of virtual velocities; for BP = length of string $- r$,

$$\therefore d_t(\text{BP}) = -d_t r = \text{virtual velocity of P},$$

$$\text{and } d_t x = \text{virtual velocity of Q};$$

$$\therefore \text{P}(-d_t r) + \text{Q}d_t x = 0;$$

$$\therefore \text{P}d_t r = \text{Q}d_t x;$$

$$\therefore \text{P}dr = \text{Q}dx.$$

1st. As an application of this general formula, let us suppose AC a parabola, and B a point in the directrix.

Then the equation of a parabola is

$$y^2 = 4m\,(\text{AM}),$$

$$= 4m\,(x - m);$$

$$\therefore r^2 = x^2 + y^2 = x^2 + 4mx - 4m^2;$$

$$\therefore rdr = xdx + 2mdx;$$

$$\therefore \frac{x + 2m}{r} = \frac{dr}{dx} = \frac{Q}{P};$$

$$\therefore \frac{P}{Q}(x + 2m) = r;$$

$$\therefore \frac{P^2}{Q^2}(x + 2m)^2 = r^2 = x^2 + 4mx - 4m^2,$$

$$= (x + 2m)^2 - 8m^2;$$

$$\therefore \left(1 - \frac{P^2}{Q^2}\right)(x + 2m)^2 = 8m^2;$$

$$\therefore x = 2m\left\{\frac{Q\sqrt{2}}{\sqrt{Q^2 - P^2}} - 1\right\}.$$

2ndly. Suppose B the focus, then the equation of the parabola becomes

$$y^2 = 4m \,.\, \text{AM (Fig. 102)},$$

$$= 4m(x + m);$$

$$\therefore r^2 = x^2 + y^2 = x^2 + 4mx + 4m^2;$$

$$\therefore r = x + 2m;$$

$$\therefore dr = dx;$$

$$\therefore P = Q;$$

wherefore, the weights must be equal, and then they will balance in every position.

3rdly. Let the curve be a quadrant of a circle (Fig. 103).

Let b be the centre, bm vertical, $Bb = a$, produce MQ to m. Then the equation of the circle is

$$a^2 = (bm)^2 + (mQ)^2,$$

$$= (BM)^2 + (Bb - MQ)^2,$$

$$= x^2 + (a - y)^2;$$

$$\therefore y = a - \sqrt{a^2 - x^2};$$

$$\therefore r^2 = x^2 + y^2 = x^2 + a^2 - 2a\sqrt{a^2 - x^2} + a^2 - x^2,$$

$$= 2a^2 - 2a\sqrt{a^2 - x^2} = 2ay;$$

$$\therefore rdr = \frac{axdx}{\sqrt{a^2 - x^2}};$$

$$\therefore \frac{\mathrm{Q}}{\mathrm{P}} = \frac{dr}{dx} = \frac{ax}{r\sqrt{a^2 - x^2}} = \frac{a\sqrt{2ay - y^2}}{\sqrt{2ay}\,.\,(a - y)},$$

by substituting for r and x their values in terms of y.

$$\therefore \frac{\mathrm{Q}}{\mathrm{P}} = \frac{a}{a - y}\cdot\left(1 - \frac{y}{2a}\right)^{\frac{1}{2}};$$

$$\therefore \frac{\mathrm{Q}^2}{\mathrm{P}^2}(a - y)^2 = a^2 - \tfrac{1}{2}\,ay;$$

$$\therefore y^2 - \left(2 - \frac{\mathrm{P}^2}{2\mathrm{Q}^2}\right)ay = \left(\frac{\mathrm{P}^2}{\mathrm{Q}^2} - 1\right)a^2;$$

$$\therefore y = a - \frac{\mathrm{P}^2}{4\mathrm{Q}^2}\cdot\left\{1 + \left(\frac{8\mathrm{Q}^2}{\mathrm{P}^2} + 1\right)^{\frac{1}{2}}\right\}.a.$$

(14). If the weight P, instead of hanging perpendicularly, rests upon a curve $a\mathrm{P}c$ (Fig. 104), we shall find at once, by the principle of virtual velocities, that

$$\mathrm{P}dx' + \mathrm{Q}dx = 0,$$

where $x' = \mathrm{B}m$, and $x = \mathrm{BM}$.

For the virtual velocity of P is $d_{,}x'$, and of Q is $d_{,}x$, and

$$\therefore \mathrm{P}d_{,}x' + \mathrm{Q}d_{,}x = 0;$$

$$\therefore \mathrm{P}dx' + \mathrm{Q}dx = 0.$$

To this we must join the following, from the nature of the machine,

$$\sqrt{x'^2 + y'^2} + \sqrt{x^2 + y^2} = l,$$

y' being equal to $\mathrm{P}m$, and l to the length of the string.

Let Ac (Fig. 105) be a circle, AC a parabola, and B a point in its directrix. Then the equation of AC is

$$y^2 = 4m(x - m),$$

and that of Ac is

$$y'^2 = 2a(x' - m) - (x' - m)^2;$$

$$\therefore l = \sqrt{x'^2 + y'^2} + \sqrt{x^2 + y^2}$$

$$= \sqrt{2a(x' - m) + 2mx' - m^2} + \sqrt{x^2 + 4mx - 4m^2};$$

$$\therefore 0 = \frac{(a + m)\,dx'}{\sqrt{2(a + m)\,x' - 2am - m^2}} + \frac{(x + 2m)\,dx}{\sqrt{x^2 + 4mx - 4m^2}}$$

$$= \frac{(a + m)\,dx'}{\text{BP}} + \frac{(x + 2m)\,dx}{\text{BQ}};$$

$$\therefore \frac{\text{P}}{\text{Q}} = -\frac{dx}{dx'} = \frac{a + m}{x + 2m} \cdot \frac{\text{BQ}}{\text{BP}}$$

$$= \frac{a + m}{x + 2m} \cdot \frac{\sqrt{x^2 + 4mx - 4m^2}}{l - \sqrt{x^2 + 4mx - 4m^2}},$$

from which equation x is to be determined.

(15). The beam CD (Fig. 106) rests with one end D upon a given inclined plane DB, and the other is suspended by a string from a fixed point A; to find its position.

Draw AB perpendicular to the plane; and let AC, CD make with AB angles equal to ϕ, θ; and let α be the inclination of DB to the horizon; let G be the centre of gravity of the beam; put CG = DG = a; AC = b, AB = c, R = the reaction at D, and T = the tension of AC. Then, if from C a line be drawn parallel to DB, we shall have, from the nature of the machine,

$$a \cos\theta + b \cos\phi = c \;.\;.\;.\;.\; (1).$$

The forces which kept the beam at rest are R, T and its own weight; resolving these parallel to the horizon, in order that the weight of the beam may disappear, we have

$$\text{R} \sin\alpha - \text{T} \sin(\phi - \alpha) = 0 \;.\;.\;.\;.\; (2).$$

Taking the moments about G, in order that the weight of the beam may disappear, we have, the moment of R about $G = R . DG \sin\theta = R . a \sin\theta$; and that of T about $G = -T . GC \sin(\theta - \phi) = -T . a \sin(\theta - \phi)$.

$$\therefore R . a \sin\theta - T . a \sin(\theta - \phi) = 0, \text{ (Art. 92).}$$

Eliminating R and T between this and equation (2), we have

$$\frac{\sin(\theta - \phi)}{\sin\theta} = \frac{R}{T} = \frac{\sin(\phi - a)}{\sin a};$$

$$\therefore \cos\phi - \cot\theta . \sin\phi = \cot a . \sin\phi - \cos\phi;$$

$$\therefore \cot\phi = \tfrac{1}{2}(\cot a + \cot\theta);$$

$$\therefore \cos^2\phi = \frac{(\cot a + \cot\theta)^2}{4 + (\cot a + \cot\theta)^2}.$$

But from equation (1) we have

$$b \cos\phi = c - a \cos\theta;$$

$$\therefore (c - a\cos\theta)^2 = \frac{b^2(\cot a + \cot\theta)^2}{4 + (\cot a + \cot\theta)^2};$$

from which equation θ is to be determined, and then ϕ from the equation

$$2\cot\phi = \cot a + \cot\theta.$$

(16). Two weights P, Q (Fig. 107) are connected by a string PAQ passing over the top of a circle situated in a vertical plane, find the position of equilibrium.

Draw CA vertical, and PM, QN horizontal; let $AP = a - s$, $AQ = a + s$, where $2a =$ the length of the string; and $r =$ the radius CB.

$$\therefore CM = r \cos ACP,$$

$$= r \cos\frac{a - s}{r},$$

$$\text{and } CN = r \cos\frac{a + s}{r}.$$

Wherefore the altitude of the common centre of gravity of P and Q above BD,

$$= \frac{Pr \cos \frac{a-s}{r} + Qr \cos \frac{a+s}{r}}{P+Q} \text{ (Art. 136)};$$

which, by Art. 171, must be a maximum or a minimum when there is an equilibrium; and, therefore,

$P \cos \frac{a-s}{r} + Q \cos \frac{a+s}{r}$ is to be a maximum or minimum;

$$\therefore \frac{P}{r} . \sin \frac{a-s}{r} - \frac{Q}{r} \sin \frac{a+s}{r} = 0 \, . . (1), \text{ by differentiating};$$

$$\therefore P \left(\sin \frac{a}{r} \cos \frac{s}{r} - \cos \frac{a}{r} \sin \frac{s}{r} \right) = Q \left(\sin \frac{a}{r} \cos \frac{s}{r} + \cos \frac{a}{r} \sin \frac{x}{r} \right);$$

$$\therefore \tan \frac{s}{r} = \frac{P-Q}{P+Q} . \tan \frac{a}{r},$$

which determines the position of equilibrium.

If it be required to ascertain whether the position be one of stable or one of unstable equilibrium, we must differentiate the left hand member of equation (1), and find the sign of the result. Its differential coefficient is

$$- \left(\frac{P}{r^2} . \cos \frac{a-s}{r} + \frac{Q}{r^2} . \cos \frac{a+s}{r} \right),$$

the sign of which is —, for the quantity within the brackets is necessarily positive, because it is equal to

$$\frac{\text{dist. of com. cent. grav. of P and Q from BD}}{r^2};$$

wherefore the altitude of the centre of gravity is a maximum, and therefore the position is one of unstable equilibrium.

(17). One end P (Fig. 108) of a beam PQ rests against a smooth vertical wall AM, and the other end is suspended by a string AQ from a point A in the wall; to find the position of equilibrium.

Let G be the centre of gravity of the beam; θ = the angle PAQ, and ϕ = the angle MPQ; x = AP; W = the weight of the beam, and R = the reaction of the wall at P. Then the beam is kept at rest by three forces, the reaction R at P, its own weight W at G, and the tension of the string AQ; wherefore, resolving these forces in a direction at right angles to AQ, in order that the unknown tension may disappear from the result, we find the resolved part of R is R cos θ, and of W is $-$ W sin θ; wherefore

$$R \cos\theta - W \sin\theta = 0, \text{ (Art. 91)} \ldots\ldots (1),$$

and taking the moments of the forces about Q, in order that the moment of the unknown tension may not appear in the result, we have the moment of R about Q = R . PQ cos ϕ, and that of W = $-$ W . GQ sin ϕ;

$$\therefore R . PQ \cos\phi - W . GQ \sin\phi = 0, \text{ (Art. 92)};$$

$$\therefore \frac{PQ . \cos\phi}{GQ . \sin\phi} = \frac{W}{R} = \frac{\cos\theta}{\sin\theta}, \text{ from (1)},$$

$$\therefore \frac{\cos\phi}{\cos\theta} = \frac{GQ}{PQ} \cdot \frac{\sin\phi}{\sin\theta},$$

$$= \frac{GQ}{PQ} \cdot \frac{\sin APQ}{\sin PAQ} = \frac{GQ . AQ}{PQ . PQ}.$$

But $\cos\phi = -\dfrac{x^2 + PQ^2 - AQ^2}{2PQ . x}$,

and $\cos\theta = \dfrac{x^2 + AQ^2 - PQ^2}{2AQ . x}$;

$$\therefore \frac{GQ . AQ}{PQ^2} = -\frac{x^2 + PQ^2 - AQ^2}{x^2 + AQ^2 - PQ^2} \cdot \frac{AQ}{PQ};$$

$$\therefore \frac{GQ}{PQ} = \frac{(AQ^2 - PQ^2) - x^2}{(AQ^2 - PQ^2) + x^2};$$

$$\therefore x^2 = \frac{PQ - GQ}{PQ + GQ}(AQ^2 - PQ^2);$$

which determines the position of the point P.

(18). A rod AB (Fig 109) is placed in a smooth hemispherical bowl, so as to rest against the edge of the bowl at P with one extremity A within; to determine the position of equilibrium.

