THE THEORY OF THE SUBMARINE TELEGRAPH & TELEPHONE CABLE

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PREFACE.

THIS book is the outcome of a desire on the part of its author to understand the phenomena which underlie the working of the submarine telegraph cable. On entering upon the study of the subject he found very little information to be available, apart from the profound theoretical researches of Oliver Heaviside. Although excellent treatises are devoted to the manufacture, laying and maintenance of cables, no reasoned explanation existed of the principles of telegraphic transmission, although to such transmission the other processes are all subservient.

The results of his investigations appeared in *The Electrician* in a serial article in 1912, and subsequent Papers have continued them. They are brought together here, with much additional matter, consisting chiefly of the chapters on the Theory of the Submarine Telephone Cable, and it is hoped that they will prove useful as an introduction to the study of this important subject.

The principles which have guided the setting forth of these results are as follow :—

(*a*) To illustrate every formula graphically in its most important cases, so as to bring out the full significance of its various parts ;

(*b*) To apply all formulæ, as far as possible, to the same typical cable or cables, so that they may be directly comparable one with another ; and

(*c*) To express all results in the ordinary practical units to render them intelligible at once to anyone who may not have followed the processes by which they were derived.

Although, as a rule, arithmetical computation is laborious, there is no other way to a full understanding of a formula, and there is no better corrective to speculation in dealing with a quantitative science.

The book makes no pretence to completeness, an approach to which in the present state of knowledge would be quite impossible. It is hoped, nevertheless, that this will not impair its usefulness, but may tend rather to stimulate research and lead towards invention and discovery which will be of lasting benefit to cable telegraphy.

The student is recommended to work out his own sets of curves for the cables and the apparatus with which he has to deal. The author will be glad to receive corrections or suggestions by means of which the book may be improved, and to hear from anyone who has found difficulty in understanding any part of the text.

He has pleasure in acknowledging his indebtedness to Dr. C. V. Drysdale, who has kindly read the proofs of the first seven chapters.

H. W. M.

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CHAPTER I.

MATHEMATICAL RECAPITULATION.

The Exponential Series—Hyperbolic Functions—Imaginary Quantities—The Circular Functions—Geometrical Representation of the Circular and the Hyperbolic Functions—Complex Quantities—Development in Trigonometric Series—Determinants.

THE EXPONENTIAL SERIES.

At an early stage in the study of cable phenomena it becomes necessary to express results in terms of what are known as *Infinite Series.* Such expressions consist of the sum of an infinite number of terms which are formed in succession according to some law. The arithmetical computation of the value of the series is carried out usually by calculating it term by term and adding the result. The most useful kind of series is that which tends to a definite limit as the number of terms is increased without limit. Such a series is said to be *convergent.* When the successive terms of a series grow rapidly smaller, the series is said to be rapidly convergent. Its arithmetical value may then be found with sufficient exactitude by taking the first few terms of the series. It is with series of this kind that we shall most frequently have to deal.

The series $1+x+\frac{x^2}{1\times 2}+\frac{x^3}{1\times 2\times 3}+\ldots$ is of so great importance in analysis that a special name is given to it. It is called the *exponential* series. The first three letters of the word exponential may be used as a convenient contraction.

Thus $$\text{exp. } x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots \text{ to } \infty \quad . \quad . \quad . \quad . \quad (1)$$

Exp. x is a *function* of x alone, *i.e.*, its value is fixed as soon as x is given. Whatever x may be, the higher terms in the series are very small, because of the rapidly increasing denominators, and are ultimately infinitely small. The ratio of the $\overline{n+1}$th term to the nth term is $\frac{x^{n+1}}{\underline{/n+1}} \div \frac{x^n}{\underline{/n}} = \frac{x}{n+1}$, so that when n is very great—ultimately infinitely great—each term is infinitely smaller than the preceding one. In approximating, therefore, to the value of exp. x, after a number of terms, greater or less, according as x is greater or less, has been taken, a great number of additional terms may be taken without affecting appreciably the first approximation to exp. x. The series *converges* to a definite limit as the number of terms is increased, and, since this is unconditionally true, the series is said to be *absolutely convergent.*

When the value unity is given to x, exp. $x=2{\cdot}71828$ to five places of decimals, as may easily be verified. This number is the base of the Naperian system of logarithms, and is customarily denoted by the letter e.

Two important properties of the exponential series are discussed below :—

(a) Every term in the series exp. x is the differential of the one following it, and the integral of the one preceding it. The series is its own differential, and $\frac{d}{dx}$ (exp. x) = exp. x.

The differential equation $\frac{dy}{dx}+ay=b$, has for solution $y=\mathrm{e}^{-ax}+\frac{b}{a}$, as may easily be verified by substitution.

In like manner $y=e^{-ax}-\frac{b^2}{a^2}$ and $y=e^{ax}-\frac{b^2}{a^2}$ are solutions of the equation $\frac{d^2y}{dx^2}-a^2y=b^2$.

(b) Exp. $x_1 \times$ Exp. x_2 = Exp. (x_1+x_2).

This may be demonstrated as follows :—Exp. x may be written $1+x+\frac{x^2}{\underline{/2}}+\mathrm{R}$, where R is the rest of the series. Hence

$$(\text{exp. } x_1-\mathrm{R}_1)\ (\text{exp. } x_2-\mathrm{R}_2)=\left(1+x_1+\frac{{x_1}^2}{\underline{/2}}\right)\left(1+x_2+\frac{{x_2}^2}{\underline{/2}}\right).$$

The product on the right hand lies between

$$1+(x_1+x_2)+\frac{(x_1+x_2)^2}{\underline{|2}}$$

and
$$1+(x_1+x_2)+\frac{(x_1+x_2)^2}{\underline{|2}}+\frac{(x_1+x_2)^3}{\underline{|3}}+\frac{(x_1+x_2)^4}{\underline{|4}}.$$

If instead of three terms of the exponential series a great number of terms had been taken to form the product, then

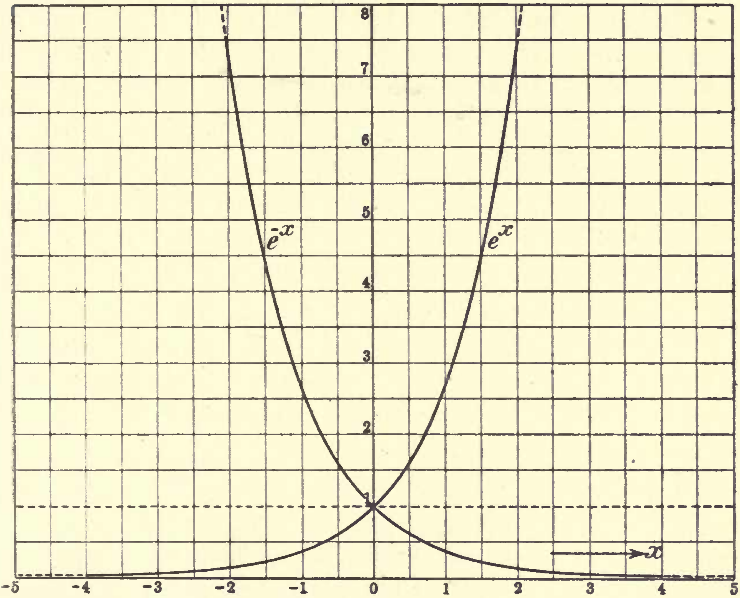

FIG. 1.—GRAPHS OF e^x AND e^{-x}, FOR REAL VALUES OF x.

R_1 and R_2 would be very small, ultimately infinitely small; also the difference between the limiting values of the product would be ultimately infinitely small, from the absolutely convergent nature of the exponential series. Hence, proceeding to the limit, when R_1 and R_2 are zero,

$$\text{exp. } x_1\times\text{exp. } x_2=1+\overline{x_1+x_2}+\frac{\overline{x_1+x_2}^2}{\underline{|2}}+\ldots \text{ to } \infty$$

$$=\text{exp. }(x_1+x_2) \quad . \quad . \quad (2)$$

By repeated multiplication,

$$\text{exp. } x_1 \times \text{exp. } x_2 \times \ldots \times \text{exp. } x_m = \text{exp. } (x_1 + x_2 + \ldots x_m).$$

When $x_1 = x_2 = \ldots = x_m = x$,

$$(\text{exp. } x)^m = \text{exp. } mx. \quad \text{Put } x = 1;$$

then $$\text{exp. } m = (\text{exp. } 1)^m = e^m \quad \ldots\ldots \quad (3)$$

Whatever value x may have, integral, fractional, positive or negative, so long as it is a real quantity it follows, as a corollary to (3), that the series exp. $x = e^x$.

In Fig. 1 the graphs of e^x and e^{-x} are drawn. Both are positive for all values of x. When x is zero $e^x = e^{-x} = 1$. As x increases, e^x rapidly increases and e^{-x} rapidly decreases. To represent a quantity the rate of increase of which is equal at every instant to the quantity itself, e^x may appropriately be chosen, and e^{-x} in a similar case of decay. If the quantity cannot be represented by a single exponential function it may be possible to represent it by the sum of a number of such functions, perhaps of an infinite series of them.

The Hyperbolic Functions.

In the exponential series

$$\text{exp. } x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots \quad \ldots\ldots \quad (1)$$

replace x by $-x$.

Then $$\text{exp. } (-x) = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \ldots, \quad \ldots\ldots \quad (4)$$

in which all the odd powers of x have changed sign. Add (1) and (4) and divide by 2. The odd powers cancel, and

$$\frac{\text{exp. } x + \text{exp. } (-x)}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots\ldots \quad (5)$$

Subtract (4) from (1) and divide by 2. The even powers cancel, and

$$\frac{\text{exp. } x - \text{exp. } (-x)}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots\ldots \quad (6)$$

The series (5) and (6), formed in the above simple manner from the exponential series, resemble it in being also of fundamental importance. They are called, for a reason which will appear below, respectively the *hyperbolic cosine*, and the *hyperbolic sine* of x. Convenient contractions are $\cosh x$ and $\sinh x$.

Thus $$\cosh x = \frac{\exp. x + \exp. (-x)}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots \quad . \quad (7)$$

and $$\sinh x = \frac{\exp. x - \exp. (-x)}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots \quad . \quad (8)$$

Hence $$\cosh x \pm \sinh x = \exp. (\pm x) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

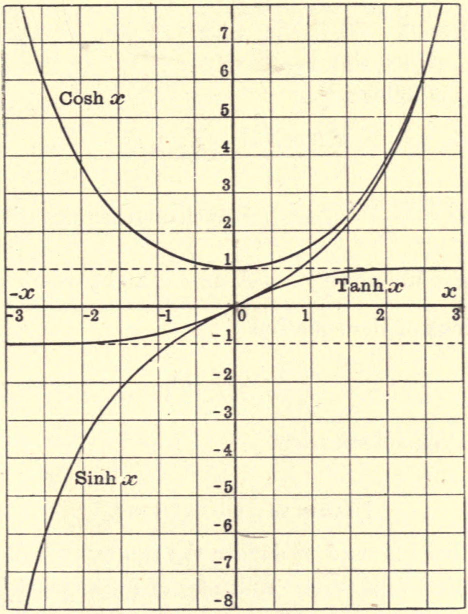

FIG. 2.—GRAPHS OF COSH x, SINH x, AND TANH x.

On squaring (7) and (8) and taking their difference it follows that

$$\cosh^2 x - \sinh^2 x = \exp. x \times \exp. (-x)$$
$$= \exp. (x - x) = \exp. 0 = 1 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The graphs of $\cosh x$, $\sinh x$, and $\tanh x = \dfrac{\sinh x}{\cosh x}$ are drawn in Fig. 2. Cosh x is an *even* function of x, *i.e.*, it does not alter in value when x changes sign. Sinh x and tanh x are both *odd* functions of x, and change sign with x. Cosh x, sinh x, and tanh x are all finite for finite values of x. From Fig. 1, at about $x=3$, exp. $(-x)$ is very small compared with exp. $(+x)$, and, correspondingly, in Fig. 2, at about $x=3$, the graphs of cosh x and sinh x very nearly coincide, and each is equal to half exp. x. Cosh x and sinh x differ by less than one part in a thousand when x is 3·8. When x is very small, cosh $x=1$. To a second approximation, $\cosh x=1+\frac{x^2}{2}$; x^4 is then neglected in comparison with x^2. Similarly, when x^3 can be neglected in comparison with x, $\sinh x=x$.

From (7) it follows that

$$\frac{d\,(\cosh x)}{dx}=\sinh x$$

and from (8), $\frac{d\,(\sinh x)}{dx}=\cosh x$, so that either curve is the slope of the other. Moreover $y=\cosh ax-\frac{b^2}{a^2}$ and $y=\sinh ax-\frac{b^2}{a^2}$ are both solutions of the equation

$$\frac{d^2y}{dx^2}-a^2y=b^2,$$

as is evident on substitution.

Imaginary Quantities.

The symbol i is used to denote the square root of negative unity,* so that $i\equiv\sqrt{-1}$. Multiples of i are called *imaginary* quantities, since the operation represented by the square root symbol cannot be carried out upon a negative quantity.

For every real quantity x there exists a corresponding imaginary quantity $\sqrt{-x^2}$ or ix.

Since i^2 by definition $=-1$, it follows that $i^3=-i$, $i^4=+1$, $i^5=i$, and so on. In general, $i^{2n}=+1$ or -1, and $i^{2n+1}=+i$ or $-i$, according to whether n is even or odd.

* Sometimes written j, but the symbol i is already well established in mathematical usage.

THE CIRCULAR FUNCTIONS.

In the definition of exp. x as equivalent to the series $1+x+\frac{x^2}{\underline{|2}}$ + &c. to ∞, no limitation was imposed on x, and the definition holds good for all values of x. Thus when x is replaced by ix, x being real,

$$\text{exp. }(ix)=1+ix+\frac{i^2x^2}{\underline{|2}}+\ldots\text{to }\infty. \quad . \quad . \quad . \quad (11)$$

The expression e^{ix} is now meaningless, but to preserve continuity it may be defined as a contracted method of writing exp. (ix), and therefore as equivalent to the exponential series —so that $e^{ix}=$exp. (ix).

Now replace x by $-ix$, and

$$\text{exp. }(-ix)=1-ix+\frac{i^2x^2}{\underline{|2}}-\frac{i^3x^3}{\underline{|3}}+\ldots\text{to }\infty. \quad . \quad . \quad . \quad (12)$$

Add (11) and (12) and divide by 2. The odd powers cancel, and

$$\frac{\text{exp. }(ix)+\text{exp. }(-ix)}{2}=1+\frac{(ix)^2}{\underline{|2}}+\frac{(ix)^4}{\underline{|4}}+\ldots\ldots$$

$$=1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Subtract (12) from (11), and divide by $2i$. Then

$$\frac{\text{exp. }(ix)-\text{exp. }(-ix)}{2i}=\frac{1}{i}\left[ix+\frac{(ix)^3}{\underline{|3}}+\frac{(ix)^5}{\underline{|5}}+\ldots\right]$$

$$=x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

To the series (13) and (14) are given the names *cosine* and *sine*, which may be contracted as in the case of the exponential series to the first three letters of the words.

$$\text{Thus } \cos x=\frac{\text{exp. }(ix)+\text{exp. }(-ix)}{2}=1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}} \quad . \quad . \quad . \quad (15)$$

and
$$\sin x=\frac{\text{exp. }(ix)-\text{exp. }(-ix)}{2i}=x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}} \quad . \quad . \quad (16)$$

On addition,

$$\cos x \pm i \sin x=\text{exp. } \pm(ix) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Squaring, and adding (15) and (16),

$$\cos^2 x+\sin^2 x=\exp.(ix)\times\exp.(-ix)=\exp.(0)=1. \quad . \quad (18)$$

The graphs of cos x, sin x and tan $x=\dfrac{\sin x}{\cos x}$, are drawn, for real values of x, in Fig. 3.

All three functions are *periodic* functions of x, of period 2π. They are so called because as x varies, the functions pass through a cycle of variations which is complete when x has altered by $\pm 2\pi$. Cos x is an even function of x, while sin x and

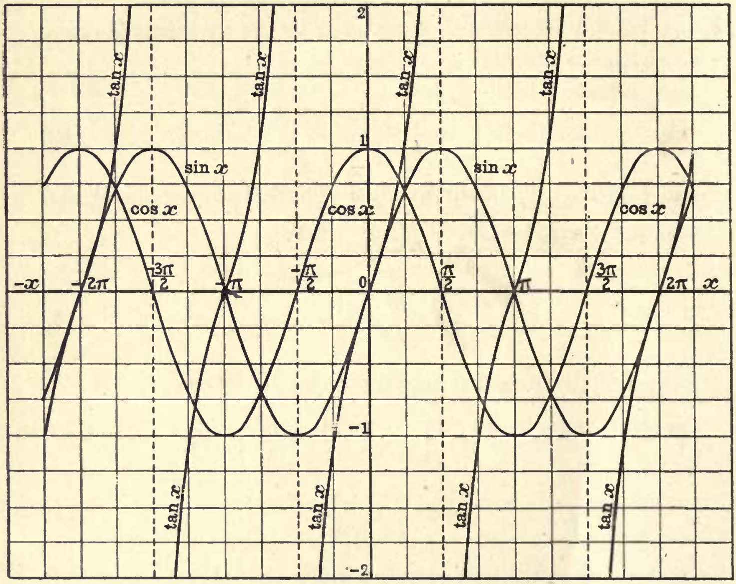

FIG. 3.—GRAPHS OF SIN x, COS x AND TAN x.

tan x are odd functions. Cos x and sin x are always finite, and vary between the limits $+1$ and -1. As x increases and passes through odd multiples of $\pi/2$, tan x rushes off to $+\infty$, reappearing on the other side of the ordinate with the value $-\infty$. If cos x is moved to the right through $\pi/2$ it coincides with sin x. Cos x may be said to be $\pi/2$ or 90 deg. in phase in advance of sin x, since when x increases from zero to $+\infty$, sin x

starts from zero at $x=0$, while $\cos x$ has already its maximum valve unity. To a first approximation $\cos x=1$; to a second approximation $\cos x=1-x^2/2$, when x is so small that x^4 can be neglected in comparison with x^2. In like manner, if x^3 can be neglected in comparison with x, $\sin x=x$. From (15) and (16) it follows that $\frac{d(\cos x)}{dx}=-\sin x$, and $\frac{d(\sin x)}{dx}=\cos x$. The sine taken negatively is the slope of the cosine, and the cosine is the slope of the sine. The equation $\frac{d^2y}{dx^2}+a^2y=b^2$ has for solutions $y=\cos a\,x+\frac{b^2}{a^2}$, and $y=\sin ax+\frac{b^2}{a^2}$.

Geometrical Representation of the Circular and the Hyperbolic Functions.

It now remains to show that the definitions (15) and (16) lead to the same functions as those which appear in the geometry of the circle, from which the name *circular* functions is derived.

Let P (Fig. 4) be a point which moves in the plane XOY so that its rectangular co-ordinates OM and MP obey the relationship

$$\mathrm{OM}=x=r\cos\phi \text{ and } \mathrm{MP}=y=r\sin\phi,$$

where r is a constant quantity, and $\cos\phi$ and $\sin\phi$ are given by (15) and (16) in terms of the variable ϕ.

Then, using (18),

$$\mathrm{OP}^2=\mathrm{OM}^2+\mathrm{MP}^2=r^2(\cos^2\phi+\sin^2\phi)=r^2.$$

Hence, $\mathrm{OP}=r$, and the locus of P is a circle of centre O and radius r. When ϕ is zero, from Fig. 3, OP coincides with OA, and when ϕ is $\pi/2$, OP coincides with OB.

As P moves round the circumference of the circle it sweeps out the area AOP. From first principles this area is equal to $\frac{1}{2}\mathrm{OA}.\mathrm{arcAP}=\frac{r}{2}r\theta=\frac{r^2\theta}{2}$, where θ is the circular measure of the angle POA. The same area may be regarded as the sum of the areas OMP and AMP. Now OMP$=\frac{1}{2}xy=\frac{r^2}{2}\cos\varphi\sin\varphi$, and

$$\text{AMP}=\int_{x=r\cos\phi}^{x=r} y\,dx=\frac{r^2}{2}(\varphi-\sin\varphi\cos\varphi).$$

Hence, adding, the total area AOP is $\frac{r^2\varphi}{2}$. Equating the two expressions, it follows that $\varphi=\theta$. Hence, $\frac{\text{OM}}{\text{OP}}=\cos\theta$, where $\cos\theta$ is defined as in (15). But $\frac{\text{OM}}{\text{OP}}$ is the geometrical definition of $\cos\theta$, and the two are therefore identical.

It has now been shown when θ is a real quantity, and the circular functions are capable of simple geometrical representation, that the definitions given in (15) and (16) coincide with the ordinary definitions of elementary trigonometry. When θ is other than real the definitions (15) and (16) still hold good, and all the properties of the *generalised* circular functions are to be derived from them.

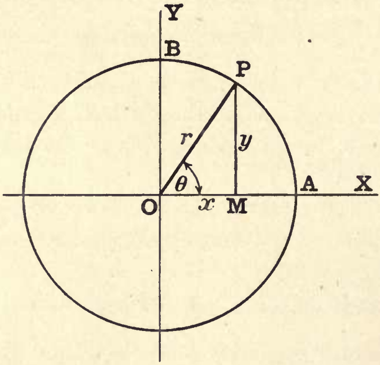

FIG. 4.—GRAPH OF ($r\cos\phi$, $r\sin\phi$).

In the case of the hyperbolic functions the relationship $\cosh^2\varphi-\sinh^2\varphi=1$ leads, when φ is real, to a geometrical representation of a somewhat similar nature. Let P be a point as in Fig. 5 of co-ordinates $x=a\cosh\varphi$ and $y=a\sinh\varphi$. Then $x^2-y^2=a^2(\cosh^2\varphi-\sinh^2\varphi)=a^2$. As φ varies, the point

P moves along the right-hand branch of a rectangular hyperbola. The area AOP swept out by OP in its movement from the initial position OA, is equal to the area MOP minus the area MAP, which is $\frac{1}{2}xy - \int_a^x y\,dx = \frac{a^2\varphi}{2}$. The quantity φ is, therefore connected in the same way with the area AOP in the case of the hyperbolic functions as in the circular functions, although φ is now no longer the angle AOP which is $\tan^{-1}\frac{y}{x}$ $= \tan^{-1}\tanh\varphi$.

The properties of the hyperbolic functions may be obtained from the geometry of the hyperbola, so long as φ is real, but

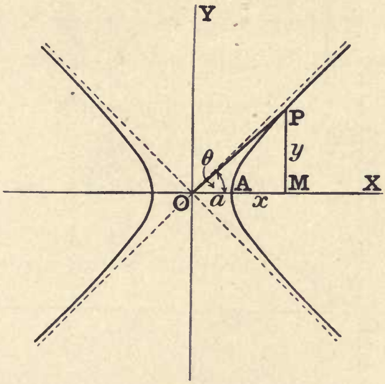

FIG. 5.—GRAPH OF (a COSH ϕ, a SINH ϕ).

this case is of little importance. Whatever value φ may have, the definitions (7) and (8) hold good, and from them all the desired relationships can be obtained.

Thus, for example—

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x,$$

and

$$\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i \sinh x.$$

Appendix III. contains the most frequently occurring properties of the hyperbolic functions, together with the corresponding properties of the circular functions, arranged in parallel columns so as to bring out the resemblance between the two.

Complex Quantities.

A *complex* quantity, as for example $x+iy$, is formed when an imaginary quantity is added to a real quantity.

A complex quantity may conveniently be represented on a diagram as in Fig. 6. Here real quantity is measured off along the horizontal axis, and imaginary quantity vertically upwards. The line OP, where P has the co-ordinates x and iy, may be looked upon as representing the complex quantity $x+iy$, since

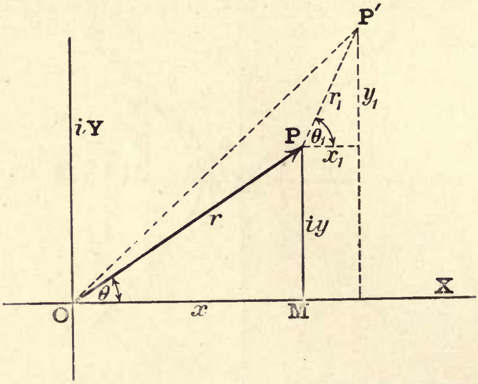

Fig. 6.—Geometrical Representation of the Complex Quantity $x+iy$.

when P is fixed both x and iy, and therefore $x+iy$, are known. Since x and y may each have an infinite number of values the complex quantity $x+iy$ has a double infinity of values, and the point P in its motion may cover the whole plane XOY.

The point P and the line OP are fixed when the distance $OP=r$, and the angle $MOP=\theta$, are given, so that the complex quantity is determined geometrically by the expression $r\underline{/\theta}$. The complex quantity may be written in any one of the equivalent forms $x+iy$, $\sqrt{x^2+y^2}\underline{/\tan^{-1}\frac{y}{x}}$, $r(\cos\theta+i\sin\theta)$, $re^{i\theta}$, and $r\underline{/\theta}$.

The length $r=\sqrt{x^2+y^2}$ is sometimes called the "modulus" of the complex quantity, and the angle θ the "argument."

Certain important properties of complex quantities are discussed below :—

(*a*) Addition of two complex quantities.

To $a+ib$ add $c+id$. The sum is $a+ib+c+id=a+c+i(b+d)$, if the ordinary rules of algebra are to be obeyed. In Fig. 6, OP′ is the geometrical representation of the sum of OP($r\underline{/\theta}$ or $x+iy$) and PP′ ($r_1\underline{/\theta_1}$ or x_1+iy_1).

(*b*) Subtraction is carried out in a similar manner.

$$\text{Thus } (a+ib)-(c+id)=(a-c)+i(b-d).$$

(*c*) Multiplication of two complex quantities.

$$\text{The product } (a+ib)(c+id)=ac-bd+i(bc+ad).$$

The right-hand side may be written—

$$\sqrt{(ac-bd)^2+(bc+ad)^2}\,\underline{/\tan^{-1}\frac{bc+ad}{ac-bd}},$$

$$=\sqrt{(a^2+b^2)(c^2+d^2)}\,\underline{/\tan^{-1}\frac{b/a+d/c}{1-bd/ac}},$$

$$=\sqrt{a^2+b^2}\sqrt{c^2+d^2}\,\underline{/\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{d}{c}}.$$

Hence the product of two complex qualities may be found by multiplying their moduli together and adding their arguments.

Concisely, if $a+ib=r_1\underline{/\theta_1}$ and $c+id=r_2\underline{/\theta_2}$, $r_1\underline{/\theta_1}\times r_2\underline{/\theta_2}=r_1r_2\underline{/\theta_1+\theta_2}$.

(*d*) To express $\dfrac{a+ib}{c+id}$ as a complex quantity,

The complex quantity $c-id$ is said to be *conjugate* to $c+id$, as it differs from it only in the sign of the imaginary part, and the product of the two is the square of their common modulus. Multiply the given expression above and below by $c-id$.

Then

$$\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{c^2+d^2}=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right).$$

As in (c), this may be written $\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}\underline{/\tan^{-1}\frac{b}{a}-\tan^{-1}\frac{d}{c}}$. Concisely, $r_1\underline{/\theta_1}\div r_2\underline{/\theta_2}=r_1/r_2\ \underline{/\theta_1-\theta_2}$.

(e) If $a+ib=c+id$, then $a=c$ and $b=d$.

For $(a+ib)-(c+id)$ must $=0$, and $(a-c)+i(b-d)=0$. As the real and the imaginary parts are quite independent they must be separately zero. Hence, $a-c=0$ and $b-d=0$, from which the proposition follows.

If two complex quantities are equal their real portions and their imaginary portions may be separately equated to each other.

(f) It is frequently desirable to represent a function of a complex quantity, as, for example, $\cos(x+iy)$, as a simple complex quantity.

By definition, $\cos(x+iy)=\frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}$,

$$=\frac{e^{ix-y}+e^{-ix+y}}{2}=\frac{(e^{ix}+e^{-ix})(e^y+e^{-y})-(e^{ix}-e^{-ix})(e^y-e^{-y})}{4},$$

on factorisation,

$$=\cos x\cosh y-i\sin x\sinh y.$$

The other formulæ of Appendix III. may be obtained in a similar manner.

By following out the processes illustrated above, any combination of complex quantities may be reduced to a simple complex quantity.

(g) If any function of $x+iy$, say $f(x+iy)$, be reduced to the form $A+iB$, then $f(x-iy)=A-iB$. For in $A+iB$, A is composed of real quantities and of imaginary quantities to an even power; these do not change sign when the sign of i is altered. On the other hand, iB contains all the odd powers of i, and iB changes sign with i.

Development in Trigonometric Series.

Let $f(x)$ denote any periodic function of x which is finite and continuous for the values of x under consideration. Such a function can always be represented as the sum of an infinite series of sines and cosines of ascending multiples of an angle. Series of this type are called *Fourier* series, after the mathematician who introduced them to mathematical physics.

Assume
$$f(x)=A_1 \sin\frac{\pi x}{c}+A_2 \sin\frac{2\pi x}{c}+ \quad . \quad . \quad . \quad . \quad . \tag{19}$$
$$=\sum_{m=o}^{m=\infty} A_m \sin\frac{m\pi x}{c}$$

When $x=0$, or $x=c$, $f(x)=0$. Let the endeavour be made to determine the A coefficients so that the series shall coincide with $f(x)$ between the limits $x=0$ and $x=c$.

Multiply both sides of (19) by $\sin\frac{n\pi x}{c}$, and integrate between 0 and c with respect to x.

The integral is $\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$

$$=\Sigma A_m\int_0^c \sin\frac{n\pi x}{c}\sin\frac{m\pi x}{c}\,dx$$

$$=\Sigma\frac{A_m}{2}\left[\frac{\sin\frac{(n-m)\pi x}{c}}{\frac{(n-m)\pi}{c}}-\frac{\sin\frac{(n+m)\pi x}{c}}{\frac{(n+m)\pi}{c}}\right]_0^c$$

$=0$, unless $n=m$.

When n very nearly coincides with m, $\overline{n-m}$ is very small, and $\sin\frac{(n-m)\pi x}{c}=\frac{(n-m)\pi x}{c}$. The first term in the square brackets becomes x, and, on putting in the limits, c.

It follows, therefore, that

$$\int_0^c f(x)\sin\frac{m\pi x}{c}dx=\frac{A_m c}{2},$$

or
$$A_m=\frac{2}{c}\int_0^c f(x)\sin\frac{m\pi x}{c}\,dx. \quad . \quad . \quad . \quad . \quad . \tag{20}$$

By means of this formula the coefficients of the sine series (19) can be obtained. A similar formula,

$$B_m = \frac{2}{c}\int_0^c f(x)\cos\frac{m\pi x}{c}\,dx$$

enables us to develop $f(x)$ in the series of cosines

$$f(x) = \tfrac{1}{2}B_0 + B_1\cos\frac{\pi x}{c} + B_2\cos\frac{2\pi x}{c} + \ldots \qquad (21)$$

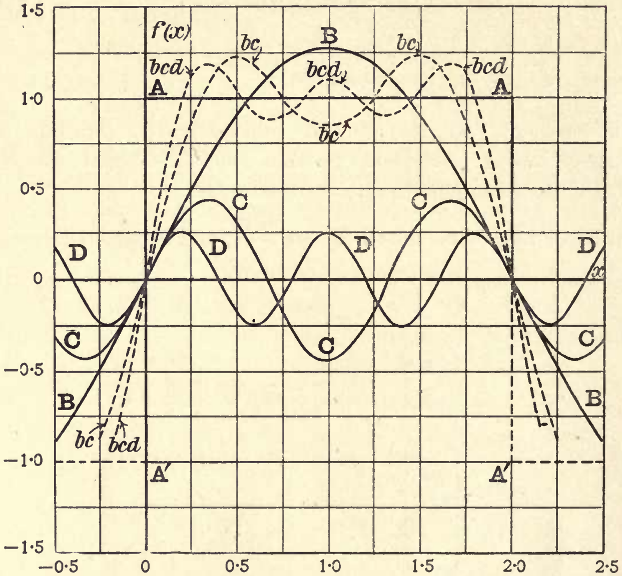

FIG. 7.—SUCCESSIVE APPROXIMATION TO $f(x)=1$ BETWEEN $x=0$ AND $x=2$.

Curve A.... $f(x)=1$. Curve B .... $\frac{4}{\pi}\sin\frac{\pi x}{2}$.

Curve C.... $\frac{4}{3\pi}\sin\frac{3\pi x}{2}$. Curve D .... $\frac{4}{5\pi}\sin\frac{5\pi x}{2}$.

Curve *bc*....B and C combined. Curve *bcd*....B, C and D combined.

Example 1.—Let $f(x)=1$ between $x=0$ and $x=c$, as in Fig. 7.

Here
$$A_m = \frac{2}{c}\int_0^c 1\times\sin\frac{m\pi x}{c}dx$$
$$= \frac{2}{m\pi}[1-\cos m\pi].$$

The even coefficients ($\cos m\pi = +1$) are all zero, and the odd coefficients are equal to $\frac{4}{m\pi}$.

Hence $$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \&c.\right]$$

$$= \frac{4}{\pi}\sum_0^\infty \frac{1}{2m+1}\sin\frac{\overline{2m+1}\pi x}{c}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The first three terms of the series are drawn in Fig. 7, taking $c=2$. Curve B is the fundamental $\left(\frac{4}{\pi}\sin\frac{\pi x}{c}\right)$, and stretches from 0 to 2, rising above the horizontal at unit height which represents $f(x)$. Curve C is the first harmonic $\left(\frac{4}{3\pi}\sin\frac{3\pi x}{c}\right)$. It cuts off the top of the fundamental and adds to it at the sides. Curve D is the third term. It tends to correct for the depression caused by C in the middle, and to supplement the total at the sides. When all the curves are added they produce the square-cornered periodic curve A′AAA′, which has unit value between the specified limits 0 and 2. Curve *bc* and Curve *bcd* are the second and third approximations respectively.

If the horizontal axis is labelled time, and the vertical axis voltage, curve A′AAA′ may be taken to represent a constant E.M.F., periodically reversed at the end of every, say, two-tenths of a second.

Example 2.—Let $f(x) = x$ between $x=0$ and $x=c$.

Here $$A_m = \frac{2}{c}\int_0^c x\sin\frac{m\pi x}{c}dx = -\frac{2c\cos m\pi}{m\pi},$$

and $$x = \frac{2c}{\pi}\left[\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c}. \ . \ .\right]$$

$$= \frac{2c}{\pi}\sum_1^\infty \frac{1}{m}\sin\frac{m\pi x}{c}\cos(m+1)\pi. \quad . \quad . \quad . \quad . \quad (23)$$

The first three terms of this series are plotted in Fig.8. The line OA is the straight line of equation $f(x) = x$, which is to be built up out of sine terms between the limits 0 and c, in this case 0 and 6. Curve B is the fundamental $\left(\frac{2c}{\pi}\sin\frac{\pi x}{c}\right)$, curve C is the first harmonic $\left(-\frac{c}{\pi}\sin\frac{2\pi x}{c}\right)$, and curve D the second harmonic $\left(\frac{2c}{3\pi}\sin\frac{3\pi x}{c}\right)$. Curves B and C combined produce curve *bc*, and curves B,C,D together produce curve *bcd*. When x is equal to c the series leaves the straight line and drops through

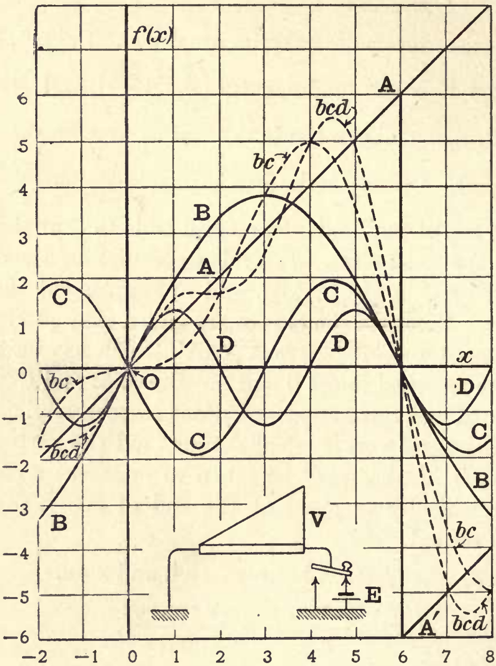

FIG. 8.—SUCCESSIVE APPROXIMATION TO $f(x)=x$, BETWEEN $x=0$ AND $x=6$.

Curve A ....$f(x)=x$. Curve B ....$\frac{12}{\pi}\sin\frac{\pi x}{6}$.

Curve C ....$-\frac{6}{\pi}\sin\frac{\pi x}{3}$. Curve D ....$\frac{4}{\pi}\sin\frac{\pi x}{2}$.

Curve *bc*....B and C combined. Curve *bcd*....B, C and D combined.

zero to an equal negative value. To the left of the origin, where x is negative, series and line coincide as far as $-c$. The series represents a saw-tooth curve which coincides with $f(x)=x$ between the limits $-c$ and $+c$.

The line OA may be taken to represent the distribution of voltage along a cable which is earthed at one end and charged at the other end by means of a battery. On removing the source of E.M.F. and putting the cable to earth the voltage drops to zero at the sending end. In the subsidence of voltage along the cable which ensues the higher harmonics disappear first. The curve of voltage successively approaches the diminished shapes of *bcd* and *bc*, finally dying away as a remnant of the fundamental, pure sine, curve B.

Example 3.—Let $f(x)=x$ from 0 to $c/2$ and $c-x$ from $c/2$ to c. Here

$$A_m=\frac{2}{c}\int_0^{c/2} x\sin\frac{m\pi x}{c}dx+\frac{2}{c}\int_{c/2}^{c}(c-x)\sin\frac{m\pi x}{c}dx=\frac{4c}{m^2\pi^2}\sin\frac{m\pi}{2}.$$

Hence

$$f(x)=\frac{4c}{\pi^2}\left[\sin\frac{\pi x}{c}-\frac{1}{3^2}\sin\frac{3\pi x}{3}+\frac{1}{5^2}\sin\frac{5\pi x}{c}-\ldots\right]$$

$$=\frac{4c}{\pi^2}\sum_1^\infty\frac{1}{m^2}\sin\frac{m\pi x}{c}\sin\frac{m\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The fundamental and the first harmonic of this series are shown in Fig. 9, curves B and C respectively. Curve A is the triangle with which the series is to coincide. Curve *bc* is formed by adding curves B and C. Except for the apex, it is already a very close approximation to the desired shape.

The right-hand half of the triangle may be taken to represent the distribution of voltage along a cable to earth at the receiving end, at the instant when it is insulated at the sending end after having been fully charged.

Example 4.—Let $f(x)=\sin\mu x$, between 0 and π.

Here
$$A_m=\frac{2}{\pi}\int_0^\pi\sin\mu x\sin mxdx$$

$$=\frac{1}{\pi}\left[\frac{\sin(\mu-m)x}{\mu-m}-\frac{\sin(\mu+m)x}{\mu+m}\right]_0^\pi$$

$$=\frac{2m}{\pi}\cdot\frac{\sin\mu\pi\cos m\pi}{\mu^2-m^2},$$

unless μ is a whole number, when the series reduces to a single term.

Hence, with the foregoing proviso,

$$\sin \mu x = \frac{2 \sin \mu\pi}{\pi} \sum_1^\infty \frac{m \sin mx \cos m\pi}{\mu^2 - m^2} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In a similar way it may be shown that

$$\cos \mu x = \frac{2 \sin \mu\pi}{\pi} \left[\frac{1}{2\mu} + \sum_1^\infty \frac{\mu \cos mx \cos m\pi}{\mu^2 - m^2} \right] . \quad . \quad (26)$$

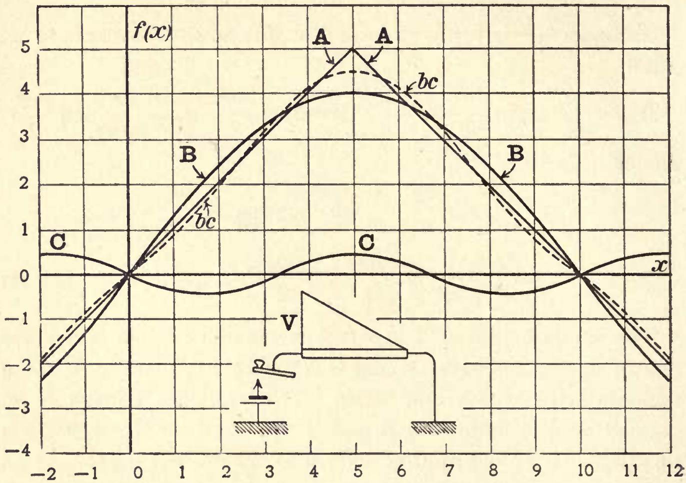

FIG. 9.—SUCCESSIVE APPROXIMATIONS TO $f(x)$, WHERE $f(x) = x$ FROM $x = 0$ TO $x = 5$ AND $f(x) = 5 - x$, FROM $x = 5$ TO $x = 10$.

Curve A....$f(x) = x$ and $5 - x$. Curve B....$\frac{40}{\pi^2} \sin \frac{\pi x}{10}$. Curve C..$-\frac{40}{9\pi^2} \sin \frac{3\pi x}{10}$.

Curve bc....B and C combined.

Since $i \sinh \mu x = \sin i\mu x$, and $\cosh \mu x = \cos i\mu x$, it follows that

$$\sinh \mu x = \frac{2 \sinh \mu\pi}{\pi} \sum_1^\infty \frac{m \sin mx \cos \overline{m-1}\pi}{\mu^2 + m^2}, \quad . \quad . \quad . \quad (27)$$

and

$$\cosh \mu x = \frac{2 \sinh \mu\pi}{\pi} \left[\frac{1}{2\mu} + \sum_1^\infty \frac{\mu \cos mx \cos m\pi}{\mu^2 + m^2} \right], \quad . \quad (28)$$

which hold for all values of μ.

Adding these two series,

$$e^{\mu x}=\frac{2\sinh\mu\pi}{\pi}\left[\frac{1}{2\mu}+\sum_{1}^{\infty}\frac{\mu\cos mx-m\sin mx}{\mu^2+m^2}\cos m\pi\right],$$

$$=\frac{2\sinh\mu\pi}{\pi}\left[\frac{1}{2\mu}-\sum_{1}^{\infty}\frac{\sin\left(mx-\tan^{-1}\frac{\mu}{m}\right)}{\sqrt{\mu^2+m^2}}\cos m\pi\right]. \qquad (29)$$

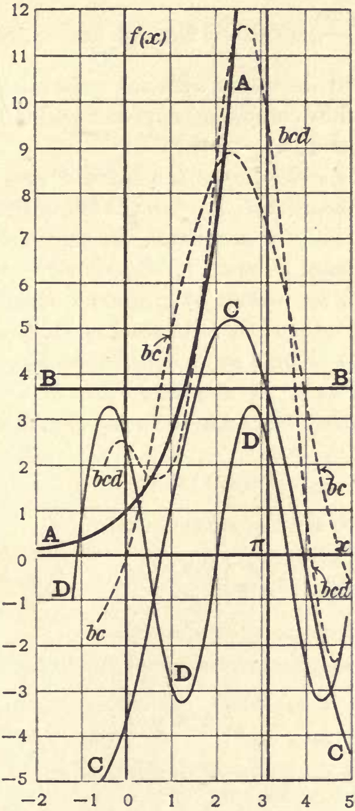

FIG. 10.—SUCCESSIVE APPROXIMATION TO $f(x)=e^x$ BETWEEN $x=0$ AND $x=\pi$.

Curve A....e^x. Curve B ....Steady term, $\frac{\sinh\pi}{\pi}$.

Curve C....1st term of sine series. Curve D ....2nd do.

Curve *bc*....B and C combined. Curve *bcd*....B, C and D combined.

The graph of this series is drawn in Fig. 10 for $\mu=1$. Curve A is e^x; curve B is the steady term $\frac{\sinh \mu\pi}{\mu\pi}$. Curves C and D are the first and second terms of the sine series. They are $\frac{\sqrt{2}}{\pi}$ $\sinh \pi \sin x$, displaced to the right through $\tan^{-1} 1 = \frac{\pi}{4}$, or 0·785, and $-\frac{2 \sinh \pi \sin 2x}{\pi\sqrt{5}}$, displaced through $\tan^{-1} \frac{1}{2}$, or 0·464. The phase displacement decreases with the order of the harmonic. Curve *bc* is formed by combining curves B and C, and curve *bcd* from curves B, C and D.

When $x=\pi$, the series leaves the curve e^x and drops to the value $e^{-\pi}$, but it coincides with e^{-x} to the left of the origin as far as $x=-\pi$. This may be seen from the sinh and cosh expansions, which become, μ being 1, respectively $-\sinh(2\pi-x)$, and $\cosh(2\pi-x)$, for values of x greater than π, while for negative value of x they still hold good as far as $x=-\pi$, since both sides of (27) change signs simultaneously, while (28) is unaffected. The sum in the one case is $[-\sinh(2\pi-x)+\cosh(2\pi-x)]$, or $e^{x-2\pi}$, and in the other case it is e^x as before.

Determinants.

Consider the two simultaneous equations

$$a_1x+b_1y=c_1,$$
$$a_2x+b_2y=c_2$$

in the two unknown quantities x and y.

Solving by substitution in the usual manner, it follows that

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1} \text{ and } y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.$$

These results may be written in the forms

$$x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and

$$y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

where the $|\,|$ signs mean "multiply across diagonally and subtract." The expressions within the $| \, |$ signs are called *determinants*. The diagonal from left to right downwards is the *principal* diagonal. The horizontal lines are called *rows* and the vertical lines *columns*. The numbers composing the determinant are its *constituents*.

The solutions of the three simultaneous equations

$$a_1x+b_1y+c_1z=d_1,$$
$$a_2x+b_2y+c_2z=d_2,$$
$$a_3x+b_3y+c_3z=d_3$$

may be written down at once, in a manner analogous to the foregoing.

For example,

$$x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This is to be read as follows:—

$$x=\left[d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} -b_1\begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} +c_1\begin{vmatrix} d_2 & b_2 \\ d_3 & b_3 \end{vmatrix}\right]$$
$$\div\left[a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} +c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right],$$
$$=[d_1(b_2c_3-b_3c_2)-b_1(d_2c_3-d_3c_2)+c_1(d_2b_3-d_3b_2)],$$
$$\div[a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)],$$

—which may be verified by solving by substitution in the ordinary way. The smaller determinants, which are formed by omitting one row and one column from the larger determinants, are called *minor* determinants, and the determinants are said to be expanded in terms of them by the aid, in this case, of the first row. The constituents in the row are to be taken alternately with positive and negative sign. Instead of the row we should obtain the same result by expanding in terms of the first column.

It is to be observed that x is the quotient of two determinants, one (the denominator) formed by writing down all the coefficients on the left-hand side of the equation and the other (the numerator) by substituting for the coefficients of x the terms on the right-hand side of the equation.

By means of the same rules the solution to a set of any number of simultaneous equations can be written down at once, and the rules can be proved to hold good generally. The use of determinants leads not only to an extremely compact form of expression for the solution, but in addition important general properties of determinants can be established, a study of which may simplify greatly the numerical calculation of the result.

Example 1.—Solve the equations

$$\begin{aligned} x \,.\, 2\cos\theta - y \qquad\qquad &= \mathrm{E}, \\ -x + y \,.\, 2\cos\theta - z &= 0, \\ -y + z \,.\, 2\cos\theta &= 0, \text{ for } x. \end{aligned}$$

Here, following the rule,

$$x = \begin{vmatrix} \mathrm{E} & -1 & 0 \\ 0 & 2\cos\theta & -1 \\ 0 & -1 & 2\cos\theta \end{vmatrix} \div \begin{vmatrix} 2\cos\theta & -1 & 0 \\ -1 & 2\cos\theta & -1 \\ 0 & -1 & 2\cos\theta \end{vmatrix}$$

$$= \mathrm{E}(4\cos^2\theta - 1) \div [2\cos\theta(4\cos^2\theta - 1) + 1 \times (-2\cos\theta)]$$

$$= \frac{\mathrm{E}(4\cos^2\theta - 1)}{4\cos\theta \,.\, \cos 2\theta}.$$

Example 2.—Evaluate the determinant

$$\delta_n = \begin{vmatrix} 2\cos\theta & -1 & . & . & . & . \\ -1 & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & 2\cos\theta \end{vmatrix}$$

in which there are n rows and n columns, and all the constituents are zero except those to the right and left of the principal diagonal.

It is evident that $\delta_1 = 2\cos\theta = \frac{\sin 2\theta}{\sin\theta}$, and $\delta_2 = 4\cos^2\theta - 1$, which may be written $\frac{\sin 3\theta}{\sin\theta}$, and so on.

It may therefore be inferred that

$$\delta_n = \frac{\sin \overline{n+1}\theta}{\sin\theta}.$$

To verify, expand the determinant by the first row. Then

$$\begin{aligned} \delta_n &= 2\cos\theta \times \delta_{n-1} - (-1 \times -1) \times \delta_{n-2}, \\ &= 2\cos\theta \times \frac{\sin n\theta}{\sin\theta} - \frac{\sin \overline{n-1}\theta}{\sin\theta}, \\ &= \frac{\sin \overline{n+1}\theta}{\sin\theta}. \end{aligned}$$

Hence, if the formula holds for δ_{n-2} and δ_{n-1}, it will hold also for δ_n. But it holds for δ_1 and δ_2 and therefore for δ_3, and from δ_2 and δ_3, for δ_4, and so on, and therefore holds generally.

Example 3.—Evaluate

$$\begin{vmatrix} E & -1 & . & . & . & . & . \\ . & 2\cos\theta & -1 & . & . & . & . \\ . & -1 & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & -1 & 2\cos\theta \end{vmatrix}$$

Expand by the first column. The determinant $=E\times\delta_n+1\times 0$
$=\dfrac{E\sin\overline{n+1}\theta}{\sin\theta}$. This result will be required later.

Addition and subtraction of determinants. The sum of the two determinants

$$\begin{vmatrix} a_1 & d & g \\ b_1 & e & h \\ c_1 & f & k \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & d & g \\ b_2 & e & h \\ c_2 & f & k \end{vmatrix}$$

is

$$\begin{vmatrix} a_1+a_2 & d & g \\ b_1+b_2 & e & h \\ c_1+c_2 & f & k \end{vmatrix}.$$

For, expanding by the first column, the last determinant is equal to

$$(a_1+a_2)\begin{vmatrix} e & h \\ f & k \end{vmatrix} -(b_1+b_2)\begin{vmatrix} d & g \\ f & k \end{vmatrix} +(c_1+c_2)\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$=a_1\begin{vmatrix} e & h \\ f & k \end{vmatrix} -b_1\begin{vmatrix} d & g \\ f & k \end{vmatrix} +c_1\begin{vmatrix} d & g \\ e & h \end{vmatrix} +a_2\begin{vmatrix} e & h \\ f & k \end{vmatrix}$$

$$-b_2\begin{vmatrix} d & g \\ f & k \end{vmatrix} +c_2\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$=\begin{vmatrix} a_1 & d & g \\ b_1 & e & h \\ c_1 & f & k \end{vmatrix} + \begin{vmatrix} a_2 & d & g \\ b_2 & e & h \\ c_2 & f & k \end{vmatrix}$$

The sum of two determinants which are alike except for a single row or column can be written down at once by adding the rows or columns in question together. A similar rule will hold for subtraction, or, conversely, a single determinant may be split up into the sum of two.

Example 4.—Evaluate

$$\Delta_n=\begin{vmatrix} 2\cos\theta-1 & -1 & . & . & . & . \\ -1 & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & 2\cos\theta-1 \end{vmatrix}$$

in which there are n^2 constituents.

This determinant differs from δ_n only in the first and last constituents. Split it up into the sum of two by the aid of the first row.

$$\Delta_n=\begin{vmatrix} 2\cos\theta & -1 & . & . & . & . \\ -1 & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & 2\cos\theta-1 \end{vmatrix}+\begin{vmatrix} -1 & -1 & . & . & . & . \\ . & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & 2\cos\theta-1 \end{vmatrix}$$

The second of these is

$$-\begin{vmatrix} 2\cos\theta & -1 & . & . & . & . \\ -1 & 2\cos\theta & -1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & 2\cos\theta-1 \end{vmatrix}$$

in which there are $\overline{n-1}^2$ terms.

Now split up each of these two determinants into the sum of other two by the aid in each case of the last row, first turning the determinant upside down, which will not affect its value.

Finally,
$$\begin{aligned}\Delta_n&=(\delta_n-\delta_{n-1})-(\delta_{n-1}-\delta_{n-2}),\\ &=\delta_n-2\delta_{n-1}+\delta_{n-2},\\ &=\frac{\sin\overline{n+1}\theta-2\sin n\theta+\sin\overline{n-1}\theta}{\sin\theta},\\ &=\frac{2\sin n\theta}{\sin\theta}(\cos\theta-1).\end{aligned}$$

Further examples of the use of determinants will occur later.

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CHAPTER II.

THE FUNDAMENTAL CABLE CONSTANTS.

Resistance—Inductance—Capacity—Effective Values—Leakance—Electrification and Absorption.

Resistance.

The *resistance* R of a conductor is a constant determined by the shape of the conductor and by the nature of the material of which it is composed. The current C in the conductor was shown by Ohm* to be connected with the P.D. V at the ends of the conductor by the simple relationship

$$C = \frac{1}{R} \times V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the electromagnetic system of units, V is of dimensions $(l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2})$, and C is of dimensions $(l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-1})$, where l is the unit of length, m is the unit of mass and t the unit of time.

It follows from (1) that the dimensions of resistance are $(l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}) \div (l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-1}) = (lt^{-1})$, which are those of a velocity. In the C.G.S. system the fundamental units l, m and t are the centimetre, gramme and second, and accordingly R may be expressed in centimetres per second.

The C.G.S. unit of potential is inconveniently small, and for practical purposes another unit, the *volt*, is employed. One volt is 10^8 C.G.S. units. The *ampere*, or practical unit of current, is one-tenth of the C.G.S. unit. Since when two of the three quantities V, C and R are known the third is fixed from equation (1) in terms of the other two, it follows that the *ohm*, or practical unit of resistance, is $10^8 \div 10^{-1} = 10^9$ cm. per second.

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* G. S. Ohm, "Die Galvanische Kette," Berlin, 1827, p. 36.

In order that it may be more easily reproduced, the ohm is legally defined by international convention as the resistance at 0°C. of a column of mercury of constant cross-section, having a mass of 14·4521 grammes and a length of 106·300 cm. The ampere is the current which, when passed through a solution of nitrate of silver in water under certain specified conditions, deposits silver at the rate of 0·00111800 gramme per second. The latest measurements by the aid of the current balance show that the ampere as above defined is less than 0·1 C.G.S. unit by $0{\cdot}01_3$ per cent. The difference between the international ohm and the absolute ohm is also quite negligible for all practical purposes. The international volt is the product of the ampere and the ohm.*

The resistance of a cylindrical conductor is directly proportional to the length of the cylinder and inversely proportional to its area of cross-section. If l be the length of the conductor and d its diameter—supposing it to be of circular section—then

$$R=\frac{4l\rho}{\pi d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ρ is a constant called the *specific resistance* or *resistivity* of the material. In the C.G.S. system it is the resistance between the opposing faces of a cube of the material 1 cm. in edge. Its reciprocal is called the *conductivity*. In submarine cable specifications the well-established custom is to measure lengths in nautical miles (n.m.) and diameters in inches, 1 n.m. being 2,029 yards, or 1,855 metres. Moreover, instead of the cross-section, it is usual to fix the weight in pounds (lbs.) per n.m., so that the specific conductivity may, if desired, be taken as that of a conductor 1 n.m. in length and 1 lb. in weight.

The conductor of a submarine cable is invariably made of high conductivity copper. The specific resistance of pure annealed copper—deduced from the results of Matthiessen's investigations, which were expressed in terms of the foot-

* The British Association (B.A.) unit of resistance, which is still occasionally met with, is 0·9866 standard ohm. The latest (N.P.L.) determination of the ohm is 1·00052 C.G.S. unit.

grain—is 1·694 microhm per centimetre cube at 60°F. This figure may be taken as a standard of conductivity, and the conductivities of other samples may be expressed in percentages of it. According to this definition, a copper conductor of 100 per cent. conductivity of length l (n.m.) and weight w (lb. per n.m.) has a resistance at 60°F. ($15\frac{1}{2}$°C.) given by the formula

$$R=\frac{1144\times l}{w}. \qquad (3)$$

The resistivity of copper increases with the temperature, and for a range of temperatures between 32°F. and 75F°. the following formula* may be used

$$R_{t^\circ}=R_{32}[1+0\cdot002386(t^\circ-32)] \qquad (4)$$

At 75°F.—at which temperature gutta-percha core is usually tested—the coefficient 1,144 in (3) must be replaced by 1,183. A frequent stipulation is that the conductivity of the copper of a submarine telegraph cable shall not fall short of 98 per cent. of Matthiessen's standard.†

Taking the average density of copper as 8·9, a solid conductor of weight w (lb. per n.m.) has a diameter d (mils, one mil= 0·001 in.) given by the formula

$$d=7\cdot36\sqrt{w}. \qquad (5)$$

The conductors of the lighter types of cable are formed by stranding seven equal wires together. The overall diameter is accordingly three times the diameter of one of the wires. The conductors of heavier types of cable (300 lb. and over) are formed of a central wire surrounded by a single layer of 10 or 12 smaller wires in a long spiral. In this way greater flexibility is given to the conductor than if it were of solid construction.

* Messrs. Clark, Forde and Taylor.

† At the meeting of the International Electrotechnical Commission at Berlin in September, 1913, the following standard of conductivity was adopted. The resistance per metre of a standard annealed copper wire of 1 sq. mm. section is taken to be 0·017241 at 20°C. and the temperature coefficient as 0·00393 per °C. The density is given as 8·89. The coefficient in (3) then becomes 1163·2 at 20°C., or, applying the temperature correction, 1181 at 75°F. The British Engineering Standards Committee adopted (1905) what is equivalent to 1185 at 75°F.

A cross-section of the deep-sea portion of a telegraph cable of heavy type is shown in Fig. 11, and in Fig. 12 is shown a continuously-loaded four-core telephone cable.

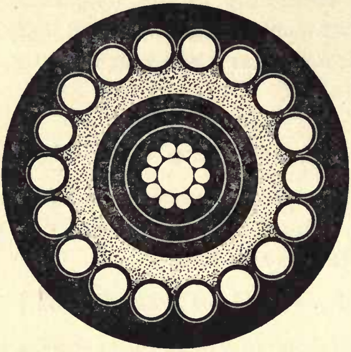

FIG. 11.—SECTION OF DEEP-SEA CABLE. (Twice Full Size).

Conductor: Centre of 0·0984 in. surrounded by 10 of 0·0415 in. Overall diameter, 0·182 in. Weight, 500 lb.
Insulation: Three coats of gutta-percha to 0·431 in. Weight, 315 lb.
Protection: Served with tanned jute yarn and sheathed with 18 galvanised steel wires each 0·083 in. diameter, compounded and taped.

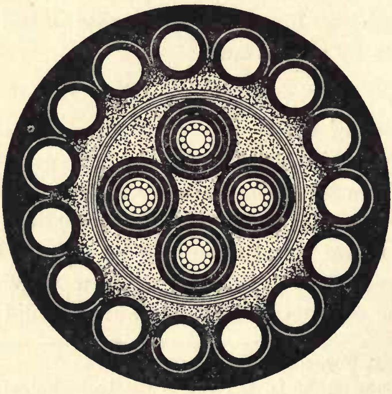

FIG. 12.—SECTION OF FOUR-CORE TELEPHONE CABLE. (Full Size).

Conductor: Centre of 0·082 in. surrounded by 12 of 0·0285 in. Overall diameter, 0·139 in. Weight, 300 lb. per nautical mile.
Loading: One layer of soft iron wire 0·012 in. diameter, 70 turns per inch.
Insulation: Gutta-percha, weighing 300 lb. per nautical mile.
Protection: The four cores laid up round a yarn centre, with yarn worming, then brass taped and sheathed with 15 galvanised steel wires, each 0·192 in. embedded in yarn and covered with two layers of yarn and compound.

INDUCTANCE.

So long as V and C are constant, equation (1) represents completely the state of affairs; but should they vary, and should the circuit be capable of storing up an appreciable amount of electromagnetic energy, equation (1) is no longer sufficient. To V is then opposed a back E.M.F. equal to the rate at which the total magnetic flux embraced by the circuit is varying. If Φ be this flux, the opposing E.M.F. is $\frac{d\Phi}{dt}$. If there is no magnetic material in the neighbourhood, the magnetic flux Φ is directly proportional to the current C, so that $\Phi = LC$, where L is a constant for the circuit, and equation (1) becomes

$$V = RC + L\frac{dC}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The constant L, as above defined, is called the *self-inductance* of the circuit. When the induced or back E.M.F. is 1 volt, and the rate of change of current with respect to time is 1 ampere per second, the inductance is of unit value and is called 1 *henry*. The dimensions of inductance are $(l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}) \div \left(\frac{l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}}{t}\right) = (l)$. L is, therefore, simply a length, and may be expressed in centimetres. The practical unit or henry is $10^8 \div 10^{-1} = 10^9$ cm. One millihenry is 1,000,000 cm.

When the magnetic flux of one circuit is interlinked with another circuit, variation in the current C_1 in the first circuit will produce a variation in the flux passing through the second circuit, and will, therefore, produce an E.M.F. $M\frac{dC_1}{dt}$ in the second circuit, equal to the rate of change of C_1 multiplied by a constant, M, for the two circuits which is called their *mutual inductance.* Conversely, an E.M.F., $M\frac{dC_2}{dt}$, is produced in the first circuit by the variation of the current C_2 in the second circuit. If there is no magnetic leakage—*i.e.*, if all the flux generated by either circuit passes through the other—M^2

$=L_1 \times L_2$, where L_1 and L_2 are the self-inductances of the circuits. In general, M^2 is less than $L_1 \times L_2$, and the quantity $1-M^2/L_1 \times L_2$ is called the leakage factor. M is measured in the same units as L.

When magnetic material is present the magnetic flux depends not only on the geometry of the circuit but also on the degree of magnetisation of the iron. The definition of L given in (6) is, in general, no longer applicable; but in telegraphy and telephony the currents employed—and therefore the magnetising forces—are usually so small that the magnetic

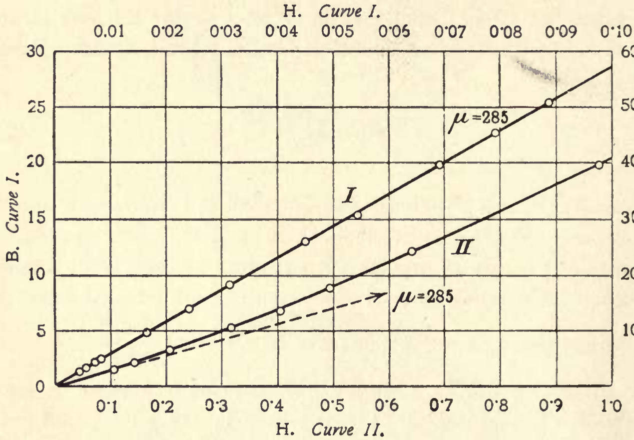

FIG. 13.—BALLISTIC TEST ON 0·2 MM. SOFT IRON WIRE.

Curve I. shows the initial values of Curve II. to a greater scale.

flux may be regarded as proportional to the current. In Fig. 13 the magnetic flux per square centimetre, B, is plotted against the magnetising force, H, for a test coil of 0·2 mm. soft iron wire, as used in telephony. It is seen that, for the sample in question, up to about $H=0{\cdot}1$ the flux is directly proportional to the magnetising force, and the permeability $\mu=B/H$ is constant. Up to about this limit, therefore, the foregoing definition of L still holds. Telephonic apparatus is specially designed to work within the range of constant permeability, otherwise distortion would be introduced.

The mutual inductance of two parallel wires of small cross-section, of distance apart b and each of length l, is given by the formula*

$$M=2l\left[\log_e\frac{2l}{b}-1+\frac{b}{l}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where the length l is great in comparison with the distance apart b. If b is in inches and l in n.m. the formula becomes, approximately

$$M\ (\text{millihenries})=l\left[0{\cdot}854\log_{10}\frac{l}{b}+4{\cdot}042\right]. \quad . \quad . \quad (8)$$

The self-inductance of a metallic return circuit consisting of two long parallel wires of axial distance apart b, diameter d and length l is given by the formula, neglecting the ends,

$$L=l\left[4\log_e\frac{2b}{d}+1\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

provided that the current is uniformly distributed over the cross-section of the conductor. From (9) it follows that

$$L\ (\text{mh. per n.m.})=1{\cdot}709\log_{10}\frac{b}{d}+0{\cdot}700. \quad . \quad . \quad (10)$$

For example, in a four-core telephone cable, in which each core is of type 160 lb. of copper ($d=0{\cdot}105$ in.), covered with 300 lb. of gutta-percha ($D=0{\cdot}392$ in.), and the four cores are laid up touching so that their axes are at the corner of a square, the axial distance apart b of the diagonal conductors is $\sqrt{2}D=0{\cdot}554$ in. Hence, from (10), $L=1{\cdot}935$ mh. per n.m. Again, from (8), the mutual inductance for 1 n.m. is $M=4{\cdot}261$ millihenries.

As the distance between the conductors increases L increases logarithmically with it. When the conductors are very far apart the formula

$$L=2l\left[\log_e\frac{4l}{d}-\tfrac{3}{4}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

*E. B. Rosa and L. Cohen:—Formulæ and Tables for the Calculation of Mutual and Self-Inductance. Bulletin of the Bureau of Standards, 5, 1908, p. 51.

may be used for a single conductor. When l is in n.m. and d in inches, this formula becomes

$$\text{L (millihenries)} = l\left[0{\cdot}854 \log_{10}\frac{l}{d} + 3{\cdot}92\right]. \quad . \quad . \qquad (12)$$

A single conductor of 500 lb. copper, of diameter 0·182 in. and length 1 n.m., has an inductance, from (12), of 4·55 millihenries.

Calculation of the inductance of a single conductor is rendered difficult by uncertainty regarding the return path. The minimum possible value would be reached if the current returned through a concentric conductor surrounding the core. The inductance of such a circuit is given by the formula

$$\text{L} = 2l\left[\log_e \frac{\text{D}'}{d} + \tfrac{1}{4}\right], \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

where D′ is the mean diameter of the outer hollow conductor. If the central conductor of 500 lb. is covered with 315 lb. of gutta-percha to an external diameter, D, of 0·423 in., D′ will be somewhat greater than D, and L will be somewhat greater than the value obtained by substituting D for D′. This value is 0·406 millihenry. If the current is supposed to return in a concentric tube in the sea water outside the cable, D′ will be somewhat greater than 0·9 in., which leads to the value 0·686 millihenry for L.

The influence of the iron sheath is to increase the inductance of the cable beyond that given by the formulæ for the unsheathed core. Experiments go to show that the inductance of an average cable exceeds that of the core alone by approximately 2 millihenries. Halving this for a single core, it may be said that the influence of the sheathing is to add about 1 millihenry to the inductance of the cable. The inductance of a single-cored cable may, therefore, be assumed to lie somewhere between $1\frac{1}{2}$ and $5\frac{1}{2}$ millihenries.

Capacity.

Let a charge of electricity, Q, given to a condenser produce a P.D., V, between the plates. The ratio Q/V is found to be a constant characterising the condenser, and is called its capacity—here denoted by the letter K; so that

$$Q=KV. \qquad \qquad (14)$$

In the electromagnetic system of units, Q is of dimensions $(l^{\frac{1}{2}}m^{\frac{1}{2}})$, and V is of dimensions $(l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2})$. The dimensions of K are, therefore, $(l^{-1}t^{2})$. When Q is 1 coulomb (ampere-second) and V is 1 volt, K is one "practical" unit of capacity, and is called 1 farad. As the farad, which is $10^{-1}\div 10^{8}$ or 10^{-9} C.G.S. unit, is inconveniently large, capacities are usually expressed in microfarads, or millionths $(\times 10^{-6})$ of a farad. One micro farad is 10^{-15} C.G.S. unit.

Differentiate each side of (14) with respect to time. Then $\frac{dQ}{dt}=K\,.\,\frac{dV}{dt}$. But $\frac{dQ}{dt}$ is the rate at which electricity is accumulating in the condenser—*i.e.*, the current C into the condenser. Hence

$$C=K\frac{dV}{dt} \qquad \qquad (15)$$

and

$$V=\frac{1}{K}\int C\,dt. \qquad \qquad (16)$$

The capacity of a condenser formed by two coaxial conducting cylinders is given by the formula*

$$K=\frac{\lambda l}{2\log_e D/d}, \qquad \qquad (17)$$

where l is the length of the cylinders in centimetres, d is the diameter of the inner cylinder, D the inner diameter of the outer, both being of circular section. This formula expresses the capacity in electrostatic units. To reduce it to electromagnetic units divide by $(3\times 10^{10})^2$, or to microfarads divide by 9×10^5. The formula then becomes

$$K\text{ (mfd. per n.m.)}=\frac{0{\cdot}0448\times\lambda}{\log_{10} D/d}. \qquad \qquad (18)$$

The constant λ is called the *specific capacity* or *dielectric constant* of the material filling the space between the cylinders.

* A. Russell, "Alternating Currents," Vol. I., p. 104.

D 2

It is the proportion which the capacity of the condenser bears to that of the same conductors with a vacuum instead of the dielectric in the space between them. For gutta-percha this number is about 3·2, its exact value depending on the nature of the mixture and on the treatment to which it has been subjected in the processes of manufacture before it is brought into the most suitable condition for wire covering. For a high-class core gutta-percha the following formula may be used :—

$$K \text{ (mfd. per n.m.)} = \frac{0{\cdot}146}{\log_{10} D/d}, \quad . \quad . \quad . \quad (19)$$

where the constant may range from 0·139 to 0·153. This constant has been gradually brought down from about 0·188 by a close study of the properties of suitable materials.

The capacity of a stranded conductor is greater than that of a solid conductor.* For practical purposes it may suffice to understand by d in formula (18) the outer diameter over the strands, or an empirical correction may be used. Care is taken in manufacture to ensure that the conductor is centrally situated in the core, since any deviation would be attended by an increase in capacity.

Taking the density of gutta-percha as unity, the weight of the dielectric for 1 n.m. of core, the outer diameter of which is D, and the inner d, inches, is given by the formula

$$W \text{ (lb.)} = 2{,}071 D^2 - \frac{w}{8{\cdot}9}. \quad . \quad . \quad . \quad . \quad (20)$$

Appendix IV. contains a list of familiar types of cable core, with diameters, resistance and capacity calculated on the basis of the formulæ which have just been given.

Effective Values.

For a variety of reasons the resistance which a conductor offers to alternating current may be different from that which a direct current experiences. This is especially the case if iron be present; the hysteresis in the iron as it is carried through a magnetic cycle and the eddy currents induced in the

* Formulæ for the calculation of the capacity of a standard conductor have been given by M. Levi-Civita. *See* H. Larose, Rapport sur l'état actuel de la télégraphie sous-marine.

iron by the varying magnetic flux producing what is equivalent to an increase in the resistance of the conductor. If W is the total watts dissipated in the conductor, of resistance R as measured by direct current, then $W = C^2R$, if there are no other losses than the ohmic heating of the conductor ; but if energy is expended in other ways the equation may be written in the form $W = C^2R_{\text{eff.}}$, where R_{eff} is the equivalent or *effective* resistance of the conductor to alternating current. In genera the suffix need not be used, as it will be understood, when dealing with alternating currents, that the values of the constants are such as would result from measurements with alternating current.

An alternating current of high frequency flowing along a straight conducting wire experiences a somewhat greater resistance than a direct current. The magnetic field, of cylindrical shape coaxial with the wire, set up by the current, produces in its growth eddy currents in the conductor which tend to oppose the current in the heart of the wire and restrict it to the surface. If the conductor is of considerable cross-section and the frequency of the alternations is high, the current may be entirely confined to a thin skin at the surface of the conductor. The increase in resistance due to "skin effect" may be calculated by the formula*

$$R_n = R_0\left(1 + \frac{n^2\pi^4 d^4}{48\rho^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

to a first approximation, where n is the frequency of the alternating current in periods per second, d is the diameter, ρ is the resistivity, and R_n and R_0 are the effective resistance and the resistance to direct current respectively. If $n = 1{,}000$, and $d = 0{\cdot}080$ in. or 2 mm. diameter, then for a copper wire where $\rho = 1{,}700$ cm. per second,

$$R_{1000} = R_0(1 + 0{\cdot}00112),$$

so that the increase in resistance is about one part in one thousand, and is quite negligible for practical purposes.

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* Lord Rayleigh, "Phil. Mag.," May 21, 1886, p. 381.

For all measurements on telegraph and telephone cables the conductors may be treated as though the currents were equally distributed over the cross-section.

Leakance.

Not only is energy lost in the conductor, but there is also loss in the insulating medium which surrounds it. This dielectric loss may be treated in a similar way. If W is the watts dissipated in the dielectric under the influence of an alternating voltage, E, then we may write

$$W=\frac{E^2}{S_{\text{eff.}}}=G_{\text{eff.}}E^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $S_{\text{eff.}}$ is a fictitious resistance the heat expended in which would be equal to the loss which is occurring in the dielectric. $G_{\text{eff.}}$, the reciprocal of $S_{\text{eff.}}$, is a fictitious conductance to which the name *leakance* is given. It may be measured in *mhos*, where 1 mho is the reciprocal of 1 ohm.

The leakance of a dielectric can be determined only by experiment, using alternating current of known frequency. Much research has been done on the question of the nature and the amount of the losses in dielectrics. There is probably no other branch of physics on which so much time and care have been expended with so little definite result. The reason is doubtless to be found in the unstable character of the majority of dielectrics. No two samples are identical, and all are in process of change with time. In many cases the effects observed must be attributed to impurities, and not to the material itself.

If the loss per cycle is independent of the speed at which the cycle is performed, the loss per second, and therefore the leakance, must be proportional to the frequency, and this is found to be nearly true. As a first approximation, the following empirical formula may be used for telegraph cable gutta-percha

$$G_{\text{mhos.}}=0{\cdot}024\,pK, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $p=2\pi n$ and K is the capacity of the core in farads.

For telephone cables special mixtures are prepared which are characterised by a much lower leakance, and for such mixtures the constant 0·002 may be substituted in (23). In any particular case the exact value can be determined only by experiment on a sample of the core.

Electrification and Absorption.

When an E.M.F. is applied to a condenser, the plates of which are separated by an insulating medium, the current which flows after the first rush of current which charges the condenser is not constant as in the case of conductors. Instead it varies continuously with time, in general decreasing. To this phenomenon is given the name *electrification*, and a steady but not excessive increase in apparent resistance is regarded as a criterion of good quality of the insulating medium.

In Fig. 14 the current curves are shown for two samples of gutta-percha core. The percentage increase or "improvement" in the resistance at the end of the second minute over that at the end of the first is taken customarily as a measure of the shape of the curve. An average value of this quantity for gutta-percha core is about 4 per cent., but lower grades may show as much as 20 or 30 per cent. In, say, half an hour the rate of current change is so slight that the current may be regarded as practically steady.

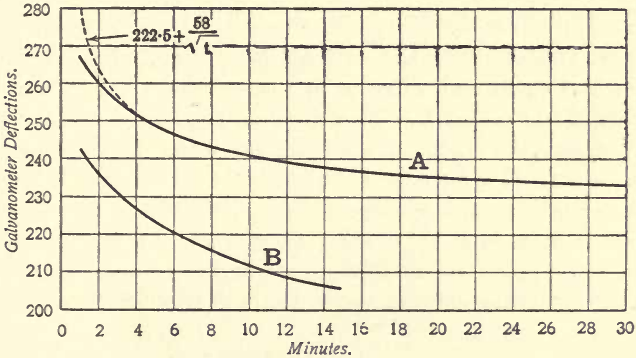

Fig. 14.—Electrification of Gutta-percha.

Curve A.—Type 661/397, as core at 75°F. Curve B.—Type 130/130, as core at 75°F.

A minimum value for the dielectric resistance which is frequently specified is 500 megohms (depending on the dimensions of the core) per n.m. after one minute's electrification, the core having been immersed in water at 75°F. for 24 hours previous to the test. Sometimes a maximum is also stipulated of, say, 2,000 megohms. In general, very high dielectric resistance is unstable.

The dielectric resistance of gutta-percha is greatly affected by temperature. To a first approximation it decreases exponentially with increase in temperature. A correction curve must be established for every new material, from which the dielectric resistance at any temperature can be read off from measurements at the standard temperature. Electrification is also dependent on the temperature, being greater at lower temperatures.

If, now—returning to the discussion of the charged condenser—the condenser be short-circuited after sufficient time has elapsed for the dielectric current to become sensibly constant, and discharged, a current flows out of the condenser which is equal and opposite to the current which seemingly charged the dielectric. To this effect the name *absorption* is given, and the charge which leaks out when the condenser is shorted is said to have been absorbed by the dielectric. Whether electrification and absorption are electrolytic or thermoelectric in nature, or whether they are to be attributed to molecular polarisation, electric endosmose or non-homogeneity of the insulating material, or to a combination of these causes, is not yet known.

The direct-current test of a material is indispensable, because of the light which it throws on the quality of the material. It would, therefore, be of the greatest advantage if it were possible at the same time to deduce from the electrification curve the leakance of the material—*i.e.*, the dielectric loss attending its use with alternating current—always provided that the phenomena have a common origin. The first step would be to obtain an empirical formula for the electrification curve. A satisfactory formula has not yet been found. The difficulty is that a formula which is a sufficiently close approximation in one case can scarcely be brought to fit another sample of the same material, much less another dielectric. Thus, in Fig. 14, a formula of the shape,

$$\text{Current (scale divisions)} = 222{\cdot}5 + \frac{58}{\sqrt{t}}$$

fits curve A except for the first two minutes, but does not fit curve B.

The reason why the reading of the dielectric resistance is stipulated to be taken at the end of one minute is that it cannot conveniently be made sooner owing to the fact that time must be given for the big rush of current on charging the condenser to pass, and for the galvanometer spot to become steady. The apparent dielectric resistance after a prolonged application of the testing voltage is of little importance in telegraphy or telephony, for such a steady state is never reached in practice, and, as will appear later, the dielectric resistance of a well-constructed cable is far too high to affect its working. On the other hand, absorption is very important, and is likely to become of even greater importance in the immediate future.

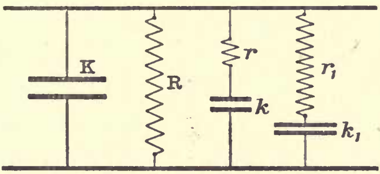

FIG. 15.—MODEL TO REPRESENT THE BEHAVIOUR OF A CONDENSER WITH IMPERFECT DIELECTRIC.

A model of a condenser may be put together as in Fig. 15 Here K is the true or geometric capacity of the condenser, and R is the true dielectric resistance, possibly measurable by the prolonged application of the testing voltage. The path $r\,k$—where r and k are both small and k is a small fraction of K—is that taken by alternating current. At very low frequencies the current flows mainly through R, but at higher frequencies the conductance of the condenser path increases, and the dielectric loss is represented by the heat expended in r. The resistance r_1 is very great, and the product r_1k_1 is also great, being measured in minutes. This branch may be supposed to produce the absorption. Alternatively, r_1 may be removed and k_1 bridged over a part of R.

It should be remembered that nothing more than a mental picture is obtained in this way. Difficulty is experienced as soon as it is attempted to make the model fit the experimental

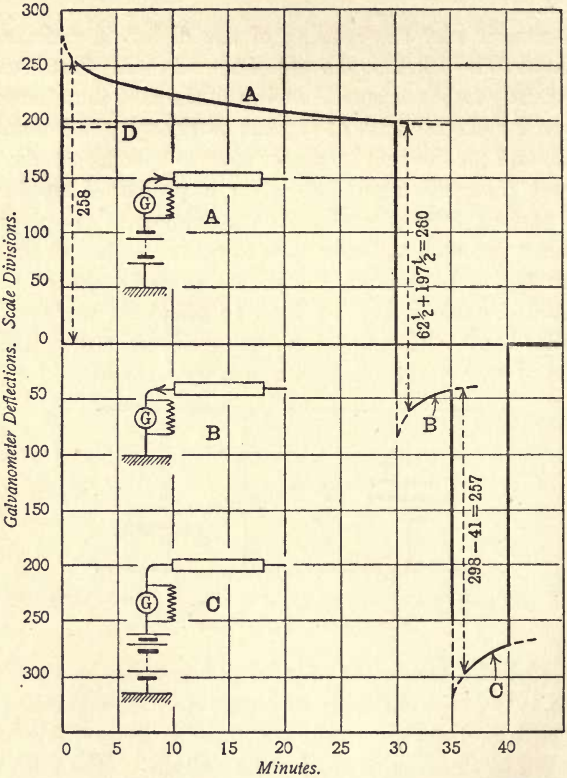

FIG. 16.—TEST ON COMPLETED LENGTH OF CABLE.

Curve A.—Negative current. Curve B.—Discharge to earth.
,, C.—Positive current. ,, D.—Sum of A and B.

results. A close approximation to the electrification curve might be obtained by choosing a great number of such branches as r_1, k_1, but at the cost of simplicity.