Let G be the centre of gravity of the rod, C the centre of the hemisphere; R, R′ the reactions at P and A, $\theta =$ the angle at which AB is inclined to the horizon = APC. The beam is kept at rest by the reactions R and R′ and its own weight at G; therefore, resolving the forces parallel to the horizon, we have the resolved part of $R = R \sin\theta$, and that of $R' = -R' \cos 2\theta$;

$$\therefore R \sin\theta - R' \cos 2\theta = 0, \text{ (Art. 91).}$$

And taking the moments about G, we have the moment of $R = R \cdot PG$; and that of $R' = -R' \cdot AG \sin\theta$;

$$\therefore R \cdot PG - R' \cdot AG \sin\theta = 0 \; . \; . \; . \text{ (Art. 92)};$$

$$\therefore \frac{\sin\theta}{\cos 2\theta} = \frac{R'}{R} = \frac{PG}{AG \sin\theta};$$

$$\therefore \frac{AG}{PG} = \frac{\cos 2\theta}{\sin^2\theta} = \frac{2\cos^2\theta - 1}{1 - \cos^2\theta};$$

$$\therefore \frac{AG}{AP} = \frac{AG}{AG + PG} = \frac{2\cos^2\theta - 1}{\cos^2\theta}.$$

Now $AP = 2CP \cos\theta$;

$$\therefore \frac{AG}{2CP \cos\theta} = \frac{2\cos^2\theta - 1}{\cos^2\theta};$$

$$\therefore \cos^2\theta - \frac{AG}{4CP} \cdot \cos\theta - \tfrac{1}{2} = 0;$$

$$\therefore \cos\theta = \frac{AG}{8CP} + \sqrt{\frac{AG^2}{64 \cdot CP^2} + \tfrac{1}{2}};$$

$$\therefore 4AP = 8CP \cos\theta = AG + \sqrt{AG^2 + 32CP^2}.$$

(19.) A beam AB (Fig. 110), of uniform thickness, rests with its lower end A on a horizontal plane DE, and its upper end on a plane inclined to the horizon, at an angle 60°. The beam makes an angle of 30° with the horizon; to find the force which must act horizontally at the foot A to prevent sliding.

Let x = the force required, R = the reaction at B, W = weight of the beam; G its centre of gravity; α = ECB, β = BAC. Then the beam is kept at rest by four forces, viz.—its own weight at G, the reactions at B and A, and the horizontal force x; resolving these in a direction parallel to the horizon, we have the revolved part of R $= -$ R $\sin\alpha$, that of $x = x$, that of W = 0, and that of the reaction at A = 0;

$$\therefore x - R\sin\alpha = 0;$$

$$\therefore x = R\sin\alpha.$$

Also, considering A as a fulcrum, the moment of W = W . AG $\cos\beta$, and that of R $= -$ R . AB sin ABR $= -$ R . AB cos ABC $= -$ R . AB $\cos(\alpha - \beta)$, the moments of the other two forces are = 0, wherefore

$$W.AG.\cos\beta - R.AB\cos(\alpha - \beta) = 0;$$

$$\therefore R = \frac{W.AG}{AB}\cdot\frac{\cos\beta}{\cos(\alpha-\beta)};$$

$$\therefore x = R\sin\alpha,$$

$$= \frac{W.AG}{AB}\cdot\frac{\sin\alpha\cos\beta}{\cos(\alpha-\beta)},$$

$$= \frac{W.AG}{AB}\cdot\frac{1}{\cot\alpha + \tan\beta},$$

$$= \frac{W.AG}{2.AG}\cdot\frac{1}{\cot 60^\circ + \tan 30^\circ},$$

$$= \frac{W}{2}\cdot\frac{1}{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}},$$

$$= \frac{W\sqrt{3}}{4}.$$

(20). A beam AB moveable in a vertical plane about a hinge at B, leans against a prop CD situated in that plane; to determine the strain upon the prop CD (Fig. 111).

Let θ = the angle ABD, a = the angle CDB, G the centre of gravity of the beam, R = the reaction at C, which will be perpendicular to AC. The beam AB is kept in equilibrium about the fulcrum B by its own weight W acting at G, and the reaction R at C; now the moment of W = $-$ W . BG cos θ, and that of R = R . BC;

$$\therefore \mathrm{R} \,.\, \mathrm{BC} - \mathrm{W} \,.\, \mathrm{BG} \cos\theta = 0;$$

$$\therefore \mathrm{R} = \mathrm{W} \,.\, \frac{\mathrm{BG}}{\mathrm{BC}} \,.\, \cos\theta.$$

And the pressure on the prop in the direction CD,

$$= \mathrm{R} \cos(\pi - \mathrm{RCD}),$$

$$= \mathrm{R} \sin \mathrm{DCB} = \mathrm{R} \sin(a + \theta),$$

$$= \mathrm{W} \,.\, \frac{\mathrm{BG}}{\mathrm{BC}} \,.\, \sin(a + \theta) \cos\theta.$$

$$\text{Now } \frac{\mathrm{BG}}{\mathrm{BC}} = \frac{\mathrm{BG}}{\mathrm{CD}} \cdot \frac{\mathrm{CD}}{\mathrm{BC}} = \frac{\mathrm{BG}}{\mathrm{CD}} \cdot \frac{\sin\theta}{\sin a};$$

therefore the pressure in the direction

$$\mathrm{CD} = \mathrm{W} \,.\, \frac{\mathrm{BG}}{\mathrm{CD}} \cdot \frac{\sin(a + \theta) \cos\theta \sin\theta}{\sin a}.$$

This force tends to compress the prop, but the force which tends to break it, is the resolved part of R, in a direction at right angles to DC, and is therefore

$$= \mathrm{R} \sin \mathrm{RCD} = \mathrm{R} \cos \mathrm{DCB} = \mathrm{R} \cos(\pi - a - \theta),$$

$$= - \mathrm{R} \cos(a + \theta),$$

$$= - \mathrm{W} \cdot \frac{\mathrm{BG}}{\mathrm{CD}} \cdot \frac{\cos(a + \theta) \cos\theta \sin\theta}{\sin a}.$$

There will be no tendency to break the prop, if DCB be a right angle, for then $\cos(a + \theta) = 0$.

In this solution we have supposed the beam AB against which the prop thrusts to be perfectly smooth; but if it should happen that the friction at C is sufficient (or if there should be a pin at C) to prevent a sliding tendency, the thrust of the prop will be wholly effective in the direction of its length, and the solution will accordingly be somewhat different from that given above.

(21). A beam AB (Fig. 112) leans against a prop CD, and the end A is prevented from sliding upon the horizontal plane AD by a string AD fastened at D; to find the tension of the string.

Let T = the tension, R the reaction of the prop at C, W = the weight of the beam, G its centre of gravity, θ = BAD. The forces which support the beam are W, T, R, and the reaction of the horizontal plane at A; wherefore, resolving these forces parallel to the horizon, we have their resolved parts respectively equal to 0, T, $-$ R sin θ, and 0;

$$\therefore \mathrm{T} - \mathrm{R} \sin \theta = 0.$$

Again, considering A as the fulcrum, their moments are respectively W . AG cos θ, 0, $-$ R . AC, and 0;

$$\therefore \mathrm{W} . \mathrm{AG} . \cos \theta - \mathrm{R} . \mathrm{AC} = 0;$$

$$\therefore \mathrm{T} = \mathrm{R} \sin \theta = \frac{\mathrm{W} . \mathrm{AG}}{\mathrm{AC}} \cdot \sin \theta \cos \theta.$$

Now $\mathrm{AC}^2 = \mathrm{AD}^2 + \mathrm{CD}^2$, $\sin \theta = \frac{\mathrm{CD}}{\mathrm{AC}}$, $\cos \theta = \frac{\mathrm{AD}}{\mathrm{AC}}$,

$$\therefore \mathrm{T} = \frac{\mathrm{W} . \mathrm{AG}}{\mathrm{AC}} \cdot \frac{\mathrm{CD} . \mathrm{AD}}{\mathrm{AC}^2}$$

$$= \mathrm{W} . \frac{\mathrm{AG} . \mathrm{CD} . \mathrm{AD}}{(\mathrm{AD}^2 + \mathrm{CD}^2)^{\frac{3}{2}}}.$$

(22). Two beams AB, AC rest against each other upon the horizontal plane ED at A, and against two smooth parallel vertical walls at B, C; to find the position of equilibrium (Fig. 113).

Let G, g be the centres of gravity of the beams; W, w their weights; θ, ϕ their inclinations to the horizon; R = the reaction of the vertical wall at B. Then the beam AB is kept in equilibrium by R, W, the mutual horizontal pressure of the beams at A against each other, and the vertical reaction of the horizontal plane ED at A; the resolved parts of these parallel to the horizon are respectively R, 0, — mutual pressure, and 0;

$$\therefore \text{R} - \text{mutual pressure} = 0;$$

$$\therefore \text{R} = \text{mutual pressure at A}.$$

Similarly, the reaction at C = the mutual pressure at A, and therefore the reaction at C = R. Again, the moments of the four forces which keep AB at rest, about A as a fulcrum, are respectively R . AB sin θ, — W . AG cos θ, 0, and 0;

$$\therefore \text{R} \,.\, \text{AB} \sin\theta - \text{W} \,.\, \text{AG} \,.\, \cos\theta = 0.$$

Similarly, $\text{R} \,.\, \text{AC} \sin\phi - w \,.\, \text{A}g \,.\, \cos\phi = 0;$

$$\therefore \frac{\text{W} \,.\, \text{AG}}{\text{AB}} \,.\, \cot\theta = \text{R} = \frac{w \,.\, \text{A}g}{\text{AC}} \,.\, \cot\phi;$$

$$\therefore \frac{\tan\phi}{\tan\theta} = \frac{\cot\theta}{\cot\phi} = \frac{w}{\text{W}} \cdot \frac{\text{A}g \,.\, \text{AB}}{\text{AG} \,.\, \text{AC}} \ldots (1).$$

Also, from the nature of the figure,

$$\text{ED} = \text{AD} + \text{AE},$$

$$= \text{AB} \cos\theta + \text{AC} \cos\phi;$$

by means of this and equation (1), both ϕ and θ may be determined.

(23). AC, BC (Fig. 114) are two beams connected by a hinge at C, and resting on two fixed points D, E, in the same horizontal line; to determine the position of equilibrium.

Let G, g be the centres of gravity; A, B the weights of the beams AC, BC; R, R′ the reactions of the point D, E; join DE, and let θ, ϕ be the inclinations of the beams to the horizon; *i.e.*, θ = CDE, ϕ = CED. Draw CH perpendicular to DE; then the beam AC is kept at rest by its own weight A at G, the reaction R at D, and the tension of the hinge at C; supposing C the fulcrum, we have the moment of A $= -$ A . CG $\cos\theta$, that of R = R . DC, and that of the tension of the hinge at C = 0,

$$\left.\begin{aligned} \therefore\ \mathrm{R}\,.\,\mathrm{DC} - \mathrm{A}\,.\,\mathrm{CG}\cos\theta &= 0 \\ \text{Similarly, } \mathrm{R}'.\,\mathrm{CE} - \mathrm{B}\,.\,\mathrm{C}g\cos\phi &= 0 \end{aligned}\right\} \ldots (1).$$

Also, when the equilibrium is once established, it will not be disturbed by supposing the hinge at C to become rigid; in which case the beams become a rigid body resting on two points D, E; and kept in equilibrium by R, R′ and its own weight; wherefore, resolving the force parallel to the horizon, we have

$$\mathrm{R}\sin\theta - \mathrm{R}'\sin\phi = 0,$$

$$\therefore \frac{\sin\theta}{\sin\phi} = \frac{\mathrm{R}'}{\mathrm{R}} = \frac{\mathrm{B}\,.\,\mathrm{C}g\cos\phi}{\mathrm{CE}}\cdot\frac{\mathrm{DC}}{\mathrm{A}.\mathrm{CG}\cos\theta}\ \text{by (1)},$$

$$= \frac{\mathrm{B}}{\mathrm{A}}\cdot\frac{\mathrm{C}g}{\mathrm{CG}}\cdot\frac{\mathrm{DC}}{\mathrm{CE}}\cdot\frac{\cos\phi}{\cos\theta},$$

$$= \frac{\mathrm{B}}{\mathrm{A}}\cdot\frac{\mathrm{C}g}{\mathrm{CG}}\cdot\frac{\sin\phi}{\sin\theta}\cdot\frac{\cos\phi}{\cos\theta};$$

$$\therefore \frac{\mathrm{A}}{\mathrm{B}}\cdot\frac{\mathrm{CG}}{\mathrm{C}g} = \frac{\sin^2\phi\cos\phi}{\sin^2\theta\cos\theta} \ldots (2).$$

Again, resolving R, R′ and the weight (A + B) of the rigid body ACB, perpendicular to the horizon, we have

$$\mathrm{R}\cos\theta + \mathrm{R}'\cos\phi - (\mathrm{A}+\mathrm{B}) = 0;$$

$$\therefore \mathrm{A}+\mathrm{B} = \mathrm{R}\cos\theta + \mathrm{R}'\cos\phi,$$

$$= \mathrm{A}\,.\,\frac{\mathrm{CG}}{\mathrm{DC}}\,.\cos^2\theta + \mathrm{B}\cdot\frac{\mathrm{C}g}{\mathrm{CE}}\cos^2\phi,$$

$$= A \cdot \frac{CG \sin\theta}{DC \sin\theta} \cdot \cos^2\theta + B \cdot \frac{Cg \sin\phi}{CE \sin\phi} \cdot \cos^2\phi,$$

$$= A \cdot \frac{CG}{CH} \cdot \sin\theta \cos^2\theta + B \cdot \frac{Cg}{CH} \cdot \sin\phi \cos^2\phi;$$

$$\therefore (A+B).CH = A.CG \sin\theta \cos^2\theta + B.Cg \sin\phi \cos^2\phi,$$

$$= A.CG\left\{\sin\theta\cos^2\theta + \frac{B.Cg}{A.CG} \cdot \sin\phi\cos^2\phi\right\},$$

$$= A.CG\left\{\sin\theta\cos^2\theta + \frac{\sin^2\theta\cos\theta}{\sin^2\phi\cos\phi} \cdot \sin\phi\cos^2\phi\right\} \text{ by (2)},$$

$$= A.CG \sin^2\theta \cos\theta (\cot\theta + \cot\phi);$$

$$\therefore (A+B).DE = (A+B)\,CH (\cot\theta + \cot\phi),$$

$$= A.CG \sin^2\theta \cos\theta (\cot\theta + \cot\phi)^2,$$

$$= A.CG \sin^2\theta \cos\theta (\cot^2\theta + 2\cot\theta\cot\phi + \cot^2\phi),$$

$$= A.CG \cos^3\theta + 2\sqrt{A.CG.B.Cg \cos^3\theta \cos^3\phi} + B.Cg \cos^3\phi;$$

$$\therefore \sqrt{(A+B).DE} = \sqrt{A.CG}. \cos^{\frac{3}{2}}\theta + \sqrt{B.Cg}. \cos^{\frac{3}{2}}\phi;$$

by means of this and equation (2), θ and ϕ may be determined.