In testing a completed length of cable it is usual to apply first one pole of the battery, say the negative, and to note the decreasing deflections of the galvanometer for, say, half an hour to an hour. The testing end of the cable is then put to earth, and the discharge curve through the galvanometer is observed for five or ten minutes, after which the opposite pole of the battery is applied for another five or ten minutes. The first part of such a test is illustrated graphically in Fig. 16. In this instance the cable is tested with the negative pole for 30 minutes, is then earthed for five minutes, and then tested for five minutes with the positive pole. Before completing the test by beginning with the positive pole the cable must be earthed for several hours until it is completely discharged. The two sets of curves should be in close agreement. According to the scheme of Fig. 15, curve A is the sum of the steady current through R and the decaying current into the concealed condenser k_1. Again, curve B is the current flowing out of k_1. When B is added algebraically to A the condenser current drops out, and there is left the steady current D of about 195 divisions. Curve C may be regarded as curve A superposed on curve B. As a check, the readings at the end of the first minute may be compared. With a smooth and steady curve of electrification and an agreement between the readings at charge and discharge as close as that in the diagram, no doubt whatever need be felt as to the soundness of the cable dielectric.

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CHAPTER III.

TRANSIENT AND PERIODIC PHENOMENA.

Growth and Decay of Current in a Circuit containing Resistance and Inductance—Resistance and Inductance with Sinusoidal E.M.F.—Growth and Decay of Current in a Circuit containing Resistance and Capacity—Resistance and Capacity with Sinusoidal E.M.F.—Charge and Discharge of a Condenser through Resistance and Inductance—Resistance, Inductance and Capacity with Sinusoidal E.M.F.—Leaky Inductance Coil—Leaky Condenser—Meshes—Starting an Alternating Current in an Inductance Coil—Starting an Alternating Current in a Resistance-Capacity Circuit—Alternating-current Start: Resistance, Inductance and Capacity in Series—Fundamental Distinction between Telegraphy and Telephony.

Growth and Decay of Current in a Circuit containing Resistance and Inductance.

The voltage V at the ends of a conductor of resistance, R, and inductance, L, is connected, according to equation (6) of Chapter II., with the current, C, through the conductor by the relationship

$$V = RC + L\frac{dC}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let us suppose that, as in Fig. 17, A, the conductor forms a closed circuit containing a source of E.M.F., E. The steady direct current which E produces is E/R. Now let the E.M.F. be suddenly removed, the circuit remaining closed, as in Fig. 17, B. Owing to the presence of inductance, the current does not fall at once to zero, but continues with diminishing strength for an appreciable time. Since V is zero, C during this time must satisfy the equation

$$\frac{dC}{dt} + \frac{R}{L}C = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is easy to verify by substitution that $C = e^{-\frac{Rt}{L}}$ is a solution of this differential equation. The rate of change of C, or slope of the curve $C=e^{-\frac{Rt}{L}}$, is at any instant related to C in the manner entailed by (2). Moreover, this expression for C is zero when $t=\infty$,—*i.e.*, according to it the current eventually dies away to nothing, as it must in fact. Nevertheless, $C=e^{-\frac{Rt}{L}}$ does not satisfy the other "terminal" condition. Reckoning time from the instant at which the E.M.F. was

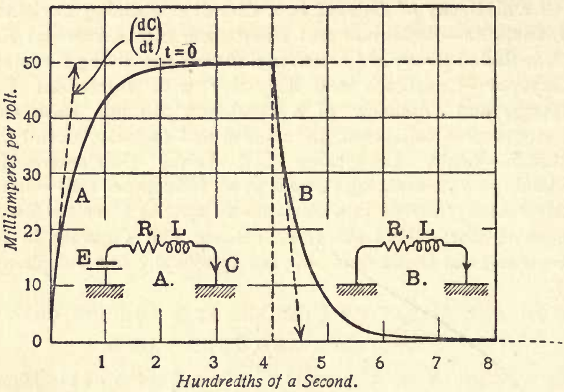

FIG 17.—CURRENT IN A CIRCUIT CONTAINING RESISTANCE AND INDUCTANCE.

R=20 ohms. L=0·1 henry.

removed from the circuit; when t is zero $C_{(t=0)}$ should be the same as the initial steady current $\frac{E}{R}$, whereas $\left(e^{-\frac{Rt}{L}}\right)_{(t=0)}=1$.

But if our tentative solution be multiplied by $\frac{E}{R}$ it will still fit the differential equation (2), and will now, in addition, satisfy both terminal conditions. The solution to (2), representing the decay of current in the coil, is therefore

$$C=\frac{E}{R}e^{-\frac{Rt}{L}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the E.M.F. E be re-inserted in the circuit after the current has fallen to zero, the current will return to its steady value, according to equation (1). Instead of (3), which was sufficient for the particular case (2) of equation (1), we must seek a more extended solution. It is obtained by subtracting the expression (3) for the decaying current from the steady current, and is

$$C=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For it is seen that this expression satisfies the differential equation (1), is zero when $t=0$, and rises to the steady value $\frac{E}{R}$ when t is infinite. Geometrically interpreted, the graph of C, plotted against t from equation (4), is a curve of the proper shape, of which the position has been fixed by two known points through which it must pass.

Fig. 17, curve A, represents the growth of current when the constant E.M.F. E is suddenly inserted in the circuit. It is plotted from (4) and Table I., taking R as 20 ohms, L as 0·1 henry and E as 1 volt. Fig. 17, curve B, represents the decay of current when the E.M.F. is suddenly removed at the end of four-hundredths of a second. Curves A and B are precisely similar, but A is drawn upwards and B downwards. In about three-hundredths of a second the current changes are practically over. The initial slope of the current curve, at zero time, is, from (1) or from (4),

$$\left(\frac{dC}{dt}\right)_{t=0}=\frac{E}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and is represented by the tangent to the curve at the beginning.

Since L is of dimensions (l) and R is of dimensions (lt^{-1}), it follows that the ratio L/R is of dimensions $(l)\div(lt^{-1})=(t)$, or is simply a time, and may be measured in seconds. For this reason the ratio L/R is called the *time-constant* of the coil. In the present instance the time-constant is 0·005 second. When that time has elapsed the current in its growth has reached the fraction $1-1/e$, or 0·632 of its final value.

Table I.—*Fig.* 17, *Curves* A *and* B.

t (seconds $\times 10^{-3}$)	1	2	3	4	5	10	15	20	30
$e^{-Rt/L}$	0·819	0·670	0·549	0·449	0·368	0·135	0·050	0·018	0·002
$\frac{E}{R}e^{-Rt/L}$ (milliamperes per volt)	40·94	33·52	27·44	22·47	18·40	6·77	2·49	0·92	0·12
$\frac{E}{R}(1-e^{-Rt/L})$ (milliamperes per volt)	9·06	16·48	22·56	27·53	31·61	43·23	47·51	49·08	49·88
$Ee^{-Rt/L}/pL$ (milliamperes per volt)	1·303	1·067	0·873	0·715	0·585	0·215	0·079	0·029	0·004

Resistance and Inductance with Sinusoidal E.M.F.

Let us now suppose that the E.M.F. is not constant, but varies continuously in a periodic manner. Since any periodic function may be replaced by an equivalent Fourier's series, it will suffice in the first instance if, for simplicity only, a single term of the series be taken as a representative example. Let us assume, therefore, that the graph of the E.M.F. when plotted against time is a pure sine curve, so that V may be represented by E sin pt, where p is a constant. The values of V repeat themselves in time T such that $\sin p(t+T)=\sin pt$—*i.e.*, when $pT=2\pi$ or $T=2\pi/p$. If there are n complete periods per second, n must equal $1/T$, and therefore p must equal $2\pi n$. T is called the *periodic time* and n is usually called the *frequency*. Since p is of dimensions (t^{-1}), pL is of dimensions (lt^{-1}), which are those of a resistance, and pL may be measured in ohms.

It is a physical law of very general application that a stress varying in a sinusoidal manner tends to give rise to a strain which is also sinusoidal in its variations and of the same periodicity as the stress. A periodic change produced in this way is called a *forced vibration* or *forced oscillation*. In the electrical case a sinusoidal E.M.F. produces a sinusoidal current of the same period, provided there is no disturbing factor, such as iron, present. It will appear presently that sufficient time must be given for the initial effects attending the introduction of the E.M.F. into the circuit to subside before the purely sinusoidal state of affairs is realised. With this assumption we may write

$$C = C_{max.} \sin (pt - \varphi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where φ, the phase difference between V and C, may have any value between $-\pi$ and $+\pi$. In order to determine $C_{max.}$ and φ, substitute this expression for C in the equation

$$L\frac{dC}{dt} + RC = E \sin pt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The left-hand side of the equation becomes

$$pLC_{max.} \cos (pt - \varphi) + RC_{max.} \sin (pt - \varphi),$$

which may be written

$$\sqrt{R^2 + p^2L^2} \, . \, C_{max.} \sin\left(pt - \varphi + \tan^{-1}\frac{pL}{R}\right).$$

Both sides of equation (7) are now identical if $\sqrt{R^2 + p^2L^2}C_{max.} = E$ and $\tan \varphi = \frac{pL}{R}$. Hence

$$C = \frac{E \sin \left(pt - \tan^{-1}\frac{pL}{R}\right)}{\sqrt{R^2 + p^2L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the voltage V had been represented by E cos pt, on following through the same reasoning the current corresponding would be found to be

$$C = \frac{E \cos \left(pt - \tan^{-1}\frac{pL}{R}\right)}{\sqrt{R^2 + p^2L^2}}. \quad . \quad . \quad . \quad . \quad (9)$$

Whether the form E sin pt or the form E cos pt be chosen to represent V is material only initially. After the oscillations have settled down into their steady purely-sinusoidal state, choice of one form or the other is merely equivalent to an arbitrary choice of zero from which to reckon time, and is analogous to the choice of the position of the origin in coordinate geometry, or to the fixing of the datum level in measuring heights.

It will be found that both forms may with great advantage be united in the single complex expression

$$V=E(\cos pt+i\sin pt)=Ee^{ipt}=E\underline{/pt}. \quad . \quad . \quad . \quad (10)$$

The choice of this form of expression for V is equivalent in the geometrical representation of complex quantities, as is evident from Fig. 18, to stating the polar co-ordinates E and pt of the line OP, which therefore represents V. Now OP is a line of *constant* length E, which rotates as t increases with the *constant* angular velocity p. Mathematically, its properties are therefore of extreme simplicity, and are much less complicated than those of either of its projections on the axes of co-ordinates. When the mathematical transformations—involved in the calculation of C from V—are complete, it remains only to reject either the real or the imaginary portions of V and C in order to obtain expressions corresponding to the physical magnitudes, which is equivalent geometrically to projecting V and C on either the vertical or the horizontal axis of co-ordinates.

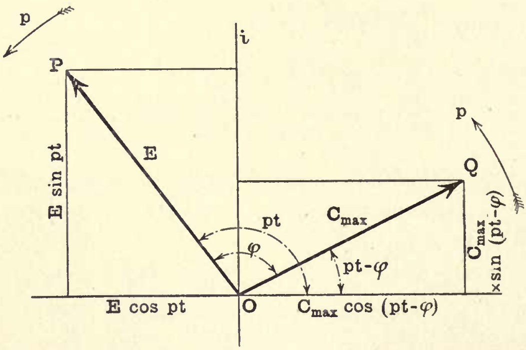

FIG. 18.—GEOMETRICAL REPRESENTATION OF SINUSOIDAL E.M.F. AND CURRENT.

Now, if V is represented by OP or $E\underline{/pt}$, C is represented by OQ or $C_{max}\underline{/pt-\varphi}$. Substitute this value of C in equation (1). Then

$$ipLC_{max.}e^{i(pt-\phi)}+RC_{max.}e^{i(pt-\phi)}=V$$

or

$$C_{max.}e^{i(pt-\phi)}[R+ipL]=V,$$

i.e.,

$$C=\frac{V}{R+ipL}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It is evident from (11) that if the form of representation for V be chosen as above, C and V are connected together by a relationship which recalls Ohm's law, and may be regarded as its analogue in alternating-current theory. The denominator $R+ipL$, which replaces R in the direct-current case, is a complex quantity, and C is to be obtained from (11) by applying the ordinary rules of Chapter I. regarding the manipulation of complex quantities. Finally, C is expressed as the sum of real and imaginary parts one or the other of which is retained. Thus

$$C=\frac{E\underline{/pt}}{R+ipL}=\frac{E\underline{/pt}}{\sqrt{R^2+p^2L^2}\underline{/\tan^{-1}\frac{pL}{R}}}$$

$$=\frac{Ee^{ipt}}{\sqrt{R^2+p^2L^2}\,.\,e^{i\tan^{-1}\frac{pL}{R}}}=\frac{Ee^{i\left(pt-\tan^{-1}\frac{pL}{R}\right)}}{\sqrt{R^2+p^2L^2}}$$

Hence, as before,

$$C=\frac{E\sin\left(pt-\tan^{-1}\frac{pL}{R}\right)}{\sqrt{R^2+p^2L^2}},\text{ if V is E }\sin pt,$$

and
$$C=\frac{E\cos\left(pt-\tan^{-1}\frac{pL}{R}\right)}{\sqrt{R^2+p^2L^2}},\text{ if V is E }\cos pt.$$

If R is small in comparison with pL, C is nearly inversely proportional to p, and therefore to the frequency n. At the same time $\tan^{-1}\frac{pL}{R}$ is nearly 90° and $C_{max.}$ is nearly at right angles to E. In the example previously discussed, where $R=20$ ohms, $L=0\cdot1$ henry and $E=1$ volt, let $p=2\pi\times1{,}000$. Then $C=\frac{1}{pL}\underline{/pt-\tan^{-1}\frac{pL}{R}}$ or $1\cdot59\,\underline{/pt-88°14'}$ milliamperes per volt.

This phase relationship is exhibited graphically in Fig. 20, A. The combination of resistance and inductance in series is called *impedance*. Thus $R+ipL$ is an impedance of which the modulus is $\sqrt{R^2+p^2L^2}$. The term *reactance* is sometimes used to denote pL. In Fig. 18, when voltage and current are projected on the vertical axis, current passes through its maximum value after (from the arrow indicating the direction of motion) the voltage has begun to decline. For this reason the current through an impedance is said to *lag* behind the voltage.*

Growth and Decay of Current in a Circuit containing Resistance and Capacity.

In Fig. 19 a resistance, R, is joined in series with a condenser of capacity K, and the circuit contains an E.M.F., E. From equation (15) of Chapter II. the current in the circuit is given by $C=K\frac{dV}{dt}$, where V is the voltage drop from one plate of the condenser to the other. Adding to V the fall in voltage RC on the resistance, the sum must be equal to the E.M.F. E, so that

$$\left.\begin{aligned} V+RC&=E \\ \frac{1}{K}\int C\,dt+RC&=E \end{aligned}\right\} \quad \text{or} \qquad (12)$$

If E is a constant E.M.F. from a battery, current ceases to flow in the circuit when the condenser is charged to the full voltage of the battery. Now suppose that E is removed and the circuit closed on itself. The condenser discharges, and the current flowing out of it is given by the equation

$$\frac{1}{K}\int C\,dt+RC=0. \qquad (13)$$

Differentiate (13) with respect to t; then

$$C/K+R\,.\,dC/dt=0. \qquad (14)$$

This equation is of the same form as (2), 1/KR replacing R/L. The solution is therefore $C=Ae^{-t/KR}$, where A is the initial value of the current. This expression represents the decay of the current; to obtain the growth of the current substitute for C in (12).

Then
$$\frac{A}{K}\int_0^t e^{-t/KR}\,dt+ARe^{-t/KR}=E,$$

* More generally, the term impedance may be used to describe the complex operator, reduced to the form $a+ib$, of a piece of apparatus consisting of any combination of resistance, inductance and capacity.

from which, on integration, A=E/R. The current into or out of the condenser is, therefore, given by

$$C = \pm E/R \, . \, e^{-t/KR} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Curve A, Fig. 19, shows the current at charge, and curve B at discharge, of a condenser of 2·5 mfd. capacity through a resistance of 2,000 ohms. The curves are of the same shape

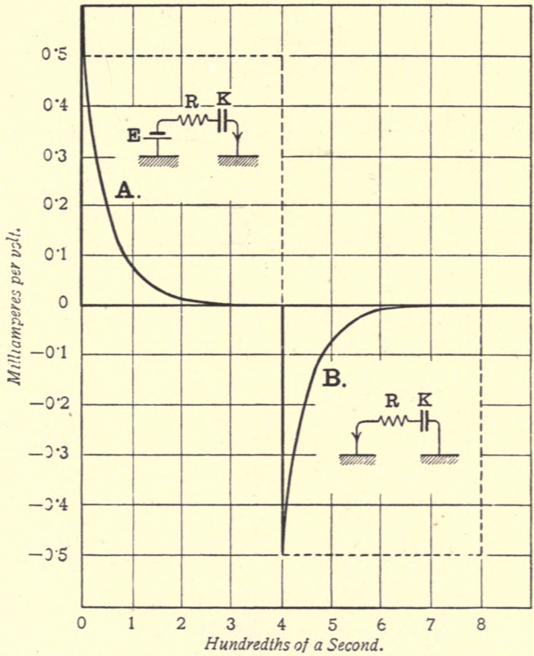

FIG. 19.—CURRENTS OF CHARGE AND DISCHARGE OF A CONDENSER THROUGH A RESISTANCE.
R=2,000 ohms. K=2·5 mfds.

as those in Fig. 17, but they are differently arranged. At the moment when E is inserted in the circuit the uncharged condenser behaves as though it were short-circuited.

Since K is of dimensions $(l^{-1}t^2)$ and R is of dimensions (lt^{-1}), the product KR is of dimensions (t), and is, like L/R in the case of resistance and inductance, the time-constant of the circuit. In this case $KR = 2{\cdot}5 \times 10^{-6} \times 2{,}000 = 0{\cdot}005$ second.

Resistance and Capacity with Sinusoidal E.M.F.

When the E.M.F. is sinusoidal, and of the form $E\underline{/pt}$, let C be written $C_{max.}\underline{/pt-\varphi}$. Now substitute $C_{max.}\,e^{i(pt-\phi)}$ for C in the equation

$$\frac{1}{K}\int C dt + RC = Ee^{ipt}, \qquad (16)$$

and it follows that

$$C = \frac{E\underline{/pt}}{R+\frac{1}{ipK}} \qquad (17)$$

Realising the denominator in the usual manner, this expression becomes

$$C = \frac{E\underline{\Big/ pt+\tan^{-1}\frac{1}{pKR}}}{\sqrt{R^2+\frac{1}{p^2K^2}}} \qquad (18)$$

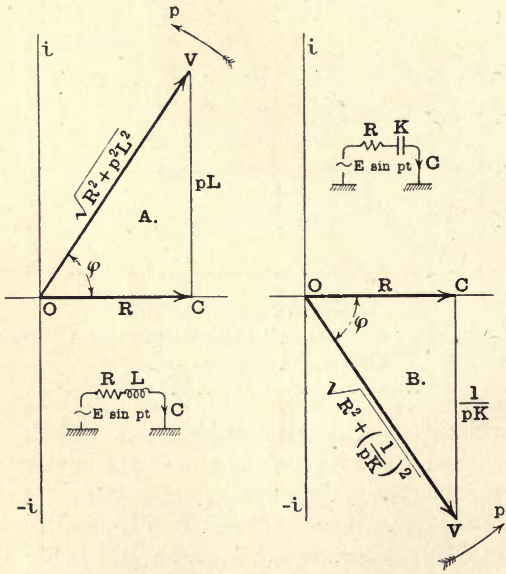

Fig. 20.—Phase Relationship: Sinusoidal Voltage and Current.
A.—Inductance coil. B.—Resistance and condenser.

In the circuit of Fig. 19, if $p=2\pi\times1{,}000$, $C=0{\cdot}5\underline{/1^\circ 49'}$ milli-amperes per volt. In this case $1/pK$ is negligible in comparison with R, and $C=\frac{E}{R}\underline{/pt+\tan^{-1}\frac{1}{pKR}}$. On the other hand, if R is small in comparison with $1/pK$ the formula becomes $C=pKE$ $\underline{/pt+\tan^{-1}\frac{1}{pKR}}$, and C is directly proportional to p, and therefore to the frequency n.

Since from (18) the angle of phase difference between V and C is positive, it follows that in Fig. 18 the line OQ drawn to represent the current in this case would be in advance of OP counter-clockwise. Projecting on the vertical axis, the current now reaches its maximum value sooner than the voltage, and the current in a condenser-resistance circuit is said to *lead* the voltage, the angle of lead being $\underline{/\tan^{-1}\frac{1}{pKR}}$. This phase relationship is exhibited graphically in Fig. 20, B. Since p is of dimensions (t^{-1}), and K is of dimensions $(l^{-1}t^2)$, it follows that $1/pK$ is of dimensions (lt^{-1}), which are those of a resistance, and $1/pK$, like pL, may be measured in ohms, or pK may be measured in mhos.

Charge and Discharge of a Condenser through Resistance and Inductance.

In the circuit of Fig. 21 an inductance coil and a condenser are joined in series to form a circuit containing an E.M.F. E. Let us suppose first that E is constant and that it is removed suddenly at the instant $t=0$, after the condenser has been fully charged, the circuit remaining closed. The equation to the discharge current is, combining the voltage V on the condenser with the ohmic drop RC and the reactance drop $L\frac{dC}{dt}$,

$$L\frac{dC}{dt}+RC+V=0. \qquad (19)$$

Differentiate (19) with respect to t and substitute C/K for dV/dt. Then

$$L\frac{d^2C}{dt^2}+R\frac{dC}{dt}+\frac{C}{K}=0. \qquad (20)$$

We may experiment with $C=e^{-\lambda t}$ on the analogy of previous successes—λ being some constant as yet undetermined—and substitute in (20). This gives

$$L\lambda^2-R\lambda+1/K=0,$$

a quadratic equation from which

$$\lambda=\frac{R}{2L}\left(1\pm\sqrt{1-\frac{4L}{KR^2}}\right). \quad . \quad . \quad . \quad . \quad (21)$$

It is evident, therefore, that $e^{-\frac{Rt}{2L}\left(1+\sqrt{1-\frac{4L}{KR^2}}\right)}$ and $e^{-\frac{Rt}{2L}\left(1-\sqrt{1-\frac{4L}{KR^2}}\right)}$ are both expressions for C which fit equation (20). Adding

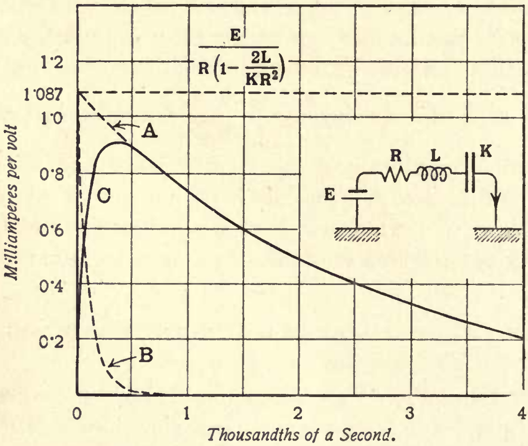

Fig. 21.—Aperiodic Charge and Discharge of a Condenser.

$K=2{\cdot}5\times10^{-6}$ farads. $L=0{\cdot}1$ henry. $R=1{,}000$ ohms. $KR=0{\cdot}0025$ seconds. $L/R=0{\cdot}0001$ second.

these, and dividing by 2, $e^{-Rt/2L}\cosh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ is a solution, and subtracting, $e^{-Rt/2L}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ is also a solution. If $4L/KR^2$ is greater than unity, the quantity under the square-root sign is negative. In that case, since $\cosh ix=\cos x$ and $\sinh ix$ $=i\sin x$, the expressions $e^{-Rt/2L}\cos\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1}$ and $e^{-Rt/2L}\sin$

$\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1}$ may be used in place of the previous ones. Out of these six typical or *normal* solutions one must be built up which will satisfy not only the differential equation but also the terminal conditions. In the present instance these conditions are (1) The current must be zero when t is infinite and the condenser is fully charged or discharged. This condition is secured by the decaying exponential factor $e^{-Rt/2L}$ which all six solutions have in common. Again (2), owing to the presence of inductance L in the circuit, the current must rise from zero when t is zero. Only $e^{-Rt/2L}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ and $e^{-Rt/2L}\sin\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ meet this condition. Since the initial voltage on the condenser at discharge is that of the battery E, it follows from (19) that (3)

$$L\left(\frac{dC}{dt}\right)_{t=0}+E=0. \qquad (22)$$

These conditions are sufficient to determine the current completely.

CASE I.—$\frac{4L}{KR^2}<1$. *Aperiodic Charge and Discharge.*

Taking $C=Ae^{-Rt/2L}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$, where A is some constant yet to be fixed, and substituting in (22), it follows that A must equal $\frac{-2E}{R\sqrt{1-\frac{4L}{KR^2}}}$. Hence finally

$$C=\frac{\pm 2Ee^{-Rt/2L}}{R\sqrt{1-\frac{4L}{KR^2}}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}, \quad . \quad . \quad (23)$$

the sign being taken positively or negatively according to whether the current is flowing into, or out of, the condenser.

Take $K=2{\cdot}5\times10^{-6}$ farads, $L=0{\cdot}1$ henry, and $R=1{,}000$ ohms.

Table II. (Fig. 21.)

t (seconds $\times 10^{-3}$).........	0·1	0·2	0·3	0·4	0·5	0·6	1·0	1·5	2·0	3·0	4·0
$e^{-t/KR} \div R(1-2L/KR^2)$	1·044	1·002	0·964	0·926	0·890	0·855	0·728	0·597	0·488	0·327	0·220
$e^{-Rt/L} \div R(1-2L/KR^2)$	0·400	0·147	0·054	0·020	0·007	0·003	...	...	...	...	...
C (milliamperes per volt)	0·644	0·855	0·910	0·906	0·883	0·852	0·728	0·597	0·488	0·327	0·220

Then $KR=0·0025$ sec. and $L/R=0·0001$ sec. Now, since $4L/KR^2=0·16$, and is therefore small in comparison with unity, $\sqrt{1-\frac{4L}{KR^2}}$ very nearly equals $1-\frac{2L}{KR^2}$, on expansion by the binomial theorem. Substituting this value in (23), the formula for C becomes approximately

$$C=\frac{E}{R\left(1-\frac{2L}{KR^2}\right)}\left[e^{-t/KR}-e^{-Rt/L}\right]. \quad . \quad . \quad . \quad (24)$$

The second of the two exponential expressions in (24) arises from the combination of resistance and inductance, and soon dies away as its time-constant is small. The other expression arises from the combination of resistance and capacity and is much more persistent for the chosen values of the constants (high value of R).

Fig. 21 is plotted from (24) and Table II. Curve A is the first term of (24) and curve B the second term. Curve C is the difference between the two and is the curve of charge of the condenser. The curve of discharge is the same curve drawn downwards. The effect of inductance is to round off the initial sharp peak of the condenser charge or discharge curve. The quantity of electricity which has accumulated in the condenser at time t is the area of curve C up to that time, and the voltage across the terminals of the condenser is that area divided by K. Expressions for Q and V may be obtained from (23) by integration. Thus

$$Q=\int_0^t Cdt=\frac{2E}{R\sqrt{1-\frac{4L}{KR^2}}}\int_0^t e^{-Rt/2L}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}\,.\,dt,$$

from which on integration, Appendix II.,

$$Q=KE-\frac{KEe^{-Rt/2L}}{\sqrt{1-\frac{4L}{KR^2}}}\left(\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}+\sqrt{1-\frac{4L}{KR^2}}\cosh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}\right)$$

$$=KE-\frac{KEe^{-Rt/2L}}{\sqrt{\frac{KR^2}{4L}-1}}\sinh\left(\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}+\tanh^{-1}\sqrt{1-\frac{4L}{KR^2}}\right). \quad . \quad . \quad (25)$$

From (25), V may be obtained by dividing by K.

CASE II.—$\frac{4L}{KR^2}=1$. *Critical Charge and Discharge.*

When $4L/KR^2$ is very nearly unity, $\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ is very small, and $\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$ very nearly equals $\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}$.

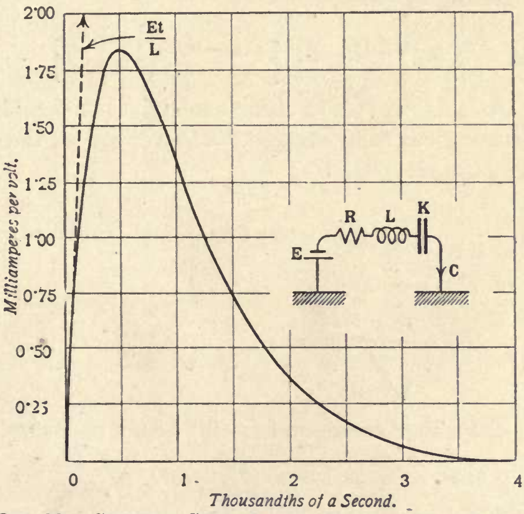

FIG. 22.—CRITICAL CHARGE AND DISCHARGE OF A CONDENSER.
$K=2{\cdot}5\times10^{-6}$ farads. $L=0{\cdot}1$ henry. $R=400$ ohms.

Hence, from (23),

$$C=\frac{Et}{L}e^{-Rt/2L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Take $K=2{\cdot}5\times10^{-6}$ farads and $L=0{\cdot}1$ henry, as before, and choose R to be 400 ohms, in order that $4L/KR^2$ may equal unity. Fig. 22 is plotted from (26) and Table III. Since the time constants are equal (0·0005 second) there is now only one exponential term in (26). The tangent to the origin is Et/L. As the factor $e^{-Rt/2L}$ diminishes the curve falls away from the tangent at the origin and is soon tangential to the horizontal axis.

Table III. (Fig. 22.)

t (seconds $\times 10^{-3}$)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$e^{-Rt/2L}$	0·819	0·670	0·549	0·449	0·368	0·247	0·135	0·050	0·018	0·002
$\frac{Et}{L}e^{-Rt/2L}$*	0·819	1·340	1·647	1·796	1·840	1·726	1·350	0·750	0·360	0·060

* Milliamperes per volt.

The quantity of electricity at any time is to be obtained from (26). Integrating by parts,

$$Q=\frac{E}{L}\int_0^t te^{-Rt/2L}dt=EK\left[1-e^{-Rt/2L}\left(1+\frac{Rt}{2L}\right)\right]. \quad . \qquad (27)$$

When $t=\infty$ this expression reduces to EK, as it should, for then the condenser is fully charged. The voltage on the condenser at any time is Q/K.

CASE III.—$\frac{4L}{KR^2}>1$. *Oscillatory Charge and Discharge.*

In this case (23) becomes

$$C=\frac{2Ee^{-Rt/2L}}{R\sqrt{\frac{4L}{KR^2}-1}}\sin\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1} \quad . \quad . \qquad (28)$$

Take $K=2{\cdot}5\times10^{-6}$ farads and $L=0{\cdot}1$ henry as before, and let $R=20$ ohms. Then, since $\frac{4L}{KR^2}=400$, $\sqrt{\frac{4L}{KR^2}-1}$ very nearly equals $\frac{2}{R}\sqrt{\frac{L}{K}}$. Hence, the formula for C may be written

$$C=E\sqrt{\frac{K}{L}}e^{-Rt/2L}\sin\frac{t}{\sqrt{KL}} \quad . \quad . \quad . \quad . \qquad (29)$$

$$=pKEe^{-Rt/2L}\sin pt, \text{ where } p=\frac{1}{\sqrt{KL}}.$$

In Fig. 23, curve A is the graph of 0·005 sin 2,000 t. Curve B is the exponential factor 0·005 e^{-100t}, the values of which may be obtained from Table III. by multiplying the times there given by 20. Curve C is obtained by combining curves A and B, being drawn as though curve A were compressed so as to lie within and be tangential to the branches of curve B. The

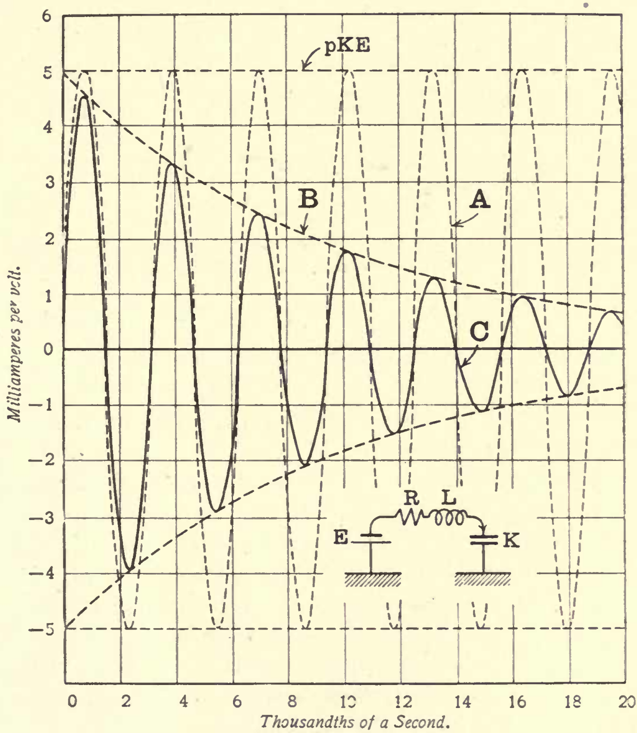

FIG. 23.—OSCILLATORY CHARGE AND DISCHARGE OF A CONDENSER.
K=2·5×10⁻⁶ farads. L=0·1 henry. R=20 ohms. $T=2\pi/p=2\pi\sqrt{KL}=0{\cdot}00314$ second.
Curve A=$pKE \sin pt$. Curve B=$pKe^{-Rt/2L}$.

current rises to a maximum, turns, and falls below the zero line, swinging backwards and forwards across it before it is finally zero. For this reason the phenomenon is said to be *oscillatory*, and, since there is no impressed periodic E.M.F. in the circuit to produce the oscillations, they are called *free* oscillations, to distinguish them from the *forced* oscillations which are produced by a periodic E.M.F.

The *natural* period of the free oscillations is fixed by the constants of the circuit. Let T be the periodic time; then from (29), $\sin\left(\frac{t+T}{\sqrt{KL}}\right)$ must equal $\sin\frac{t}{\sqrt{KL}}$, or $\frac{T}{\sqrt{KL}}=2\pi$, and

$$T=2\pi\sqrt{KL}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If R is large enough to necessitate consideration, (28) may be used in place of (29). Proceeding in the same way, the periodic time is found then to be

$$T=\frac{2\pi\sqrt{KL}}{\sqrt{1-\frac{KR^2}{4L}}}=2\pi\sqrt{KL}\,.\left(1+\frac{KR^2}{8L}\right), \quad . \quad . \quad (31)$$

nearly, on expanding the denominator by the binomial theorem. In the present instance $\frac{KR^2}{8L}=0{\cdot}00125$, so that the effect of the presence of resistance is to increase the periodic time by a little over one part in one thousand. The frequency of the oscillations, which is the reciprocal of the periodic time, is reduced in the same proportion.

Although resistance affects the periodic time but slightly, its influence on the amplitude of vibration is considerable. If R were zero the factor $e^{-Rt/2L}$ would be unity for all values of t, and the oscillations would be represented by curve A of Fig. 23. Such oscillations of constant amplitude are usually called persistent or *continuous* oscillations. But when resistance is present the oscillations die away, in proportion as the factor $e^{-Rt/2L}$ decays, and are termed *damped* oscillations, $e^{-Rt/2L}$ being called the damping factor and R/2L the damping coefficient. The ratio of consecutive current maximum to current minimum is $e^{-Rt/2L}\div e^{-\frac{R}{2L}(t+\pi)}=e^{R\pi/2L}$, which is a constant quantity—*i.e.*, the amplitude of the oscillations falls off according to a geometric progression.

As no circuit in practice is entirely self-contained and free from loss of energy, whether through resistance or in some other way, free oscillations are necessarily always more or less damped, and continuous oscillations must be forced. By charging and discharging the condenser at regular intervals a train or succession of damped oscillations may be produced.

The quantity of electricity stored up in the condenser during the first semi-period of charging may be obtained from (29). Integrating from time $t=0$ to time $t=\mathrm{T}/2$, and applying Appendix II.,

$$\mathrm{Q}_1=\mathrm{E}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}\int_0^{\mathrm{T}/2} e^{-\mathrm{R}t/2\mathrm{L}} \sin \frac{t}{\sqrt{\mathrm{KL}}} \,.\, dt$$

$$=\mathrm{EK}\left[e^{-\frac{\mathrm{R}\pi}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}+1\right]=\mathrm{EK}\left[e^{-\frac{\mathrm{R}\pi}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}+1\right] \quad (32)$$

approximately.

The quantity, flowing out, in the second semi-period is

$$\mathrm{Q}_2=-\mathrm{EK}\left[e^{-2\frac{\mathrm{R}\pi}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}+e^{-\frac{\mathrm{R}\pi}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}\right],$$

and so on. The total at the end of m semi-periods is

$$\mathrm{EK}\left[1-e^{-\frac{m\mathrm{R}\pi}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}} \,.\, \cos m\pi\right].$$

When m is infinite, $\mathrm{Q}=\mathrm{EK}$, and the condenser is fully charged. The voltage on the condenser at any instant is Q/K.

Resistance, Inductance and Capacity with Sinusoidal E.M.F.

If the E.M.F. is sinusoidal and of the form $\mathrm{E}e^{ipt}$, the current which it produces may be found very easily from the equation

$$\mathrm{C}=\frac{\mathrm{E}\angle pt}{\mathrm{R}+ip\mathrm{L}+\frac{1}{ip\mathrm{K}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where the denominator is the total symbolic or generalised resistance of the circuit. The denominator may be written $\mathrm{R}+i\left(p\mathrm{L}-\frac{1}{p\mathrm{K}}\right)$, from which it follows at once in the usual manner that

$$\mathrm{C}=\frac{\mathrm{E}\angle pt-\tan^{-1}\left(\frac{p\mathrm{L}}{\mathrm{R}}-\frac{1}{p\mathrm{KR}}\right)}{\sqrt{\mathrm{R}^2+\left(p\mathrm{L}-\frac{1}{p\mathrm{K}}\right)^2}} \quad . \quad . \quad . \quad . \quad (34)$$

When p is such that $p^2\mathrm{LK}=1$, or $p\mathrm{L}=\frac{1}{p\mathrm{K}}$, the current is given by

$$\mathrm{C}=\frac{\mathrm{E}\angle pt}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The current is then the same as though there were no inductance or capacity present. In this case the circuit is said to be tuned to or in *resonance* with the frequency of the impressed E.M.F. The natural frequency of the circuit is, from (31), $\dfrac{\sqrt{1-\dfrac{KR^2}{4L}}}{2\pi\sqrt{KL}}$, and the circuit resonates to a frequency $1/2\pi\sqrt{KL}$, from which it follows that natural and resonating frequency are indentical only when R is zero.

Let R, L be a coil of resistance 2 ohms and inductance 0·1 henry, and let K be a condenser of capacity 1 mfd. The de-

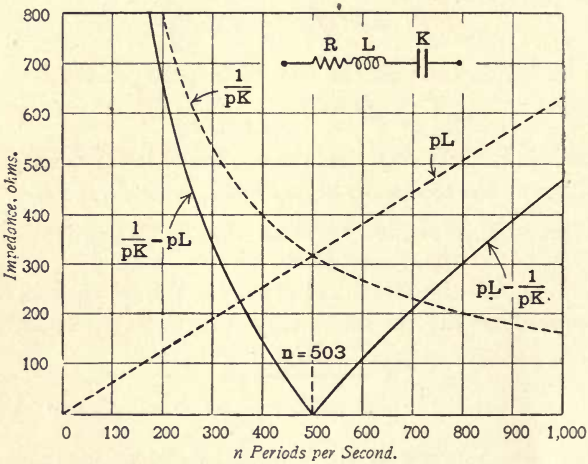

FIG. 24.—IMPEDANCE OF CIRCUIT TO ALTERNATING CURRENT OF VARIED FREQUENCY.

R=2 ohms. L=0·1 henry. K=1 mfd.

nominator of (34), which is the modulus of the impedance of the circuit, is plotted for various frequencies in Fig. 24 from Table IV. When R is small, the impedance is very nearly $\pm\left(pL-\dfrac{1}{pK}\right)$ and may be obtained by taking the difference of the two curves representing these quantities, except in the immediate neighbourhood of the point of resonance. Fig. 25 is plotted from Table V., taking R=6 ohms, L=0·5 henry and K=1 mfd. From the diagram it is seen that when the reactance voltage

pLC and the condenser voltage C/pK are equal they cancel, being in opposition, and there is left the ohmic drop RC which is equal to E.

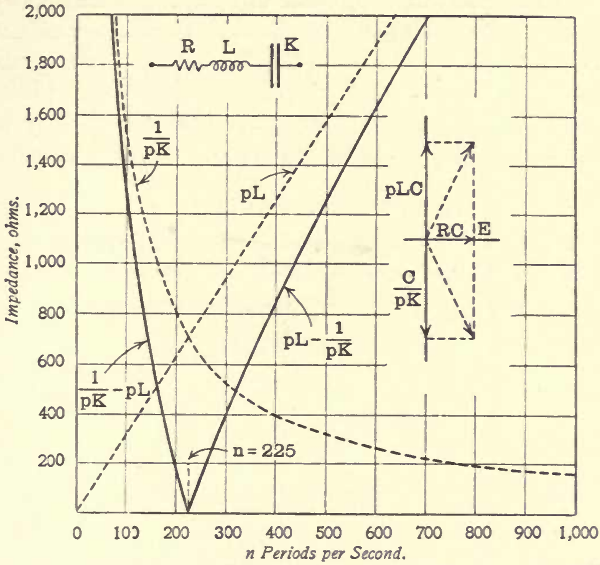

FIG. 25.—IMPEDANCE OF CIRCUIT TO ALTERNATING CURRENT OF VARIED FREQUENCY.

R=6 ohms. L=0·5 henry. K=1 mfd.

Table IV. (Fig. 24.)

n.	*p*L.	1/*p*K.	$pL \sim \frac{1}{pK}$	*n.*	*p*L.	1/*p*K.	$pL \sim \frac{1}{pK}$
100	63	1,592	1,529	600	377	265	112
200	126	796	670	700	440	227	213
300	189	531	342	800	503	199	304
400	251	398	147	900	565	177	388
500	314	318	4	1,000	628	159	469

Table V. (Fig. 25.)

n.	*p*L.	1/*p*K.	$pL \sim \frac{1}{pK}$	*n.*	*p*L.	1/*p*K.	$pL \sim \frac{1}{pK}$
100	314	1,592	1,278	500	1,571	318	1,253
200	628	796	168	600	1,885	265	1,620
300	993	531	462	700	2,199	227	1,972
400	1,257	398	859	800	2,513	199	2,314

It is evident from these figures that in order that resonance may be sharp the dip in the impedance curve must be acute. In other words, pL and $1/p$K must be great in comparison with R, or pL/R and $1/p$KR must be great in comparison with unity. This result will be produced if (1) the inductance time constant of the circuit L/R is great (as in Fig. 25) and the capacity time constant KR is small; or (2) if p is great. This second condition is best satisfied in wireless telegraphy, and consequently resonating circuits play a greater part there than anywhere else.

Leaky Inductance Coil.*

The method which has been explained for the calculation of simple sine voltages and currents by means of complex

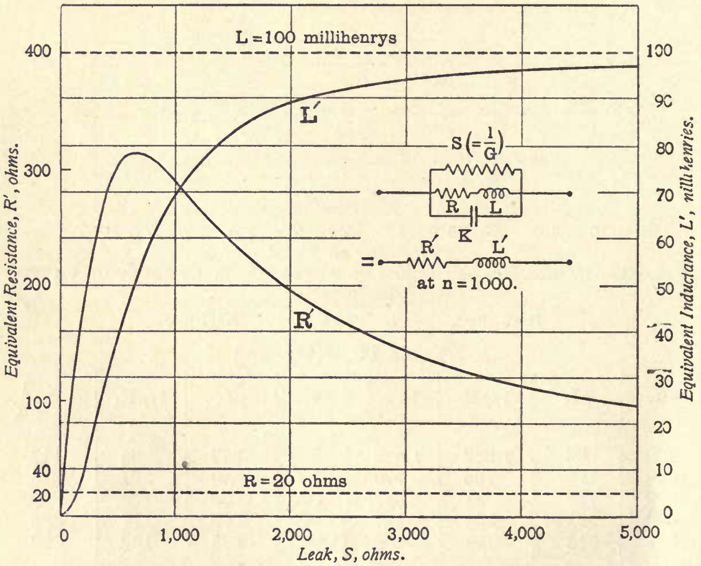

Fig. 26.—Equivalent Resistance and Inductance of a Leaky Coil.
R=20 ohms. L=0·1 henry. n=1,000 p.p.s. K=0.

quantities will suffice for all simple circuits. An interesting application is to the case of a leaky inductance coil. In Fig. 26

* A. Campbell and J. L. Eckersley, *The Electrician*, Dec. 10, 1909, p. 350.

the inductance coil R, L has a leak of resistance S ohms, or conductance G mhos, across it. The small condenser K may be regarded as a first approximation to the distributed capacity of the coil. Let R′ and L′ be the equivalent resistance and inductance of the circuit, *i.e.*, let R′, L′ be a simple coil without leak or capacity in which the same current would be produced by the same voltage as in the combination, so that no disturbance would be produced if it were substituted for the other. Then, by the rule for combining resistances in parallel,

$$R'+ipL'=\frac{1}{G+\dfrac{1}{R+ipL}+ipK}$$

$$=\frac{1}{\left(G+\dfrac{R}{R^2+p^2L^2}\right)+ip\left(K-\dfrac{L}{R^2+p^2L^2}\right)},$$

from which it follows that

$$R'=\frac{G+\dfrac{R}{R^2+p^2L^2}}{\left(G+\dfrac{R}{R^2+p^2L^2}\right)^2+p^2\left(K-\dfrac{L}{R^2+p^2L^2}\right)^2} \qquad (36)$$

and

$$L'=\frac{\dfrac{L}{R^2+p^2L^2}-K}{\left(G+\dfrac{R}{R^2+p^2L^2}\right)^2+p^2\left(K-\dfrac{L}{R^2+p^2L^2}\right)^2}. \quad . \quad . \qquad (37)$$

If G is zero and K is small, and if R^2 may be neglected in comparison with p^2L^2, these equations reduce to

$$R'=R(1+2p^2KL) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

and

$$L'=L(1+p^2KL), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

which enable the corrections to the resistance and inductance of a coil for distributed capacity to be calculated.

If K is zero and R^2 is neglected as before in comparison with p^2L^2,

$$R'=\frac{R+p^2L^2G}{1+2RG+p^2L^2G^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

and

$$L'=\frac{L}{1+2RG+p^2L^2G^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

When G is small, these expressions reduce to

$$R' = R + p^2 L^2 G, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$L' = L(1 - 2RG - p^2 L^2 G^2). \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If the degree of approximation in (43) were the same as in (42), the equation would be simply $L' = L$. The effect of a slight leak is more marked on the equivalent resistance than on the equivalent inductance.

Take $R = 20$ ohms, $L = 0{\cdot}1$ henry, and let S vary from zero upwards. Fig. 26 shows the corresponding variations of R' and L', calculated from formulæ (40) and (41). Unless S is very small, R' is always greater than R. The effect of the shunt is to increase the apparent resistance of the coil. At the same time the apparent inductance is reduced. In the construction of standard inductance coils and of loading coils for telephone circuits great care must be taken to secure high insulation.

Leaky Condenser.

A leaky condenser may be treated in a similar way. In Fig. 27, the condenser K is shunted by the resistance $S(=1/G)$, and the combination is supposed to be replaced by the con-

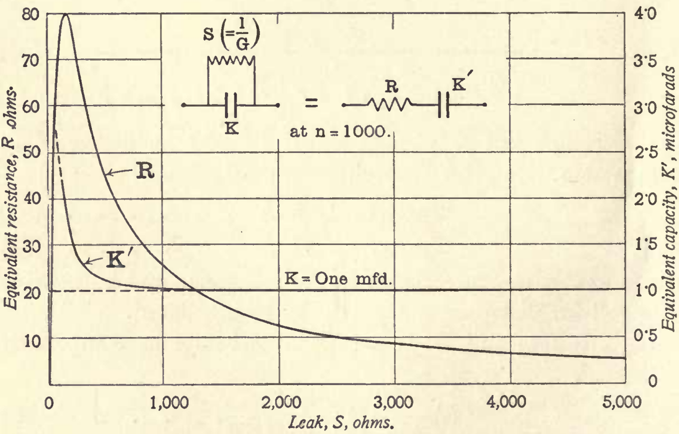

Fig. 27.—Condenser with Leak equivalent to Condenser with Series Resistance.

K=1 mfd. n=1,000 p.p.s.

denser K′ with the resistance R in series with it. Equating the symbolic impedances,

$$R+\frac{1}{ipK'}=\frac{1}{G+ipK}=\frac{G-ipK}{G^2+p^2K^2}.$$

Hence $$R=\frac{G}{G^2+p^2K^2}=\frac{S}{1+p^2K^2S^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

and $$K'=\frac{G^2+p^2K^2}{p^2K}=K\left(1+\frac{1}{p^2K^2S^2}\right). \quad . \quad . \quad . \quad . \qquad (45)$$

Take $K=1\times10^{-6}$ farad and $p=2\pi n=2\pi\times1{,}000=6{,}283$. Fig. 27 shows the variations of R and K′ for values of S from zero to 5,000 ohms, and therefore for G/pK from infinity down to 0·0318. Except for very low values of S, the series resistance is much less than the shunt resistance, and is approximately inversely proportional to the shunt or leak. For low values of the leak, the equivalent condenser K′ is greater than K, but when S is at all great the two capacities are practically identical.

MESHES.

It now remains to show how the method which has been explained for the calculation of the periodic current in any simple circuit may be extended to cover the case of more than one circuit, especially when the circuits combine to form a network. As an illustration take the Wheatstone bridge of Fig. 28 and suppose that it is required to find the current through the galvanometer for any arrangement of the arms of the bridge. In the first place let R_1, R_2, S_1, S_2 be simple resistances, and let E be the E.M.F. of a battery; also, let R_g be the resistance of the galvanometer and B that of the battery. Let C_3 denote the current in the battery diagonal, C_1 the current in the arm R_1, and C_2 the current in the arm R_2. Then C_3-C_1 must be the current in the arm S_1, C_1-C_2 the current through the galvanometer, and C_2-C_3 the current in the arm S_2, all as indicated by the arrows. It is clear that the same distribution of currents would result if we imagine the currents to circulate quite independently, C_3 being supposed confined to the path BS_1S_2, C_1 to the path $S_1R_1R_g$, and C_2 to the path $S_2R_gR_2$, as indicated by the circular arrows. The currents C_1, C_2 and C_3

are called *cyclic* currents, and the circuits in which they flow are *meshes* of the network.

The voltage drop in any branch is to be obtained from Ohm's law by multiplying the current in it by the resistance. The total drop in any mesh must be equal to the E.M.F. there, or be zero if there is no E.M.F. Hence for each of the three meshes a linear equation may be written down—

$$\begin{aligned} B \, . \, C_3+S_1(C_3-C_1)+S_2(C_3-C_2) &= E, \\ R_1 \, . \, C_1+R_g(C_1-C_2)+S_1(C_1-C_3) &= 0, \\ R_2 \, . \, C_2+S_2(C_2-C_3)+R_g(C_2-C_1) &= 0. \end{aligned}$$

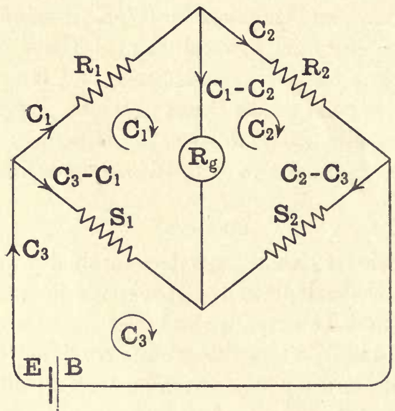

FIG. 28.—EQUILIBRIUM OF WHEATSTONE BRIDGE.
$R_1S_2=R_2S_1$.

On re-arrangement,

$$\begin{array}{rrr} -S_1 \, . \, C_1 & -S_2 \, . \, C_2+(B+S_1+S_2) \, . \, C_3=E, & \\ (R_1+R_g+S_1) \, . \, C_1 & -R_g \, . \, C_2 & -S_1 \, . \, C_3=0, \\ -R_g \, . \, C_1+(R_2+S_2+R_g)C_2 & & -S_2 \, . \, C_3=0. \end{array}$$

These equations can now be solved for C_1, C_2 and C_3 by the aid of determinants in the manner described in Chap. I. Any current is the quotient formed by two determinants, the lower of which contains all the coefficients of the currents, and the upper differs only in having the terms on the right-hand sides of the equations in place of the column formed by the coefficients of the current in question. The current through the galvanometer, which it was required to find, is the difference between C_1 and C_2. Alternatively, we may write $C_g = C_1 - C_2$ or $C_1 = C_g + C_2$ and substitute for C_1. Thus, rewriting the equations—

$$\begin{array}{rll} -S_1 \cdot C_g & -(S_1+S_2)\,C_2 + (B+S_1+S_2)\,C_3 = E, & \\ (R_1+R_g+S_1) \cdot C_g & +(R_1+S_1)\,C_2 \qquad\qquad -S_1 \cdot C_3 = 0, & \\ -R_g \cdot C_g & +(R_2+S_2)\,C_2 \qquad\qquad -S_2 \cdot C_3 = 0, & \end{array}$$

from which

$$C_g = \begin{vmatrix} E & -S_1-S_2 & B+S_1+S_2 \\ 0 & R_1+S_1 & -S_1 \\ 0 & R_2+S_2 & -S_2 \end{vmatrix} \div \begin{vmatrix} -S_1 & -S_1-S_2 & B+S_1+S_2 \\ R_1+R_g+S_1 & R_1+S_1 & -S_1 \\ -R_g & R_2+S_2 & -S_2 \end{vmatrix}$$

which becomes on evaluation of the numerator,

$$C_g = \frac{E(S_1R_2 - S_2R_1)}{\Delta}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

where Δ is the determinant forming the denominator and containing all the coefficients of the currents as its constituents. It is evident that the condition for C_g to be zero is that S_1R_2 should equal S_2R_1.

If, now, the E.M.F. is not constant but is of sinusoidal form, E must be replaced by $E\underline{/pt}$, and the same analysis holds good as before. Furthermore, the arms of the bridge may now be generalised resistances, and (46) will still be true.

Thus let the R arms be replaced by inductance coils R_1L_1 and R_2L_2, and for the S arms let capacities K_1 and K_2 be substituted.

Then $$C_g = \frac{E\underline{/pt}\left[\dfrac{R_2+ipL_2}{ipK_1} - \dfrac{R_1+ipL_1}{ipK_2}\right]}{\Delta_1} \quad . \quad . \quad . \quad . \qquad (47)$$

For equilibrium, $$\frac{R_2+ipL_2}{ipK_1} = \frac{R_1+ipL_1}{ipK_2}.$$

Equating real and imaginary quantities,

$$R_1K_1 = R_2K_2$$

and $$L_1K_1 = L_2K_2,$$

or $$\frac{R_1}{R_2} = \frac{L_1}{L_2} = \frac{K_2}{K_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

Any other combination of resistances, inductances and capacities may be treated in a similar way.

It is seen that the numerator of C_g contains a complex factor, both the real portion and the imaginary portion of which must separately be zero before C_g is zero. In general two conditions must be satisfied before equilibrium can be established when the arms of the bridge are not simple resistances.

In the practical adjustment of the bridge it is therefore essential that we should be able to vary two of its arms quite independently. This adjustment is most conveniently carried out by means of a variable resistance, and a variable inductance or capacity. With the introduction of such apparatus in recent years alternating-current measurements have been greatly facilitated.

When the E.M.F. is periodic but not of pure sine form it may be resolved into a Fourier series, consisting of a fundamental and its train of harmonics. Every term is then of pure sine form, and to it the conditions of (46) apply. Two cases now arise; either the conditions for equilibrium do not contain the frequency, as for example those illustrated in (48), or they do contain the frequency explicitly. In the former case the conditions of equilibrium are the same for every term of the series, and therefore for the whole series and for the periodic E.M.F. In the latter case the conditions are satisfied by only one term of the series at a time, and balance can be obtained only for a single frequency. Two courses may then be pursued in the second case: either (1) the testing E.M.F. must be purified, or (2) a detecting instrument must be used which responds only to a single frequency. In most cases although the conditions for balance do not contain the frequency explicity they do so implicitly, because the arms of the bridge vary somewhat with the frequency. The desirability of a source of pure sine E.M.F. for testing purposes is therefore manifest.

Since a single impulse may be regarded as being repeated at long intervals, and therefore as part of a periodic change, it follows that if the conditions of equilibrium do not involve the frequency—and therefore hold for any periodic E.M.F.—

they hold also for any impulse, such, for example, as is produced by the sudden insertion of a constant E.M.F. into the circuit. The same limitations as to the invariability of the constants of the arms of the bridge apply also to this case.

Starting an Alternating Current in an Inductance Coil.

Suppose that the sinusoidal E.M.F. Ee^{ipt} is introduced at the instant $t=0$ in the circuit of Fig. 29, containing both resistance

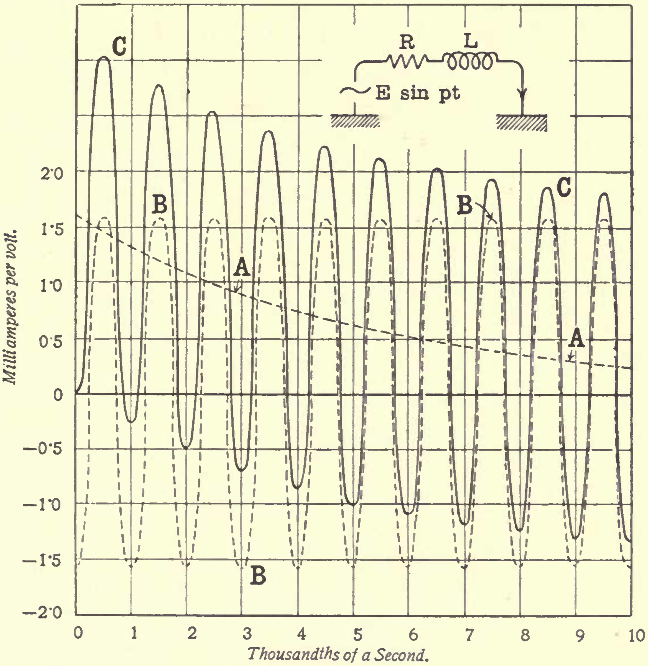

Fig. 29.—Starting an Alternating Current in an Inductance Coil Switch closed at Zero of E.M.F.

R=20 ohms. L=0·1 henry. n=1,000. pL=628·3.
A.—Transient component. B.—Periodic component. C.—Total current.

and inductance. The instantaneous value of the E.M.F. at the moment of introduction is E, which, according to (4) produces a current having at time t_1 later the value

$$\frac{E}{R}(1-e^{-Rt_1/L}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

Suppose that the sinusoidal variation in the E.M.F. now takes place in successive small stages, each occupying a short interval of time dt. The increment in the E.M.F. in the first interval of time dt is $[d(\mathrm{E}e^{ipt})]_{t=0}$, or $ip\mathrm{E}dt$.

This produces a current which at time t_1 is

$$\frac{ip\mathrm{E}dt}{\mathrm{R}}(1-e^{-\mathrm{R}(t_1-dt)/\mathrm{L}}). \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The current at time t_1 produced by the next element of E.M.F. is

$$\frac{ip\mathrm{E}e^{ipdt}\,.\,dt}{\mathrm{R}}(1-e^{-\mathrm{R}(t_1-2dt)/\mathrm{L}}), \quad . \quad . \quad . \quad . \quad (51)$$

and so on. Adding up all these currents, and making dt infinitely small, the total current at time t_1 is given by

$$\mathrm{C}_{t_1}=\frac{\mathrm{E}}{\mathrm{R}}(1-e^{-\mathrm{R}t_1/\mathrm{L}})+\int_0^{t_1}\frac{ip\mathrm{E}e^{ipt}dt}{\mathrm{R}}(1-e^{-\mathrm{R}(t_1-t)/\mathrm{L}}). \quad . \quad (52)$$

The integral is

$$\frac{\mathrm{E}}{\mathrm{R}}\left[e^{ipt_1}-1-\frac{ip}{ip+\mathrm{R}/\mathrm{L}}(e^{ipt_1}-e^{-\mathrm{R}t_1/\mathrm{L}})\right],$$

so that, dropping the suffix to t, no longer required now that the integration is complete,

$$\mathrm{C}_t=\mathrm{E}\frac{e^{ipt}-e^{-\mathrm{R}t/\mathrm{L}}}{\mathrm{R}+ip\mathrm{L}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

It is seen from (53) that the expression for the current consists of two parts. The second of these, $\frac{e^{-\mathrm{R}t/\mathrm{L}}}{\mathrm{R}+ip\mathrm{L}}$, has a real exponential factor, which makes it vanish eventually when t is infinite. This part we may term the *transient component* of the current. The other part, $\frac{e^{ipt}}{\mathrm{R}+ip\mathrm{L}}$, is recognised as the current produced by a sinusoidal E.M.F. after the initial disturbances have subsided. It is the *periodic component* of the current, and does not die away, but passes through a cycle of changes.

Since $$\frac{-\mathrm{E}e^{-\mathrm{R}t/\mathrm{L}}}{\mathrm{R}+ip\mathrm{L}}=\frac{\mathrm{E}(ip\mathrm{L}-\mathrm{R})e^{-\mathrm{R}t/\mathrm{L}}}{\mathrm{R}^2+p^2\mathrm{L}^2},$$

we have, on neglecting first the real and then the imaginary part of the complex numerator,

$$C\text{ (transient, corresponding to E sin } pt) = \frac{pL \cdot Ee^{-Rt/L}}{R^2+p^2L^2}, \quad . \quad (54)$$

and

$$C\text{ (transient, corresponding to E cos } pt) = \frac{-REe^{-Rt/L}}{R^2+p^2L^2}. \quad (55)$$

When R is small compared with pL these formulæ become

$$C_{(E \sin pt)} = \frac{Ee^{-Rt/L}}{pL}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and

$$C_{(E \cos pt)} = \frac{-REe^{-Rt/L}}{p^2L^2}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

In the first case the current is inversely proportional to the frequency, and in the second to the square of the frequency. The transient is, therefore, a minimum, and the current most quickly reaches its steady periodic state, when the E.M.F. is introduced at the moment when it is passing through its maximum value.

Take R=20 ohms, L=0·1 henry, and n=1,000, as in Fig. 29, which is obtained very easily from Table I. The switch is here supposed to make contact as the E.M.F. is passing through its zero valve. The current shown in curve C is the sum of curves A and B, and rises initially above its steady periodic value, curve B, which it does not reach until the transient component, curve A, has completely died away. The periodic component is 90 deg. in phase behind the volts, and is given by

$$C\text{ (periodic)} = -\frac{E \cos pt}{pL}. \quad . \quad . \quad . \quad . \quad . \quad (58)$$

When the switch is closed just as the E.M.F. has its maximum value the transient component as given by (57) is R/pL times, or only one-thirtieth of, its former value. The current is therefore practically periodic from the start, and is given by

$$C = \frac{E \cos\left(pt - \frac{\pi}{2}\right)}{pL} = \frac{E \sin pt}{pL}, \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which is curve B moved through 90 deg. to the left.

By combining the transients in the two cases the transient component for any epoch of introduction of the E.M.F. may be obtained. For let the E.M.F. be E sin $(pt+\varphi)$, which has the value E sin φ at zero time. Then $E \sin(pt+\varphi) = \cos\varphi\, E \sin pt + \sin\varphi \,.\, E \cos pt$, and, therefore,

C (transient, E sin $(pt+\varphi)$

$$=(pL\cos\varphi - R\sin\varphi)\,.\,\frac{Ee^{-Rt/L}}{R^2+p^2L^2}. \qquad (60)$$

The transient component is therefore zero when $pL\cos\varphi - R\sin\varphi$ is zero, or when

$$\tan\varphi = pL/R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

When pL is large compared with R, φ is very nearly 90 deg. and E cos pt produces minimum transient.

Starting an Alternating Current in a Resistance-Capacity Circuit.

In the circuit of Fig. 19 let the E.M.F. E^{ipt} be introduced as before at the instant $t=0$. Following out the reasoning just explained, using in this case formula (15) for the charging current by a direct E.M.F. it follows that

$$C_{t_1} = \frac{E}{R}e^{-t_1/KR} + \frac{ipE}{R}\int_0^{t_1} e^{ipt}\,.\,e^{-\frac{t_1-t}{KR}}\,.\,dt,$$

$$= \frac{E}{R}e^{-t_1/KR} + \frac{ipE}{R}\,.\,\frac{e^{ipt_1} - e^{-t_1/KR}}{ip+\frac{1}{KR}},$$

from which $$C = \frac{Ee^{ipt}}{R+\frac{1}{ipK}} + \frac{Ee^{-t/KR}}{R(1+ipKR)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

The transient component is the second of these expressions. Hence

$$C \text{ (transient, due to E sin } pt) = \frac{-pKEe^{-t/KR}}{1+p^2K^2R^2}, \quad . \quad . \quad . \qquad (63)$$

and $$C \text{ (transient, due to E cos } pt) = \frac{Ee^{-t/KR}}{R(1+p^2K^2R^2)}. \quad . \qquad (64)$$

If $p^2K^2R^2$ is great compared with unity, the formulæ become

$$C_{(E \sin pt)} = -\frac{E}{R}\,.\,\frac{e^{-t/KR}}{pKR}, \quad . \quad . \quad . \quad . \quad . \qquad (65)$$

and $$C_{(E \cos pt)} = \frac{E}{R}\,.\,\frac{e^{-t/KR}}{p^2K^2R^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66)$$

By combining (63) and (64) the general expression for the transient current may be obtained. Thus—

$$C_{[E \sin (pt+\phi)]}=\left(-pK \cos \varphi+\frac{\sin \varphi}{R}\right)\frac{Ee^{-t/KR}}{1+p^2K^2R^2}. \quad . \quad (67)$$

The transient component is zero when $-pK \cos \varphi+\frac{\sin \varphi}{R}$ is zero, or when

$$\tan \varphi=pKR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

When pKR is great, φ is very nearly 90 deg., and E cos pt produces minimum transient.

Take R = 2,000 ohms and K = 2·5 mfd., and let $p = 2\pi\times 1{,}000 = 6{,}283$. Then $pKR = 31{\cdot}416$ and $p^2K^2R^2$ is great in comparison with unity, and therefore formula (65) may be used. The periodic component is

$$\frac{E/\underline{pt}}{\sqrt{R^2+\frac{1}{p^2K^2}}\;/\underline{\tan^{-1}\frac{-1}{pKR}}}=\frac{E\,/\underline{pt+\tan^{-1}\frac{1}{pKR}}}{R}, \quad . \quad (69)$$

nearly. The total current when the E.M.F. is introduced as it is passing through its zero value is therefore

$$\frac{E}{R}\left[\sin\left(pt+\tan^{-1}\frac{1}{pKR}\right)-\frac{e^{-t/KR}}{pKR}\right]. \quad . \quad . \quad . \quad (70)$$

It is evident from (70) that the effect of the transient component is to make the current just after switching on slightly less than that to which it afterwards settles down, but pKR in the present instance is so great that the transient component bears only a small proportion to the periodic. From (66) it is seen that the transient component is still smaller when the switch is closed at the moment when the E.M.F. is passing through its maximum value, while the periodic component is naturally charged only in phase, so that the current in this case is very nearly purely periodic from the start.

If, on the other hand, pKR is small in comparison with unity, the amplitude of the periodic component is pKE, and the transient components are $-pKEe^{-t/KR}$ and $\frac{E}{R}e^{-t/KR}$. If p is small the greatest transient is produced by switching on at 90 deg., as is also evident from (68). If pKR is small because KR is small the transient soon decays, and if pKR is small because p is small the periodic time is great compared with the time of decay of the transient. In either case the transient affects only the first few periods of the periodic current.

Alternating-current Start—Resistance, Inductance, and Capacity in Series.

In the circuit of Fig. 30, consisting of a resistance, an inductance, and a capacity joined in series, let the E.M.F. Ee^{ipt} be introduced as before at zero time. The current into the condenser due to a steady E.M.F. E is in this case given by formula (23),

$$C=\frac{2Ee^{-Rt/2L}}{R\sqrt{1-\frac{4L}{KR^2}}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}=\frac{2Ee^{-at}}{Rb}\,.\sinh abt\,. \quad (71)$$

where $a=\frac{R}{2L}$ and $b=\sqrt{1-\frac{4L}{KR^2}}$.

Proceeding as before, the current at time t_1 due to the first element of E.M.F. is

$$\frac{2Ee^{ipdt}\,.\,ipdt}{Rb}\,e^{-a(t_1-dt)}\sinh ab(t_1-dt),$$

$$=\frac{E}{Rb}e^{ipdt}\,.\,ipdt[e^{-a(1-b)(t_1-dt)}-e^{-a(1+b)(t_1-dt)}],$$

and the current due to all the elements is

$$\frac{Eip}{Rb}\int_0^{t_1}e^{ipt}\,dt[e^{-a(1-b)(t_1-t)}-e^{-a(1+b)(t_1-t)}],$$

$$=\frac{Eip}{Rb}\left[\frac{e^{ipt_1}-e^{-a(1-b)t_1}}{a(1-b)+ip}-\frac{e^{ipt_1}-e^{-a(1+b)t_1}}{a(1+b)+ip}\right].$$

The transient part of this expression is

$$-\frac{2Eipe^{-at}}{Rb}\left[\frac{(a+ip)\sinh abt+ab\cosh abt}{[a(1-b)+ip][a(1+b)+ip]}\right]=$$

$$\frac{-2Eipe^{-at}}{Rb}\left[\frac{\{(a+ip)\sinh abt+ab\cosh abt\}\left\{\left(\frac{1}{LK}-p^2\right)-2aip\right\}}{\left(\frac{1}{LK}-p^2\right)^2+(2ap)^2}\right],$$

since $a^2(1-b^2)=1/LK$.

The transient current due to E sin pt is obtained from the imaginary part of this expression, and is

$$C_{(E\sin pt)}=\frac{-2Epe^{-at}}{Rb}\cdot\frac{a\left(\frac{1}{LK}+p^2\right)\sinh abt+ab\left(\frac{1}{LK}-p^2\right)\cosh abt}{\left(\frac{1}{LK}-p^2\right)^2+(2ap)^2},$$

$$=\frac{-Ee^{-Rt/2L}}{R^2+\left(pL-\frac{1}{pK}\right)^2}\left[\left(pL+\frac{1}{pK}\right)\frac{\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}}{\sqrt{1-\frac{4L}{KR^2}}},\right.$$

$$\left.-\left(pL-\frac{1}{pK}\right)\cosh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KR^2}}\right]. \qquad (72)$$

When $4L/KR^2$ is greater than unity,

$$C_{(E\sin pt)}=\frac{-Ee^{-Rt/2L}}{R^2+\left(pL-\frac{1}{pK}\right)^2}\left[\left(pL+\frac{1}{pK}\right)\frac{\sin\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1}}{\sqrt{\frac{4L}{KR^2}-1}},\right.$$

$$\left.-\left(pL-\frac{1}{pK}\right)\cos\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1}\right], \qquad (73)$$

and when $\frac{4L}{KR^2}=1$,

$$C_{(E\sin pt)}=\frac{-Ee^{-Rt/2L}}{R^2+\left(pL-\frac{1}{pK}\right)^2}\left[\left(pL+\frac{1}{pK}\right)\cdot\frac{Rt}{2L}-pL+\frac{1}{pK}\right]. \qquad (74)$$

If, in addition, $pL=\frac{1}{pK}$, formula (74) becomes

$$C_{(E\sin pt)}=\frac{-ptEe^{-Rt/2L}}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

In (73) put $pL=\frac{1}{pK}$, and we obtain

$$C_{(E\sin pt)}=\frac{-2pLEe^{-Rt/2L}}{R^2\sqrt{\frac{4L}{KR^2}-1}}\sin\frac{Rt}{2L}\sqrt{\frac{4L}{KR^2}-1}, \quad . \qquad (76)$$

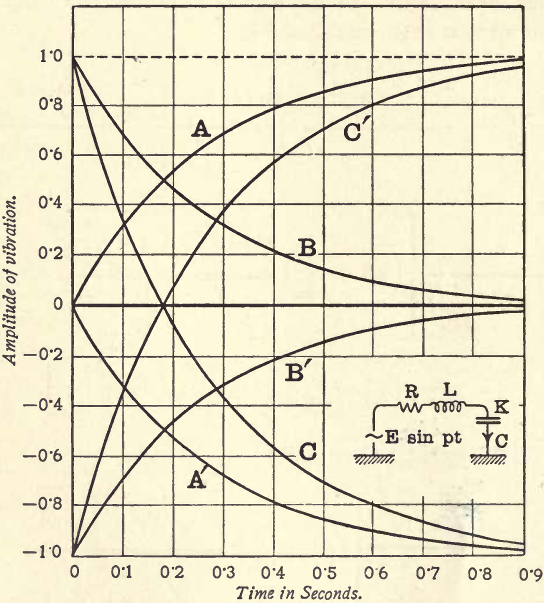

FIG. 30.—AMPLITUDE OF DEFLECTION OF TUNED VIBRATION GALVANOMETER.

Curves A, A′=Current made. Curves B, B′=Current broken. Curves C, C′=Current reversed. L/R=0·13 second.

and if $4L/KR^2$ is great in comparison with unity,

$$C_{(E\sin pt)}=\frac{-Ee^{-Rt/2L}}{R}\sin pt. \quad . \quad . \quad . \quad . \quad . \qquad (77)$$

This is the transient which is present at the starting up of a periodic E.M.F. in a circuit tuned to the same frequency, such as are the sending and receiving circuits in radiotelegraphy. The total current by the addition of (35) is

$$C=\frac{E}{R}(1-e^{-Rt/2L})\sin pt. \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The amplitude of the oscillation contains the transient factor $1-e^{-Rt/2L}$. In order that this factor may decay rapidly and the current start up quickly the ratio $2L/R$ must be small. The conditions for sharp tuning and for quick rise to maximum current are, therefore, incompatible. This question would, however, only become of importance if the time of unit signal were comparable with $2L/R$.

The quantity of electricity, Q, in the condenser at time t may be obtained from (78) by integration. Thus

$$Q=\frac{E}{R}\int_0^t(1-e^{-Rt/2L})\sin pt,$$

$$=\frac{E}{R}\left[-\frac{\cos pt}{p}-\frac{e^{-Rt/2L}\left(-\frac{R}{2L}\sin pt-p\cos pt\right)}{\frac{R^2}{4L^2}+p^2}\right]_0^t,$$

from Appendix II.,

$$=\frac{E}{R}\left[\frac{-\cos pt+1}{p}+\frac{e^{-Rt/2L}\sin\left(pt+\tan^{-1}\frac{2Lp}{R}\right)}{\sqrt{\frac{R^2}{4L^2}+p^2}}-\frac{p}{\frac{R^2}{4L^2}+p^2}\right],$$

and if p be great in comparison with $R/2L$,

$$Q=\frac{E}{pR}[e^{-Rt/2L}-1]\cos pt. \quad . \quad . \quad . \quad . \quad (79)$$

The amplitude of the charge in the condenser rises, when t is great, to a steady value, which is E/pR. The charge in the condenser produced by a direct E.M.F. would be KE. Hence

$$\frac{Q\text{ (alternating)}}{Q\text{ (direct)}}=\frac{E}{pR}\times\frac{1}{KE}=\frac{1}{pK\,.\,R}=\frac{pL}{R}. \quad . \quad (80)$$

The mechanical analogue of a capacity, inductance and resistance circuit is a moving-coil system in which the angular deflection corresponds to the charge Q, the moment of inertia of the coil to the inductance L, the restoring couple to the reciprocal of the capacity (1/K), and a frictional couple proportional to the angular velocity of the coil to the resistance R. Hence all the formulæ of this paragraph can be translated immediately and applied to the study of the moving-coil galvanometer.

In a certain vibration galvanometer* the ratio of the deflection produced by an alternating current, with which the galvanometer was adjusted to be in tune, to the deflection produced by an equal direct current was found to be very nearly proportional to the frequency, and to be given by the formula $0{\cdot}8\times n$. Dividing this ratio by p, it follows from (80) that $L/R=0{\cdot}13$ second. Substituting this value for L/R in (79), the deflection at any instant is obtained. The result is shown in Fig. 30. The curves A, A′ represent the growth of the amplitude of deflection, or the breadth of the band of light reflected from the mirror, and curves B, B′ the decay. The curves C, C′ are constructed from the others, and show the effect of reversing the current. The separate periodic oscillations, of frequency from 100 to 1,000 per second, are too close together to admit of representation in the figure.

Fundamental Distinction Between Telegraphy and Telephony.

The problems which confront the student of telephone theory are in one respect more easy to handle than those which arise in cable telegraphy. It is true that in the study of the telephone cable it is necessary to take account of dielectric and of iron losses, and of discontinuities in the cable itself, with which, up to the present, the telegraph engineer has not been troubled ; but, on the other hand, the theory which is used by the telephone engineer is much simplified by the assumption of a purely periodic state of affairs. In his calculations he pre-supposes that the periodic E.M.F. which occasions the transmission of energy from one end of the line to the other has

* W. Duddell, "Proc. Phys. Soc.," 21, p. 782, 1910.

been sustained for so long a time (compared with the time constant of the line and apparatus) that everywhere current and voltage have settled down into a steady state of oscillation. In other words, he neglects entirely the transient phenomena which accompany the switching in and out of the periodic transmitting E.M.F.

This procedure is justified by the fact that such calculations agree with carefully made speech tests. It is argued that the ear responds to each individual component of the complex oscillation, irrespective of its phase relationship to the fundamental. It is the relative amplitude alone which gives the sound its character. The ear acts, as it were, like a frequency teller, in which the loudness of the components of the received oscillation is measured by the extent of vibration of the reeds. The transmitting qualities of a telephone line may thus be tested by the way in which it acts towards a range of frequencies which is typical of those occurring in conversation. Still more simply, such a single *mean frequency* may be selected as will give a good average for most practical purposes.

But this power of analysing a low frequency wave motion is not possessed by the eye. Moreover, in passing through a long submarine telegraph cable alternating current would be so far reduced as to be quite insensible at the receiving end. Signalling is of necessity carried on by the transient phenomena which accompany the interruptions of direct current. It may be said, therefore, that, whereas the science of telephony deals with periodic phenomena, telegraphy is exclusively concerned with transient phenomena.

This fundamental distinction between telephony and telegraphy has not always been borne in mind. The attempt to apply elementary transient theory to telephony led to the imposition of the KR law, by which the progress of long-distance telephony was retarded. In like manner an attempt to apply purely periodic theory to telegraphy would be equally disastrous, and has, in fact, already led to confusion of thought. It has given rise to such expressions as telegraphic frequency, which, on examination, is seen to be almost a contradiction in terms.

The telephone engineer is concerned with volume of speech, and he desires to produce maximum current at the receiving end. Articulation is, for the present at any rate, of secondary importance. On the other hand, as will be shown later, the history of telegraphic improvement in speed has been the continual sacrifice of size of signal for shape—*i.e.*, the telegraph engineer is concerned with articulation first and volume second.

The aims of the two branches of transmission are different, and the theory or language in which they are expressed is different. But no hard-and-fast line can be drawn between them. They find common ground in the cable or line, the properties of which are manifested in the phenomena of transmission. Sooner or later the telephone engineer must enter upon the study of transient phenomena, by which his results may be profoundly affected. In like manner, for the telegraph engineer the best introduction to the study of transient phenomena lies in telephone theory. And in time to come, should alternating current act in telegraphy as the transmitting agent, as at present in radiotelegraphy, it may be found that many of the greatest difficulties in the way have already been encountered in telephony and overcome.

PART II.—PURELY PERIODIC PHENOMENA—THE TELEPHONE CABLE.

CHAPTER IV.

THE TELEGRAPHIC EQUATION AND ITS PERIODIC SOLUTION.

Derivation of the Telegraphic Equation—Solution of the Equation for a Purely Periodic E.M.F.—Infinitely Long Cable—Expression of the Attenuation Constant and the Wave-Length Constant in Terms of the Fundamental Constants—"Standard" Telephone Cable—Submarine Telephone Cable—Loaded Cables—The Spiral Diagram—Spiral Diagram for a Loaded Cable—Models to Represent Wave Propagation—The Distortionless Cable—Influence of Leakance.

DERIVATION OF THE TELEGRAPHIC EQUATION.

In Fig. 31 is represented diagrammatically a single telegraph conductor of length l, and of resistance R, inductance L, capacity to earth K, and leakance G, all per unit length of conductor. In Fig. 32 is shown the corresponding "metallic return" circuit, which it is necessary to use in telephony in order to avoid overhearing and other inductive effects. The outgoing and return conductors are alike, and are laid up in a long spiral.

In the case of the single conductor the return path through earth is of such enormous cross-section that, despite the indifferent conductivity of the materials of which it is composed, its resistance is practically zero, provided that good contact is made to earth and there are no peculiarities in the formation of the subsoil in the immediate neighbourhood. In the submarine telegraph cable this required excellence of contact is secured by connecting the earth wire to the iron sheathing of the cable. When inductive disturbances are to be feared near the cable hut the shore end of the cable may be provided with two cores, as in a telephone cable, one of which is connected to the sheathing some distance out at sea.