(24). A paraboloid, formed by the revolution of a given parabola about its axis, is placed with its convex surface upon a horizontal plane; to determine the position of equilibrium.

Let $4m$ be the latus rectum of the generating parabola ABC (Fig. 115). P the point of the paraboloid in contact with the plane, PN vertical, PM perpendicular to AC. Then the solid is kept in equilibrium by the reaction at P in the direction PN, and its own weight, at its centre of gravity; and, therefore (Art. 128), the centre of gravity must be in PN, and consequently it is at N.

Now MN $= 2m$, and AN $= \frac{2}{3}$ AC (Ex. 16, page 132); let θ be the inclination of the axis AC to the horizon;

$$\therefore \text{MP} = \frac{\text{MN}}{\tan\theta} = 2m\cot\theta,$$

$$4m\,.\,\text{AM} = \text{MP}^2 = 4m^2\cot^2\theta;$$

$$\therefore \text{AM} = m\cot^2\theta;$$

$$\therefore \tfrac{2}{3}\text{AC} = \text{AN} = m\cot^2\theta + 2m = m\,\text{cosec}^2\theta + m;$$

$$\therefore \text{cosec}\,\theta = \left(\frac{2}{3}\cdot\frac{\text{AC}}{m} - 1\right)^{\frac{1}{2}}.$$

If $\frac{2\text{AC}}{3m}$ be not > 1, or if AC be not $> \frac{3}{2}m$, the solid can only rest in equilibrium with its axis AC in a vertical position.

(25). A solid composed of a cone and hemisphere of equal bases, placed base to base, rests with the convex surface of the hemisphere in contact with a horizontal plane; having given the radius of the hemisphere, to determine the dimensions of the cone (Fig. 116).

Let ABC be the cone, BCD the hemisphere, AD their common axis; g, G their centres of gravity; P the point on which the body rests. Then, since the solid is supported by the reaction at P in the direction PO, its centre of gravity must be in that line (Art. 128), and it is at O; wherefore, by Art. 134,

$$(\text{hemisphere})\,.\,\text{GO} - (\text{cone})\,.\,g\text{O} = 0 \;\ldots\; (1).$$

Now the cone and hemisphere are respectively $\frac{1}{3}$ and $\frac{2}{3}$ of their circumscribing cylinders, and as these cylinders have a common base, viz.—the base of the hemisphere and cone,

$$\therefore \text{cone} : \text{hemisphere} :: \tfrac{1}{3}\,\text{AO} : \tfrac{2}{3}\,.\,\text{DO},$$

$$:: \text{AO} : 2\text{DO}.$$

Also GO = $\frac{3}{8}$DO, (Ex. 120, page 130), and gO = $\frac{1}{4}$AO, therefore substituting in (1), we have

$$2DO \cdot \tfrac{3}{8} \cdot DO = AO \cdot \tfrac{1}{4} \cdot AO;$$

$$\therefore 3DO^2 = AO^2;$$

$$\therefore 4DO^2 = AO^2 + DO^2,$$
$$= AO^2 + CO^2 = AC^2;$$

$$\therefore 2DO = AC,$$

$$\text{or } BC = AC,$$

wherefore the triangle ABC is equilateral.

Since the normal to the hemisphere at any point passes through O, the solid will be in equilibrium upon any point of the hemisphere.

(26). A solid generated by the revolution of a given curve about its axis, is placed with its convex surface upon a horizontal plane; to determine the position of equilibrium.

Let APB (Fig. 115) be the generating curve, AC its axis, P the point in contact with the plane, PN a vertical, which is a normal and may be shewn, as in Prob. 24, to pass through the centre of gravity of the solid. The question is therefore reduced to finding the normal which passes through the centre of gravity. Let θ = the inclination of the axis AC to the horizontal plane, that is, to the tangent at P. Draw PM perpendicular to AC, and put x = AM, y = MP.

$$\therefore \tan\theta = d_x y, \text{ by Diff. Calc.,}$$

$$\text{also, } MN = y d_x y, \text{ by Diff. Calc.;}$$

$$\therefore AN = x + y d_x y.$$

But, by Art. 181, AN = $\dfrac{\int_x (xy^2)}{\int_x (y^2)}$, the integrals to be taken so as to include the whole solid.

Hence, then, the position of the solid will be known from the two equations

$$\left.\begin{aligned} \frac{\int_x(xy^2)}{\int_x(y^2)} &= x + yd_xy, \\ \tan\theta &= d_xy, \end{aligned}\right\}$$

and the equation of the generating curve.

Ex. Suppose the solid to be a hemispheroid, generated by the revolution of a quadrant of an ellipse about its major axis.

$$\text{Here } y^2 = \frac{b^2}{a^2}(2ax - x^2);$$

$$\therefore \int_x y^2 = \frac{b^2}{a^2}(ax^2 - \tfrac{1}{3}x^3) + \text{C},$$

$$= \tfrac{2}{3}ab^2, \text{ from } x = 0 \text{ to } x = a,$$

$$\text{and } \int_x(xy^2) = \frac{b^2}{a^2}(\tfrac{2}{3}ax^3 - \tfrac{1}{4}x^4) + \text{C},$$

$$= \tfrac{5}{12}\,.\,a^2b^2, \text{ from } x = 0 \text{ to } x = a,$$

$$\text{and } yd_xy = \frac{b^2}{a^2}(a - x);$$

$$\therefore \frac{\tfrac{5}{12}a^2b^2}{\tfrac{2}{3}ab^2} = x + \frac{b^2}{a^2}(a - x);$$

$$\therefore \tfrac{5}{8}a = \frac{b^2}{a} + \left(1 - \frac{b^2}{a^2}\right)x;$$

$$\therefore x = \frac{(\tfrac{5}{8}a^2 - b^2)a}{a^2 - b^2}.$$

$$\text{But } \tan\theta = d_xy,$$

$$= \frac{b}{a}\cdot\frac{a - x}{\sqrt{2ax - x^2}},$$

$$= \frac{3ab}{\sqrt{(11a^2 - 8b^2)(5a^2 - 8b^2)}},$$

by substituting for x its value.

If $\frac{a}{b}$ be less than $\left(\frac{8}{5}\right)^{\frac{1}{3}}$ the solid can only rest on its vertex A.

(27). A solid of any form whatever is placed with its convex surface upon a horizontal plane; to determine the position of equilibrium.

Let $z = f(x, y)$ be the equation of its surface referred to three rectangular co-ordinate axes; and let p, q be the partial differential coefficients of z with respect to x and y respectively; also, let $\dot{x}\ \dot{y}\ \dot{z}$ be the co-ordinates of the centre of gravity of the solid, which are known by the formulæ of Art. 182. Then the horizontal plane on which the body rests will be a tangent plane to the surface at the point of contact, and the normal at that point will pass through the centre of gravity. If therefore a, β, γ be the inclinations of the co-ordinate axes of x, y, z to the horizon, or (which is the same) the inclinations of the tangent plane to the axes; then $90° - a$, $90° - \beta$, $90° - \gamma$ will be the inclinations of the vertical normal through the centre of gravity to the co-ordinate axes, and therefore

$$\left.\begin{aligned} \sin a &= \frac{-p}{\sqrt{1 + p^2 + q^2}} \\ \sin \beta &= \frac{-q}{\sqrt{1 + p^2 + q^2}} \\ \sin \gamma &= \frac{1}{\sqrt{1 + p^2 + q^2}} \end{aligned}\right\} \quad \ldots (1).$$

and the equations of this normal are

$$\left.\begin{aligned} \dot{x} - x + p(\dot{z} - z) &= 0 \\ \dot{y} - y + q(\dot{z} - z) &= 0 \end{aligned}\right\} \quad \ldots\ldots (2);$$

from these two and the equation $z = f(x, y)$ the values of x, y, z must be determined, and then substituted in equations (1).

Ex. If the solid be the eighth part of a sphere, the equation of its surface will be

$$a^2 = x^2 + y^2 + z^2,$$

whence we find $p = -\frac{x}{z}$, $q = -\frac{y}{z}$. Also, by Ex. 18, page 136, $\dot{x} = \dot{y} = \dot{z} = \frac{3}{8}a$; wherefore, substituting in (2),

$$\tfrac{3}{8}a - x - \frac{x}{z}(\tfrac{3}{8}a - z) = 0,$$

$$\tfrac{3}{8}a - y - \frac{y}{z}(\tfrac{3}{8}a - z) = 0;$$

$$\therefore x = y = z = \frac{a}{\sqrt{3}};$$

$$\therefore p = q = -1;$$

$$\therefore \sin\alpha = \sin\beta = \sin\gamma = \frac{1}{\sqrt{3}} = .5773503,$$

$$= \sin 35^\circ\ 15'\ 52'';$$

$$\therefore \alpha = \beta = \gamma = 35^\circ\ 15'\ 52''.$$

(28). To determine the nature of the equilibrium when a body rests upon a curve surface.

Let bAc (Fig. 117) be the body, and BAC the surface on which the body rests. Draw the normal DAd at the point of contact A, this will necessarily pass through G the centre of gravity of the incumbent solid, (Art. 128). Let now the solid be slightly disturbed from its position of rest, by causing it to roll along the surface AB; and let $b'\,P\,c'$ be its new position; a', G', d' being respectively the positions taken by A, G, and d. Then, as every point of aP has been in its turn in contact with a corresponding point of AP, therefore

$$AP = aP.$$

Let now Pp be a vertical, then the body cannot be in equilibrium, unless Pp pass through G' (Art. 128); but if it does not, then the body will go back to its original position, or depart farther from it, according as G′ falls to the right or left of Pp; that is, the position bAc will be *stable, unstable*, or *neuter*, according as G′ falls to the *right* or *left* of Pp, or coincides with it; that is, according as ap is greater than, less than, or equal to aG′. To express these conditions more conveniently, draw a normal DPp' at P, then AD, ad' are ultimately the centres of curvature of the arcs AB, Ab at A; denote them by R, r respectively; and let θ = the angle PDA; and θ' = the angle P$d'a$;

$$\therefore \text{PA} = \text{R}\theta, \text{ and } pa = r\theta';$$

$$\therefore \text{R}\theta = r\theta';$$

$$\therefore \frac{\theta'}{\theta} = \frac{\text{R}}{r}.$$

$$\text{But } \frac{d'p}{d'\text{P}} = \frac{\sin d'\text{P}p}{\sin d'p\text{P}} = \frac{\sin d'\text{P}p}{\sin \text{P}pa};$$

$$\therefore \frac{d'p}{r} = \frac{\sin\theta}{\sin(\theta+\theta')} = \frac{\theta}{\theta+\theta'} \text{ ultimately,}$$

$$= \frac{r}{r+\text{R}};$$

$$\therefore d'p = \frac{r^2}{r+\text{R}};$$

$$\therefore ap = r - d'p = \frac{\text{R}r}{\text{R}+r},$$

wherefore the position will be *stable, unstable*, or *neuter*, according as

$$\text{AG is} < > \text{ or } = \frac{\text{R}r}{\text{R}+r},$$

$$\text{or } \frac{1}{\text{AG}} > < \text{ or } = \frac{1}{r} + \frac{1}{\text{R}}.$$

Remark. If the surface BAC be concave instead of convex, as is supposed in the above demonstration, then R is negative, and the conditions are

$$\frac{1}{AG} > < \text{ or } = \frac{1}{r} - \frac{1}{R}.$$

If the surface of support be a plane, then R is infinite, and $\frac{1}{R} = 0$, and the conditions are

$$AG < > \text{ or } = r.$$

Ex. What segment of a paraboloid will rest in a position of neuter equilibrium upon a spherical surface whose radius is R?

Let $4m$ be the latus rectum of the parabola, by the revolution of which the paraboloid is generated; then $r =$ the radius of curvature at the vertex $= 2m$; and, by Ex. 16, page 132, $AG = \frac{2}{3}x$,

$$\therefore \frac{1}{\frac{2}{3}x} = \frac{1}{2m} + \frac{1}{R};$$

$$\therefore x = \frac{3mR}{R + 2m};$$

which gives the length of the axis of the parabolic segment.

(29). A body P (Fig. 118) rests upon a curve line AB, being acted on by given forces in the plane of the curve; to determine the position of equilibrium.

Let Ox, Oy be the rectangular co-ordinates to which the curve is referred by its equation, draw PM parallel to Oy, and PR a normal at P. Resolve the forces which act on P into others parallel to the axes Ox, Oy; and let X, Y be their respective sums; then we may consider P as kept at rest by the action of three forces X, Y, and the reaction R in the direction PR; but we may resolve the latter into

$$-\,Rd_s y \text{ parallel to the axis } Ox,$$

$$\text{and } Rd_s x \text{ parallel to the axis } Oy,$$

where $x = OM$, $y = MP$, and $s = AP$; for $d_s y$, $d_s x$

are the cosines of the angles which PR makes with Ox, Oy;

$$\left.\begin{aligned}\therefore\ 0 &= X - Rd_s y\\ \text{and } 0 &= Y + Rd_s x\end{aligned}\right\} \ldots\ldots (1).$$

Multiply the first by dx, the second by dy, and add,

$$\therefore\ 0 = Xdx + Ydy \quad \ldots\ldots (2),$$

which is the condition of equilibrium.

If the pressure sustained by the curve be required, we have

$$R^2 = R^2 d_s x^2 + R^2 d_s y^2,$$
$$= X^2 + Y^2;$$
$$\therefore\ R = \sqrt{X^2 + Y^2}.$$

Ex. 1. Suppose AB to be a parabola whose axis is Oy, and that the particle is acted on by gravity in the direction PM, and by a force tending from Oy, and proportional to the distance from Oy.

Let g denote the force of gravity, and μx the force tending from Oy;

$$\therefore\ X = \mu x, \text{ and } Y = -g;$$

also the equation of the curve is

$$x^2 = 4my;$$

$$\therefore\ xdx = 2mdy,\ \therefore\ dy = \frac{xdx}{2m},$$

$$\text{and } 0 = Xdx + Ydy,$$
$$= \mu xdx - g \cdot \frac{xdx}{2m};$$

$$\therefore\ 0 = \mu - \tfrac{1}{2}\frac{g}{m};$$

$$\therefore\ 4m = \frac{2g}{\mu};$$

that is, there cannot be an equilibrium unless the latus rectum of the parabola be equal to $\frac{2g}{\mu}$, and when this condition is satisfied the particle will remain in equilibrium on any point of the curve.

Ex. 2. A body P rests on the surface of a prolate spheroid (Fig. 119), and is attracted towards the foci S and H, with forces respectively varying as $(\mathrm{SP})^m$ and $(\mathrm{HP})^n$; to find the position of equilibrium.

Let the force in the direction $\mathrm{SP} = \alpha\,(\mathrm{SP})^m$, and that in the direction $\mathrm{HP} = \beta\,(\mathrm{HP})^n$. Let the centre C be the origin of co-ordinates $x = \mathrm{CM}$, $y = \mathrm{PM}$; and, therefore, $\mathrm{SP} = a + ex$ and $\mathrm{HP} = a - ex$; a, e the semi-major axis and eccentricity. Now the resolved parts of $\alpha\,(\mathrm{SP})^m$ in the direction

$$\text{of } x = -\,\alpha(\mathrm{SP})^m . \frac{\mathrm{SM}}{\mathrm{SP}} = -\,\alpha(a + ex)^{m-1}(ae + x),$$

$$\text{of } y = -\,\alpha(\mathrm{SP})^m . \frac{\mathrm{PM}}{\mathrm{SP}} = -\,\alpha(a + ex)^{m-1} y;$$

and, similarly, the resolved parts of $\beta\,(\mathrm{HP})^n$ in the direction

$$\text{of } x = \beta\,(a - ex)^{n-1}(ae - x),$$

$$\text{of } y = -\,\beta\,(a - ex)^{n-1} y;$$

$$\therefore \mathrm{X} = -\,\alpha\,(a+ex)^{m-1}(ae+x) + \beta(a-ex)^{n-1}(ae-x),$$

$$\mathrm{Y} = -\,\alpha\,(a+ex)^{m-1}\,y - \beta(a-ex)^{n-1}\,y.$$

$$\text{Now } 0 = \mathrm{X}dx + \mathrm{Y}dy,$$

$$= \mathrm{X}dx - \mathrm{Y}\frac{b^2x}{a^2y}\,dx,$$

$$\text{for because } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \;\therefore dy = -\frac{b^2x}{a^2y}\,dx;$$

$$\therefore 0 = a^2\mathrm{X}y - b^2\mathrm{Y}x;$$

$$\therefore 0 = -a^2\alpha\,(a+ex)^{m-1}(ae+x)y + a^2\beta(a-ex)^{n-1}(ae-x)y$$
$$+\, b^2\alpha\,(a+ex)^{m-1}\,xy + b^2\beta(a-ex)^{n-1}\,xy;$$

$$\therefore \alpha (a+ex)^{m-1} a^2(ae+e^2x) = \beta(a-ex)^{n-1} a^2(ae-e^2x);$$

$$\therefore \alpha (a+ex)^m = \beta(a-ex)^n,$$

from which equation the value of x is to be determined. This equation expresses that the forces are equal at the point P.

(30). A body P rests upon a curve surface, being acted on by given forces in any directions; to determine the position of equilibrium.

Let $z = f(x, y)$ be the equation of the surface on which P rests; p, q the partial differential coefficients of z with regard to x, y respectively. Resolve the forces which act on P parallel to the three co-ordinate axes, and let X, Y, Z be their respective sums, and R the reaction of the surface at P, which takes place in the direction of a normal. Then we may consider P as kept at rest by the four forces X, Y, Z and R. Now the resolved part of R in the direction

$$\text{of } x = \frac{Rp}{\sqrt{1+p^2+q^2}},$$

$$\text{of } y = \frac{Rq}{\sqrt{1+p^2+q^2}},$$

$$\text{of } z = \frac{-R}{\sqrt{1+p^2+q^2}};$$

$$\therefore 0 = X + \frac{Rp}{\sqrt{1+p^2+q^2}},$$

$$0 = Y + \frac{Rq}{\sqrt{1+p^2+q^2}},$$

$$0 = Z - \frac{R}{\sqrt{1+p^2+q^2}};$$

wherefore, by dividing the first by p and the second by q, we have

$$\frac{X}{p} = \frac{Y}{q} = -Z \ldots (1),$$

which combined with equation of the given surface will give the point P.

The pressure on the surface is evidently equal to

$$R = \sqrt{X^2 + Y^2 + Z^2}.$$

Ex. A body rests on an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and is attracted towards the co-ordinate planes by forces which are respectively proportional to its distances from them; to determine the position of equilibrium.

Let αx, βy, γz be the respective forces;

$$\therefore X = -\alpha x,\ Y = -\beta y,\ Z = -\gamma z;$$

also, by differentiating the equation of the given surface,

$$p = -\frac{c^2 x}{a^2 z} \text{ and } q = -\frac{c^2 y}{b^2 z};$$

$$\therefore \alpha x . \frac{a^2 z}{c^2 x} = \beta y \cdot \frac{b^2 z}{c^2 y} = \gamma z \text{ from (1)};$$

$$\therefore \alpha a^2 = \beta b^2 = \gamma c^2;$$

these two conditions must be satisfied, otherwise there cannot be an equilibrium at all, and when they are satisfied the body will rest on any point of the given surface.

On the Funicular Polygon.

(31). ABCDEF (Fig. 120) is a cord, supposed devoid of weight, suspended from two points A, F in a horizontal line; at the knots B, C, D, E . . . . . . weights W_1, W_2, W_3, W_4 . . . . are hung; to determine the proportion of these weights that it may hang in a given form. This is called the funicular polygon.

From A draw A*c*, A*d*, A*e*, A*f* respectively parallel to the portions BC, CD, DE, EF of the cord; and denote the respective inclinations of AB, BC, CD . . . . . to the horizontal line AF by $\alpha, \beta, \gamma, \delta$ . . . .; draw BM vertical. Then B is kept at rest by the tension of AB, BC and the weight W_1, which forces are respectively parallel to the sides BA, A*c*, *c*B of the triangle AB*c*, and are therefore proportional to them. Therefore W_1 is proportional to B*c*. In the same manner W_2 is proportional to *cd*; and they are on the same scale, for in both A*c* represents the tension of BC.

$$\therefore \frac{W_1}{W_2} = \frac{Bc}{cd} = \frac{BM - cM}{cM - dM},$$

$$= \frac{AM \tan \alpha - AM \tan \beta}{AM \tan \beta - AM \tan \gamma},$$

$$= \frac{\tan \alpha - \tan \beta}{\tan \beta - \tan \gamma}.$$

$$\text{Similarly, } \frac{W_2}{W_3} = \frac{\tan \beta - \tan \gamma}{\tan \gamma - \tan \delta}, \ldots\ldots$$

It appears, therefore, that any one of the weights is proportional to the difference of the tangents of the angles at which the two sides of the polygon, which form the angle at which it is suspended, are inclined to the horizon.

Remark. The angles MAe, MAf, which are situated above the line AF, are to be accounted negative.

The horizontal tension of any string is represented by AM, for it is the resolved part of the lines AB, Ac, Ad . . . . . which represent the whole tensions; and this horizontal tension : any weight (W_2 suppose) :: AM : cd : : 1 : $\tan\beta - \tan\gamma$.

The tension of any string BC : the horizontal tension :: Ac : AM :: AM $\sec\beta$: AM :: $\sec\theta$: 1.

(32). If AB, BC, CD . . . ., in the preceding figure, instead of being lines devoid of weight, be heavy beams of wood, or bars of metal, connected at the joints A, B, C, D . . . . . by hinges, we must consider each beam as exerting by means of its weight vertical forces at its extremities. Thus, if w_1, w_2, w_3 . . . . be the weights of AB, BC, CD . . . . we may consider BC as exerting equal pressures $\frac{1}{2}w_2$ at B and C in a vertical direction, the centre of gravity of the beam being supposed at its middle point; in like manner AB exerts a vertical pressure equal to $\frac{1}{2}w_1$ at B, and therefore we may consider $W_1 + \frac{1}{2}(w_1 + w_2)$ as the whole weight suspended at B. Similarly, the weights to be considered as suspended at C, D, ... are respectively

$$W_2 + \tfrac{1}{2}(w_2 + w_3)\,;\; W_3 + \tfrac{1}{2}(w_3 + w_4)\,;\; \ldots\ldots$$

and these weights are to be used instead of those given in the preceding article.

Their considerations are intimately connected with the construction of suspension bridges.

If W_1, W_2, W_3 . . . . . are evanescent, then the weights to be considered as suspended are $\frac{1}{2}(w_1 + w_2)$, $\frac{1}{2}(w_2 + w_3)$ . . . .; and if the beams are all equal, each of these become equal to w_1.

On the Catenary.

(33). To deduce the equation of the catenary from the properties of the funicular polygon.

Def. A catenary is the curve assumed by a fine chain or flexible string when suspended from its extremities.

Let AOF (Fig. 121) be the catenary, which we may consider a funicular polygon, whose sides are the equal indefinitely small links of the chain of which it is composed. Let PQ be one of the links; O the lowest point; OK vertical; PM, QN perpendicular to OK; Pp parallel to MN; x = OM, y = PM, OP = s, MN = δx, pQ = δy, PQ = δs, T = the horizontal pressure which we have seen is the same for every beam or link; W = the weight of a portion of the chain whose length is l;

$$\therefore \text{ the weight of PQ} = \frac{W\delta s}{l},$$

and since the weights of the links are equal, the weights to be considered as suspended at P, Q .... are each equal to

$$\frac{W\delta s}{l}.$$

Wherefore if θ be the inclination of PQ to the horizon, and $\theta + \delta\theta$ that of the next link, we have

$$T : W\,.\frac{\delta s}{l} :: 1 : \tan(\theta + \delta\theta) - \tan\theta,$$

$$:: 1 : \sec^2\theta\,.\,\delta\theta \text{ ultimately};$$

$$\therefore \sec^2\theta\,.\,\delta\theta = \frac{W}{T}\cdot\frac{\delta s}{l};$$

$$\therefore \sec^2\theta = \frac{W}{T}\cdot\frac{\delta_\theta s}{l}$$ by taking the limits, by supposing the links indefinitely small.

$$\therefore \tan\theta = \frac{W}{T}\cdot\frac{s}{l}, \text{ by integration.}$$

$$\text{But } \tan\theta = PQp = \frac{Pp}{Qp} = \frac{dx}{dy};$$

$$\therefore \frac{dx}{dy} = \frac{W}{T}\cdot\frac{s}{l}.$$

This expression may be rendered more simple by writing a for $\frac{Tl}{W}$; then

$$\frac{dx}{dy} = \frac{s}{a};$$

$$\therefore ds = \sqrt{dx^2 + dy^2},$$

$$= dx\left(1 + \frac{a^2}{s^2}\right)^{\frac{1}{2}};$$

$$\therefore dx = \frac{sds}{\sqrt{a^2 + s^2}};$$

$$\therefore x + C = \sqrt{a^2 + s^2}.$$

To determine the value of the constant we must observe that $x = 0$, when $s = 0$, and therefore $C = a$;

$$\therefore x + a = \sqrt{a^2 + s^2};$$

$$\therefore s^2 = 2ax + x^2,$$

the equation required.

To determine the meaning of a, we have

$$W : T :: l : a,$$

and therefore as l is the length of chain whose weight is W, a must be the length of chain whose weight is equal to the horizontal tension.

(34). The tension at the point P : the horizontal tension T :: sec θ : 1. (Prob. 31, Rem.)