In the case of the twin conductor the resistance per unit length of twin conductor is twice that of the single conductor. At the same time the capacity, now from core to core, is half the capacity in the former case, being equivalent to two equal capacities in series. It is here assumed that the cores are

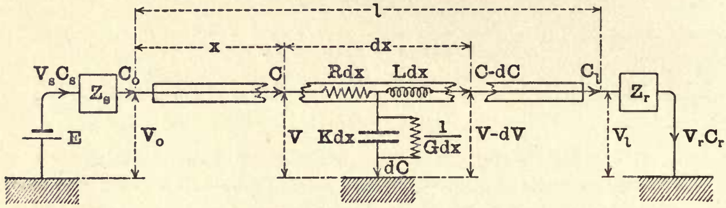

FIG. 31.—CABLE WITH EARTH RETURN.

immersed in water, and in testing this point must receive attention, otherwise capacity and leakance will depend on the state of the intervening medium. The inductance of the twin conductor is in general less than twice that of the single conductor, but if the cores are linked together by a magnetic

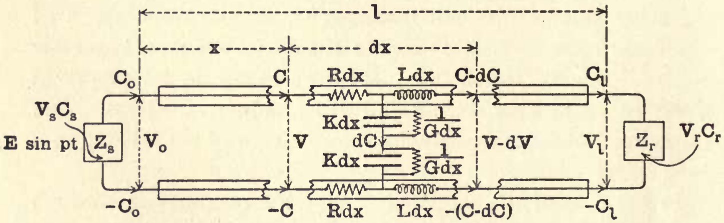

FIG. 32.—TWIN CABLE, LOOP, OR METALLIC RETURN CIRCUIT.

flux, as when loading coils are present, their separate inductances will be less than half. In any case, as will appear, all the formulæ which are established here apply equally well to both forms of circuit, provided that the constants of the metallic circuit are specified *per unit length of twin conductor.* With this proviso, it is immaterial for most purposes whether we represent the circuit by a single or by a double conductor.

Let the conductor of Fig. 31 be connected to earth through sending apparatus Z_s and receiving apparatus Z_r at the ends, and at the sending end let there be present a source of E.M.F. varying in any manner. The sending E.M.F. produces a current in the cable, part of which travels along the conductor to the receiving end, and part disappears to earth through the capacity-leakance path and never reaches the receiving end. When the cable is long, or when alternating current of high frequency is used, the proportion of current which passes to earth may be so great that practically none may reach the receiving end. To express this loss by the way the term *attenuation* is used, and the current is said to be attenuated in transmission.

If the sending E.M.F., although periodic, is not purely sinusoidal, it may be resolved into its periodic components; these will experience not only varied attenuation depending on the frequency, but will also bear phase relationships to each other which differ at the receiving end from those at the sending end. As a consequence, the current produced by a composite E.M.F. is said to suffer *distortion* in transmission.

To find the current C and voltage V at any point distant, x, from the sending end of the cable, one way, of which simple examples have already been given, in which we may proceed is, first, to obtain the differential equation of the circuit; second, to find an integral of the equation; and, third, to adapt the integral so that it may fit the terminal conditions. The differential equation may be obtained from a consideration of the properties of an infinitesimal portion or element of the circuit. Just as any small portion of the graphs of C or V when plotted against x may be regarded as approximately a straight line, so we may expect the current and voltage of an element of the conductor to be linked up with the constants of the conductor in a simple (linear) manner.

Consider an element of the conductor of length dx distant x from the sending end. The total resistance, inductance capacity and leakance of the element are Rdx, Ldx, Kdx and Gdx respectively. The voltage to earth (or from core to core) drops along the element from left to right, and the main current C—with the exception of the small portion dC, which escapes to earth—flows in the same direction. The difference in voltage between the ends of the element is $-dV$, the negative sign indicating that V decreases as x, the independent variable, increases.

Now, if dx is small, dC is small, and the current C from left to right through the element of the conductor is sensibly uniform. Equation (6) of Chapter II. may, therefore, be applied at once to it. Thus, equating voltage drop and back E.M.F.,

$$-dV = Rdx \,.\, C + Ldx \,.\, \frac{dC}{dt}.$$

or

$$-\frac{dV}{dx} = RC + L\frac{dC}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To the current dC which escapes to earth equation (15) of Chap. II. may be applied, understanding by V the mean voltage to earth throughout the element. To dC is also to be given the negative sign, for the same reason as before. Hence adding the leakance current,

$$-dC = Gdx \,.\, V + Kdx \,.\, \frac{dV}{dt},$$

or

$$-\frac{dC}{dx} = GV + K\frac{dV}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are the fundamental equations of the circuit. They may be combined into one. Thus, differentiating (1) with respect to t,

$$-\frac{d^2V}{dxdt} = R\frac{dC}{dt} + L\frac{d^2C}{dt^2},$$

and, differentiating (2) with respect to x,

$$-\frac{d^2C}{dx^2} = G\frac{dV}{dx} + K\frac{d^2V}{dxdt}.$$

Hence, substituting for $\frac{d^2V}{dxdt}$ from the former equation, and re-arranging with the help of (1), finally

$$\frac{d^2C}{dx^2} = GRC + (GL+KR)\frac{dC}{dt} + KL\frac{d^2C}{dt^2} \quad . \quad . \quad . \quad . \quad (3)$$

In like manner it may be shown that

$$\frac{d^2V}{dx^2} = GRV + (GL+KR)\frac{dV}{dt} + KL\frac{d^2V}{dt^2}. \quad . \quad . \quad . \quad (4)$$

Equation (3) or (4), for C or V, is usually called the telegraphic equation, because it was first deduced for the telegraph cable, but since no stipulation has been made regarding the sending E.M.F., it is characteristic of the cable, and applies equally well to all forms of transmission.

The expressions, whatever they may be, for the current and the voltage along a cable of uniformly distributed constants must always satisfy (3) and (4), but the actual form which they may take will depend on a variety of circumstances, such as the shape of the sending E.M.F. and the nature of the connections at the ends of the cable.

Solution of the Telegraphic Equation for a Purely Periodic E.M.F.

Consider the two simultaneous equations (1) and (2)—

$$-\frac{dV}{dx}=RC+L\frac{dC}{dt}$$

and
$$-\frac{dC}{dx}=GV+K\frac{dV}{dt}.$$

Let the sending E.M.F. be purely sinusoidal, of the form Ee^{ipt} Then V and C are also sinusoidal and of the same period. Hence, as in Chap. III., we may write

$$-\frac{dV}{dx}=(R+ipL)C, \quad \ldots \quad (5)$$

$$-\frac{dC}{dx}=(G+ipK)V, \quad \ldots \quad (6)$$

and
$$\therefore \quad \frac{d^2V}{dx^2}=(R+ipL)(G+ipK)V,$$

$$=P^2V, \text{ say}, \quad \ldots \quad (7)$$

and
$$\frac{d^2C}{dx^2}=(R+ipL)(G+ipK)C,$$

$$=P^2C, \quad \ldots \quad (8)$$

where, in both cases, $P=\sqrt{(R+ipL)(G+ipK)}$. . . . . (9)

The equations (5), (6), (7) and (8) do not involve t. They show how the amplitude and the phase of V and C vary as the position in the cable at which they are measured is altered.

A particular solution of (7) is evidently $V=e^{Px}$; and $V=e^{-Px}$ is also a solution. Adding together these two, each multiplied by a constant as yet undetermined, it follows that

$$V_x = Ae^{Px} + Be^{-Px} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

is also a solution, and, since it contains *two* independent constants, according to the theory of differential equations* it may be regarded as the complete solution of (7), an equation of the *second* degree.

It remains to choose the constants A and B in such a way that the solution may satisfy the terminal conditions.

From (5),

$$C = \frac{-1}{R+ipL}\frac{dV}{dx} = \frac{-P}{R+ipL}(Ae^{Px} - Be^{-Px})$$

on substituting for V from (10). Since $P=\sqrt{(R+ipL)(G+ipK)}$, it follows that $\dfrac{R+ipL}{P} = \sqrt{\dfrac{R+ipL}{G+ipK}}$. Now $\dfrac{1}{G+ipK}$, the reciprocal of a conductance, is equivalent to an impedance, and therefore $\sqrt{\dfrac{R+ipL}{G+ipK}}$ is the square root of an impedance squared, and is itself an impedance, of complex or generalised form. It is called the *characteristic* impedance of the cable, and is denoted by the symbol Z_0, so that

$$Z_0 = \sqrt{\frac{R+ipL}{G+ipK}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence

$$C_x = -\frac{1}{Z_0}(Ae^{Px} - Be^{-Px}). \quad . \quad . \quad . \quad (12)$$

At the sending end, where x is zero, let the voltage V_0 and current C_0 be supposed to be known. Then, from (10) and (12), putting $x=0$,

$$V_0 = A + B, \text{ since } e^{\pm 0} = 1,$$

and $$-Z_0C_0 = A - B.$$

Hence $$2A = V_0 - Z_0C_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and $$2B = V_0 + Z_0C_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

---

* In general, on differentiating the solution twice, two new equations are formed, making three in all, and from three equations two, and only two, constant quantities can be eliminated by substitution to form again the differential equation.

A and B are now fixed. Substituting their values in (10) and (12)

$$2V_x=(V_0-Z_0C_0)e^{Px}+(V_0+Z_0C_0)e^{-Px},$$

and $$-2Z_0C_x=(V_0-Z_0C_0)e^{Px}-(V_0+Z_0C_0)e^{-Px}.$$

Re-arranging,

$$2V_x=V_0(e^{Px}+e^{-Px})-Z_0C_0(e^{Px}-e^{-Px}),$$

and $$2C_x=C_0(e^{Px}+e^{-Px})-(V_0/Z_0)(e^{Px}-e^{-Px}),$$

which may be written

$$\left.\begin{aligned}V_x&=V_0\cosh Px-Z_0C_0\sinh Px,\\ C_x&=C_0\cosh Px-(V_0/Z_0)\sinh Px,\end{aligned}\right\}\quad .\ .\ .\ . \quad (15)$$

These expressions constitute the required solutions of the telegraphic equation for a sinusoidal source. By means of them the amplitude and the phase of voltage and current at any point in the cable can be calculated.

On transposing the equations they may be written in the equivalent forms,

$$\left.\begin{aligned}V_0&=V_x\cosh Px+Z_0C_x\sinh Px,\\ C_0&=C_x\cosh Px+(V_x/Z_0)\sinh Px,\end{aligned}\right\}\quad .\ .\ .\ . \quad (16)$$

in which the sending voltage and current are expressed in terms of the voltage and current at distance x from the sending end. This form is sometimes more useful and is the one usually adopted in practical calculations.

Infinitely Long Cable.

Let x be very great. Then $\cosh Px=\sinh Px=\frac{e^{Px}}{2}$ ultimately and the equations (15) become

$$\left.\begin{aligned}2V_x&=(V_0-Z_0C_0)e^{Px},\\ \text{and}\qquad 2C_x&=\left(C_0-\frac{V_0}{Z_0}\right)e^{Px}.\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ . \quad (17)$$

As x increases, V_x and C_x diminish, and e^{Px} increases. The terms within the brackets in (17) must, therefore, diminish as x increases, and be zero when x is infinite. It follows that in an infinitely long cable the relationship between the sending end voltage and current is simply

$$C_0=\frac{V_0}{Z_0}.\quad .\ .\ .\ .\ .\ .\ .\ .\ . \quad (18)$$

The characteristic impedance, Z_0, is therefore the ratio of volts to amperes at the sending-end of the infinitely long cable, and for this reason it is also called the *sending-end* impedance.

Let, now, V_x and C_x as given by (15) in terms of V_0 and C_0, be the voltage and current at distance x along a cable of infinite length. Then from (18), $V_0=Z_0C_0$, which, on substitution, leads to

$$V_x=V_0 \cosh Px-V_0 \sinh Px,$$
$$=V_0e^{-Px}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and in like manner, $C_x=C_0e^{-Px}$. . . . . (20)

Hence, in a cable of infinite extent away from the sending end, the voltage and current at any point may be obtained by multiplying the values at the sending-end by a simple exponential factor. The quantity P, which multiplied by the distance constitutes the exponent of this factor, may be called the *Propagation* constant. It is a complex quantity and can, therefore, be reduced to the form $\alpha+i\beta$, where

$$P=\alpha+i\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Then $$V_x=V_0e^{-(\alpha+i\beta)x}=V_0e^{-\alpha x}\overline{|\beta x},$$

and $$C_x=C_0e^{-(\alpha+i\beta)x}=C_0e^{-\alpha x}\overline{|\beta x}.$$

When the cable is infinitely long, and the terminal effects can be neglected, voltage and current suffer attenuation according to a simple exponential law. The quantity α is called the *attenuation* constant, and the factor $e^{-\alpha x}$, by which the amplitudes of the sending voltage and current must be multiplied to give the amplitudes of the voltage and current at any distance x, is called the *attenuation factor.*

If for V_0 we write Ee^{ipt}, then

$$V_x=Ee^{-\alpha x}\underline{/pt-\beta x},$$
$$=Ee^{-\alpha x}[\cos (pt-\beta x)+i \sin (pt-\beta x)].$$

Neglecting the real part of V_0, which is equivalent to reckoning time from the instant at which V_0 passes through its zero value, and retaining only the imaginary part, it follows when

that $$\left.\begin{array}{l} V_0=E \sin pt \\ V_x=Ee^{-\alpha x} \sin (pt-\beta x). \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The change of phase of V_x is directly proportional to the distance from the sending-end, and is obtained by multiplying x by the constant β. If x has a value λ such that $\beta\lambda = 2\pi$, then V_x is in phase with the sending voltage, and it is also in phase for successive multiples of λ. It is evident, therefore, that

$$\lambda = \frac{2\pi}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

is the distance between points along the cable where the voltages (and the currents) are in the same phase, *i.e.*, it is equal to the distance from crest to crest or from hollow to hollow. For this reason β is called the *wave-length* constant.

Expression of α and β in Terms of the Fundamental Constants.

It is now necessary to connect the newly-defined derived constants α and β with the fundamental constants of the cable, R, K, L and G.

By definition,

$$P = \sqrt{(R + ipL)(G + ipK)} = \alpha + i\beta.$$

Squaring both sides, and equating real and imaginary quantities,

$$\alpha^2 - \beta^2 = RG - p^2LK,$$

and

$$2\alpha\beta = p(LG + KR).$$

Now

$$\begin{aligned} \alpha^2 + \beta^2 &= \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}, \\ &= \sqrt{(RG - p^2LK)^2 + p^2(LG + KR)^2}, \\ &= \sqrt{(R^2 + p^2L^2)(G^2 + p^2K^2)}. \end{aligned}$$

Hence, adding

$$\left.\begin{aligned} 2\alpha^2 &= \sqrt{(R^2 + p^2L^2)(G^2 + p^2K^2)} + (RG - p^2LK) \\ \text{and} \quad 2\beta^2 &= \sqrt{(R^2 + p^2L^2)(G^2 + p^2K^2)} - (RG - p^2LK) \end{aligned}\right\} \quad (24)$$

through which equations α and β are now known in terms of R, K, L, G and p.

Certain particular cases of these formulæ are of interest. Suppose, in the first place, that L and G are zero, as is very nearly the case in a non-loaded or plain submarine cable. The formulæ then reduce to

$$\alpha = \beta = \sqrt{\tfrac{1}{2}pKR}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If G alone is zero, the formulæ (24) reduce to

$$\left.\begin{aligned} 2\alpha^2 &= pK(\sqrt{R^2+p^2L^2}-pL) \\ \text{and} \qquad 2\beta^2 &= pK(\sqrt{R^2+p^2L^2}+pL) \end{aligned}\right\} . \quad . \quad . \quad . \quad (26)$$

When pL is so small compared with R that p^2L^2 may be neglected in comparison with R^2, (26) may be written

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{1}{2}pK(R-pL)} \\ \text{and} \qquad \beta &= \sqrt{\tfrac{1}{2}pK(R+pL)} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Comparing (27) with (25), it is seen that the effect of the insertion of a small amount of inductance in a cable originally devoid of inductance is equivalent to reducing the resistance of the cable by an amount pL proportional to the added inductance.

When L and G are zero the formula for Z_0 reduces to

$$Z_0 = \sqrt{\frac{R}{ipK}} = \sqrt{R/pK} \,\underline{\backslash\pi/4}. \quad . \quad . \quad . \quad . \quad (28)$$

In a long cable without inductance or leakance the current leads the voltage by an angle $\pi/4$ or 45 deg.

"Standard" Telephone Cable.

Inasmuch as the ultimate object of the telephone engineer is to reproduce at the distant end the speech communicated at the sending end, it is desirable to have a standard circuit with which the transmitting qualities of other circuits and apparatus may be compared experimentally. A standard of this kind was established in this country in the year 1905 as the result of agreement between the Post Office and the late National Telephone Company. Not only was the type of cable fixed but also the terminal apparatus to be used in conjunction with it. The selected cable is an air-space paper-covered cable having a conductor weighing 20 lb. per mile. The electrical constants are defined as being

$$\left.\begin{aligned} R &= 88 \text{ ohms} \\ K &= 0{\cdot}054 \text{ mfd.} \\ G &= 5\times10^{-8} \text{ mhos} \end{aligned}\right\} \text{per loop mile.}$$

If a length of another type of cable be such that it produces the same effect in transmission as 1 mile of the standard cable, it may be said to be *one standard mile* in length.

Let us apply the preceding formula to this cable. Neglecting G, we have from (25)

$$\alpha=\beta=\sqrt{\pi\times 800\times 0{\cdot}054\times 10^{-6}\times 88},$$
$$=0{\cdot}1093 \text{ per loop mile,}$$

when the frequency is 800 periods per second.

In obtaining this value of α the inductance is neglected. If we assume it to be very nearly one millihenry per mile, then on substitution in (27), it follows that

$$\alpha=\sqrt{\pi\times 800\times 0{\cdot}054\times 10^{-6}\times(88-5{\cdot}0265)},$$
$$=0{\cdot}1061 \text{ per mile,}$$

and $$\beta=\sqrt{\pi\times 800\times 0{\cdot}054\times 10^{-6}\times(88+5{\cdot}0265)}$$
$$=0{\cdot}1124 \text{ per mile.}$$

The attenuation constant of standard cable is therefore 0·106 per mile, or 0·122 per n.m. at 800 periods per second. The variation of the attenuation constant with frequency for standard cable is exhibited in Fig. 33. The parabolic curve A shows the relationship between the attenuation and wave-length constants and the frequency when the inductance of the cable is neglected. Curves B and C show the amount of divergence which is caused by the assumption of an inductance of one millihenry per mile. The higher the frequency the greater is the effect of the inductance in lowering the attenuation constant and in raising the wave-length constant.

Not only does the standard cable furnish a measure in terms of which other cables may be expressed, but it also enables a definition to be given of what may be called commercial speech. It is found by experiment that the greatest distance through which transmission satisfactory for normal working can be obtained is about 43 miles of standard cable. If, therefore, the total length of any line in standard miles is less than 43, satisfactory transmission may be expected.

Since $43\times 0{\cdot}106=4{\cdot}562=\max.\ \alpha L$, it follows that the limit of commercial speech may be regarded as that length of cable which gives a total attenuation of 4·5 and an attenuation factor of $e^{-4\cdot5}$ or 0·01 approximately. The apparatus is here supposed in perfect order. To allow for possible disturbances and

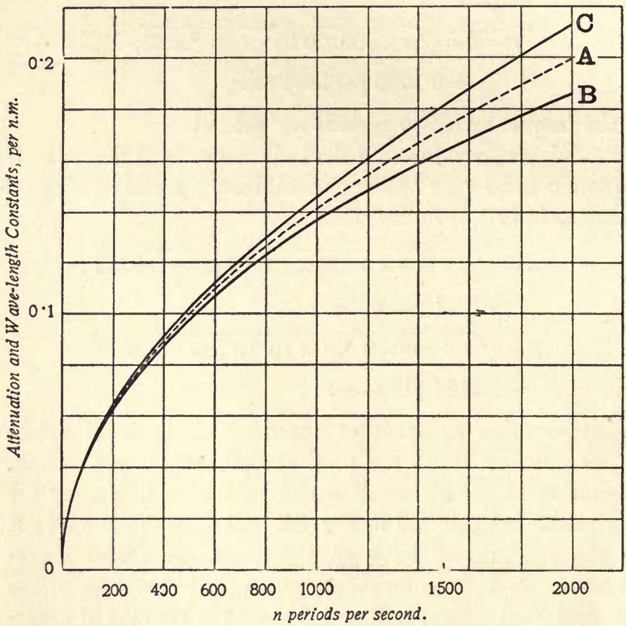

FIG. 33.—VARIATION WITH FREQUENCY OF ATTENUATION AND WAVE-LENGTH CONSTANTS.—"STANDARD" TELEPHONE CABLE.

R=88 ohms; K=0·054 mfd. per mile loop.
Curve A, α, β; L=0.
" B, α; L=0·001 per mile loop.
" C, β; L=0·001 " "

render good and easy communication available at all times, the value 2·5 may be adopted as the greatest permissible attenuation* in the design of a telephonic circuit.

* The Second International Conference of Postal Telegraph Engineers decided upon the following scale for overhead lines :—

αL	2·5	Very good.
"	3·5	Good.
"	4·8	Commercial limit.

See "E.T.Z.," 32, 1911, p. 390.

Submarine Telephone Cable.

Now, consider another type of cable, say, a plain core consisting of a stranded conductor weighing 160 lb. of copper covered with 150 lb. of gutta-percha per n.m. as used in the Anglo-Belgian telephone cable of 1911, and in the Anglo-Irish telephone cable of 1913. The resistance and capacity of this type of cable are given as 14·2 ohms and 0·157 mfd. per n.m. loop.* From (25) neglecting inductance,

$$\alpha_{800} = \beta_{800} = 10^{-3}\sqrt{2514 \times 0{\cdot}157 \times 14{\cdot}2}$$

$$= 0{\cdot}0749 \text{ per n.m.}$$

Hence one n.m. of this type of submarine telephone cable produces the same attenuation as $0{\cdot}122 \div 0{\cdot}0749 = 1{\cdot}633$ n.m. of standard cable, or the equivalent of 1 n.m. of this type of cable is $0{\cdot}0749 \div 0{\cdot}106 = 0{\cdot}706$ standard mile.

It must not be supposed that these lengths of cable could replace each other indifferently in a telephone circuit without affecting the transmission. Before such a substitution could be made it would be necessary to consider what takes place at the junctions of dissimilar types of cable, a matter which must be deferred to a later chapter. Nevertheless the conception of the standard cable possesses the great advantage of making a direct appeal to the mind which the attenuation constant does not.

Loaded Cables.

In loaded cables the inductance L is artificially increased, and both pL and pK are usually great compared with R and G respectively. Formula (24) for α may, therefore, be written—taking pL and pK outside the square root—

$$2\alpha^2 = p^2\mathrm{LK}\left[\sqrt{\left(1+\frac{\mathrm{R}^2}{p^2\mathrm{L}^2}\right)\left(1+\frac{\mathrm{G}^2}{p^2\mathrm{K}^2}\right)} - \left(1-\frac{\mathrm{RG}}{p^2\mathrm{LK}}\right)\right].$$

Expand the square root by the binomial theorem, and neglect

* P.O. Eng. "Journal," April, 1912, p. 55.

$(R^2/p^2L^2)^2$ and $(G^2/p^2K^2)^2$ and higher powers of the small quantities R^2/p^2L^2 and G^2/p^2K^2.

Then $2\alpha^2 = p^2LK\left[\left(1+\frac{R^2}{2p^2L^2}\right)\left(1+\frac{G^2}{2p^2K^2}\right)-1+\frac{RG}{p^2LK}\right]$,

$$\therefore \quad \alpha^2 = \frac{p^2LK}{2}\left[\frac{R^2}{2p^2L^2}+\frac{G^2}{2p^2K^2}+\frac{RG}{p^2LK}\right]$$

and $$\alpha = \frac{R}{2}\sqrt{\frac{K}{L}}+\frac{G}{2}\sqrt{\frac{L}{K}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This formula is a very simple and convenient one for calculating the attenuation constant of a loaded cable.* It is to be observed that it consists of two parts—(*a*) the resistance term $(R/2)\sqrt{K/L}$, and (*b*) the leakance term $(G/2)\sqrt{L/K}$. If the leakance is so small that the latter term may be neglected, the formula becomes simply

$$\alpha = \frac{R}{2}\sqrt{\frac{K}{L}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

When G is zero, to a second degree of approximation as regards pL and R, the formula may be written

$$\alpha = \frac{R}{2}\left(1-\frac{R^2}{8p^2L^2}\right)\sqrt{\frac{K}{L}} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The formula for β corresponding to (29) for α may be obtained in a similar way from (24) which reduces to

$$\beta^2 = \left(\frac{R}{2}\sqrt{\frac{K}{L}}-\frac{G}{2}\sqrt{\frac{L}{K}}\right)^2 + p^2LK.$$

Now, the term within brackets is less than α, which itself for a loaded cable is small in comparison with β, since the effect of loading is to increase β at the same time as α is diminished. The formula for β, therefore, reduces very nearly to

$$\beta = p\sqrt{LK}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

It is important to observe that (29) does not contain p. *In a heavily loaded cable the attenuation is independent of the fre-*

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* F. Breisig, " E.T.Z." 29, 1908, p. 588.

quency. Moreover, from (32) since the wave-length $\lambda=2\pi/\beta$, it follows that

$$\lambda=\frac{2\pi}{p\sqrt{LK}}=\frac{1}{n\sqrt{LK}} \text{ or } n\lambda=\frac{1}{\sqrt{LK}}.$$

Now $n\lambda=v$, where v is the velocity of propagation of the wave, and therefore v is independent of p. Hence all waves are transmitted with the same velocity, and the components of a complex wave conserve their phase relationships in passing along the cable. *The loaded cable is free from distortion.* Both these conclusions remain true only so long as the quantities R, K, L, and G are themselves independent of the frequency.

From (32) it is seen that β is unaffected within limits by alterations in R and G, whereas the resistance and the leakance terms in α are respectively directly proportional to these quantities. Any change in the effective values of R and G influences α but leaves β comparatively unaffected.

Differentiate (29) with respect to L, and equate to zero. Then

$$0=\frac{d\alpha}{dL}=\frac{1}{2L}\left[-\frac{R}{2}\sqrt{\frac{K}{L}}+\frac{G}{2}\sqrt{\frac{L}{K}}\right] \quad . \quad . \quad . \quad (33)$$

The attenuation constant is, therefore, a minimum when sufficient inductance is added to make both terms equal. It is then given by $\alpha=R\sqrt{\frac{K}{L}}$, where $L=\frac{KR}{G}$. But in obtaining this formula any change in R and K consequent on the introduction of L is neglected. In general R is unavoidably increased at the same time as L and therefore the best value of L and the lowest attenuation constant are not reached until after the critical value of L given by (33) is attained.

Taking the same cable as before, where R=14·2 ohms and K=0·157 mfd. per n.m. loop, let L vary from zero up to 100 millihenries per n.m. loop and let us calculate α and β for a frequency of 800 p.p.s. Formulæ (26) may be used for this purpose, or more easily (27) when L is very small, and (29), (31) and (32) when L is large. The results are exhibited in Fig. 34. Curve A, for the attenuation constant, drops sharply for a slight inductance from the value 0·075 per n.m., which it has when L is zero. Curve B, representing the wave-length constant, rises steeply from the same initial value. At 100 millihenries per n.m. it has the value $\beta = p\sqrt{LK} = 0{\cdot}6299$ per n.m. and even at 10 mh. per n.m. it is already 10 times α. Up to one millihenry per n.m., Curve A is represented with sufficient exactitude by the formula $\alpha = \sqrt{\frac{1}{2}pK(R - pL)}$; and

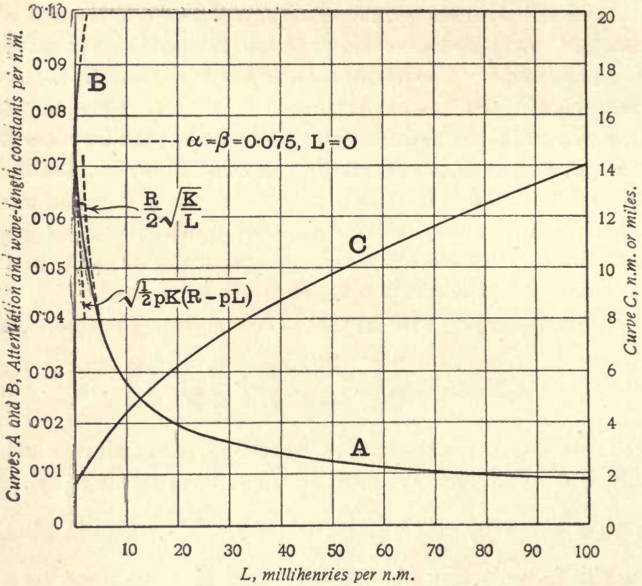

FIG. 34.—INFLUENCE OF ADDED INDUCTANCE ON ATTENUATION CONSTANT.

$n = 800$. Type 160/150. R = 14·2 ohms. K = 0·157 mfd.
Curve A.—Attenuation Constant. Curve B.—Wave-length Constant.
Curve C.—N.m. equivalent to one n.m. of standard cable, or miles equivalent to one standard mile.

again when L is greater than 5 millihenries per n.m.—as in loaded cables—the formula $\alpha = \frac{R}{2}\sqrt{\frac{K}{L}}$ fits the curve closely. Curve C represents the number of n.m. of the cable which would produce the same attenuation as one n.m. of standard cable, or the number of miles equivalent to one standard mile.

From Fig. 34 it is seen that a slight amount of inductance is sufficient to produce a big change in the attenuation and wave-length constants of the cable. As the inductance of the cable depends on the size and distance apart of the conductors and on the sheathing, there may be considerable uncertainty, in the absence of measurement, regarding these constants for the plain cable. In the case of the loaded cable the added inductance is usually enough to swamp any irregularity due to uncertainty regarding the inductance of the cable itself, and, moreover, as is evident from the figure, the greater the inductance the less sensitive are α and β to slight changes in it.

An approximation to the inductance may be obtained from formula (12) of Chapter II. For 160/150 core, d may be taken as 0·105 in. (*see* Appendix IV.), and D as 0·2866 in. Hence $\log_{10} D/d = 0{\cdot}4361$, and $L = 1{\cdot}45$ mh. per n.m. For this value of L, $\alpha = 0{\cdot}0585$ and $\beta = 0{\cdot}0958$ per n.m.

The Spiral Diagram.

The distribution of voltage and current along the cable may conveniently be represented by the aid of a diagram as in Fig. 35, which consists of an end elevation on the left as viewed from the sending-end of the cable, and a side elevation on the right, in which the direction of transmission is from left to right. The line OE is drawn to represent the sending E.M.F. of, say, 5 volts. If this line is made to rotate counter-clockwise with the angular velocity p, its projection on the vertical axis OY will represent the instantaneous value of the sending E.M.F., E sin pt. The voltage at distance x along the cable is given by

$$V_x = Ee^{-\alpha x} \sin (pt - \beta x).$$

Taking β as 0·0958, then $\beta x = \pi/6$, or 30 deg., when $x = 5{\cdot}465$ n.m. If there were no attenuation the point H at the end of the radius vector OH, which is 30 deg. behind OE, when projected to the right, so as to cut the ordinate through 5·47 n.m., would give the point H′ on the side elevation of the voltage distribution curve. In this way the curve $V_0 \sin (pt - \beta x)$, representing the unattenuated voltage is obtained. But when x is 5·47 n.m., $e^{-\alpha x}$ is 0·7264, and the line OJ, where $OJ = 5 \times 0{\cdot}7264 = 3{\cdot}632$, is the actual or attenuated

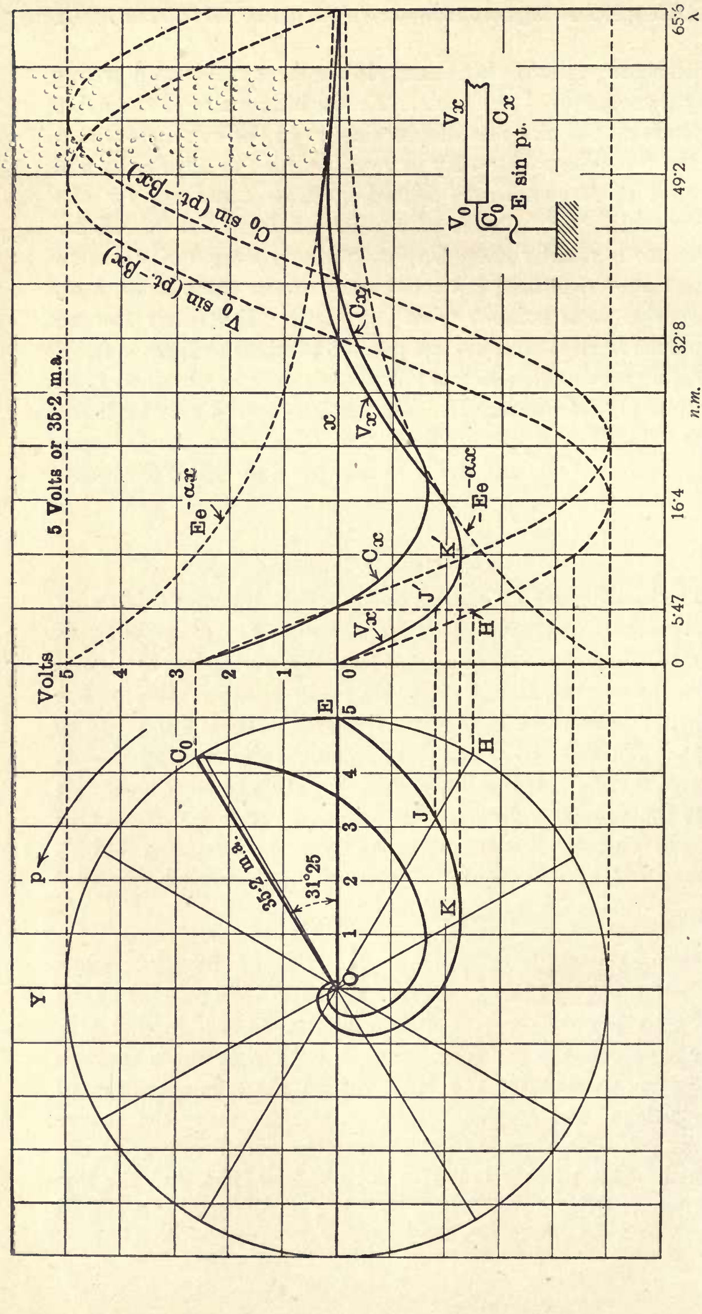

FIG. 35.—SPIRAL DIAGRAM—INFINITE CABLE, TYPE 160/150.

$n=800$; $\alpha=0{\cdot}0585$; $\beta=0{\cdot}0958$; $\lambda=65{\cdot}6$ n.m.; $Z_0=142{\cdot}2\backslash 31°25$.

voltage at that distance. By projecting J to the right, J′ is obtained, and in this way the curve V_x is constructed. The spiral EJK–, in this case an equiangular or logarithmic spiral, gives the voltage and phase at any point along the cable. As OE moves round counter-clockwise the end of the curve V_x is lifted up, and its points of contact with the fixed curves, (Ee^{-ax} and $-Ee^{-ax}$), between which it is constrained to lie, move to the right, the wave travelling from left to right.

To obtain the current, the equation is $C_0 = V_0/Z_0$.

$$\text{Now,}\quad Z_0 = \frac{\sqrt[4]{R^2+p^2L^2}}{\sqrt{pK}} \Big/ \underline{\tfrac{1}{2}\tan^{-1}\frac{pL}{R} - \frac{\pi}{4}} = 142 \cdot 2/\overline{31^\circ\,25},$$

the inverted sign being used to denote a negative angle.

Hence, $C_x = 0 \cdot 0352 \sin(pt - \beta x + 31^\circ\,25')$. The line OC_0 drawn at an angle 31° 25′ in advance of OE may be taken to represent the sending current and be labelled 35·2 milliamperes. The same spiral as for the volts displaced through the angle EOC_0, will then serve for the current also, and the current curve on the right will oscillate between the same fixed limits. The curves extend to infinity on the right, although only the first wave-length is shown.

Spiral Diagram for a Loaded Cable.

Now, suppose that the cable is loaded so as to bring its inductance up to 0·1 henry per n.m., being otherwise left unaltered. From Fig. 34, a is now reduced to 0·008896, and β is 0·6298. Hence $\lambda = 9 \cdot 976$ n.m. and $\beta x = 30^\circ$ when $x = 0 \cdot 8313$. For this value of x, e^{-ax} equals 0·9926. The voltage diagram is constructed in the same way as before, and is shown in Fig. 36. The spiral now makes a great number of turns, instead of only a single one, as in the last case, before it practically reaches the origin, and the waves, of which two are shown to the right, are much more closely packed together.

To obtain the current, we have $Z_0 = \sqrt{\dfrac{R+ipL}{ipK}}$. Since $(pL)^2 = (502 \cdot 65)^2$, which is great compared with R^2, $(14 \cdot 2)^2$, it follows that $Z_0 = \sqrt{\dfrac{ipL}{ipK}}$ merely, $= \sqrt{\dfrac{L}{K}} = 791 \cdot 8$. Actually

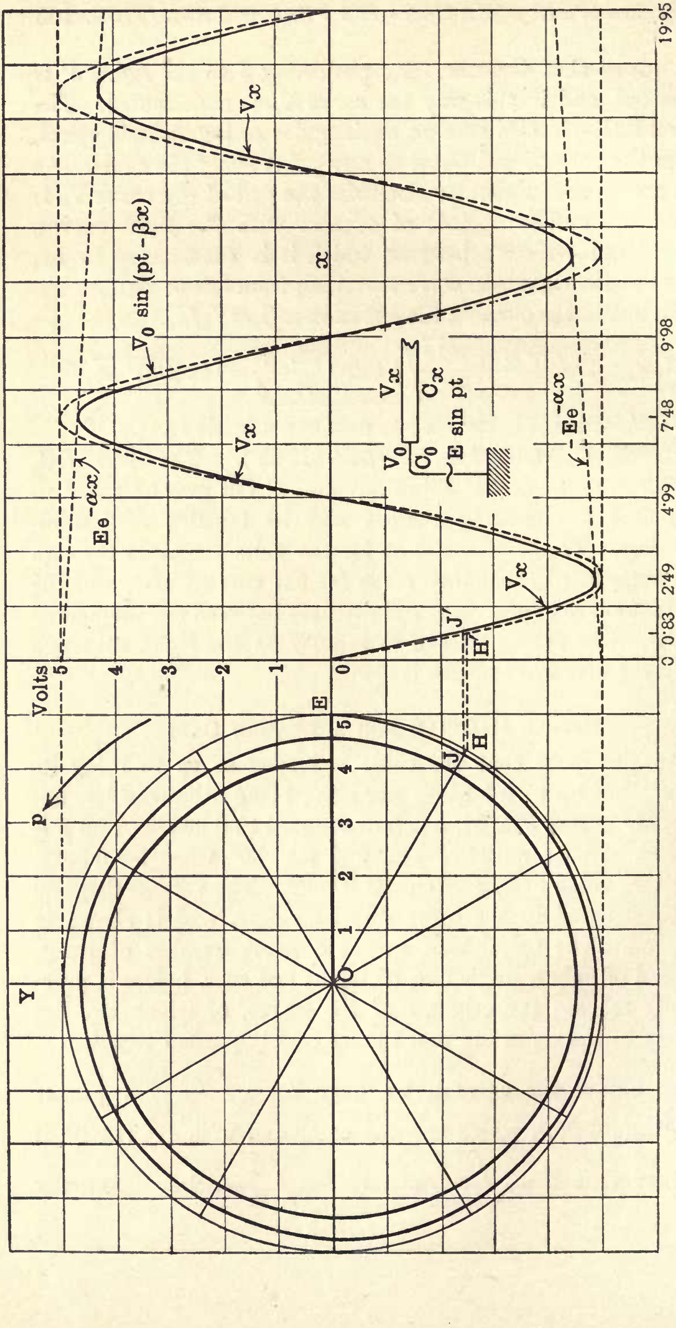

FIG. 36.—SPIRAL DIAGRAM—CABLE LOADED WITH 0·1 HENRY PER N.M.

$\alpha=0{\cdot}0089$; $\beta=0{\cdot}6298$; $\lambda=9{\cdot}98$ n.m.; $Z_0=798\angle 0^\circ 47'$.

$Z_0 = 798{\cdot}3\,\overline{|0^\circ 49'}$. Neglecting the small phase displacement, the same curve will represent the current as the voltage, but to a different scale in which OC represents $5 \div 798{\cdot}3$, $= 6{\cdot}263$ milliamperes.

Models to Represent Wave Propagation.

A simple model for illustrating the propagation of waves in cables may be constructed in the following manner.* A thin wire is weighted by soldering to it rectangular pieces of tinplate all of the same size and at equal distances from each

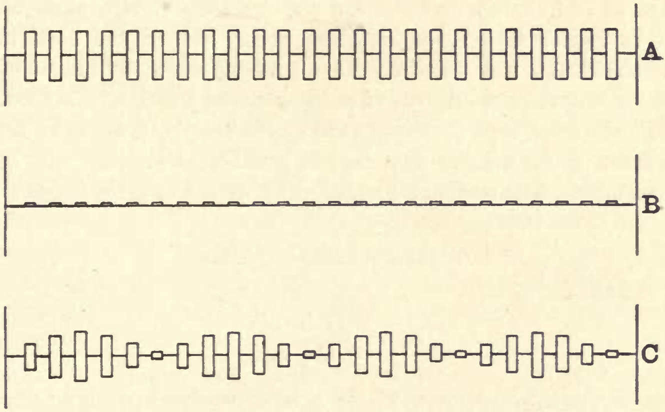

Fig. 37.—Model to represent Wave-propagation.
A and B.—Loaded wire at rest. C.—Wire under torsional vibration.

other. The wire is clamped at one end, from which it hangs vertically, or it may be placed horizontally and clamped at both ends. In the position of rest, Fig. 37, A and B, the pieces of tinplate are all in the same plane. By various means, such as by moving one of the ends to and fro, the wire can be set into torsional vibration. If the wire be thin and the strip proportionately heavy, the period may be several seconds, and the oscillations once started will last several seconds. Viewed from a distance, the strips appear black against a white back-

* H. Barkhausen, "Phys. Zeitschr.," 14, 1913, p. 62. *See* also H. Hausrath, *ibid.* p. 1,045, who uses a steel tape, and W. H. Julius, "Zeitschr. f.d. Physik. u. Chem. Unterricht," 1907, p. 87.

ground, and of length depending on the angle through which they are twisted. The lower end may be clamped (cable to earth) or left free (end open), or another wire of different constants may be attached to it, and the reflections of waves investigated as they travel along the wire (*see* Chap. VI.). The model permits of an experimental study of wave-propagation which may even be quantitative in character. Another model,* which permits of the study of forced vibrations, is constructed by attaching one end of a cord to a crankshaft which is oscillated by an electric motor in such a manner that the end of the cord acquires a circular motion in a plane at right angles to its length. The other end of the cord is attached to a nut which travels on a long screw, by means of which any required degree of tension can be put upon the cord. The vibrating cord appears to the eye to be spindle-shaped. By adjusting the tension any desired number of sections can be obtained. The cord can be loaded by means of glass beads.

The Distortionless Cable.

In the formula

$$Z_0=\sqrt{\frac{R+ipL}{G+ipK}},$$

let $R/G=L/K=\mu$, say, where μ is some constant quantity. Then $Z_0=\sqrt{\mu}$, which does not contain p, and is, therefore, independent of the frequency. At the same time the formula for the propagation constant becomes

$$P=\sqrt{(R+ipL)(G+ipK)}=(G+ipK)\sqrt{\mu}=\alpha+i\beta,$$

whence $$\alpha=G\sqrt{\mu}=\sqrt{RG}=R\sqrt{\frac{K}{L}}, \quad . \quad . \quad . \quad . \quad (34)$$

and $$\beta=pK\sqrt{\mu}=p\sqrt{KL}.$$

It is to be observed that the expression for β is the same as that for a loaded cable (32), while the expression for α is twice the corresponding expression for a loaded cable with zero leakance. The same remarks apply in this case as in the

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* J. A Fleming, "Proc." Phys. Soc., 26, 1913, p. 61.

loaded case. Waves of all frequencies are attenuated alike, and are propagated with the same velocity. The essential difference is that in the present instance there is no assumption as to high frequency, or to relatively high values of pL compared with R, and pK with G, as in the former case. Provided that the condition

$$\mathrm{KR} = \mathrm{LG} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

holds, the propagation is the same for all frequencies. For this reason equation (35) was termed by Heaviside the *distortionless* condition,* and a cable with its constants in this proportion is called a distortionless cable.

Since the attenuation constant for a loaded cable is given by the formula (29)—

$$\alpha = \frac{\mathrm{R}}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}} + \frac{\mathrm{G}}{2}\sqrt{\frac{\mathrm{L}}{\mathrm{K}}},$$

in which the second term is usually much smaller than the first, it follows that α for a cable which is merely loaded is less than for one in which the distortionless condition holds. If G could be made to vanish altogether, α would be only half the distortionless value, and the speaking range would be twice as great.

In the distortionless cable waves of all frequencies are affected alike, and therefore a wave of complex shape retains its shape during transmission through the cable, although its size is reduced. A square-topped wave, for example—such as is formed by reversals of a battery E.M.F.—remains a square-topped wave although it shrinks as it passes along the cable. In the theoretical study of cable transmission the distortionless cable is invaluable. But in telephonic transmission the distortionless state is not required, for satisfactory speech will be obtained even although the lower frequencies are distorted, because they do not play an important part in the reproduction. Since volume of speech is of more importance than character it is preferable to make G as small as possible, even although articulation would be improved by the introduction of a leak distributed along the cable.

* "Electromagnetic Theory," Vol. I., p. 366.

Influence of Leakance.

In the complete formula (24) for a, let G be small in comparison with pK. Then

$$\begin{aligned} 2a^2 &= \sqrt{(R^2+p^2L^2)(G^2+p^2K^2)}+(RG-p^2LK) \\ &= pK(\sqrt{R^2+p^2L^2}-pL)+RG \\ &= 2a_0^2+RG, \quad \ldots \ldots \ldots \ldots \quad (36) \end{aligned}$$

where a_0 is the value which a would have if G were zero. From (36) the influence of moderate values of G may be calculated. Take R = 14·2 ohms, and K = 0·157 mfd. per n.m. loop, as before.

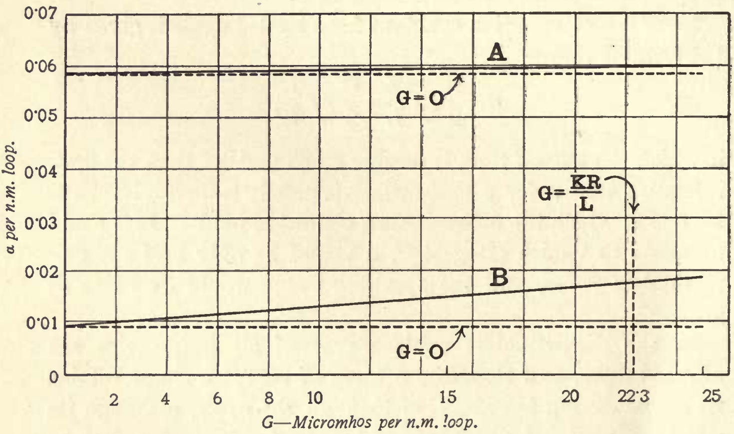

Fig. 38.—Influence of Leakance on Attenuation Constant.

Curve A.—Plain cable. R = 14·2 ohms; K = 0·157 mfd.; L = 0·00145 henry.
,, B.—Loaded cable. L = 0·1 henry. All per n.m. loop.

Then $a_0 = 0{\cdot}0585$, when L = 1·45 mh.; and $a_0 = 0{\cdot}0089$, when L = 100 mh. The influence of various values of G from 0 to 25×10^{-6} mhos per n.m. loop is shown in Fig. 38, as calculated from (36), which may be written $a^2 = a_0^2 + RG/2$, or when G is small

$$a = a_0\left(1+\frac{RG}{4a_0^2}\right) = a_0 + \frac{RG}{4a_0}. \quad \ldots \quad (37)$$

Curve A is the attenuation constant for L = 1·45 mh. per n.m. loop. It deviates only slightly from the dotted straight line, with which it would coincide if G were zero. When L is great the formula for the attenuation constant is

$$a=\frac{R}{2}\sqrt{\frac{K}{L}}+\frac{G}{2}\sqrt{\frac{L}{K}},$$

which is plotted in curve B. The increase in a due to leakance is now absolutely greater, and proportionately very much greater, than for the unloaded cable. When $G=22{\cdot}3\times10^{-6}$ mhos the cable is distortionless and a is twice the value which it has when G is zero. When $G/K=120$, $G=18{\cdot}84\times10^{-6}$ and $a=0{\cdot}0164$. When $G/K=12$, $G=1{\cdot}88\times10^{-6}$ and $a=0{\cdot}0096$. The importance of low leakance in securing low attenuation is thus clearly evident. Curve B approaches curve A, and would eventually cut it if the leakance were great enough. A very leaky cable would be rendered worse for transmission purposes by loading.

This peculiarity is more manifest on land lines, where capacity is lower and where high insulation is more difficult to maintain, than on cables. Thus in the formula $a=\frac{R}{2}\sqrt{\frac{K}{L}}+\frac{G}{2}\sqrt{\frac{L}{K}}$ let K be reduced to one-ninth part of its original value. The first term in a is then one-third, and the second is three times what it was before. The relative importance of the leakance term is therefore increased nine times. A point is soon reached at which increase of inductance ceases to be advantageous, unless the insulation resistance of the circuit can be increased.

References.

T. H. Blakesley, The Electrician, Vol. XIV., p. 510; Vol. XV., 1885, pp. 22, 58.

C. V. Drysdale, The Electrician, 60, p. 277, 1907.

F. Breisig, "E.T.Z.," XXIX., 1908, p. 588.

O. Heaviside, "Electromagnetic Theory," Vol. I.

CHAPTER V.

THE METHODS OF LOADING CABLES.

Early Proposals—The Krarup Method—Calculation of Inductance of Core—Calculation of Effective Resistance of Core—Design of a Continuously-loaded Cable—Some Continuously-loaded Telephone Cables—Coil-loading—Calculation of the Inductance of a Loading Coil—Design of a Loading Coil—Effective Resistance—The Lake Constance Coil Cable—The Anglo-French Cable of 1910—Superimposing—The Anglo-Belgian Coil Cable—The Anglo-Irish Coil Cable—Comparison between Continuous and Coil-loading—Single-core Telephone Cables.

Early Proposals.

When the importance of inductance in telephonic transmission had begun to obtain recognition, various proposals were made for securing an artificial increase in the inductance of cables, in which type of conductor the effects of distributed capacity are most pronounced. Efforts were directed in the first place towards continuous loading. One suggestion was to mix finely divided iron with the dielectric whilst in the plastic condition. But the increase of inductance which can be produced in this way is small, too small in fact to be of significance for propagation. Moreover the capacity is increased in proportion to the amount of metal present. Again, long before any appreciable inductive effect is obtained the dielectric resistance is reduced too far for safety. The fatal objection is the great danger that a chain of particles in contact may form a weak spot in the insulation and lead sooner or later to a fault and breakdown.

The objection to the use of iron as a conductor in place of copper is the high effective resistance which it presents to alternating currents of telephonic frequency. Moreover, the increased bulk of the core which it would necessitate rules it out for cable purposes. Proposals have been made from time to time to deposit iron electrolytically on copper, but so far the experiments do not appear to have given the expected results. All other methods of twisting and stranding conductors round each other or round central cores are found on examination to lead to an increased resistance and an increased cost which are out of proportion to the increase of inductance which can be obtained in this way. It will appear below that the attenuation constant of a continuously-loaded cable is much more sensitive to an increase of resistance than to an increase of inductance. Great care must therefore be taken that any gain in inductance is not outweighed by a simultaneous augmentation of resistance.

The Krarup Method.

The plan which is followed at the present day in loading cables on the continuous principle is based exclusively on some

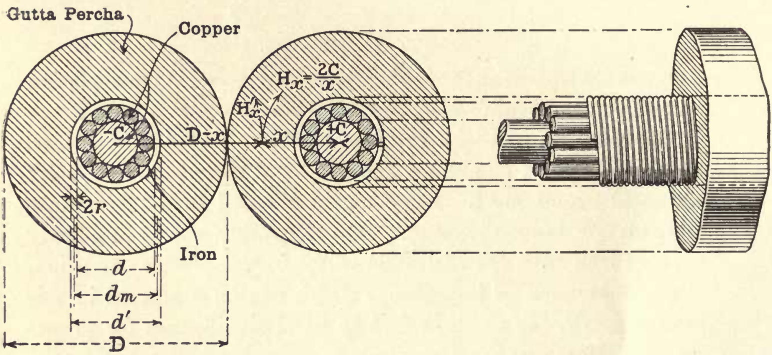

Fig. 39.—Twin Cores of Telephone Cable, loaded continuously with one whipping of Iron Wire.

D = Diameter of core, d = Diameter of stranded conductor, d' = Diameter over iron whipping, d_m = Mean diameter of whipping, r = Radius of iron wire, $d' = d + 4r$, $2d_m = d + d'$.

form of iron whipping. This method was introduced by the late Prof. Krarup, of Copenhagen, whose name it bears. A fine iron wire, usually 8 mils (0·2 mm.), or sometimes 12 mils in diameter, is wound in close spirals round the conductor, as in Fig. 39, in from one to three layers. The magnetic flux generated by the current in the cable conductor takes the form of tubes encircling the conductor and tending to coincide with the iron whipping. The magnetic resistance of the path of the flux is much reduced by the presence of the highly permeable iron, and therefore the flux and the inductance are much increased. As the wire is necessarily spiralled and not in rings these tubes of force must be supposed to snap over at some point from one step in the spiral to the next, and magnetic resistance is unavoidably introduced at this point. One advantage in the use of wire is that the operations of wire-drawing produce a structure in the wire which is attended with increased permeability in the direction of its length. Instead of wire thin tape might be used, which would expedite the processes of whipping, but would lead to a result less desirable electrically, on account of the increased effective resistance of the core, due to eddy currents, which it is found to entail.

Formulæ have been given* which enable the inductance and effective resistance of such whipped cores to be calculated, the permeability of the iron being supposed known. Alternatively the permeability of the iron may be obtained by comparison of the calculated value of the inductance with the measured, and when the value so found is substituted in the formula for effective resistance the increase due to eddy currents can be calculated.

Calculation of Inductance of Core.

Let L be the inductance per centimetre length of the whipped conductor, and let Φ be the flux produced per centimetre length by the current C in the conductor. When all three are in c.g.s. units,

$$LC = \Phi. \quad \quad (1)$$

The greatest part of the flux is in the iron whipping. Let d_m be the mean diameter of a single turn, supposed for the present to be a closed ring of wire, the radius of section of which is r. The magnetic conductivity of the ring is

$$\frac{\mu \cdot \pi r^2}{\pi d_m} = \frac{\mu r^2}{d_m}, \quad \quad (2)$$

* A. Larsen, "E.T.Z.," 29, 1908, p. 1030.

where μ is the permeability. The magnetomotive force is $4\pi C$. Hence the flux for one turn is

$$4\pi C \times \frac{\mu r^2}{d_m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the turns of wire touch, the number per centimetre length of the conductor is $1/2r$ and the flux per centimetre is

$$\frac{4\pi C\mu r^2}{d_m} \times \frac{1}{2r} = \frac{2\pi C\mu r}{d_m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Consider, next, the flux in the dielectric. The intensity of the magnetic force at distance x from the axis is (Fig. 39)

$$H_x = \frac{2C}{x},$$

and for the other conductor

$$H'_x = \frac{2C}{D-x}.$$

The flux per centimetre in the dielectric beyond the iron whipping is therefore

$$\int_{d'/2}^{D/2} (H_x + H'_x)dx = \int_{d'/2}^{D/2} \left(\frac{2C}{x} + \frac{2C}{D-x}\right)dx = 2C \log \frac{2D-d'}{d'}.$$

Hence, adding together the flux in the iron and the flux in the dielectric, which are the chief components of the total flux, approximately

$$\Phi = \frac{2\pi C\mu r}{d_m} + 2C \log \frac{2D-d'}{d'},$$

from which the inductance of the core is to be obtained, as in (1), by dividing by C. This formula gives a first approximation to the inductance of a single core, and since the inductance of whipped twin core is practically twice that of whipped single core, the principal part of the flux being in the iron, the inductance of a loop may be obtained by multiplying by two.* Hence

$$L = \frac{4\pi\mu r}{d_m} + 4 \log \frac{2D-d'}{d'} \text{ (c.g.s. units per cm. loop)} \quad . \quad (5)$$

$$= \frac{1{\cdot}26\mu r}{d_m} + 0{\cdot}92 \log_{10} \frac{2D-d'}{d'} \text{ (mh. per km. loop)} \quad . \quad (6)$$

$$= \frac{2{\cdot}33\mu r}{d_m} + 1{\cdot}71 \log_{10} \frac{2D-d'}{d'} \text{ (mh. per n.m. loop)}. \quad . \quad (7)$$

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* "E.T.Z.," 25, 1904, p. 160.

The more extended formulæ of Larsen take account of the flux between the turns of iron wire, which need not be in contact, and make allowance for the magnetic resistance introduced by the snapping-across of the tubes of force from one ring to the next. On account of the uncertainty regarding μ, the importance of these formulæ lies chiefly in the light they throw on the manner in which L varies with the dimensions of conductor and whipping. It will suffice to take account of space between the turns by multiplying the first term in (5) by $2r/(2r+a)$ where a is the axial distance apart of successive turns.

Example. In the Elba-Piombino Telephone Cable of 1910,

$$d=2{\cdot}1, \qquad r=0{\cdot}1$$
$$D=6{\cdot}5, \qquad d'=d+4r=2{\cdot}5,$$
$$d_m=\frac{d+d'}{2}=2{\cdot}3,$$

where all dimensions are in millimetres.

From (6), taking μ as 120,

$$L=6{\cdot}57+0{\cdot}57$$
$$=7{\cdot}14 \text{ mh. per km. loop.}$$

If there are only 40 turns per centimetre of conductor instead of a maximum of 50, then $2r/(2r+a)=40/50=0{\cdot}8$

and
$$L=6{\cdot}57\times0{\cdot}8+0{\cdot}57$$
$$=5{\cdot}83 \text{ mh. per km. loop.}$$

As the first term of (5) is much the greater of the two considered, it is seen that L is very nearly (a) directly proportional to μ, and (b) to the radius of the iron wire, but (c) is inversely proportional to the mean diameter of the layer of wire.

If there were no iron whipping, the self-inductance of the return circuit, supposing two cores laid up side by side and touching, would be (from (9), Chap. II.),

$$L=10^{-1}\left[4\log_e\frac{2D}{d}+1\right]$$
$$=0{\cdot}829 \text{ mh. per km.}$$

The self-inductance of a single conductor would be (from (11), Chap. II.),

$$L=0\cdot2\left[\log_e \frac{4\times10^5}{d}-\tfrac{3}{4}\right]$$

$$=2\cdot742 \text{ mh. per km.}$$

The mutual inductance of the two conductors would be (from (7), Chap. II.),

$$M=0\cdot2\left[\log_e \frac{2\times10^5}{D}-1\right]$$

$$=2\cdot327 \text{ mh. per km.}$$

Now $$L_{(\text{loop})}=2L_{(\text{single core})}-2M$$

$$=0\cdot830 \text{ mh. per km.}$$

which agrees with, and forms a check on, the value found just above by direct calculation.

The effect of the single iron whipping is seen to be to increase the inductance from 0·83 to 5·83 mh. per km. of loop. It is because the mutual inductance of the two conductors is so much reduced by the shielding action of the iron whipping that the self-inductance of the return circuit may be considered as twice that of the single, whipped, conductor.

Calculation of Effective Resistance of Core.

As the current in the conductor varies it produces proportional variations in the magnetic flux in the iron, the magnetisation being supposed to be so weak that the permeability is sensibly constant. These variations in flux give rise to eddy currents in cylindrical paths coaxial with the iron wire. It is shown by Larsen that these eddy currents do not appreciably distort the magnetic field, which may be regarded as uniformly distributed over the cross-section of the iron wire.

When current and magnetic flux are both increasing as indicated by the larger arrows in Fig. 40, the direction of the eddy currents round the iron wire is indicated by the smaller arrows. Consider, now, any elementary hollow cylinder of current, of which the inner radius is x, the thickness of wall is dx, and the length is πd_m. The resistance of this cylinder to the current in it is

$$\frac{2\pi x \cdot \rho}{\pi d_m \cdot dx} = \frac{2x\rho}{d_m \cdot dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where ρ is the specific resistance of the iron in C.G.S. units. The E.M.F. produced by the variation of the flux inside the elementary tube is

$$E_x = \frac{-d\Phi_x}{dt}.$$

Now Φ for one turn has been shown to be (3)

$$\frac{4\pi C\mu r^2}{d_m},$$

and, therefore, if the flux is uniformly distributed over the

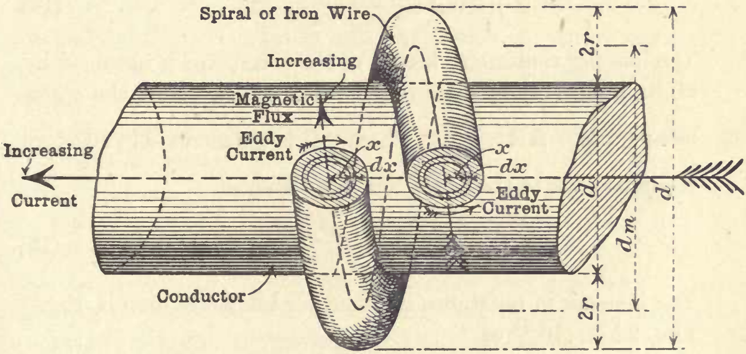


FIG. 40.—EDDY CURRENTS IN WHIPPED CONDUCTOR.

cross-section of the iron, the part of the flux inside the elementary cylinder must be

$$\Phi_x = \frac{4\pi C\mu r^2}{d_m} \times \frac{x^2}{r^2} = \frac{4\pi x^2 \mu C}{d_m}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Putting $C = C_{max}.\sin pt$, it follows that

$$E_x = \frac{-p \cdot 4\pi x^2 \mu}{d_m}.C_{max}.\cos pt. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The elementary eddy-current dI_x which (11) produces is obtained by dividing it by (9)—

$$dI_x = \frac{-p \cdot 2\pi x \mu dx}{\rho}.C_{max}.\cos pt. \quad . \quad . \quad . \quad (12)$$

The loss in watts in the elementary cylinder is obtained by taking the mean value of the product of (11) and (12) over a whole period.

Thus, $$dW_x = \frac{1}{T}\int_0^T E_x\, dI_x \,.\, dt.$$

$$= \frac{8\pi^2 p^2 x^3 \mu^2 dx}{\rho d_m} \cdot \frac{1}{T}\int_0^T C^2_{\text{max}} . \cos^2 pt \,.\, dt$$

$$= \frac{8\pi^2 p^2 x^3 \mu^2 dx}{\rho d_m} \cdot \frac{C^2_{\text{max.}}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence the total loss over all the cross-section of the iron wire is

$$\int_0^r dW_x = \frac{2\pi^2 p^2 \mu^2 r^4}{\rho d_m} \cdot \frac{C^2_{\text{max.}}}{2} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The loss per centimetre length of the conductor is obtained by multiplying (14) by $1/2r$ if the wires touch. Now the watts lost per cm. $= \Delta R \times \frac{C^2_{\text{max.}}}{2}$, where ΔR is the increase in effective resistance due to the eddy currents. Hence

$$\Delta R = \frac{2\pi^2 p^2 \mu^2 r^4}{\rho d_m \,.\, 2r} = \frac{\pi^2 p^2 \mu^2 r^3}{\rho d_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The increase in resistance in ohms per kilometre loop is, therefore, $2\Delta R \times 10^{-13}$ or

$$R_{\text{eff}} - R_0 = \frac{2 \times 10^{-13} \pi^2 p^2 \mu^2 r^3}{\rho d_m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where ρ is in ohms per centimetre cube, and r and d_m are in centimetres.

In the example previously given, take μ as before to be 120, and let $\rho = 13$ microhms per centimetre cube.

Then $$R_{1000} - R_0 = \frac{2 \times 10^{-13} \pi^2 \times (6283)^2 \times (120)^2 \times (0{\cdot}01)^3}{13 \times 10^{-6} \times 0{\cdot}23}$$

$$= 0{\cdot}375 \text{ ohm.}$$

If there are only 40 turns per centimetre instead of a possible maximum of 50 turns, then $R_{\text{eff.}} - R_0$ will be reduced to $0{\cdot}375 \times 0{\cdot}8 = 0{\cdot}300$ ohm. The effective resistance of the whipped core at various frequencies, as calculated from (16), is contained in Table VI.

Table VI.—*Increase in R due to Eddy Currents. 40 turns of 0·2 mm. Wire per Centimetre.*

n(p.p.s.)	0	200	750	1,000	2,000
$R_n - R_0$	...	0·01	0·17	0·30	1·20
R_n (ohms per kilometre loop)	13·20	13·21	13·37	13·50	14·14

Design of a Continuously-loaded Cable.

The formulæ which have just been obtained will be found serviceable when it is required to arrive at a suitable design for a continuously-loaded cable. Thus, (a) from (16), it is seen that the eddy current increase in effective resistance varies as the square of the frequency ($n^2 = p^2/4\pi^2$). Although the formulæ do not admit of the calculation of hysteresis losses, they enable these losses to be separated from the eddy current losses. The hysteresis losses may be taken to be independent of the speed with which the cycle is performed, and therefore as the same for every cycle and proportional to the number of cycles per second, *i.e.*, to n. By making a series of measurements of effective resistance at various frequencies and obtaining an empirical equation for the curve connecting them in the form $y = a + bn + cn^2$, the components of the increase in resistance may be traced to their respective sources. Since the hysteresis loss depends on the degree of magnetisation care must be taken not to use too great a testing current. In measurements on coils the limit is frequently set at 1 milliampere, and if the same limit be used in tests on lengths of loaded cables no serious hysteresis losses need be anticipated.

Again, (b), the eddy current loss is inversely proportional to the resistance of the material forming the whipping. This suggests the use of an alloy which shall combine the qualities of high permeability for low magnetising forces with high resistivity.* In the selection of such a material consideration should be given to the deterioration which it must undergo

* *See* N.P.L. report, 1914.

in the processes of whipping. In other words, the materials must be compared when in position on the wire and not before.

Furthermore, (*c*), the eddy current loss varies from (16) as the cube of the radius of the iron wire. It is therefore essential to use fine wire for the purpose of whipping. The limit of size is determined by the cost of the wire, and in the case of alloys by the difficulty of drawing the wire down to the required diameter. The thicker the wire the less it may be expected to suffer from stretching, and the more from bending round the conductor. In the case of soft iron wire 0·2 mm. (8 mil) is usually taken as a suitable diameter, and somewhat more in the case of alloys (0·3 mm. to 0·4 mm.). Supposing a multiple head to be placed on the covering machine, so as to put on all whippings simultaneously, the rate of forward travel of the conductor is proportional, and the time required is inversely proportional, to the diameter of the iron wire used in whipping, assuming the same peripheral speed for the bobbins.

Again, (*d*), both inductance and eddy current losses are inversely proportional to the mean diameter of the iron whipping. The steady resistance of the conductor is, on the other hand, inversely proportional to the square of its diameter. If the diameter be doubled, the inductance is only halved, or may be greater than half, since the wire suffers less the greater the diameter of the conductor round which it is bent, while at the same time the resistance is reduced to one-quarter. The heavier the conductor the greater is the proportion which the added inductance bears to the resistance, but the greater is also the proportion which the increase in effective resistance due to eddy currents bears to the steady resistance.

Finally, (*e*), the inductance is proportional to the permeability of the iron, but the eddy current loss to the square of the permeability. Hence an increase in permeability such as, for example, might be effected by annealing the wire after winding it on the conductor, might not be altogether beneficial, since the accompanying increase in the effective resistance would tend to counteract the improvement produced by the increased inductance. Such would be especially the case with cables of heavy type and low resistance, where the effective resistance must be kept down.

The best design for the core of a continuously-loaded cable can only be reached after a careful study on the lines indicated of the results of accurate measurements on a range of suitably chosen short sample lengths of core which are representative of what can be attained with the available materials in practical working conditions.

Particulars of some Continuously-loaded Telephone Cables.

In Tables VII. and VIII. are brought together the particulars available regarding a few typical cables which are loaded on the continuous plan. Most of the early cables are laid round the coasts of Denmark, in which country this form of loading had its origin.

The electrical data in Table VIII. are not, strictly speaking, comparative, inasmuch as they were obtained by different observers in various conditions of the cables, and by tests which are not always free from objection. Thus (1) there is no mention of the magnitude of the testing current, although the effective resistance is in great measure dependent on the degree of magnetisation of the iron whipping; nor (2) is, with one exception, any value given for the leakance, a quantity the importance of which had only begun to be realised at the time when these measurements were made. Nevertheless they form an excellent starting point for the design of a new cable, which may be obtained frequently by slight modification of that which comes nearest in electrical values to the requirements. Moreover, certain general conclusions can be drawn from a consideration of the tables. It is clear, for example, that the effective capacity is in every case considerably less than the capacity measured in the ordinary way by single charge and discharge. This difference might be put down to lack of complete penetration by the sea-water to the gutta-percha covered core, but the same effect is noticeable in the lead-sheathed cables. It is not observed in the plain cable, and must be attributed to some shielding action of the iron

Table VII.—*Mechanical Data*

Cable.	Year of laying.	Length n.m.	No. of cores.	Copper *d* in.	Iron wire, mils.	Dielectric D in.	Copper lb. per n.m.	Iron lb. per n.m.	Dielectric lb. per n.m.	References.
Elsinore–Helsingborg.	1902	2·85	4	(0·094)	1×7·87	0·323	(127)	(34)	(191)	"Journ. Télég.," Vol. 29, 1905, p. 187.
Fehmarn–Laaland No. 1......	1902	10·38	4	(0·159)	1×11·81	...	(364)	(83)	...	Impregnated paper, lead sheathed.
Greetsiel (Emden)–Borkum	1903	15·85	4	(0·094)	1×11·81	...	(127)	(58)	...	Air space paper, lead sheathed.
Cuxhaven–Heligoland.......	1903	40·50	2+2/2	(0·154)	1×11·81	...	(437)	(91)	...	Solid paper, lead sheathed.
Seeland-Samsö-Jutland ...	1904	8·93 +11·02	4	(0·129)	3×7·87	0·354	285	(168)	(195)	Copper centre of 0·089 surrounded by three strips each 0·094 in. ×0·020 in.
Korsör–Nyborg	1906	10·30	8	(0·129)	3×7·87	0·354	(285)	(168)	(195)	"E.T.Z.," Vol. 29, 1908, p. 586.
Along coast of Iceland & between Faroe Islands	1907	...	2	(0·112)	1×19·7	0·315	(182)	(118)	(159)	"E.T.Z.," *Ibid.*
Fehmarn–Laaland No. 2 ...	1907	10·41	4	0·125	3×7·87	...	(262)	(163)	...	Air-space, paper, lead sheathed. Copper centre of 0·085 in., with three copper strips each 0·094 in. ×0·020 in. "E.T.Z.," *Ibid.*
Elba–Piombino................	1909	7·10	4	0·084	1×7·87	0·255	101	28	116	—
French P.O. Channel.........	1911	23·42	4	0·131	1×12·0	0·412	300	(78)	(300)	"Lum. Elect.," 1912, p. 323. Centre of 0·106 in., with five copper strips.
Adriatic, Vienna, Dalmatia circuit	1913	31·80	4	(0·104)	3×7·87	0·331	(200)	(140)	(180)	"E.T.Z.," Vol. 34, 1913, p. 295, copper tapes, special gutta.
Vancouver–Victoria..........	1913	26·30	4	0·139	1×12·0	0·414	300	(83)	300	THE ELECTRICIAN, Vol. 71, 1913, p. 822. Centre of 0·082 in., with 12 outers of 0·0285 in.

The figures in parentheses are calculated and approximate. The weights refer to single-cores.

whipping. The difference between the capacities increases with the frequency.

In Fig. 41, Curve A shows the dependence of the effective

Table VIII.—*Electrical Data.*
All n.m. loop values.

Cable.	P.p.s. n.	Ohms.		Mfds.		Mh. L.	Micro-mhos G.	a	Ohms. Z_0.
		R_0.	R_{eff}.	K_0.	K_{eff}.				
Elsinore–Helsingborg .	900	16·98	17·58	0·162	0·152	9·84	...	0·0340	(260\8°46)
Fehmarn–LaalandNo.1	900	6·36	9·58	0·151	0·133	9·30	...	0·0184	(267\5°10)
Greetsiel (Emden)-Borkum	900	18·06	22·16	0·069	(0·062)	14·82	...	0·0225	(497\7°24)
Cuxhaven–Heligoland.	900	5·06	6·76	0·085	(0·075)	7·96	...	0·0104	(328\4°16)
Seeland–Samsö–Jutland	900	7·96	8·36	0·250	0·223	16·16	...	0·0156	(270\2°37)
Korsör–Nyborg.........	765 1,225	7·98	8·65 9·42	0·254	0·237	17·3	...	0·0199 0·0245	(271\2°58) (271\2°1)
Along coast of Iceland and between Faroe Islands	...	...	...	...	...	...	...	...	...
Fehmarn-LaalandNo.2	796 955	8·92	...	0·087	0·077	18·2	...	0·0124 0·0134	... ...
Elba–Piombino.........	1,000	22·45	24·88	0·172	0·169	9·76	...	0·0492	(250\11°3)
French P.O. Channel...	1,000	(7·75)	8·53	...	0·176	13·5	19·1	0·0184	(278\2°23)
Adriatic, Vienna, Dalmatia Circuit	Good speech through 2 × 5 9 km. of cable.								
Vancouver–Victoria ...	800	8·0	...	...	0·175	...	...	0·019	...

The figures in parentheses are calculated and approximate.

resistance of the core on the frequency, for one of the cables; Curve B the variation with frequency of a, as calculated from the formula $a=(R/2)\sqrt{K/L}$, neglecting leakance; and Curve C the actual measured values of a.* Curve C is steeper and runs at a higher level than Curve B. The difference between the two may be ascribed in greater part to the dielectric losses.

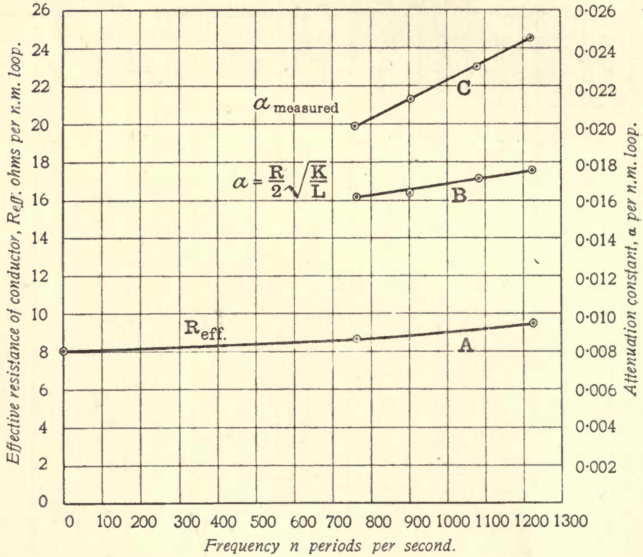


FIG. 41.—KORSÖR–NYBORG CONTINUOUSLY-LOADED CABLE.

Curve A = Variation of effective resistance of core with frequency.
,, B = Calculated attenuation constant, neglecting leakance.
,, C = Attenuation constant, as measured.

COIL LOADING.

Instead of attempting to distribute the loading uniformly throughout the length of the cable, the cable may be cut and the loading may be inserted discontinuously in lumps at intervals which are not too great. This method was readily applicable in its early stages to the loading of lines on land where the junctions are easily accessible. Now, the telephone circuits of this country are necessarily of moderate length, and

* "E.T.Z.," Vol. XXIX, 1908, p. 588.

at the time when this form of loading was first proposed the need for loading was not pressingly felt; moreover, the opinions of authorities were unfavourable to its introduction here.* It was left, therefore, for other countries to take up these proposals, to demonstrate their soundness by experiment, and to carry them out in practice.† After coil loading had been proved a success on land the further step was taken of applying it to submarine cables. If the difficulties in the way of inserting coils in cables are great the benefits of loading are not less great, because of the distortion and attenuation which even a moderate length of cable produces.

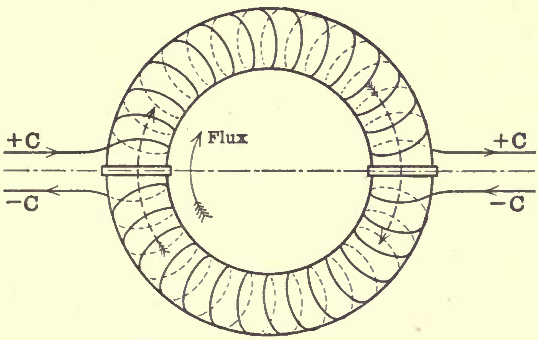


FIG. 42.—DOUBLY-WOUND COIL IN TWIN CIRCUIT.

The insertion of loading coils necessarily increases the conductor resistance, but if the time-constant (L/R) of the coils is high the effect of the added inductance should more than compensate for the increased resistance. To secure high time-constant the coils are wound on an iron core. In order that the eddy-current loss in the iron may be as small as possible, the iron core is finely subdivided and consists of fine iron wire. Since at any point of a telephone circuit the currents in the

* O. Heaviside, THE ELECTRICIAN, Nov. 3, 1893.

† M. I. Pupin, "Trans.," Am.I.E.E., 17, 1900, p. 245. G. A. Campbell, "Phil. Mag.," 5, p. 313. The method of shunt or transformer loading proposed by S. P. Thompson (THE ELECTRICIAN, Vol. XXXI., 1893., pp. 439,473) does not seem ever to have been put to the test of experiment.

twin conductors are equal and in phase, the two coils of the separate conductors may be combined and wound on the same iron core,* as in Fig. 42. In this way the inductance of the circuit is increased by the mutual inductance between the windings, or the same inductance can be secured for a lower effective resistance.

Calculation of the Inductance of a Loading Coil.

In order to facilitate the design of loading coils it is desirable to obtain a formula which will express the inductance of a coil in terms of its dimensions. The design of a loading coil for submarine cables is based on somewhat different considerations from those which guide the design of loading coils for land lines.

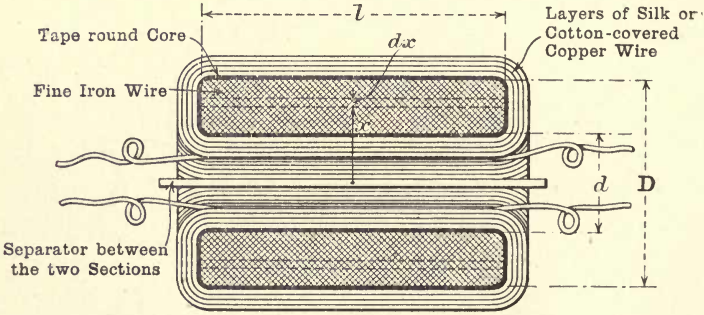


Fig. 43.—Longitudinal Section through Loading Coil.

For land use the space occupied is a minor consideration, and the coil is designed to produce the greatest electrical efficiency without regard to size. In the submarine cable the coil has to pass within the sheathing wires. Space considerations are therefore of paramount importance, and electrical efficiency has to be sacrificed accordingly.

Let D be the outer diameter of the iron cylinder forming the core, d the inner diameter, and l the length, as in Fig. 43. If the core is uniformly wound the lines of magnetic force are circles with centres on the axis of the cylinder. A circle drawn outside the coil is not linked with the windings: the line

* M. I. Pupin, British Patent 16,529, of 1902.

integral of the magnetic force round the circle is consequently zero, and the magnetic force along the circumference is everywhere zero. It follows that the magnetic flux of a symmetrically-wound coil with core of cylindrical shape is entirely confined to the interior of the coil. For submarine coils surrounded by sheathing wires it is important that there should be no stray flux to cut the sheathing wires, and there set up eddy currents which would lead to great increase in the effective resistance of the coil.

Consider the elementary cylinder of core, shown dotted in Fig. 43, of inner radius x, height l, and thickness dx. The magnetic conductivity is $\mu l dx/2\pi x$. Now the magnetomotive force within the core is $4\pi NC$ where N is the number of turns in the winding. Hence the flux through the cylindrical ring is

$$\Phi_x = 4\pi NC \times \mu l dx/2\pi x$$
$$= 2NC\mu l dx/x.$$

The total flux is

$$\Phi = \int_{d/2}^{D/2} \Phi_x dx = 2NC\mu l \log D/d.$$

Hence $L = N\Phi/C$, since there are N turns linked with the flux,

$$= 2N^2 \mu l \log_e D/d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In this formula no account has been taken of any insulation on the iron wire, the effect of which is to reduce the sectional area in the proportion $\pi n^2 \div (2n + 2\tau)^2$, where r is the radius of the iron wire and τ the radial thickness of the covering. When this correction is made

$$L = 2 \times 10^{-9} \times N^2 \mu l \cdot \frac{\pi n^2}{(2n+2\tau)^2} \log_e D/d, \quad . \quad (19)$$

where L is in henries and l in cm.