But $\tan\theta = \frac{dx}{dy}$, $\therefore \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{dx^2}{dy^2}$

$$= 1 + \frac{s^2}{a^2} = \left(\frac{x+a}{a}\right)^2;$$

$\therefore$ tension at P : horizontal tension :: $\frac{x+a}{a} : 1$,

$$:: x + a : a.$$

Now a is the length of a portion of the chain whose weight is equal to the horizontal tension, and therefore $x + a$ is the weight of a portion of the chain whose length is equal to the tension of the chain at P; wherefore produce KO to B, so that OB may be equal to a, and draw BC perpendicular and PL parallel to KB; and PL will represent the tension at P, for it is equal to BM $= a + x$. BC is called the *directrix* of the catenary.

(35). The equation of the catenary will be simplified by taking B for the origin instead of O; for then $x =$ BM $= a +$ OM; and since

$$(a + \text{OM})^2 = a^2 + s^2;$$
$$\therefore x^2 = a^2 + s^2.$$

If the relation between x and y be required, we have

$$s = \sqrt{x^2 - a^2} \quad \ldots\ldots \quad (1);$$

therefore

$$\sqrt{dx^2 + dy^2} = ds = \frac{xdx}{\sqrt{x^2 - a^2}};$$

$$\therefore dy = dx \,.\left\{\frac{x^2}{x^2 - a^2} - 1\right\}^{\frac{1}{2}},$$

$$= \frac{adx}{\sqrt{x^2 - a^2}};$$

$\therefore y = a \log_e\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$, by integration;

$$\therefore e^{\frac{y}{a}} = \frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}};$$

$$\therefore e^{-\frac{y}{a}} = \frac{x}{a} - \left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}};$$

$$\therefore e^{\frac{y}{a}} + e^{-\frac{y}{a}} = \frac{2x}{a} \quad \ldots\ldots\ldots\ldots \quad (2).$$

If the relation between y and s be required, we have from (1)

$$\left(\frac{s}{a}\right)^2 = \left(\frac{x}{a}\right)^2 - 1;$$

$$\therefore \left(\frac{2s}{a}\right)^2 = \left(\frac{2x}{a}\right)^2 - 4,$$

$$= e^{\frac{2y}{a}} - 2 + e^{-\frac{2y}{a}};$$

$$\therefore \frac{2s}{a} = e^{\frac{y}{a}} - e^{-\frac{y}{a}} \quad \ldots\ldots\ldots \quad (3).$$

(36). The equation of the catenary may also be deduced independently of the funicular polygon.

Let PT (Fig. 122) be a tangent to the curve at the point P; then after the chain has assumed its form of equilibrium, it will not be disturbed by supposing the part PO to become rigid; in which case the rigid body PO is kept at rest by the action of its own weight in the direction MT; the tension of PT in the direction TP, and the horizontal tension at O in the direction PM, which forces being in the directions of the sides of the triangle MTP are also proportional to them;

$$\therefore \text{weight of OP} : \text{tension at O} :: \text{MT} : \text{MP},$$

$$:: \frac{\text{MT}}{\text{MP}} : 1,$$

$$\text{or } s : a :: \tan\theta : 1;$$

$$\therefore \frac{dx}{dy} = \tan\theta = \frac{s}{a},$$

the same equation as before deduced.

(37). We have supposed the density of the chain to be uniform, but if this be not the case, let the density at P be ρ; then the weight of an element δs

$$= g\rho\delta s, \text{ (Art. 122)};$$

and, therefore, the weight of OP $= g\int(\rho ds) = g\int_s(\rho d_s s)$; and since, as above shewn,

$$\text{weight of OP : tension of O :: } \tan\theta = \frac{dx}{dy} : 1;$$

$$\therefore g\int_s(\rho d_s s) : \text{T} :: \frac{dx}{dy} : 1 :: 1 : d_s y;$$

$$\therefore \frac{\text{T}}{d_s y} = g \cdot \int_s(\rho d_s s).$$

We may simplify this equation by supposing a = the length of chain of density 1, whose weight is equal to the tension T;

$$\therefore \text{T} = ga;$$

$$\therefore \frac{a}{d_s y} = \int_s(\rho d_s s);$$

$$\therefore -\frac{a d_s^2 y}{(d_s y)^2} = \rho d_s s, \text{ by differentiation} \ldots (1);$$

$$\therefore \frac{a(d_s s)^2}{(d_s y)^2} = \rho \cdot \frac{(d_s s)^3}{-d_s^2 y},$$

$$\text{or } a \,.\, (d_y s)^2 = \rho \times (\text{radius of curvature});$$

$$\therefore \rho = \frac{a \,.\, \sec^2 \text{ of the inclination}}{\text{radius of curvature}} \ldots (2);$$

for $d_y s$ = secant of the inclination of the tangent at P to the horizon. Equation (2) gives the density at any point P, in order that the chain may hang in a given form; thus, in order that a chain may hang in form of a semi-circle, the density at any point must vary as the square of the secant of the inclination of the tangent at that point to the horizon. On the contrary, equation (1) will give the form when the density is known. As an example, suppose the density at any point to be proportional to the tension at the same point.

Referring to the figure, we have

$$\mathrm{T} : \text{tension at P} :: \mathrm{MP} : \mathrm{PT} :: 1 : \frac{\mathrm{PT}}{\mathrm{MP}},$$

$$:: \quad 1 \quad : \sec\theta :: 1 : \frac{ds}{dy}.$$

Wherefore ρ the tension at P

$$\propto \frac{ds}{dy} = \frac{a}{b}\frac{ds}{dy} \text{ suppose};$$

whence, from equation (1),

$$-\frac{a d_x^2 y}{(d_x y)^2} = \frac{a}{b}\, d_x s \,.\, \frac{ds}{dy} = \frac{a}{b}\, d_x s \,.\, \frac{d_x s}{d_x y};$$

$$\therefore -\frac{b d_x^2 y}{1 + (d_x y)^2} = d_x y;$$

$$\therefore b \cot^{-1}(d_x y) = y + \mathrm{C}, \text{ by integration.}$$

To determine the constant C, we observe that at the lowest point, $y = 0$, $d_x y = \infty$, and $\cot^{-1}(d_x y) = 0$, and therefore $\mathrm{C} = 0$.

$$\therefore \cot^{-1}(d_x y) = \frac{y}{b};$$

$$\therefore d_x y = \cot\frac{y}{b};$$

$$\therefore \frac{\sin\frac{y}{b} \cdot d_x y}{\cos\frac{y}{b}} = 1 \ldots . (1);$$

$$\therefore \log_e \cos\frac{y}{b} = \mathrm{C} - \frac{x}{b},$$

and if we take the lowest point for the origin of co-ordinates, $x = 0$ when $y = 0$, and therefore $C = 0$;

$$\therefore \log_e \cos \frac{y}{b} = -\frac{x}{b};$$

$$\therefore \cos \frac{y}{b} = e^{-\frac{x}{b}},$$

$$\text{or } \sec \frac{y}{b} = e^{\frac{x}{b}}.$$

This is the form of a chain that is equally able to bear its own weight in every part; the density or thickness at any point varies as

$$\frac{ds}{dy}, \text{ or as } \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}}.$$

But, from equation (1),

$$d_x y = \cot \frac{y}{b};$$

$$\therefore \text{thickness or density} \propto \left(1 + \tan^2 \frac{y}{b}\right)^{\frac{1}{2}},$$

$$\propto \sec \frac{y}{b},$$

$$\propto e^{\frac{x}{b}}.$$

(38). To find the catenary when the chain is acted on by a force tending to a fixed centre.

Let C (Fig. 123) be the centre of force, O the lowest point of the curve; PY a tangent at P, and CY a perpendicular upon it; $CP = r$, $CY = p$, $F =$ the force acting on P in the direction PC, $t =$ the tension at P, $CM = x$; the remainder of the notation as before. The mass of $PQ = \rho\delta s$, and the force which acts upon it in the direction MC is $F\frac{x}{r}$, and therefore the weight of PQ in that direction

$$= \rho F \frac{x\delta s}{r}$$

and, therefore, the weight of OP in that direction

$$= \int \left(\rho F \frac{xds}{r}\right).$$

Similarly, the weight of OP in the direction PM

$$= \int \left(\rho F \frac{yds}{r}\right).$$

And the forces which keep OP in equilibrium are these two weights, and the tensions at A and P; wherefore by resolving them in the directions of x and y, we have

$$t \frac{dx}{ds} - \int \left(\rho F \frac{xds}{r}\right) = 0,$$

$$t \frac{dy}{ds} - \int \left(\rho F \frac{yds}{r}\right) - T = 0,$$

$$\therefore \int \left(\rho F \frac{xds}{r}\right) = t\, d_s x,$$

$$\int \left(\rho F \frac{yds}{r}\right) + T = t\, d_s y.$$

By differentiating these equations with regard to s, we find

$$\left.\begin{aligned} \rho F \frac{x}{r} &= t\, d_s^2 x + d_s t\, d_s x \\ \rho F \frac{y}{r} &= t\, d_s^2 y + d_s t\, d_s y \end{aligned}\right\} \ldots (1).$$

Multiply these respectively by $d_s x$, $d_s y$, and add, observing that $d_s x\, d_s^2 x + d_s y\, d_s^2 y = 0$, because $(d_s x)^2 + (d_s y)^2 = 1$,

$$\therefore \rho F d_s r = d_s t;$$

$$\therefore \textstyle\int_r (\rho F) + C = t.$$

Again, multiply equations (1) by y and x respectively, and subtract;

$$\therefore 0 = t\,(y d_s^2 x - x d_s^2 y) + d_s t\,.\,(y d_s x - x d_s y);$$

$$\therefore C' = t(yd_s x - xd_s y), \text{ by integration,}$$
$$= tp, \text{ by the Differential Calculus ;}$$
$$\therefore t = \frac{C'}{p};$$
$$\therefore \frac{C'}{p} = \int_r(\rho F) + C.$$

This equation will give us the form of the curve when the law of the force and density are known. By differentiating it, we have

$$-\frac{C'dp}{p^2} = \rho F dr;$$
$$\therefore F \propto \frac{d_r p}{\rho p^2},$$

which gives the law of force, that a chain of given density may hang in a given form. Also,

$$\rho \propto \frac{d_r p}{F p^2},$$

which gives the law of density or thickness, that a chain suspended from two points, and being acted upon by a given force, may hang in a given form.

If the force be repulsive, F must be accounted negative.

(39). To find the catenary when the chain is acted on by any forces in its own plane.

Using the figure and notation of the last problem, resolve the forces on P in directions parallel to the co-ordinate axes, and let X, Y be the components. Then, proceeding exactly as before, we find instead of (1) these equations

$$-\rho X = t\, d_s^2 x + d_s t\, d_s x,$$
$$-\rho Y = t\, d_s^2 y + d_s t\, d_s y.$$

Multiply by $d_s x$, $d_s y$, and add;

$$\therefore -\rho(\text{X}d_s x + \text{Y}d_s y) = d_s t;$$

$$\therefore -\int \rho(\text{X}dx + \text{Y}dy) + \text{C} = t \quad \ldots \quad (1).$$

Multiply y and x, and subtract

$$-\rho(\text{X}y - \text{Y}x) = t(yd_s^2 x - xd_s^2 y) + d_s t . (yd_s x - xd_s y);$$

$$\therefore \int_s \rho(\text{Y}x - \text{X}y) + \text{C}' = t(yd_s x - xd_s y) \quad \ldots \quad (2),$$

and by eliminating t between (1) and (2) we shall have the equation of the curve.

On Roofs and Bridges.

(40). If the whole figure of problem 31, be inverted or turned round the horizontal line AF through an angle of 180°, as in Fig. 124, we shall find the same relations between the weights as before; it will also appear, from the same reasoning, as in Prob. 32, that the weights to be considered as hanging from B, C, D .... are the same as there investigated. In this state the problem contains the whole theory of roofs, arches, and bridges. If ABCDEF be considered as a roof, of which AB, BC .... are the beams, then the horizontal thrust at A and F tending to push out the walls on which the roof is erected, is represented by AM, on the same scale as that wherein Bc represents the weight to be supposed suspended from B, it is therefore equal to

$$\frac{\text{W}_1 + \frac{1}{2}(w_1 + w_2)}{\tan \alpha - \tan \beta}.$$

This thrust is usually prevented from taking effect upon the walls by inserting the ends A, F of the beams AB, FE into another AF called the *tie-beam*, which is thus made to sustain the whole thrust; at other times the walls are prevented from bulging by buttresses, or shores, built against them.

If it were required to construct a roof of given span with given beams, which has to support given weights, we must take an equal number of smaller proportional beams, and connect them by strings or pins at the joints, so as to allow them to move freely, and load them with proportional weights. Then if this model be suspended from its extremities at a proportional distance, as in problem 31, it will assume the required form, which we have merely to turn round AF through an angle of 180°, and it will be a perfect model of the required roof; and will possess the property of being in equilibrium in every part. In such a roof there will be no unnecessary strain on any part of the materials of which it is constructed, and consequently no part will require to be unnecessarily strong. In this simple manner we may also obtain the model of a bridge of given span, by taking a great number of very short beams to represent the arch stones, and connecting them as before. If when we suspend this model-string of arch stones loaded with weights proportional to what (in the place they occupy in the bridge) they will have to sustain, we find that the bridge would be too lofty, we must remove the points of suspension farther apart, until we have obtained the proper altitude. This method will give us a bridge, in perfect equilibrium in every part, and in which there is, therefore, no injurious strain, no useless strength, nor dangerous weakness in any part.

On Balances.

(41). A *balance* is any instrument invented for the purpose of comparing the heaviness of different bodies; that is, for ascertaining their *weights*.

The *common balance* (Fig. 125), consists of an inflexible rod AB, called the beam, resting upon a fulcrum C at its middle point; from its extremities A, B are suspended two equal scales D, E by means of fine chains or strings. The fulcrum C and the points of support are in the same straight line, but the centre of gravity of the beam is a little below C. In this state the balance when unloaded ought to rest with its beam AB in a horizontal position. If a weight be put into one of the scales, the common centre of gravity of the scale and its load will be in the vertical passing the point of support, (Art. 128); and therefore we may transmit both the scale and its load to the point of support. Wherefore, when weights are placed in the scales, we may suppose them placed immediately at A and B, and therefore the balance becomes a straight lever whose fulcrum is C; and since the arms AC, BC are equal, there will be an equilibrium when the weights are equal (Art. 196). If the weights are unequal, let G (Fig. 126) be the centre of gravity of the beam AB in the oblique position assumed in consequence of the inequality of the weights. Let w be the weight of the beam, which by Art. 131 we may suppose to be placed at G; S the weight of each of the equal scales; P, W the weights in D and E respectively; $\theta =$ the inclination of the beam to the horizon. Then the machine is kept at rest by four parallel forces, viz. $S + P$ at A, S + W at B, w at G and the reaction of the fulcrum at C; the perpendiculars from C upon the directions of these forces are AC cos θ, CB cos θ, GC sin θ, and zero; therefore, by Art. 198,

$$(S + P) . AC \cos\theta + w . GC \sin\theta = (S + W) . BC \cos\theta;$$

$$\therefore P . AC + w . GC \tan\theta = W . BC,$$

by dividing by cos θ, and observing that AC = BC;

$$\therefore \tan\theta = \frac{W - P}{w} \cdot \frac{AC}{GC}.$$

The sensibility of a balance consists in the beam attaining considerable obliquity, when the difference between P and W is extremely small; and therefore the obliquity attained by different balances when loaded with the same weights, might be taken as a measure of their respective sensibilities. As W − P is constant in this case, and as θ is very nearly equal to tan θ, we may use

$$\frac{AC}{w . GC}$$

as the measure of the stability.

A different measure of stability is however generally used. Which may be thus explained. Let δ be the difference between W and P which produces a *given* (which is the same for all balances) very minute appreciable deviation θ';

$$\therefore \theta' \text{ or } \tan\theta' = \frac{\delta}{w} \cdot \frac{AC}{GC};$$

$$\therefore \delta = w \cdot \frac{GC}{AC} \cdot \theta',$$

the ratio of the whole pressure P + W + 2S + w (Art. 198) on the fulcrum to this weight is taken as the measure of the stability, or neglecting θ' in this measure which is constant, and using $2P + 2S + w$, for the pressure on the fulcrum, the fraction

$$\frac{P + S + \frac{1}{2}w}{w} \cdot \frac{AB}{GC}$$

is the measure generally employed. From either of these measures we derive the following general results:—

That the sensibility of a balance is increased,

1st. By increasing the length of the beam.

2ndly. By diminishing the distance of its centre of gravity from the fulcrum.

3rdly. By diminishing its weight.

For further information on subjects connected with the common balance, the reader is referred to Captain Kater's Treatise on Machines.

(42). On the *Steelyard*, or *Roman Balance*. This instrument is a lever AB (Fig. 127) with unequal arms AC, CB; the fulcrum being C. As it is commonly constructed, the longer arm AC preponderates over the shorter CB; let therefore G be the centre of gravity of the beam AB, at which point we may suppose its weight w collected. And let P be a given weight suspended from p, and Q the body to be weighed from B. Then (Art. 197)

$$P . Cp + w . CG = W . CB;$$

$$\therefore W = \frac{P . Cp + w . CG}{CB};$$

$$\propto P . Cp + w . CG,$$

$$\propto Cp + \frac{w}{P} . CG.$$

Now let D be such a point that when P is suspended from D, it just balances the beam,

$$\therefore P . CD = w . CG;$$

$$\therefore CD = \frac{w}{P} CG;$$

$$\therefore W \propto Cp + CD \propto Dp.$$

It appears therefore, that the weight W is proportional to the distance of *p* from D. If we suppose then, that when *p* is at E, W is one pound, then making EF, FH, HI . . . . . each equal to DE; when *p* is at F, H, I . . . . . W will be 2lbs., 3lbs., 4lbs., ... respectively, and we may number the points E, F, H . . . . 1, 2, 3 . . . respectively; and if the spaces DE, EF . . . . be subdivided into sixteen equal parts, each of them will correspond to one ounce, and we shall be able to ascertain W with corresponding accuracy by sliding the weight P along the arm AC until it comes into such a position as to balance W, and then reading off its place, which will be the number of pounds and ounces which express its weight.

The advantage of this balance is, that it requires but one weight P, and the pressure on the fulcrum, on which the friction depends, being equal to P + W, is less than in the common balance so long as the substance to be weighed is heavier than P; on the contrary, however, when the substance to be weighed is not so heavy as P, the pressure on the fulcrum is greater than in the common balance, and consequently the friction, which diminishes the sensibility of the machine, is greater; and, therefore, for the determination of small weights the common balance is to be preferred, both on account of the diminution of friction and also because small weights can be more accurately subdivided than small spaces on the arm.

(43). On the *Danish Balance*. This instrument consists of a lever AB (Fig. 128), at one end A of which is fastened a given weight A, and at the other B a dish D to receive the substance to be weighed. The fulcrum or point of support C is made to slide along AB until the beam is horizontal, and by its place on the graduated beam AB the weight of the substance put into the scale-pan is determined. The method of graduating the beam AB may be thus investigated. Let G be the centre of gravity of the instrument (including the beam, weight A, and scale-pan* D), P its weight; W the weight in the scale D. Then we may suppose P applied at G (Art. 131), and since there is an equilibrium between P and W about the fulcrum C,

$$\therefore W.BC = P.CG = P.(BG - BC),$$
$$= P.BG - P.BC;$$
$$\therefore BC = \frac{P.BG}{P+W}.$$

Wherefore, when W has the values 0, 1, 2, 3lbs., ... and if P be $n$lbs., BC has the values $\frac{n.BG}{n}, \frac{n.BG}{n+1}, \frac{n.BG}{n+2}, \frac{n.BG}{n+3}$, ... which quantities are in harmonical progression, because their reciprocals are in arithmetical progression. The divisions 0, 1; 1, 2; 2, 3; .... may be again subdivided, if necessary, and when this beam is thus prepared, the weight W may be ascertained with as much facility as in the common steelyard; but the disadvantage of this balance is, that as the weight increases the intervals between the divisions become smaller, and consequently it is not so well adapted for determining large weights as small ones.

* The scale-pan is here supposed to be transmitted to B.

On Elastic Strings.

(44). Strings made of certain substances are found to be elastic; that is, they admit of being lengthened by the application of forces to their extremities, and regain their original dimensions, or nearly so, when the forces are removed. Spiral springs composed of steel wire, such as the one exhibited in Fig. 129, are found to possess the same property in a remarkable degree. The connection between the force which stretches a string, or a spring of the kind here mentioned, and the increase of length cannot be investigated from mathematical considerations, but is to be determined entirely by experiments.

Let MN (Fig. 130) be a very smooth horizontal table; AB an elastic string or spring laid upon it and fastened at A; W a weight stretching the string by means of a thread passing over the pulley C, whose position is such that ABC coincides with the table. Then, if W stretches the string to b, and another weight W′ stretches it still farther to b', it is found that

$$Bb : Bb' :: W : W';$$

that is, the excess of a given elastic string or spiral spring above its natural length is proportional to the weight which stetches it.

(45). Hence it follows that this excess is, in different strings of the same make and materials, proportional to their lengths. For the tension of a string being the same in every part, if we divide the string into any number of equal parts, the increase of length in each part will be equal, and therefore the increase of the whole string will be proportional to the number of these equal parts which it contains; that is, to its length. Consequently, upon the whole, the increase of length of a string is proportional to

(its length) × (weight which stretches it).

Wherefore, if L be the natural length of a string, and l its length when stretched by a weight W,

$$l - \mathrm{L} \propto \mathrm{L}.\mathrm{W} = \mathrm{C}.\mathrm{LW};$$

where C denotes a constant dependent on the material, thickness and make of the string.

(46). Suppose the string AB (Fig. 131), whose length is a, to be suspended vertically from one end A, and stretched by its own weight w only; to determine the increase of its length.

In AB take any points P, Q very near to each other, and when the string is stretched let b, p, q, a be the points corresponding to B, P, Q, A; $x = \mathrm{BP}$, $\delta x = \mathrm{PQ}$, $y = bp$, $\delta y = pq$. Then δx is stretched into δy by the weight of bp or BP which $= \dfrac{wx}{a}$;

$$\therefore \delta y - \delta x = \mathrm{C}.\delta x.\frac{wx}{a};$$

therefore, dividing by δx, and taking the limits,

$$d_x y - 1 = \mathrm{C}\cdot\frac{wx}{a};$$

$$\therefore y - x = \frac{\mathrm{C}}{2}\cdot\frac{wx^2}{a}, \text{ by integration};$$

$$\therefore ab - \mathrm{AB} = \frac{\mathrm{C}}{2}\cdot\frac{wa^2}{a} = \tfrac{1}{2}\mathrm{C}wa.$$

Hence the increase is one half of what it would be, if AB were stretched upon a horizontal table by a weight equal to its own weight.

(47). If a weight W be now suspended from b, we can determine the further increase of length.

For the weight which stretches pq is, in that case,

$$W + \frac{wx}{a};$$

$$\therefore \delta y - \delta x = C.\delta x.\left(W + \frac{wx}{a}\right);$$

$$\therefore d_x y - 1 = C\left(W + \frac{wx}{a}\right);$$

$$\therefore y - x = C\left(Wx + \frac{wx^2}{2a}\right);$$

$$\therefore ab - AB = CWa + \tfrac{1}{2}Cwa.$$

Of this increase the part $\frac{1}{2}Cwa$ we have seen is due to the weight of the string, and therefore CWa, the part due to the weight W, is the same as if the string had no weight. Hence when a string is stretched by several forces, each one produces as great an increase of length as it would do if the other forces did not act.

(48). Two weights P, Q (Fig. 132), resting on two inclined planes AB, AC, are connected by a given elastic string; to find the position of equilibrium.

Let α, β be the inclinations of AB, AC, and θ that of PQ to the horizon; a = the natural length of PQ; T = its tension. Then P is kept in equilibrium on the plane AB by T acting in the direction PQ;

$$\therefore T \cos APQ = P \sin \alpha, \text{ (Art. 215).}$$

But $APQ = \alpha - \theta$;

$$\therefore T = \frac{P \sin \alpha}{\cos(\alpha - \theta)}.$$

$$\text{Similarly, } T = \frac{Q \sin \beta}{\cos(\beta + \theta)};$$

$$\therefore P\frac{\cos(\beta+\theta)}{\cos\theta\sin\beta} = Q\frac{\cos(a-\theta)}{\cos\theta\sin a};$$

$$\therefore P(\cot\beta - \tan\theta) = Q(\cot a + \tan\theta);$$

$$\therefore \tan\theta = \frac{P\cot\beta - Q\cot a}{P+Q},$$

$$\text{and } PQ = a + C.a.T$$

$$= a\left(1 + \frac{C.P\sin a}{\cos(a-\theta)}\right).$$

From which PQ is known, and thence AP and AQ by means of the triangle APQ, whose angles are all known.

(49). Two equal weights P, Q (Fig. 133) are connected by an elastic string, whose natural length is BC; to find the nature of the curves BP, CQ, on which they will always rest in equilibrium with the string parallel to the horizon; the plane of the curves being vertical.

It is manifest, since the weights are equal, that the curves must also be equal. Bisect BC in A, and draw AM vertical; $AB = AC = a$, $AM = x$, $MP = MQ = y$, $T =$ the tension of PQ;

$$\therefore PQ - BC = C.BC.T,$$

$$\text{or } 2y - 2a = C.2a.T;$$

$$\therefore y - a = CaT.$$

But P being sustained upon the curve BP by its gravity P and the force T, we have by Prob. 29,

$$Pdx - Tdy = 0;$$

$$\therefore T = Pd_y x;$$

$$\therefore y - a = CaPd_y x;$$

$$\therefore (y-a)^2 = 2CaPx, \text{ by integration,}$$

which is the equation of a parabola. Hence BP, CQ are two semi-parabolas, whose vertices are B, C.

(50). Let a cord PAQ (Fig. 134) passing over a smooth cylinder, be acted upon by two equal forces P, Q; to find the pressure of the cord upon the cylinder.

Since the tension and the curvature of the cord are the same in every part in contact with the cylinder, the pressure upon every point will be the same. Let AB be a very small element; join CA, CB (C being the centre of curvature), and let AT, BT be tangents at A and B. Then AB is kept at rest by the tensions at A and B, and the reactions of the cylinder at every point of AB. All these equal reactions may be supposed to take place at the middle point D, (that being the place of their resultant). These three forces are in the directions TA, AD, DT respectively, which pass through the point T, and may therefore be transmitted to it. Wherefore, putting the angle ACB $= \theta$, the angle ATB $= 180° - \theta$, and since the pressure DT is the resultant of the two tensions at A, B (Art. 20), we have

$$(\text{pressure on AB})^2 = P^2 + 2P.P\cos(180° - \theta) + P^2 \text{ (Art. 26)},$$

for P is equal to each of the tensions;

$$\therefore (\text{pressure on AB})^2 = 2P^2(1 - \cos\theta),$$

$$= 4P^2 \sin^2\frac{\theta}{2},$$

$$= 4P^2\left(\frac{\theta}{2}\right)^2 \text{ ultimately},$$

$$= P^2\theta^2;$$

$$\therefore \text{pressure on AB} = P\theta.$$

And as the pressure on every equal element is the same, the pressure on any arc of the cylinder is equal to the product of the tension into the angle (expressed in terms of the radius) subtended by the arc at the centre.