The Design of a Loading Coil.

As the formula for L depends on a number of independent variables, the best shape and size of coil are most suitably determined by experiment. A series of coils should be constructed showing a range of variations in d, D, l and N, from the results of tests on which L may be plotted against any chosen independent variable. Certain working rules may nevertheless be deduced from a general consideration of the formulæ.

(1) Let μ, D and d be fixed, and suppose that N and l vary. If for simplicity the winding be supposed to consist of a single layer, its resistance will be approximately proportional to

$$R=N(2l+D-d). \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now
$$L=2N^2\mu l \log_e D/d,$$

and therefore by substitution for l from (20)

$$=\mu N[R-N(D-d)] \log_e D/d.$$

Hence for given R, L is a maximum when $dL/dN=0$.
This condition gives $2N(D-d)=R$,

and, from (20),
$$l=\frac{D-d}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The resistance of the subsequent layers will be greater in proportion than that expressed by (20). But as a first approximation (21) may be taken to hold, and as a consequence the cross-section of the core should be square.

A core of square cross-section would be quite suitable for land lines, where space is a secondary consideration, but not for insertion in cables, as it would be too bulky. A compromise must be attempted, and the core must be drawn out axially just so far as is rendered necessary by the mechanical problem of inserting the coil within the sheathing wires. Any departure from the short squat form of coil is attended with sacrifice of electrical efficiency, and should it be possible to relax the mechanical conditions by any means the result would be an improved coil and a better cable.

The outer diameter D of the core is decided by the greatest permissible overall diameter of the coil. The inner diameter d is limited by the necessity of drawing all the turns of wire symmetrically through the heart of the coil. Too small a ratio $d:D$ would mean too unequal a distribution of the magnetic force, and an intensification of the iron losses in the interior of the ring. The number of turns N and the size of the winding wire are regulated by the increase which they produce in the bulk of the coil, and by the proportion which the resistance of the last turn of wire bears to the first. The size of the winding wire should not exceed, say, No. 20 on account of skin effect; wire of smaller size would not be so robust, and, moreover, more space would be occupied by insulation.

(2) Formula (19) shows how important it is that the insulating covering of the iron wire used to form the core should be as thin as possible. Take, for example, No. 42 wire, of diameter 4 mils, and suppose it to be covered with silk to bring up the diameter to $5\frac{1}{2}$ mils. Then $2r=4$ mils and $2r+2\tau=5\frac{1}{2}$ mils. L is therefore reduced in the proportion $(2r)^2 : (2r+2\tau)^2$ or 64 : 121. In other words the inductance of the coil would be halved by putting even a thin layer of silk insulation on the iron wire. The thinner the wire the greater proportionately is the space that is occupied by insulating material. A material which possesses advantages in this connection is some form of insulating enamel, but its use would be an obstacle in the way of subsequent annealing of the core.

Effective Resistance of a Loading Coil.

Formulæ may be obtained for the effective resistance of a loading coil by a method similar to that which was employed in the case of the whipped conductor. Thus in (14) the loss in watts in a single turn of iron wire of mean diameter d_m has been shown to be

$$W=\frac{2\pi^2p^2\mu^2r^4}{\rho d_m}\cdot\frac{C^2_{\text{max.}}}{2},$$

where ρ is the specific resistance of the material. In the coil of Fig. 43 there are N turns in the copper winding and $l/(2r+2\tau)$ turns of iron wire in every layer of the core, while d_m extends from d to D.

Hence the total loss in the iron

$$=\frac{l}{2r+2\tau}\int_{d/2}^{D/2}\frac{2\pi^2p^2\mu^2r^4}{\rho d_m}\cdot\frac{d\left(\frac{d_m}{2}\right)}{2r+2\tau}\cdot\frac{N^2C^2_{\text{max.}}}{2}$$

$$=\frac{\pi^2p^2\mu^2r^4l}{\rho(2r+2\tau)^2}\log_e D/d\cdot\frac{N^2C^2_{\text{max.}}}{2}$$

and $$\Delta R=\frac{10^{-18}\pi^2p^2\mu^2r^4lN^2}{\rho(2r+2\tau)^2}\log_e D/d, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where R is in ohms, ρ in ohms per centimetre cube, and l in centimetres.

It follows, using (19), that

$$\Delta R = \frac{10^{-9} p^2 \mu \pi r^2}{2\rho} . L. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For a given value of L, the eddy current loss measured in ohms of increased coil resistance depends on p, μ, r, and ρ. Hence, at a given frequency, ΔR depends only on μ, r and ρ, that is to say, on the permeability, gauge and resistance of the iron wire used to construct the core. In order to reduce the eddy-current loss to a minimum, attention must, therefore, be concentrated on the iron wire, as the loss, for a given value of the inductance, is entirely independent of the shape and size of the coil. High permeability is desirable in order that L may be great; but r and ρ are at disposal independently of L. In other words, the gauge of the wire should be as small as is practical, with the limitations set forth above of cost and insulation, and the resistivity of the material should be as high as it is possible to obtain. Any improvement in these directions in the state of the iron would make itself felt at once in the time constant of the coil, and therefore in the efficiency of the cable for a given cost, or in the cost of the cable for a given speaking value.

The Lake Constance Coil Cable.

The first coil-loaded cable to be laid under water was that across Lake Constance in the year 1906. The length of the cable is approximately 12 km. The depth at which it lies is great, and reaches a maximum of 250 metres, or 140 fathoms, the pressure corresponding to which is about 25 atmospheres. As provision required to be made for seven twin circuits, a lead-covered cable was decided upon, and to enable the lead cover to resist the hydraulic pressure, a spiral of steel wire was placed inside it. The cable was then protected and sheathed with iron wires in the usual manner. At the place where the coils are inserted the cable is unavoidably made larger, and the coils, of anchor-ring shape, are placed so as to be coaxial with the cable, being separated from each other by rubber buffers.* The transition from the cylindrical part of the coil container to the cylindrical part of the cable is attained through the intermediary of a conical part, which at the same time forms a stopper to prevent the penetration of water in the case of a fault; this is effected by completely filling up the conical part and replacing the hygroscopic paper covering of the conductors with rubber. The coil pieces are inserted at every 500 m. and contain alternately three and four coils. The replacement of the sheathing after the insertion of the coil pieces was carried out partly by machine and partly by hand.

The conductor (of strips) has a sectional area of 1·77 sq. mm. and is equivalent to a solid conductor of 1·5 mm. diameter (d=0·059 in. or 64½ lb. per n.m.). The resistance specified for the conductor was 20 ohms per kilometre loop; and for the coil, 20 ohms effective, making a total of 40 ohms. The combined resistance actually attained was 33·5 ohms per kilometre loop (62·1 ohms per n.m. loop). The other electrical constants are contained in Table IX.

Table IX.—*Electrical Data. All per n.m. loop.*

Cable.	n.	R coil +cable).	K.	L.	G.	a.	Z_0.
Friedrichshafen-Romanshorn	900	62·1	0·072	0·39	...	0·0134	$(2328\backslash\overline{0°48})$

It is believed that this cable is no longer in operation.†

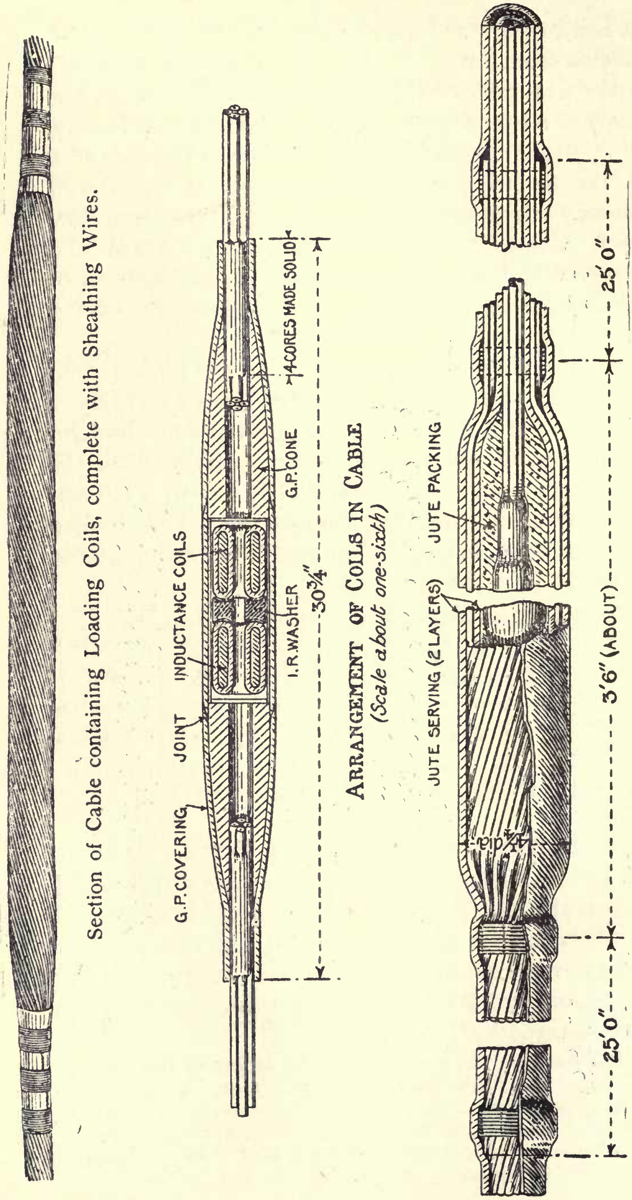
The Anglo-French Channel Cable.

The first coil-loaded submarine telephone cable was that laid in 1910 across the English Channel between Abbot's Cliff and Grisnez.‡ This cable is 21 n.m. in length, and contains four gutta-percha covered cores. These consist of 160 lb. of copper per nautical mile, with 300 lb. of gutta-percha, a type of core

* British Patent, 25,306 of 1906.

† A full description of the manufacture and laying of this cable is given in "E.T.Z.," 28, 1907, p. 661; The Electrician, 59, 1907, p. 217.

‡ For a full description of this cable *see* Major W. A. J. O'Meara. "Journ." I.E.E., 46, 1911, p. 309, and discussion thereon. We are indebted to the "Journal" of the Institution of Electrical Engineers for the illustration of the Anglo-French Cable given in Fig. 44.



Section of Cable containing Loading Coils, complete with Sheathing Wires.

ARRANGEMENT OF COILS IN CABLE
(Scale about one-sixth)

METHOD OF SHEATHING OVER COILS
(Scale about one-sixth)

FIG. 44.—ARRANGEMENT OF LOADING COILS IN ANGLO-FRENCH CABLE OF 1910.

which had been used for previous Continental cables, but is now obsolete. Loading coils of 0·1 henry inductance are inserted at every nautical mile. The method of protecting the coils is shown in Fig. 44. The two coils, one for each of the diagonally-opposite pairs of cores, are contained in a gutta-percha tube. Into the ends of this tube hollow cones of gutta-percha are slipped, and over them the tube is tooled down and united to them and to the four cores which are made solid with gutta-percha.* Each pair of cores passes alternately through the centre of the coil to which it is not to be joined. The coils are wrapped in tinfoil to prevent a gradual transference of moisture from the gutta-percha to the more or less hygroscopic material with which the coil is impregnated, and thus to prevent the gradual fall in insulation from winding to winding which would otherwise occur. The cable sheathing is continued over the coil section, but, owing to the increased bulk of the latter, the sheathing would only cover the coil if the pitch were lessened, which would lead to a loss in strength. Accordingly, a second group of sheathing wires is placed round the coil between the interstices of the first, and these at their ends are caused to overrun the first sheathing wires to which they are securely lashed. The whole arrangement is designed to combine strength with flexibility.† Further particulars are contained in Tables X. and XI.

Table X.—*Mechanical Data.*

Cable.	Date.	Length, n.m.	No. of cores.	Copper, *d* in.	G.P., D in.	Copper, lb. per n.m.	G.P., lb. per n.m.
Anglo-French	1910	20·0	4	0·106	0·390	160	300

Table XI.—*Electrical Data. All per n.m. Loop.*

—	R_0, ohms.	R_{750}, ohms.	L_{750}, millihenries.	K, microfarads.	*a*.	Z_0.
Cable ...	14·95	14·95	2·0	0·138	0·0166	$858\backslash 0°54$
Coil	2·25	6·0	100·0	...	...	...

* *See* Brit. patent 25,306, of 1906.

† Brit. patent 5,547, of 1907.

With the completion of this cable the difficulties, which might have seemed insuperable, in the way of coil loading in submarine cables were all overcome. It should be observed that the responsibility for the design as well as for the success of the cable lay with the contractors. The greatness of their achievement can be appreciated to the full perhaps only by those who are actually engaged in the design, manufacture and laying of submarine cables.

The attenuation to be attained was limited by the stipulation that the volume of speech transmitted should be at least equal to that through one-seventh of the same length of standard cable. Taking the attenuation constant of standard cable at 750 p.p.s. as 0·1187, the maximum attenuation constant allowed works out at 0·0169. The value of the attenuation constant given in the table can be taken as only approximate, as the methods of testing by which it was obtained were not free from objection. Alternative tenders were invited for continuous loading, but the proportions of copper to gutta-percha (160/300) are not suited to the most economical design of continuously loaded cable. The reason given for the restriction was that this was the type of core which had been employed in the previous telephone cable, and that should the coils fail and require to be cut out the resulting plain cable would be as good as the existing one. Nevertheless, in that case, as will be shown, a much cheaper cable, say of type 107/150, lightly loaded on the continuous plan would have given better transmission. In the later cables the type 160/150 has been adopted.

Superimposing.

By a suitable arrangement of the end apparatus, three telephonic circuits can be made out of two. In Fig. 45 the circuits A and B are loop circuits, to which the transmitting and receiving apparatus is coupled up through transformers. If, now, the conductors of the circuits A and B are joined in parallel to apparatus at SS, so that each pair forms one of the conductors of a new circuit, C, it is clear that conversation may be conducted simultaneously on all three circuits without mutual interference. The circuits A and B are called the primary or transformer circuits, and the circuit C the superimposed or phantom circuit. In practice it is found necessary to insert transformers also at SS in order that the balance may be maintained with sufficient exactitude.

In a plain or continuously-loaded submarine telephone cable the superimposed circuit has half the resistance and twice the capacity of one of the primary circuits, and therefore its attenuation constant is the same as that of a primary circuit.

From Figs. 42 and 45 it is clear that, while the current in the primary circuit produces fluxes in the same direction in

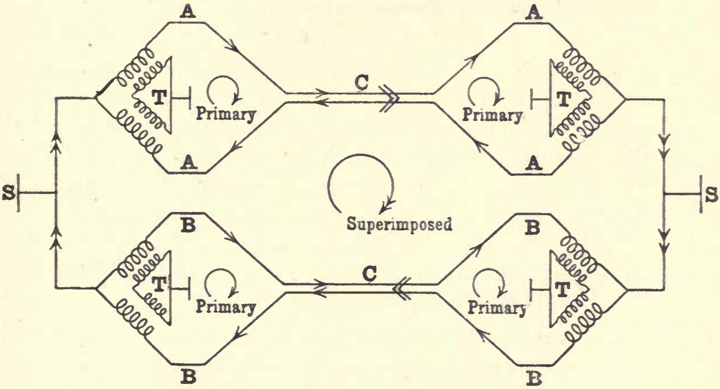


FIG. 45.—SUPERIMPOSED WORKING TWO CIRCUITS ARRANGED TO FORM THREE.

A and B=Primary or transformer circuits, currents $\gtrless$
C=Superimposed or phantom circuit, current >> <<

both halves of a loading coil, on the other hand the current in the superimposed circuit, in which the windings are in parallel and not in series, produces fluxes in opposing directions in the two halves, which therefore tend to neutralise each other. If it is desired to form a loaded superimposed circuit from two coil-loaded primary circuits, special coils must be inserted for the purpose.

In Fig. 46, coils A and B are ordinary loading coils. Their windings are joined in parallel, and then used as though they were a single conductor to wind coil C. The object of dividing the windings into quarters over this coil is to secure a more symmetrical distribution of the field.* A loading coil for the superimposed circuit requires careful balancing in order that there may be no stray magnetic flux, which would enormously increase the effective resistance of the coil. If the loading is to be of the same proportions in the superimposed circuit as in the primary circuits, the resistance and inductance of the superimposed coil should be half that of a primary coil. The superimposed coil is unavoidably somewhat bulkier than the primary coils. All three may be placed beneath the same gutta-percha covering in submarine cables.

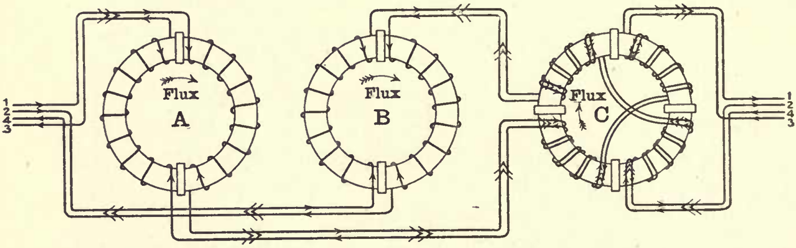


FIG. 46.—ARRANGEMENT OF COIL WINDINGS FOR SUPERIMPOSED WORKING.

A, B=Coils in primary (transformer) circuits, currents ≶
C=Coil in superimposed (phantom) circuit, current >> <<

Although by superimposing an extra circuit is gained, which is especially advantageous in a submarine cable where the number of circuits is strictly limited, there are certain disadvantages which must be borne in mind. Thus (*a*) the speaking values of the circuits are reduced by the transformer losses ; (*b*) in the case of coil-loaded cables the speaking value of the primary circuits is further reduced by the ohmic resistance of the coils required to load the superimposed circuit ; and (*c*) unless the conditions are quite symmetrical, there is danger of overhearing between the superimposed and the

* M. J. Pupin, Brit. pat. No. 16,529, 1902.

primary circuits. The superimposed circuit is highly sensitive to disturbance, which renders it difficult to maintain on land lines; but this objection does not hold to the same extent on cables, although there the dangers of overhearing are greater.*

THE ANGLO-BELGIAN COIL CABLE.

This cable was laid in the year 1911 between St. Margaret's Bay (Dover) and La Panne (Belgium). It is a four-core cable of length 50 n.m. The copper is seven strand to a weight of 160 lb. per nautical mile. The cable differs from the Anglo-French mainly in three respects :—

1. The weight of gutta-percha was reduced to one-half, from 300 lb. to 150 lb. per nautical mile.

2. The limiting valve of the attenuation constant was to be "such that none of the circuits should have a standard-cable equivalent of more than 9 miles, excluding such losses as can be eliminated by adding lengths of standard cable to each end of the loaded loop."

3. The loading scheme was to be designed so as to permit of loaded superimposed working.

The losses referred to may be taken to be terminal losses, although the addition of standard cable at the ends would not eliminate them. Neglecting end effect and taking the attenuation constant of standard cable as 0·1030 per statute mile at 750 p.p.s., the attenuation constant per nautical mile at 750 p.p.s. of the new cable was consequently to be not greater than

$$\frac{0{\cdot}1030 \times 9}{50} = 0{\cdot}01853.$$

The attenuation constant stipulated for the Anglo-French cable on the same basis was 0·01696 per nautical mile at 750 p.p.s. But only half the weight of dielectric was allowed for the new cable, so that the conditions were not less severe, even without the superimposed loading.

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* For other possible combinations of circuits *see* F. Jacob, Brit. patent No. 231, of 1882.

The electrical tests on this cable resulted as follows*:—

R_0 (conductor alone) (per n.m. loop)	14·2 ohms at 15°C.
K_0 (primary circuit) ,, ,, ,, ...	0·157 mfd.
K_0 (superimposed ditto) ,, ,, ,, ...	0·314 mfd.
L (primary circuit coil)	0·1 henry.
R_0 (inductive and non-inductive coil in primary circuit)...................................	6·6 ohms.
R_{800} (ditto) with 1 milliampere at 15°C.	11·5 ohms.
L (coil in superimposed circuit)	0·05 henry.
R_0 (superimposed circuit coil)	3·3 ohms.
R_{800} (ditto) with 1 milliampere at 15°C...........	4·6 ohms.
K (of coil in primary circuit)	0·004 mfd.
K (of coil in superimposed circuit)...................	0·0001 mfd.

For the core of this cable the usual dielectric conditions were relaxed, with a view to securing specially low leakance, the importance of which in heavily-loaded cables has already been emphasised, and it was stated that alternatively any substitute for pure gutta-percha possessing superior electrical qualities to it for the required purpose, should be permissible, subject to reasonable guarantees as to mechanical strength, pliability and durability being furnished. Accordingly, a special mixture, or so-called "improved" gutta-percha, was prepared, having a specially low leakance. The ratio G/K is given as 12, at 800 p.p.s. and 15°C.

The attenuation constant was measured at different frequencies by means of the Franke machine, and the results are contained in Table XII.

Table XII.—*Variation of Attenuation Constant* (*per n.m. loop*) *of Anglo-Belgian* (1911) *Cable, with Frequency.*

n	$p=2\pi n$.	Primary circuit.	Superimposed circuit.
478	3,000	0·0157	0·0156
637	4,000	0·0165	0·0161
796	5,000	0·0177	0·0173
955	6,000	0·0192	0·0185
1,114	7,000	0·0204	0·0195

The sending-end impedance (Z_0) varied irregularly with the frequency, and averaged about 800 ohms for the primary circuit and 400 ohms for the superimposed circuit. As regards the question of overhearing between the three circuits, it was found to be impossible to avoid overhearing entirely between

* "Post Office Engineers' Journal," V., 1912, p. 55.

the transformer circuits and the phantom circuit, although the transformer circuits relatively to each other are free from it. The amount of overhearing was observed in London when the circuits were looped at the distant end of the cable at La Panne. It was equal in loudness to speech conducted through 60 miles of standard cable, taking 46 miles as the commercial limit of speech.

The Anglo-Irish Coil Cable.

This cable is similar in most respects to that just described. It forms the sea link in the main trunk route between Ireland and the southern half of Britain. It is approximately 64 n.m. in length. The copper conductor is, as before, seven strand, weighing 160 lb. per nautical mile, of d=0·170 in. The dielectric is in three coats to 0·285 in., and weighs 150 lb. per nautical mile. Loading coils are inserted at intervals of 1 n.m., and covered with extra sheathing over a space of 92 ft. The coils are of inductances 0·1 henry in the primary circuit and 0·05 henry in the superimposed circuit. Each completed set of coils was tested before insertion at a pressure of 15 cwt. per square inch. The aerial lines from Dublin and Manchester, to which the cable is connected, are of 600 lb. copper per statute mile. The overhearing between the superimposed circuit and one of the primary circuits was specified, from experience with the Belgian cable, to be not greater than 65 miles of standard cable. Further electrical data are given in Table XIII., and the results of tests on the completed cable in Table XIV.

Table XIII.

—	R_0 cable alone.	$R_{eff.}$ at 5,000, coils.	K mfd.	L henry.	G/K.
Primary circuit	14·2	6·8	0·166	0·100	15
Superimposed ,, ...	7·1	3·2	0·320	0·050	15

Table XIV.

—	$p=2\pi n$.	a	Z_0
Primary	3,000	0·0138	665 −5° 45′
	5,000	0·0150	690 −2° 40′
	7,000	0·0168	695 −2° 40′
Superimposed	5,000	0·0150	446 0° 52′

The efficiency of the coils is stated to be such that the ratio of effective resistance to inductance is 48. It follows, therefore, that $R_{eff.}=4{\cdot}8$ ohms for a 0·1 henry coil, probably at $p=5{,}000$. Subtracting this from 6·8 (Table XIII.), the effective resistance of the coils in the primary circuit, the difference 2·0 ohms represents the steady resistance by which the superimposed coil increases the resistance of the primary circuit. Its effective resistance is 2·4 ohms (48×0·05), which subtracted from 3·2 ohms (Table XIII.) leaves 0·8 ohm as the steady resistance which the primary coils add to the superimposed circuit.

The efficiency of the Anglo-Belgian coils is not stated, but a plausible arrangement of the windings may be obtained as follows. Let a superimposed coil be wound by splitting the wire of the primary coil in two, and reducing the bulk of the iron to make room for the extra insulation required. The single windings of the coils would then be of resistance 1·1 ohms for the primary coil and 2·2 ohms for the superimposed coil, making a total of 3·3 ohms for a superimposed coil and two primaries, and 6·6 ohms for primary and superimposed coil in series.

This improvement in the loading coils would suffice to account for the reduction in the attenuation constant from 0·0177 (Anglo-Belgian) to 0·0150 (Anglo-Irish).

The attenuation constant of a coil cable arranged for superimposed working may be written in the form

$$\alpha=\overset{\text{A}}{\frac{R}{2}\sqrt{\frac{K}{L}}}+\overset{\text{B}}{\frac{R_c}{2}\sqrt{\frac{K}{L}}}+\overset{\text{C}}{\frac{R_s}{2}\sqrt{\frac{K}{L}}}+\overset{\text{D}}{\frac{G}{2}\sqrt{\frac{L}{K}}},$$

where R is the resistance of the core, R_c that of the coil, and R_s that of the superimposed coil. In Fig. 47 curve A is the first term of this summation. Curve B is the second term, drawn as a straight line through the values at 0 and 800 p.p.s., and thus involving the assumption of linear dependence of the coil's effective resistance on the frequency. Curve C is the contribution from the superimposed coil. Cuive D is that due to leakance, the assumption of linear dependence being made in this case also for lack of more definite information. Curves A and B combined give the attenuation of the coil loaded core, and curves A, B and D combined show the influence of leakance. The addition of curve C to the sum of these three shows the increase in attenuation brought about by the insertion of the coil for loading the superimposed circuit.

Curve E connects the actual measured values. It lies somewhat above the calculated values, and appears to tend upwards. The difference may arise from a variety of causes, but without

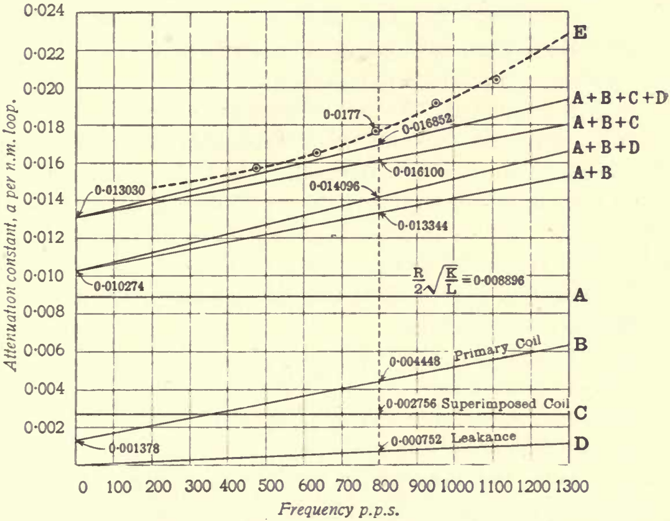

FIG. 47.—SYNTHETIC COMPARISON OF CALCULATED ATTENUATION CONSTANT WITH MEASURED, FOR ANGLO-BELGIAN CABLE.

Curve A = Core alone with inductance.
„ C = Superimposed coil, steady resistance.
„ A+B = Core and primary coils.
„ A+B+C = Do + superimposed coils.
A+B+C+D = Do + leakance.
Curve B = Primary coil, effective resistance.
„ D = Leakance term.
„ E = Measured values.

further investigation it would be impossible to fix them more precisely. The assumptions on which the calculated curve is based may require revision; the testing current in the completed cable may have been greater, at any rate at the sending end, than that used in testing the coils; or there may be some other contributing cause of loss which has not been taken into account. In these circumstances the agreement must be taken as sufficient.

By constructing curves in this way with the best material available we can obtain a good idea of the relative importance of the various parts which go to compose the attenuation constant.

Comparison Between Continuous and Coil Loading.

A matter of great interest, and one which is frequently discussed, is the comparative merit of the two systems of loading. It may be attempted to decide the question on the basis of the formulæ which have been given for the attenuation of a cable core, taking into account the weight and cost of the material required to produce it. Nevertheless the use of these formulæ is hardly desirable for any other than the straightforward purpose for which they were devised. The endeavour to extrapolate from them—or to invert them and deduce best possible values—may lead to impossible results, the suppositions on which they are based being usually true only within narrow limits. For this reason it is perhaps better to discuss the question in general terms.

The purposes for which submarine telephone cables are required are mainly two.

1. To connect outlying districts, especially islands with the main network of telephone land lines.

2. To form the connecting link between the telephone systems of two countries separated by sea.

As an example, take the case of telephonic communication between London and Paris. The construction of this circuit previous to the laying of the first loaded cable in 1910 is shown in Table XV.*

Instead of being of type 160/300, which was adopted in order that the whole circuit might have minimum KR for a given cost, the submarine cable may be replaced by a lighter type, and both may be loaded continuously as in Table XVI.

From Tables XV. and XVI., Table XVII. in constructed. It shows a variety of alternative ways in which the same circuit could be built up.

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* W. H. Preece.

Table XV.—*London–Paris Line.**

297 statute miles.

Aerial.	Submarine.	Aerial.
London–St. Margaret's. 83 statute miles. 400 lb. per statute mile. 2·25 ohms. do. 0·015 mfd. do. Total 186·75 ohms.	St. Margaret's–Sangatte. 21 n.m. 160/300 quad. 7 ohms per n.m. } single 0·27 mfd. do. } core. Total ...... 147 ohms.	Sangatte–Paris. 190 statute miles + a few miles underground. 600 lb. per statute mile. 1·5 ohms do. 0·015 mfd. do. Total ... 285 ohms.

Grand total .... { 619 ohms. 9·765 mfd. 6,044 KR.

Speaking is said to be very good.

Table XVI.—*Estimated Constants of Various Types of Core. All per n.m. loop at 1,000 p.p.s.*

Core.	R ohms.	L millihenries	K mfd.	G micromhos	a
160/300	15·5	1·93	0·128	28	0·057
Do. with one layer of 0·2 mm. iron wire	15·8	3·86	0·137	30	0·048
107/150	23·0	1·84	0·139	30	0·081
Do. with one layer of 0·2 mm. iron wire	23·4	8·81	0·156	34	0·053

Certain interesting conclusions may be drawn from Table XVII. :—

1. If the limit of commercial speech be taken as reached when the attenuation is 2·5, it is clear that all five circuits will give good speech. It is only when it is desired to connect up outlying districts to the ends of the circuit that it becomes necessary to take the fifth type, or coil-loaded cable, or to design an equivalent continuously-loaded cable. The cost of the cable itself may then be small compared with that of the networks which it unites.

2. A cable of type 107/150 with one layer of iron wire will give better results than a cable of type 160/300 not loaded, at

* W. H. Preece.

Table XVII.—*Alternative Forms of Circuit.*

Type of circuit.	Attenuation.				
83 miles of 400 lb. aerial	0·33	0·33	0·33	0·33	0·33
190 miles of 600 lb. aerial	0·53	0·53	0·53	0·53	0·53
21 n.m. of 160/300	1·20	...	...	...	...
Do. with one layer of 0·2 mm. iron wire	...	1·01	...	...	...
21 n.m. of 107/150	...	...	1·70	...	...
Do. with one layer of 0.2 mm. iron wire	...	...	...	1·11	...
G.P.O. specification for 1910 cable	...	...	...	...	0·36
Totals	2·06	1·87	2·56	1·97	1·22
Construction ...	Circuit as it was before 1910	If conductor of sub. cable were whipped.	If conductor of sub. cable were replaced by 107/150	If conductor of sub. cable were replaced by 107/150 whipped.	If conductor of sub. cable (160/300) were coil loaded.

approximately half the cost. It is clearly imperative that *all* telephone cables be loaded in some way or other. The loading may be on the continuous plan, except where extremely low attenuation is necessary, and then coil loading should be used if possible.

3. The improvement produced in the cable of type 160/130 by a single layer of iron wire is not so marked. The reason is that the increase in inductance is less (Table XVI.) because of the greater diameter of the conductor on which the iron is wound. With a light type of conductor the speaking range may be doubled by the addition of a single layer of iron wire, and that without any appreciable increase in the cost of the cable.

4. The insertion of coils necessarily increases the resistance of a cable. Continuous loading, on the other hand, increases both the effective resistance and the capacity of the cable. In the Anglo-Irish cable the ratio of coil inductance to resistance

is given as 0·021. The increase in effective resistance caused by whipping is at present somewhat uncertain in the absence of measurements with limited testing current. Assuming an increase of 1 ohm for an inductance of 19 millihenries per n.m. on a 300 lb. conductor, which would appear to be possible of attainment, it will be seen that the two time constants are not far apart in the present transitional stage of development.

5. For superimposed working continuous loading has one distinct advantage, inasmuch as a continuously loaded four-core cable is available at once for superimposed working, since each core carries its own loading. The superimposed circuit has inductance in the same proportion to its resistance already contained in it. On the other hand, the insertion of coils in the superimposed circuit of a coil-loaded cable lowers by their ohmic resistance the efficiency of the primary circuit. In the Anglo-Irish cable the effective resistance of the coils in the primary circuit is in this way increased to 6·8 ohms, making $L/R=0{\cdot}015$.

6. To secure the best results the core must be specially designed for the kind of loading which it is to undergo. For coil loading resistance is high, otherwise the added coil resistance would play proportionally too great a part, while capacity is low. In the continuously-loaded cable the resistance must be as low as possible, which means a heavy conductor (preferably built up of a solid centre surrounded by copper strips) and a high capacity. The equitable way is to adopt the type of core which is best suited to the kind of loading proposed, then to adjust the loading to give the best result, and finally to compare the costs and attenuation constants so produced.

7. The capacity of the continuously-loaded cable may be brought down to equality with that of the coil cable by increasing the weight of the dielectric covering. Now the dielectric is the most costly part of the cable, and as a consequence the continuously-loaded cable of equal capacity would inevitably be more expensive than the coil-loaded cable. To place a minimum below which the thickness of dielectric must not be reduced may have the effect of cutting continuous loading out of competition. The question reduces itself to

balancing one risk against another. In the continuously-loaded cable the risk is of reducing the weight of dielectric below the limit at which the cable may be manufactured, laid, and, still more, recovered in safety, especially in a tropical climate. In the coil-loaded cable the risk is due to possible weakness introduced by the junctions, which may conceivably lead to trouble, as, for example, in picking up an old cable the sheathing of which is not in the best condition.

8. The cost of the cable cannot be taken as merely that of the core alone. To the cost of the coil cable must be added the cost of the coils, of their insertion, testing and sheathing. When the speed at which cotton and silk covering machines work and the cheapness of their product are remembered, the expense of whipping the conductor of a cable with multi-headed machines need not be great in comparison with the heavy cost of the dielectric covering, given that the manufacturer has the enterprise to put down the necessary plant. The time taken need also not be excessive inasmuch as work may run concurrently with the other manufacturing processes.

9. To sum up: Every telephone cable should be loaded; if short or of light type, continuously; if long and if minimum cost is essential, on the coil principle, unless for reasons of safety, when continuous-loading may be preferred.

Single-core Telephone Cables.

Hitherto the design of submarine telephone circuits has followed the practice on land in providing a metallic return. Now a bifilar cable requires to be made cylindrical with some kind of packing before being sheathed, and it is, therefore, preferable to use four conductors, and obtain two, or three, circuits without much further increase in weight. Nevertheless such a four-core cable if of low attenuation constant would unavoidably be heavier than any telegraphic cable, especially if continuously-loaded, and the problem of laying and repairing it in deep water would be difficult. It has, therefore, been proposed more than once to use a single core for telephone cables. To avoid inductive disturbances arising from power circuits the cable would have two cores for some distance out to sea, where one of the cores would be joined to the sheathing wires, as is the practice with telegraph cables.

The chief disadvantage of this form of construction lies in the fact that the external field is not zero as in a symmetrical bifilar conductor, but cuts the sheathing wires. As a consequence the effective resistance of the conductor is considerably increased. Experiments on whipped conductors show a difference of 40 to 45 per cent. between effective resistance and steady resistance. Allowing 20 per cent. for whipping, there remains 20 to 25 per cent. as the increase brought about by the sheathing. Transformers would be required at the ends to link up the bifilar circuits with the unifilar, and further slight losses would be introduced by them.

Although the single-core telephone cable is so far no more than a proposal, it would appear to possess striking advantages, and may provide a means of extending greatly the range of submarine cable telephony.

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CHAPTER VI.

DISCONTINUITIES AND REFLECTIONS.

Cable of Finite Length—Variation of Impedance with Length of Cable ; Quasi-infinite Cable—Impedance of a Short Cable—Sending-end Impedance of Loaded Cable—Circuit of Three Sections—Conditions for Symmetry ; Single Junction—Cable with Single Land Line Extension—Cable with Extensions at Both Ends—Cable with Apparatus at Ends—Cable with Apparatus Midway—The Pupin Criterion—Calculation of Equivalent Cable—Shunt Loading—Application to Practical Case.

Cable of Finite Length.

Up to the present the cable has been supposed to extend uninterruptedly to an infinitely great distance from the sending-end. It is now necessary to consider how the effects of discontinuities in the cable can be taken into account. In the first place consider what happens when the cable is either free, or is put to earth at the receiving end. Turning to the fundamental equations (15) of Chapter IV., let the cable be cut at distance l, and let the end be *insulated from earth.* Then $C_l=0$, and from the second equation it follows that

$$C_0=\frac{V_0}{Z_0 \coth Pl}. \quad \quad (1)$$

Hence on substitution in the first equation and rearrangement,

$$V_{x\,(\text{end free})}=\frac{V_0 \cosh P(l-x)}{\cosh Pl}, \quad \quad (2)$$

and similarly,

$$C_{x\,(\text{end free})}=\frac{V_0 \sinh P(l-x)}{Z_0 \cosh Pl} \quad \quad (3)$$

If the end be now *put to earth,* $V_l=0$, and

$$C_0=\frac{V_0}{Z_0 \tanh Pl} \quad \quad (4)$$

At the same time,

$$V_{x(\text{end earthed})}=\frac{V_0 \sinh P(l-x)}{\sinh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$C_{x(\text{end earthed})}=\frac{V_0 \cosh P(l-x)}{Z_0 \sinh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The spiral diagrams may be constructed for these two cases, by the help of Table XVIII., in a manner similar to that which was employed for the infinite cable. Suppose that the cable of Chapter IV. without loading, of type 160/150, is cut at 49·2 miles, which from Fig. 35 is seen to be three-quarters of a wave-length from the sending end, where the voltage is 270 deg. behind that at the origin. When the end is insulated the spiral diagram takes the shape which is shown in Fig. 48.

The values of $\sinh P(l-x)$ and $\cosh P(l-x)$ in Table XVIII. are obtained by the aid of the formulæ (Appendix III.)

$$\begin{aligned}\sinh P(l-x)&=\sinh[\alpha(l-x)+i\beta(l-x)]\\&=\sinh \alpha(l-x)\cos\beta(l-x)+i\cosh\alpha(l-x)\sin\beta(l-x)\end{aligned}$$

and

$$\begin{aligned}\cosh P(l-x)&=\cosh[\alpha(l-x)+i\beta(l-x)]\\&=\cosh \alpha(l-x)\cos\beta(l-x)+i\sinh\alpha(l-x)\sin\beta(l-x),\end{aligned}$$

which are then converted into the $r/\underline{\theta}$ form.

Comparing Fig. 48 with Fig. 35 of Chapter IV. it is seen that the voltage at the free end of the cable is double that at the same distance along the infinite cable. That this must be so is evident from formula (2), which may be written

$$V_x=\frac{e^{P(l-x)}+e^{-P(l-x)}}{e^{Pl}+e^{-Pl}},$$

and which on multiplication above and below by e^{-Pl}, gives

$$V_x=\frac{e^{-Px}+e^{-P(2l-x)}}{1+e^{-2Pl}}=\frac{e^{-\alpha x}/\underline{\beta x}+e^{-\alpha(2l-x)}/\underline{\beta(2l-x)}}{1+e^{-2\alpha l}/\underline{2\beta l}} \quad . \quad . \quad (7)$$

When l is great, the denominator is nearly unity, and when $x=l$ the numerator is $2e^{-\alpha l}/\underline{\beta l}$. It follows that when the end is free the voltage is double that at the same distance along an infinite cable. Since $e^{-\alpha(2l-x)}$ is the amplitude of a wave which has travelled a distance equal to the whole length of the cable and back to the position x from the sending-end, the prolongation to the right of the curve Ee^{-ax} of Fig. 35 may be supposed bent back and added to the curve to the left, but in so doing attention must be paid to phase, as is evident from (7).

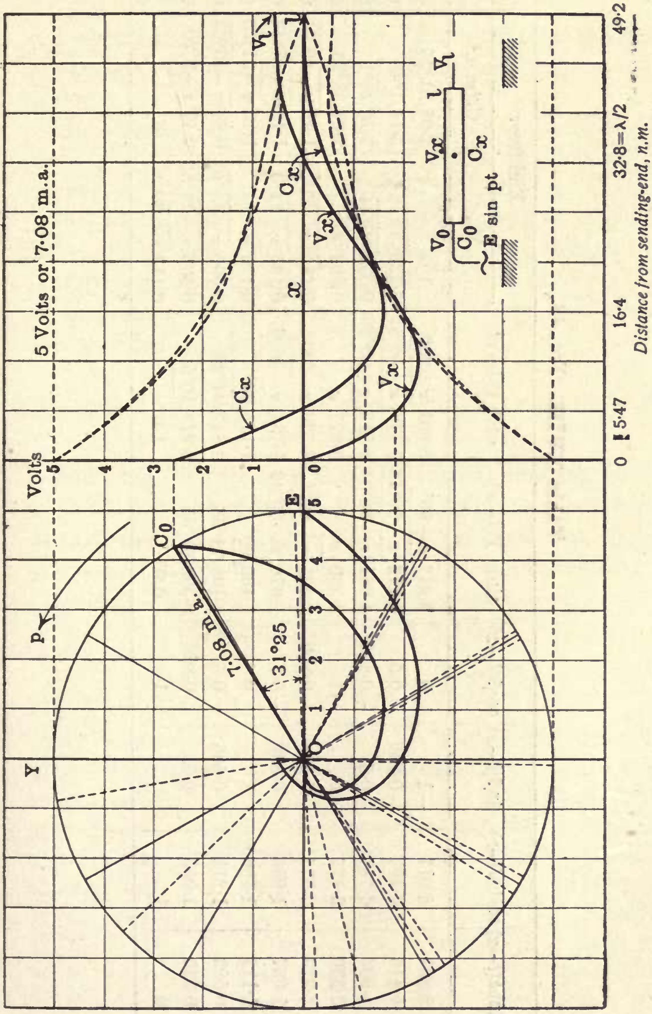

Fig. 48.—Spiral Diagram. Cable 160/150. End Free at 49·2 n.m.

$\alpha=0{\cdot}0585$, $\beta=0{\cdot}0958$, $\lambda=65{\cdot}6$ n.m., $Z_0=142{\cdot}2\rceil 31°25$

Table XVIII. Fig. 48.

x(n.m)	sinh$\alpha(l-x)$	cosh$\alpha(l-x)$	sin$\beta(l-x)$	cos$\beta(l-x)$	sinh P$(l-x)$	cosh P$(l-x)$	End free. V_x.	End free. C_x(m.a.)	End earthed. V_x.	End earthed. C_x(m.a.)
0	8·861	8·917	−1	0	8·917/π+90°	8·861/π+90°	1\0	7·08/31°25	1/0	6·99
5·465	6·416	6·493	−0·866	−0·5	6·473/π+60°18	6·436/π+59°42	0·726\30°18	5·14/1°43	0·726\29°42	5·08
10·93	4·634	4·741	−0·5	−0·866	4·661/π+30°34	4·715/π+29°26	0·532\60°34	3·70\28°1	0·523\59°26	3·72
16·40	3·330	3·477	0	−1	3·330/π	3·477/π	0·392\90°	2·64\58°35	0·370\90°	2·74\58°35
21·87	2·373	2·575	0·5	−0·866	2·425/π−32°4	2·526/π−28°1	0·285\π−61°59	1·92\π−89°21	0·272\π−57°56	1·99\86°36
27·34	1·657	1·935	0·866	−0·5	1·870/π−63°42	1·731/π−56°0	0·195\π−34°0	1·48\π−57°43	0·210\π−26°18	1·37\π−65°25
32·79	1·113	1·496	1	0	1·496/90	1·113/90°	0·126\π	1·19\π−31°25	0·168\π	0·88\π−31°25
38·26	0·683	1·211	0·866	0·5	1·103/71°58	0·847/44°20	0·096\π+45°40	0·88\π−13°23	0·124\π+18°2	0·67\π+14°15
43·72	0·325	1·055	0·5	0·866	0·598/61°55	0·928/10°5	0·105\π+79°55	0·47\π−3°20	0·067\π+28°5	0·73\π+48°30
49·19	0	1	0	1	0/0°	1/0°	0·113\π+90	0	0\π+90	0·79\π+58°35

The voltage at the free end of the cable is reflected without change of phase, and the amplitude is doubled. On the other hand the current is given by the formula

$$C_x = \frac{V_0(e^{-Pl} - e^{-P(2l-x)})}{Z_0(1+e^{-2Pl})}, \quad . \quad . \quad . \quad (8)$$

and when x is equal to l, C_l is zero. The current at the free end is reflected with change of sign, and the amplitude is reduced to zero.

The corresponding diagram when the distant end is put to earth is given in Fig. 49. Voltage is now reflected with, and current without, change of phase. The spirals for C_x and V_x in Fig. 49 are the same as the spirals for V_x and C_x respectively in Fig. 48, although altered in position and to a different scale, as is evident on comparing equations (6) and (5) with (2) and (3), and remembering that in this case Pl is large, so that sinh Pl nearly equals cosh Pl (Table XVIII.).

Variation of Impedance with Length of Cable—Quasi-Infinite Cable.

From Fig. 48 it is seen that the current entering the cable when the distant end is open (7·08 m.a.) is greater than when the distant end is closed (6·99 m.a.) for the particular frequency in question, and both are less than the current entering when

Table XIX.—Z_0=142·2\31°25.

l.	sinh Pl.	cosh Pl.	Z_0 tanh Pl.	Z_0 coth Pl.
0	0	1	0	∞
5·465	0·598/61°55	0·928/10°5	91·63/20°25	220·7\83°15
10·93	1·103/71°58	0·847/44°20	185·2/3°47	109·2\59°3
16·40	1·496/90	1·113/90°	191·1\31°25	105·8\31°25
21·87	1·870/π−63°42	1·731/π−56°0	153·6\38°67	131·6\23°43
27·34	2·425/π−32°4	2·526/π−28°1	136·5\35°28	148·1\27°22
32·79	3·330/π	3·477/π	136·2\31°25	148·5\31°25
38·26	4·661/π+30°34	4·715/π+29°26	140·6\30°17	143·8\32°33
43·72	6·473/π+60°18	6·436/π+59°42	143·0\30°49	141·4\32°1
49·19	8·917/3π/2	8·861/3π/2	143·1\31°25	141·3\31°25

the cable is infinitely long. The apparent impedance of the cable with open end is, from (1), $Z_0 \coth Pl$, and with closed end, from (4), $Z_0 \tanh Pl$. The variations of these quantities for different values of l may be obtained from Table XVIII.,

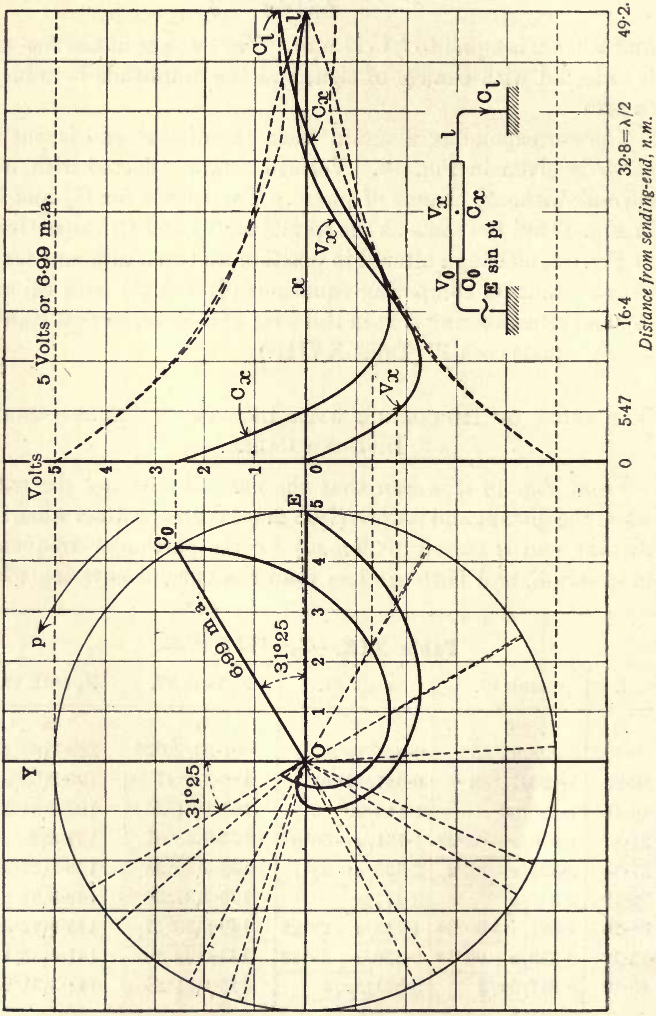

FIG. 49.—SPIRAL DIAGRAM. CABLE 160/150. END TO EARTH AT 49·2 N.M.

which contains sinh $P(l-x)$ and cosh $P(l-x)$ for values of $l-x$ from zero to 49·2 n.m., and from these tanh Pl and coth Pl may be obtained by division.

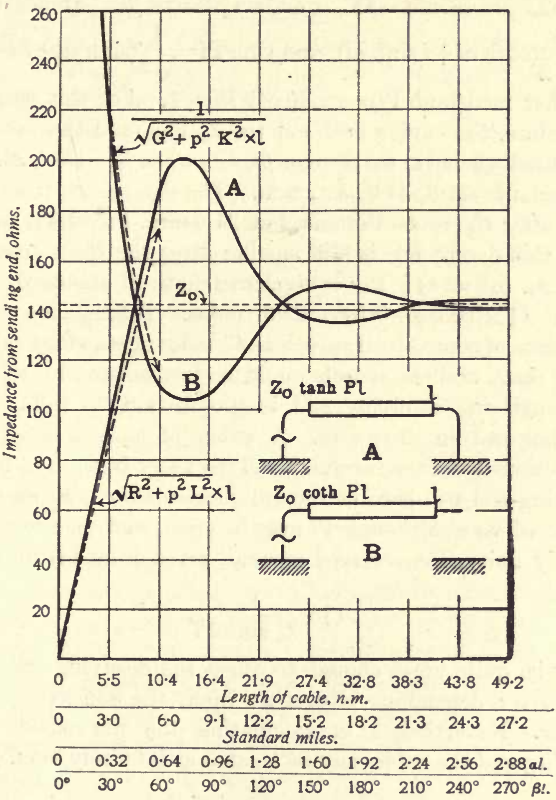

FIG. 50.—IMPEDANCE MEASURED FROM THE SENDING-END OF VARIED LENGTH OF CABLE; TYPE 160/150.

Curve A = Distant end closed; mod (Z_0 tanh Pl).
" B = Distant end open; mod (Z_0 coth Pl).
" $\alpha=0{\cdot}0585$. $\beta=0{\cdot}0958$. $\lambda=65{\cdot}6$ n.m. $Z_0=142\backslash\overline{31°25'}$.

The variations in the modulus of the impedance are contained in Table XIX., and are shown in Fig. 50. The initial value, for a very short length of cable, is zero when the end is closed, and infinitely great when the end is open. The impedance in both cases oscillates about a fixed value of 142·2 ohms (Z_0), which it reaches when the length of the cable is very great. When $l=8{\cdot}2$, $\beta l=0{\cdot}7854=45°$, and $\sin\beta l=\cos\beta l$. Hence $\cosh Pl$ $=\frac{1}{\sqrt{2}}(\cosh al+i\sinh al)$, and $\sinh Pl=\frac{1}{\sqrt{2}}(\sinh al+i\cosh al)$, so that mod($\tanh Pl$)=mod($\coth Pl$)=1. For this value of l, therefore, the curves both cut the Z_0 line, and this crossing is repeated whenever $\sin\beta l=\cos\beta l$, *i.e.*, when $\beta l=135°$, 225°, &c., or when $l=24{\cdot}6$, 41·0, &c., n.m. The divergence from the Z_0 line after the second crossing at 24·6 n.m. ($al=1{\cdot}44$) is small, and this divergence is still smaller after the third crossing at 41 n.m. ($al=2{\cdot}4$); the equivalent length of standard cable is then $41\times0{\cdot}0585\div0{\cdot}106=22{\cdot}6$ miles. Taking the limiting distance of commercial speech as 43 miles of standard cable it is seen that excellent speech would be transmitted by the cable, although the receiving end is too distant to influence the sending-end in any way. A cable of such a length that reflections from the receiving end die away before reaching the sending-end will here be termed *quasi-infinite.* From another point of view, although Pl may be great, and therefore $\tanh Pl$ nearly unity, the received current, given from (6) by putting $x=l$,

$$C_l=\frac{V_0}{Z_0\sinh Pl'}$$

may be quite great enough to affect the receiving instrument, a matter dependent altogether upon the sensitivity of the latter. Nevertheless in calculating the phenomena *at the sending end* the cable may be treated as infinitely long.

Impedance of a Short Cable.

When Pl is small, $\sinh Pl=Pl$, nearly, and $\cosh Pl=1$. Hence

$$Z_0\tanh Pl=Z_0\times Pl=\sqrt{\frac{R+ipL}{G+ipK}}\times l\sqrt{(R+ipl)(G+ipK)}$$

$$=l(R+ipL)=l\sqrt{R^2+p^2L^2}/\tan^{-1}pL/R \quad . \quad (9)$$

At the same time

$$Z_0\coth Pl=Z_0/Pl=\frac{1}{l(G+ipK)}=\frac{1}{l\sqrt{G^2+p^2K^2}/\tan^{-1}pK/G} \quad (10)$$

It is evident therefore that when the end of a short piece of cable is closed or to earth the current which takes the side path through the capacity of the cable is so small that only the resistance and inductance of the cable are felt at the sending end. Conversely when the short piece is open or free at its distant end, it behaves like a condenser with a negligible amount of resistance (and inductance) in its plates.

Curves showing the variation of (9) and (10) with l are plotted also in Fig. 50. The first is a straight line through the origin, the second a rectangular hyperbola, and both cut the Z_0 line in the same point, which is such that $Pl=1$, or $l=1/P$. Up to the point where curves A and B cut the Z_0 line (8·2 n.m. giving an attenuation of 0·48), and even somewhat beyond, $Z_0 \tanh Pl$ may be replaced by $(R+ipL)l$, and $Z_0 \coth Pl$ by $1/(G+ipK)l$ without great inaccuracy. A cable of such a length that this substitution is possible (say al less than 0·5 or l less than 5 miles of standard cable) may be termed a *short* cable. It is equivalent to a single inductive coil when the end is closed, or to a shunted condenser when the end is open. The constants of the cable may be regarded as localised instead of being distributed along the cable, and the current is the same at any instant throughout the whole of the circuit.

If the frequency is increased the propagation constant of the plain cable is increased. Hence $\tanh Pl$ approaches unity for a smaller value of l than before, and since β is greater, the oscillations about the Z_0 line take place more rapidly as l is increased. The higher the frequency therefore, the smaller the length of cable that may be regarded as quasi-infinite, and the smaller the length of cable that may be treated as short.

SENDING-END IMPEDANCE OF LOADED CABLE.

In a similar manner the influence of loading may be demonstrated. Take for example the Anglo-Irish cable of **1913**, of which the constants are :—

$Z_0=690\overline{\left(2°40.\right.}$

$a=0·0150.$ $\qquad \lambda=9·753$ n.m.

$\beta=0·6442.$ $\qquad p=5,000.$

Table XX.—a=0·015 ; Z_0=690\2°40.

βl.	l.	al.	sinh Pl.	Cosh Pl.	Z_0 tanh Pl.	Z_0 coth Pl.
30°	0·813	0·0122	0·5002/88°48	0·8660/0°24	399/85°44	1,195\91°4
45°	1·219	0·0183	0·7072/88°57	0·7072/1°3	690/85°14	690\90°34
60°	1·625	0·0244	0·8665/89°11	0·5006/2°25	1,194/84°6	399\89°26
90°	2·438	0·0366	1·0007/π/2	0·0366/π/2	18,875\2°40	25\2°40
135°	3·657	0·0549	0·7091/π−86°52	0·7081/π−3°8	691\86°24	689/81°4
3π/2	7·315	0·1097	1·0060/3π/2	0·1099/3π/2	6,317\2°40	75\2°40
27π/2	65·834	0·9873	1·5283/3π/2	1·1557/3π/2	913\2°40	522\2°40

For moderate values of l, al is now small, and sinh al is also small, while cosh al is nearly unity. Hence, tanh Pl is very great when βl=90°, and is very small when $\beta l=\pi$. Conversely, coth Pl is great when $\beta l=\pi$, and is very small when βl=90°. The fluctuations of the impedance about the Z_0 line are very great initially, as is clear from Table XX. Since a is small in this case, l must be large before the curves approach their limiting value. To convert n.m. of cable into miles of standard cable it is necessary to multiply l in this case by 0·015÷0·106=0·1415, and, therefore, the whole length of the cable is only 64×0·1415=9·05 standard miles, giving an attenuation of 0·96. For the same attenuation, or same length in standard miles, the degree of approximation to the Z_0 line is necessarily the same whether the cable is loaded or not, but as the wave-length is less the total number of fluctuations is much greater when the cable is loaded.

Circuit of Three Sections.

Only in rare instances may a telephonic circuit be taken as consisting of the cable alone. Usually the circuit is completed through land lines at the ends of the cable. It is now necessary to take into consideration such discontinuities of structure, and also the influence of end apparatus, and of loading coils inserted at intervals in the line of the cable.

Let the circuit consist of three sections as in Fig. 51, diagram A. For simplicity write sinh 1 for sinh P′l', and Z_1 for the characteristic impedance (Z_0') of the first section, and so on. Now apply the fundamental equations of Chapter IV. to each section in turn :—

Then
$$\left.\begin{aligned}
V_0&=V_s \cosh 1-Z_1C_s \sinh 1\\
C_0&=C_s \cosh 1-(V_s/Z_1) \sinh 1\\
V_l&=V_0 \cosh 2-Z_2C_0 \sinh 2\\
C_l&=C_0 \cosh 2-(V_0/Z_2) \sinh 2\\
V_r&=V_l \cosh 3-Z_3C_l \sinh 3\\
C_r&=C_l \cosh 3-(V_l/Z_3) \sinh 3
\end{aligned}\right\} \quad . \quad . \quad (11)$$

In the six equations (11) there are eight unknown quantities. Of these eight, four may be eliminated by substitution from one equation to the other, leaving two equations in the

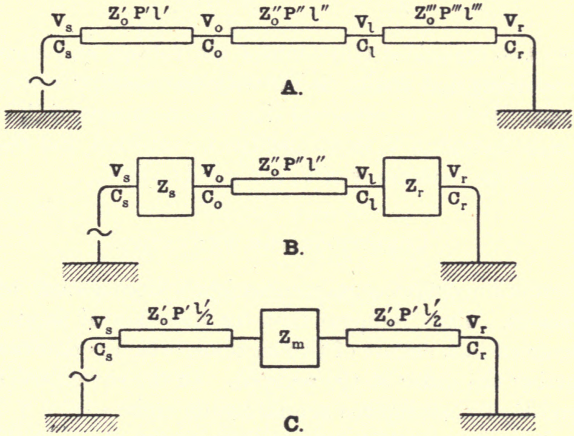

FIG. 51.—THREE SECTIONS IN SERIES.

Diagram A=Three cables. Diagram B=Cable and end apparatus. Diagram C=Cable and apparatus midway.

remaining four quantities, say, V_r, C_r and V_s, C_s. Thus, on substituting from the first and second equations in the third and fourth :—

and
$$\left.\begin{aligned}
V_l&=V_s[\cosh 1 \cosh 2+(Z_2/Z_1) \sinh 1 \sinh 2]\\
&\quad -C_s[Z_1 \sinh 1 \cosh 2+Z_2 \cosh 1 \sinh 2]\\
C_l&=C_s[\cosh 1 \cosh 2+(Z_1/Z_2) \sinh 1 \sinh 2]\\
&\quad -V_s\left[\frac{\sinh 1 \cosh 2}{Z_1}+\frac{\sinh 2 \cosh 1}{Z_2}\right]
\end{aligned}\right\}, \qquad (12)$$

and on substituting these values in the fifth and sixth equations

$$\left.\begin{aligned}
V_r=&V_s[(Z_2/Z_1)\sinh 1 \sinh 2 \cosh 3+(Z_3/Z_1)\sinh 1 \cosh 2 \sinh 3\\
&\quad+(Z_3/Z_2)\cosh 1 \sinh 2 \sinh 3+\cosh 1 \cosh 2 \cosh 3]\\
&-C_s[Z_1 \sinh 1 \cosh 2 \cosh 3+Z_2 \cosh 1 \sinh 2 \cosh 3\\
&\quad+Z_3 \cosh 1 \cosh 2 \sinh 3+(Z_1Z_3/Z_2)\sinh 1 \sinh 2 \sinh 3]\\
\text{and}\qquad&\\
C_r=&C_s[(Z_1/Z_2)\sinh 1 \sinh 2 \cosh 3+(Z_1/Z_3)\sinh 1 \cosh 2 \sinh 3\\
&\quad+(Z_2/Z_3)\cosh 1 \sinh 2 \sinh 3+\cosh 1 \cosh 2 \cosh 3]\\
&-V_s\left[\frac{\sinh 1 \cosh 2 \cosh 3}{Z_1}+\frac{\cosh 1 \sinh 2 \cosh 3}{Z_2}\right.\\
&\quad\left.+\frac{\cosh 1 \cosh 2 \sinh 3}{Z_3}+(Z_2/Z_1Z_3)\sinh 1 \sinh 2 \sinh 3\right].
\end{aligned}\right\} \quad (13)$$

The two equations (13) connect the voltage and current at the receiving end of the circuit with the sending voltage and current. It is to be observed that the coefficient of C_s in C_r is the same as that of V_s in V_r with the Z's inverted, and the same is true of the coefficients of V_s in C_r and C_s in V_r.

Conditions for Symmetry.

Certain important consequences follow from a study of these equations. Thus, suppose that the direction of transmission is reversed in a circuit of two sections, which is equivalent to inverting the order of the sections. In the equations (12) for V_l and C_l, 1 must replace 2, and 2 must replace 1.

In following out this process it is seen that the coefficients of C_s in V_l and of V_s in C_l are unaltered. These coefficients are symmetrical in 1 and 2. At the same time the coefficients of V_s in V_l and of C_s in C_l change places. If voltages and currents are to be unaffected it is necessary that these two coefficients should be equal. This leads at once to the condition

$$\frac{Z_2}{Z_1}=\frac{Z_1}{Z_2}, \text{ or } Z_1^2=Z_2^2, \quad . \quad . \quad . \quad . \qquad (14)$$

which is the same as

$$\frac{R_1+ipL_1}{G_1+ipK_1}=\frac{R_2+ipL_2}{G_2+ipK_2},$$

and leads, on multiplying out, to

$$\left.\begin{aligned} &R_1G_2-R_2G_1=p^2(L_1K_2-L_2K_1) \\ \text{and} \quad &L_1G_2-L_2G_1=K_1R_2-K_2R_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

If the conditions (15) are to hold for all frequencies, the factor by which p^2 is multiplied must vanish.

$$\left.\begin{aligned} \text{Hence,} \quad &L_1K_2=L_2K_1 \\ \text{and} \quad &R_1G_2=R_2G_1 \\ \text{while} \quad &L_1G_2-L_2G_1=K_1R_2-K_2R_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

If G_1 and G_2 are zero, these equations reduce to

$$\frac{R_1}{R_2}=\frac{L_1}{L_2}=\frac{K_1}{K_2}.$$

In any case the conditions (15) will be satisfied identically, if

$$\frac{R_1}{R_2}=\frac{L_1}{L_2}=\frac{G_1}{G_2}=\frac{K_1}{K_2}.$$

When $Z_1=Z_2$, the equations (12) become

$$V_l=V_s \cosh (1+2)-C_sZ_1 \sinh (1+2),$$
$$C_l=C_s \cosh (1+2)-(V_s/Z_1) \sinh (1+2),$$

where by cosh (1+2) is meant $\cosh (P'l'+P''l'')$. It is evident therefore that the composite cable of two sections may be replaced by a single cable having the same characteristic impedance, and of propagation constant P and length l, such that $Pl=P'l'+P''l''$.

Cable with Single Land Line Extension.

It has just been shown that if the characteristic impedances of the sections of a composite line are equal the line may be replaced by a single cable. If this condition is not satisfied the circuit is unsymmetrical, and identical substitution is no longer possible. But in every case it is permissible to calculate what length of any single cable will produce the same attenuation at a stipulated frequency, although the phase relationships are not reproduced.

In the diagram of Fig. 52 let the composite line of two sections Z_0', P′, l' and Z_0'', P″, l'' be replaced by that length l of standard cable at which the same entering current C_s would have suffered the same attenuation as in the composite line. The lengths l' and l'' of the two sections can each be expressed in miles of standard cable, and when the sum of these equiva-

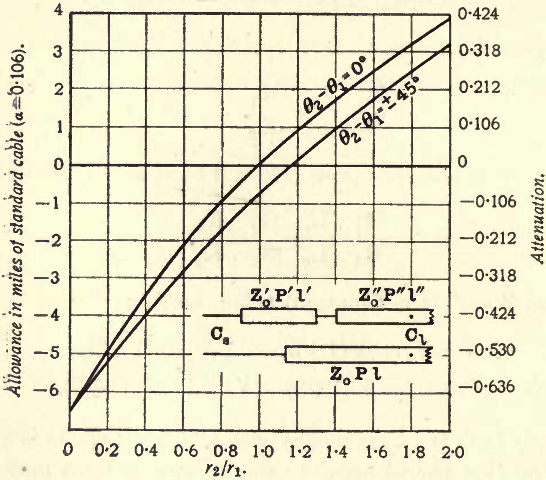

FIG. 52.—ALLOWANCE FOR JUNCTION OF CABLE AND SINGLE LAND-LINES EXTENSION ; BOTH QUASI-INFINITE.

$Z_0'=r_1/\underline{\theta_1}$; $Z_0''=r_2/\underline{\theta_2}$

$al-a''l''-a'l'=\log_e \bmod \frac{1}{2}(1+Z_0''/Z_0')$.

lent lengths is subtracted from l, the remainder, whether positive or negative, may be termed the *allowance*, in miles of standard cable, for reflections at the junctions of the two sections.* To eliminate the influence of the receiving end the second section is supposed prolonged to infinity.

* The allowance is here taken as positive when there is a loss in transmission, but the signs may be reversed. On this point *see* A. J. Aldridge, J.I.E.E., LI., 1913, p. 392.

In equations (12) it follows that $V_l = Z_2 C_l$, and therefore,

$$Z_2 C_l = V_s[\cosh 1 \cosh 2 + (Z_2/Z_1) \sinh 1 \sinh 2] - C_s[Z_1 \sinh 1 \cosh 2 + Z_2 \cosh 1 \sinh 2]$$

and
$$C_l = C_s[\cosh 1 \cosh 2 + (Z_1/Z_2) \sinh 1 \sinh 2] - V_s\left[\frac{\sinh 1 \cosh 2}{Z_1} + \frac{\sinh 2 \cosh 1}{Z_2}\right]$$

from which on elimination of V_s,

$$C_l = \frac{Z_1 C_s(\cosh 2 - \sinh 2)}{Z_2 \sinh 1 + Z_1 \cosh 1} = \frac{C_s e^{-P''l''}}{\cosh P'l' + \frac{Z_0''}{Z_0'} \sinh P'l'} \quad . \quad (19)$$

In the infinitely long standard cable, $C_l = C_s e^{-Pl}$, from which it follows that

$$e^{-Pl} = \frac{e^{-P''l''}}{\cosh P'l' + \frac{Z_0''}{Z_0'} \sinh P'l'},$$

and, therefore,

$$e^{Pl - P''l''} = \cosh P'l' + (Z_0''/Z_0') \sinh P'l'. \quad . \quad (20)$$

Neglecting phase relationships, (20) may be written

$$e^{al - a''l''} = \text{mod}\left(\cosh P'l' + \frac{Z_0''}{Z_0'} \sinh P'l'\right),$$

from which

$$l - \frac{a''l''}{a} = \frac{1}{a} \log_e \text{mod}\left(\cosh P'l' + \frac{Z_0''}{Z_0'} \sinh P'l'\right). \quad . \quad . \quad (21)$$

When l' is great, and $\sinh P'l' = \cosh P'l' = e^{P'l'}/2$, this equation reduces to

$$l - \frac{a''l''}{a} - \frac{a'l'}{a} = \frac{1}{a} \log_e \text{mod}\ \tfrac{1}{2}\left(1 + \frac{Z_0''}{Z_0'}\right). \quad . \quad . \quad . \quad (22)$$

Now, $a''l''/a$ and $a'l'/a$ are the equivalent lengths in miles of standard cable of the two sections of cable, and therefore

$$\frac{1}{a} \log_e \text{mod}\ \tfrac{1}{2}\left(1 + \frac{Z_0''}{Z_0'}\right)$$

is the allowance in standard miles for the junction when the sections are both quasi-infinite. Alternatively,

$$\log_e \text{mod}\ \tfrac{1}{2}\left(1 + \frac{Z_0''}{Z_0'}\right)$$

is the additional attenuation introduced by the junction. This expression is zero when $Z_0''/Z_0'=1$, and is positive or negative according to whether Z_0''/Z_0' is $>$ or <1. Taking $e^{P'l'}$ outside the bracket, equation (21) may be written

$$l-\frac{a''l''}{a}-\frac{a'l'}{a}=\frac{1}{a}\log_e \bmod\left[\frac{Z_0'+Z_0''}{2Z_0'}+\frac{Z_0'-Z_0''}{2Z_0'}e^{-2P'l'}\right]. \quad (23)$$

When l' is great, $e^{-2P'l'}$ is small, and (23) degenerates to (22).

If Z_0' be written $r_1\underline{/\theta_1}$ and Z_0'' be written $r_2\underline{/\theta_2}$,

then
$$\frac{Z_0'+Z_0''}{2Z_0'}=\tfrac{1}{2}\left(1+\frac{r_2}{r_1}\underline{/\theta_2-\theta_1}\right)$$

and
$$\bmod\frac{Z_0'+Z_0''}{2Z_0'}=\tfrac{1}{2}\sqrt{1+\left(\frac{r_2}{r_1}\right)^2+\frac{2r_2}{r_1}\cos(\theta_2-\theta_1)}.$$

Hence the allowance is

$$\frac{1}{2a}\log_e \tfrac{1}{4}\left[1+\left(\frac{r_2}{r_1}\right)^2+\frac{2r_2}{r_1}\cos(\theta_2-\theta_1)\right] \quad . \quad . \quad . \quad (24)$$

Now θ_1 and θ_2 vary from $\overline{\backslash 45^\circ}$ for a plain cable up to zero for a heavily loaded cable, and therefore $\cos(\theta_2-\theta_1)$ may vary from $1/\sqrt{2}$ to 1. If at the same time r_2 is less than or not much greater than r_1, the allowance is negative, indicating a gain in transmission from the one cable to the other. The loss in transmission in the reverse direction would be

$$\frac{1}{2a}\log_e \tfrac{1}{4}\left[1+\left(\frac{r_1}{r_2}\right)^2+\frac{2r_1}{r_2}\cos(\theta_1-\theta_2)\right] \quad . \quad . \quad . \quad (25)$$

in which r_1/r_2 replaces r_2/r_1 of (24). It is evident, therefore, that unless $r_1=r_2$, the loss in general is not equal to the gain.

In Fig. 52 the ratio r_2/r_1 is supposed to vary from zero up to 2, while $\theta_2-\theta_1$ has the values 0° and $+45^\circ$. The allowance is negative up to $r_2/r_1=1$, is $+^{ve}$ or $-^{ve}$, according to the phase difference, from $r_2/r_1=1$ to $r_2/r_1=1{\cdot}16$; and after that is positive. When there is no change of phase, transmission across the junction is accompanied by a gain or a loss according as r_2/r_1, is $<$ or $>$ 1. When $r_2/r_1=1$, it is clear that a change of phase is always accompanied by a gain in transmission.

To apply these formulæ to a practical case, take $Z_0''=690 \overline{\backslash 2^\circ 40}$, as given for the Anglo-Irish Cable of 1913, and let the extension be an aerial line of 600 lb. copper per mile, for which $Z_0'=594\backslash\overline{4°30}$. Then $Z_0''/Z_0'=1{\cdot}162/\underline{1°50}$, and $\frac{Z_0'+Z_0''}{2Z_0'}=1{\cdot}081/\underline{0°59}$. The allowance for the single reflection is, therefore, in this case $1/\alpha \log_e 1{\cdot}081$ or 0·734 standard mile, producing an attenuation of 0·078.

The manner in which the allowance varies as the length of the extension is increased from zero to infinity is shown in Fig. 53, which is plotted from (21) and Table XXI. The allow-

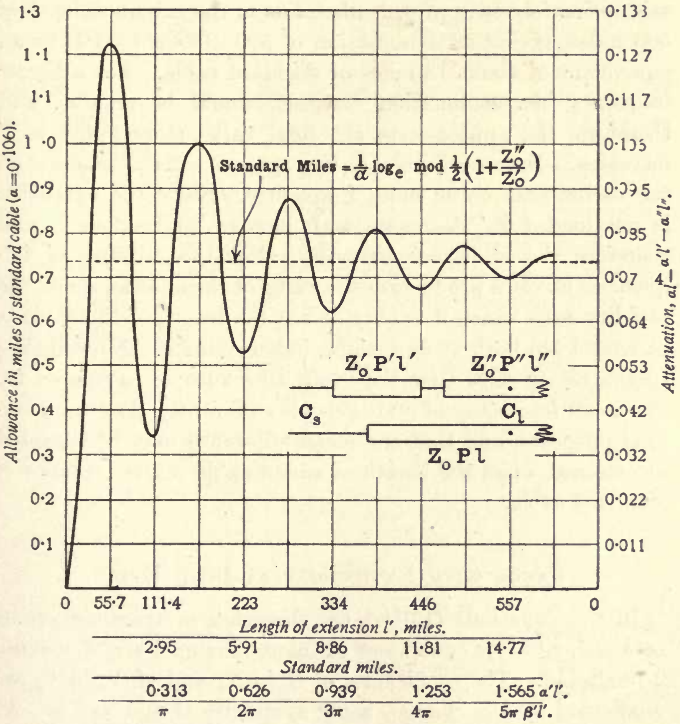

FIG. 53.—ALLOWANCE FOR JUNCTION OF QUASI-INFINITE CABLE WITH LAND-LINE EXTENSION, OF 600 LB. COPPER PER MILE AND VARIED LENGTH l'.

$Z_0''=690\backslash\overline{2°40}$; $Z_0'=594\backslash\overline{4°30}$; $\frac{Z_0''}{Z_0'}=1{\cdot}162/\underline{1°50}$.
$\alpha'=0{\cdot}00281$; $\beta'=0{\cdot}0282$ per mile; $\lambda'=222{\cdot}8$ miles.

Table XXI.—Fig. 53. $a=0{\cdot}106$; $a'=0{\cdot}00281$.

$\beta' l'$	l'	$a' l'$	sinh $a'l'$	cosh $a'l'$	sin$\beta' l'$	cos $\beta' l'$	Mod*	$1/a\log_e$mod	$a'l'/a$	Difference†
30°	18·6	0·0522	0·0522	1·0014	0·5	0·866	1·0877	0·792	0·492	0·301
45°	27·9	0·0783	0·0783	1·0031	0·707	0·707	1·1554	1·362	0·738	0·624
60°	37·1	0·1043	0·1045	1·0055	0·866	0·5	1·2247	1·911	0·983	0·983
90°	55·7	0·1565	0·1571	1·0123	1	0	1·3332	2·713	1·477	1·237
180°	111·4	0·3130	0·3181	1·0494	0	−1	1·410	3·300	2·953	0·347
270°	167·1	0·4695	0·4869	1·1123	−1	0	1·779	5·435	4·430	1·005
360°	222·8	0·6260	0·6677	1·2024	0	1	1·978	6·434	5·906	0·528

ance oscillates about the mean value (22) to which it may be said ultimately to approach when l' is in the neighbourhood of 500 miles, giving an attenuation of $500\times0{\cdot}00281=1{\cdot}4$, or an equivalent of about 13 miles of standard cable. For a higher frequency the attenuation constant would be greater, and therefore, the approach to the final value more rapid as l' increases. Moreover, since β' is greater, $\lambda'=2\pi/\beta'$ is less and the oscillations occur more frequently. Since the extension is not loaded Z_0' decreases with increase of frequency, and therefore the allowance increases. For the mixture of frequencies in voice production the value of the allowance and the rapidity with which it is obtained as the length of the cable is increased are both at first sight indeterminate. Nevertheless speech tests‡ show that the mean allowance as calculated for the mean frequency of 800 cycles is sufficiently close for practical purposes, and that the mean allowance may be regarded as attained when the length of added cable reaches 8 miles of standard cable.

Cable with Extensions at Both Ends.

In the equations (13) let the direction of transmission be reversed, which is equivalent to interchanging 1 and 3, leaving 2 unaltered. The coefficients of C_s in V_r and of V_s in C_r are unaffected by the change, being symmetrical in 1 and 3. At the same time the coefficients of V_s in V_r and of C_s in C_r change places. If the equations are to remain unaffected these two

* Mod[cosh $P'l' + (Z_0''/Z_0')$ sinh $P'l'$]
† Miles of standard cable.
‡ A. J. Aldridge, *l.c.*

coefficients must be equal. This condition leads at once to the relationship,

$$\left(\frac{Z_2}{Z_1}-\frac{Z_1}{Z_2}\right)\sinh 1 \sinh 2 \cosh 3+\left(\frac{Z_3}{Z_1}-\frac{Z_1}{Z_3}\right)\sinh 1 \cosh 2 \sinh 3$$

$$+\left(\frac{Z_3}{Z_2}-\frac{Z_2}{Z_3}\right)\cosh 1 \sinh 2 \sinh 3=0;$$

or

$$\left(\frac{Z_2}{Z_1}-\frac{Z_1}{Z_2}\right)\coth 3+\left(\frac{Z_3}{Z_1}-\frac{Z_1}{Z_3}\right)\coth 2+\left(\frac{Z_3}{Z_2}-\frac{Z_2}{Z_3}\right)\coth 1=0. \quad (26)$$

This equation is obviously satisfied if $Z_1=Z_2=Z_3$. It is also

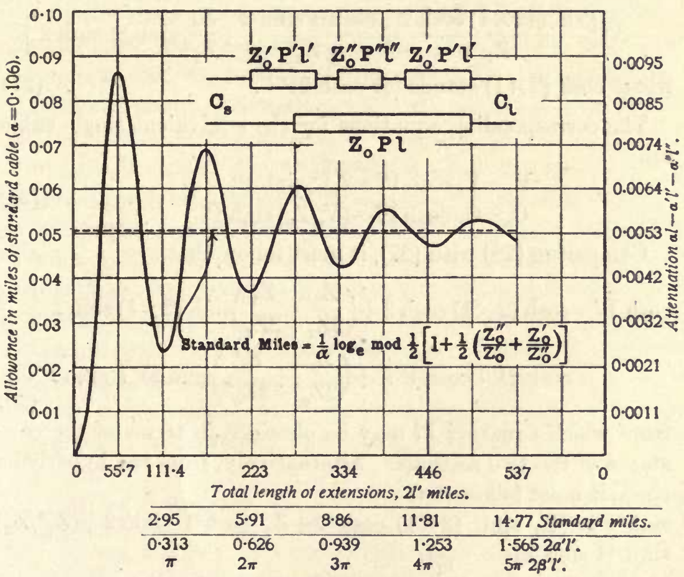

FIG. 54.—ALLOWANCE FOR JUNCTION OF QUASI-INFINITE CABLE WITH LAND-LINE EXTENSION. LENGTH OF EXTENSIONS, 2l′, VARIED.

$Z_0''=690\overline{\backslash 2°40}$; $Z_0'=594\overline{\backslash 4°30}$; $\frac{1}{2}\left(\frac{Z_0''}{Z_0'}+\frac{Z_0'}{Z_0''}\right)=1{\cdot}01074/\underline{0°16{\cdot}4}$
$\alpha'=0{\cdot}00281$; $\beta'=0{\cdot}0282$ per mile.
$\lambda'=222{\cdot}8$ miles.

satisfied if the sections 1 and 3 are identical, as is seen when 1 is substituted for 3. Hence, if for example, the central section is a cable and the end sections are land lines both of the same type and length, as shown in Fig. 54, the circuit will be symmetrical and the cable with its extensions may be replaced, formally at any rate, by a single cable.