(51). An elastic ring CD (Fig. 135) is placed round a vertical cone and descends by its own weight; required the position of equilibrium.

Let AB be the axis of the cone; CD the position of equilibrium of the ring; a the radius of the ring when unstretched, $AB = x$, $\alpha =$ the angle CAB, $T =$ the tension of the ring. Then the original length of the ring $= 2\pi a$, and the stretched length $= 2\pi . x \tan \alpha$, for $BC = x \tan \alpha$;

$$\therefore 2\pi x \tan \alpha - 2\pi a = C . 2\pi a . T;$$

$$\therefore T = \frac{x \tan \alpha - a}{aC}.$$

Therefore the pressure of the ring upon the cone, in a horizontal plane,

$$= T . 2\pi = \frac{x \tan \alpha - a}{aC} . 2\pi, \text{ (Prob. 50).}$$

and this being the force which, acting horizontally, sustains the ring upon the inclined plane, viz.—the surface of the cone, whose inclination to the horizon is $90° - \alpha$, we have, by Art. 216,

$$2\pi . \frac{x \tan \alpha - a}{aC} = W . \tan (90° - \alpha),$$

$$= W \cot \alpha,$$

where W denotes the weight of the ring;

$$\therefore BC = x \tan \alpha = a + \frac{aCW}{2\pi} \cot \alpha,$$

$$= a \left(1 + \frac{CW}{2\pi} \cot \alpha\right),$$

which determines the position of equilibrium.

(52). *White's Pulley.* In the common systems of pullies, each pulley has its own independent centre of motion, and consequently, as they all move with different velocities and with different degrees of pressure, some of them will be liable to greater wear than others, which will very much tend to increase the friction and other inequalities and resistances; which will greatly diminish the efficiency of the machine. To obviate these difficulties, Mr. James White invented a moufle (Fig. 136), consisting of two blocks A, B; into which grooves were cut, the radii of those in the lower block being as the numbers 1, 3, 5 . . . . and the radii of those in the upper block being as the numbers 2, 4, 6 . . . . Now, suppose the lower block to be raised through one inch, then each of its strings will be shortened one inch, and therefore the circumference of the pulley BA_1 describes one inch; that of AA_1, two inches; that of BB_2, three inches, and so on; which numbers being proportional to the radii of the respective pullies, they will all move with the same angular velocity; and, consequently, each block instead of being composed of separate pullies may consist of one solid piece of wood or metal, containing the grooves before mentioned. The disadvantage of this system is, that if the cord be at all elastic the friction becomes very great, on account of the tension not being the same in every part.

(53). *Hunter's Screw.* We have seen (Art. 218) that the advantage of a screw increases in proportion as the distance between the threads diminishes, and as the length of the lever at which the power act increases; therefore, by making the threads of the screw sufficiently fine, we may increase the advantage as much as we please; but there is a limit to the fineness of the thread; for as all the weight is borne upon them, if they are too fine they will not be sufficiently strong to bear the load laid upon them. If we, on the contrary, increase the length of the arm of the lever, for the purpose of increasing the advantage of the screw, the power will have to describe an inconveniently large space. To obviate these natural defects, and increase the advantage to any degree, Mr. Hunter invented the screw in Fig. 137; A and B are two common screws, of which A is also a hollow screw to admit B, which is fastened to the moveable plane D of wood or metal. If D, d be the distances between two threads of the screws A, B respectively; then, while the power describes one circumference, A descends through D, and B ascends in A through d, and the space descended by the plane D is $D - d$; for when A descends it carries B along with it, though B is at the same time ascending in A. Wherefore, by Art. 193,

$$P . (\text{circumf. described by } P) = W . (D - d);$$

$$\therefore \frac{W}{P} = \frac{\text{circumf. described by } P}{D - d}.$$

Now we can make D and d as nearly equal as we please without diminishing the strength of the machine, and therefore the advantage of this screw admits indefinite increase.

(54). It appears from Art. 212, that the advantage of a wheel and axle is

$$\frac{\text{rad. of wheel}}{\text{rad. of axle}},$$

which might theoretically be augmented ad libitum, either by increasing the radius, or by diminishing that of the axle. But by the former means, the power would practically have to describe an inconveniently large space, and the machine would become cumberous; and, in the latter case, it would be too weak to bear the pressure of the weight upon its axle. To remedy these inconveniences, and at the same time to increase the advantage in any requisite degree, the form of Fig. 138, has been given to it; where A is the wheel, B and C two axles of unequal radii, firmly fixed to each other, and having the same axis. The cord BDC as P descends is wound upon the axle B with the larger radius, and is at the same time unwound from the axle C with the smaller radius; it passes under a pulley D, to which the weight W is attached. Let R be the radius of the wheel, $r\,r'$ those of the axles B, C. Then when the machine turns once round, P descends through $2\pi R$, and the length of the cord wound upon B is $2\pi r$, and the length unwound at the same time from C is $2\pi r'$; wherefore, upon the whole, the length of cord hanging down from the axles is diminished by

$$2\pi r - 2\pi r';$$

and, therefore, as there are two parts of the cord, W has ascended through

$$\pi r - \pi r'.$$

Wherefore, by Art. 193,

$$P : W :: \pi r - \pi r' : 2\pi R,$$

$$:: \quad r - r' \quad : 2R;$$

$$\therefore \text{ the advantage} = \frac{W}{P} = \frac{2R}{r - r'};$$

and as we can diminish the denominator of this fraction as much as we please, without weakening the materials of the machine, there is no limit to the advantage of it, except what arises from the very great length of cord that must be used in raising W through a very small space.

(55). *The Genou.* This instrument is represented in its simplest form in Fig. 139, where AF is the profile of a frame in which the rods AB, BC work. AB is moveable about a fixed axis passing through A; it is connected with BC by a compass joint at B; and the other end C of BC, by means of a pin passing through it, is compelled to move in the vertical groove EF. The power is applied at G, a point in AB, in the plane of the rods ABC. It causes B to come nearer to AF; and, consequently, C presses downwards upon any obstacle opposed to it. It is obvious this machine is only applicable in those cases in which C is required to descend through a small space, as in printing, where it presses the paper upon the type.

Let W = the reaction at C, P the power applied horizontally at G, θ = the angle BAF, a = AB, b = BC, c = AG, and let GP intersect AF in p. Then $Gp = c \sin\theta$, and therefore the virtual velocity of P

$$= d_t(Gp) = c \cos\theta \,.\, d_t\theta.$$

Also $AF = a\cos\theta + b\cos BCA$, therefore the virtual velocity of F or W

$$= d_t(AF) = -a\sin\theta \, d_t\theta - b\sin BCA . d_t(BCA).$$

$$\text{Now } \sin BCA = \frac{a}{b}\sin\theta;$$

$$\text{and } \therefore \cos BCA . d_t(BCA) = \frac{a}{b}\cos\theta \, d_t\theta;$$

$\therefore$ the virtual velocity of W

$$= -a\sin\theta \,.\, d_t\theta - a\sin\theta . \frac{a}{b} . \frac{\cos\theta \, d_t\theta}{\cos BCA},$$

$$= -\left(1 + \frac{a}{b} . \frac{\cos\theta}{\cos BCA}\right) . a\sin\theta \,.\, d_t\theta.$$

Wherefore, by Art. 192, the advantage of the machine

$$= \frac{W}{P} = \frac{c\cos\theta \, d_t\theta}{\left(1 + \frac{a}{b} . \frac{\cos\theta}{\cos BCA}\right) . a\sin\theta \, d_t\theta},$$

$$= \frac{c}{a} \cdot \frac{b \cos BCA \,.\, a \cos \theta}{a \sin \theta \, (a \cos \theta + b \cos BCA)},$$

$$= \frac{AG \,.\, Cb \,.\, bA}{AB \,.\, Bb \,.\, AC},$$

where Bb is drawn parallel to GP.

(56). A combination of wheels and axles may be used instead of the machine in Prob. 54, when that is inconvenient and great advantage is required. Fig. 140 represents a combination of three of these mechanical powers. An endless strap passes over the axle a and the wheel B, and another strap passes over the axle b and the wheel C. If two successive wheels are required to turn in opposite directions, the strap must be crossed as between A and B in the figure; when the wheels are to turn in the same direction, the strap must not be crossed. B and C are turned by the friction of the straps upon their surfaces; and hence it is manifest, that if the force to be overcome by any wheel be greater than the friction of its strap, the strap will slip round without carrying the wheel with it, and the action of the machine will cease. Wherefore, in order to make the friction upon the surfaces of the wheels and axles as great as possible, they are covered with leather, which is nailed or glued on them; and both this leather and the concave side of the straps are suffered to be in a rough state; the friction is also increased by crossing the straps.

To calculate the advantage of this combination, denote the tension of the strings d and e by T, T′; then since P balances the tension T on the axle a, we have, by Art. 212,

$$\frac{T}{P} = \frac{\text{rad. of wheel A}}{\text{rad. of axle } a}.$$

$$\text{Similarly, } \frac{T'}{T} = \frac{\text{rad. of wheel B}}{\text{rad. of axle } b},$$

$$\text{and } \frac{W}{T'} = \frac{\text{rad. of wheel C}}{\text{rad. of axle } c};$$

and, therefore, by multiplying these equations together, we have

$$\frac{W}{P} = \frac{\text{product of radii of all the wheels}}{\text{product of radii of all the axles}}.$$

(57). To calculate the friction of the straps in the last problem.

Let C (Fig. 141) be the centre of the wheel (or axle) APB over which passes the strap TAPBT'. Let AB be the arc of contact, PQ a very small element of it. T, T' the tensions of AT, BT'; $t =$ the tension at P, $t + \delta t =$ that at Q; ACP $= \theta$, PCQ $= \delta\theta$, $f =$ the coefficient of friction. Then, by Prob. 50,

the pressure on PQ $= t\delta\theta$ ultimately;

$\therefore$ the friction on PQ $= f \,.\, t\delta\theta$.

But the tension at Q $=$ tension at P $-$ friction of PQ;

$\therefore$ friction on PQ $= t - (t + \delta t)$,

$= -\delta t$;

$\therefore \delta t = -ft\,\delta\theta$;

$\therefore \frac{d_\theta t}{t} = -f$, by dividing by $t\delta\theta$ and taking the limits;

$\therefore \log t = -f\theta + C$ by integration.

Now at A, $t = T$ and $\theta = 0$;

$\therefore \log T = C$,

and at B, $t = T'$ and $\theta =$ ACB.

$$\therefore \log T' = -f . ACB + \log T;$$

$$\therefore T' = T . e^{-f . ACB};$$

$$\therefore T - T' = T(1 - e^{-f . ACB}),$$

a quantity which is equal to the whole friction, and increases as the angle ACB increases, and will evidently be the greatest when the strap is crossed.

(58). *Toothed Wheels.* By far the most general modification under which wheels and axles are used in practical Mechanics, is that of toothed wheels.

Let A, *a* (Fig. 142) be the centres of two wheels BC, *bc*; upon the circumferences of which let teeth or cogs D, E, F; *d*, *e*, *f*; of any proposed form, be raised at equal distances all round; in order that this may be possible, the radii of the two wheels must be in proportion to the number of teeth that are to be constructed upon them. If one of the wheels (*bc* for instance) be turned round its axis *a*, its teeth will press upon the teeth of the other wheel BC, and turn it round its axis A in a contrary direction, and as two corresponding teeth F, *f* separate from each other in consequence of the motion, two others D, *d* come in contact; and thus the wheel *a* is enabled to produce a continuous motion in the wheel A. Similar teeth are constructed upon the axles of each wheel, and the axle so prepared is called a *pinion*, and its teeth are called *leaves*. From the nature of the wheel and axle it is manifest that motion is communicated to each wheel, in this modification, by a pinion in which it runs as in Fig. 143, where P descending turns with it the pinion *a* which turns the wheel B, and this carries with it the pinion *b* which turns the wheel C and axle *c*, and raises the weight W. In this case, as in problem 56, it is clear that

$$\frac{W}{P} = \frac{\text{product of the radii of the wheels}}{\text{product of the radii of the axles}},$$

$$= \frac{\text{radius of A}}{\text{radius of } c} \times \frac{\text{product of number of teeth in the wheels}}{\text{product of number of leaves in the pinions}}.$$

Here there are no teeth in A and *c*, on which account we have not reduced their radii to equivalent numbers of teeth.

(59). In the description of toothed wheels we have said that the teeth or cogs are to be of any proposed form, because in fact they are commonly made in any form that meets the fancy of the maker. It must not be imagined, however, that all forms are equally advantageous, as we shall easily understand by referring to Fig. 144, and tracing the actions of the teeth upon each other during their motion. Suppose *bc* to begin to turn round, and let us trace the actions of *d* and D. When *d* first comes in contact with D, the latter presses against the side of *d* in a single line of points, very near the extremity of *d*, in the direction of a normal to the side of *d*, that is, in the direction *p*D perpendicular to the radius *ad*. Therefore, drawing A*p* parallel *ad*; the action of *d* may be transmitted to *p*, and its efficiency varies

as A*p*.