When 1 is substituted for 3 in the equations (13) they reduce to

$$V_r=V_s\left[\cosh(2\times 1)\cosh 2+\left(\frac{Z_2}{2Z_1}+\frac{Z_1}{2Z_2}\right)\sinh(2\times 1)\sinh 2\right]$$

$$-C_s\left[Z_1\sinh(2\times 1)\cosh 2+Z_2\cosh^2 1\sinh 2+\frac{Z_1^{\,2}}{Z_2}\sinh^2 1\sinh 2\right]$$

$$C=C_s\left[\cosh(2\times 1)\cosh 2+\left(\frac{Z_2}{2Z_1}+\frac{Z_1}{2Z_2}\right)\sinh(2\times 1)\sinh 2\right]$$

$$-V_s\left[\frac{\sinh(2\times 1)\cosh 2}{Z_1}+\frac{\cosh^2 1\sinh 2}{Z_2}+\frac{Z_2}{Z_1^{\,2}}\sinh^2 1\sinh 2\right],$$

where cosh (2×1) stands for cosh $2P'l'$. (27)

The corresponding equations for the equivalent single cable are

$$\left.\begin{aligned}V_r&=V_s\cosh Pl-Z_0C_s\sinh Pl,\\ C_r&=C_s\cosh Pl-(V_s/Z_0)\sinh Pl.\end{aligned}\right\}\quad .\ .\ .\ .\quad (28)$$

Comparing (28) with (27), it must follow that

$$\cosh Pl=\cosh(2\times 1)\cosh 2+\left(\frac{Z_2}{2Z_1}+\frac{Z_1}{2Z_2}\right)\sinh(2\times 1)\sinh 2$$

$$=\cosh 2P'l'\cosh P''l''+\left(\frac{Z_0''}{2Z_0'}+\frac{Z_0'}{2Z_0''}\right)\sinh 2P'l'\sinh P''l'',\qquad (29)$$

from which equation Pl may be obtained in terms of the constants of the two sections. Alternatively, from the hyperbolic sines, it must follow that

$$\sinh^2 Pl=[Z_1\sinh(2\times 1)\cosh 2+Z_2\cosh^2 1\sinh 2+(Z_1^{\,2}/Z_2)\sinh^2 1\sinh 2]$$

$$\times\left[\frac{\sinh(2\times 1)\cosh 2}{Z_1}+\frac{\cosh^2 1\sinh 2}{Z_2}+\frac{Z_2}{Z_1^{\,2}}\sinh^2 1\sinh 2\right],$$

and this equation may readily be shown by transformation to be identical with (29).

Certain cases of equation (29) are of special interest.

(1) Let $Z_0''=Z_0'$. Then

$$\cosh Pl=\cosh(2P'l'+P''l'')$$

and $Pl=2P'l'+P''l''$, as was to be expected.

(2) Let l', l'', and therefore l, be great. Equation (29) then reduces to

$$e^{Pl}=\tfrac{1}{2}e^{2P'l'}\,.\,e^{P''l''}+\tfrac{1}{2}\left(\frac{Z_0''}{2Z_0'}+\frac{Z_0'}{2Z_0''}\right)e^{2P'l'}\,.\,e^{P''l''}$$

Hence
$$e^{Pl-2P'l'-P''l''}=\tfrac{1}{2}+\tfrac{1}{2}\left(\frac{Z_0''}{2Z_0'}+\frac{Z_0'}{2Z_0''}\right).$$

Taking logarithms,

$$l-\frac{2a'l'}{a}-\frac{a''l''}{a}=\frac{1}{a}\log_e \operatorname{mod}\tfrac{1}{2}\left[1+\tfrac{1}{2}\left(\frac{Z_0''}{Z_0'}+\frac{Z_0'}{Z_0''}\right)\right]. \quad . \quad (30)$$

If the single cable is standard cable of length sufficient to reproduce the attenuation in the other three conductors, formula (30) expresses the allowance for the junctions in standard miles. Comparing (30) with (22), it is seen that the effect of the second junction is to replace Z_0''/Z_0' by $\frac{1}{2}$ $(Z_0''/Z_0' + Z_0'/Z_0'')$. Since the arithmetic mean of a quantity and its reciprocal has a minimum value when the quantity is unity, it follows in the case when Z_0''/Z_0' is wholly real, as for instance, when both sections are loaded, that the allowance for the two junctions is positive and has its minimum value when $Z_0''=Z_0'$, in which case it is zero. The effect of the two junctions is thus to produce a loss in transmission. According to whether the ratio of the characteristic impedances is greater or less than unity in the first case will the allowance be merely reduced, or increased and made positive, by the second junction.

(3) Let l'' be great, and let l' vary. Then l is also great, and finally

$$l-\frac{a''l''}{a}=\frac{1}{a}\log_e \operatorname{mod}\left[\cosh 2P'l'+\left(\frac{Z_0''}{2Z_0'}+\frac{Z_0'}{2Z_0''}\right)\sinh 2P'l'\right] \quad (31)$$

Take $Z_0''=690\overline{|2°40}$ and $Z_0'=594\overline{|4°30}$ as before. Fig. 54 is plotted from (31) by the help of Table XXII.

Here $Z_0''/Z_0'=1{\cdot}162/\underline{1°50}$ and $Z_0'/Z_0''=0{\cdot}8606\overline{|1°50}$. Hence $\frac{1}{2}+\frac{1}{4}(Z_0''/Z_0'+Z_0'/Z_0'')=1{\cdot}0054/\underline{0°8}$, and $1/a \log_e 1{\cdot}0054=0{\cdot}0505$ standard miles. The effect of the addition of land lines at both ends instead of only at one end is to reduce the allowance from

Table XXII.—Fig. 54. $\alpha=0{\cdot}106$; $\alpha'=0{\cdot}00281$.

$2\beta'l'$	$2l'$	$2\alpha'l'$	sinh $2P'l'$	mod*	$(1/\alpha)\log_e$mod	$\alpha'l'/\alpha'$	Difference
30°	18·6	0·052	0·503/84°51	1·055	0·503	0·492	0·011
45°	27·9	0·078	0·711/86°34	1·085	0·770	0·738	0·031
60°	37·1	0·104	0·872/86°34	1·117	1·041	0·984	0·057
90°	55·7	0·157	1·012/π/2	1·180	1·563	1·477	0·086
180°	111·4	0·313	0·318/π	1·371	2·977	2·953	0·024
270°	167·1	0·470	1·112/3π/2	1·611	4·499	4·430	0·069

0·734 to 0·050 standard miles, as is seen when Figs. 53 and 54 are compared.

(4) Now let l' be great, and let l'' vary from zero upwards. The formula (29) for the allowance becomes

$$l-2\alpha'l'/\alpha=1/\alpha\log_e \text{mod}\left[\cosh P''l''+\left(\frac{Z_0''}{2Z_0'}+\frac{Z_0'}{2Z_0''}\right)\sinh P''l''\right] \quad (32)$$

For the Anglo-Irish cable, $\alpha''=0{\cdot}0150$, and $\beta''=p\sqrt{KL}=5\sqrt{0{\cdot}166\times0{\cdot}1}=0{\cdot}6442$ per nautical mile, when $p=5{,}000$. Hence $\lambda''=9{\cdot}753$ n.m. The total length of the cable is 64 n.m. or 6·56 wave-lengths. The values of cosh $P''l''$ and sinh $P''l''$ may be taken from Table XX. The variations of the allowance with the length l'' of the cable are plotted in Fig. 55. As l'' increases, the allowance approaches the same constant value (0·050 standard mile) as in Fig. 54. Since the wave-length is less the total number of fluctuations before the steady value is reached is greater.

An expression of the form

$$\cosh Pl+(r/\underline{\theta})\sinh Pl$$

may be written

$$\cosh(al+i\beta l)+(a+ib)\sinh(al+i\beta l),$$

which becomes on expansion and rearrangement

$$\cosh al\cos\beta l+a\sinh al\cos\beta l-b\cosh al\sin\beta l$$
$$+i(\sinh al\sin\beta l+b\sinh al\cos\beta l+a\cosh al\sin\beta l).$$

The modulus of this complex quantity reduces after simplification to

$$\sqrt{[\tfrac{1}{2}(a^2+b^2)(\cosh 2al-\cos 2\beta l)+\tfrac{1}{2}(\cosh 2al+\cos 2\beta l)+a\sinh 2al-b\sin 2\beta l]}$$

* mod$[\cosh 2P'l'+\frac{1}{2}(Z_0''/Z_0'+Z_0'/Z_0'')\sinh 2P'l']$

The extra attenuation introduced by the cable and junction or junctions is, therefore, of the form

$$\tfrac{1}{2}\log_e \tfrac{1}{2}[(1+r^2)\cosh 2al+2a\sinh 2al$$
$$+(1-r^2)\cos 2\beta l-2b\sin 2\beta l].$$

The quantity in the square brackets consists of two parts; one, aperiodic, which increases continuously with l; and a

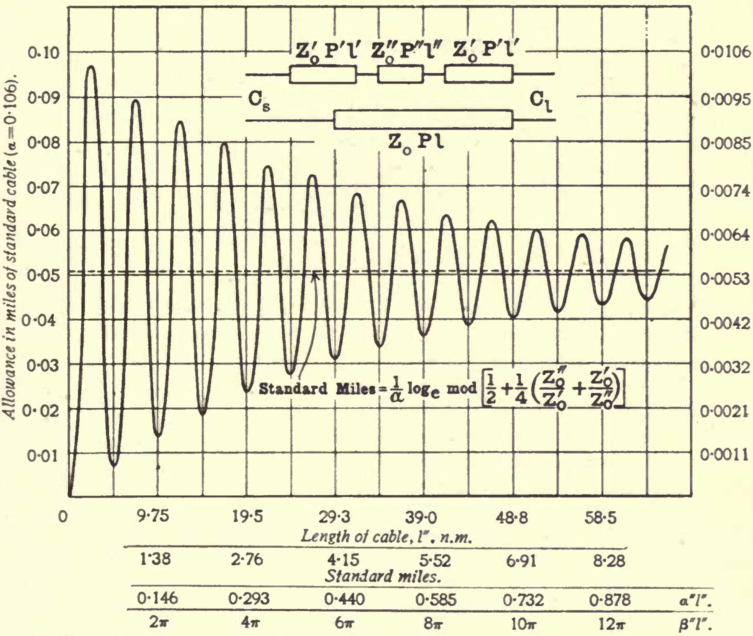

Fig. 55.—Allowance for Junctions of Cable with Quasi-infinite Land-lines. Length of Cable, l'', Varied.

$Z_0''=690\backslash\overline{2°40}$; $Z_0'=594\backslash\overline{4°30}$.
$\alpha''=0{\cdot}0150$; $\beta''=0{\cdot}6442$ per n.m.; $\lambda''=9{\cdot}753$ n.m.

second, which varies periodically as l increases. When l is great $\cosh 2al=\sinh 2al=e^{2al}/2$, and the periodic portion is small in comparison. The formula then simplifies to

$$al+\tfrac{1}{2}\log_e \tfrac{1}{4}[1+r^2+2a],$$

which should be compared with (24). When plotted against

l it represents a straight line making an angle a with the horizontal axis, and cutting the vertical axis in a point distant $\frac{1}{2}\log_e\frac{1}{4}[1+r^2+2a]$ from the origin. It is to be observed that the wave-length of the periodic fluctuations in the longer formula is $\pi/\beta=\lambda/2$. In other words, the fluctuations pass through two complete cycles as l increases by one wave-length, which is also clear from Figs. 53, 54 and 55.

Cable with Apparatus at the Ends.

Returning to (13) let the leakance and capacity of the first and third sections of the circuit be zero. These two sections then become simply pieces of apparatus of impedance $(R_1+ipL_1)l'$, which may be designated Z_s, and $(R_2+ipL_2)l''$, or Z_r, as represented in Fig. 51, Diagram B.

Now when Pl is reduced without limit, $\dfrac{\sinh Pl}{Z_0}$ approximates to $l(G+ipK)$ and $Z_0 \sinh Pl$ to $l(R+ipK)$. Hence in (13), when the first and third sections have degenerated into apparatus, the substitutions Z_s for $Z_1 \sinh 1$, and Z_r for $Z_3 \sinh 3$ must be made, while $\dfrac{\sinh 1}{Z_1}=\dfrac{\sinh 3}{Z_3}=0$, and $\cosh 1=\cosh 3=1$. The equations then become

$$\left.\begin{aligned} V_r=&V_s[\cosh P''l''+(Z_r/Z_0'')\sinh P''l''] \\ &-C_s[(Z_s+Z_r)\cosh P''l''+(Z_0''+Z_sZ_r/Z_0'')\sinh P''l''] \\ C_r=&C_s[\cosh P''l''+(Z_s/Z_0'')\sinh P''l''] \\ &-(V_s/Z_0'')\sinh P''l'' \end{aligned}\right\} \quad . \quad (33)$$

The condition for symmetry is now simply that Z_s should equal Z_r. If Z_0, P, and l are the constants of the equivalent cable, as in Fig. 56, then, as before, comparing the coefficients of (33) with those of (28), it follows that

$$\cosh Pl=\cosh P''l''+(Z_r/Z_0'')\sinh P''l''. \quad . \quad . \quad (34)$$

When $P''l''$ and Pl are large, this relationship reduces, neglecting phase angles, to

$$l-a''l''/a=(1/a)\log_e \operatorname{mod}(1+Z_r/Z_0'') \quad . \quad . \quad . \quad (35)$$

Write $Z_r = Z_s = r_s\underline{/\theta_s}$ and $Z_0'' = r_2\underline{/\theta_2}$. Then

$$1+Z_r/Z_0'' = 1+(r_s/r_2)\underline{/\theta_s-\theta_2} = 1+(r_s/r_2)\cos(\theta_s-\theta_2) + i(r_s/r_2)\sin(\theta_s-\theta_2).$$

Hence $\operatorname{mod}(1+Z_r/Z_0'') = \sqrt{1+(r_s/r_2)^2+(2r_s/r_2)\cos(\theta_s-\theta_2)}$ and the allowance is

$$\frac{1}{2a}\log_e[1+(r_s/r_2)^2+(2r_s/r_2)\cos(\theta_s-\theta_2)] \quad . \quad . \quad . \quad (36)$$

This expression depends both on the ratio of the moduli, r_s/r_2, and on the difference of the angles, $\theta_s-\theta_2$. Now θ_s may vary from $\overline{\backslash 90^\circ}$ to $\underline{/90^\circ}$, and θ_2 may vary, as before, from $\overline{\backslash 45^\circ}$ for a plain cable to zero for a loaded cable. Hence $\theta_s-\theta_2$ may vary from $\overline{\backslash 90^\circ}$ to $\underline{/135^\circ}$. Let the ratio r_s/r_2 vary at the same time from 0 to 2. The allowances corresponding to these values of r_s/r_2 and to differences of phase 0°, $\pm 45°$, $\pm 90°$, 120° and 135° are plotted in Fig. 56. The expression (36) is similar to (24), but the quantity of which the logarithm is taken is four times as great. They differ, therefore, in a constant term $(1/2a)\log_e\frac{1}{4} = (-1/a)\log_e 2 = -0{\cdot}6932/a = -6{\cdot}54$ standard miles. The curves for 0° and $\pm 45°$ in Fig. 52 are displaced by this amount vertically downwards, but are otherwise identical with the corresponding curves of Fig. 56.

The allowance is a minimum for a given value of $\theta_s-\theta_2$ when $r_s/r_2+\cos(\theta_s-\theta_2)=0$, as is easily seen on differentiation of (36). Since r_s/r_2 is positive, it follows that $\theta_s-\theta_2$ must be greater than $\pi/2$, and, moreover, that r_s/r_2 cannot be greater than unity, if a minimum value is to exist. Substituting by the aid of this relationship in (36), it becomes

$$\left.\begin{aligned}\text{Allowance}_{\text{(minimum)}} &= \frac{1}{2a}\log_e\left(1-\frac{r_s^2}{r_2^2}\right)\\ &= \frac{1}{a}\log_e\sin(\theta_s-\theta_2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

This expression is zero when r_s is zero, is negative as r_s increases from zero, and is imaginary when r_s is greater than r_2. Alternatively it is zero when $\theta_s-\theta_2=\pi/2$, and is negative when $\theta_s-\theta_2$ is greater than $\frac{\pi}{2}$; as is also evident from Fig. 56.

To calculate the value of (37) in a practical case, take first the plain cable of type 160/150, for which $Z_0''=142{\cdot}2\backslash\overline{31°25}$. The minimum allowance is obtained when

$$r_s/142{\cdot}2=-\cos(\theta_s+31°25).$$

It is evident that θ_s may range from 58°35, which is the least value for which the cosine of the total angle is negative, up to 90°, which is the greatest possible value of θ_s. When $\theta_s=58°35$, $r_s=0$, and when $\theta_s=90°$, $r_s/r_2=0{\cdot}5213$. Hence the minimum

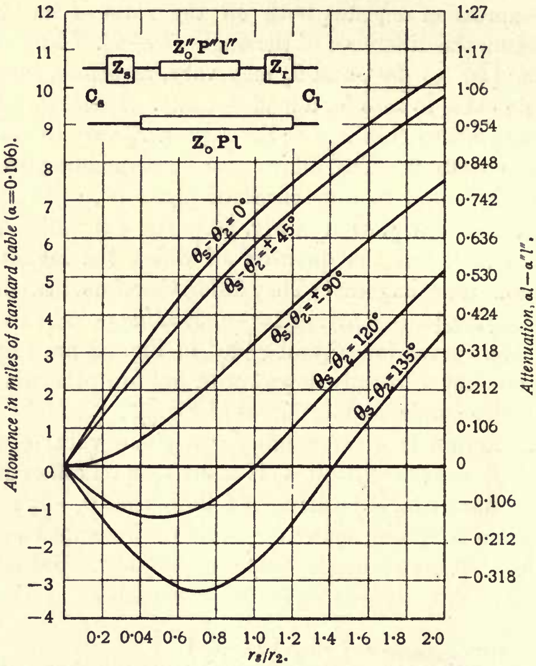

FIG. 56.—ALLOWANCE FOR APPARATUS AT THE ENDS OF QUASI-INFINITE CABLE.

$Z_r=Z_s=r_s/\underline{\theta_s}$; $Z_0''=r_2/\underline{\theta_2}$; $al-a''l''=\log_e \text{mod}\,(1+Z_s/Z_0'')$.

value of the allowance, substituting in (37), is $-1{\cdot}495$ standard miles, or the minimum attenuation is $-0{\cdot}158$. This could be attained with an inductance coil of negligible resistance, and of reactance $pL_s=74{\cdot}1$ ohms, equal to an inductance, at $n=800$, of 14·7 millihenries.

For the loaded cable considered in the previous examples, where $Z_0=690\overline{\backslash 2°40}$, taking $\theta_s=90°$ as before, $\cos(\theta_s-\theta_2)=-0{\cdot}0465$. Hence $r_s=32{\cdot}1$ ohms, or $L_s=6{\cdot}38$ millihenries, which would give a minimum allowance of $-0{\cdot}0104$ standard miles, or a minimum attenuation of $-0{\cdot}0011$.

In practice such small inductances could not be constructed with a negligible resistance component, and, moreover, if in the form of transformers at the ends of the cable they would not permit of the transmission of sufficient energy from the sending apparatus to the receiver. A compromise must, therefore, be arranged in which the apparatus is designed with a view to this requirement as well as to minimum attenuation.

Cable with Apparatus Midway.

Suppose, now, that as in Fig. 51 (C), the second section is simply a piece of apparatus, Z_m, and that the first and third sections are alike, and each of length $l/2$, as in the diagram of Fig. 57. The equations for this case will be obtained from (13) by replacing $Z_2 \sinh 2$ by Z_m, $\cosh 2$ by unity, and by making $\dfrac{\sinh 2}{Z_2}$ zero.

This procedure leads to the equations

$$\left.\begin{aligned} V_r&=V_s\left[\cosh P'l+\frac{Z_m}{2Z_0'}\sinh P'l\right]\\ &\quad-C_s\left[Z_0'\sinh P'l+\frac{Z_m}{2}(\cosh P'l+1)\right]\\ C_r&=C_s\left[\cosh P'l+\frac{Z_m}{2Z_0'}\sinh P'l\right]\\ &\quad-(V_s/Z_0'^2)\left[Z_0'\sinh P'l+\frac{Z_m}{2}(\cosh P'l-1)\right]\end{aligned}\right\} \quad (38)$$

The equations of an equivalent, unbroken cable of constants P and Z_0, and also of length l, are

$$V_r=V_s\cosh Pl-Z_0\,C_s\sinh Pl,$$
$$C_r=C_s\cosh Pl-(V_s/Z_0)\sinh Pl.$$

If the relationship between voltage and current are undisturbed by the substitution, then it follows as before, on equating coefficients, that*

$$\left.\begin{aligned} &\cosh Pl = \cosh P'l + \frac{Z_m}{2Z_0'}\sinh P'l \\ \text{and} \quad &Z_0 = Z_0'\sqrt{\frac{2Z_0'\sinh P'l + Z_m(\cosh P'l+1)}{2Z_0'\sinh P'l + Z_m(\cosh P'l-1)}} \end{aligned}\right\} . \qquad (39)$$

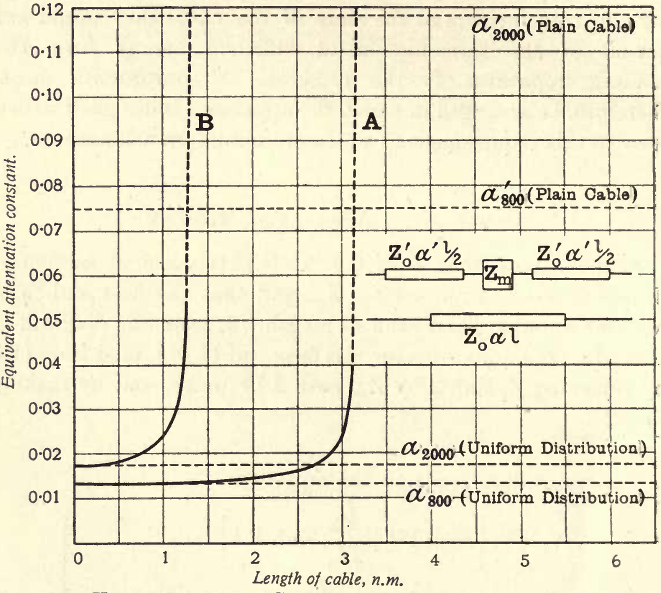

FIG. 57.—VARIATION WITH SPACING OF EQUIVALENT ATTENUATION CONSTANT.

Cosh $Pl = \cosh P'l + (Z_m/2Z_0')\sinh P'l$.

Curve A.—$n=800$. ${}_{800}Z_m = Z_cl = 508 \times l/89°11$. ${}_{800}Z_0' = 134{\cdot}1\backslash 45°$.

" B.—$n=2{,}000$. ${}_{2000}Z_m = Z_cl = 1{,}257 \times l/89°21{\cdot}2$. ${}_{2000}Z_0' = 84{\cdot}83\backslash \pi/4$.

In the first equation of (39), if l be great, and therefore if $P'l$ and Pl are great, the exponentials may be substituted for the hyperbolic functions, and the equation reduces, as before, to

$$l - \frac{\alpha' l}{\alpha} = \frac{1}{\alpha}\log_e \operatorname{mod}(1 + Z_m/2Z_0'). \quad . \quad . \quad . \quad . \qquad (40)$$

* These equations were first given by G. A. Campbell, "Phil. Mag.," 5, 1903, p. 319.

Writing $r_m/\underline{\theta_m}$ for Z_m, and $r'/\underline{\theta'}$ for Z_0', this equation becomes

$$l-\frac{a'l}{a}=\frac{1}{2a}\log_e\left[1+\left(\frac{r_m}{2r'}\right)^2+\frac{r_m}{r'}\cos\left(\theta_m-\theta'\right)\right]. \quad . \quad . \quad (41)$$

If Z_m is a highly inductive coil, θ_m is approximately 90°, and θ' for a plain cable may be $\backslash$45°. Hence $\theta_m-\theta'$ may be as great as 135°, and therefore $\cos(\theta_m-\theta')$ may be negative. It follows that for values of r_m/r' up to not much >1, the allowance may be negative. The formulæ (40) and (41) are the same as (35) and (36), except that $Z_m/2$ replaces Z_s, and therefore $r_m/2$ replaces r_s. It is a matter of indifference, so far as the allowance is concerned, whether the apparatus is placed in the middle or is split up in halves, and placed at the ends of the cable, but the phase relationships are not the same in the two cases, as is clear when the coefficients in equations (33) and (38) are compared.

Fig. 56 serves for (41), as well as for (36), when the above substitution is made. The allowance is negative when $\theta_m-\theta'=135°$ for values of $r_m/r'(=2r_s/r_2)$ up to 2·8. In general the allowance is positive, and the introduction of the coil causes a loss in transmssion. Thus for the plain cable of type 160/150, where $Z_0'=142\overline{\backslash 31°25}$, if Z_m be a coil of 7·1 ohms and 0·1 henry, $\frac{r_m}{r'}/\underline{\theta_m-\theta'}=\frac{503}{142}/\underline{90°+31°25}$ nearly $=3·54/\underline{121°25}$, from which the allowance is 3·9 standard miles. It is evident from an inspection of Fig. 56 that the improvement in transmission, due to the insertion of a single coil, or to a series of coils which are far apart, is confined within narrow limits.

The Pupin Criterion.

Having studied equations (39) for the case when l is long, let us now examine it when l is short. Converting all functions to hyperbolic sines, the equation may be written in the form

$$\sinh^2\frac{Pl}{2}=\sinh^2\frac{P'l}{2}+\frac{Z_m}{4Z_0'}\sinh P'l.$$

Now when l is small, $\sinh P'l = P'l$ nearly, and $\sinh \frac{P'l}{2} = P'l/2$, so that the equation becomes

$$\sinh^2 \frac{Pl}{2} = \left(\frac{P'l}{2}\right)^2 + \frac{Z_m}{4Z_0'} \cdot P'l.$$

Substituting for P_0' and Z_0'; and $R_m + ipL_m$ for Z_m,

$$\sinh^2 \frac{Pl}{2} = \frac{l^2}{4}(R' + ipL')(G' + ipK') + \frac{l^2}{4}\left(\frac{R_m}{l} + ip\frac{L_m}{l}\right)(G' + ipK')$$

$$= \frac{l^2}{4}(G' + ipK')[(R' + R_m/l) + ip(L' + L_m/l)].$$

If, now, Pl is so small that $\sinh Pl/2$ is sensibly the same as $Pl/2$, the equation may be written

$$\frac{l^2}{4}(R + ipL)(G + ipK) = \frac{l^2}{4}(G' + ipK')[(R' + R_m/l) + ip(L' + L_m/l)]$$

which is satisfied when $R = R' + R_m/l$,

$$L = L' + L_m/l,$$

$$G = G', \text{ and } K = K'.$$

The constants of the loaded cable may then be regarded as distributed, and their value per unit length may be added to those of the cable.

The criterion $\sinh Pl/2 = Pl/2$ may be written

$$\sinh \frac{al}{2} \cos \frac{\beta l}{2} + i \cosh \frac{al}{2} \sin \frac{\beta l}{2} = \frac{al}{2} + i\frac{\beta l}{2}.$$

For a loaded cable a is small and β is great, *e.g.*, for the Anglo-Belgian cable $a = 0{\cdot}0087$ and $\beta = 0{\cdot}6298$. Writing $al/2$ for $\sinh al/2$ and 1 for $\cosh al/2$, the equation becomes

$$\frac{al}{2} \cos \frac{\beta l}{2} + i \sin \frac{\beta l}{2} = \frac{al}{2} + i\frac{\beta l}{2},$$

for which

$$\left.\begin{aligned} &\cos \frac{\beta l}{2} = 1 \\ \text{and} \quad &\sin \frac{\beta l}{2} = \frac{\beta l}{2} \end{aligned}\right\} \quad \ldots\ldots \quad (42)$$

The criterion expressed by the second of these equations was obtained by Pupin in 1899.* To apply it to a particular case, take the Anglo-Belgian cable of 1912, for which $\beta=0{\cdot}6298$ as above, and l, equal to the distance apart of the coils, is 1 n.m. Then $\beta l/2=0{\cdot}3149$ and $\sin \beta l/2=0{\cdot}3097$. The difference between the two is 1·7 per cent., which may be taken as quite small enough for practical purposes, and the coil constants may be regarded as distributed along the cable.

Since the effect of loading is to increase the wave-length constant, β for the loaded cable is greater than β' for the plain cable, which is also greater than α', and the stipulation made above that $\sinh \mathrm{P}'l/2$ should be sensibly equal to $\mathrm{P}'l/2$ is included in (42). Since $\sin\frac{\beta l}{2}=\frac{\beta l}{2}\left(1-\frac{\beta^2l^2}{24}\right)$, neglecting higher powers than the cube, and $\cos\frac{\beta l}{2}=1-\frac{\beta^2l^2}{8}$, it is seen that the equations (42) are both included in the statement that $\beta^2l^2/8$ should be small compared with unity.

Calculation of Equivalent Cable.

When the criterion (42) is not satisfied we must return to equations (39), and calculate the equivalent value of the attenuation αl from them. In carrying out this process the right-hand side is reduced to the form $a+ib$, and the equation becomes

$$\cosh(\alpha+i\beta)l=a+ib,$$

from which

$$\cosh \alpha l \cos \beta l=a,$$

and

$$\sinh \alpha l \sin \beta l=b.$$

Eliminating βl between these two equations by the relationship $\cos^2\beta l+\sin^2\beta l=1$, a single equation in αl is obtained—

$$a^2\sinh^2\alpha l+b^2\cosh^2\alpha l=\sinh^2\alpha l\cosh^2\alpha l,$$

or

$$\cosh^4\alpha l-(a^2+b^2+1)\cosh^2\alpha l+a^2=0.$$

Regarding this equation as a quadratic in $\cosh^2\alpha l$ and solving, it follows that

$$\cosh^2\alpha l=\frac{(a^2+b^2+1)\pm\sqrt{(a^2+b^2+1)^2-4a^2}}{2}, \quad . \quad (43)$$

* M. I. Pupin, "Trans." Am.I.E.E., 16, p. 93, 1899, and 17, p. 445, 1900.

N 2

from which cosh al can be obtained by extracting the square root. The expression on the right-hand side of (43) may also be written in the form $\frac{1}{4}[\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}]^2$ as is easily verifiable.

Hence, finally,

$$\cosh al = \left.\frac{\sqrt{(1+a)^2+b^2}\pm\sqrt{(1-a)^2+b^2}}{2}\right\} \quad . \quad . \quad (44)$$

Of the two roots, the one with the negative sign is either less than zero or negative, which values are not permissible for cosh al.

To illustrate the calculation, take the cable of type 160/150, for which $R'=14{\cdot}2$ ohms and $K'=0{\cdot}157$ mfd. per n.m. Neglecting L' and G', $a'=\beta'=0{\cdot}07485$ and $Z_0'=134{\cdot}14\overline{\backslash 45^\circ}$ at 800 p.p.s.

Let coils be inserted in the line of the cable at intervals of 1 n.m., of constants

$$R_c=7{\cdot}1 \text{ ohms, and } L_c=0{\cdot}1 \text{ henry, at } 800 \text{ p.p.s.}$$

Then $Z_c=502{\cdot}7\,\underline{/89^\circ 11}$ and $Z_c/2Z_0'=1{\cdot}874\,\underline{/\pi-45^\circ 49}$.

If the same extent of loading is to be carried out by coils spaced at l n.m. instead of 1 n.m., the impedance of each coil must be $Z_m=Z_c\times l$. Table XXIII. contains the calculated values of a, the equivalent attenuation constant, for increasing length of coil section and, therefore, greater spacing.

Table XXIII.—Fig. 57, Curve A. $n=800$.

l	$a'l=\beta'l$	Sinh $P'l$	$Z_m/2Z'_0$	Cosh Pl	Cosh al	a
1·5	0·1123	$0{\cdot}1588\,\underline{/45^\circ 14{\cdot}4}$	2·811	+0·5536, 0·0171	1·00022	0·0139
2·0	0·1497	$0{\cdot}2117\,\underline{/45^\circ 25}$	3·748	+0·2065, 0·0280	1·00040	0·0140
2·5	0·1871	$0{\cdot}2647\,\underline{/45^\circ 40{\cdot}1}$	4·685 $\underline{/\pi-45^\circ 49}$	−0·2402, 0·0382	1·00084	0·0164
2·75	0·2059	$0{\cdot}2912\,\underline{/45^\circ 48{\cdot}5}$	5·153	−0·5009, 0·0426	1·00120	0·0178
3·0	0·2246	$0{\cdot}3176\,\underline{/45^\circ 58}$	5·622	−0·7860, 0·0458	1·00275	0·0247
3·3	0·2470	$0{\cdot}3494\,\underline{/46^\circ 10{\cdot}5}$	6·184	−1·1617, 0·0475	1·16536	0·1720

Fig. 57 is plotted from Table XXIII. Up to 1 n.m. between the coils the equivalent attenuation constant is practically the same as that calculated on the assumption of even distribution from the formula,

$$a=\frac{R'+R_c}{2}\sqrt{\frac{K'}{L'}}=0{\cdot}0133.$$

Thereafter it begins to diverge, and this divergence increases with great rapidity after $l=3$.

At a frequency of 2,000 p.p.s. R_c may be taken as 14·2 ohms, from which $Z_c=1257/\underline{89°21{\cdot}2}$. The other quantities do not vary appreciably. Also

$$Z_0'=84{\cdot}83\,\overline{\backslash\pi/4} \text{ and } Z_c/2Z_0'=7{\cdot}407/\underline{\pi-45°39{\cdot}7}.$$

Curve B is plotted from Table XXIV.

Table XXIV.—Fig. 57, Curve B. $n=2{,}000$

l	$\alpha'l=\beta'l$	Sinh $P'l$	$Z_m/2Z'_0$		Cosh Pl	Cosh al	a
0·8	0·0947	0·1340/45°8·8	5·926	$/\pi-45°38{\cdot}8$	0·2070, 0·0159	1·00015	0·0208
1·1	0·1302	0·1841/45°19·1	8·148		−0·5003, 0·0255	1·00047	0·0276
1·2	0·1420	0·2008/45°23·3	8·889		−0·7852, 0·0282	1·00100	0·0370
1·5	0·1775	0·2511/45°36	11·111		−1·7895, 0·0338	1·78995	0·7908

Curve B leaves the horizontal, the position of which is calculated from the formula $a=\frac{R'+R_c}{2}\sqrt{\frac{K'}{L'}}=0{\cdot}0178$, much sooner than Curve A. The higher the frequency the closer must the spacing be if the assumption of even distribution is to hold good. In the present instance it is clear that 1 n.m. spacing is sufficient. If the coils are spaced more closely together they are a more frequent source of weakness, and the time constant of the individual coil is reduced. If spaced further apart the loading is more concentrated and the time constant of the coils may be greater for the same total inductance, but the coils are bulkier and more difficult to enclose within the sheathing wires.

From Fig. 57 it is clear that if the spacing be greater than 1·3 n.m. all frequencies higher than 2,000 are eliminated, whilst if the spacing be greater than 3·2 n.m. all frequencies higher than 800 are eliminated. The results of speech tests on loaded cables, arranged so as to permit of variable spacing, show that the highest indispensable frequency lies between 800 and 1,100, but that for first-rate transmissions frequencies up to 1,500 p.p.s. should be retained.*

SHUNT LOADING.

In obtaining the Pupin Criterion the apparatus was supposed to be inserted midway in the cable section in the form of a series impedance. It may equally well take the form of a conductance to earth.† To consider this case return to the general equations (13), and suppose that the central section shrinks to a conductance to earth $Z_m = G_m + ipK_m$. This may be obtained by making the central section of cable of zero resistance and inductance, and writing $G_m = G_2 l_2$ and $K_m = K_2 l_2$. Then Z_2 and sinh 2 are both zero ; cosh 2 is unity, and $\frac{\sinh 2}{Z_2}$ is Z_m.

The equations now become

$$V_r = V_s[(Z_3/Z_1)\sinh 1 \sinh 3 + Z_3 Z_m \cosh 1 \sinh 3 + \cosh 1 \cosh 3]$$
$$-C_s[Z_1 \sinh 1 \cosh 3 + Z_3 \cosh 1 \sinh 3 + Z_1 Z_3 Z_m \sinh 1 \sinh 3]$$
$$C_r = C_s[Z_1 Z_m \sinh 1 \cosh 3 + (Z_1/Z_3)\sinh 1 \sinh 3 + \cosh 1 \cosh 3]$$
$$-V_s\left[\frac{\sinh 1 \cosh 3}{Z_1} + Z_m \cosh 1 \cosh 3 + \frac{\cosh 1 \sinh 3}{Z_3}\right]. \quad (45)$$

If the first and second sections are alike and each of length $l/2$, as in Fig. 58, these equations reduce to

$$V_r = V_s\left[\cosh Pl + \frac{Z_0 Z_m}{2}\sinh Pl\right]$$
$$-C_0\left[Z_0 \sinh Pl + \frac{Z_0{}^2 Z_m}{2}(\cosh Pl - 1)\right]. \quad . \quad (46)$$
$$C_r = C_s\left[\frac{Z_0 Z_m}{2}\sinh Pl + \cosh Pl\right] - V_0\left[\frac{\sinh Pl}{Z_0} + \frac{Z_m}{2}(\cosh Pl + 1)\right].$$

As before, let P and Z_0 be the constants of the equivalent cable loaded without discontinuity. Then

$$\left.\begin{aligned} \cosh Pl &= \cosh P'l + \frac{Z_0' Z_m}{2}\sinh P'l \\ \text{and} \quad Z_0 &= Z_0'\sqrt{\frac{2Z_0' \sinh P'l + Z_0'^2 Z_m(\cosh P'l - 1)}{(2/Z_0')\sinh P'l + Z_m(\cosh P'l - 1)}} \end{aligned}\right\}. \quad (47)$$

* B. S. Cohen and G. M. Shepherd, "Jour." I.E.E., May 9, 1907.
† S. P. Thompson.

These are the expressions for a conductance to earth which correspond to the Campbell expressions (39) for a series impedance. From them the attenuation produced by any distribution of leaks can be calculated, as was done in the case of coils, or a criterion may be obtained as to the limits within which the distribution may be regarded as uniform.

Let, as before, $P'l$ be small, so that to a first approximation $\sinh P'l = P'l$, and write (47) as

$$\sinh^2\frac{Pl}{2} = \sinh^2\frac{P'l}{2} + \frac{Z_0'Z_m}{2}\sinh P'l$$
$$= \frac{P'^2l^2}{4} + \frac{Z_0'Z}{4}P'l$$
$$= \frac{l^2(R'+ipL')(G'+ipK')}{4} + \frac{l^2(R'+ipL')(G_m/l+ipK_m/l)}{4}$$
$$= \frac{l^2}{4}(R+ipL)[(G'+G_m/l)+ip(K'+ipK_m/l)].$$

Hence, if Pl is so small that $\sinh Pl_2$ is sensibly the same as $Pl/2$, the equation becomes

$$(R+ipL)(G+ipK) = (R'+ipL')[G'+G_m/l)+ip(K'+K_m/l)],$$

which is satisfied if

$$R=R',\ L=L',\ G=G'+G_m/l, \text{ and } K=K'+K_m/l.$$

In other words, the capacity and leakance of the apparatus may be added to the line constants as though they were uniformly distributed along the cable. The criterion is the same as for series loading.

Application to Practical Case.

Let the cable have an inductive leak to earth of R_m ohms and L_m henries per unit length, in addition to the ordinary capacity K' and leakance G'. In the formula for the propagation constant, $G'+ipK'$ must now be replaced by

$$G'+ipK'+\frac{1}{R_m+ipL_m}=G+ipK.$$

Hence, equating real quantities and imaginary,

$$\left.\begin{aligned} G&=G'+\frac{R_m}{R_m^2+p^2L_m^2} \\ \text{and}\qquad K&=K'-\frac{L_m}{R_m^2+p^2L_m^2}\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

For heavy loading, R_m^2 is small compared with $p^2L_m^2$, and

$$\left.\begin{aligned} G&=G'+\frac{R_m}{p^2L_m^2} \\ \text{while} \qquad pK&=pK'-\frac{1}{pL_m} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

It is to be observed that G is always increased by the presence of the shunt coil, whereas K is decreased. When $p^2=1/L_mK'$, K is zero, and the circuit resonates to the sending E.M.F. The circuit does not transmit all frequencies alike, but is selective and may be tuned.

Taking the core of type 160/150 as before, where $R=R'=14{\cdot}2$ ohms and $K'=0{\cdot}157$ mfd., suppose that coils of 0·1 henry are placed in shunt at intervals of 1 n.m. apart, instead of in series. Then R_m is small compared with $p^2L_m^2$, and, neglecting the natural leakance and inductance of the cable, the formula for the attenuation constant becomes simply

$$a=\sqrt{\frac{R}{2}\left(pK'\sim\frac{1}{pL_m}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Curve A, Fig. 58, is plotted from Table XXV. It resembles the resonance curves of Chapter III. When $p=1/\sqrt{KL}$ or $n=1{,}270$, the attenuation constant is zero, to a first approximation. Curve A lies in general much above Curve B, the attenuation constant with series loading, and only for a constricted space in the region of the resonating point does it come beneath it; nevertheless, it is better over a certain range than Curve C for no loading, but much worse for low frequencies, which would disappear altogether in transmission. At low frequencies the attenuation constant approximates to the dotted line, the equation to which is

$$a=\sqrt{\frac{R}{2pL_m}},$$

and at higher frequencies to the line of equation,

$$a=\sqrt{\frac{RpK'}{2}}.$$

This type of circuit would be most useful in a case where it is desired to transmit a single frequency only.

Table XXV. FIG. 58, CURVE A.

n	p	$10^6 \times pK'$	$10^6/pl_m$	$10^6 pK' \left(\backsim \frac{1}{pl_m} \right)$	a^2	a
200	1,257	197	7,957	7,760	0·0551	0·235
400	2,513	394	3,978	3,583	0·0254	0·160
600	3,770	591	2,653	2,061	0·0146	0·121
800	5,026	789	1,989	1,200	0·0085	0·092
1,000	6,283	986	1,591	604	0·0043	0·065
1,200	7,539	1,184	1,326	142	0·0010	0·032
1,300	8,167	1,282	1,224	58	0·0004	0·020
1,400	8,796	1,381	1,137	244	0·0017	0·042
1,600	10,053	1,578	995	583	0·0041	0·064
1,800	11,311	1,776	884	892	0·0063	0·080
2,000	12,567	1,973	796	1,177	0·0084	0·092

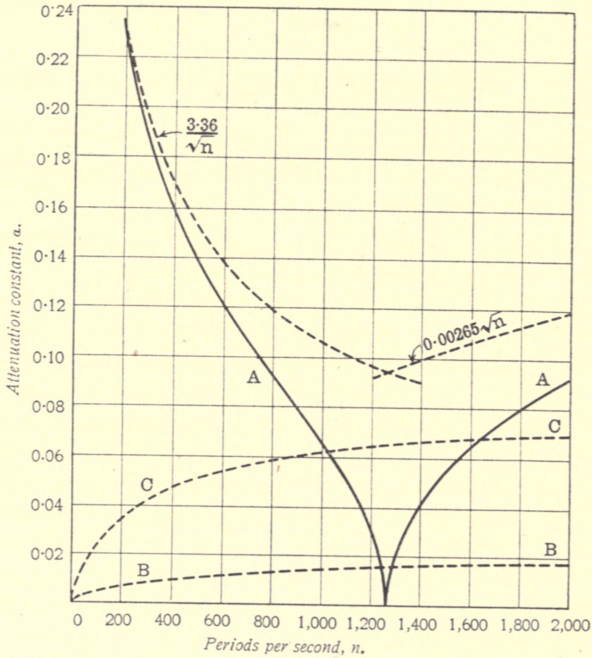

FIG. 58.—ATTENUATION CONSTANT OF SHUNT-LOADED CABLE.

Type 160/150. R=14·2. K=0·157×10⁻⁶. R_c=7·1 at 800. L_c=0·1.
Curve A. Coils in shunt at every n.m.
,, B. Coils in series at every n.m.
,, C. Cable without coils.

By increasing the loading the point of resonance could be moved to the position of mean speech frequency. But the tuning would then be even more marked. If the resistances of the loading coils and the inductance and leakance of the cable itself were taken into account the curve would be somewhat modified, especially at the resonating point where it would not fall to zero. Other interesting cases which may be treated similarly are when the shunts contain a series condenser and when series and shunt loading are combined.

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CHAPTER VII.

ALTERNATING CURRENT MEASUREMENTS.

Wheatstone Bridge for Alternating Currents—Variable Standard of Self-Inductance—Variable Standard Resistance—Source of Alternating Current: Vibrating Wire—The Microphone Hummer—The Sine-wave Alternator—Detecting Instruments—Vibration Galvanometers—Simple Theory of the Vibration Galvanometer—Measurement of Capacity—Concealed Capacity—Standard Variable Mutual Inductances—Theory of Mutual Inductance Bridge—Measurements on Completed Cables—Method of Calculation: Example—Direct Method—High-Frequency Ammeters—The Franke Machine—Drysdale-Tinsley Alternating-Current Potentiometer—Comparison of the Different Methods.

As in the case of direct currents, the methods which are used in measurement with alternating currents may be divided into two classes, according to whether they rest on a null or a deflectional basis. Null methods, employing as a rule some form of bridge or potentiometer, have many advantages, and are to be preferred wherever practicable. In deflectional methods it is either the heating or the dynamometric effect of the current that is measured, and if the source is impure and contains harmonics these will contribute their share to the total effect, and thus tend to vitiate the result. In null methods the influence of the harmonics if present is merely to mask the position of balance and render difficult its exact location. Their presence is evident, and endeavours may either be made to remove them, or, by the use of tuned detecting instruments responding to the fundamental, the work may be carried on as though they were not present, even although the source is far from pure.

Wheatstone Bridge for Alternating Currents.

When it is desired to measure the inductance of a piece of apparatus several methods may be adopted. Much the easiest plan, provided that a standard inductance coil is available, is to compare the inductances of the two in the arms of a Wheatstone bridge, as in Fig. 59. The battery of direct-current measurements is replaced by some source of alternating current, and a telephone receiver is substituted for the galvanometer. The ratio arms are conveniently simple resistances.

The conditions for balance are easily obtained. From Chapter III. they may be written down at once, using the

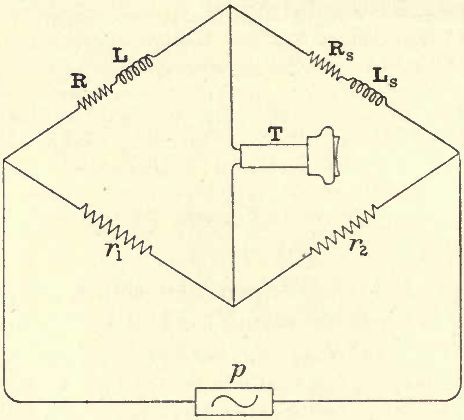

Fig. 59.—Wheatstone Bridge for Comparing Inductances.

R, L = Unknown resistance and inductance.
R_s, L_s = Standard resistance and inductance.
r_1, r_2 = Resistance ratio arms.
T = Telephone.

complex impedances of the bridge arms in the ordinary bridge formula—

$$(R+ipL)r_2=(R_s+ipL_s)r_1. \quad . \quad . \quad . \quad . \quad (1)$$

where the frequency of the alternating source is $n=p/2\pi$. Equating real quantities and imaginary quantities, it follows that

and $$\left.\begin{array}{l} Rr_2=R_sr_1 \\ Lr_2=L_sr_1 \end{array}\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Two conditions require to be satisfied for balance, and therefore two adjustments require to be made before silence is obtained in the telephone. Most conveniently, R_s and L_s are adjustable and the ratio arms are fixed. But if an adjustable inductance is not available the ratio arms r_1 and r_2 must be altered as well as R_s until balance is found. If the impedances are low, the required fine adjustment of the ratio arms may be obtained by the use of an ordinary slide wire.

The resistances r_1 and r_2 should be devoid of inductance. If wound bifilarly they are free from this defect, but are liable to contain distributed capacity, which at high frequency would lead to serious inaccuracy. They are best wound in a single layer on flat mica plates and afterwards varnished with shellac and baked, the direction of winding being continually reversed so as to neutralise induction.*

In all bridge measurements, in order that the most accurate results may be obtained, it is highly desirable to arrange the apparatus so that the ratio arms are equal, otherwise various unsuspected inequalities may enter. Thus, suppose for example that mutual inductance M exists between the standard coil and the unknown. Writing down the mesh equations in the usual manner and solving, the conditions for equilibrium are easily found to be

$$r_2 R = r_1 R_s$$

and

$$r_2(L \pm M) = r_1(L_s \pm M),$$

from which it is clear that the presence of mutual inductance between the coils does not affect the balance if the ratio arms are equal. The exact equality of the ratio arms may be checked by interchanging them, whereupon the position of balance should remain unaltered.

Variable Standard of Self-Inductance.

To meet the requirement of equality it is necessary that the standard inductance L_s should be capable of adjustment in order that it may be brought to equality with the unknown inductance L. A convenient range in measuring loading coils would be from zero up to 250 millihenries. The adjustment

* Feussner, "E.T.Z.," 20, p. 611, 1899. It is advisable to check the values of the resistances from time to time.

may either be continuous, or in steps with a smaller supplementary adjustable standard in series to fill up the gaps. This adjustable standard may take the form of two coils, one of which is fixed and the other is movable, and either pivoted so as to turn inside the fixed coil,* or arranged to slide on top of it. The two are connected up in series by flexible leads, and their total self-inductance is

$$L_s = L_1 + L_2 + 2M,$$

where L_1 is the inductance of the fixed coil, L_2 that of the moving coil and M is their mutual inductance. When the two coils are coaxial M is a maximum. As the axis of the moving coil is turned so as to be inclined to the axis of the fixed coil—assuming that the rotating form of construction is adopted—M decreases and passes through zero when the axes are at right angles, after which it is negative. It is seen that it is only a portion of L_s which is variable. The resistance of the combination throughout the movement is constant, and the bridge balance is unaffected on this account, resistance and inductance adjustments being therefore quite independent.

Attached to the moving coil is a pointer which passes over a scale. The inductance is calibrated by comparing it with a range of fixed standard inductances at points sufficiently close together on the scale.

Older types of variable inductance contained much metal work, which it is desirable to avoid as far as possible owing to the inaccuracies produced by eddy currents excited in adjacent conductors when measurements are made at telephonic frequencies. Especially is it necessary to avoid the presence of iron, and when metal fittings are indispensable they should be as small as possible and preferably made of resistance material.

* M. Wien, "Ann." d. Phys., 57, p. 249, 1896. In this instrument the range is from 0·6 to 120 millihenries. The fixed coil has four windings, and the inner two; they are joined together by flexible conductors in the axis of rotation. The windings are not short-circuited but cut out by a switch. The ranges are chosen to overlap. For a full description of an instrument of this kind, *see* G. Shepherd, "Telephone Journal," April, 1909.

The coils of fixed standard inductances are wound in square channels cut in some such rigid material as marble, or, in the sub-standards, ebonite, or teak which has been impregnated with paraffin wax.* The wire is of stranded copper, silk covered and shellacked, or thoroughly impregnated with paraffin wax by boiling, so as to secure the highest possible insulation.† In order that skin effect may not be appreciable, the copper wire should not be of greater diameter than, say, 0·036 in. (No. 20 S.W.G.).‡ The diameter of the coil and the side of the square section are chosen so as to make R a minimum for the given inductance and size of wire.§ The leads to the apparatus should, wherever possible, be twinned in order to reduce their self-inductance to a minimum.

Variable Standard Resistance.

The adjustable resistance R_3 may also be capable of variation in steps, and is preferably of dial pattern. The same remarks regarding construction apply here as to the ratio arms. All metal should be avoided as far as possible, and to eliminate capacity the higher resistances may be made up of sets of coils in series, wound bifilarly on porcelain bobbins, or on split metal formers; or the wire may be woven with silk threads into a fabric or gauze about $7\frac{1}{2}$ in. wide, which is mounted on porcelain insulators and subdivided where required.|| Alternatively, the resistances may be wound on mica plates, the direction of winding being frequently reversed. To counteract residual inductance, probably not entirely removed by the reversals in winding, a compensating device may be employed whereby, as each resistance coil is switched in, a coil of the same shape, but of copper wire, is cut out. The increase in the resistance is the difference between those of the two coils, while the path

* M. Wien, "Ann." d. Phys., 58, p. 553, 1896.

† Chap. III., p. 66.

‡ Stranded enamelled wire is used by the Phys. Techn. Reichsanstalt. For the increase in effective resistance due to coiling, *see* F. Dolezalek, "Ann." d. Phys., 12, p. 1142, 1903.

§ The Electrician, Vol. LXXV., p. 64 and p. 98, 1915.

|| Duddell-Mather.

of the current is unaltered in shape, and therefore the inductance is unaffected.*

To obtain the fine gradations necessary to secure an exact adjustment of the bridge, the steps may be filled in by the addition in series of a carbon plate or carbon cloth rheostat. As this form of resistance is somewhat uncertain in value, and requires independent measurement after every adjustment, it is preferable to use a loop of resistance wire. A simple form is shown in Fig. 60. Here the loop of non-oxidisable wire, soldered at the ends to the terminals T T, is kept stretched by the screws S S. The bridge piece B moves along a scale parallel to the wires, and short-circuits the loop by means of a double spring clip. The instrument may be calibrated as a low resistance up to, say, 1 ohm by means of the potential

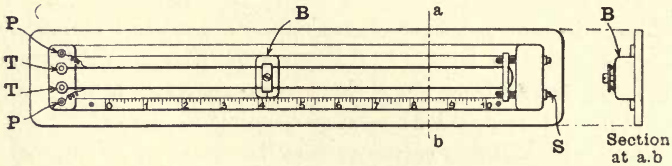

FIG. 60.—ADJUSTABLE LOW RESISTANCE.

T=Current terminals.
P=Potential terminals.
S=Screws for tightening wire loop.
B=Short-circuiting bridge.

terminals P P, and the varying inductance of the loop, although slight, may be measured and allowed for in exact work. Alternatively, the same principle as described above may be used in this case also, according to which a copper loop would replace the resistance-wire loop, and thus prevent any change in the inductance.

SOURCE OF ALTERNATING CURRENT—VIBRATING WIRE.

As a source of alternating current some form of interrupter may be used, of which one of the simplest is the tuning-fork interrupter. It consists of a tuning fork, one limb of which carries a style just dipping into mercury, and thereby making and breaking the circuit of a battery. The tuning fork is kept

* The Campbell constant inductance rheostat and resistance box are based on this principle.

in vibration by the agency of an electromagnet in series with the battery. This kind of instrument suffers from the defect that the amplitude of the vibration is small, and therefore the adjustment of the break is delicate. It is preferable to replace the tuning fork by a vibrating wire.*

In Fig. 61 the steel wire W, under tension, is kept in vibration by the electromagnet M, which is energised by the battery B_1. The circuit of B_1 is made and broken as the platinum† style dips into and is pulled out of the mercury cup C_1. To reduce sparking at the contact the surface of the mercury may be kept covered with alcohol, and the magnet winding should be of low inductance and consist of a few turns of wire. The vibrating wire will then run for hours at a time without re-

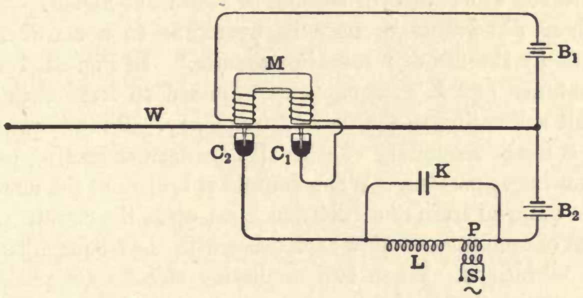

FIG. 61.—VIBRATING WIRE INTERRUPTER.
W=Vibrating wire. M=Magnet. B_1=Wire battery. B_2=Local battery.
C_1, C_2=Mercury cups. L, K=Tuning circuit. P, S=Transformer.

quiring attention. It should be strongly clamped, preferably in a steel frame, with a fine screw adjustment at one end, by means of which the tension, and therefore the period of the wire, can be adjusted. The battery may conveniently be a small 4-volt portable accumulator. Alternatively, a copper wire carrying a current may be used in place of a steel wire, with a view to reducing sparking by breaking a circuit of less inductance than that containing the attracting magnet.‡

* M. Wien, "Ann." d. Phys., 42, p. 593; 44, p. 681, 1891.

† A nickel style and paraffin oil are used by the N.P.L.

‡ Arons, "Ann." d. Phys., 66, p. 1177, 1898. E. Orlich, "E.T.Z.," 24, p. 502, 1903.

By including a transformer in the circuit, it could be used to supply current alternating with the wire frequency to the testing bridge. In Fig. 61 the supply circuit is shown as an independent circuit, made and broken by the contact C_2, an arrangement which is more convenient for purposes of regulation.

The chief advantages of the wire interrupter are (*a*) the freedom from variation of its period when the wire is left to itself, and (*b*) the ease and delicacy with which the period can be adjusted. Its disadvantages are (*a*) the impurity of the wave which it supplies, rendering necessary the use of a tuned detecting instrument in the other diagonal of the bridge; and (*b*) the limited range in frequency which can be obtained, from 100 to 200 vibrations per second, or somewhat higher.

These disadvantages may be overcome to a considerable extent by the use of resonating circuits.* In Fig. 61, L is an inductance and K is a capacity adjusted to form with L a circuit resonating to the desired frequency. P is the primary and S is the secondary of a small transformer leading to the measuring apparatus. When contact is broken at the mercury cup a damped train of oscillations is set up in the circuit. The trains of oscillations follow each other with the frequency of the wire vibrations. When two oscillating circuits are placed in series in the circuit of the mercury contact, and the secondaries of the transformers are connected so as to oppose each other, continuous waves are produced. Using a coil of 40 millihenries inductance and 3 ohms effective resistance, together with capacities from 2·64 mfd. down to 0·07 mfd., a range of frequencies from 500 up to 3,000 p.p.s. can be obtained in this way.

THE MICROPHONE HUMMER.

In the microphone hummer the vibrating part, instead of making and breaking contact, as in the wire interrupter, is caused to vary the pressure in a microphone. There is thus produced an alternating current, by means of which the vibrations are sustained. In Fig. 62 a very simple and portable form

* B. S. Cohen, "Phil. Mag.," p. 480, September, 1908.

of microphone hummer is shown.* From a steel membrane is suspended a light microphone consisting of a sack containing carbon granules. The membrane is attracted by a steel cylinder which is surrounded by a magnetising coil. Suppose that the membrane approaches the cylindrical magnet, then the resistance of the microphone is lowered by the increased pressure, and the current from the local battery increases. An induced current flows in the secondary circuit of the transformer, which is connected up in such a way that the current increases the magnetism of the cylinder, and thus assists the

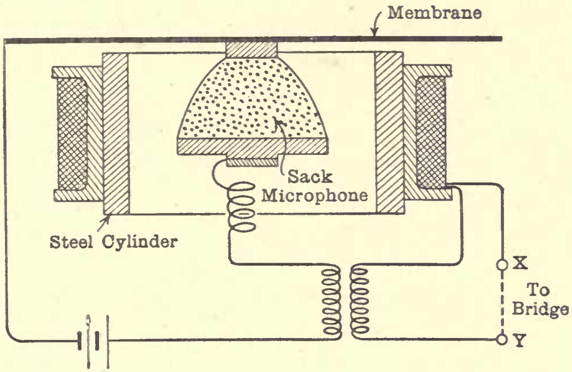

FIG. 62.—SIMPLE FORM OF MICROPHONE HUMMER.

motion of the membrane. Leads are taken to the testing bridge at XY, across which a condenser may be placed to obtain resonance and improve the wave form. Frequencies from 300 to 600 or higher are obtainable by varying the thickness of the membrane.

For high frequencies a different form of construction must be adopted. The vibrating part may consist of a rod of mild steel† supported horizontally at nodal points, as in Fig. 63, and

* F. Dolezalek, "Zeitschr. f. Instkde.," 23, p. 242, 1903.

† A. Campbell, "Proc." Roy. Soc., A., Vol. LXXVIII., p. 208, September, 1906. For earlier microphone hummers see also R. Appleyard, "Elec. Rev.," 1890, Vol. XXVI., pp. 57 and 656; J. E Taylor, "Journ." Inst. E.E., 1901, Vol. XXXI., p. 396.

carrying at one end a light microphone shunted by a condenser, K_1. The secondary of the transformer T is connected to the coil of a polarised telephone magnet, M. A sharp tap on the rod suffices to set it in vibration, when the pulsating current through the microphone produces an alternating current round M, which maintains the vibration. The testing current is taken from another secondary winding of T. A mild steel rod 2·5 cm. in diameter and 23·7 cm. long gives 2,000 p.p.s. With shorter rods frequencies of 3,000 and 4,000 can be attained.

The chief advantages of this form of microphone hummer, in addition to portability, are (1) the purity of the tone which

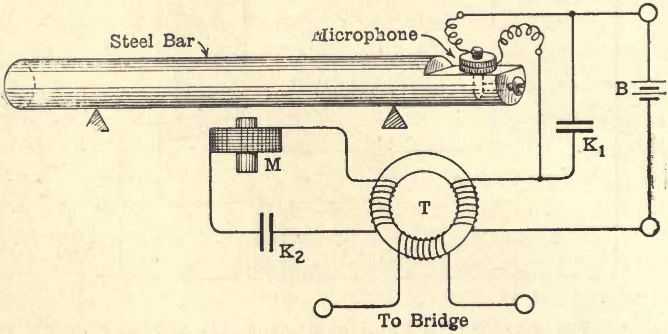

FIG. 63.—CONNECTIONS OF CAMPBELL MICROPHONE HUMMER.

M=Magnet. T=Transformer. K_1=Microphone condenser.
K_2=Tuning condenser, B=Local battery.

it is capable of yielding ; (2) the invariability of the frequency, which can be known as a consequence to a high degree of accuracy. The purity of the note can be increased by the insertion of a condenser, K_2, to resonate with the inductance of M and the transformer winding. Its disadvantages are (1) the limited output, which is (2) liable to fluctuation during the course of a test. It is, therefore, desirable to have in series with the bridge some form of current measurer, especially when iron is present in the bridge and the degree of magnetisation requires to be kept under control.

The Sine-wave Alternator.

The production of high-frequency alternating current by mechanical means is a problem which has occupied much attention in wave telegraphy. In telephonic measurements the need for such a generator is no less, but the difficulties are not so great. The frequency required is much lower and the output much less. There is, on the other hand, the additional requirement that the wave-form should be nearly sinusoidal, because the lower frequencies of telephony do not permit of the same facilities for purification by resonance.

The earlier machines were of the inductor type. This form has the advantage of dispensing with moving coils and contact brushes. The rotor is usually built up out of thin soft-iron stampings with teeth projecting round the rim. The whole is carefully balanced, and forms a flywheel, by means of which steadiness of motion is assisted. The teeth of the rotor pass in front of the unsaturated core of a horseshoe magnet, the magnetic circuit of which is consequently alternately closed and opened. The fluctuations in the magnetic field produced in this way give rise to an E.M.F. in a second winding on the magnet core, which is led to the measuring bridge.* In another form† soft-iron stampings are let into slots in a gunmetal disc, which forms the rotor and passes between the poles of a horseshoe magnet. The magnet tips, on which the armature coils are placed, must also be built up of soft-iron stampings to reduce loss by eddy-currents to a minimum.

By giving the magnet poles a suitable shape the wave-form can be improved. Nevertheless, the output is limited. For this reason the inductor principle has been abandoned in the more recent forms of alternator, which are of the multipolar, rotating field magnet type of generator. One of the latest machines‡ is designed to give 10 watts at any frequency from

* F. Dolezalek, "Ann." d. Phys., 23, p. 240, 1903. The rotor has 120 teeth, and at 4,000 revs. gives 8,000 p.p.s. The maximum useful output is 15 watts.

† M. Wien, "Ann." d. Phys., 4, 1901, p. 425. See also 66, p. 871, 1898, and A. Franke, "E.T.Z.," 12, p. 447, 1891.

‡ W. Duddell, "Proc." Phys. Soc., Vol. XXIV., 1912, p. 172.

100 to 2,000 p.p.s. The stator is wound as a gramme ring, and requires a large exciting current; the air-gap is considerable and the wave consequently little distorted by load. The rotor, which is of 20 cm. diameter and has 30 poles, runs at 8,000 revs. per min. The output at the higher frequencies is $\frac{1}{2}$ kw. There a slight third harmonic present.*

The advantages of the sine-wave alternator are (1) its large and steady output, and (2) the possibility of securing much higher frequencies with yet adequate strength by resonating the harmonics. The frequency range is (3) well under control and can be varied continuously; moreover (4), it can be accurately measured by counting the number of revolutions per minute and multiplying by a constant factor according to the number of poles or teeth.

The great disadvantage of the sine-wave alternator is the tendency of the frequency to fluctuate slightly. In some bridge measurements balance is dependent on the frequency, variations in which render very difficult the process of approximating to silence in the telephone. Moreover, changes of frequency upset the adjustment of resonating circuits if such are used, as well as of tuned detecting instruments in the other bridge diagonal.

The alternator should be coupled to a motor which is driven from a storage battery; on a power circuit, fluctuations in voltage produced when heavy machines are switched in and out may cause sudden alterations of speed at a critical moment.

As a further security some kind of automatic governor is desirable. This may take the form of a contact on the motor shaft which works in synchronism with a tuning fork. Should the motor speed increase or decrease the synchronism is disturbed and a relay actuated which decreases or increases the

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* Other forms of generator have been described by B. G. Lamme (inductor type, 10,000 cycles), "Trans." Am.I.E.E., Vol. XXIII, 1904, p. 417, and Hartmann-Kempf (rotating permanent horseshoe magnets), "Phys. Zeitschr.," 10, 1909, p. 1018). See also R. A. Fessenden, The Electrician, 61, 1908, p. 441, and E. F. W. Alexanderson, (100,000 cycles), The Electrician, July, 1909; "Trans." Am.I.E.E., Vol. XXVIII, Part I, 1909, p. 399.

driving current of the motor and restores synchronism. Many other forms of governor have been described. The control need not necessarily be automatic if a sensitive speed indicator, preferably electrical, is available.* The assistance of a second observer is then required.

DETECTING INSTRUMENTS.

In the adjustment of the bridge to balance, some form of detecting instrument is necessary in the bridge diagonal, which shall take the place of the galvanometer in direct-current measurements, and indicate the presence of current in that diagonal or of P.D. across it. For this purpose an ordinary telephone receiver offers great advantages. It is (1) cheap, portable and readily obtainable ; (2) on account of the delicacy of response by the ear to the faintest sounds, it is extremely sensitive ; (3) when the current wave is impure it may to a certain extent be analysed by the ear, and after a little practice it becomes possible to concentrate attention only on one component of the wave. Its chief disadvantage is that it is not a tuned instrument. A purely sinusoidal source of testing current is, therefore, necessitated in exact measurements. A fine balance would otherwise be difficult to secure, inasmuch as, although the fundamental were reduced to zero, total silence would not be produced in the telephone, which would still respond to an unbalanced harmonic. The exact point of disappearance of the fundamental would thus be disguised.

This disadvantage may be removed to some extent by the exertion of pressure on the membrane from a sharp point, such as that of a screw passing through the ebonite cover of the receiver, or by some other similar adjustable device. The position at which the point presses upon the membrane is chosen by trial, so that the membrane may be constrained to vibrate only with the single frequency to which it is desired that it should respond. This procedure is especially serviceable for the higher frequencies. The ordinary membrane may require to be replaced by one of stouter make.

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* Rosa, " Bull." of Stand., 3, p. 557, 1907.

In the Wien optical telephone* the membrane is corrugated. At its centre is fixed a small iron disc which is free to vibrate between the poles of a horseshoe magnet. The poles are wound to take the alternating current to be measured. The vibrations of the disc are communicated by a pin to a small mirror supported by a spring. The adjustment to the required periodicity is made by choosing a suitable membrane and an appropriate size of iron disc, and by varying the damping by altering the position of the magnets.

The desire for better tuning has led to the introduction of instruments which are capable of exact tuning and are intended in use to resonate at the testing frequency.

Vibration Galvanometers.

The earlier forms of these instruments may be considered as adaptations of the Thomson galvanometer, inasmuch as the moving part consists of a piece of iron or steel and the current to be measured passes through fixed coils. Later forms of vibration galvanometer have been of the moving-coil type.

The Rubens† form employs the torsional vibrations of a stretched wire, the frequency of which can be varied by altering the length of the wire or its diameter. Attached to the wire are small pieces of iron which are attracted by electromagnets in such a manner as to be set into vibration.

For use with higher frequencies the instrument was redesigned by M. Wien.‡ The moving system is smaller, and consists of a brass wire to which a piece of tin, 4 mm. by 2 mm., is soldered, carrying three small iron magnets and a small mirror. The field magnet is of ring shape, and has a flexible core 10 cm. in diameter and of section 4 mm. by 3 mm. Tuning is accomplished by adjusting the position of nodal pieces on the wire and by bending the core. A sensitivity of 1 mm./m. for 3×10^{-7} amperes at 500 p.p.s. and $1{\cdot}5\times10^{-6}$ amperes at 1,000 p.p.s. is attainable.

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* "Ann." d. Phys., 42, 1891, p. 593; 44, 1891, p. 681; 47, 1892

† Rubens, "Ann." d. Phys., 56, p. 27, 1895.

‡ M. Wien, "Ann." d. Phys., 4, p. 439, 1901.

In the Drysdale-Tinsley vibration galvanometer, which is suitable for frequencies up to 200 per second, the moving system consists of a small piece of soft iron suspended horizontally by a vertical silk fibre and carrying a light mirror. The system is clamped between the poles of a polarising permanent magnet, and is kept in vibration by the deflecting action of a coil of wire which is traversed by the alternating current that is being measured. To bring the movement into resonance the magnetic field is shunted by a movable cross piece, the exact position of which is adjusted in tuning by means of a milled head. The great advantage of this type of instrument is the opportunity which it presents of choosing a coil to suit the conditions of testing.

In the Campbell* vibration galvanometer, one form of which is shown in Fig. 64, a very narrow coil is held in the air-gap of a permanent magnet by means of a bifilar suspension, the tension of which can be altered by an adjustable spiral spring. Two milled heads and suitable gearing permit of the adjustment of the effective length and also of the tension of the bifilar suspension. The moments of inertia of coil and mirror are reduced to a minimum compatible with mechanical and optical considerations. To tune the instrument alternating current is sent through it, conveniently by purposely putting the bridge out of balance, and the length of the bifilar suspension is adjusted until the light spot widens out into a long band. Thereupon by fine adjustment of the tension the band is made to attain its maximum length, and the tuning is correct. In a recent model† the suspension is unifilar instead of bifilar. With an effective resistance of 70 ohms at 100 p.p.s. the sensitivity is of the order of 50 mm. at 1 metre for 1 microampere.

The movement of the Duddell‡ vibration galvanometer resembles that of the bifilar oscillograph in consisting of two wires, stretched in the narrow gap of a permanent magnet, across which is fixed a small mirror, but whereas the damping in the oscillograph is designed to make the motion aperiodic,

* A. Campbell, Phys. Soc. "Proc.," 20, 1907, p. 626.

† "Proc.," Phys. Soc., 25, 1913, p. 203, and 26, 1914, p. 120.

‡ "Proc.," Phys. Soc., 21, 1910, p. 774.

in the vibration galvanometer the damping is intended to be as small as possible. The period of the instrument is adjusted by alteration of the length and tension of the wires.

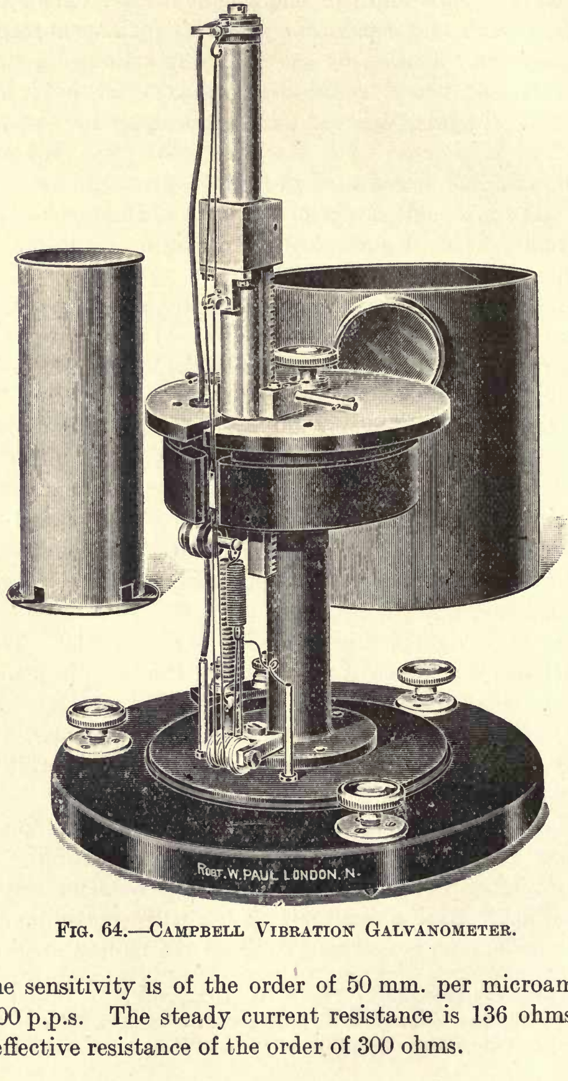

FIG. 64.—CAMPBELL VIBRATION GALVANOMETER.

The sensitivity is of the order of 50 mm. per microampere at 100 p.p.s. The steady current resistance is 136 ohms and the effective resistance of the order of 300 ohms.

In the moving-coil type of vibration galvanometer the back E.M.F. is considerable, unless the coil is of few turns, and may greatly increase the effective impedance of the instrument and lower the volt sensitivity accordingly. In the bifilar type the back E.M.F. is comparatively low. The greatest defect of the latter type is the small size of mirror which it is necessary to use.

The chief advantages of vibration galvanometers are (1) the sharp tuning of which they are susceptible, and by which the sensitivity to their natural period is enormously increased, rendering it permissible to regard the testing current as consisting of current of that period alone; (2) the ease in adjustment of the frequency; and (3) their robustness, which arises from the strained suspension, and their lack of sensitivity to direct current rushes. Their chief disadvantage is the way in which the sensitivity falls off at the higher frequencies. Above 1,000 p.p.s it is preferable to use a tuned telephone.

Simple Theory of the Vibration Galvanometer.

Let θ be the angular deflection of the coil at any instant when the current through it is $I_{max.} \sin pt$. Then $d\theta/dt$ is the velocity of motion, and $d^2\theta/dt^2$ is the acceleration. Since the sum of the reaction torques must balance the action torque producing the motion, the equation of motion may be written down at once in the form

$$mk^2\frac{d^2\theta}{dt^2}+b\frac{d\theta}{dt}+c\cdot\theta=g\cdot I_{max.}\sin pt, \quad . \quad . \quad . \quad (3)$$

where mk^2 is the moment of inertia of the coil movement, b is the damping couple, c is the restoring torque arising from the suspension, and g is a factor of proportion. The assumption is made that the damping whether electromagnetic or due to air friction is directly proportional to the velocity of motion. Now, in a circuit containing resistance, inductance and capacity in series, in which an alternating E.M.F. is acting, the voltages may be equated in a similar manner.

Thus $$\frac{Q}{K}+L\frac{dC}{dt}+RC=E\sin pt,$$

or $$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{K}\cdot Q=E\sin pt. \quad . \quad . \quad . \quad . \quad (4)$$

The equations (3) and (4) are similar if mk^2 is replaced by L, b by R, and c by 1/K as pointed out in Chapter III. The results obtained for the electrical circuit may be applied directly to the motion of the vibrating coil.

Thus, when the circuit resonates to the frequency of the applied E.M.F., it follows from equation (79) of Chapter III. that

$$Q=\frac{E/pt}{pR} \text{ or from (3), } \theta=\frac{gI_{max.}/pt}{pb}, \quad . \quad . \quad . \quad (5)$$

and, therefore, when the coil suspension is tuned to be in resonance with the alternating current through it, the amplitude of vibration is inversely proportional to the product of the damping and the frequency. It is evident that to secure high sensitivity the damping must be reduced to a minimum, and, moreover, that as the frequency increases the sensitivity of the vibration galvanometer falls off in inverse proportion.

Again, from (31), Chapter III., the natural or free period of the electrical circuit is given by

$$T=\frac{2\pi\sqrt{KL}}{\sqrt{1-\frac{KR^2}{4L}}},$$

and, therefore, the free period of the coil is given by

$$T=\frac{2\pi mk^2}{\sqrt{mk^2 \cdot c-b^2/4}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the damping is small, as it must be in the vibration galvanometer, (6) reduces to

$$T=2\pi\sqrt{\frac{mk^2}{c}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It follows that the periodic time of the coil's motion is inversely proportional to the square root of the control. Now the direct-current sensitivity is inversely proportional to the control. Hence it follows that the direct-current sensitivity is proportional to the square of the periodic time, or is inversely proportional to the square of the frequency.

Since the tuned alternating-current sensitivity varies inversely as the frequency, the ratio which it bears to the direct current must be proportional to the frequency. This result is obtained otherwise in equation (80) of Chapter III., where the ratio is shown to be $p\mathrm{L}/\mathrm{R}$ which is the same in the mechanical analogue as pmk^2/b. The higher the frequency, *i.e.*, the horter the wires or the greater the tension on them, the greater the moment of inertia, and the smaller the damping, the greater is the sensitivity of the instrument to alternating current as compared with direct current.

Measurement of Capacity.

Condensers may be compared for capacity in a manner similar to that adopted for inductance coils, by placing them

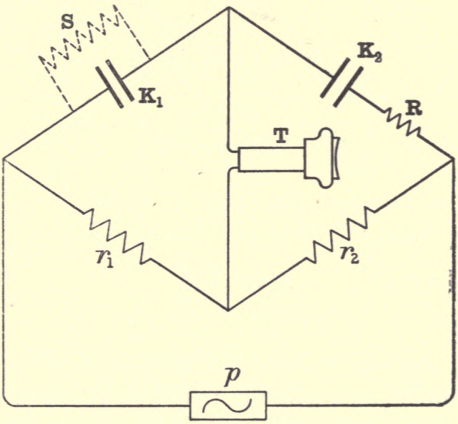

Fig. 65.—Measurement of Leakance by the Wien Series Resistance Method.

K_1, S=Unknown capacity and equivalent leak.
K_2, R=Standard capacity and series resistance.

in adjacent arms of a Wheatstone bridge. For this purpose a subdivided condenser is of great service. It is preferably built up of mica plates to ensure high insulation and small internal losses. Just as in comparing inductance coils both the inductance and effective resistance must be balanced at the same time, so in comparing condensers the effective insulation resistance must be balanced as well as the capacity. This resistance is, however, generally too high to affect the balance appreciably, and special methods must be used when it is desired to measure the effective conductance, or leakance as it is called. One of the most convenient and widely used is Wien's series resistance method,* in which the apparent leak in the condenser is balanced by a resistance in series with the standard condenser. In Fig. 65 the effective resistance, S, of the condenser K_1, is balanced by the adjustable resistance R in series with the adjustable standard condenser K_2. From the equations of equilibrium of the Wheatstone bridge it follows that

$$r_1\left(R+\frac{1}{ipK_2}\right)=\frac{r_2}{1/S+ipK_1},$$

from which and

$$\left.\begin{aligned} r_1R&=\frac{r_2/S}{1/S^2+p^2K_1^2} \\ \frac{r_1}{pK_2}&=\frac{r_2pK_1}{1/S^2+p^2K_1^2}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

Now $1/S^2$ is very small compared with $p^2K^2{}_1$.

The equations (8), therefore, reduce to

and

$$\left.\begin{aligned} r_1K_1&=r_2K_2 \\ p^2&=\frac{1}{RSK_1K_2}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9)$$

approximately, from which K and S can be found.

If r_1 and r_2 are equal, K_1 and K_2 are also very nearly equal. If S is large, R must be small, and the advantage of the method lies in this fact—a small resistance free from capacity, and readily under control, is used to balance the much greater, apparent, resistance of the condenser.

Take $K_1=K_2=1$ mfd., and let S be 5,000 ohms. Then R=5 ohms nearly, when the frequency is 1,000. The variation of R with S is shown graphically in Fig. 27 of Chapter III.

Alternatively the Carey-Foster method may be used, in which the capacity is compared with a mutual inductance.

* M. Wien, "Ann. d. Phys.," *l.c.*

In Fig. 66, K is the unknown capacity; r_1, r_2, r_3 and r_4 are resistances; L and N are inductive coils, and M is their mutual inductance. Calling the cyclic currents C_1, C_2 and C_3, the E.M.F. equations are

$$(r_4+ipN)C_1+r_1(C_1-C_2)-ipMC_3=Ee^{ipt}$$
$$G(C_3-C_2)+(r_2+ipL)C_3-ipMC_1=0$$
$$(r_3+1/ipK)C_2+r_1(C_2-C_1)+G(C_2-C_3)=0.$$

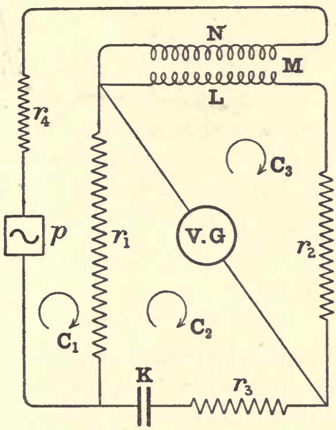

FIG. 66.—MEASUREMENT OF CAPACITY BY CAREY FOSTER'S METHOD.

K=M/r_1, r_2. L, N=Coils of mutual inductance standard.
K=Unknown condenser. r_1, r_2, r_3, r_4=Resistances. V.G=Vibration galvanometer.

Write C for C_2-C_3 and substitute for C_3. Then

$$(r_4+ipN+r_1)C_1-(r_1+ipM)C_2+ipM.C=Ee^{ipt}$$
$$-Mip.C_1+(r_2+ipL)C_2-(r_2+G+ipL)C=0$$
$$-r_1C_1+(r_1+r_3+1/ipK)C_2+G.C=0.$$

Hence

$$C=\begin{vmatrix} r_4+ipN+r_1 & -r_1-ipM & Ee^{ipt} \\ -ipM & r_2+ipL & 0 \\ -r_1 & r_1+r_3+1/ipK & 0 \end{vmatrix} \div \Delta,$$

For C to be zero the numerator must vanish. Expanding the determinant it follows that

$$-ipM(r_1+r_3+1/ipK)+r_1(r_2+ipL)=0$$

from which $$r_3=r_1(L/M-1)$$

and $$K=M/r_1r_2.$$

K is therefore known in terms of M and the resistances r_1 and r_2. If K be in microfarads and M be in millihenries, the equation becomes

$$K=\frac{M\times 1{,}000}{r_1r_2}.$$

Balance may be obtained either by adjusting the resistances, when M is fixed, or more conveniently by the use of a variable mutual inductance standard as described below, in which case L is constituted by the fixed coils and N by the adjustable coils, while r_3 is variable. Balance is obtained by adjusting r_3 and M. Most conveniently $r_1\ r_2=1{,}000$, and the reading of M in millihenries gives the value of K directly in microfarads.*

CONCEALED CAPACITY.

In the measurement of leakance as in all exact determination of capacity the greatest difficulty is that arising from hidden capacities in the bridge. These are found where the connecting wires or parts of the apparatus approach each other or any object which forms part to greater or less extent of the earthed surroundings, such as the table on which the apparatus rests. The presence of such capacities is made manifest by the disturbance of the balance which is produced when any metallic portion of the bridge is touched with the hand. From Fig. 27, Chapter III., it is clear that in the series resistance method of measuring leakance the condenser leak can vary within wide limits without affecting the capacity balance. Conversely a slight inaccuracy in the capacity balance will produce a big error in the measurement of leakance. The difficulty is especially marked when the condensers are small as is the case when

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* A. Campbell, Phys. Soc. "Proc.," Oct., 1907. "Phil. Mag.," Jan., 1908, p. 155. "The Electrician," Feb. 7, 1908.

a short length of cable is being tested. Various means may be adopted to overcome it. The point of junction of the equal ratio arms may be put to earth, and conducting shields may be placed round the telephone and the arms of the bridge. In addition a specially designed transformer may be interposed between the source of testing current and the bridge.* This transformer should have the highest possible insulation, and may consist of two ebonite cylinders covered each with a layer of varnished silk-covered wire. One cylinder slips inside the other without touching it, and between them is a third cylinder covered with a layer of tinfoil which is connected to earth. In this way it is secured that the bridge shall have no potential to earth.†

Standard Variable Mutual Inductances.

In the design and manufacture of continuously-loaded cables it is desirable to be in a position to measure the inductance and the effective resistance of short lengths of core or cable. The results of such measurements afford valuable information regarding the state of the iron whipping, and permit of variation and experiment at a minimum of cost. Work of this kind is greatly facilitated if a variable mutual inductance is available, by the aid of which the small self-inductances in question can be accurately measured. In the Campbell variable mutual inductance standard‡ shown diagrammatically in Fig. 67, there are two coaxial primary coils in series, P and P_1. In a parallel plane midway between them moves eccentrically a coil, S_1, which forms part of the secondary, and by means of which the continuous variation in the mutual inductance is secured. The position of the moving coil relative to the fixed primary coils is indicated by a pointer which moves over a semi-circular scale. The other part of the secondary winding, S, is fixed, and is made of stranded wire. By means of a switch one or

* A. Campbell.

† For alternative methods *see* K. W. Wagner and A. Wertheimer, "Phys. Zeitschr.," 13, p. 368, 1912; C. E. Hay, "Journ." I.P.O. Eng., Nov., 1912.

‡ A. Campbell, "Proc." Phys. Soc., 21, p. 69, 1908.

more of the sections of S can be placed in series with S_1, so as to increase the mutual inductance by equal increments. The main object of having two primary coils is to produce a rectilinear magnetic field in which slight vertical displacements of S_1 shall be of little account. One winding of the primaries is connected to the terminals 0 and $\times 1$; all ten are in series across the terminals 0 and $\times 10$. The range of the instrument may thus be multiplied tenfold. For short lengths of core of low resistance, the lower range is more convenient on account

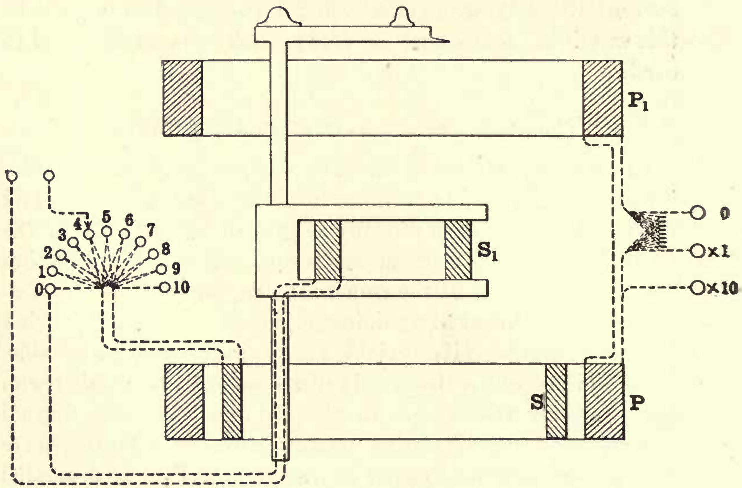

FIG. 67.—WINDING DIAGRAM FOR CAMPBELL MUTUAL INDUCTANCE STANDARD.

P, P_1 = Primary coils. S, S_1 = Secondary coils.

of its lower impedance. For telephone work the coil S_1 may conveniently have a mutual inductance of 100 microhenries with the single primary winding, and each of the turns of S may have the same. The range of the instrument is then from zero to 1·1 millihenry which may be increased to 11 millihenries by using the stranded turns of the primary in series.

A variable mutual inductance standard possesses certain advantages over a self-inductance standard: (*a*) the absolute values can be calculated with greater certainty from the geometric dimensions ; (*b*) the inductance is not so liable to vary with the frequency ; (*c*) the value of the inductance can be varied continuously from negative through zero to positive. The great advantage in practical working is that the mutual inductance is simply proportional to the number of turns in the secondary if the primary be fixed. The windings of the secondary may, therefore, be placed in series and added as though they were resistances. The employment of stranded conductors enables this advantage to be utilised to the full.

Theory of Mutual Inductance Bridge.

In Fig. 68, the ratio arms are r_1 and r_2. L_1 and L_2 are the inductances of the two primary coils. M is the mutual in-

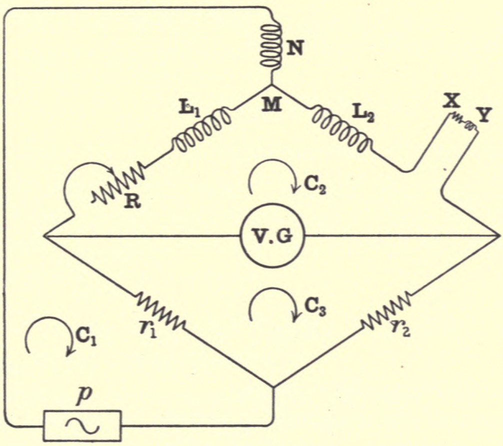

Fig. 68.—Mutual Inductance Bridge.

V.G=Vibration galvanometer. R=Adjustible resistance.
r_1, r_2=Ratio arms. XY=Unknown coil.
L_1, L_2, N=Coils of inductance standard.

ductance between the primary and secondary as read on the scale of the standard, and N is the self-inductance of the secondary. The unknown coil is of resistance X and inductance Y. R is an adjustable resistance.

Let C_1, C_2 and C_3 be the cyclic currents in the three meshes of the bridge, and let A and G be the impedances of the alternator and vibration galvanometer (or telephone) respectively. The resistance of the coils of the mutual inductance standard may be supposed included in A, R and X without affecting the argument. The E.M.F. equations are then

$$\left.\begin{aligned}&(A+ipN)C_1+(ipL_1+R)(C_1-C_2)-ipM.C_2\\&\qquad\qquad+r_1(C_1-C_3)=Ee^{ipt},\\&(R+ipL_1)(C_2-C_1)-ipM\,.\,C_1+(ipL_2+X\\&\qquad\qquad+ipY)C_2+G(C_2-C_3)=0,\\&r_1(C_3-C_1)+G(C_3-C_2)+r_2C_3=0.\end{aligned}\right\}\quad . \quad (10)$$

Write C for C_2-C_3 and substitute for C_3. Then

$$\left.\begin{aligned}&(A+ipN+ipL_1+R+r_1)C_1-(ipL_1+R\\&\qquad\qquad+ipM+r_1)C_2+r_1C=Ee^{ipt},\\&-(R+ipL_1+ipM)C_1+(R+ipL_1+ipL_2\\&\qquad\qquad+X+ipY)C_2-GC=0,\\&-r_1.C_1+(r_1+r_2)C_2-(r_1+r_2+G)C=0.\end{aligned}\right\}\quad . \quad (11)$$

Hence, solving by determinants,

$$C=\begin{vmatrix}A+ipN+ipL_1+R+r_1 & -ipL_1-R-ipM-r_1 & Ee^{ipt}\\ -R-ipL_1-ipM & \begin{matrix}R+ipL_1+ipL_2\\+X+ipY\end{matrix} & 0\\ -r_1 & r_1+r_2 & 0\end{vmatrix}\div\Delta$$

$$=Ee^{ipt}[-(R+ipL_1+ipM)(r_1+r_2)+r_1(R+ipL_1$$
$$+ipL_2+X+ipY)]\div\Delta \quad . \quad (12)$$

which must be zero for balance.

Equating real and imaginary quantities, it follows that

$$\left.\begin{aligned}r_2R&=r_1X\\ \text{and}\qquad M(r_1+r_2)&=r_1L_2-r_2L_1+r_1Y.\end{aligned}\right\}\quad . \quad . \quad . \quad (13)$$

The measurement is as a rule most accurate when the ratio arms are equal. Equations (13) then reduce to

$$\left.\begin{aligned}R&=X\\ \text{and}\qquad 2M&=L_2-L_1+Y.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If, in addition $L_2=L_1$, the unknown inductance Y is obtained at once from the reading of the standard by multiplying it by two

For the measurement of inductances beyond the range of the standard, as when, for example, a number of loading coils have to be tested which are all of the same nominal inductance, it is most convenient to use a standard self-inductance of the nominal inductance of the loading coil, and place it in the left arm of the bridge. Twice the reading of the mutual inductance standard positive or negative will then show how much the unknown inductance is greater or less than the standard coil.

Measurements on Completed Cables.

The current entering a cable, when the voltage at the sending end is V_s, is given by the equation

$$C_s=\frac{V_s}{Z_0 \tanh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

provided that the distant end is closed. If it be free, the equation must be replaced by

$$C_s=\frac{V_s}{Z_0 \coth Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Let the cable be connected up in the alternating-current Wheatstone bridge, and suppose that its impedance is measured as though it were that of a simple piece of apparatus (1) with distant end closed and (2) with distant end open Let these impedances be Ω_1, and Ω_2.

Then $$\Omega_1=Z_0 \tanh Pl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and $$\Omega_2=Z_0 \coth Pl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Multiply and extract the square root. It follows that

$$Z_0=\sqrt{\Omega_1\Omega_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Divide and extract the square root. Then

$$\tanh Pl=\sqrt{\frac{\Omega_1}{\Omega_2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now $$\tanh Pl=\frac{e^{Pl}-e^{-Pl}}{e^{Pl}+e^{-Pl}}=\frac{e^{2Pl}-1}{e^{2Pl}+1}$$

Hence $$e^{2Pl}=\frac{1+\tanh Pl}{1-\tanh Pl} \text{ and } P=\frac{1}{2l}\log_e\frac{1+\tanh Pl}{1-\tanh Pl} \quad . \quad (21)$$

Finally $$R+ipL=P\times Z_0$$
and $$G+ipK=P\div Z_0.$$

Suppose, first, that l is small.

Then $$\text{Sinh } Pl=Pl \text{ nearly, and } \cosh Pl=1.$$

Hence $$\Omega_1=Pl\times Z_0=(R+ipL)l$$
and $$\Omega_2=Pl\div Z_0=(G+ipK)l.$$

If the cable is very short, it may evidently be regarded as a simple inductance coil when the further end is closed, and as a leaky condenser when the distant end is open. R, L, G and K are then to be obtained directly from the bridge balance. The value of l up to which this approximation holds good is best obtained by plotting a diagram as in Fig. 50, Chapter VI.

When l lies beyond this limit, as will ordinarily be the case, the formulæ (19) and (20) can be used to calculate the derived constants Z_0 and P, and from them the fundamental constants R, L, G and K can be obtained. It is clear also from Fig. 50, Chapter VI., that with further increase of l the effect of opening and closing the distant end is eventually no longer discernible at the sending end, where the impedance is sensibly Z_0. This takes place before the speaking limit is reached; the cable is *quasi-infinite* in length. There is, therefore, sooner or later, a limit to the length of every cable at which this method cease to be applicable.

The process of calculation is best illustrated by an example.*

Method of Calculation—Example.

Suppose that measurements are made on a cable of length 23·42 nautical miles, in the first place with the distant end closed. Using equal ratio arms, let the cable be balanced in the bridge by an arm of resistance 241·4 ohms and inductance 34·37 millihenries when the frequency of the testing current is 1,000 p.p.s. Again, when the distant end is open, let balance

* For further illustrations see di Pirro, "Journ. Télég.," 31, 1907, p. 25.

be found with a condenser of 0·486 mfd. shunted by a resistance of 346 ohms. Then, since $p=6{,}283$, it follows that

$$\Omega_1=241{\cdot}4+ip\times0{\cdot}03437=241{\cdot}4+i\times216{\cdot}0$$
$$=323{\cdot}9\underline{/41^\circ\ 49}$$

and

$$\Omega_2=\frac{1}{1/346+ip\times0{\cdot}486\times10^{-6}}$$
$$=\frac{10^6}{2{,}890+i\times3{,}054}$$
$$=163{\cdot}5-i\times172{\cdot}7=237{\cdot}8\overline{\backslash 46^\circ 35}.$$

Hence

$$Z_0=\sqrt{\Omega_1\Omega_2}=277{\cdot}5\overline{\backslash 2^\circ 23}$$

and

$$\tanh \mathrm{P}l=\sqrt{\frac{\Omega_1}{\Omega_2}}=1{\cdot}167\underline{/44^\circ 12}=0{\cdot}8366+i\times0{\cdot}8135.$$

Again

$$1+\tanh \mathrm{P}l=1{\cdot}8366+i\times0{\cdot}8135=2{\cdot}009\underline{/23^\circ 54}$$

and

$$1-\tanh \mathrm{P}l=0{\cdot}1634-i\times0{\cdot}8135=0{\cdot}8297\overline{\backslash 78^\circ 39}$$

and, therefore,

$$\frac{1+\tanh \mathrm{P}l}{1-\tanh \mathrm{P}l}=2{\cdot}4213\underline{/102^\circ 33}=2{\cdot}4213e^{i(2m\pi+102^\circ 33)}$$

where m may be any whole number, since $e^{2m\pi i}=1$.
Hence

$$\mathrm{P}l=\tfrac{1}{2}\log_e\frac{\tanh \mathrm{P}l+1}{\tanh \mathrm{P}l-1}$$
$$=0{\cdot}4421+i(m\pi+51^\circ 16\tfrac{1}{2}),$$

from which

$$al=0{\cdot}4421 \text{ and } a=0{\cdot}01887$$

also

$$\beta l=m\pi+51^\circ 16\tfrac{1}{2}=m\times3{\cdot}1416+0{\cdot}8950. \quad . \quad . \quad (22)$$

To determine m we proceed as follows. The capacity of the cable, K, may be taken as very nearly the value previously obtained by ballistic methods, say 0·176 mfd.; moreover, L is known approximately from the design of the cable and the inductance which it is desired to attain. Suppose that it has been estimated at 13 millihenries.

Table XXVI.—*Constants Calculated from Measurements on* 23·42 *n.m. of Cable.*

n p.p.s.	Per n.m. loop.							λ n.m.	Z_0 ohm.
	R ohm.	L m.h.	G mho.	K farad.	α.	β.	P.		
1,000	8·76	13·5	$21{\cdot}9 \times 10^{-6}$	$0{\cdot}176 \times 10^{-6}$	0·0189	0·307	$0{\cdot}307\ \underline{/86^\circ 29}$	20·5	$278\ \overline{\backslash 2^\circ 23}$

Then $\qquad \beta l = pl\sqrt{\text{KL}}$ approximately

$$=6283 \times 23{\cdot}42 \times 10^{-3}\sqrt{0{\cdot}176 \times 0{\cdot}013}$$

$$=7{\cdot}04 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Comparing (22) with (23), the nearest integral value of m is 2,

hence $\qquad \beta l = 6{\cdot}2832 + 0{\cdot}8950 = 7{\cdot}1782$

and $\qquad \beta = 0{\cdot}3065.$

Thus $\qquad \text{P} \doteq \alpha + i\beta = 0{\cdot}01887 + i \times 0{\cdot}3065 = 0{\cdot}3071\,\underline{/86^\circ 29},$

and $\qquad \text{R} + ip\text{L} = \text{P} \times \text{Z}_0$

$$=85{\cdot}21\,\underline{/84^\circ 6} = 8{\cdot}759 + i \times 84{\cdot}77,$$

from which R=8·76 ohms and L=13·5 millihenries,

also $\qquad \text{G} + ip\text{K} = \text{P} \div \text{Z}_0$

$$=1106{\cdot}7 \times 10^{-6}\,\underline{/88^\circ 52}$$

$$=10^{-6}(21{\cdot}89 + i \times 1106{\cdot}5),$$

from which G=21·9 micromhos, and K=0·176 mfd.

These results are brought together in Table XXVI. above.

Direct Method.

The method which has just been explained for determining the constants of a cable has certain disadvantages. As pointed out, it breaks down if the cable is long, for then the effect of closing or opening the circuit at the distant end is inappreciable at the transmitting end. Again (2), the constants are obtained by a somewhat roundabout calculation, as a consequence of which the measurements are not to be depended on for R and G, although they give accurate results for L and K. That is because the quantities R and G are small in comparison with pL and pK, and consequently very great accuracy would be required in the determination of P and Z_0 before $R+ipL$ and $G+ipK$ could be known sufficiently closely for variations in R and G to be appreciable. Lastly (3), it affords no guide as to the actual current that is being transmitted during the test unless separate current measurements are made.

These objections have led to the desire for some method which would allow of the direct determination of the attenuation constant. One proposal may be described as follows.* The characteristic impedance of a loaded cable, Z_0, is approximately $\sqrt{L/K}$, in which L is known from the design of the cable or from measurements on short lengths, and K may be taken from the results of the usual ballistic tests. Thus, taking as before $K=0{\cdot}176$ mfd., say, and $L=13$ millihenries.

$$Z_0=\sqrt{\frac{0{\cdot}013}{0{\cdot}176\times10^{-6}}}=272 \text{ ohms.}$$

Now join up a resistance of 272 ohms at the end of the cable and measure the apparent impedance of the cable and resistance in the Wheatstone bridge in the manner described above. Suppose that the result obtained is $277{\cdot}5\,\overline{\backslash 2°23}$, which is the same as

$$\frac{1}{1/277{\cdot}8+ip\times0{\cdot}02385\times10^{-6}}.$$

Let, now, the end resistance be adjusted to 277·8 ohms, across which is placed a condenser of 0·024 mfd. capacity, and measure again in the bridge. If the same impedance is found as before the true value of Z_0 is known to be $277{\cdot}5\,\overline{\backslash 2°23}$. Should the new measurement be slightly different, the end apparatus must be readjusted and the process repeated until the end apparatus is identical with the impedance as measured at the sending end.

If an impedance, Z_0, were connected up to a cable at the receiving end, currents and voltages would be the same throughout the cable as though it were infinitely long, since all reflections would be eliminated by the end apparatus. The sending current is then

$$C_s=\frac{V_s}{Z_0}$$

* Devaux-Charbonnel, "Jour. Télég.," Vol. XXXV., 1911, p. 125. Also "Lum. Elect.," 14, 1911, p. 163.

and the received current is

$$C_r = \frac{V_s e^{-Pl}}{Z_0}.$$

The ratio of the two is

$$C_s/C_r = e^{Pl} = e^{al} \underline{/\beta l},$$

from which

$$a = \frac{1}{l} \log (C_s/C_r), \quad . \quad . \quad . \quad . \quad (24)$$

and a is obtained directly by simple measurements of the sent and received currents.

HIGH-FREQUENCY AMMETERS.

The ammeters by means of which the currents are measured in the method just described are supposed to have such low resistances as not to affect the distribution of current and voltage. A suitable instrument for this purpose is the Duddell thermo-galvanometer.*

In this instrument a very light bismuth-antimony thermo-couple, attached to a single loop of silver wire and a glass stem carrying a small mirror, is suspended by a fine quartz fibre in the field of a permanent magnet. Below is a wire filament or platinum grid, which is heated by the current to be measured. Heat radiated from the grid causes the junction to rise in temperature, and the accompanying thermo-electric current produces a proportional deflection of the couple and mirror. The couple is suitably enclosed in a heavy metal case, so as to shield it from sudden changes of temperature. As with all thermal instruments, the deflections are proportional to the square of the current. It is advisable to calibrate the instrument with direct current, and for this purpose it is convenient to arrange a throw-over switch so that immediately an alternate current is measured the same deflection can be repeated by means of a direct current, a suitable ammeter being in series with the direct-current E.M.F. and the thermo-galvanometer. The chief advantages of the instrument are (*a*) its sensitiveness and (*b*) its negligible self-inductance and capacity. With a heater of resistance 4 ohms a deflection of 10 mm. at 1 m. is given

* "Phil. Mag.," July, 1904.

by a current of 350 microamperes; with a heater of 100 ohms the current required is reduced to 70 microamperes. In cases where portability is to be preferred to sensitiveness a direct-reading form or thermo-ammeter may be used, in which the movement is pivoted instead of being suspended, and carries a pointer moving over a scale.

An alternative means of measuring feeble alternating currents of high frequency is furnished by the barretter.* The principle of this instrument is not thermo-electric, but it depends upon the change of resistance of a conductor which accom-

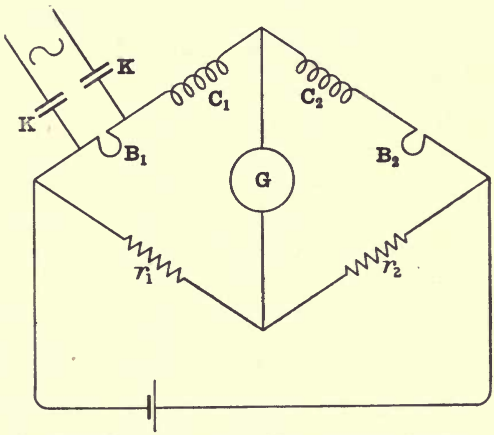

FIG. 69.—BARRETTERS IN WHEATSTONE BRIDGE.

B_1, B_2=Barretters. r_1, r_2=Ratio arms.
C_1, C_2=Choke coils. K, K=Block condensers.
G=Direct-current galvanometer.

panies a rise of temperature. A fine wire is heated by the alternating current, and the change of resistance is measured by direct current. In Fig. 69 the barretters B_1 and B_2 are in adjacent arms of a Wheatstone bridge, B_2 being used to counterbalance slight changes in resistance produced by fluctuations of temperature in the apparatus as a whole. The two are as nearly as possible of the same resistance and temperature coefficient, and are mounted close together. G is the

* Paalzow and Rubens, "Ann." d. Phys., 37, p. 529, 1889.

direct-current galvanometer, and r_1, r_2 are the ratio arms. K, K are condensers to prevent the direct current from leaving the bridge, and C_1, C_2 are choking coils to restrict the alternating current to the filament of B_1. This instrument has been developed by Béla Gàti,* who employs a fine gold wire measuring only 0·0005 mm. in diameter. A limit to the magnification is caused by the necessity for restricting the bridge current if the wire is not to fuse; it must not exceed $2\frac{1}{2}$ milli-amperes in the case of the 0·0005 mm. wire. With a barretter of 350 ohms and a pointer galvanometer of 300 ohms of sensitivity such that 1 deg.$=10^{-7}$ amperes, a deflection of 50 deg. for 31 micro-amperes and 5 deg. for one micro-ampere of high-frequency current is obtainable.

The Franke Machine.

With thermal instruments it is possible to measure only the root-mean-square current, and no indication is furnished of phase relationship. To eliminate this defect special instruments have been devised, of which the Franke machine was one of the earliest to be introduced.† It consists of a high-frequency alternator of inductor type, with two electrically independent armatures, the positions of which are adjustable relatively to a rotating field. The axis of the machine is placed vertically, as in Fig. 70. It is driven by a directly-coupled motor, M, which is placed beneath. The object of the vertical design is to economise space, bring the adjustments to a convenient level and render the machine portable. The magnetic field is formed by two coaxial wrought-iron cylinders, which are securely fixed on the rotating shaft by means of brass rings. Of these cylinders, the outer, F, is seen in the figure. The space between them is occupied by a magnetising solenoid, the exciting current for which is led in by means of slip-rings on the

* "Phys. Zeitschr.," 10, 1909, p. 897. The form of bridge used differs from that in the sketch. In the Cohen barretter ordinary 24-volt carbon lamps are used. "Journ." I.E.E., May, 1907.

† A. Franke, "E.T.Z.," 1891, p. 447. For a full description of the latest improvements in this apparatus see The Electrician, Oct. 24, 1913, p. 88.

shaft. At top and bottom of the cylinders are mounted wrought-iron rings, with rounded teeth which project radially inwards and outwards towards each other. Into the gaps between the

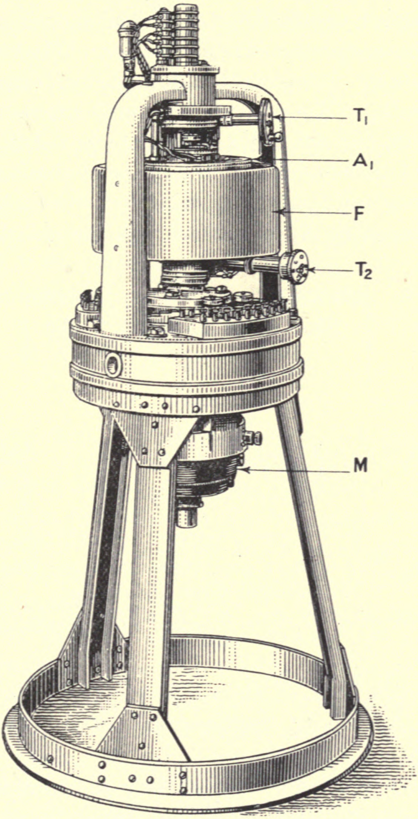

FIG. 70.—FRANKE HIGH-FREQUENCY GENERATOR.

M=Driving motor. A_1=Upper armature.
T_1, T_2=Upper and lower micrometer screws. F=Field magnets.

poles so formed dip two stationary armatures, of which the upper, A_1, is partly visible in the figure.

As the magnet system rotates an alternating E.M.F. is induced in the armatures. The frequency is exactly the same in both the armatures, and is N times the number of revolutions per second, where N is the number of teeth in the magnets, each armature ring having 2N slots carrying a zigzag winding. As the armature windings are on insulating material, the magnetic field is relatively perfectly at rest, and no magnetisations or eddy currents are produced in its rotation. The lower armature can be turned through a measurable angle by means of the micrometer screw T_2. The upper armature is not free to turn, but can be slid up and out of the field by means of the micrometer screw T_1. It is evident that the phase of the E.M.F. induced in A_2 by the rotating magnets, as well as the magnitude of the E.M.F. induced in A_1, is perfectly under control. If A_2 be

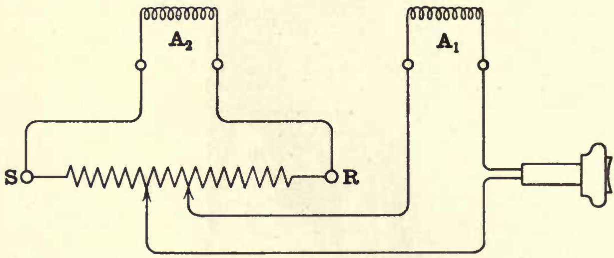

FIG. 71.—SCHEME OF CONNECTIONS FOR FRANKE HIGH-FREQUENCY GENERATOR.

A_1 = Upper armature. A_2 = Lower armature. SR = Telephone line or apparatus.

turned through 1/2N of the whole circumference, the E.M.F. induced in A_2 will be displaced by 180 deg. in phase with respect to A_1.

The method of using the apparatus is shown in Fig. 71. Here A_2 is the lower armature supplying current to a telephone line or piece of apparatus, SR. Potential leads from any two points of the line are joined through a telephone receiver to the upper armature A_1. The phase of the E.M.F. in A_2 and the magnitude of that in A_1 are adjusted until there is silence in the telephone. Then the potential drop in the telephone line between the potential leads is equal to the E.M.F. of A_1, which is known from its position by previous calibration. This calibration may be carried out by potentiometer methods or by taking advantage of the multiple windings on the armatures, which can be balanced against each other.

In the later design of apparatus, as illustrated, the rotating magnets are of increased size, so as to give increased power and greater uniformity of speed. The number of teeth N is brought up to 40, and the frequency attainable is over 2,000 p.p.s. A commutator on the shaft is connected to a reed indicator, from which the frequency is read and controlled by resistance in the motor circuit or by mechanical breaking. Special provision is made that no rotating motion shall accompany the up and down motion of armature A_2. Approximately three turns of the hand wheel raise A_2 1 mm. out of the field. For a phase displacement of one complete cycle 36 revolutions of the spindle T_2 are required, and angles can be measured accurately to within two minutes of arc.

Drysdale-Tinsley Alternating-current Potentiometer.

Instead of carrying out the phase adjustment on the alternator, it may be performed in the bridge itself. This is the principle on which the apparatus to be described is based. In the ordinary direct-current potentiometer E.M.F.s are compared by balancing them against the drop in potential in a standard wire carrying a steady current. Two alternating E.M.F.s of the same frequency may be compared in a similar manner, using a vibration galvanometer in place of a direct-current galvanometer as criterion of balance, always provided that the currents are in phase with the standard which will not in general be the case.

To bring the E.M.F.s into phase a special device called a phase-shifting transformer is necessary. This piece of apparatus resembles in appearance an induction motor. The stator has two windings at right angles, and the rotor a single diametral winding. Current is applied to the stator from a two-phase supply in such a manner as to produce a pure rotating field. If, now, the rotor be turned through any angle the E.M.F. induced in it will be of constant magnitude during the

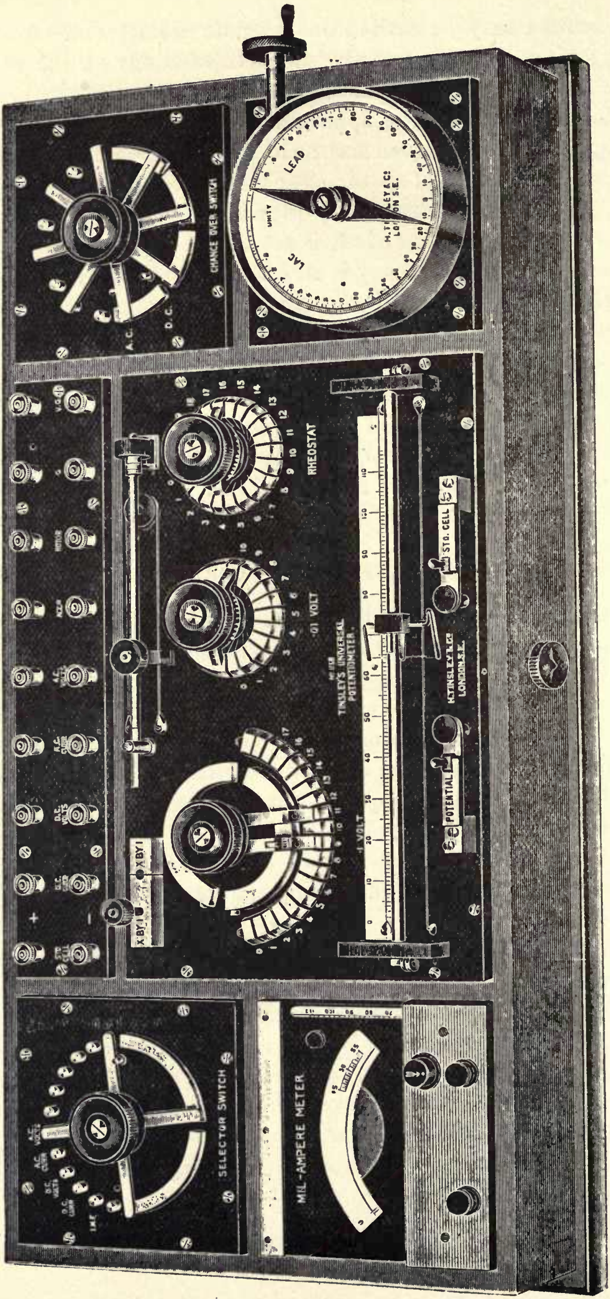

FIG. 72.—DRYSDALE-TINSLEY ALTERNATING-CURRENT POTENTIOMETER.

movement, and the phase displacement will be proportional to the angle through which the rotor has been turned. The current from the rotor may, therefore, be used to replace that from the secondary battery in direct-current potentiometer measurements, the phase being adjusted to agree with that of the E.M.F to be measured.

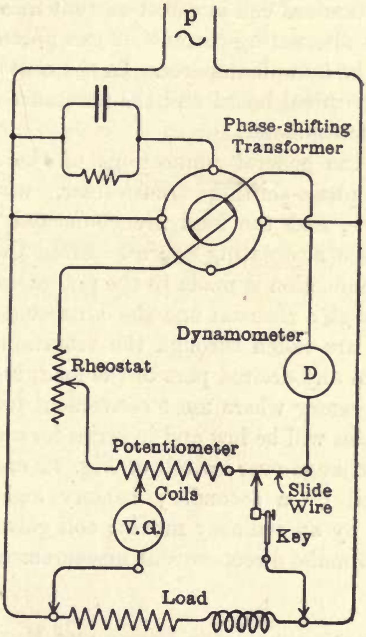

FIG. 73.—SCHEME OF CONNECTIONS FOR DRYSDALE-TINSLEY ALTERNATING-CURRENT POTENTIOMETER.

As in general the source of supply will be a single-phase circuit, a special phase-splitting device, consisting of a condenser and series resistance, is placed across the stator windings. Capacity and resistance are adjusted according to the frequency of the supply.

The complete apparatus is shown in Fig. 72. The phase-shifting transformer is in the right-hand corner. The doubleended pointer enables the position of the rotor to be read off, and the milled head permits of accuracy in setting. Above is the switch for changing over from direct to alternating-current measurements. In the left-hand upper corner is a switch enabling the particular circuit to be selected on which measurements are desired. Below is the milliamperemeter, which performs the function in alternating-current measurements of the standard cell in direct-current measurements.

The normal alternating-current in the potentiometer coils is adjusted to be 50 milliamperes. In the centre of the apparatus are the terminal board and the resistance coils and slide wire of the potentiometer.

In Fig. 73 the general connections of the apparatus are shown. The phase-shifting transformer, with its phase-splitting device, and the load are connected up across the terminals of the alternating source. From the rotor of the transformer connection is made to the potentiometer coils and slide wire through a rheostat and the dynamometer. Potential tapping leads are taken through the vibration galvanometer and a key from any desired part of the load, resistances being inserted if necessary where most convenient for this purpose. These resistances will be low and in series for current measurements. The change-over switch of Fig. 72 enables the rotor to be replaced by a secondary battery and the vibration galvanometer by an ordinary moving-coil galvanometer when it is desired to make direct-current measurements.

Comparison of the Different Methods.

The choice of the apparatus required in any particular instance depends to a great extent on the nature of the test which is to be performed. Bridge methods are limited to the measurement of a generalised resistance in some shape or other. They do not permit of the measurement of currents and voltages, for which the potentiometer is peculiarly suitable. When observations have to be made on transmission in a cable or network of conductors, or on the efficiency of an alternating-current source, one of the two potentiometer methods which have been developed up to the present is, therefore, desirable. On the other hand, the application of the potentiometer to the measurement of generalised resistances is limited by the degree of accuracy with which the angle can be read off. This is especially marked in cases where the angle is nearly 0 deg. or 90 deg., as when residual inductance, or the resistance of a highly inductive coil and the leakance of a condenser, are to be measured. In certain cases considerations of portability and cost of duplications may decide the question. Such arise, for example, when it is necessary to take simultaneous readings of current at widely separated places, and it may then be advisable to fall back on deflectional methods.

REFERENCES.

For the comparison of theory with experiment by measurements on artificial cables, *see* C. V. Drysdale, THE ELECTRICIAN, August 1, 1913, and A. E. Kennelly and F. W. Lieberknecht, THE ELECTRICIAN, Vol. LXXIII., September 25, 1914, p. 965; *Trans.* Am.I.E.E., Vol. XXXII., Part II., 1913, p. 1283.

PART III.—PURELY TRANSIENT PHENOMENA. THE TELEGRAPH CABLE.

CHAPTER VIII.

THE TELEGRAPHIC EQUATION AND ITS TRANSIENT SOLUTION.

INTRODUCTION.

Within recent years the growth of the telephone industry has been accompanied by corresponding growth in the theory of telephonic transmission. Many investigators have assisted in bringing the subject to its present high state of development; but, with the exception of the work of Oliver Heaviside, the theory of the submarine telegraph cable remains very much where it was left by Sir William Thomson in 1855. Nevertheless, the problems which await solution in the telegraphic case are no less in scientific interest and commercial importance than those which have been solved in telephony.

The commercial value of a telegraph cable is measured by its working speed. Increase of speed may be brought about by changes in the apparatus used at the ends of the cable, or in the construction of the cable itself. Inasmuch as the object is a quantitative improvement, it is highly desirable that experiment should be preceded by a preliminary calculation of the magnitude of the result which is anticipated. The aim of the following treatment of the subject is to provide the means for this purpose, and by developing the general theory of the cable to suggest in what directions improvements may hopefully be sought. It is shown how telephonic theory must naturally be transformed and extended in order that it may cover the telegraphic case.

In the theory of the telegraph cable the fundamental problem is to determine the electrical state of the cable at any point and time after an electrical change has been impressed on one end of the conductor, and to ascertain how this state is affected by alterations in the cable and in its auxiliary apparatus. This problem has been studied for many years, and solutions of particular cases have been obtained from time to time. It will now be shown that the problem is capable of complete mathematical and numerical solution. The more complicated the conditions are the more complicated is the solution, but the process is in every case the same, and it is arithmetically straightforward and simple.

The Telegraphic Equation in its Simplest Form.

In the case of a long submarine cable worked at a low speed, the determining factors are the resistance R and the capacity K.

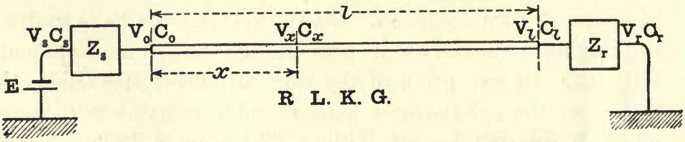

Fig. 74.—Cable and Apparatus Arranged for Simplex Working.

The value of L is unknown; for a single conductor it is difficult to calculate, and it has never been measured.* G is also indefinite; it is not merely the reciprocal of the steady-current

* *See* Devaux-Charbonnel, "Comptes Rendus," 143, p. 113, 1906, and Chapter II.

dielectric resistance, but it includes also the effect of absorption. As L and G are thought to influence the speed of signalling only slightly, they may be left out of account in the first place. The equations (1) and (2) of Chap. IV. then reduce to

$$\mathrm{RC}=-\frac{d\mathrm{V}}{dx} \quad \ldots\ldots\ldots \quad (1)$$

and

$$-\frac{d\mathrm{C}}{dx}=\mathrm{K}\frac{d\mathrm{V}}{dt} \quad \ldots\ldots\ldots \quad (2)$$

From these equations it follows by differentiation that

$$\left.\begin{aligned}\mathrm{KR}\frac{d\mathrm{V}}{dt}&=\frac{d^2\mathrm{V}}{dx^2}\\ \text{and}\qquad \mathrm{KR}\frac{d\mathrm{C}}{dt}&=\frac{d^2\mathrm{C}}{dx^2}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

The differential equation (3) is the telegraphic equation in its simplest form, as it was obtained by Sir William Thomson in 1855.* Since then it has been studied by Heaviside, Poincaré Picard, Boussinesq and other mathematicians.† It expresses the voltage or current at any time and place along the cable in terms of the constants of the cable. Kelvin saw that it was the same equation as that of the linear motion of heat in a conductor, a problem already treated in detail by Fourier. The solutions which had been found for the heat case could, therefore, be applied at once to the telegraphic problem. Proceeding in this way, he showed that when both ends of the cable were put directly to earth, the current at the receiving end of the cable produced by the application of a constant E.M.F., E, could be expressed by the infinite series,

$$\mathrm{C}_r=\frac{\mathrm{E}}{\mathrm{R}l}\left[1+2\sum_{m=1}^{\infty}e^{-\frac{m^2\pi^2t}{\mathrm{KR}l^2}}.\cos m\pi\right], \quad \ldots\ldots \quad (4)$$

where the summation extends over all integral values of m from 1 to infinity.

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* Sir W. Thomson, Roy. Soc., "Proc.," 7, p. 383, 1855. Math. and Phys. Papers, 2, p. 62, 1884.

† A. G. Webster, "Theory of Electricity and Magnetism," London, p. 540, 1897. *See* also Poincaré, "Ecl, Electr.," 40, p. 121, 1904.

Deduction of the Kelvin Solution.

The expression for the arrival current which has just been given may be obtained directly from the differential equation without a knowledge of the heat solution. Suppose that the E.M.F. has been applied to the cable, and maintained sufficiently long for all transient effects to have died away. Then, since there is no leakage, the slope of potential along the cable must be a straight line falling from E at the sending end, to zero at the receiving end. Taking as usual the sending end at $x=0$ and the receiving end at $x=l$, inversely as in the diagram of Fig. 8, Chap. I., the equation to the steady voltage is easily seen to be $V=E(1-x/l)$. Expanding $1-x/l$ in a Fourier sine series in the manner explained in Chap. I., the steady voltage at any point distant x along the cable is represented by the expression

$$V=E(1-x/l)=\frac{2E}{\pi}\sum_1^\infty\frac{1}{m}\sin\frac{m\pi x}{l} \quad . \quad . \quad . \quad . \quad (5)$$

Throughout its variations before the steady state is reached the voltage must satisfy the differential equation (3). Now, the expression

$$\frac{2E}{m\pi}e^{-\frac{m^2\pi^2t}{KRl^2}}\,.\sin\frac{m\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is a solution of (3), as may readily be verified. Moreover, it is zero when t is infinite, and is the mth term of (5) when t is zero. Adding all such terms together, it follows that the voltage in the cable at any time after application of the battery is represented by the expression

$$V=E\left(1-\frac{x}{l}\right)-\frac{2E}{\pi}\sum_1^\infty\frac{1}{m}e^{-\frac{m^2\pi^2t}{KRl^2}}\,.\sin\frac{m\pi x}{l} \quad . \quad . \quad . \quad (7)$$

For every term in it satisfies the differential equation, it is zero when t is zero, and it rises to $E(1-x/l)$ when t is infinite, *i.e.*, it fulfils all the conditions required.

From (1) the current may be obtained by differentiation of (7).

Thus

$$C=-\frac{1}{R}\frac{dV}{dx}=\frac{E}{Rl}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2t}{KRl^2}}\,.\cos\frac{m\pi x}{l}\right] \quad . \quad . \quad (8)$$

At the receiving end where $x=l$, the current is

$$C_r=\frac{E}{Rl}\left[1+2\sum_{1}^{\infty}e^{-\frac{m^2\pi^2t}{KRl^2}}.\cos m\pi\right],$$

which is the required series.

Case of a Long Submarine Cable.

It is customary to denote the quantity $\frac{\pi^2 t}{KRl^2}$ by τ, and to take τ as a new time variable. This presents advantages from the point of view of theory, inasmuch as the series in (4) is then the same for every cable. But as it is important that an exact idea should be obtained of the magnitude of the quantities involved, time will be measured here throughout in seconds, potential in volts or millivolts and current in microamperes.

To apply the solution to a particular case, take that of the San Francisco-Honolulu cable, laid by the India Rubber, Gutta Percha, and Telegraph Works Company in 1903 for the Commercial Pacific Company. The necessary data are

$l=2{,}276{\cdot}4$ n.ms.

$R=2{\cdot}1856$ ohms per n.m. at sea temperature.

$K=0{\cdot}3842$ mfds. per n.m.

whence $KRl^2=4{\cdot}3514$ seconds, $Rl=4{,}975$ ohms and $Kl=874{\cdot}6$ mfds. Equation (4) then becomes

$$C_r\begin{pmatrix}\text{microamperes per volt}\\ \text{of sending battery}\end{pmatrix}=201\left[1+2\sum_{m=1}^{\infty}e^{-2{\cdot}268m^2t}.\cos m\pi\right].\quad(9)$$

The "KR" Law.

The "KR" law may be expressed in various ways. "We may infer [from (4)] that the time required to reach a stated fraction of the maximum strength of current at the remote end will be proportional to KRl^2." "The equation shows that two submarine wires will be similar, provided the squares of their lengths vary as the times divided by KR, or the time of any electrical operation is proportional to KRl^2."*

* Sir W. Thomson, Papers, Vol. II., p. 71.

The " KR " law, as above stated, is of very limited application. *It pre-supposes the cable to be without inductance and leakance, and both ends to be put to earth without signalling or receiving apparatus.* Estimates based upon it of the working speed of cables under actual conditions are, therefore, liable to be greatly in error.* It will be shown later how it must be extended in order to take these conditions into account.

The Kelvin Arrival Curve.

The values of C_r as given by (9) are plotted from Table XXVII. in Fig. 75, curve A. The broken curve B is the arrival curve for a cable of which the resistance is 20 per cent. greater than that corresponding to A. The curve C is for one of which the capacity is 20 per cent. less than that corresponding to A.

Kelvin took the quantity $a=\frac{KRl^2}{\pi^2}\log_e\frac{4}{3}=0{\cdot}1268$ second as unit of time. " The curve so nearly coincides with the line of abscissæ at first as to indicate no sensible current until the interval of time corresponding to a has elapsed, although, strictly speaking, the effect at the remote end is instantaneous. After the interval a the current very rapidly rises, and after $4a$ more attains to half its full strength. After $10a$ from the com-

Table XXVII. (Fig. 75.)

t seconds.			Microamperes per volt of sending battery.	
A.	B.	C.	A and C.	B.
0·1	0·12	0·08	0	0
0·25	0·3	0·2	12·2	9·8
0·4	0·48	0·32	49·3	39·4
0·5	0·6	0·4	76·0	60·8
0·6	0·72	0·48	99·7	79·8
0·75	0·9	0·6	128·1	102·5
1·0	1·2	0·8	159·4	127·5
1·5	1·8	1·2	187·6	150·1
2·0	2·4	1·6	196·7	157·4
3·0	3·6	2·4	200·5	160·4

* Report of the Pacific Cable Committee, " Elect. Rev.," 44, p. 955, 1899.

mencement it has attained so nearly its full strength that the further increase would be probably insensible."*

It must not be supposed, as is often done, that a is a constant of theoretical importance, or that there is any discontinuity of slope at the point $t=a$. To examine this question it is necessary to plot the curve at the beginning to a greater scale. Now, the series in (4) is only slowly convergent for small

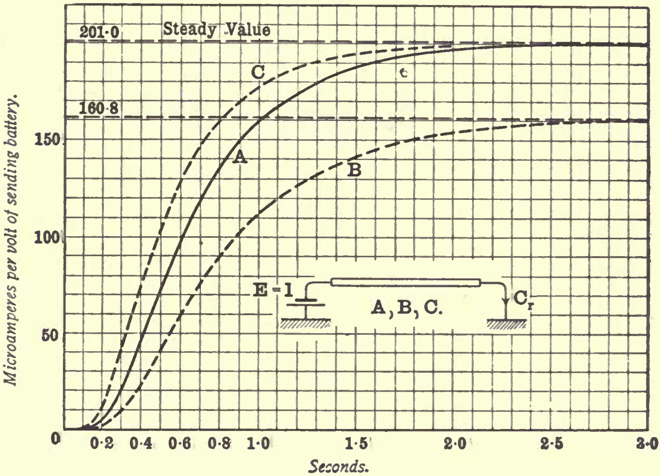

FIG. 75.—ARRIVAL CURRENT. ENDS EARTHED.

Curve A. San Francisco–Honolulu Cable. l=2276.4 n.ms.

	Rl.	Kl.	KRl^2.
Curve A	4,975	874·6	4·352
„ B	5,970	874·6	5·222
„ C	4,975	699·7	3·482

values of t, and the convergence grows worse the smaller the value of t. It becomes a matter of considerable labour to calculate C accurately for small values of t. It is, therefore, convenient to replace (4) by another expression which lends itself more readily to numerical evaluation when t is small.

* Sir W. Thomson, Papers, Vol. II., p. 71.

Alternative Expression for the Arrival Curve.

To obtain this use the identity*

$$q\sum_{n=-\infty}^{n=+\infty} e^{-\pi(p+nq)^2}=1+2\sum_{m=1}^{m=\infty} e^{-\frac{m^2\pi^2}{q^2}}\cos\frac{2m\pi p}{q}. \quad . \quad (10)$$

Put $$p=\frac{q}{2} \text{ and } q^2=\frac{KRl^2}{t}.$$

Then the right-hand side of (10) becomes the same as the series in (4) and the left-hand side gives

$$C_r=E\sqrt{\frac{K}{R\pi t}}\sum_{n=-\infty}^{n=\infty} e^{-\frac{(2n+1)^2KRl^2}{4t}}=2E\sqrt{\frac{K}{R\pi t}}\sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2KRl^2}{4t}} \quad (11)$$

The series (11) converges very rapidly for small values of t. If all terms except the first be neglected the expression for C_r is

$$C_r=2E\sqrt{\frac{K}{R\pi t}}e^{-\frac{KRl^2}{4t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The curve

$$C_r \text{ (microamperes per volt)}=\frac{473\cdot 1}{\sqrt{t}}e^{-\frac{1\cdot 088}{t}} \quad . \quad . \quad (13)$$

is plotted from Table XXVIII. in Fig. 76.

Table XXVIII. (Fig. 76).

t.	C_r	t.	C_r	t.	C_r
0·00	0	0·475	69·49	1·30	179·93
0·05	$7{\cdot}5\times 10^{-7}$	0·50	75·92	1·40	184·20
0·08	$2{\cdot}1\times 10^{-3}$	0·55	88·25	1·45	186·01
0·09	$8{\cdot}9\times 10^{-3}$	0·575	94·06	1·475	186·83
0·10	$2{\cdot}8\times 10^{-2}$	0·60	99·63	1·50	187·61
0·11	$7{\cdot}2\times 10^{-2}$	0·65	110·03	1·60	189·82
0·12	0·158	0·675	114·88	1·70	192·49
0·13	0·305	0·70	119·48	1·80	194·22
0·14	0·534	0·75	128·06	1·90	195·60
0·15	0·865	0·775	132·04	1·95	196·18
0·175	2·26	0·80	135·78	1·975	196·44
0·20	4·59	0·85	142·67	2·00	196·69
0·25	12·18	0·90	148·89	2·10	197·57
0·275	17·26	0·95	154·43	2·20	198·26
0·30	22·98	0·975	157·04	2·30	198·82
0·35	35·69	1·00	159·38	2·40	199·26
0·375	42·44	1·10	167·84	2·50	199·61
0·40	49·26	1·15	171·31	3·00	200·55
0·45	62·84	1·20	174·57	4·00	200·95

* Sir W. Thomson, Papers, Vol. II., p. 48.

It is seen that there is no definite point at which the rapid increase in the current can be said to begin. The quantity a is thus only of historical interest. *In a cable consisting of resistance and distributed capacity only an* "interval of silence" *does not exist.* In the present instance the expression (13) may

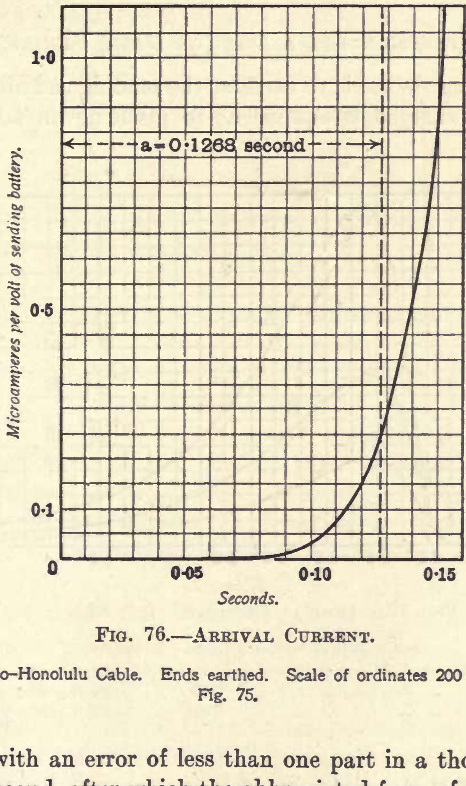

FIG. 76.—ARRIVAL CURRENT.

San Francisco–Honolulu Cable. Ends earthed. Scale of ordinates 200 times that of Fig. 75.

be used with an error of less than one part in a thousand up to $t=1$ second, after which the abbreviated form of (4)—*i.e.*,

$$C_r=\frac{E}{Rl}\left[1-2e^{-\frac{\pi^2 t}{KRl^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (14)$$

may be used with the same degree of accuracy, and Table XXVIII. was calculated from these expressions.

By differentiating (12) twice, $\frac{d^2C_r}{dt^2}=\frac{C_r}{16t^4}[12t^2-12t\,.\,KRl^2+(KRl^2)^2]$. Hence, $\frac{d^2C_r}{dt^2}=0$ when $t=0{\cdot}0917\times KRl^2$. In the present instance, $t=0{\cdot}399$ second, and this is the time when the slope of the arrival curve is steepest.

Expression for a Dot (or Dash) Signal.

Connecting the cable to earth at the sending end after it has been fully charged is equivalent to inserting an additional,

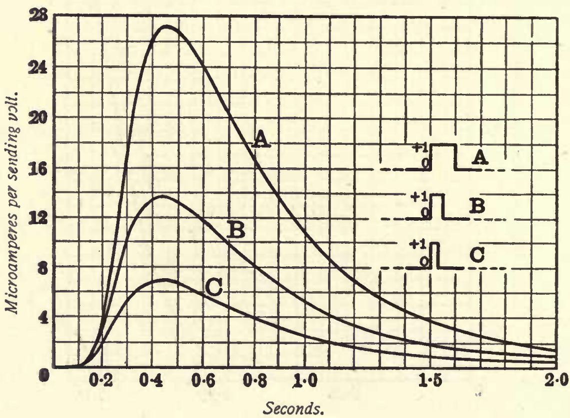

Fig. 77.—Arrival Current. Dot Signal.

San Francisco–Honolulu Cable. Ends earthed.

Curve A.—Contact lasting for 0·1 second $=KRl^2\div\ 43{\cdot}5$
„ B. „ „ 0·05 „ $=KRl^2\div\ 97{\cdot}0$
„ C. „ „ 0·025 „ $=KRl^2\div 194{\cdot}0$

negative, E.M.F. equal to the first at the sending end. Hence the curve representing the decay of the current at the receiving end is the same as the arrival curve, but with ordinates drawn downwards from the steady value. The curve for a dot lasting time t_1 is therefore to be obtained by subtracting from the first curve a second, similar, curve displaced through a distance t_1. From equation (4) it thus follows that the formula for a dot may be written

$$C_r = \frac{2E}{Rl}\left[\sum_{m=1}^{\infty} e^{-\frac{m^2\pi^2 t}{KRl^2}} \cos m\pi - \sum_{m=1}^{\infty} e^{-\frac{m^2\pi^2(t-t_1)}{KRl^2}} . \cos m\pi\right] \ldots (15)$$

The subtraction is most conveniently performed graphically.

Fig. 77 shows the form of a "dot" when the time of contact is (A) 0·10 second, (B) 0·05 second, (C) 0·025 second. According to Crehore and Squier,* a siphon recorder requires 42·5 microamperes to produce a suitable deflection. The shorter the time of contact the lower and flatter are the curves.

Comparison of Arrival Curves.

From the foregoing it is evident that, given the arrival curve, the working speed can be obtained. This is done by superposing the arrival curves positively or negatively to build up the received signals for a range of different rates of sending. The signals so formed are then to be submitted to an expert telegraphist to determine the limit of legibility, and the corresponding speed is known. The problem is, therefore, reduced to finding the arrival curve under the various conditions of working and making this resemble the original square-topped sending voltage as nearly as possible. Since the arrival curves have shape as well as size they cannot be compared by any simpler means. It may be of advantage to sacrifice size for shape—as is the case when signalling condensers are used—the limit being fixed by the sensitivity of the recorder.

Effect of End Apparatus.

To obtain the arrival curve under practical conditions it is necessary to take account of the apparatus at the ends. The foregoing simple theory then no longer applies. The effect of resistance at the ends of a cable has been treated by Heaviside,†

* A. C. Crehore and G. O. Squier, The Electrician, Vol. LXVI, p. 328, 1910.

† O. Heaviside, "Phil. Mag.," Vol. III., p. 211, 1877; "Electrical Papers," Vol. 1, p. 61, 1892.

who has also given the approximate solution for the case of the use of signalling condensers* ; this case has also been discussed by Devaux-Charbonnel† by considering them as equivalent to a prolongation of the cable itself.

The exact solution for apparatus of any kind may be obtained by development in a series of Normal Functions. This method has been treated exhaustively by Heaviside.‡ The coefficients of the terms of the series are obtained by means of the conjugate property of two normal functions. Recently this method has been used by K. W. Wagner.§

The method of normal functions fits peculiarly the problem of the decay of an initial state when some constraint is removed, such as the subsidence of a distribution of temperature. But the telegraphic case is that of the sudden application of a source to a system initially devoid of energy, and it requires a more direct and—owing to the great complexity of the apparatus that may be used in connection with the cable—a more powerful method of treatment than does the heat problem.

The method worked out in Chapter IX. is a general one, which can be applied to all conditions of apparatus and cable constants. First of all the simply periodic solution is found in the form in which it is required by the telephone engineer. This is done by elimination from a set of simple simultaneous equations. This periodic solution is then transformed in a manner to be described, and the required solution for the telegraphic case—that is to say, for a sudden rise in voltage at the sending end of the line—is obtained as the sum of a series, the terms of which are found from the roots of a trigonometrical equation. It is obvious that a mere formal solution would be of little practical value ; it is, therefore, fortunate that the resultant series are not only easy to obtain, but are also easy of numerical computation.

In the remainder of this chapter the periodic solution is

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* O. Heaviside, " Phil. Mag.," Vol. XLVII., p. 426, 1874. "Electrical Papers," Vol. 1, p. 47, 1892.

† Devaux-Charbonnel, " Ecl. Electr.," Vol. XXXI., p. 124, 1902.

‡ O. Heaviside, "Electrical Papers," Vol. I., p. 141, 1892.

§ K. W. Wagner, " Phys. Zeitschr.," Vol X., p. 865, 1909.

discussed with Breisig's extension to express the telegraphic solution as the sum of a series of periodic terms.

Attenuation of a Simple Sine Oscillation in a Long Submarine Cable.

In order to show the effect of attenuation on the amplitude of the received current when sine oscillations of different frequencies are transmitted through a long submarine cable, Fig. 78, Curve A, has been drawn from Table XXIX.

The amplitudes of the received currents are shown as ordinates and the frequencies as abscissæ. The ends of the cable are connected directly to earth.

For a cable as in Fig. 74, with apparatus of impedance Z_s at the sending end, and of impedance Z_r at the receiving end :—

$$\left.\begin{array}{l} V_s = Z_s C_s + V_0 \\ V_l = Z_r C_r + V_r = Z_r C_r, \text{ since } V_r = 0 \\ C_s = C_0 \text{ and } C_l = C_r. \end{array}\right\} \quad . \quad . \quad . \quad (19)$$

Again, in the fundamental transmission equations (15) of Chap. IV.—

$$V_x = V_0 \cosh Px - Z_0 C_0 \sinh Px$$

and

$$C_x = C_0 \cosh Px - (V_0/Z_0) \sinh Px$$

—writing l for x, it follows that

$$\left.\begin{array}{ll} & V_l = V_0 \cosh Pl - Z_0 C_0 \sinh Pl \\ \text{and} & C_l = C_0 \cosh Pl - (V_0/Z_0) \sinh Pl \end{array}\right\} \quad . \quad . \quad . \quad (20)$$

Hence by substitution from (19) in (20)

$$C_r = \frac{V_s}{(Z_s + Z_r) \cosh Pl + \left(\dfrac{Z_s Z_r}{Z_0} + Z_0\right) \sinh Pl} \quad . \quad . \quad . \quad (21)$$

This equation is of great importance in telegraphy. It gives the periodic current C_r in the receiving apparatus Z_r, corresponding to a periodic voltage V_s impressed on the cable through apparatus Z_s, when the length l, the propagation

Table XXIX.—(*Fig.* 78, *Curve A.*)

Frequency, *n*.	$Z_0\sqrt{\frac{\pi}{4}}$	$al=\beta l$.	Sinh al.	Amplitude of received current.
1	951·5	3·697	21·42	49·07
2	673·0	5·227	93·11	15·96
3	549·0	6·403	301·8	6·035
4	476·0	7·394	813·1	2·584
5	425·6	8·266	1,943	1·209
6	388·4	9·057	4,292	0·600
7	359·6	9·782	8,854	0·314
8	336·5	10·455	17,340	0·171
9	317·2	11·094	32,886	0·0959
10	300·9	11·693	59,866	0·0555

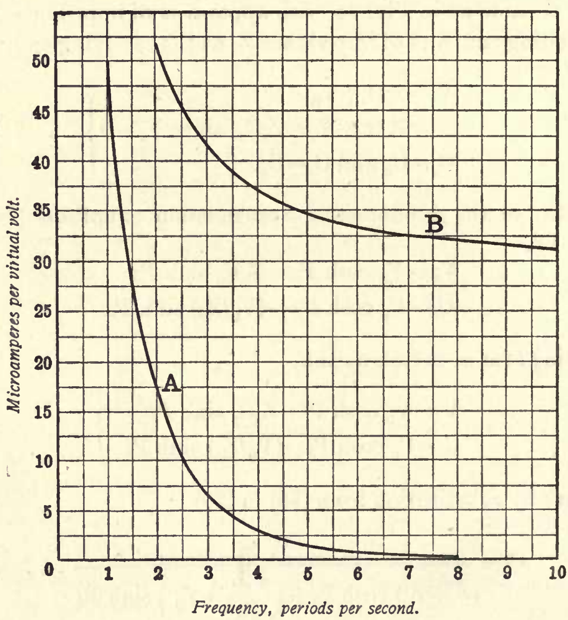

Fig. 78.—Amplitude of Sine Current Received at Various Frequencies.

Curve A.—San Francisco–Honolulu Cable. Ends earthed.
„ B.—Ditto, assumed loaded with 0·1 henry per n.m.

constant P, and the characteristic Z_0, of the cable, are known. When Pl is large, $\cosh Pl = \sinh Pl$,

and
$$C_r = \frac{V_s}{\left\{Z_s + Z_r + Z_0 + \left(\frac{Z_s Z_r}{Z_0}\right)\right\} \sinh Pl} \quad . \quad . \quad . \quad . \quad (22)$$

Table XXIX. is calculated from the equation

$$C_r = \frac{V_s}{Z_0 \sinh Pl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which is obtained from (22) by putting $Z_s = Z_r = 0$. For the simple cable without inductance or leakance

$$Z_0 = \sqrt{\frac{R}{ipK}} = \sqrt{\frac{R}{pK}} \Big| \frac{\pi}{4}, \quad . \quad . \quad . \quad . \quad (24)$$

and
$$P = a + ia = (1+i)\sqrt{\tfrac{1}{2}pKR} = \sqrt{pKR} \Big/ \frac{\pi}{4}. \quad . \quad . \quad (25)$$

Hence

$$C_r \text{ (microamperes per virtual volt of sending E.M.F.)} = \frac{10^6 \sin\left(pt - al + \frac{\pi}{4}\right)}{\sqrt{\frac{R}{pK}} \sinh l\sqrt{\tfrac{1}{2}pKR}}. \quad . \quad (26)$$

The amplitude of the received current in microamperes is

$$\frac{10^6}{\sqrt{\frac{R}{pK}} \sinh l\sqrt{\tfrac{1}{2}pKR}}.$$

The received current falls off very rapidly as the frequency is increased. If 42·5 microamperes is required to form a signal of suitable size, this current would be given by 50 virtual volts at the sending end at a frequency of 5½ periods per second. At higher frequencies a much more sensitive instrument would be necessary in order to obtain any indication at the receiving end.

In order to show the influence of loading on the received current, Curve B, Fig. 78, is drawn from Table XXX. In this case the cable is supposed to have an inductance of 0·1 henry per nautical mile, without leakance. Here the longer formulæ (26) and (11) of Chapter IV. are used to calculate α, β and Z_0.

Table XXX.—(*Fig.* 78. *Curve B.*)

Frequency, n.	Z_0.	al.	l.	Sinh al.	Amplitude of received current.
2	722·7 $\angle 30°3$	3·977	6·875	26·67	51·89
4	587·3 $\angle 20°31$	4·523	12·093	46·06	36·97
6	548·5 $\angle 15°3$	4·698	17·465	54·87	33·23
8	532·8 $\angle 11°45$	4·767	22·923	58·77	31·94
10	524·8 $\angle 9°36$	4·813	28·452	61·52	30·94

The characteristic impedance Z_0 is increased, and sending voltage and current are brought more into phase: α is smaller, and β is greater, than without loading. The higher the frequency the greater is the effect of loading on sinh al, and therefore on the amplitude of the received current. For $n=5$ the ratio of the amplitudes is 29 to 1.

Experimental Test of a Cable with a Periodic E.M.F.

Measurements have been made by Crehore and Squier* of the amplitude of the signals received through the Coney

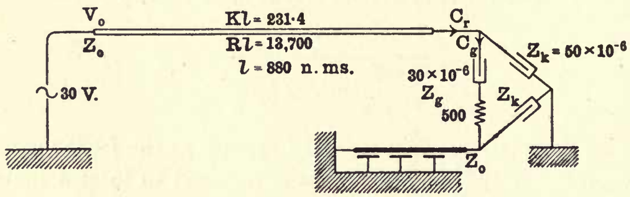

Fig. 79.—Coney Island Cable.

Island cable at different frequencies of sending. The sending E.M.F. of 30 virtual volts was derived from an alternator in New York, and a siphon recorder was used as the receiving instrument in Canso. Signals were sent direct to the cable

* A. C. Crehore and G. O. Squier, Amer.Inst.E.E., Trans., 17, p. 385, 1900.

at New York, and received with the usual duplex arrangements in Canso, as shown in Fig. 79.

At the receiving end the current divides into two parts, one through the condenser Z_k and the other through the recorder and condenser Z_g, and afterwards through the artificial cable Z_0 and the condenser Z_k in parallel. The total impedance Z_r at the receiving end is therefore

$$Z_r = \cfrac{1}{\cfrac{1}{Z_k} + \cfrac{1}{Z_g + \cfrac{1}{\cfrac{1}{Z_0} + \cfrac{1}{Z_k}}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The total received current is, from (22),

$$C_r = \frac{V_s}{(Z_0 + Z_r) \sinh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and of this the fraction

$$C_g = \cfrac{C_r Z_k}{Z_k + Z_g + \cfrac{1}{\cfrac{1}{Z_0} + \cfrac{1}{Z_k}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

is the recorder current.

Hence
$$C_g = \frac{V_s}{\{Z_g + 2Z_0 + (Z_0 Z_g)/Z_k\} \sinh Pl} \quad . \quad . \quad . \quad (30)$$

This curve is plotted from Table XXXI. in Fig. 80.

The straight line is the experimental result found by Crehore and Squier, which has the equation

$$2a = -2{\cdot}97n + 20{\cdot}64 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where a is the amplitude of motion of the recorder in hundredths of an inch, and n the frequency. As the sensitivity of the recorder is not stated it is impossible to fit the curves together other than by trial. Suppose that a current of 3 microamperes is sufficient to produce a double amplitude of two hundredths of an inch; then the calculated curve must be drawn as represented.

Table XXXI.—(*Fig.* 80. *Curve B.*)

n.	$Z_0\overline{\underline{\|45°}}$.	$Z_k\overline{\underline{\|90°}}$.	Z_g.		Sinh *al.*	Amplitude of C_g microamperes per virtual volt.
1	3,070	3,183	5,329	84°37	11·74	6·165
2	2,171	1,592	2,700	79°20	43·37	2·392
3	1,772	1,061	1,837	74°13	118·26	1·043
4	1,535	795·8	1,417	69°20	275·57	0·493
5	1,373	636·6	1,173	64°46	576·8	0·254
6	1,253	530·5	1,016	60°31	1,137·8	0·135
7	1,160	454·7	908	56°35	2,115·2	0·075
8	1,085	397·9	831	52°59	3,762	0·044

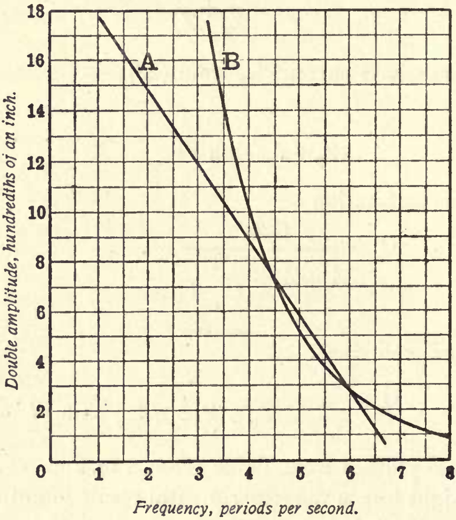

FIG. 80.—CONEY ISLAND CABLE. FREQUENCY AND DOUBLE AMPLITUDE OF RECORDER WITH 30 VIRTUAL VOLTS AT SENDING END.

Curve A. Observed. Curve B. Calculated.

The observed curve does not agree with the calculated curve so well as might be hoped. This may be due to the fact that the E.M.F. supplied by the alternator was far from sinusoidal, or from the inability of the siphon recorder to follow the oscillations at the higher frequencies.* It is highly desirable

* *See* F. A. Taylor, "Journal" I.E.E., 28, p. 500, 1899.

that this valuable experiment should be repeated with apparatus to which no objection could be taken.*

Representation of a Square-topped E.M.F. by a Fourier's Series.

If the E.M.F. which is applied to the end of the cable is rectangular in shape instead of simply sinusoidal, it may be analysed into its Fourier components. Thus

$$V_s = \frac{4E}{\pi}\sum_0^\infty \frac{1}{2m+1}\sin\frac{(2m+1)\,.\,2\pi t}{T} \quad . \quad . \quad . \quad (32)$$

represents the rectangular E.M.F. which is built up of the simple sine oscillations, the first three of which are represented,

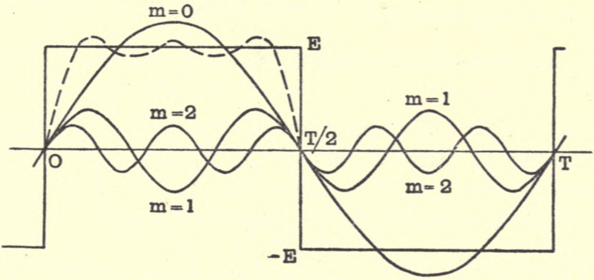

Fig. 81.—Decomposition of a Square-topped E.M.F. into its Sinusoidal Components.

together with their summation curve, in Fig. 81. This rectangular E.M.F. may be derived from a battery connected between the sending end of the cable and earth, the poles of the battery being commutated at the end of every T/2 seconds.

The above reasoning applies to every term in (32). Accordingly the current received at the end of a line without apparatus is

$$C_r = \frac{V_s}{Z_0 \sinh Pl} = \frac{4E}{\pi}\sum_0^\infty \frac{\sin\dfrac{(2m+1)2\pi t}{T}}{(2m+1)Z_0 \sinh Pl} \quad . \quad . \quad . \quad (33)$$

* Tests have been made by H. Tinsley on an artificial cable, using the alternate-current potentiometer; The Electrician, August 1, 1913.

Take E=1, and T=0·1, corresponding to a time of contact of 0·05 second. Then, since $\alpha l=\beta l$, and both are great, sinh Pl may be written $e^{\alpha l(1+i)}$, and (33) becomes

$$C_r=\frac{4}{\pi}\sum_0^\infty\frac{\sin(2m+1)\,.\,20\pi t}{(2m+1)Z_0\sinh Pl}$$

$$=\frac{4}{\pi}\sum_0^\infty\frac{e^{-l\sqrt{\frac{1}{2}pKR}}\sin\left(\overline{2m+1}\,.\,20\pi t+\frac{\pi}{4}-l\sqrt{\tfrac{1}{2}pKR}\right)}{(2m+1)\sqrt{\frac{R}{pK}}}. \quad (34)$$

When equation (34) is applied to the San Francisco-Honolulu cable the result is

C_r (microamperes per volt of sending battery) $=0{\cdot}0353\ \sin(20\pi t+94°58)+0{\cdot}0390$
$\times 10^{-4}\sin(60\pi t-35°23)+0{\cdot}0822\times10^{-7}\sin(100\pi t$
$-13°33)+\ldots.$ (35)

It is seen that the amplitude of the first harmonic is only one ten-thousandth part of that of the fundamental. Hence it follows that the square-topped oscillation (32) at the sending end of the line becomes the pure sine oscillation

C_r (microamperes per volt of sending battery) $=0{\cdot}0353\sin(20\pi t+94°58)$. . (36)

at the receiving end of the line.

The conclusion must therefore be drawn that *in a long submarine cable, whatever the shape of the sending* PERIODIC *E.M.F., the received current is purely sinusoidal.* On account of the extreme attenuation, only the fundamental survives to emerge at the receiving end. For this reason it has been supposed possible to apply the periodic theory directly to the case of telegraphic signalling. This method has been adopted by Hockin,* Kennelly,† Béla Gáti,‡ and other writers.

Consideration does not seem to justify this view. In practical telegraphy the sending E.M.F. never consists of simple

* C. Hockin, "Journal" Soc.T.E., 5, p. 432, 1876.
† A. E. Kennelly, THE ELECTRICIAN, 54, p. 224, 1904.
‡ Béla Gáti, "Elekt. und Maschinenbau," 27, p. 847, 1909.

reversals, and although the square corners of the sending E.M.F. are rounded off in transmission, the received current is essentially not of sine shape, nor is it built up of portions of sine curves. It will appear later that it may be represented as the sum of a series of components which decay according to an exponential law. The fundamental distinction lies, as explained in Chap. III., in the fact that the telegraph engineer has to deal with the transient phenomena of the system, not with the forced oscillations which concern the telephone engineer.

Transient and Alternating-current Phenomena in a Telegraph Cable.

Let one pole of a battery be connected to the sending end of the San Francisco-Honolulu cable, and after an interval of 0·10 second let the sending end be put to earth. After a further interval of 0·10 second has elapsed let the same pole of the battery be applied again, and so on—*i.e.*, let a series of dots be sent into the cable, each dot and each space lasting one-tenth of a second. The shape of the received current due to the making of the first contact is given by (9); for the subsequent earthing of that contact by (9) taken negatively with t decreased to $t-0{\cdot}10$, and so on for successive makes and earthings. These arrival curves are most conveniently added or subtracted graphically. Fig. 82 shows the result. Curve A is the ordinary arrival curve representing the effect of the first contact. Curve B represents the train of dots. The cable does not settle down into its final state of oscillation (of amplitude 1·209 microamperes, Table XXIX., Fig. 78, for the train of dots, and 2·418 for the train of reversals) about the mid position until the effect of the first contact has died away—*i.e.*, until the end of about $2\frac{1}{2}$ seconds, during which time 12 dot signals have been sent. Now, in telegraphy not more than five dots are sent in succession, so that with this rate of signalling the cable never reaches a state of pure sinusoidal oscillation. *The periodic theory applies only after the transient* *phenomena have died away.** Fig. 82, C, shows the effect of a train of reversals; instead of putting the sending end to earth so as to make a space it is here connected to the opposite pole of the battery to form a dash, as in (32). The foregoing result is independent of the speed of signalling, and is quite general. It may be stated thus: *When a periodic E.M.F. is introduced*

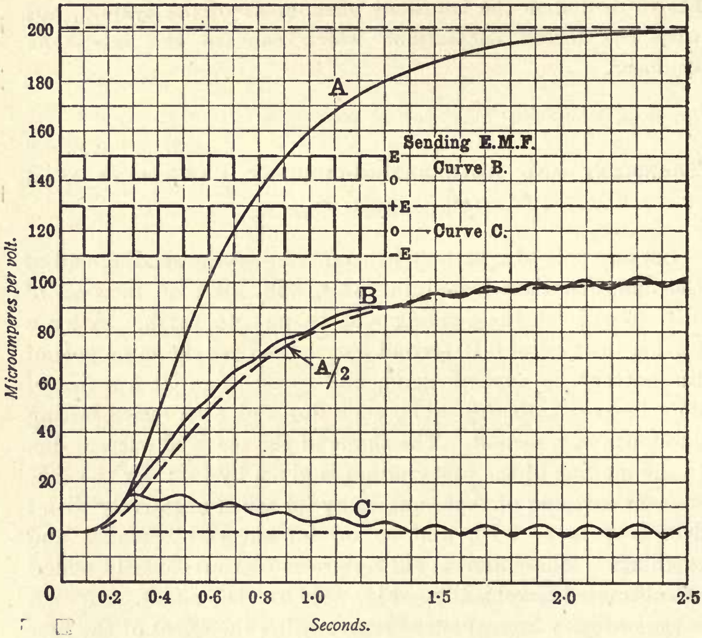

FIG. 82.—ARRIVAL CURRENT. ENDS EARTHED.

Curve A. Single contact.
" B. Train of dots, at the rate of five per second.
" C. Train of reversals.

in an electric circuit the time that elapses before the purely periodic oscillation is established is the same as that in which the arrival curve approximates to its steady value.

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* This is one of the difficulties which confront the proposal made from time to time to telegraph through long cables by means of high-frequency alternating currents. *See* K. W. Wagner, "E.T.Z.," 31, p. 163, 1910.

Now, consider a length of "standard" telephone cable of constants—

$$l=20 \text{ n.m.}$$

$$\left.\begin{array}{l} R=101{\cdot}5 \text{ ohms} \\ K=\ \ 0{\cdot}0623 \text{ mfd.} \end{array}\right\} \begin{array}{l} \text{per n.m.} \\ \text{loop} \end{array} \quad \left.\begin{array}{l} 50{\cdot}8 \text{ ohms} \\ 0{\cdot}125 \text{ mfd.} \end{array}\right\} \begin{array}{l} \text{per n.m.} \\ \text{single core.} \end{array}$$

and apply equation (10). The received current is 99 per cent. of its maximum value when $e^{\frac{\pi^2}{Rl^2}}=200$, or $t=1{\cdot}36\times10^{-3}$ second. Thus, in the telephonic case, *the cable is so short* that the current rises almost instantaneously to its steady value. If the frequency is 800 periods per second only $1{\cdot}36\times800\times10^{-3}=1{\cdot}09$ periods are required for the current to reach within 1 per cent. of the steady value. It is evident that in the telephonic case the state of periodic oscillation is reached almost within the first period. In other words, in the telephonic case the transient phenomena disappear so quickly that they can be left out of account.

General Solution by Periodic Series.

It being thus clear that the simple assumption of sine law transmission is insufficient for the solution of the telegraphic case, some other way must be sought of attacking the problem. The way that suggests itself naturally is to make the speed of sending very low so that the current has time in every instance to reach its steady value. It is seen from Fig. 75, Curve A, that when the transmitting key has been held down for T/2 seconds, where T is greater than 6, the current has, for all practical purposes, reached its steady value. If, now, the poles be commutated, a negative current will flow to line; commutate again, and so on, at each reversal maintaining contact for T/2 seconds, so that the current each time reaches, or very nearly reaches, its steady value. The resultant current at the far end is periodic, of period T seconds. In fact

$$C_r=\frac{4E}{\pi}\sum_0^\infty\frac{\sin(2m+1)2\pi\frac{t}{T}}{(2m+1)Z_0\sinh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

where T=6 seconds, is the equation to the received current.