But as the wheel *bc* continues turning, the point of contact D slides along the side of *d*, and thus produces a very strong friction, and consequently rapid wear both of the side of *d* and of the edge of the tooth D. This goes on till *d* and D come into the position *e* and E, when their sides are for a moment in contact, and then the efficiency of *d* in turning D varies

as AD.

When the teeth *d* and D leave this position a similar action to what has just been described commences, only it is in a reverse order; and the edge of the tooth *d* presses against and rubs the side of the tooth D.

It appears then, with teeth of the form of those in this figure,—

1st. That the efficiency of the pressure which one tooth exerts upon another, and consequently the motion produced, is ***very irregular***, being in one position proportional to A*p*, and in another to AD.

2ndly. That the edges of the teeth are subject to very rapid wear in consequence of ***rubbing*** with a single line of points in contact with the sides of the teeth of the other wheel, which latter is thereby also very soon worn hollow, and the whole rendered useless.

3rdly. That in consequence of the rubbing of the teeth against each other much of the power is rendered ineffective.

4thly. That since there are favourable and unfavourable positions, the power must be sufficient to move the weight in the most unfavourable position with the requisite degree of celerity; and consequently, when the machine is in the most favourable position there will be a great excess of power which will cause the machine to move much too rapidly, and often produce fractures; nothing in fact having so great a tendency to tear asunder the parts of a machine and render it useless as an irregular motion of this kind.

From these considerations it will at once be evident that the best form of the teeth will be, when,—

1st. The teeth of one wheel press upon those of the other in such a direction that the efficacy may be uniform; that is, such that the perpendiculars upon that direction from A and *a* are of constant lengths.

2ndly. The teeth of one wheel do not *rub* but *roll* upon those of the other.

3rdly. The motion of one tooth upon another is uniform.

When these conditions are fulfilled, it is also necessary that the distances of the axes of the wheels should be such that as great a number of teeth may be in contact at one time as possible, and that there may be no jolting nor violence of any kind when two teeth separate or come in contact. These precautions will diminish the chances of fracture very much.

Many forms of teeth have been proposed fulfilling one or more of these conditions, but it seems to be agreed on that the following is the best.

(60). Let ABD (Fig. 145) be a given wheel on which it is proposed to erect teeth; and let AB be the proposed breadth of a tooth. Upon AD wrap a string and fasten it at D. Then unwrap it, beginning at A, and its extremity A will trace out the curve Aa called the involute of the circle AD. In a similar manner, describe the involute Bb intersecting the former in C; then ACB will be the tooth required, which may be taken as a pattern of all the others to be formed upon the wheel. In a similar manner the leaves of the pinion may be found, by first constructing a pattern by means of the involute of its circumference. Let PL be a position of the thread whose extremity generates the involute Aa; then we may suppose the point L to be fixed for an instant, and therefore P will begin to describe an arc of a circle whose centre is L, and therefore PL is a normal to the curve AC, and OL the perpendicular upon this normal is constant. In the same manner it may be shewn, that the normals to the leaves of the pinion are all constant and equal to the radius of the pinion. Wherefore, since the leaves of the pinion press against the teeth of the wheel in the directions of normals at the points of contact, and the perpendiculars on these directions are always the same, the action will be uniform, and consequently the motion will be uniform also.

Let A, *a* (Fig. 146) be the centres of two toothed wheels, P the point of contact of two teeth C, *c*; L, *l* the line of pressure which is a tangent to both circles, cutting A*a* in B. This line is fixed in position, because the circles are given in magnitude and position; and therefore B is a fixed point. Also, the angular motion of P about l = the angular velocity of the wheel *a*, and therefore the linear motion of P along the tooth $cP = lP$. angular velocity of wheel *a*. Similarly, the linear motion of P along the tooth $CP = LP \times$ angular velocity of the wheel A. Now the angular velocities of A and *a*, are proportional to $\frac{1}{AL}$ and $\frac{1}{al}$ respectively.

Hence $\frac{\text{velocity of P along CP}}{\text{velocity of P along } cP} = \frac{LP}{AL} \cdot \frac{al}{lP} = \frac{al}{AL} \cdot \frac{LP}{lP}$.

As this ratio is not one of equality, the teeth will not roll but rub;* and consequently there will be some friction; the distance of the centres A, *a*, may however be so adjusted as to cause the fraction

$$\frac{al}{AL} \cdot \frac{LP}{lP}$$

---

* *Every* author I have been able to consult upon this point, affirms that the teeth *roll* upon each other, and do not *rub*; as, however, after several revisals, I have not discovered any error in the investigation of the matter given in the text, I am obliged to differ from them.

not to differ much from unity, and then the friction will be but small.

Notwithstanding this defect, the form of the teeth here detailed possesses very great advantages over every other, on account of the uniformity of action, and consequent uniformity of effect, which it produces.

(61). In the construction of wheel-work, there is yet one thing of practical importance to be observed. If the number of leaves in a pinion be an aliquot part of the number of teeth in the wheel in which it runs, a leaf of the pinion will always come in contact with the same set of teeth in the wheel; for instance, if there be 5 leaves in the pinion, and 30 teeth in the wheel, a leaf of the pinion will rub upon every 5th tooth in the wheel, and there will be 6 teeth which are rubbed by the same leaf, and by no other. Hence, unless both the pinion and wheel are quite accurately constructed, and each of perfectly homogeneous materials,—things which it is almost impossible should occur in practice,—there will be a considerable inequality of wear; and consequently the machine will be sooner worn out than it otherwise would. In order to remedy this, the numbers of leaves and teeth ought to be prime to each other, and then each leaf of the pinion will in its turn come in contact with each tooth of the wheel, and perfect uniformity of wear will be ensured.

(62). The *Endless Screw*. This machine, represented in Fig. 147, consists of a screw A whose axis is BC; and a wheel and axle D, E; the wheel being furnished with teeth exactly fitting the threads of the screw. The screw is turned by means of the winch CP, and its thread instead of pressing against a nut, press against the teeth of the wheel, and force them forward; each turn of the screw or winch, advancing the wheel one thread of the screw; or, which is the same, one tooth of the wheel. The winch must therefore be turned round as many times as there are teeth in the wheel, in order to turn the axle E once round. Wherefore, putting R for the radius of the circle described by the power P; r for that of the axle E, and n for the number of teeth in the wheel D; the circumference described by $P = 2\pi R$, and therefore the space described in one turn of the wheel D, is

$$2n\pi R.$$

But the space ascended by W in the same time = the circumference of the axle E

$$= 2\pi r.$$

Consequently, by Art. 193,

$$\frac{W}{P} = \frac{2n\pi R}{2\pi r} = n \cdot \frac{R}{r}.$$

(63). *Limits of Equilibrium.* Friction, as observed in Chapter VII, opposes the commencement of motion, and consequently if P and W are in equilibrium on any machine supposed free from friction; we might, when friction acts, diminish P, and still W would be prevented by the friction from beginning to move; we may go on diminishing P until W has such an excess of weight that if P be any more diminished W will overcome the friction and begin to move; this value of P is the least that can maintain equilibrium, and is therefore called the ***inferior limit*** of P.

On the contrary, if instead of diminishing P we had increased it continually, we should have arrived at such a value, that if it were any more increased it would overcome the friction and put the machine in motion; this is called the *superior limit* of P.

(64). To find the limits of equilibrium on the inclined plane.

Let AB (Fig. 74) be the plane, AC its base, BC its altitude; f = the coefficient of friction; therefore the friction $= f\text{R}$, and for the inferior limit this force acts upon W in the direction WB. W is therefore kept at rest by four forces P, R, fR and W acting respectively in the direction WP, WR, WB and WG; wherefore, first resolving these parallel to AB, and then perpendicular to AB, we have separately, by Art. 40,

$$\text{P} \cos \text{PWB} + \text{R} \cos \text{RWB} + f\text{R} \cos 0° - \text{W} \cos \text{AWG} = 0,$$

and

$$\text{P} \sin \text{PWB} + \text{R} \sin \text{RWB} + f\text{R} \sin 0° - \text{W} \sin \text{AWG} = 0,$$

which equations, by using the notation of Art. 215, become

$$\text{P} \cos \theta + f\text{R} - \text{W} \sin \text{A} = 0,$$

$$\text{P} \sin \theta + \text{R} \quad - \text{W} \cos \text{A} = 0;$$

$$\therefore \text{P} = \text{W} . \frac{\sin \text{A} - f . \cos \text{A}}{\cos \theta - f . \sin \theta},$$

by eliminating R. The superior limit will be obtained by changing f into $-f$, and is therefore

$$\text{P} = \text{W} . \frac{\sin \text{A} + f \cos \text{A}}{\cos \theta + f \sin \theta}.$$

These expressions may be put under very simple forms by using α for the inclination of the plane in the last experiment of Art. 222; for then, by Art. 223, $f = \tan \alpha$, and therefore the inferior value of P, becomes

$$W \cdot \frac{\sin A - \tan a \cdot \cos A}{\cos \theta - \tan a \cdot \sin \theta},$$

$$= W \cdot \frac{\cos a \sin A - \sin a \cos A}{\cos a \cos \theta - \sin a \sin \theta},$$

$$= W \cdot \frac{\sin (A - a)}{\cos (\theta + a)}.$$

Similarly, the superior value of P is

$$W \cdot \frac{\sin (A + a)}{\cos (\theta - a)}.$$

In a similar manner we may determine the limits of equilibrium in any proposed machine whatever.

(65). A body W rests upon a horizontal plane AB (Fig. 148); required the direction WP in which the least possible power P must act in order to move it.

Put angle PWB $= \theta$, and denote the coefficient of friction by $\tan a$; therefore the friction acts in the direction WA, and is equal to fR; and to find the superior limit of P, we observe that this may be deduced from the last problem by supposing $A = 0$;

$$\therefore P = \frac{W \sin a}{\cos (\theta - a)},$$

which will be the least possible, when $\theta - a = 0$, or $\theta = a$, and

$$\therefore P = W \sin a$$

is the least possible limit of P. And since $\theta = a$, it appears that "the best angle of draught is exactly that obliquity which should be given to a road in order to enable a carriage to move of itself.

"This obliquity is sometimes called the *angle of repose*, and is that angle which determines the proportion of the friction to the pressure in the second method, explained in

Art. 222. The more rough the road is, the greater will this angle be; and therefore it follows, that on bad roads the obliquity of the traces to the road should be greater than on good ones. On a smooth Macadamized way, a very slight declivity would cause a carriage to roll by its own weight; hence, in this case, the traces should be nearly parallel to the road.

"In rail roads, for like reasons, the line of draught should be parallel to the road, or nearly so." *

(66). *Friction Wheels* are a contrivance for diminishing the effect of friction when the pressure is very great.

A and B (Fig. 149) are two equal wheels, whose axes are a, b; which are so situated that the line ab is horizontal; C is another wheel sustaining a great pressure, and having one end c of its axle resting upon the wheels A, B. The other end of its axle is supported on two other wheels, similar and equal to A, B, which are also fixed upon the same axles as A and B. To calculate the friction in this construction, let f be the coefficient of friction of a and b; and f' the coefficient of rolling friction of c, which by Art. 230 is very much less than f; R = radius of A or B, r the radius of their axles a, b; and r' the radius of the axle c; W = the weight of A and its axle, which is also that of B and its axle; W′ = the weight of C, its axle and load.

The friction of the two ends of $a = f(W + \frac{1}{2}W')$ nearly; and its moment

$$= f(W + \tfrac{1}{2} W')\, r;$$

and, therefore, the whole moment opposing the motion of C

$$= f(2W + W')r,$$

* Captain Kater's Machines.

which is equivalent to a friction $f(2W + W') \cdot \frac{r}{R}$ acting at the circumference of c; wherefore the whole friction at the circumference of $c = f'W' + f(2W + W')\frac{r}{R}$. Now if there were no friction wheels, the friction opposing c would be

$$fW',$$

and ratio of the two is

$$\frac{f'}{f} + \left(2\frac{W}{W'} + 1\right)\frac{r}{R};$$

which is very small, because f', W and r are respectively very much smaller than f, W′ and R. And, consequently, this contrivance does very much diminish the friction, providing the wheels A, B are not too heavy nor too small.

In practice W is very much less than W′, and consequently we may use the expression

$$\frac{f'}{f} + \frac{r}{R}$$

as the measure of the advantage gained by the use of friction wheels.

(67). *Roberval's Balance.* This machine consists of four straight rods AB, B*b*, *ba*, *a*A (Fig. 150), forming a parallelogram in a vertical plane, and being connected by compass joints at B, *b*, *a*, A; at C and D the middle points of the rods AB and *ab* there are fixed axes about which they are moveable; GE, FH are two rods rigidly connected with A*a* and B*b*, from which the equal weights P and Q are suspended. The peculiarity of this balance is, that P and Q will be in equilibrium from whatever points of the rods GE and FH they are suspended. To prove this property, suppose the machine to be put in motion; then if A ascends, B will descend through an equal space; and as AB ba must necessarily continue to be a parallelogram, Aa and Bb will continue parallel to CD, and therefore each vertical; wherefore E will ascend and F will descend though spaces respectively equal to those described by A and B, and therefore equal to each other. It is also manifest, since Aa and Bb continue vertical during the motion, that GE and FH move parallel to themselves, and consequently the space ascended by P is equal to that descended by Q, wherefore they satisfy the equation of Art. 193, and are consequently in equilibrium in every position by Art. 194.

As the two parts of this balance, to the right and left of the points of support C, D, are perfectly equal and similar, à priori reasoning would lead us to imagine that P and Q could not balance unless placed in corresponding positions.

THE END.

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