Only the portion from $t=0$ to $t=T/2$ seconds is required to give the form of the arrival curve. This approximate method becomes an exact one when T is made very great. But then the series is very slowly convergent, and the labour of calculating a sufficient number of terms is great. T should, therefore, be chosen as small as is compatible with the degree of accuracy required.

The Arrival Curve Built up as the Sum of a Number of Periodic Terms.

The method just explained was used by Breisig* to calculate the arrival curve of the Emden-Vigo cable. As an illustration, let it be applied to calculate the arrival curve already drawn in Fig. 75, Curve A. In order that the new curve may be of the same size as the old, take E/2 instead of E as the battery voltage, since a change from $-E/2$ to $+E/2$ is equivalent to applying a voltage $+E$ to the uncharged cable. T is at disposal; choose it to be $2\times 2{\cdot}771$ seconds. This value of T makes $\beta l = l\sqrt{\pi KR/T} = \pi/2$. And since $\lambda = 2\pi/\beta$, it follows that $\lambda = 4l$, or the wave-length of the fundamental is four times the length of the cable. Sinh Pl is then a pure imaginary for the fundamental and its fourth harmonic, which is a slight arithmetical simplification. Using the constants correspond-

Table XXXII.

m.	$Z_0\lfloor 45°$	al	Sinh al.	Cosh al.	Sinh Pl.	
0	2,239·9	1·5708	2·5008	2·5092	2·5092	90° 0
1	1,293·3	2·7206	7·5618	7·6277	7·573	155°43
2	1,001·7	3·5124	16·750	16·779	16·756	201°18
3	846·6	4·1559	31·897	31·913	31·909	238° 9
4	746·6	4·7124	55·665	55·665	55·665	270° 0
5	675·4	5·2094	91·50	...	...	...
6	621·2	5·6638	144·12	...	...	...
7	578·4	6·0841	219·4	...	...	...
8	543·2	6·4767	324·9	...	...	...
9	513·9	6·8466	470·3	...	...	...
10	488·8	7·1981	663·5	...	...	...

* F. Breisig, "E.T.Z.," 21, p. 1046, 1900.

ing to the San Francisco-Honolulu cable, the values given in Table XXXII. are obtained.

After $m=5$, sinh $al=$cosh al, and sinh $Pl=$sinh $al(1+i)$ $=$sinh $al \times e^{-al}$.

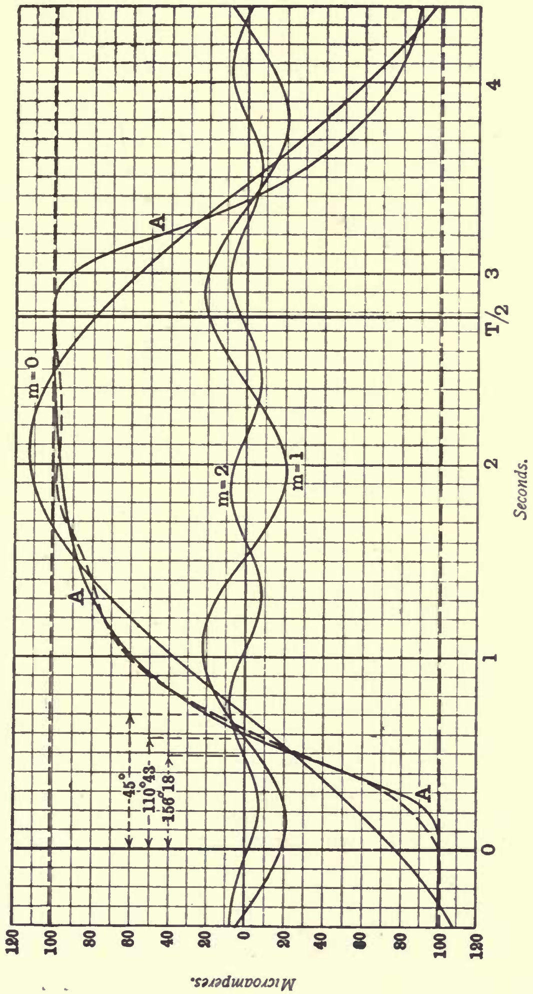

FIG. 83.—ATTENUATION AND PHASE DISPLACEMENT OF FUNDAMENTAL AND FIRST AND SECOND HARMONICS OF SQUARE-TOPPED E.M.F. ENDS EARTHED.

As a final result,

$$C_r \text{ (microamperes)} = 113{\cdot}28 \sin(1{\cdot}134t - 45°) + 21{\cdot}665 \sin(3{\cdot}402t - 110°\ 43) + 7{\cdot}5865 \sin(5{\cdot}670t - 156°\ 18) + 3{\cdot}3667 \sin(7{\cdot}938t - 193°\ 9) + 1{\cdot}7016 \sin(10{\cdot}206t - 225°) + 0{\cdot}93665 \sin(12{\cdot}474t - 253°\ 31) + 0{\cdot}5470 \sin(14{\cdot}742t - 279°\ 32) + 0{\cdot}3345 \sin(17{\cdot}010t - 303°\ 36) + 0{\cdot}21217 \sin(19{\cdot}278t - 326°\ 7) + 0{\cdot}13864 \sin(21{\cdot}546t - 347°\ 19) + 0{\cdot}0928 \sin(23{\cdot}814t - 367°\ 26) + \&c. \quad (37)$$

This is plotted in Fig. 83 for the fundamental, and its first two harmonics, and these added together give the broken curve. If the ordinates from all the terms of (37) were added together they would give Curve A, which is the same as the ordinary arrival curve Fig. 75, A. The higher the frequency of the harmonic the greater its attenuation and phase displacement.

Effect of Loading.

Suppose, now, that an inductance of 0·1 henry is added per nautical mile to the San Francisco-Honolulu cable. The formulæ (26) and (11) of Chapter IV. must now be used to calculate α, β and Z_0 respectively, putting $G=0$ and $L=0{\cdot}1$. In this way Table XXXIII. is obtained.

Table XXXIII.

m.	*p.*	Z_0.		*al.*	*βl.*	Sinh *Pl.*	
0	1·134	2,241·7	43°31	1·5308	1·6124	2·419	92°10
1	3·402	1,301·1	40°34	2·5176	2·9398	6·163	168°17
2	5·670	1,018·1	37°43	3·089	3·9928	10·98	228°53
3	7·938	873·2	35°1	3·478	4·9647	16·21	284°26
4	10·206	784·2	32°28	3·758	5·9045	21·42	338°17

As a final result, for the loaded cable,

$$C_r \text{ (microamperes)} = 117{\cdot}42 \sin(1{\cdot}134t - 44°19) + 26{\cdot}465 \sin(3{\cdot}402t - 127°43) + 11{\cdot}391 \sin(5{\cdot}670t - 191°10) + 6{\cdot}4254 \sin(7{\cdot}938t - 249°25) + 4{\cdot}2105 \sin(10{\cdot}206t - 305°49) + \&c. \quad (38)$$

This is plotted in Fig. 84, again for the fundamental and the first and second harmonics. The broken curve is obtained by adding the ordinates of those three curves together.

The effect of loading is (*a*) to diminish the attenuation, (*b*) to bring the components more into phase, as they are in the

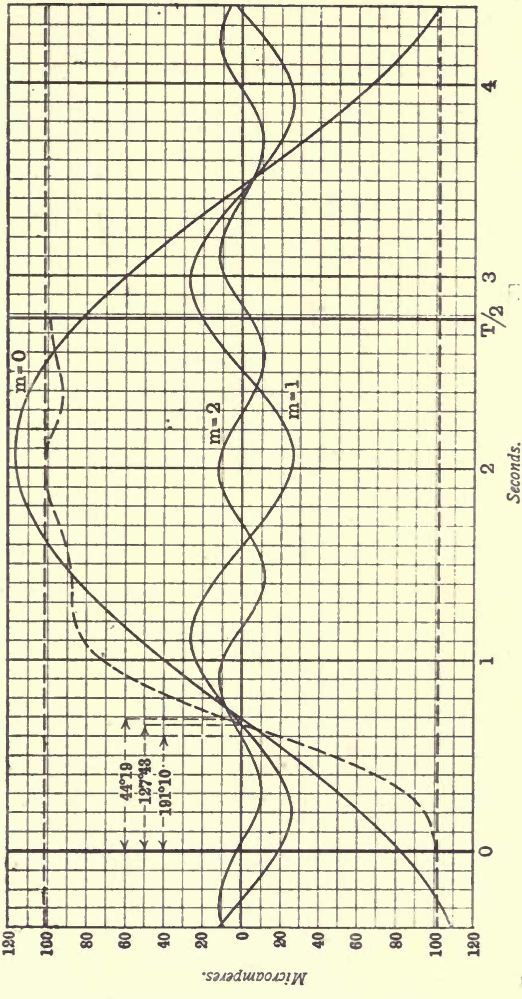

FIG. 84.—ATTENUATION AND PHASE DISPLACEMENT OF FUNDAMENTAL AND FIRST AND SECOND HARMONICS OF SQUARE-TOPPED E.M.F. ENDS EARTHED. ASSUMED LOADED WITH 0·1 HENRY PER NAUTICAL MILE.

sending E.M.F. illustrated in Fig. 81. The resultant arrival curve, which may be obtained to any required degree of accuracy by taking a sufficient number of the terms of (38), is considerably steeper than the arrival curve without loading,* and more closely resembles the square-topped sending E.M.F.

The Distortionless Circuit.

The formulæ for α and β in a loaded cable, if LG be put equal to KR, reduce to

$$\alpha=\sqrt{RG}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and

$$\beta=p\sqrt{LK}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and the velocity of transmission

$$=n\lambda=\frac{n\,.\,2\pi}{\beta}=p/\beta=1/\sqrt{LK}. \quad . \quad . \quad . \quad . \quad (41)$$

It is evident that in these circumstances oscillations of every frequency would be attenuated alike, and would be transmitted with the same velocity. There is no phase displacement, and a square-topped sending E.M.F. gives rise to a square-topped arrival curve.†

If now, as in the case just considered, L=0·1 henry, then for the above condition to hold

$$G=KR/L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

or

$$G=\frac{0{\cdot}3842\times10^{-6}\times2{\cdot}1856}{0{\cdot}1},$$
$$=8{\cdot}397\times10^{-6}\text{ mhos.}$$

which is equivalent to a leak to earth of 119,100 ohms per nautical mile. With this leakance there would be no distortion, although the attenuation would be great. It will be shown later that the steady value of the current corresponding to a prolonged contact with a continuously distributed leakance of G mhos per nautical mile is

$$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh\sqrt{RGl^2}}. \quad . \quad . \quad . \quad . \quad (43)$$

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* For experiments on loading an artificial cable, *see* B. Davies, "Journal" I.E.E., 46, p. 377, 1911.

† *See* Chapter IV., p. 106.

When the foregoing values of R, G and l are substituted, the current which would be received for 1 volt of sending battery is $C_r = 0{\cdot}2193$ microampere. The receiving instrument would have to give for this current a deflection of suitable size to right and to left of the centre line, and would require to be more sensitive and of shorter period than the type of recorder at present in use. The question as to how far the conductor with lumped inductances and leaks would approximate, as regards telegraphic signals, to the distortionless conductor with uniform distribution must be left for another occasion (Chap. XVI.).

Theoretical Deductions from the General Solution by Periodic Series.

The method just explained is extremely powerful, and by means of it the solution in any case, however complicated, can be derived. The trouble lies in the arithmetical work of calculating the series, which is usually considerable, especially when convergence is slow. It is of great value for the light which it throws on the way in which the cable works, and for the theoretical deductions of which it permits.

Thus, from equations (21) and (32) it follows that

$$C_r = \frac{2E}{\pi}\sum_0^\infty \frac{\frac{1}{2m+1}\sin(2m+1)\frac{2\pi t}{T}}{(Z_s+Z_r)\cosh Pl + \left(\frac{Z_sZ_r}{Z_0}+Z_0\right)\sinh Pl} \quad . \quad (44)$$

will give the arrival curve when sending and receiving apparatus is present. From (44) it is evident that (a) by symmetry, if Z_s and Z_r be interchanged—*i.e., if the apparatus at the ends of the cable be changed end for end—the signals are unaltered;* (b) if $Z_s = Z_r$, the denominator is $2Z_r \cosh Pl + \frac{Z_r^2}{Z_0} + Z_0 \sinh Pl$. If $Z_s = 0$, and Z_r is doubled, the denominator is $2Z_r \cosh Pl + Z_0 \sinh Pl$. Hence, since these expressions differ, *if the impedance of the apparatus at one end be doubled the effect is not* *the same as if the impedances at both ends were put in series at one end.* The volts at the receiving end are given by

$$V_l = C_r Z_r = \frac{2E}{\pi} \sum_0^\infty \frac{\frac{1}{2m+1} \sin (2m+1) \frac{2\pi t}{T}}{\left(1+\frac{Z_s}{Z_r}\right) \cosh Pl + \left(\frac{Z_0}{Z_r}+\frac{Z_s}{Z_0}\right) \sinh Pl} . \quad (45)$$

It is seen that inference (*a*) does not apply to the received voltage. Thus, although the voltages at the receiving end of the cable differ in the two cases the current through the receiving apparatus is the same.

THE GENERALISED KR LAW.

A cable of constants L′, R′, K′, G′, and with apparatus Z'_s and Z'_r, will have as its arrival current

$$C'_r = \frac{2E}{\pi} \sum_0^\infty \frac{\frac{1}{2m+1} \sin (2m+1) \frac{2\pi t}{T}}{(Z'_s + Z'_r) \cosh P'l' + \left(\frac{Z'_s Z'_r}{Z'_0} + Z'_0\right) \sinh P'l'} . \quad (46)$$

In order that C'_r and C_r may be of the same shape, corresponding terms in (44) and (46) must bear the same proportion to one another. This necessitates

$$P'l' = Pl,$$

or $\quad l'\sqrt{(R'+ip_m L')(G'+ip_m K')} = l\sqrt{(R+ip_m L)(G+ip_m K)},$

where p_m is the value of p in the $(m+1)^{\text{th}}$ terms of (44) and (46). Equating real and imaginary quantities,

$$l'^2(R'G' - p^2_m L'K') = l^2(RG - p^2_m LK)$$
$$l'^2(L'G' + K'R') = l^2(LG + KR),$$

which must hold for the infinite number of values of p.

Hence $\quad \frac{R'l'}{Rl} = \frac{L'l'}{Ll} = \frac{Kl}{K'l'} = \frac{Gl}{G'l'} = \mu$, say.

From the formula for Z_0 it is seen that these conditions make $Z'_0 = \mu Z_0$. Further, if $Z'_s/Z_s = Z'_r/Z_r = \mu$—that is if all the resistances and inductances constituting Z'_s are μ times those of Z_s and all the capacities and shunts $1/\mu$ times, the denominators of (44) and (46) are in the proportion $1:\mu$. These results may be summarised as follows :—

Two cables of constants, R, L, K, G and R′, L′, K′, G′, with sending apparatus of impedances Z_s, Z'_s and receiving apparatus of impedances Z_r, Z'_r respectively, will transmit signals of the same shape, provided that

$$\frac{\mathrm{R}'l'}{\mathrm{R}l}=\frac{\mathrm{L}'l'}{\mathrm{L}l}=\frac{\mathrm{K}l}{\mathrm{K}'l'}=\frac{\mathrm{G}l}{\mathrm{G}'l'}=\mu=\frac{Z'_s}{Z_s}=\frac{Z'_r}{Z_r},$$

and the size of the signals will be in the proportion μ to 1.

In particular, when L, G, L′, G′, Z_s, Z'_s, Z_r, Z'_r, are all zero, the equations reduce to

$$\frac{\mathrm{R}'l'}{\mathrm{R}l}=\frac{\mathrm{K}l}{\mathrm{K}'l'}=\mu,$$

or $\mathrm{K}'\mathrm{R}'l'^2=\mathrm{KR}l^2$, which is the simplest, or Kelvin, form of the "KR" law, and applies, as has been said, to the particular case of a cable supposed devoid of inductance and leakance, without apparatus, and earthed at both ends.

Kelvin dealt only with the received signals, but the above general form of the KR law may, from equation (89) to follow, be shown to hold good for the signals at the sending end as well.

S 2

CHAPTER IX.

THE PART PLAYED BY THE SIGNALLING APPARATUS.

Periodic Solution of the Telegraphic Equation for Infinitely Slow Sending —The Telegraphic Solution obtained by Differentiation of the Telephonic Solution : The Expansion Theorem—Applications of the Method : Resistance at One End of a Cable—Best Resistance for Recorder—Condenser at One End of a Cable—Condenser at Both Ends : Double Block—Dot Signal with Double Block—Condenser and Recorder—Condenser at an Intermediate Point—Voltage Arrival Curve : Receiving End Insulated—Arrival Curve of Voltage on a Pure Resistance—Arrival Curve of Voltage : Single Condenser —Arrival Voltage : Double Block—Voltage on an Inductance Coil : Theory of the Inductive Shunt.

PERIODIC SOLUTION—INFINITELY SLOW SENDING.

It is evident that the approximate method, described in the last chapter, of calculating the shape of the arrival curve, would become an exact one if the speed of sending were infinitely low. The steady state would then have time to establish itself after every change, and the built-up curve would agree exactly with the curve due to a single prolonged contact. This corresponds to making T infinitely great, and it is required to find what C_r then becomes. This problem may be solved as follows :—

Instead of the Fourier's series (32) for the sending voltage, use a form of Fourier's integral,* and put

$$V_s = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} f(\mu) \cos p(\mu - t) d\mu. \quad . \quad . \quad (47)$$

This integral represents $f(t)$ for all values of t, with certain

* Fourier, " Theory of Heat," p. 351, 1878.

limitations as to continuity. In the present instance, let $f(t)=+\mathrm{E}$ when t is positive; and let $f(t)=-\mathrm{E}$ when t is negative, and substitute in (47); then $f(\mu)=-\mathrm{E}$ from $-\infty$ to 0, and $+\mathrm{E}$ from 0 to $+\infty$. μ and p are integration variables.

Hence

$$f(t)=\frac{\mathrm{E}}{2\pi}\int_{-\infty}^{+\infty} dp\left\{\int_0^{\infty}\cos p(\mu-t)d\mu-\int_{-\infty}^{0}\cos p(\mu-t)d\mu\right\}$$

$$=\frac{\mathrm{E}}{\pi}\int_{-\infty}^{+\infty}\frac{\sin pt}{p}dp. \quad \quad (48)$$

Since (32) may be written

$$\frac{8\mathrm{E}}{\mathrm{T}}\sum_m\frac{\sin p_m t}{p_m}, \text{ where } p=\frac{(2m+1)2\pi}{\mathrm{T}},$$

it is seen that the definite integral (48) may be regarded as the limit of the series (32) when the time of sending is made infinitely great. More generally, write

$$f(t)=\frac{\mathrm{E}}{\pi i}\int_{-\infty}^{+\infty}\frac{e^{ipt}}{p}dp, \quad \quad (49)$$

and consider $f(t)$ as the real part only of (49). But since at the point $t=0$, $f(t)$ has a finite discontinuity, it follows from the conditions of continuity that the integral (47) has a value equal to half the sum of the two values of the function $f(t)$ at the point of discontinuity—*i.e.*, it is equal to $\frac{1}{2}[\mathrm{E}+(-\mathrm{E})]$, and is zero. Hence, in order that it may represent the voltage E applied at $t=0$, the term $+\mathrm{E}$ must be added to it.

This makes $$\mathrm{V}_s=\mathrm{E}+\frac{\mathrm{E}}{\pi i}\int_{-\infty}^{+\infty}\frac{e^{ipt}}{p}dp. \quad \quad (50)$$

Now, as in equation (21), the received current for a periodic E.M.F. is given by an expression of the form

$$\mathrm{C}_r=\frac{\mathrm{V}_s}{\varphi(ip)}. \quad \quad (51)$$

and dividing, as in (33), every element of the integral (50) by the corresponding value of $\varphi(ip)$,

$$C_r = \frac{E}{\varphi(o)} + \frac{E}{\pi i}\int_{-\infty}^{+\infty} \frac{e^{ipt}}{p\varphi(ip)}dp = \frac{E}{\varphi(o)} + \frac{E}{\pi i}\int_{-\infty i}^{+\infty i} \frac{e^{yt}}{y\varphi(y)}dy \quad . \quad . \quad (52)$$

where $y = ip$, and $\varphi(o)$ is the value of $\varphi(ip)$ when p is zero. To integrate (52), express $\frac{1}{\varphi(y)}$ as the sum of a series of partial fractions, thus

$$\frac{f(y)}{\varphi(y)} = \sum_m \frac{f(y_m)}{(y-y_m)\varphi'(y_m)}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where $y_1, y_2 \ldots y_m$ are the roots of $\varphi(y)=0$, and $\varphi'(y) = \frac{d}{dy}\{\varphi(y)\}$. Substituting (53) in (52),

$$C_r = \frac{E}{\varphi(o)} + \frac{E}{\pi i}\int_{-\infty}^{+\infty} \sum_m \frac{e^{y_m t}dy}{y_m(y-y_m)\varphi'(y\;)} = \frac{E}{\varphi(o)} + \frac{E}{\pi i}\sum_m \frac{e^{y_m t}}{y_m\varphi'(y_m)}\int_{-\infty}^{+\infty}\frac{dy}{y-y_m}$$

$$= \frac{E}{\varphi(o)} + E\sum_m \frac{e^{y_m t}}{y_m\varphi'(y_m)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

This is the required expression for C_r.

THE TELEGRAPHIC SOLUTION OBTAINED BY DIFFERENTIATION OF THE TELEPHONIC SOLUTION—THE EXPANSION THEOREM.

The theorem to which the above reasoning has led, and of which (54) is the mathematical expression, is the Expansion Theorem of Heaviside,* who made use of it to rationalise his operational solutions. The foregoing method of deduction shows it to be the natural expression of the effect of a single instantaneous change of voltage at one end of the line, and leads to the following method of application.

* Stated without proof, "Electromagnetic Theory," 2, p. 127.

Write down the periodic solution for a particular frequency, $\frac{p}{2\pi}$, *in the form*

$$C_r = \frac{Ee^{yt}}{\varphi(y)}, \text{ where } y = ip.$$

Find $\varphi'(y)$ *by differentiation of* $\varphi(y)$. *Substitute in it the values of the roots of* $\varphi(y)=0$, *and obtain the exponential series representing the arrival curve for the case under consideration.* The periodic solution, which is of great value in itself, can always be obtained by the solution of a set of simple simultaneous equations. The series have as particular cases series of the Fourier type—of which (4) is an example—to which they degenerate when the apparatus is simplified or removed.

Applications of the Method—Resistance at One End of a Cable.

The way in which the method is to be applied is best made clear by means of examples. Suppose that a cable has a resistance, R_r, at one end and no other apparatus.

Then, from (21),

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl} \quad \ldots \ldots \ldots \quad (55)$$

is the periodic solution.

This is
$$C_r = \frac{Ee^{ipt}}{R_r \cosh l\sqrt{RipK} + \sqrt{\frac{R}{ipK}} \sinh l\sqrt{RipK}}$$

$$= \frac{Ee^{yt}}{R_r \cosh l\sqrt{RKy} + \sqrt{\frac{R}{Ky}} \sinh l\sqrt{RKy}}.$$

Then $\varphi(y) = R_r \cosh l\sqrt{RKy} + \sqrt{\frac{R}{Ky}} \sinh l\sqrt{RKy}$,

which is to be equated to zero. In order that the equation may have real roots, write xi for $l\sqrt{\mathrm{RK}y}$. Then $y=\dfrac{-x^2}{\mathrm{KR}l^2}$, and

$$y\varphi'(y)=\frac{x}{2}f'(x),$$

where $$f(x)=\mathrm{R}_r\cos x+\frac{\mathrm{R}l}{x}\sin x.$$

Also $$f(x)=0,\ \text{is}\ \tan x=\frac{-x\mathrm{R}_r}{\mathrm{R}l}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

And $$f'(x)=-\mathrm{R}_r\sin x-\frac{\mathrm{R}l}{x^2}\sin x+\frac{\mathrm{R}l}{x}\cos x,$$

$$=\frac{\mathrm{R}l}{x}\left[\frac{1}{\cos x}-\frac{\sin x}{x}\right],$$

on substituting from (56).

Hence $$\frac{x}{2}f'(x)=\frac{\mathrm{R}l}{2}\left[\frac{1}{\cos x}-\frac{\sin x}{x}\right].$$

The required expression for C_r is therefore

$$\mathrm{C}_r=\frac{\mathrm{E}}{\mathrm{R}l+\mathrm{R}_r}+\sum_x\frac{2\mathrm{E}e^{-\frac{x^2t}{\mathrm{KR}l^2}}}{\mathrm{R}l\left(\dfrac{1}{\cos x}-\dfrac{\sin x}{x}\right)} \quad . \quad . \quad . \quad . \qquad (57)$$

where the x's are given by equation (56). The steady term in (57) is obtained by putting $x=0$, $\cos x=1$, and $\dfrac{\sin x}{x}=1$, in $f(x)$. In particular, when $\mathrm{R}_r=0$, equation (56) gives $\tan x=0$, or $x=m\pi$, and C_r degenerates to

$$\mathrm{C}_r=\frac{\mathrm{E}}{\mathrm{R}l}+\sum_m\frac{2\mathrm{E}e^{-\frac{m^2\pi^2t}{\mathrm{KR}l^2}}}{\mathrm{R}l}\cos m\pi,$$

which is the formula for the Kelvin arrival curve, (4).

The roots of equation (56) are to be found approximately by means of a diagram as in Fig. 85, and their exact values by means of tables of tangents. The first few roots are sufficient, and they are contained in Tables XXXIV. and XXXV. for the values $\mathrm{R}_r=\mathrm{R}l/10$ and $\mathrm{R}_r=\mathrm{R}l$.

Table XXXIV.—$R_r = Rl/10$, $\tan x = -x/10$. (*Fig.* 85, *Curve A.*)

x.		Cos x.	Sin x.
2·863	$\pi-15°\ 58$	$-0·9614$	$+0·2751$
5·761	$2\pi-29°\ 57$	$+0·8665$	$-0·4992$
8·708	$3\pi-41°\ 3$	$-0·7541$	$+0·6567$
11·703	$4\pi-49°\ 29$	$+0·6497$	$-0·7602$
14·733	$5\pi-55°\ 50$	$-0·5616$	$+0·8274$
17·793	$6\pi-60°\ 40$	$+0·4899$	$-0·8718$

Table XXXV.—$R = Rl$, $\tan x = -x$. (*Fig.* 85, *Curve B.*)

x.		Cos x.	Sin x.
2·029	$\pi-63°\ 46$	$-0·4420$	$+0·8970$
4·914	$2\pi-78°\ 30$	$+0·1994$	$-0·9799$
7·975	$3\pi-82°\ 51$	$-0·1225$	$+0·9922$
11·09	$4\pi-84°\ 51$	$+0·0898$	$-0·9960$

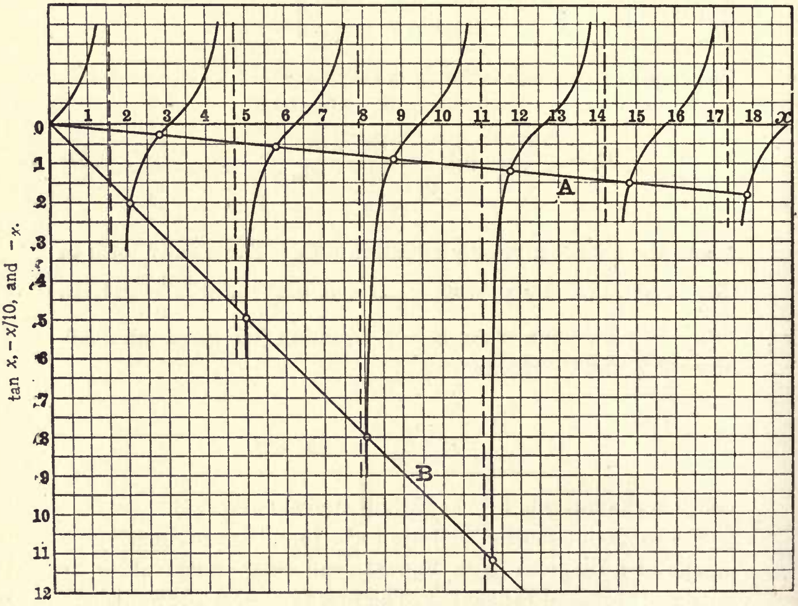

Fig. 85.

Position of the roots of tan $x = -x/10$....Curve A.
,, ,, tan $x = -x$ .... ,, B.

Table XXXVI.—(*Fig.* 86, *Curve B.*)

t secs.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
0+	182·70	182·70	182·70	182·70	182·70	182·70	182·70	182·70	182·70	182·70
x_1-	292·97	242·68	200·98	166·47	137·90	94·60	53·75	20·96	8·17	1·24
x_2+	151·08	70·47	32·87	15·33	7·15	1·56	0·16	...	...	...
x_3-	50·22	8·79	1·54	0·27	0·05	...	...	...	...	...
x_4+	10·79	0·46	0·02	...	...	...	...	...	...	...
x_5-	1·49	0·01	...	...	...	...	...	...	...	...
x_6+	0·13	...	...	...	...	...	...	...	...	...
*C_r	0·02	2·15	13·07	31·29	51·90	89·66	129·11	161·74	174·53	181·46

Table XXXVII.—(*Fig.* 86, *Curve C.*)

t secs.	0·1	0·12	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
0+	100·50	100·50	100·50	100·50	100·50	100·50	100·50	100·50	100·50	100·50
x_1-	135·19	123·00	111·89	101·78	92·60	76·64	57·70	35·95	22·41	8·70
x_2+	44·24	25·40	14·58	8·37	4·81	1·58	0·30	0·02	...	...
x_3-	11·25	2·61	0·60	0·14	0·03	...	...	...	...	...
x_4+	2·12	0·13	0·07	...	...	...	...	...	...	...
x_5-	0·36	...	...	...	...	...	...	...	...	...
*C_r	0·06	0·42	2·66	6·95	12·68	25·44	43·10	64·57	78·09	91·80

* Microamperes per sending volt.

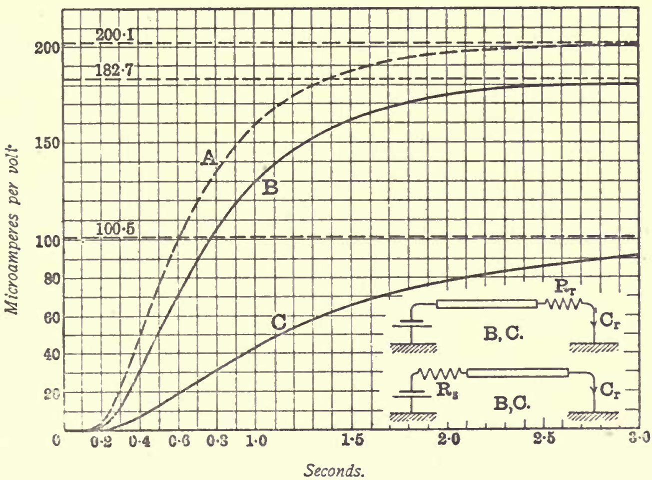

Fig. 86.—Arrival Current. Cable with Resistance at Sending or Receiving End.

Curve A......R_r=0.

Curve B......R_r=R*l*/10=497·5 ohms. Curve C......R_r=R*l*=4975 ohms.

In Tables XXXVI. and XXXVII. are to be found the values of C_r calculated from (57), and they are plotted in Fig. 86, Curve B, and Fig. 86, Curve C.

The smaller the values of t, the more slowly convergent is the series for C_r, and for this reason the value of C_r corresponding to $t=0 \cdot 1$ second may be slightly in error.

Fig. 86, Curve A, is the ordinary arrival curve for the cable without apparatus. The effect of a considerable resistance in retarding the arrival current is very marked.

Best Resistance for Recorder.

Equation (55) is

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl}.$$

Suppose that the sensitivity of the recorder is proportional to the square root of its resistance R_r. The deflection shown by the recorder is then proportional to

$$\frac{\sqrt{R_r}}{Z_r \cosh Pl + Z_0 \sinh Pl}.$$

Now, if Pl be greater than 4, $\sinh Pl = \cosh Pl$ very nearly, and the deflection is proportional to

$$\frac{\sqrt{R_r}}{(Z_r + Z_0) \sinh Pl}.$$

It is required to find what value of R_r makes this expression greatest. Now

$$\frac{\sqrt{R_r}}{Z_r + Z_0} = \frac{\sqrt{R_r}}{R_r + ipL_r + \sqrt{\frac{R}{2pK}} - i\sqrt{\frac{R}{2pK}}},$$

where L_r is the inductance of the recorder. The modulus of this complex quantity is

$$\frac{\sqrt{R_r}}{\sqrt{\left(R + \sqrt{\frac{R}{2pK}}\right)^2 + \left(pL_r - \sqrt{\frac{R}{2pK}}\right)^2}}$$

and this is to be made a maximum. Square, and differentiate and equate to zero. Then

$$R_r = \sqrt{\frac{R}{pK} + p^2L_r^2 - 2pL_r\sqrt{\frac{R}{2pK}}}. \quad . \quad . \quad (58)$$

This is the value of R_r which gives the greatest deflection on the recorder for a particular frequency, $p/2\pi$, provided the assumption made as to the relation between the sensitivity and the resistance of the recorder is true.

When $$L_r = 0,\ R_r = \sqrt{\frac{R}{pK}} = \frac{Rl}{\sqrt{pKRl^2}}. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If N be the number of words per minute and τ be the time of an element, Hockin* writes

$$\tau = \frac{60}{23N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and when $\frac{2\pi}{\tau}$ is substituted for p in (59),

$$R_r = \frac{0{\cdot}64Rl}{\sqrt{N \cdot KRl^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This formula is usually quoted as giving the best resistance of the recorder. But from the mode of derivation adopted above it is seen that *it is only the recorder resistance which gives the greatest deflection with a simple sine E.M.F. of stated frequency*, and the best resistance under the actual conditions of signalling may be something entirely different.

The most suitable resistance for the recorder could be found empirically by plotting the arrival curves as in Fig. 86 for a variety of recorder resistances. The resistance which will give the greatest *steady* deflection is, with the same assumption as to variation of sensitivity with resistance, that which makes $\frac{\sqrt{R}}{R_r + Rl}$ a maximum. This implies that $R_r = Rl$, or the resistance of the galvanometer equals that of the line. But, as is shown in Fig. 86, C, such a resistance would introduce too great a retardation of the signals.

* C. Hockin, "Journal" Soc. Tel. Eng., V., p. 432, 1876.

CONDENSER AT ONE END OF A CABLE.

In equation (55)

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl},$$

let Z_r now be a condenser of capacity K_r. Then

$$Z_r = \frac{1}{ipK_r} = \frac{1}{yK_r} = \frac{-KRl^2}{x^2K_r}.$$

This gives
$$f(x) = \frac{-KRl^2}{x^2K_r}\cos x + \frac{Rl}{x}\sin x,$$

and the equation for x is $f(x)=0$, or

$$\tan x = \frac{Kl}{K_r}\cdot\frac{1}{x}. \quad \ldots \ldots \ldots \quad (62)$$

Also
$$f'(x) = \frac{2KRl^2}{x^3K_r}\cos x + \frac{KRl^2}{x^2K_r}\sin x - \frac{Rl}{x^2}\sin x + \frac{Rl}{x}\cos x$$

$$= \frac{Rl}{x^2}\sin x + \frac{Rl}{x}\cdot\frac{1}{\cos x},$$

on substitution from (62). Hence

$$C_r = \sum_x \frac{2Ee^{\frac{-x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x} + \frac{1}{\cos x}\right]} \quad \ldots \ldots \ldots \quad (63)$$

where x is given by (62). There is no steady term.

Let $K_r = Kl/10 = 87{\cdot}5$ mfds., and, in another example, let $K_r = Kl/20 = 43{\cdot}7$ mfds. The position of the roots of the corresponding equations, $\tan x = 10/x$, and $\tan x = 20/x$, may be obtained from Fig. 87, and they are contained in Tables XXXVIII. and XXXIX. Fig. 88 is plotted from Tables XL. and XLI. After $t=1{\cdot}5$ sec. the curves are represented to less

Table XXXVIII.—$K_r = \frac{Kl}{10}$, $\tan x = \frac{10}{x}$ (*Fig.* 87, *Curve A*).

x,		Tan x.	sin x.	Cos x.
1·429	81° 52	6·997	+0·9899	+0·1415
4·306	π+66° 42	2·322	−0·9184	−0·3955
7·228	2π+54° 8	1·383	+0·8104	+0·5959
10·200	3π+44° 26	0·9804	−0·7001	−0·7141
13·214	4π+37° 7	0·7568	+0·6034	+0·7974
16·260	5π+31° 36	0·6152	−0·5240	−0·8517

Table XXXIX.—$K_r = \frac{Kl}{20}$, $\tan x = \frac{20}{x}$ (*Fig.* 87, *Curve B*).

x.		tan x.	sin x.	cos x.
1·4961	85° 43·3	13·368	+0·9972	+0·07459
4·492	π+77° 21·0	4·452	−0·9757	−0·2190
7·495	2π+69° 27·0	2·668	+0·9364	+0·3510
10·512	3π+62° 16·5	1·903	−0·8852	−0·4652
13·542	4π+55° 54·0	1·4770	+0·8281	+0·5606
16·587	5π+50° 20·0	1·2058	−0·7698	−0·6383

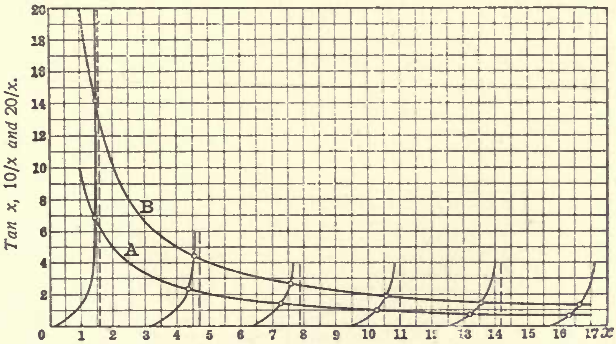

FIG. 87.—POSITION OF THE ROOTS OF

Tan x=10/x......Curve A.
Tan x=20/x...... „ B.

than 1 per cent. by the first term of the series (63), and the upper has approximately twice the height of the lower. The time taken to reach the steady value, which in this case is zero, is very much greater than when both ends of the cable are to earth.

Table XL.—(*Fig.* 88, *Curve A*).

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0	4·0
x_1+	49·42	47·15	44·99	42·93	40·97	37·29	32·40	25·62	20·26	12·67	7·93
x_2-	95·77	62·53	40·84	26·67	17·41	7·43	2·07	0·25	0·03	...	...
x_3+	66·53	20·03	6·03	1·81	0·55	0·05	...	...	...	...	...
x_4-	25·05	2·29	0·21	0·02	...	...	...	...	...	...	...
x_5+	5·48	0·10	...	...	...	...	...	...	...	...	...
x_6-	0·77	...	...	...	...	...	...	...	...	...	...
*C_r	0·16	2·46	9·97	18·05	24·10	29·91	30·33	25·37	20·23	12·67	7·93

* Microamperes per volt of sending battery.

Table XLI.—(*Fig.* 88, *Curve B*.)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0	4·0
x_1+	27·13	25·77	24·48	23·25	22·09	19·93	17·08	13·20	10·21	6·10	3·65
x_2-	52·85	33·24	20·91	13·15	8·27	3·27	0·81	0·08	0·01	...	...
x_3+	37·17	10·22	2·81	0·77	0·21	0·08	...	...	...	...	...
x_4-	14·22	1·12	0·09	...	...	...	...	...	...	...	...
x_5+	3·32	0·05	...	...	...	...	...	...	...	...	...
x_6-	0·45	...	...	...	...	...	...	...	...	...	...
†C_r	0·01	1·68	6·29	10·87	14·03	16·74	16·27	13·12	10·20	6·10	3·65

† Microamperes per sending volt.

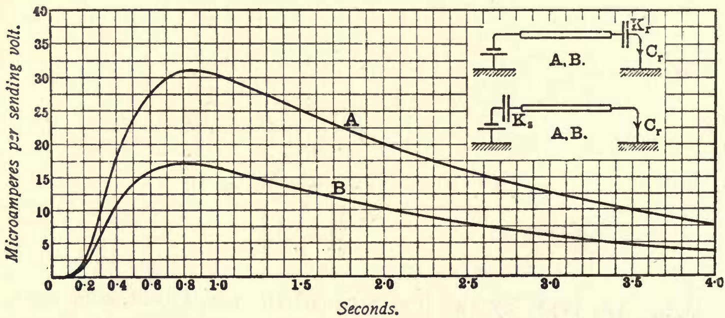

Fig. 88.—Arrival Current. Cable with Condenser at One End.

C irve A .....$K_r = Kl/10 = 87·5$ mfds.
„ B......$K_r = Kl/20 = 43·7$ „

Condensers at Both Ends—Double Block.

The received current for a periodic source is given by (21)

$$C_r = \frac{V_s}{(Z_s+Z_r)\cosh Pl + \left(\dfrac{Z_s Z_r}{Z_0}+Z_0\right)\sinh Pl.}$$

Making the substitutions $\dfrac{Rl}{xi}$ for Z_0, $\dfrac{KRl^2}{-x^2K_r}$ for Z_r, and $\dfrac{KRl^2}{-x^2K_s}$ for Z_s, the denominator becomes

$$f(x) = \frac{-KRl^2}{x^2}\left(\frac{1}{K_s}+\frac{1}{K_r}\right)\cos x$$

$$+\left(-\frac{KRl^2}{K_s}\cdot\frac{KRl^2}{K_r}\cdot\frac{1}{Rl}\cdot\frac{1}{x^3}+\frac{Rl}{x}\right)\sin x.$$

The equation for x is $f(x)=0$, or]

$$\tan x = \frac{\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}}{x-\dfrac{1}{x}\cdot\dfrac{Kl}{K_s}\cdot\dfrac{Kl}{K_r}}. \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Hence

$$C_r = \sum_x \frac{2Ee^{\frac{-x^2t}{KRl^2}}}{xf'(x)}$$

$$= \sum_x \frac{2Ee^{\frac{-x^2t}{KRl^2}}}{Rl\left[\dfrac{\sin x}{x}+\dfrac{1}{\cos x}+\left(\dfrac{\sin x}{x}-\dfrac{1}{\cos x}\right)\dfrac{1}{x^2}\cdot\dfrac{Kl}{K_s}\cdot\dfrac{Kl}{K_r}\right]} \quad (65)$$

This may also be written

$$C_r = \sum_x \frac{2Ee^{\frac{-x^2t}{KRl^2}}}{Rl\left[\dfrac{2\sin x}{x}+\dfrac{1}{x}\left(\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}\right)\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{x}\right)\right]}. \quad (66)$$

When $K_s = K_r$,

$$C_r = \sum_x \frac{Ee^{\frac{-x^2t}{KRl^2}}}{Rl\left[\dfrac{\sin x}{x}+\dfrac{1}{x}\dfrac{Kl}{K_r}\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{x}\right)\right]} \quad . \quad . \quad . \quad . \quad (67)$$

When $K_s = \infty$—*i.e.*, when the sending condenser is short-circuited—(65) degenerates to (63). When $K_s = K_r = \infty$, (66) and (67) degenerate with (64) to (4). The roots of (64) are shown in Fig. 89 for $K_s = K_r = Kl/10 = 87{\cdot}5$ mfds., and $K_s = K_r = Kl/20 = 43{\cdot}7$ mfds., and they are contained in Tables XLII. and

Table XLII.—$K_s = K_r = Kl/10$, $\tan x = \dfrac{20}{x - \dfrac{100}{x}}$ (*Fig.* 89, *Curve A*).

x.		Tan x.	Sin x.	Cos x.
2·6272	$\pi - 29°\ 27$	$-\ 0{\cdot}5646$	$+0{\cdot}4917$	$-0{\cdot}8708$
5·307	$2\pi - 55°\ 55$	$-\ 1{\cdot}4779$	$-0{\cdot}8282$	$+0{\cdot}5604$
8·067	$3\pi - 77°\ 47$	$-\ 4{\cdot}6187$	$+0{\cdot}9774$	$-0{\cdot}2116$
10·909	$3\pi + 85°\ 1$	$+11{\cdot}468$	$-0{\cdot}9962$	$-0{\cdot}0867$
13·819	$4\pi + 71°\ 47$	$+\ 3{\cdot}0385$	$+0{\cdot}9499$	$+0{\cdot}3126$
16·782	$5\pi + 61°\ 34$	$+1{\cdot}8469$	$-0{\cdot}8794$	$-0{\cdot}4761$

Table XLIII.—$K_s = K_r = Kl/20$, $\tan x = \dfrac{40}{x - \dfrac{400}{x}}$ (*Fig.* 89, *Curve B*).

x.		Tan x.	Sin x.	Cos x.
2·8577	$\pi - 16°\ 16$	$-0{\cdot}2918$	$+0{\cdot}2801$	$-0{\cdot}9600$
5·726	$2\pi - 31°\ 57$	$-0{\cdot}6237$	$-0{\cdot}5292$	$+0{\cdot}8485$
8·612	$3\pi - 46°\ 36$	$-1{\cdot}058$	$+0{\cdot}7266$	$-0{\cdot}6871$
11·521	$4\pi - 59°\ 53$	$-1{\cdot}724$	$-0{\cdot}8650$	$+0{\cdot}5018$
14·461	$5\pi - 71°\ 43$	$-3{\cdot}027$	$+0{\cdot}9495$	$-0{\cdot}3137$

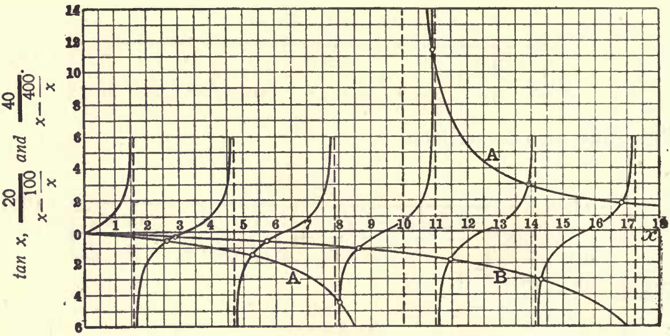

Fig. 89.—Position of the Roots of

$\tan x = \dfrac{20}{x - \dfrac{100}{x}}$......Curve A. $\tan x = \dfrac{40}{x - \dfrac{400}{x}}$......Curve B.

XLIII. By the help of these, Tables XLIV. and XLV. have been calculated from (65), and the corresponding arrival curves are drawn in Fig. 90. The curves for double block are quite different in shape from those with a single condenser. Most noticeable is the disappearance of the long tailing-off shown

Table XLIV.—(*Fig.* 90, *Curve A.*)

t seconds.		0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0	4·0
x_1	+	18·66	15·92	13·59	11·59	9·89	7·20	4·48	2·03	0·92	0·19	0·04
x_2	—	39·99	20·93	10·96	5·74	3·00	0·82	0·12	...	...	...	...
x	+	31·70	7·11	1·59	0·36	0·08	...	...	...	...	...	...
x_4	—	13·00	0·84	0·05	...	...	...	...	...	...	...	...
x_5	+	3·06	0·04	...	...	...	...	...	...	...	...	...
x_6	—	0·44	...	...	...	...	...	...	...	...	...	...
*C_r		0·01	1·30	4·17	6·21	6·97	6·38	4·36	2·03	0·92	0·19	0·04

Table XLV.—(*Fig.* 90, *Curve B.*)

t seconds.		0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0	4·0
x_1	+	6·07	5·03	4·17	3·46	2·87	1·97	1·12	0·44	0·17	0·03	0·00
x_2	—	13·12	6·18	2·91	1·37	0·64	0·14	0·01	...	...	...	...
x_3	+	10·55	1·92	0·35	0·06	0·01	...	...	...	...	...	...
x_4	—	4·41	0·21	0·01	...	...	...	...	...	...	...	...
x_5	+	1·07	0·01	...	...	...	...	...	...	...	...	...
*C_r		0·16	0·57	1·60	2·15	2·24	1·83	1·11	0·44	0·17	0·03	0·00

* Microamperes per sending volt.

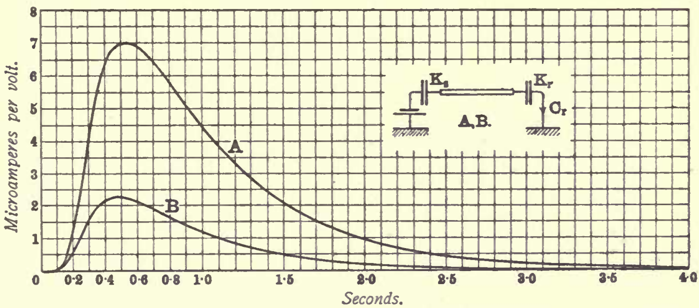

FIG. 90.—ARRIVAL CURRENT. DOUBLE BLOCK.

Curve A......$K_s = K_r = Kl/10 = 87·5$ mfds

„ B......$K_s = K_r = Kl/20 = 43·7$ „

in Fig. 88, and the curves come down sharply to the zero line. The greatest value to which the current attains is less than with a single condenser. Curve A, Fig. 90, for 87·5 mfds. +87·5 mfds., may be compared with Curve B, Fig. 88, for 43·7 mfds., and it is seen that the curve for the two condensers separated by the cable is not the same as the curve which is obtained when they are both placed in series at one end. This proposition was proved above in the general case of any kind of apparatus.

Dot Signal with Double Block.

The arrival curves which have just been obtained for the cable with condensers forming double block may be combined

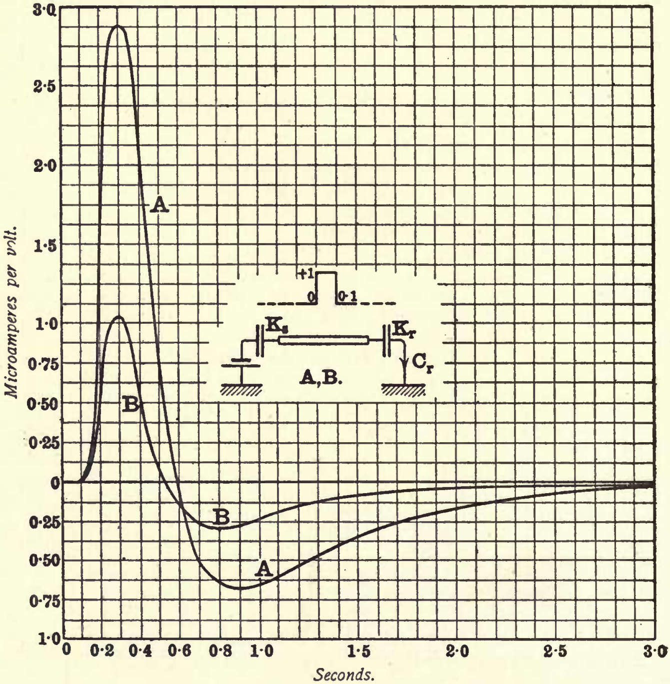

Fig. 91.—Arrival Current. Dot Signal Lasting 0·1 second. Double Block.

Curve A......Condensers $K_s = K_r = Kl/10 = 87{\cdot}5$ mfds
" B......Condensers $K_s = K_r = Kl/20 = 43{\cdot}7$ "

in the manner already described so as to form the arrival curve for a dot signal. This is shown in Fig. 91 for a dot lasting one-tenth of a second, and the curves should be compared with the corresponding one in Fig. 77, Curve A. The greatest height of the curve is attained sooner with condensers than without, but the height is only about one-tenth as great. These curves have been drawn from Fig. 90 by a graphical subtraction. A more accurate representation can be obtained if desired by calculating a table similar to Table XXVIII. The slope of the arrival Curve A in Fig. 75 for the cable without apparatus is always positive, whilst that of the curves in Fig. 90 is at first positive and then negative. Consequently the dot curves in Fig. 77 are altogether above the zero line; but those in Fig. 91 pass after a time below the zero line and become negative. The algebraic sum of the areas enclosed by the two branches of the curve and any time ordinate represents the charge in the receiving condenser at that time.

Condenser and Recorder.

In the periodic solution

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl}$$

write $R_r + \dfrac{1}{ipK_r}$ for Z_r. R_r is the resistance of the recorder, and its inductance is neglected. This leads to the equation

$$\tan x = \frac{-R_r}{Rl}x + \frac{Kl}{K_r} \cdot \frac{1}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Also $$f(x) = \left(R_r - \frac{KRl^2}{x^2K_r}\right)\cos x + \frac{Rl}{x} \cdot \sin x$$

and $$f'(x) = \frac{Rl}{x^2}\sin x + \frac{2R}{x}\cos x + \frac{Rl}{x}\,\frac{1}{\cos x}, \text{ from (68),}$$

or $$\frac{Rl}{x\cos x} - \frac{Rl\sin x}{x^2} + \frac{2KRl^2}{x^3K_r}\cos x.$$

Hence $$C_r = \Sigma_x \frac{2E e^{\frac{-x^2t}{KRl^2}}}{Rl\left[\dfrac{\sin x}{x} + \dfrac{1}{\cos x} + \dfrac{2R_r}{Rl}\cos x\right]} \quad . \quad . \quad . \quad (69)$$

This may also be written

$$C_r=\sum_x \frac{2\mathrm{E}e^{\frac{-x^2t}{\mathrm{KR}l^2}}}{\mathrm{R}l\left[\dfrac{1}{\cos x}-\dfrac{\sin x}{x}+\dfrac{2\mathrm{K}l}{\mathrm{K}_r}\cdot\dfrac{\cos x}{x^2}\right]}. \quad . \quad . \quad (70)$$

When $\mathrm{R}_r=0$, (69) degenerates to (63). When $\mathrm{K}_r=\infty$, (70) becomes the series in (57). The position of the roots of (68)

Table XLVI.—$\mathrm{R}_r=\mathrm{R}l/10$, $\mathrm{K}_r=\mathrm{K}l/20$, $\tan x=20/x-x/10$.

x.		Tan x.	Sin x.	Cos x.
1·4953	85° 40½	13·197	0·9972	0·07542
4·469	$\pi+76°$ 4	4·031	−0·9706	−0·2408
7·384	$2\pi+63°$ 5	1·970	0·8917	0·4527
10·183	$3\pi+43°$ 25	0·9462	−0·6873	−0·7264
12·835	$4\pi+15°$ 23	0·2751	0·2653	0·9642
15·461	$5\pi-14°$ 10	−0·2524	0·2447	−0·9696

Table XLVII.—(*Fig.* 92, *Curve A.*)

t seconds.		0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0	4·0
x_1	+	27·39	26·02	24·72	23·48	22·30	20·12	17·25	13·34	10·32	6·17	3·69
x_2	−	57·51	36·34	22·97	14·51	9·17	3·75	0·92	0·09	0·01	...	...
x_3	+	47·45	13·55	3·87	1·11	0·32	0·03	...	...	...	...	...
x_4	−	23·34	2·15	0·20	0·02	...	...	...	...	...	...	...
x_5	+	7·30	0·17	...	...	...	...	...	...	...	...	...
x_6	−	1·37	0·01	...	...	...	...	...	...	...	...	...
*C_r		−0·08	1·24	5·42	10·06	13·45	16·40	16·33	13·25	10·31	6·17	3·69

* Microamperes per sending volt.

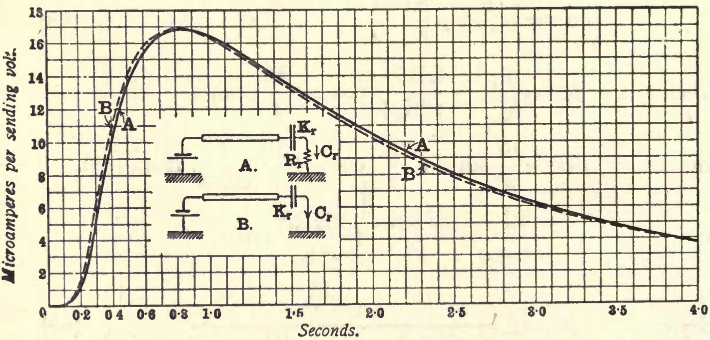

FIG. 92.—ARRIVAL CURRENT. INFLUENCE OF RECORDER.

Curve A (with condenser and recorder)... $\mathrm{K}_r=\mathrm{K}l/20=43{\cdot}7$ mfds.; $\mathrm{R}_r=\mathrm{R}l/10=497{\cdot}5$ ohms.
" B (with condenser only)............... $\mathrm{K}_r=\mathrm{K}l/20=43{\cdot}7$ mfds.

may be found by applying Fig. 85, Curve A, to Fig. 87, Curve B. They are contained in Table XLVI. Fig. 92 is plotted from (69) and Table XLVII. Fig. 92, Curve B, is the arrival curve for the condenser alone, reproduced from Fig. 88, Curve B, in order to show the influence of the recorder. The effect of the recorder resistance is to retard slightly the rise and the fall of the arrival curve, but the effect is not so marked as in Fig. 86, Curve B, for the cable and recorder alone. In like manner the influence of the recorder resistance on the curves of Fig. 90 might be shown.

Condenser at an Intermediate Point.

It is interesting to see what would happen if the condenser, instead of being placed at one end, were inserted somewhere along the line of the cable. Suppose that the position is distant y from the sending end and $l-y$ from the receiving end.

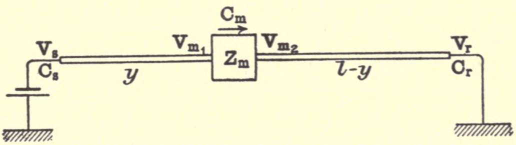

Fig. 93.—Apparatus at an Intermediate Point.

V_{m_1} is the voltage on the sending side of the inserted impedance Z_m and V_{m_2} the voltage on the receiving side; C_m is the current through the impedance. The periodic equations are, from (15) of Chapter IV.,

$$\left.\begin{aligned} V_{m_1} &= V_s \cosh Py - Z_0 C_s \sinh Py \\ C_m &= C_s \cosh Py - (V_s/Z_0) \sinh Py \\ 0 &= V_{m_2} \cosh P(l-y) - Z_0 C_m \sinh P(l-y) \\ C_r &= C_m \cosh P(l-y) - (V_{m_2}/Z_0) \sinh P(l-y) \end{aligned}\right\} \quad .(71)$$

and $V_{m_1} - V_{m_2} = Z_m C_m$.

From the five homogeneous equations (71), C_r is obtained in terms of V_s by elimination of the four quantities V_{m_1}, V_{m_2}, C_s and C_m. Thus

$$C_r=\frac{V_s}{Z_m \cosh Py \cosh P(l-y)+Z_0 \sinh Pl} \quad . \quad . \quad . \quad (72)$$

If $y=l-y$, that is, if the impedance Z_m be at the middle point of the cable, then

$$C_r=\frac{V_s}{Z_m \cosh^2 \frac{Pl}{2}+Z_0 \sinh Pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Suppose that Z_m is a condenser of capacity K_m. Proceeding as before, the equation for x is

$$f(x)=\frac{KRl^2}{-x^2K_m} \cos^2 \frac{x}{2}+\frac{Rl}{x} \sin x=0,$$

$$\left.\begin{array}{ll} \text{or} & \cos \frac{x}{2}=0 \\ \text{and} & \tan \frac{x}{2}=\frac{1}{2x} \cdot \frac{Kl}{K_m} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

$$f'(x)=\frac{2KRl^2}{x^3K_m} \cos^2 \frac{x}{2}+\frac{KRl^2}{x^2K_m} \cos \frac{x}{2} \sin \frac{x}{2}-\frac{Rl}{x^2} \sin x+\frac{Rl}{x} \cos x,$$

which becomes $\frac{-Rl}{x}$, or $\frac{Rl}{x^2} \sin x+\frac{Rl}{x}$, when (74) is substituted in it. Hence

$$C_r=\sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[1+\frac{\sin x}{x}\right]}-\sum_{m=0}^{\infty} \frac{2Ee^{-\frac{(2m+1)^2\pi^2t}{KRl^2}}}{Rl} \quad . \quad . \quad . \quad (75)$$

is the required expression.

If K_m be put equal to $Kl/40$, then (74) is $\tan y=10/y$, where $y=x/2$. The y roots of (74) are then the same as the x roots in Table XXXVIII. They are contained in Table XLVIII.

Table XLVIII.—$K_m = Kl/40$, tan $x/2 = 20/x$.

x.		sin x.
2·858	163° 44	0·2801
8·612	$2\pi+133°$ 24	0·7266
14·456	$4\pi+108°$ 16	0·9496
20·400	$6\pi+$ 88° 52	0·9998

Table XLIX.—(*Fig.* 94, *Curve A.*)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	303·43	251·51	208·47	172·80	143·23	98·38	56·03	21·92	8·58	1·31
x_2 +	67·41	12·26	2·23	0·41	0·07	...	...	...	...	...
x +	3·09	0·03	...	...	...	...	...	...	...	...
x^3 +	0·03	...	...	...	...	...	...	...	...	...
$m=0$, −	320·37	255·37	203·56	162·26	129·33	82·17	41·62	13·39	4·31	0·45
$m=1$, −	52·19	6·78	0·88	0·11	0·01	...	...	...	...	...
$m=2$, −	1·39	...	...	...	...	...	...	...	...	...
$m=3$, −	0·01	...	...	...	...	...	...	...	...	...
C_r*	0·00	1·65	6·26	10·84	13·96	16·21	14·41	8·53	4·27	0·86

* Microamperes per sending volt.

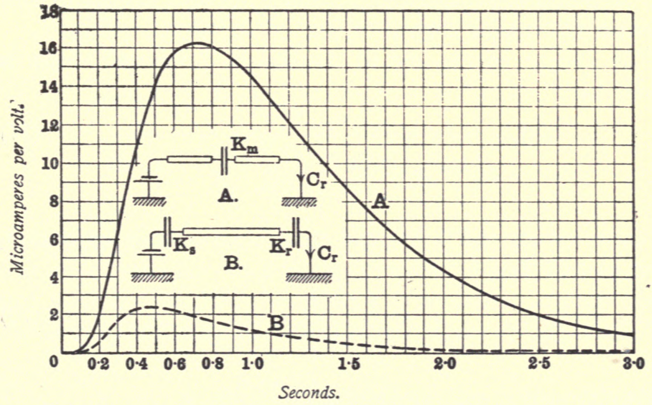

Fig. 94.—Arrival Current. Cable with Intermediate Apparatus.

Curve A....With single condenser K_m at the middle, $K_m = Kl/40 = 21{\cdot}9$ mfds.

,, B....With condensers K_s and K_r at the ends, $K_s = K_r = Kl/20 = 43{\cdot}7$ mfds..
K_s and K_r in series $= 21{\cdot}9$ mfds.

Fig. 94 is drawn from (75) and Table XLIX. After $t=0{\cdot}4$ second, (75) becomes, to less than 1 per cent.,

$$C_r=\frac{2E}{Rl}\left[0{\cdot}9107e^{-\frac{(2\cdot858)^2t}{KRl^2}}-e^{-\frac{\pi^2t}{KRl^2}}\right]. \quad . \quad . \quad (76)$$

In Fig. 94, Curve A is the arrival curve for the single condenser at the middle of the cable, and Curve B is reproduced from Fig. 90 for the sake of comparison. The arrival curve for the single condenser at the middle resembles in type more closely the curves of Fig. 90 for double block than those of Fig. 88 for a single condenser, and the current rises to a much greater value. This result opens up a promising field of investigation, but it must be left for the present in this general survey. The curves for double block and a single condenser at the middle, or for a number of condensers along the line of the cable, could be obtained in a similar manner.

Voltage Arrival Curve—Receiving End Insulated.

The curves of arrival for the voltage at the receiving end of the cable can be obtained in a manner precisely similar to that used for the current arrival curves.

The voltage at the end of the line in the periodic case is given by

$$V_l=Z_rC_r=\frac{V_s}{\left(1+\frac{Z_s}{Z_r}\right)\cosh Pl+\left(\frac{Z_s}{Z_0}+\frac{Z_0}{Z_r}\right)\sinh Pl}. \quad . \quad . \quad (77)$$

If the receiving end is free and there is no sending apparatus, $Z_r=\infty$, and $Z_s=0$.

Then
$$V_r=\frac{V_s}{\cosh Pl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Here $f(x)=\cos x$, and the roots are $\frac{2m+1}{2}\pi$.

Also $f'(x)=-\sin x$, $\frac{xf'(x)}{2}=-\frac{x\sin x}{2}=-\frac{(2m+1)}{4}\pi\sin\frac{2m+1}{2}\pi$

and $$V_r = E - \sum_0^{\infty} \frac{4Ee^{-\frac{(2m+1)^2\pi^2 t}{4KRl^2}}}{(2m+1)\pi \sin \frac{2m+1}{2}\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Fig. 95, A is plotted from (79) and Table L.

The curve of arrival voltage with receiving end free rises much more slowly to its steady value than does the arrival

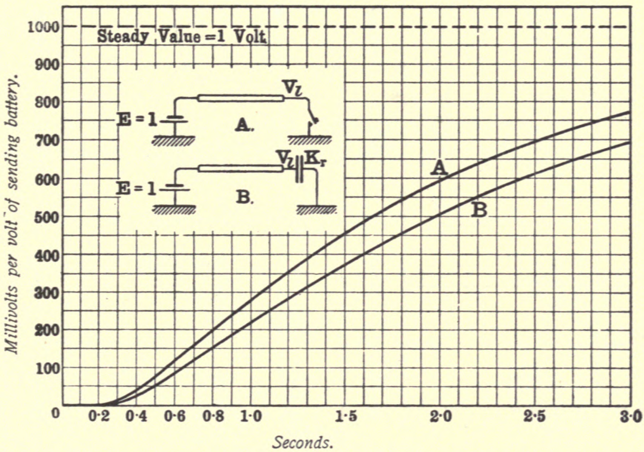

FIG. 95.—ARRIVAL VOLTAGE. END FREE AND EARTHED THROUGH A CONDENSER.

Curve A.... Receiving-end free.
„ B.... Receiving-end earthed through a condenser.
$K^r = Kl/10 = 87{\cdot}5$ mfds.

curve of current with receiving end to earth, Fig. 75, Curve A. After time one second it is evident from Table L. that the curve is represented by the first term of the series (79), *i.e.*,

$$V_r = E\left(1 - \frac{4}{\pi} e^{-\frac{\pi^2 t}{4KRl^2}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

which should be compared with the corresponding expression (10) for the arrival current with end earthed.

Table L.—(*Fig.* 95, *Curve A.*)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00
$m=0$, −	1,203·10	1,136·70	1,074·10	1,014·80	958·90	856·00	722·18	543·80	409·60	232·34
$m=1$, +	254·78	152·94	91·81	55·12	33·09	11·92	2·58	0·20	0·02	...
$m=2$, −	61·70	14·95	3·62	0·88	0·21	0·01	...	...	...	...
$m=3$, +	11·30	0·70	0·04	...	...	...	...	...	...	...
$m=4$, −	1·43	0·01	...	...	...	...	...	...	...	...
$m=5$, +	0·12	...	...	...	...	...	...	...	...	...
$m=6$, −	0·01	...	...	...	...	...	...	...	...	...
V (millivolts per sending volt)	−0·04	1·98	14·13	39·44	73·98	155·91	280·40	456·40	590·42	767·66

Table LI.—(*Fig.* 95, *Curve B.*)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00	1,000·00
x_1, −	1,203·30	1,148·15	1,095·70	1,045·35	997·40	907·99	788·78	623·82	493·35	308·55
x_2, +	256·98	167·79	109·57	71·55	46·73	19·93	5·55	0·66	0·08	...
x_3, −	63·38	19·08	5·74	1·73	0·52	0·05	...	...	...	...
x_4, +	11·98	1·10	0·10	0·01	...	...	...	...	...	...
x_5, −	1·60	0·03	...	...	...	...	...	...	...	...
x_6, +	0·14	...	...	...	...	...	...	...	...	...
V_l (millivolts per sending volt)	0·82	1·63	8·23	24·48	48·81	111·89	216·77	376·84	506·73	691·45

Arrival Curve of Voltage on a Pure Resistance.

If the receiving end is connected to earth through a resistance, R_r, and there is no sending apparatus, Z_r in (77) equals R_r, and Z_s is zero. Then

$$V_l=\frac{V_s}{\cosh Pl+\frac{Z_0}{Z_r}\sinh Pl}. \quad . \quad . \quad . \quad . \quad (81)$$

$$f(x)=\cos x+\frac{Rl}{xR_r}\sin x=0.$$

$$\therefore \quad \tan x=-x\,.\,\frac{R_r}{Rl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

This is equation (56) again, and the roots for $R_r=Rl/10$ and $R_r=Rl$ are contained in Tables XXXIV. and XXXV.

$$f'(x)=-\sin x-\frac{Rl}{x^2R_r}\sin x+\frac{Rl}{xR_r}\cos x,$$

which from (56)

$$=\frac{\cos x}{x}-\frac{1}{\sin x}$$

$$\therefore \quad V_l=\frac{E}{1+\frac{Rl}{R_r}}+\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}}{\cos x-\frac{x}{\sin x}}. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

The arrival voltage might otherwise have been obtained from the current through the resistance. Thus, from (57)

$$C_r=\frac{E}{Rl+R_r}+\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{1}{\cos x}-\frac{\sin x}{x}\right]};$$

$$\therefore V_l \quad =R_rC_r=\frac{E\,.\,R_r}{Rl+R_r}+\sum_x\frac{2R_re^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{1}{\cos x}-\frac{\sin x}{x}\right]},$$

which, by the aid of (56), is seen to be the same as (82). If C_r be measured in microamperes, then, to obtain V_l in millivolts, C_r in Table XXXVI. must be multiplied by 0·4975, since $R_r=497·5$ ohms. In this way, Fig. 96, Curve A is obtained, which is of the same shape as Fig. 86, Curve B.

Arrival Curve of Voltage.—Single Condenser.

Let the receiving end of the cable be connected to earth through a condenser of capacity K_r. Then in (81) Z_r is to be put equal to $\frac{KRl^2}{-x^2K_r}$.

Hence $$f(x)=\cos x-\frac{K_r}{Kl}x\sin x,$$

and $$\tan x=\frac{1}{x}\cdot\frac{Kl}{K_r}.$$

This is equation (62) again.

$$f'(x)=\frac{-1}{\sin x}-\frac{\cos x}{x},$$

by substitution from (62).

Hence $$V_l=E-\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}}{\frac{x}{\sin x}+\cos x}, \quad . \quad . \quad . \quad . \quad (83)$$

which corresponds to (63).

Fig. 95, Curve B is drawn from (83) and Table LI. for a condenser $K_r=Kl/10=87·5$ mfds., and it should be compared with the corresponding current (Curve A in Fig. 88). The voltage curve (Fig. 95, Curve B) is the time integral of the current curve (Fig. 88, Curve A)—*i.e.*, the area bounded by the current curve and any time ordinate is the quantity of electricity in the condenser, and this, divided by the capacity K_r of the condenser, is equal to the voltage on the condenser at that time. Whereas the current increases to a maximum and falls again, the voltage on the condenser increases to a steady value, which is equal to the E.M.F. of the sending battery. The rise in the voltage takes place more slowly over a condenser than when the end is insulated, as in Fig. 95, Curve A.

Arrival Voltage.—Double Block.

Suppose, now, that there are condensers at both ends of the cable. Then, in (77),

$$V_l = \frac{V_s}{\left(1+\frac{Z_s}{Z_r}\right)\cosh Pl + \left(\frac{Z_s}{Z_0}+\frac{Z_0}{Z_r}\right)\sinh Pl},$$

the substitutions $\frac{KRl^2}{-x^2K_s}$ for Z_s and $\frac{KRl^2}{-x^2K_r}$ for Z_r must be made.

Hence $\quad f(x) = \left(1+\frac{K_r}{K_s}\right)\cos x + \left(\frac{Kl}{K_s}\cdot\frac{1}{x} - x\frac{K_r}{Kl}\right)\sin x,$

and $\quad \tan x = \dfrac{1+\frac{K_r}{K_s}}{x\frac{K_r}{Kl} - \frac{1}{x}\frac{Kl}{K_s}},$

which is equation (64), and the roots are given in Tables XLII. and XLIII.

$$f'(x) = -\left(1+\frac{K_r}{K_s}\right)\sin x - \left(\frac{Kl}{K_s}\cdot\frac{1}{x^2} + \frac{K_r}{Kl}\right)\sin x$$
$$+\left(\frac{Kl}{K_s}\cdot\frac{1}{x} - x\frac{K_r}{Kl}\right)\cos x,$$

which, by substitution from (64),

$$= \frac{1}{x}\frac{Kl}{K_s}\left(\frac{1}{\cos x} - \frac{\sin x}{x}\right) - \frac{xK_r}{Kl}\left(\frac{1}{\cos x} + \frac{\sin x}{x}\right).$$

Hence, finally

$$V_l = \frac{E\,.\,K_s}{K_s+K_r+K_l} - \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{\frac{x^2K_r}{Kl}\left(\frac{1}{\cos x} + \frac{\sin x}{x}\right) + \frac{Kl}{K_s}\left(\frac{\sin x}{x} - \frac{1}{\cos x}\right)} \quad (84)$$

When $K_s = \infty$ this reduces to (83).

Fig. 96, Curve B, is drawn from (84) and Table LII. The corresponding current curve for $K_s = K_r = Kl/10$ is Curve A, Fig. 90. The arrival curve of voltage rises very much more quickly to its steady value, which is, however, only one-twelfth of the voltage applied at the sending end, than does the curve for a single condenser, Fig. 95, Curve B.

Table LII.—(*Fig.* 96, *Curve B.*)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33
x_1 —	134·49	114·74	97·92	83·56	71·30	51·92	32·26	14·60	6·61	1·35
x_2 +	70·63	37·00	19·37	10·14	5·31	1·46	0·21	...	...	...
x_3 —	24·24	5·43	1·22	0·27	0·06	...	...	...	...	...
x_4 +	5·43	0·35	0·02	...	...	...	...	...	...	...
x_5 —	0·80	0·01	...	...	...	...	...	...	...	...
x_6 +	0·08	...	...	...	...	...	...	...	...	...
$*V_l$	—0·06	0·50	3·58	9·64	17·28	32·87	51·28	68·73	76·72	81·98

* Millivolts per sending volt.

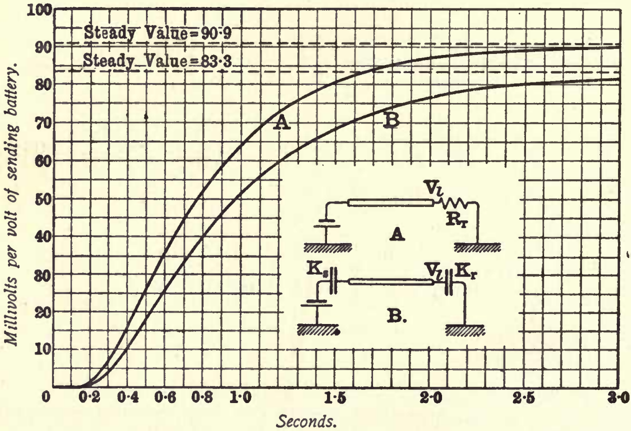

FIG 96.—ARRIVAL VOLTAGE. CABLE WITH APPARATUS.

Curve A....Receiving-end earthed through a resistance. $R_r = Rl/10 = 497·5$ ohms.
" B....Double block. $K_s = K_r = Kl/10 = 87·5$ mfds.

As a check on the accuracy of the formulæ, (84) may be obtained by integration from (65).

Thus $$V_l=\frac{1}{K_r}\int_0^t C_r \,.\, dt.$$

Substitute for C_r from (65), and

$$V_l=\sum_x\frac{2E}{K_r}\int_0^t\frac{e^{-\frac{x^2t}{KRl^2}}dt}{Rl\left[\frac{\sin x}{x}+\frac{1}{\cos x}+\left(\frac{\sin x}{x}-\frac{1}{\cos x}\right)\frac{1}{x^2}\frac{Kl}{K_s}\cdot\frac{Kl}{K_r}\right]}.$$

When the integration is carried out (84) is obtained.

Voltage on an Inductance Coil.—Theory of the Inductive Shunt.

It has been found by experiment that a great improvement in the received signals results if the recorder be shunted by an inductance coil, and this plan is now generally adopted. To examine this case, let Z_r in (81) be an inductance coil of resistance R_r and inductance L_r, so that $Z_r=R_r+ipL_r$. Then

$$f(x)=\cos x+\frac{Rl}{x\left(R_r-\frac{x^2L_r}{KRl^2}\right)}\sin x,$$

and $$\tan x=\frac{xR_r}{Rl}\left(\frac{x^2L_r}{KRl^2\,.\,R_r}-1\right). \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Also $$f'(x)=-\frac{1}{\sin x}+\frac{2R_r}{Rl}\cdot\frac{\cos^2 x}{\sin x}+\frac{3\cos x}{x},$$

by help of (85). Hence

$$V_l=\frac{E}{1+\frac{Rl}{R_r}}+\sum_x\frac{Ee^{-\frac{x^2t}{KRl^2}}}{\frac{x}{2\sin x}\left(\frac{2R_r}{Rl}-1\right)-\frac{xR_r}{Rl}\sin x+\frac{3}{2}\cos x}, \quad (86)$$

where x is given by (85).

The position of the roots of equation (85) can be obtained from Fig. 97. When L_r is large the tangent curve is cut in all its positive branches. When L_r is smaller the points of intersection with the first positive branch are imaginary. These imaginary points of intersection give rise to a complex root of equation (85). When L_r is made still smaller the point of intersection with the second positive branch passes to the first negative branch, and the imaginary points of intersection move to the right. When L_r is very small equation (85) becomes (56), the roots of which are shown in Fig. 85.

Let $R_r = Rl/10$, and (85) becomes

$$\tan x = \frac{x}{10}\left(\frac{x^2 L_r}{KRl^2 \, . \, R_r} - 1\right).$$

An interesting case is when $\frac{L_r}{R_r} = KRl^2$, or the time constant, $\frac{L_r}{R_r}$, of the coil equals the time constant, KRl^2, of the cable. Equation (85) is then

$$\tan x = \frac{x}{10}(x^2 - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

The real roots of this equation are contained in Table LIII.

To find the complex root of (87), write $x = a + ib$, and separate real quantities from imaginary. As a result, equation (87) gives rise to the simultaneous equations,

$$\left.\begin{array}{l} A \cot a + B \tanh b = 1 \\ A \tan a - B \coth b = -1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where $$A = \frac{a}{10}(a^2 - 3b^2 - 1),$$

and $$B = \frac{b}{10}(3a^2 - b^2 - 1).$$

On solving these equations by a process of trial and error, $a + ib$ is found to be $2{\cdot}0105 + i \times 1{\cdot}143$. This value is to be sub-

Table LIII.—$R_r = Rl/10$; $L_r = KRl^2 \times R_r$. (*Fig.* 97, *Curve B.*)

x.		Sin x.	Cos x.
4·6053	$\pi+83°$ 52	−0·99430	−0·10684
7·8327	$2\pi+88°$ 47	+0·99978	+0·02123
10·988	$3\pi+89°$ 34	−0·99997	−0·00756
14·134	$4\pi+89°$ 48	+0·99999	+0·00349
17·277	$5\pi+89°$ 53	−1·00000	−0·00204

Table LIV.—$R_r = Rl/10$; $L_r = \dfrac{KRl^2 \,.\, R_r}{\pi^2}$. (*Fig.* 97, *Curve C.*)

x.		Sin x.	Cos x.
3·14159	π	0	−1
7·588	$2\pi+74°$ 45	+0·96479	+0·26303
10·913	$3\pi+85°$ 16	−0·99659	−0·08252
14·100	$4\pi+87°$ 53	+0·99932	+0·03693
17·259	$5\pi+88°$ 52	−0·99980	−0·01978

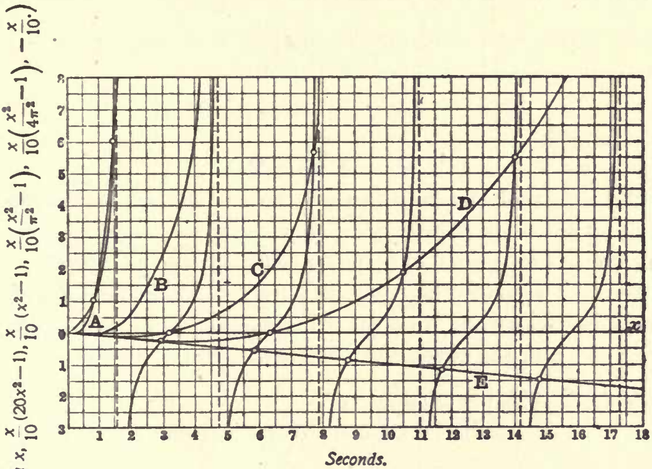

FIG. 97.—POSITION OF THE ROOTS OF

$$\text{Tan } x = \frac{xR_r}{Rl}\left(\frac{x^2}{KRl^2} \cdot \frac{L_r}{R_r} - 1\right).$$

Curve A....$R_r = \dfrac{Rl}{10}$, $\dfrac{L_r}{R_r} = 20$.

„ B....$R_r = \dfrac{Rl}{10}$, $\dfrac{L_r}{R_r} = KRl^2$.

„ C....$R_r = \dfrac{Rl}{10}$, $\dfrac{L_r}{R_r} = \dfrac{KRl^2}{\pi^2}$.

„ D....$R_r = \dfrac{Rl}{10}$, $\dfrac{L_r}{R_r} = \dfrac{KRl^2}{4\pi^2}$.

„ E....$R_r = \dfrac{Rl}{10}$, $L_r = 0$.

stituted in (86), and the corresponding term is $\left(\frac{H+iJ}{M+iN}\right)$, where

$$H=e^{-\frac{(a^2-b^2)t}{KRl^2}}\left(\sin a\cosh b\cos\frac{2abt}{KRl^2}+\cos a\sinh b\sin\frac{2abt}{KRl^2}\right),$$

$$J=e^{-\frac{(a^2-b^2)t}{KRl^2}}\left(\cos a\sinh b\cos\frac{2abt}{KRl^2}-\sin a\cosh b\sin\frac{2abt}{KRl^2}\right),$$

$$M=\frac{-9a}{20}+\frac{a}{20}\cos 2a\cosh 2b+\frac{b}{20}\sin 2a\sinh 2b+\frac{3}{4}\sin 2a\cosh 2b.$$

$$N=\frac{-9b}{20}+\frac{b}{20}\cos 2a\cosh 2b-\frac{a}{20}\sin 2a\sinh 2b+\frac{3}{4}\cos 2a\sinh 2b.$$

$\frac{H-iJ}{M-iN}$ is the conjugate term, and when the two terms are combined the resultant real term is $\frac{2(HM+JN)}{M^2+N^2}$. When the value $2{\cdot}0105+i\times 1{\cdot}143$ for the complex root is substituted in this expression, the damped oscillation,

$$0{\cdot}6623e^{-0{\cdot}62855t}\sin(1{\cdot}0561t-37°27)$$

is obtained.

Fig. 98, Curve A is plotted from Table LV.

The effect of the inductance is to lift the arrival voltage high up above its steady value, to which it re-descends according to a sine curve of exponentially decreasing amplitude. After $t=2$ seconds the curve is purely oscillatory, the exponential terms arising from the real roots of (87) having entirely died out, and the equation to the arrival curve is then

$$V_l \text{ (millivolts per sending volt)}$$
$$=90{\cdot}91+662{\cdot}3e^{-0{\cdot}62855t}\sin(1{\cdot}0561t-37°27).$$

Another interesting case is when the time constant of the coil is equal to the time constant of the cable divided by π^2—*i.e.*, when $L_r=\frac{KRl^2}{\pi^2}R_r$. In this case equation (85) is

$$\tan x=\frac{x}{10}\left(\frac{x^2}{\pi^2}-1\right).$$

Now, for the root $x=\pi$ of this equation the denominator of (86) is infinitely great, and the corresponding term drops out.

Table LV.—(*Fig.* 98, *Curve A.*)

t seconds.	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91
x_1 +	285·64	175·45	107·77	66·19	40·66	15·34	3·56	0·31	0·03	...
x_2 −	62·84	15·35	3·75	0·91	0·22	0·01	...	...	...	...
x_3 +	11·38	0·71	0·04	...	...	...	...	...	...	...
x_4 −	1·43	0·01	...	...	...	...	...	...	...	...
x_5 +	0·12	...	...	...	...	...	...	...	...	...
Oscillatory	−324·08	−250·08	−181·30	−118·06	−60·62	+36·44	+138·41	+206·88	+187·24	+58·94
V_l (millivolts per sending volt)	−0·30	1·63	13·67	38·13	70·73	142·68	232·88	298·10	278·18	149·85

Table LVI.—(*Fig.* 98, *Curve B.*)

Steady	0·1	0·02	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
t seconds.	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91	90·91
x_1 +	0	...	...	...	...	...	...	...	...	...
x_2 −	76·45	20·36	5·42	1·44	0·38	0·03	...	...	...	...
x_3 +	11·59	0·75	0·05	...	...	...	...	...	...	...
x_4 −	1·48	0·02	...	...	...	...	...	...	...	...
x_5 +	0·12	...	...	...	...	...	...	...	...	...
Oscillatory	−25·05	−69·33	−72·72	−57·34	−37·25	−7·48	+3·64	+0·12	−0·08	+0·00
V_l (millivolts per sending volt)	−0·36	1·95	12·82	32·13	53·28	83·40	94·55	91·03	90·83	90·91

The complex root is found to be $4{\cdot}669+i\times 1{\cdot}988$, and the real roots are contained in Table XXVIII. The oscillatory term is

$$-0{\cdot}29980\times e^{-4{\cdot}1010t}\sin(4{\cdot}2656t-17^\circ 12).$$

Fig. 98, B is plotted from Table LVI. With this smaller

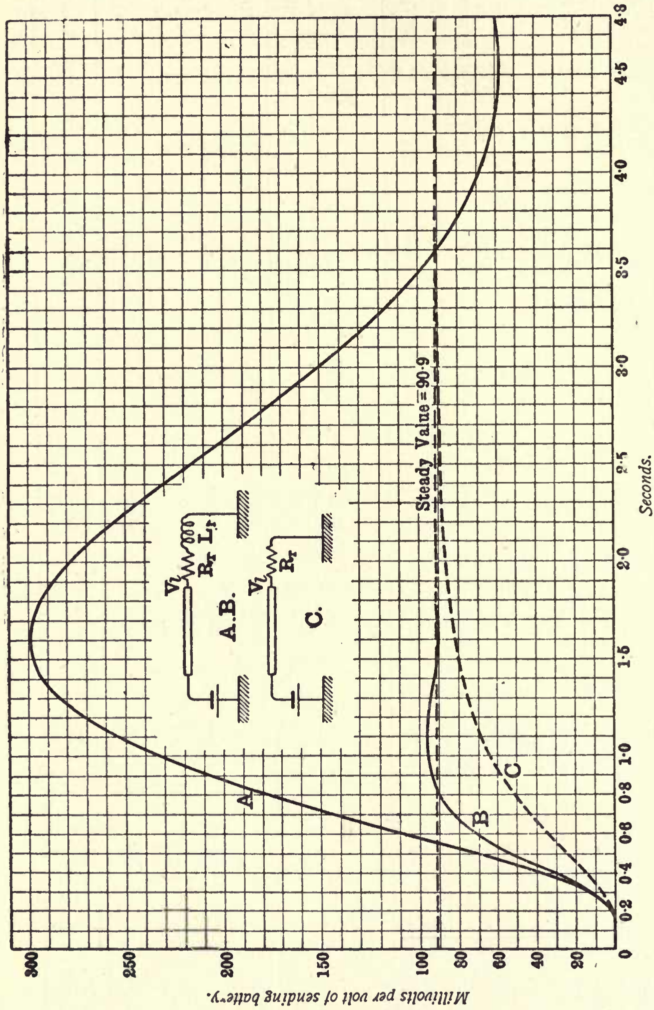

FIG. 98.—ARRIVAL VOLTAGE ON INDUCTANCE COIL.

Curve A.... $R_r = Rl/10 = 497{\cdot}5$ ohms. $L_r = R_r \cdot KRl^2 = 2{,}165$ henries.

Curve B.... $R_r = Rl/10 = 497{\cdot}5$ ohms. $L_r = \frac{R_r KRl^2}{\pi^2} = 219$ henries.

Curve C.... $R_r = Rl/10 = 497{\cdot}5$ ohms. $L_r = 0$.

value of the inductance the oscillatory term dies away more quickly. Fig. 98, C is a reproduction of Fig. 96 for a coil devoid of inductance.

Since when $L_r = \frac{KRl^2 . R_r}{\pi^2}$ the term in (86) corresponding to the first real root of (85) vanishes, this value of L_r may be regarded as a critical one. For, the first term is the slowest to decay, and until it has died out the curve cannot reach its steady value. If L_r be greater than the critical value the voltage curve shoots up far beyond the steady value; if L_r be less than the critical value the rise in voltage is too slow. Hence is derived the following simple rule for obtaining the closest approximation to a square-topped arrival curve: *The time constant of the inductance shunt should equal the time constant of the cable divided by* π^2. As regards the resistance of the shunt, the same considerations apply as in the case of the best resistance for the recorder. The smaller the resistance of the shunt the better, and the limit is determined by the sensitivity of the recording instrument.

CHAPTER X.

SENDING PHENOMENA AND THE INFLUENCE OF LEAKS.

Sending Currents and Voltages: The Artificial Cable—Infinitely Long Cable—Finite Cable: No End Apparatus—Sending through a Resistance—Sending through a Condenser—Sending through a Resistance and a Condenser—Voltage on Cable with Sending Condenser—Effect of Absorption—Single Leak in a Cable—Simple Leak at the Middle of the Cable—Condenser Leak—Leak with Double Block: Dot Signal—Continuously Distributed Leakance—Inductive Leak—Application of the Formulæ to any Cable—Use of Arrival Curves to Build up a Message—Conclusion.

SENDING CURRENTS AND VOLTAGES.—THE ARTIFICIAL CABLE.

On account of their importance in duplex working the phenomena at the sending end of the cable deserve almost as much consideration as those at the receiving end.

In Fig. 74, from equations (19) and (20), by elimination of all the unknowns except C_s and V_s, in a manner similar to that used in finding C_r, it follows that

$$C_s = \frac{V_s(Z_0 \cosh Pl + Z_r \sinh Pl)}{Z_0(Z_r \cosh Pl + Z_0 \sinh Pl) + Z_s(Z_r \sinh Pl + Z_0 \cosh Pl)}. \quad (89)$$

This equation for the sending-end current corresponds to (21) for the receiving end, and is likewise of fundamental importance. It gives the sending current C_s for a simple sine voltage, V_s, and it requires now to be transformed to the telegraphic case of the sudden application of a battery voltage $+E$. The process is the same as before, and a few of the more important applications will here be discussed. But in the next paragraph the effect of making the cable infinitely long will first be considered. Again, similar considerations to those applied to (46) are applicable to (89). The following deduction can, therefore, be made. *An artificial cable of constants* R′, K′, L′, G′, l', *will represent to scale* $1/\mu$ *a cable of constants* R, K, L, G, l, provided that

$$\frac{\mathrm{R}l}{\mathrm{R}'l'}=\frac{\mathrm{L}l}{\mathrm{L}'l'}=\frac{\mathrm{K}'l'}{\mathrm{K}l}=\frac{\mathrm{G}'l'}{\mathrm{G}l}=\mu.$$

As a first approximation, when L, L′, G, G′ are neglected, the relationship is

$$\frac{\mathrm{R}l}{\mathrm{R}'l'}=\frac{\mathrm{K}'l'}{\mathrm{K}l}=\mu.$$

The question of the degree of equivalence between a real cable and an artificial cable built up discontinuously of condensers and resistances must be left for another occasion.*

Infinitely Long Cable.

When the cable is infinitely long, $\cosh \mathrm{P}l=\sinh \mathrm{P}l=\infty$, and (89) reduces to

$$\mathrm{C}_s=\frac{\mathrm{V}_s}{\mathrm{Z}_s+\mathrm{Z}_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

If signals are sent direct to line, $\mathrm{Z}_s=0$, and

$$\mathrm{C}_s=\frac{\mathrm{V}_s}{\mathrm{Z}_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

The corresponding heat problem is that of the linear diffusion of heat into an infinite solid, when one plane face is kept at a constant temperature, and the solution is well known. It has the form

$$\mathrm{C}_s=\mathrm{E}\left(\frac{\mathrm{K}}{\mathrm{R}\pi t}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

The corresponding curve is plotted in Fig. 99, C. On account of the absence of self-induction, the current is at first infinitely great. When the inductance of the cable is taken into account the solution of (91) is

$$\mathrm{C}_s=\mathrm{E}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}\left[1-\frac{\mathrm{R}t}{2\mathrm{L}}+\frac{1\cdot 3}{(\lfloor 2)^2}\left(\frac{\mathrm{R}t}{2\mathrm{L}}\right)^2-\frac{1\,.\,3\,.\,5}{(\lfloor 3)^2}\left(\frac{\mathrm{R}t}{2\mathrm{L}}\right)^3+\ldots\right], \quad (93)$$

* Chap. XIII.

which is a convergent series in ascending powers of the time.* An alternative solution is

$$C_s = E\sqrt{\frac{K}{R\pi t}}\left[1 + \frac{1^2}{\underline{|1}}\cdot\frac{L}{4Rt} + \frac{1^2 3^2}{\underline{|2}}\cdot\left(\frac{L}{4Rt}\right)^2 + \frac{1^2 \cdot 3^2 \cdot 5}{\underline{|3}}\left(\frac{L}{4Rt}\right)^3 + \cdots\right], \quad (94)$$

which is a divergent series in descending powers of the time.*

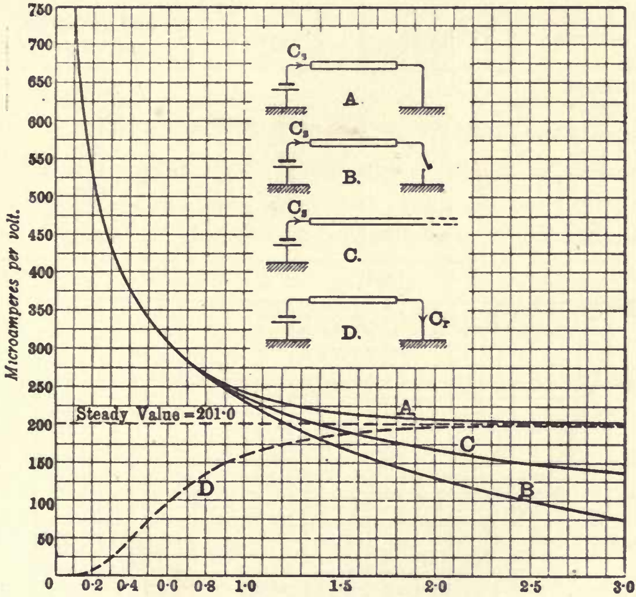

FIG. 99.—CABLE WITHOUT APPARATUS.

Sending Current.—Curve A....Receiving-end earthed.
" B....Receiving-end free.
" C....Cable infinitely long—

$$C_s = \frac{236 \cdot 6}{\sqrt{t}} \text{ microamperes per volt.}$$

Arrival Current.—Curve D....Receiving-end earthed.

It is to be taken as far as its smallest term. In Fig. 100, Curve A, the curve is drawn from Table LVII., which represents (93) and (94). The San Francisco–Honolulu cable is supposed

* O. Heaviside, " Electromagnetic Theory," Vol. 2, p. 292.

in this case to possess an inductance of 50 millihenries per nautical mile without any change being made in its other constants. Curve B is the same as that marked B in Fig. 99, and represents the sending current for the ordinary cable without loading. The curves differ considerably at the very beginning, where the inductance makes itself felt. After

Table LVII.—(*Fig.* 100, *Curve A.*) $4R/L = 174{\cdot}85$; $R/2L = 21{\cdot}856$.

t (seconds).	$174{\cdot}85 \times t$.	$21{\cdot}856t$.	C_s (microamperes per volt).
0·0000	...	0	2,772
0·0057	...	$\frac{1}{8}$	2,459
0·0114	...	$\frac{1}{4}$	2,190
0·0229	...	$\frac{1}{2}$	1,783
0·0458	...	1	1,289
0·0915	...	2	823
0·1144	20	...	742
0·1716	30	...	590

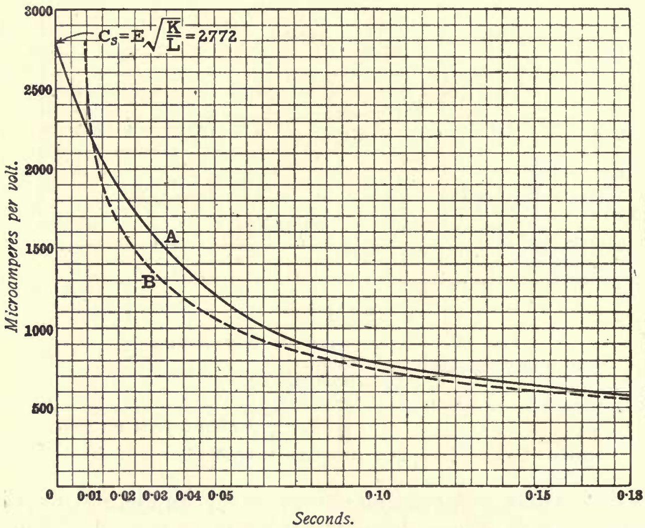

FIG. 100.—SENDING CURRENT.

Curve A....Assumed loaded with 0·050 henry per nautical mile.
,, B....No loading.

$t=0{\cdot}57$ second, they differ by less than 1 per cent. The current at the moment of making contact is found by putting $t=0$ in (93). This makes $C_s\ (t=0)=E\sqrt{\frac{K}{L}}=2{,}772$ microamperes per volt.

The solutions for the case when Z_s is a simple resistance, or a simple condenser, in series with a cable of infinite length, have been obtained by Heaviside* by the use of his operational methods. When Z_s consists of both a resistance and a condenser the problem is considerably more complicated, and it has been attacked by Gaye† and Heaviside.

Finite Cable—No-End Apparatus.

Returning to the cable of finite length, suppose that there is no sending or receiving apparatus. Then in (89), if the cable is earthed at the receiving end, Z_s and Z_r are to be made zero. Hence

$$C_s=\frac{V_s}{Z_0 \tanh Pl}. \qquad (95)$$

If the cable is free at the receiving end, $Z_r=\infty$, and

$$C_s=\frac{V_s}{Z_0 \coth Pl}. \qquad (96)$$

In (95), $$f(x)=\frac{Rl}{x}\tan x=0.$$

This gives $$x=m\pi, \text{ and } f'(x)=\frac{Rl}{x\cos^2 x},$$

In (96), $$f(x)=\frac{Rl}{-x}\cot x=0,$$

and $$x=\frac{2m+1}{2}\pi\,;\quad f'(x)=\frac{Rl}{x\sin^2 x}.$$

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* O. Heaviside, "Electromagnetic Theory," Vol. 2, pp. 37, 40.

† W. Gaye, The Electrician, Vol. 53, p. 905, 1904.

Hence the required expressions for C_s are

$$C_s=\frac{E}{Rl}+\sum_1^\infty \frac{2Ee^{-\frac{m^2\pi^2 t}{KR^2}}}{Rl}, \quad . \quad . \quad . \quad . \quad . \quad (97)$$

and

$$C_s=\sum_0^\infty \frac{2Ee^{-\frac{(m+1)^2\pi^2 t}{4KRl^2}}}{Rl}. \quad . \quad . \quad . \quad . \quad . \quad (98)$$

Equation (97) is similar to (4), the Thomson solution for the arrival current without apparatus, but every term in the series has the positive sign.

Curves A and B, Fig. 99, have been plotted from (97) and (98) and Tables LVIII. and LIX.

Table LVIII.—(*Fig.* 99, *Curve A.*)

t (secs.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0
$m=1$	320·4	255·4	203·6	162·3	129·3	82·2	41·6	13·4	4·3	0·5
$m=2$	162·3	65·5	26·4	10·7	4·3	0·7	0·1	...	...	...
$m=3$	52·2	6·8	0·9	0·1	...	...	...	...	...	...
$m=4$	10·7	0·3	...	...	...	...	...	...	...	...
$m=5$	1·4	...	...	...	...	...	...	...	...	...
$m=6$	0·1	...	...	...	...	...	...	...	...	...
C_s*	748·1	529·0	431·9	374·1	334·6	283·9	242·7	214·4	205·3	201·5

Table LIX.—(*Fig.* 99, *Curve B.*)

t (secs).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$m=0$	379·8	358·9	339·1	320·4	302·7	270·3	228·0	171·7	129·3	73·4
$m=1$	241·3	144·8	87·0	52·2	31·3	11·3	2·4	0·2	...	...
$m=2$	97·3	23·6	5·7	1·4	0·3	...	...	...	...	...
$m=3$	25·0	1·6	0·1	...	...	...	...	...	...	...
$m=4$	4·1	...	...	...	...	...	...	...	...	...
$m=5$	0·4	...	...	...	...	...	...	...	...	...
C *	747·9	528·9	431·9	374·0	334·3	281·6	230·4	171·9	129·3	73·4

* Microamperes per volt.

The current is infinitely great at first, because all impedance at the sending end has been neglected, and falls off rapidly. The influence of the distant end does not manifest itself until about three-quarters of a second has elapsed, at which point the curves for end "free" and end "earthed" begin to diverge.

It is interesting to observe that the curve for the infinitely long cable is exactly midway between the other two. *Just as there is a characteristic time—depending on the* KRl^2 *of the cable—for the current to approximate to its steady value, so at the sending end there is a characteristic time that elapses before the influence of the distant end becomes perceptible.* The area enclosed by Curve B and any time ordinate is the charge in the cable at that time when the distant end is free. Similarly, the area below Curve A and to the left of Curve D and any time ordinate, is the charge in the cable at that time, when the distant end is earthed. Expressed analytically, from (4) and (97),

$$Q_t = Q_\infty + \frac{2E}{Rl}\int \sum_1^\infty \left[e^{-\frac{m^2\pi^2 t}{KRl^2}} - e^{-\frac{m^2\pi^2 t}{KRl^2}}\cos m\pi\right]dt,$$

$$= Q_\infty - \frac{4E}{\pi^2}Kl\sum_0^\infty \frac{e^{-\frac{(2m+1)^2\pi^2 t}{KRl^2}}}{(2m+1)^2}.$$

Q_∞ is the ultimate charge in the cable, and equals $\frac{E}{2} \cdot Kl$.

Hence the charge at any time t, when the distant end is earthed, is given by

$$Q_t = \frac{E}{2}Kl\left[1 - \frac{8}{\pi^2}\sum_0^\infty \frac{e^{-\frac{(2m+1)^2\pi^2 t}{KRl^2}}}{(2m+1)^2}\right]. \quad . \quad . \quad . \quad (99)$$

The total charge in the cable when the distant end is free is $E \cdot Kl$, which is twice as great as when the distant end is earthed. Hence the area between Curve B and the time axis is twice the area between the Curves A and D.

Sending through a Resistance.

In (89) let Z_r be zero.

Then
$$C_s = \frac{V_s}{Z_s + Z_0 \tanh Pl}. \quad . \quad . \quad . \quad . \quad . \quad (100)$$

This corresponds to the case of distant end earthed. If the distant end be free, $Z_r = \infty$, and

$$C_s = \frac{V_s}{Z_s + Z_0 \coth Pl}. \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Suppose that Z_s is a pure resistance, R_s. The equation for x is

$$\tan x = \frac{-xR_s}{Rl}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

and the corresponding expansion is

$$C_s = \frac{E}{Rl + R_s} + \sum_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{Rl \sec^2 x + R_s}. \quad . \quad . \quad (103)$$

And when the receiving end is insulated,

$$\cot x = \frac{xR_s}{Rl}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

and

$$C_s = \Sigma \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{Rl \operatorname{cosec}^2 x + R_s}. \quad . \quad . \quad . \quad . \quad (105)$$

Fig. 101 is plotted from Tables LX., LXI., and LXII. for $R_s = Rl/10$, and $R_s = Rl$.

Table LX.—$R_s = Rl/10 = 497{\cdot}5$ ohms. (*Fig.* 101, *Curve A.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	182·8	182·8	182·8	182·8	182·8	182·8	182·8	182·8	182·8	182·8
x_1 +	281·8	233·5	193·4	160·2	132·7	91·0	51·7	20·2	7·9	1·2
x_2 +	133·1	62·1	29·0	13·5	6·3	1·4	0·1	...	...	...
x_3 +	37·9	6·6	1·2	0·2	...	...	...	...	...	...
x_4 +	7·0	0·3	...	...	...	...	...	...	...	...
x_5 +	0·9	...	...	...	...	...	...	...	...	...
x_6 +	0·1	...	...	...	...	...	...	...	...	...
C_s*	643·6	485·3	406·4	356·7	321·8	275·2	234·6	203·0	190·7	184·0

Table LXI.—$R_s = Rl/10 = 497{\cdot}5$ ohms. (*Fig.* 101, *Curve B.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	342·3	326·7	311·7	297·4	283·8	258·4	224·4	177·5	140·4	87·8
x_2 +	204·2	133·4	87·1	56·9	37·2	15·9	4·4	0·5	0·1	...
x_3 +	74·6	22·5	6·8	2·0	0·6	0·1	...	...	...	...
x_4 +	17·2	1·6	0·1	...	...	...	...	...	...	...
x_5 +	2·6	...	...	...	...	...	...	...	...	...
x_6 +	0·2	...	...	...	...	...	...	...	...	...
C_s*	641·1	484·2	405·7	356·3	321·6	274·4	228·8	178·0	140·5	87·8

Table LXII.—$R_s = Rl$. (*Fig.* 101, *Curve C.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	100·5	100·5	100·5	100·5	100·5	100·5	100·5	100·5	100·5	100·5
x_1 +	59·8	54·4	49·5	45·0	40·9	33·9	25·5	15·9	9·9	3·8
x_2 +	8·8	5·1	2·9	1·7	1·0	0·3	0·1	...	...	...
x_3 +	1·4	0·3	0·1	...	...	...	...	...	...	...
x_4 +	0·2	...	...	...	...	...	...	...	...	...
C_s*	170·7	160·3	153·0	147·2	142·4	134·7	126·1	116·4	110·4	104·3

* Microamperes per volt.

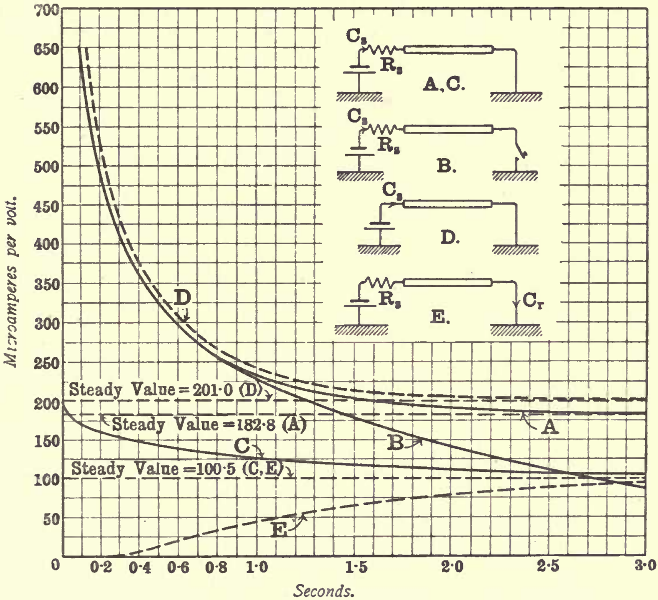

FIG. 101.—SENDING CURRENT, WITH RESISTANCE BEFORE CABLE.

Curve A....$R_s = Rl/10 = 497·5$ ohms, distant end earthed.
„ B....Ditto, distant end free.
„ C....$R_s = Rl = 4{,}975$ ohms, distant end earthed.
„ D....$R_s = 0$, ditto.
„ E....Arrival current, corresponding to C.

The roots of equations (102) and (104) are contained in Tables XXXIV., XXXV. and XXXVIII. For the purpose of comparison, Curve D is reproduced from Fig. 99, Curve A. The effect of a moderate resistance is to diminish slightly the sending current. The effect becomes very marked when the resistance is great, as in Curve C. As before, the effect of the distant end is not noticeable until about three-quarters of a second has elapsed. When $R_s = Rl/10$ the initial value of the current is 2,010 microamperes.

Sending Through a Condenser.

From (100), when the sending apparatus Z_s is a condenser of capacity K_s,

$$\tan x = \frac{Kl}{xK_s} \quad \ldots \ldots \ldots \quad (106)$$

and the expansion is

$$C_s = \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\sec^2 x + \frac{\tan x}{x}\right]} \quad \ldots \quad (107)$$

When the distant end is free, (101) gives

$$\text{Cot}\, x = \frac{-Kl}{xK_s} \quad \ldots \ldots \ldots \quad (108)$$

and the expansion is

$$C_s = \sum \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\operatorname{cosec}^2 x - \frac{\cot x}{x}\right]} \quad \ldots \quad (109)$$

The roots of the equations (106) and (108) are given in Tables XXXVIII., XXXIX. and XXXIV. for $K_s = Kl/10$ and $K_s = Kl/20$. Fig. 102, shows curves A, B and C, representing (107) and (109). They are plotted from Tables LXIII., LXIV. and

Table LXIII.—$K_s = Kl/10 = 87{\cdot}5$ mfds. (*Fig.* 102, *Curve A.*) *Distant End Earthed.*

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	6·99	6·67	6·36	6·07	5·79	5·27	4·58	3·62	2·87	1·79
x_2 +	37·87	24·73	16·15	10·55	6·89	2·94	0·82	0·10	0·01	...
x_3 +	38·98	11·74	3·53	1·06	0·32	0·03	...	...	...	...
x_4 +	17·88	1·64	0·15	0·01	...	...	...	...	...	...
x_5 +	4·47	0·08	...	...	...	...	...	...	...	...
x_6 +	0·65	...	...	...	...	...	...	...	...	...
C_s*	106·84	44·86	26·19	17·69	13·00	8·24	5·40	3·72	2·88	1·79

Table LXIV.—$K_s = Kl/10 = 87·5$ mfds. (*Fig.* 102, *Curve B.*) *Distant End Free.*

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	23·09	19·12	15·84	13·12	10·87	7·46	4·24	1·65	0·64	0·10
x_2 +	43·45	20·27	9·45	4·41	2·06	0·45	0·05	...	...	...
x_3 +	28·71	5·03	0·88	0·15	0·03	...	...	...	...	...
x_4 +	9·59	0·41	0·02	...	...	...	...	...	...	...
x_5 +	1·82	0·01	...	...	...	...	...	...	...	...
x_6 +	0·21	...	...	...	...	...	...	...	...	...
C_s*	106·87	44·84	26·19	17·68	12·96	7·91	4·29	1·65	0·64	0·10

Table LXV.—$K_s = Kl/20 = 43·7$ mfds. (*Fig.* 102, *Curve C.*) *Distant End Earthed.*

t (secs.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	2·02	1·92	1·83	1·73	1·65	1·49	1·27	0·98	0·76	0·46
x_2 +	11·58	7·28	4·58	2·88	1·81	0·72	0·18	0·02	...	...
x_3 +	13·05	3·59	0·99	0·27	0·07	0·03	...	...	...	...
x_4 +	6·61	0·52	0·04	...	...	...	...	...	...	...
x_5 +	1·81	0·03	...	...	...	...	...	...	...	...
x_6 +	0·28	...	...	...	...	...	...	...	...	...
C_s*	35·35	13·34	7·44	4·88	3·53	2·24	1·45	1·00	0·76	0·46

* Microamperes per volt.

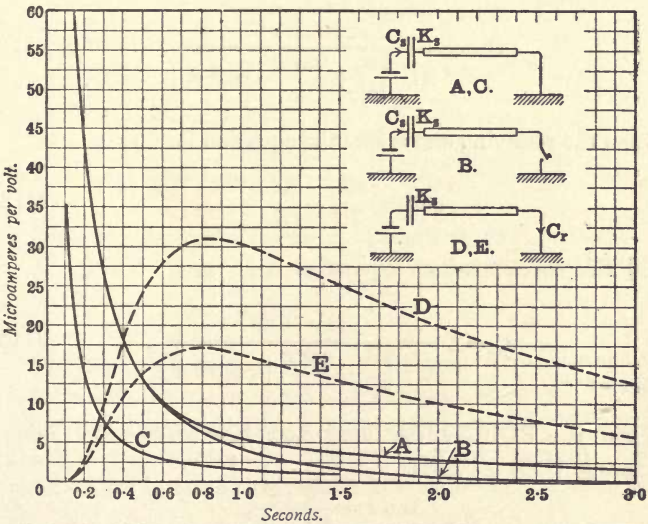

FIG. 102.—SENDING CURRENT. CABLE WITH SENDING CONDENSER.

Curve A....$K_s = Kl/10 = 87·5$ mfds., distant end to earth.
" B....Ditto. Distant end free.
" C....$K_s = Kl/20 = 43·7$ mfds. Distant end to earth.
" D....Arrival current, corresponding to A.
" E....Ditto, corresponding to C.

LXV. The effect of the 87·5 mfds. condenser is to cut down the sending current very considerably. It is to be observed that the curve with distant end free approaches the zero line more rapidly than the curve with distant end to earth. At $t=0{\cdot}4$ second, where Curves A and D cut, the received and sending currents are equal. The difference between the areas of Curves A and D is the charge in the condenser.

Sending Through a Resistance and a Condenser.

If, in addition to the condenser K_s, a resistance, R_s, be placed in series before the cable, then, in (100), $R_s-\dfrac{KRl^2}{x^2K_s}$ must be substituted for Z_s. Hence the equation to determine x is, when the distant end is to earth,

$$\tan x=\frac{-x}{Rl}\left(R_s-\frac{KRl^2}{x^2K_s}\right). \quad . \quad . \quad . \quad . \quad (110)$$

The corresponding expansion is

$$C_s=\sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\dfrac{2R_s}{Rl}+\dfrac{\tan x}{x}+\sec^2 x\right]}. \quad . \quad . \quad . \quad . \quad (111)$$

When the receiving end is free the equation for x is

$$\cot x=\frac{xR_s}{Rl}-\frac{Kl}{xK} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

and the corresponding expansion is

$$C_s=\sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\dfrac{2R_s}{Rl}-\dfrac{\cot x}{x}+\operatorname{cosec}^2 x\right]}. \quad . \quad . \quad (113)$$

When $R_s=Rl/10=497{\cdot}5$ ohms and $K_s=Kl/20=43{\cdot}7$ mfds., equation (110) becomes

$$\tan x=\frac{-x}{10}+\frac{20}{x},$$

the roots of which are contained in Table XLVI. Fig. 103, Curve A, is plotted from Table LXVI. Curve B is reproduced from Fig. 102, C.

Table LXVI.—(*Fig.* 103, *Curve A.*)

t (secs.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	2·07	1·96	1·86	1·77	1·68	1·52	1·30	1·01	0·78	0·47
x_2 +	13·84	8·75	5·53	3·49	2·21	0·90	0·22	0·02	...	...
x_3 +	21·48	6·13	1·75	0·50	0·14	0·01	...	...	...	...
x_4 +	16·95	1·56	0·14	0·01	...	...	...	...	...	...
x_5 +	7·04	0·16	...	...	...	...	...	...	...	...
x_6 +	1·33	0·01	...	...	...	...	...	...	...	...
C_s*	62·71	18·57	9·28	5·77	4·03	2·43	1·52	1·03	0·78	0·47

* Microamperes per volt.

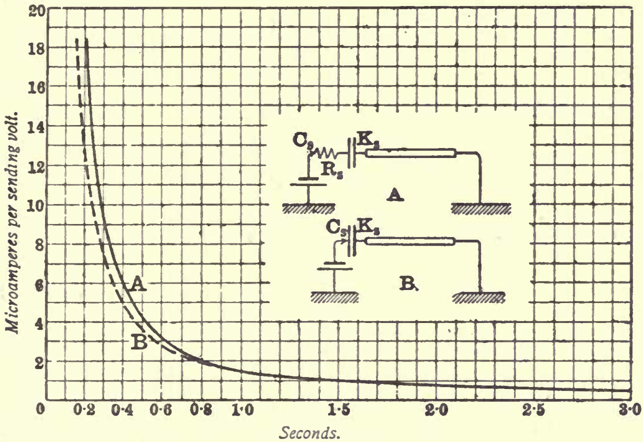

Fig. 103.—Sending Current. Receiving End to Earth.

Curve A....With condenser $K_s = Kl/20 = 43·75$ mfds., and Resistance $R_s = Rl/10 = 497·5$ ohms.

" B....With condenser alone.

The effect of the resistance placed in series with the condenser is to render the curve of sending current less steep. Without resistance the current into condenser and cable is infinitely great at the moment of contact (inductance zero). When resistance is inserted the initial current is finite; in the present instance it is 2,010 microamperes per volt. As time increases the two curves cross, and the resistance + condenser curve is the upper, as shown in the diagram. *Thenceforth the current with added resistance is greater than without.* With further lapse of time the curves run together, and by time $t=0\cdot 9$ sec. they are very nearly coincident. The areas enclosed by the two curves and the axes are equal, and represent the charge in the condenser. The importance of the small rheostat inserted at the apex of the duplex bridge in adjusting the duplex balance is well known.*

Voltage on Cable with Sending Condenser.

Since the voltage V_0 on the end of the cable is connected with the sending voltage by the equation

$$V_s-V_0=Z\,C_s,$$

it follows, from (89), that

$$V_0=V_s-Z_sC_s=\frac{V_sZ_0(Z\ \cosh Pl+Z_0\sinh Pl)}{Z_0(Z_r\cosh Pl+Z_0\sinh Pl)+Z_s(Z_r\sinh Pl+Z_0\cosh Pl)}\cdot\quad(114)$$

When the distant end is earthed, $Z_r=0$, and (114) becomes

$$V_0=\frac{V_s}{1+\frac{Z_s}{Z_0}\coth Pl}\cdot\quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

When the distant end is free $Z_r=\infty$, and

$$V_0=\frac{V_s}{1+\frac{Z_s}{Z_0}\tanh Pl}\cdot\quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

As an example, the case when Z_s is a simple condenser must suffice. Here, for Z_s, $\frac{KRl^2}{-x^2K_s}$ has to be substituted.

The equation for x is

$$\cot x=x\frac{K_s}{Kl}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

* Charles Bright, "Engineering," 58, p. 838, 1894.

Table LXVII.—$K_s = Kl/10 = 87{\cdot}5$ mfds. (*Fig.* 104, *Curve A.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	170·4	162·5	155·1	148·0	141·2	128·5	111·7	88·3	69·8	43·7
x_2 +	101·6	66·4	43·3	28·3	18·5	7·9	2·2	0·3	...	...
x_3 +	37·1	11·2	3·4	1·0	0·3	...	...	...	...	...
x_4 +	8·6	0·8	0·1	...	...	...	...	...	...	...
x_5 +	1·3	...	...	...	...	...	...	...	...	...
x_6 +	0·1	...	...	...	...	...	...	...	...	...
V_0*	319·1	240·9	201·9	177·3	160·0	136·4	113·9	88·6	69·8	43·7

Table LXVIII.—$K_s = Kl/10 = 87{\cdot}5$ mfds. (*Fig.* 104, *Curve B.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	90·9	90·9	90·9	90·9	90·9	90·9	90·9	90·9	90·9	90·9
x_1 +	140·2	116·1	96·2	79·7	66·0	45·3	25·7	10·0	3·9	0·6
x_2 +	65·7	30·6	14·3	6·7	3·1	0·7	0·1	...	...	...
x_3 +	18·9	3·3	0·6	0·1	...	...	...	...	...	...
x_4 +	3·5	0·1	...	...	...	...	...	...	...	...
x_5 +	0·4	...	...	...	...	...	...	...	...	...
V_0*	319·6	241·0	202·0	177·4	160·0	136·9	116·7	100·9	94·8	91·5

* Millivolts per volt.

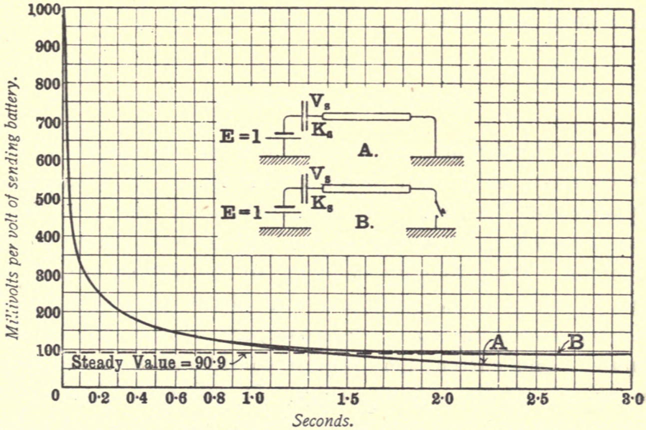

FIG. 104.—VOLTAGE ON CABLE, WITH CONDENSER.

$K_s = Kl/10 = 87{\cdot}5$ mfds.
Curve A....Receiving end to earth.
„ B....Ditto, free.

and the expansion is

$$V_0 = \sum_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{\left[1 + \frac{x}{\sin x \cos x}\right]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

When the distant end is free, the equation for x is

$$\tan x = \frac{-xK_s}{Kl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and the expansion is

$$V_0 = \frac{E}{1 + \frac{Kl}{K_s}} + \sum_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{\left[1 - \frac{x}{\sin x \cos x}\right]} \quad . \quad . \quad . \quad (118)$$

The roots of equations (106) and (108) are contained in Tables XXXVIII. and XXXIV., for the value $K_s = Kl/10 = 87 \cdot 5$ mfds. Fig. 104 is plotted from (117) and (118) by the aid of Tables LXVII. and LXVIII. It is seen that the voltage on the sending end of the cable falls off very rapidly. In this case the curves do not begin to separate appreciably until about one second has elapsed.

Effect of Absorption.

It is of importance to observe how far the shapes of the curves which have been found are influenced by absorption. Suppose that the total dielectric resistance of the cable is at first 1 megohm, increasing at the end of one minute to 10 megohms. Then, roughly speaking, to the current in Fig. 99 must be added an absorption current of the order of 1 microampere, decreasing to $\frac{1}{10}$ microampere at the end of one minute. This current may require consideration in adjusting the balance of the duplex bridge. It will be evident later, when continuous leakance is considered, how the exact value of the current may be obtained, and it will also be seen that even a greater value of the absorption would not affect appreciably the signals at the receiving end.

SINGLE LEAK IN A CABLE.

A question which has been much discussed is the effect of a leak in increasing the speed of working of a cable. Suppose that the leak Z_m divides the cable into two sections of lengths y and $l-y$, as in Fig. 105. Then $C_s = C_0$ and $C_l = C_r$, and for a periodic source the following equations hold :—

$$\left.\begin{aligned}
V_s &= V_0 + Z_s C_0 \\
V_y &= Z_m C_m \\
V_l &= Z_r C_r \\
V_y &= V_0 \cosh Py - Z_0 C_0 \sinh Py \\
C_y &= C_0 \cosh Py - (V_0/Z_0)\,.\,\sinh Py. \\
V_l &= V_y \cosh P(l-y) - Z_0(C_y - C_m) \sinh P(l-y) \\
C_r &= (C_y - C_m)\cosh P(l-y) - (V_y/Z_0)\sinh P(l-y)
\end{aligned}\right\} \quad (119)$$

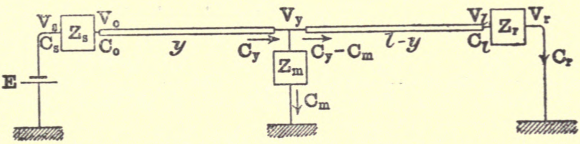

FIG. 105.—SINGLE LEAK IN A CABLE.

From these seven equations in eight unknown quantities the ratio of any two can be obtained, say V_s and C_r. Hence, by elimination and simplification,

$$C_r = V_s \div \left(\begin{aligned} &(Z_s + Z_r)\cosh Pl + \left(Z_0 + \frac{Z_s Z_r}{Z_0}\right)\sinh Pl \\ &+ \frac{(Z_0 \sinh Py + Z_s \cosh Py)\,(Z_0 \sinh P(l-y) + Z_r \cosh P(l-y))}{Z_m} \end{aligned} \right).$$

In particular when $Z_s = Z_r = 0$, (120)

$$C_r = \frac{V_s}{Z_0 \sinh Pl + \dfrac{Z_0^2}{Z_m} \sinh Py \sinh P(l-y)} \quad . \quad . \quad (121)$$

The general expression (120) is not symmetrical in Z_s and Z_r; that is to say, the signals received depend on the direction of sending.

SIMPLE LEAK AT THE MIDDLE OF THE CABLE.

Consider first the case of a simple resistance leak, R_m, in a cable earthed at both ends. When the leak is at the middle, $y=l-y$, and (121) becomes

$$C_r=\frac{V_s}{Z_0 \sinh Pl+\frac{Z_0^2}{Z_m}\sinh^2\frac{Pl}{2}}. \quad . \quad . \quad . \quad . \quad (122)$$

Proceeding in the usual manner,

$$f(x)=\frac{Rl}{x}\sin x+\frac{R^2l^2}{x^2R_m}\sin^2\frac{x}{2}=0$$

$$\therefore \qquad \sin\frac{x}{2}=0,$$

$$\text{or} \qquad \tan\frac{x}{2}=\frac{-2xR_m}{Rl}. \quad . \quad . \quad . \quad . \quad . \quad (123)$$

$$f'(x)=-\frac{Rl}{x^2}\sin x+\frac{Rl}{x}\cos x-\frac{2R^2l^2}{x^3R_m}\sin^2\frac{x}{2}+\frac{R^2l^2}{x^2R_m}\sin\frac{x}{2}\cos\frac{x}{2}.$$

When $\quad \sin\frac{x}{2}=0, \qquad f'(x)=\frac{Rl}{x},$

and when $\quad \tan\frac{x}{2}=-\frac{2xR_m}{Rl}, \quad f'(x)=\frac{Rl}{x^2}\sin x-\frac{Rl}{x}.$

Hence the required expansion is

$$C_r=\frac{E}{Rl+\frac{R^2l^2}{4R_m}}+\sum_1^\infty\frac{2Ee^{-\frac{(2m\pi)^2t}{KRl^2}}}{Rl}+\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left(\frac{\sin x}{x}-1\right)}. \quad (124)$$

Let $4R_m=Rl$, or $R_m=1,244$ ohms. Then (123) becomes $\tan\frac{x}{2}=-\frac{x}{2}$, and the value of $\frac{x}{2}$ may be taken from Table XXXV. Fig. 106, Curve A, is plotted from (124) and Table LXIX. Fig. 106, Curve B, is the same as Curve A in Fig. 75, drawn to half scale for the purpose of comparison.

It is seen that the effect of the leak, while it cuts down the steady value of the current to one-half its former value, is to

Table LXIX.—(*Fig.* 106, *Curve A.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48
x_1 —	231·33	159·33	109·74	75·58	52·05	24·70	8·07	1·25	0·19	...
x_2 —	42·00	4·56	0·50	0·05	...	...	...	...	...	...
x_3 —	1·14	...	...	...	...	...	...	...	...	...
$m=1$ +	162·26	65·49	26·44	10·67	4·31	0·70	0·05	...	...	...
$m=2$ +	10·67	0·28	0·01	...	...	...	...	...	...	...
C_r*	−1·06	+2·36	16·69	35·52	52·74	76·48	92·46	99·23	100·29	100·48

Table LXX.—(*Fig.* 106, *Curve C.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	201·01	201·01	201·01	201·01	201·01	201·01	201·01	201·01	201·01	201·01
x_1 —	303·43	251·51	208·47	172·80	143·23	98·38	56·03	21·92	8·58	1·31
x_2 —	67·41	12·26	2·23	0·41	0·07	...	...	...	...	...
x_3 —	3·09	0·03	...	...	...	...	...	...	...	...
x_4 —	0·03	...	...	...	...	...	...	...	...	...
$m=1$ +	162·26	65·49	26·44	10·67	4·31	0·70	0·05	...	...	...
$m=2$ +	10·67	0·28	0·01	...	...	...	...	...	...	...
C_r*	−0·02	+2·98	16·76	38·47	62·02	103·33	145·03	179·09	192·43	199·70

* Microamperes per volt.

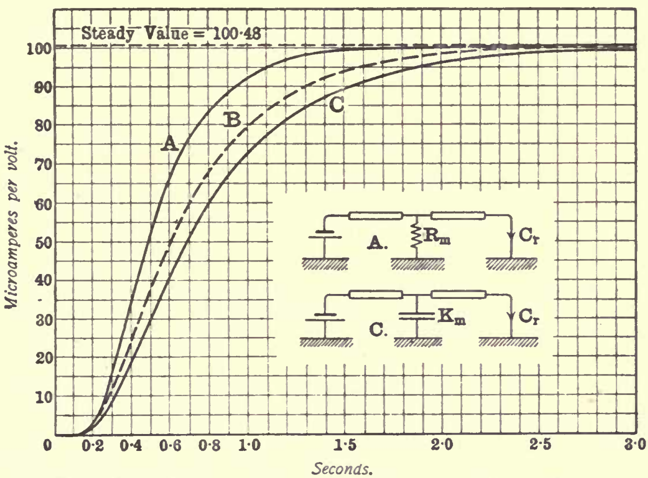

FIG. 106.—ARRIVAL CURRENT. CABLE WITH LEAK MIDWAY.

Curve A....Leak of $R_m=Rl/4=1{,}244$ ohms.
,, B....No leak ; $R_m=\infty$. Ordinates half actual size.
,, C....Condenser leak. $K_m=Kl/10=87{\cdot}5$ mfds. Ordinates half actual size.

improve considerably the shape of the arrival curve. By using a still smaller leak a still greater improvement in shape could be attained, with consequent increase in speed of working, provided that the sensitivity of the recording apparatus could be increased so as to keep pace with the diminution in the arrival current.

Condenser Leak.

If the leak Z_m consist simply of a condenser, K_m, then in (122) $\frac{-KRl^2}{x^2K_m}$ must be substituted for Z_m. Hence

$$C_r=\frac{E}{Rl}-\sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x}+1\right]}+\sum_1^\infty \frac{2Ee^{-\frac{(2m\pi)^2t}{KRl^2}}}{Rl}, \quad . \quad (125)$$

where

$$\tan\frac{x}{2}=\frac{Kl}{K_m}\cdot\frac{2}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

The roots of (126) when $K_m=Kl/10$, and $\tan\frac{x}{2}=10\cdot\frac{2}{x}$, are twice the roots contained in Table XXXVIII. Fig. 106, Curve C, is plotted, to scale one-half, from (125) and Table LXX.

The effect of the condenser leak is, as might have been anticipated, to render the arrival curve less steep.

Leak with Double Block—Dot Signal.

Suppose that the leak is at the middle of the cable. Then in (120) y must be put equal to $l/2$. Suppose, also, that Z_s and Z_r are equal. Then

$$C_r=\frac{V_s}{2Z_r\cosh Pl+\left(Z_0+\frac{Z_r^2}{Z_0}\right)\sinh Pl+\frac{(Z_0\sinh Pl/2+Z_r\cosh Pl/2)^2}{Z_m}}. \quad (127)$$

When Z_s and Z_r are condensers of capacity K_r, and Z_m is a resistance, R_m, on substituting as usual, $f(x)$ becomes

$$f(x)=Rl\left[-\frac{2Kl}{x^2K_r}\cos x+\frac{\sin x}{x}-\frac{\sin x}{x^3}\left(\frac{Kl}{K_r}\right)^2+\frac{Rl}{R_m}\left(\frac{\sin x/2}{x}-\frac{\cos x/2}{x^2}\frac{Kl}{K_r}\right)^2\right]=0.$$

Hence it follows that the equation for the x roots is

$$\left[\tan\frac{x}{2}-\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{K}_r}\right]\left[\tan\frac{x}{2}\left(\frac{2\mathrm{K}l}{\mathrm{K}_r}+\frac{\mathrm{R}l}{\mathrm{R}_n}\right)+2x-\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{K}_r}\cdot\frac{\mathrm{R}l}{\mathrm{R}_m}\right]=0. \quad (128)$$

Again
$$f'(x)=\mathrm{R}l\left[\frac{2\mathrm{K}l}{x^2\mathrm{K}_r}\sin x+\frac{4\mathrm{K}l}{x^3\mathrm{K}_r}\cos x-\frac{\sin x}{x^2}+\frac{\cos x}{x}\right.$$
$$+\frac{3\sin x}{x^4}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)-\frac{\cos x}{x^3}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2+\frac{2\mathrm{R}l}{\mathrm{R}_m}\left(\frac{\sin x/2}{x}\right.$$
$$-\left.\frac{\cos x/2}{x^2}\frac{\mathrm{K}l}{\mathrm{K}_r}\right)\left(\frac{\cos x/2}{2x}-\frac{\sin x/2}{x^2}+\frac{\sin x/2}{2x^2}\cdot\frac{\mathrm{K}l}{\mathrm{K}_r}\right.$$
$$\left.\left.+\frac{2\cos x/2}{x^3}\cdot\frac{\mathrm{K}l}{\mathrm{K}_r}\right)\right].$$

When $\tan\frac{x}{2}=\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{K}_r}$,

$$\frac{xf'(x)}{2}=\frac{\mathrm{R}l}{2}\left[\sin x\left\{\frac{2\mathrm{K}l}{x\mathrm{K}_r}-\frac{1}{x}+\frac{3}{x^3}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right\}+\cos x\left\{\frac{4\mathrm{K}l}{x^2\mathrm{K}_r}+1-\frac{1}{x^2}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right\}\right],$$

and when $\tan\frac{x}{2}\left(\frac{2\mathrm{K}l}{\mathrm{K}_r}+\frac{\mathrm{R}l}{\mathrm{R}_m}\right)+2x-\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{R}_r}\frac{\mathrm{R}l}{\mathrm{R}_n}=0,$

$$\frac{xf'(x)}{2}=\frac{\mathrm{R}l}{2}\left[\sin x\left\{\frac{1}{x}-\frac{1}{x^3}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right\}-\cos x\left(\frac{2}{x^2}\frac{\mathrm{K}l}{\mathrm{K}_r}\right)-\left\{1+\frac{2}{x^2}\frac{\mathrm{K}l}{\mathrm{K}_r}+\frac{1}{x^2}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right\}\right].$$

Hence the required expansion for C_r is

$$\mathrm{C}_r=\frac{2}{\mathrm{R}l}\sum_x\frac{\mathrm{E}e^{-\frac{x^2t}{\mathrm{KR}l^2}}}{\sin x\left[\frac{2\mathrm{K}l}{x\mathrm{K}_r}-\frac{1}{x}+\frac{3}{x^3}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right]+\cos x\left[\frac{4\mathrm{K}l}{x^2\mathrm{K}_r}+1-\frac{1}{x^2}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right]}$$

$$\left(\text{where }\tan\frac{x}{2}=\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{K}_r}\right)$$

$$-\frac{2}{\mathrm{R}l}\sum_x\frac{\mathrm{E}e^{-\frac{x^2t}{\mathrm{KR}l^2}}}{\sin x\left[\frac{1}{x^3}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2-\frac{1}{x}\right]+\cos x\,.\,\frac{2}{x^2}\frac{\mathrm{K}l}{\mathrm{K}_r}+\left[1+\frac{2}{x^2}\frac{\mathrm{K}l}{\mathrm{K}_r}+\frac{1}{x^2}\left(\frac{\mathrm{K}l}{\mathrm{K}_r}\right)^2\right]} \quad (129)$$

$$\left(\text{where }\tan\frac{x}{2}\left(\frac{2\mathrm{K}l}{\mathrm{K}_r}+\frac{\mathrm{R}l}{\mathrm{R}_m}\right)+2x-\frac{1}{x}\frac{\mathrm{K}l}{\mathrm{K}_r}\cdot\frac{\mathrm{R}l}{\mathrm{R}_m}=0\right).$$

The expressions are long, but the work is quite straightforward. Particular cases already considered are when K_r or R_m is infinitely great. These are given in (124) and (67). Thus, when $R_m = \infty$, (129) becomes

$$C_r = \sum_x \frac{Ee^{-\frac{x^2 t}{KRl^2}}}{\frac{Rl}{1+\cos x}\left(\frac{\sin x}{x}+1\right)} + \sum_x \frac{Ee^{-\frac{x^2 t}{KRl^2}}}{\frac{Rl}{1-\cos x}\left(\frac{\sin x}{x}-1\right)} \tag{130}$$

$$\left(\text{where } \tan\frac{x}{2} = \frac{1}{x}\frac{Kl}{K_r}\right) \qquad \left(\text{where } \tan\frac{x}{2} = \frac{-xK_r}{Kl}\right),$$

which is another form of (67).

When $Kl = 10K_r$, and $Rl = 4R_m$, the equations for the roots become $\tan\frac{x}{2} = \frac{10}{x}$, the roots of which are contained in Table LXXI., and $\tan\frac{x}{2} = \frac{5}{3x} - \frac{x}{12}$, the roots of which are contained in Table LXXII. At the same time (129) becomes

$$C_r{}^* = \frac{2\times 10^6}{Rl}\left[\sum_x \frac{e^{-\frac{x^2 t}{KRl^2}}}{\sin x\left(\frac{19}{x}+\frac{300}{x^3}\right)+\cos x\left(1-\frac{60}{x^2}\right)}\right.$$

$$\left. - \sum_x \frac{e^{-\frac{x^2 t}{KRl^2}}}{\sin x\left(\frac{100}{x^3}-\frac{1}{x}\right)+\cos x\,\frac{20}{x^2}+\left(1+\frac{120}{x^2}\right)}\right]. \tag{131}$$

Table LXXI.—*Roots of* $\tan\frac{x}{2} = \frac{10}{x}$.

x.		Sin x.	Cos x.
2·6276	$\pi - 29° \, 27$	0·49157	− 0·87083
8·0672	$\pi - 77° \, 47$	0·97735	−0·21161
13·819	71° 47	0·94985	+0·31271
19·786	53° 38	0·80524	+0·59295

* Microamperes per volt.

Table LXXII.—*Roots of* $\tan \frac{x}{2} = \frac{5}{3x} - \frac{x}{12}$.

x.		Sin x.	Cos x.
1·5310	87° 42	0·99919	0·040132
5·8766	2π − 23° 18	− 0·39555	0·91845
11·232	4π − 76° 28	− 0·97223	0·23401
17·005	6π − 105° 40	− 0·96285	− 0·27004

Table LXXIII.—*Roots of* $\tan \frac{x}{2} = \frac{20}{11x} - \frac{x}{22}$.

x.		Sin x.	Cos x.
1·6191	π − 87° 14	+0·9988	−0·04827
6·2896	2π + 0° 22	+0·00640	+1·0000
11·833	4π − 42° 2	−0·6696	+0·7428
17·631	6π − 69° 50	−0·9387	+0·3448

Table LXXIV.—(*Fig.* 107, *Curve A.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	18·664	15·925	13·588	11·595	9·894	7·203	4·476	2·025	0·916	0·187
x_2 +	31·691	7·104	1·592	0·357	0·080	0·004	...	...	...	...
x_3 +	3·067	0·038	...	...	...	...	...	...	...	...
x_4 +	0·035	...	...	...	...	...	...	...	...	...
x_1 −	4·778	4·527	4·290	4·065	3·852	3·458	2·942	2·248	1·717	1·002
x_2 −	37·258	16·848	7·618	3·445	1·558	0·319	0·030	...	...	...
x_3 −	11·040	0·608	0·034	...	...	...	...	...	...	...
x −	0·365	...	...	...	...	...	...	...	...	...
C_r*	0·016	1·084	3·238	4·442	4·564	3·430	1·504	− 0·223	− 0·801	− 0·815

Table LXXV.—(*Fig.* 107, *Curve B.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1 +	6·069	5·031	4·170	3·456	2·865	1·968	1·121	0·438	0·172	0·026
x_2 +	10·549	1·919	0·349	0·063	0·012	...	...	...	...	...
x_3 +	1·060	0·009	...	...	...	...	...	...	...	...
x_4 +	0·014	...	...	...	...	...	...	...	...	...
x_1 −	1·447	1·362	1·283	1·208	1·137	1·008	0·841	0·622	0·461	0·252
x_2 −	12·324	4·965	2·001	0·806	0·325	0·053	0·003	...	...	...
x_3 −	3·791	0·152	0·006	...	...	...	...	...	...	...
x_4 −	0·130	...	...	...	...	...	...	...	...	...
C_r*	0·000	0·480	1·229	1·505	1·415	0·907	0·277	−0·184	−0·289	−0·226

* Microamperes per volt.

Fig. 107 is plotted from (131) and (132) and Tables LXXIV. and LXXV.

When $Kl=20K_r$, and $Rl=4R_m$, the equations for the roots become $\tan\frac{x}{2}=\frac{20}{x}$, the roots of which are twice those given in Table XXXVIII., and $\tan\frac{x}{2}=\frac{20}{11x}-\frac{x}{22}$, in the other part

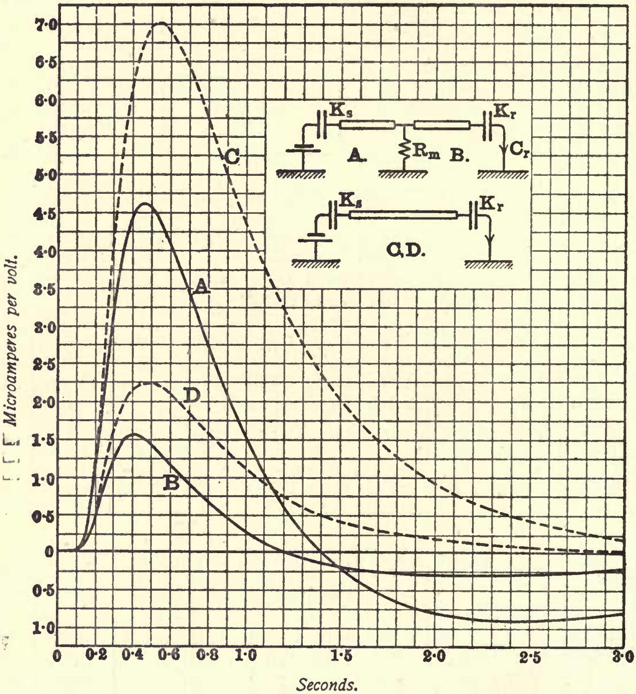

FIG. 107.—ARRIVAL CURRENT. DOUBLE BLOCK AND LEAK

Curve A.... $\begin{cases} K_s = K_r = Kl/10 = 87{\cdot}5 \text{ mfds.} \\ R_m = Rl/4 = 1{,}244 \text{ ohms.} \end{cases}$

" B.... $\begin{cases} K_s = K_r = Kl/20 = 43{\cdot}7 \text{ mfds.} \\ R_m = Rl/4 = 1{,}244 \text{ ohms.} \end{cases}$

Double Block; No Leak.

" C......$K_s = K_r = Kl/10$. $R_m = \infty$.

" D......$K_s = K_r = Kl/20$. $R_m = \infty$.

of (129), the roots of which are given in Table LXXIII. At the same time (129) becomes

$$C_r = \frac{2\times10^6}{Rl}\left[\sum_x \frac{e^{-\frac{x^2t}{KRl^2}}}{\sin x\left(\frac{39}{x}+\frac{1{,}200}{x^3}\right)+\cos x\left(1-\frac{320}{x^2}\right)}\right.$$

$$\left.-\sum_x \frac{e^{-\frac{x^2t}{KRl^2}}}{\sin x\left(\frac{400}{x^3}-\frac{1}{x}\right)+\cos x\frac{40}{x^2}+\left(1+\frac{440}{x^2}\right)}\right]. \quad (132)$$

Curves C and D (Fig. 107) are reproduced from Fig. 90 Curves A and B, for comparison.

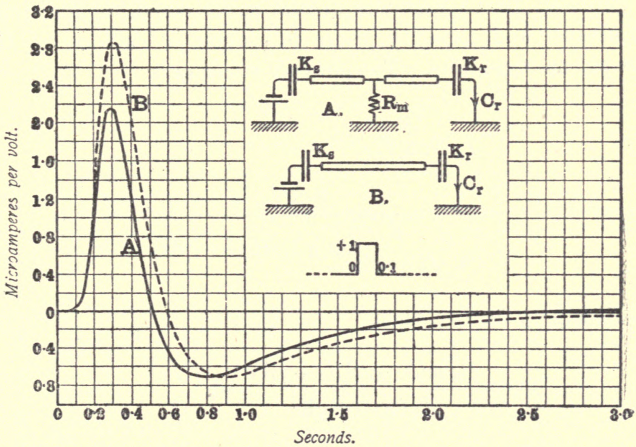

FIG. 108.—ARRIVAL CURRENT. DOT SIGNAL, LASTING 0·1 SECOND.

Double block with leak.

Curve A.... $\begin{cases} K_s = K_r = Kl/10 = 87{\cdot}5 \text{ mfds.} \\ R_m = Rl/4 = 1{,}244 \text{ ohms.} \end{cases}$

Double block; no leak.

" B.... $\begin{cases} K_s = K_r = Kl/10 = 87{\cdot}5 \text{ mfds.} \\ R_m = \infty. \end{cases}$

The arrival curve with leak reaches its greatest height sooner than that without leak, but the greatest value is less than without leak. Thus, if the voltage were proportionally increased, so as to render the greatest heights the same, Curve A would be considerably steeper than Curve C. After a time the curve with leak decreases to zero and becomes negative, recalling in this respect the effect of curbing on condenser signalling. In Fig. 108, Curve A, is shown the arrival current for a dot signal lasting 0·1 sec., with condensers equal to one-tenth part of the cable capacity, and Fig. 108, Curve B, is Fig. 91, Curve A, reproduced for the sake of comparison. The greatest depths are equal, but the curve without leak has the greater height.

Continuously Distributed Leakance.

Taking the simplest case of a cable earthed at both ends, let the problem be to find the arrival curve when, in addition to resistance and capacity, there is present a leakance of G mhos per nautical mile. The periodic solution is

$$C_r = \frac{V_s}{Z_0 \sinh Pl}$$

$$= \frac{Ee^{ipt}}{\sqrt{\dfrac{R}{G+ipK}} \sinh l\sqrt{R(G+ipK)}}.$$

Substituting y for ip; then

$$\phi(y) = \frac{Rl \sinh l\sqrt{R(G+yK)}}{l\sqrt{R(G+yK)}} = 0.$$

This gives $l\sqrt{R(G+yK)} = m\pi i$; or, squaring,

$$y = -\frac{m^2\pi^2}{KRl^2} - \frac{G}{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

Again, using (133)

$$\phi'(y) = \frac{Rl}{2} \cdot \frac{K}{G+yk} \cosh l\sqrt{R(G+yK)}$$

$$= -\frac{Rl}{2} \cdot \frac{KRl^2}{m^2\pi^2} \cos m\pi.$$

The steady term is obtained from $\phi(0)$. Hence, finally,

$$C_r = \frac{E}{\frac{R}{G}\sinh l\sqrt{RG}} + \sum_1^\infty \frac{2Ee^{-\left(\frac{m^2\pi^2}{KRl^2}+\frac{G}{K}\right)t}}{Rl\left(1+\frac{GRl^2}{m^2\pi^2}\right)\cos m\pi}. \quad (134)$$

When G=0 this reduces to (4).

Fig. 109, Curve A, is drawn from (134) and Table LXXVI. for

Table LXXVI.—(*Fig.* 109, *Curve A.*)

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	105·73	105·73	105·73	105·73	105·73	105·73	105·73	105·73	105·73	105·73
$m=1$, −	201·21	145·08	104·66	75·48	54·44	28·32	10·62	2·07	0·40	0·02
$m=2$, +	133·59	48·80	17·83	6·51	2·38	0·32	0·02	...	...	...
$m=3$, −	45·14	5·30	0·62	0·07	0·01	...	...	...	...	...
$m=4$, +	9·40	0·23	0·01	...	...	...	...	...	...	...
$m=5$, −	1·23	...	...	...	...	...	...	...	...	...
$m=6$, +	0·10	...	...	...	...	...	...	...	...	...
$m=7$, −	0·01	...	...	...	...	...	...	...	...	...
C_r*	1·23	4·38	18·29	36·69	53·66	77·73	95·13	103·66	105·33	105·71

* Microamperes per volt.

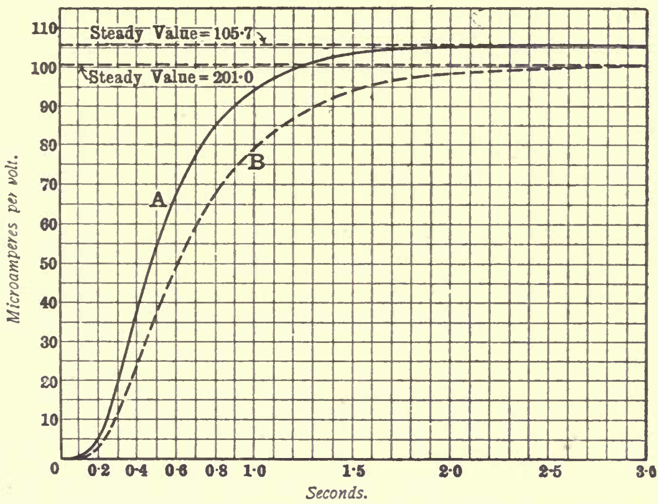

Fig. 109.—Arrival Current. Cable with Ends Earthed.

Curve A....With continuous leakance. Insulation resistance=2·6 megohms per nautical mile.

„ B....No leakance, ordinates half actual size.

the value of $G=K=0{\cdot}3842\times10^{-6}$ mhos, or a leak to earth of 2·60 megohms per nautical mile, and a total insulation resistance of 1,143 ohms, instead of the actual value of perhaps 10 megohms. The arrival curve resembles that of the case previously considered, of a single leak of 1,244 ohms at the middle of the cable. The steady current is somewhat greater, but the curve is not quite so steep. Fig. 109, Curve B, is the same as Curve A in Fig. 75, reproduced to a scale of one-half. More complicated cases may be treated in a similar way. *It is seen that the insulation resistance of the submarine telegraph cable is of the order ten thousand times greater than it need be.*

Electrical and mechanical durability is the essential quality of the insulating material, and the absolute value of the dielectric resistance is of small importance compared with the steadiness with which the insulation is maintained.

Inductive Leak.

Returning to (122), let Z_m consist of a coil of resistance, R_m, and inductance L_m at the middle of the cable. Proceeding as before, the equation for x is

$$f(x)=\frac{Rl}{x}\sin x+\frac{R^2l^2}{x^2\left(R_m-\dfrac{x^2L_m}{KRl^2}\right)}\sin^2\frac{x}{2}=0.$$

This gives $\sin\frac{x}{2}=0,$

or
$$\tan\frac{x}{2}=-\frac{2x}{Rl}\left(R_m-\frac{x^2L_m}{KRl^2}\right). \quad . \quad . \quad . \quad . \quad (135)$$

$$f'(x)=\frac{Rl}{x}, \text{ if } \sin\frac{x}{2}=0,$$

$$=\frac{3Rl}{x}\sin x-\frac{Rl}{x}+\frac{4R_m}{x}(1+\cos x),$$

if (135) holds.

$$\therefore\ C_r=\frac{E}{Rl+\dfrac{R^2l^2}{4R_m}}+\sum_1^\infty\frac{2Ee^{-\frac{(2m\pi)^2t}{KRl^2}}}{Rl}+\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\dfrac{3}{x}\sin x-1+\dfrac{4R_m}{Rl}(1+\cos x)\right]}. \quad (136)$$

Let R_m be put as before equal to $Rl/4$, or 1,244 ohms, and let $4L_m = R_m \,.\, KRl^2$, or $L_m = 1{,}354$ henrys. Equation (135) then becomes

$$\tan\frac{x}{2} = -\frac{x}{2}\left(1 - \frac{x^2}{4}\right). \quad . \quad . \quad . \quad . \quad (137)$$

The real roots of this equation are contained in Table LXXVII.

Table LXXVII.—*Roots of* $\tan\frac{x}{2} = \left(\frac{x}{2}\right)^3 - \frac{x}{2}$.

x.		Sin x.	Cos x.
9·4146	$3\pi - 1°\ 9$	0·02007	−0·9998
15·704	$5\pi - 0°\ 14$	0·00407	−1·0000
21·99	$7\pi - 0°\ 6$	0·00175	−1·0000

From an examination of the similar case illustrated in Fig. 98, it is evident that (137) has a pair of complex roots in the first quadrant. To find them, write $a + ib$ for $x/2$; then (137) becomes

$$\left.\begin{aligned} aA \cot a + bB \tanh b &= 1 \\ bB \coth B - aA \tan a &= 1 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (138)$$

where $A \equiv a^2 - 3b^2 - 1$ and $B \equiv 3a^2 - b^2 - 1$.

When (138) is solved by a method of trial and error, $a + ib$ is found to be $1{\cdot}4245 + i \,.\, 0{\cdot}4449$. Hence $x = 2{\cdot}8490 \pm i \,.\, 0{\cdot}8898$, and when these values are substituted in (136) the final expression is

$$C_r{}^* = \frac{2\times 10^6}{Rl}\left[\frac{1}{4} + e^{-\frac{4\pi^2 t}{KRl^2}} + 1{\cdot}0860\, e^{-\frac{7{\cdot}3248t}{KRl^2}} \sin\left(\frac{5{\cdot}0708t}{KRl^2} - 43°6'.4\right)\right.$$

$$\left. - \left(1{\cdot}0066\, e^{-\frac{88{\cdot}63}{KRl^2}t} + 1{\cdot}0008\, e^{-\frac{246{\cdot}70}{KRl^2}t} + 1{\cdot}0002\, e^{-\frac{483{\cdot}6}{KRl^2}t}\right)\right]. \; (139)$$

Fig. 110, A is plotted from (139) and Table LXXVIII.

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* Microamperes per volt.

Table LXXVIII.—(*Fig.* 110, *Curve A.*)

t (seconds.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48	100·48
$m=1$, +	172·93	65·78	26·45	10·67	4·31	0·70	0·05	...	...	...
x_1 −	52·78	6·88	0·90	0·12	0·02	...	...	...	...	...
x_2 −	1·39	0·02	...	...	...	...	...	...	...	...
x_3 −	0·01	...	...	...	...	...	...	...	...	...
Oscillatory	−219·05	−154·70	−103·26	−62·85	−31·78	+8·50	+32·53	+29·32	+15·07	+1·09
C_r (microamperes per volt)	0·18	4·66	22·77	48·18	72·99	109·68	133·06	129·80	115·55	101·57

Fig. 110, Curve B is Fig. 106, A, for a leak without inductance. Fig. 110, Curve C is the same as Curve A in Fig. 75, reproduced to scale one-half.

It is seen that the effect of the inductance is to lift the arrival curve up over the steady value to which it afterwards sinks. The steepness of the curve would lead one to expect a considerable increase in the speed of signalling were this plan adopted with a coil of the right dimensions.

Application of the Formulæ to Any Cable.

The numerical results which are given in tables in the preceding pages have been calculated for the sake of uniformity for the particular case of the San Francisco – Honolulu cable; but they are applicable at once to a cable of any type, provided that the necessary alterations are made. Thus, if the arrival curve is required for a cable with sending and receiving condensers, the equation to the received current is (65), or

$$C_r = \sum_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{Rl\left[\frac{\sin x}{x} + \frac{1}{\cos x} + \left(\frac{\sin x}{x} - \frac{1}{\cos x}\right)\frac{1}{x^2}\frac{Kl}{K_s}\cdot\frac{Kl}{K_r}\right]},$$

where x is given by

$$\tan x = \frac{\dfrac{Kl}{K_s} + \dfrac{Kl}{K_r}}{x - \dfrac{1}{x}\dfrac{Kl}{K_s} \cdot \dfrac{Kl}{K_r}}.$$

Let the constants of the new cable be K′, R′ and l', and the sending and receiving condensers K_s' and K_r' respectively.

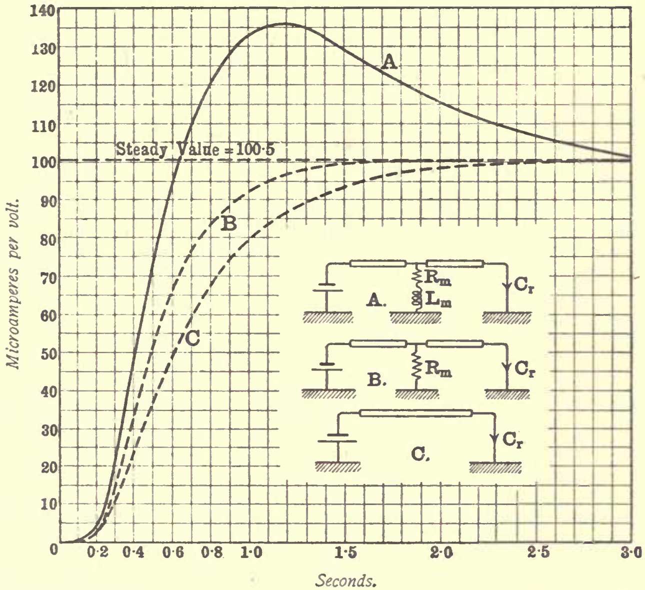

FIG. 110.—ARRIVAL CURRENT. WITH AND WITHOUT INDUCTIVE LEAK.

With inductive leak.

Curve A.... $\begin{cases} R_m = R/4 = 1{,}244 \text{ ohms.} \\ L_m = \dfrac{Rl.KRl^2}{16} = 1{,}354 \text{ henries.} \end{cases}$

„ B......Leak of 1,244 ohms., without inductance.

„ C......No leak, ordinates half actual size.

Then the equation for x is unaltered if the capacities of the new condensers bear the same proportions to the capacity of the new cable as was the case before—*i.e.*, if

$$\frac{K_s'}{K'l'} = \frac{K_s}{Kl} \text{ and } \frac{K_r'}{K'l'} = \frac{K_r}{Kl}.$$

Table LXXIX.

Cable.	Date.	Length, n.m.	R ohms, n.m.	K mfds, n.m.	Rl ohms.	Kl mfds.	KRl^2 seconds.	Copper, lb. per n.m.	G.P. lb. per n.m.
A. Commercial Pacific, San Francisco–Honolulu section	1903	2,276	2·186	0·384	4,975	875	4·35	500	315
B. Anglo-American Atlantic	1894	1,848	1·834	0·420	3,388	776	2·63	650	400
C. German Atlantic No. 2, Borkum–Fayal section	1903	1,916	2·724	0·404	5,218	774	4·04	400	280
D. Pacific, Fanning Island–Fiji section	1902	2,043	5·352	0·365	10,936	746	8·15	220	180

Table LXXX.—*Table LXXX. is calculated from Table LXXIX., making use of Table XLIV.*

A. San Francisco–Honolulu.		B. Anglo-American Atlantic.		C. German Atlantic.		D. Pacific.	
t (seconds).	C_r (microamps. per volt).	$t\times\frac{2\cdot63}{4\cdot35}$	$C_r\times\frac{4,975}{3,388}$	$t\times\frac{4\cdot04}{4\cdot35}$	$C_r\times\frac{4,975}{5,218}$	$t\times\frac{8\cdot15}{4\cdot35}$	$C_r\times\frac{4,975}{10,936}$
0·1	0·01	0·060	0·01	0·093	0·01	0·187	0·00
0·2	1·30	0·121	1·91	0·186	1·24	0·375	0·59
0·3	4·17	0·181	6·12	0·279	3·98	0·562	1·90
0·4	6·21	0·242	9·12	0·371	5·92	0·749	2·83
0·5	6·97	0·302	10·24	0·464	6·65	0·936	3·17
0·7	6·38	0·423	9·37	0·650	6·08	1·311	2·90
1·0	4·36	0·604	6·40	0·928	4·16	1·873	1·98
1·5	2·03	0·906	2·98	1·392	1·94	2·809	0·92
2·0	0·92	1·208	1·35	1·857	0·88	3·745	0·42
3·0	0·19	1·812	0·28	2·785	0·18	5·618	0·09
4·0	0·04	2·416	0·06	3·713	0·04	7·490	0·02

The numerator of C_r becomes $2Ee^{-\frac{x^2t}{K'R'l'^2}}$. But if t be put equal to $\frac{K'R'l'^2}{KRl^2}t'$, the numerator of C_r is $2Ee^{-\frac{x^2t'}{KRl^2}}$, in which the alteration is thrown on the time. The denominator of C_r is altered in the proportion $\frac{R'l'}{Rl}$. Hence it is only necessary to multiply the times by $\frac{K'R'l'^2}{KRl^2}$, and to multiply the corresponding received currents by $\frac{Rl}{R'l'}$.

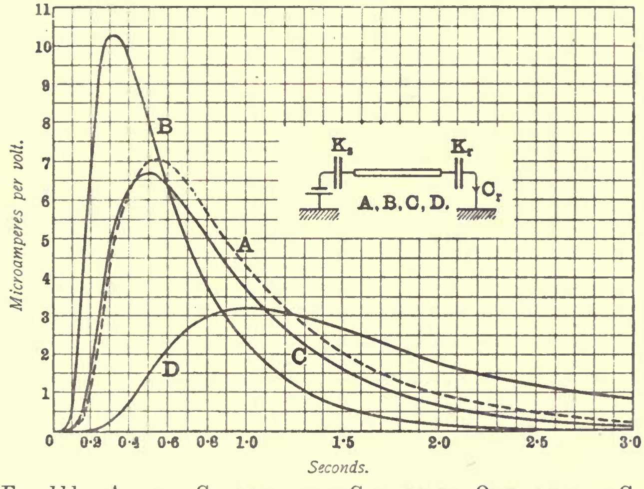

FIG. 111.—ARRIVAL CURRENT, WITH CONDENSERS ONE-TENTH OF CABLE CAPACITY.

Curve A....San Francisco–Honolulu Cable. $K_s = K_r = Kl/10 = 87{\cdot}5$ mfds.
„ B....Anglo-American Atlantic. $K_s = K_r = Kl/10 = 77{\cdot}6$ mfds.
„ C....German-Atlantic No. 2. $K_s = K_r = Kl/10 = 77{\cdot}4$ mfds.
„ D....Pacific. $K_s = K_r = Kl/10 = 74{\cdot}6$ mfds.

Three representative cables are (1) the 1894 Anglo-American Atlantic, (2) the Borkum–Fayal section of the second German Atlantic, and (3) the Fanning Island–Fiji section of the Pacific Cable. The data necessary are contained in Table LXXIX.

Table LXXX. is plotted in Fig. 111, which shows how the resistance of the cable affects the "height," and the product of resistance and capacity the "length," of the curve of the received current. The other tables may be altered in a similar manner to fit any desired case.

Use of Arrival Curves to Build up a Message.

Let t seconds be the element of time in sending a dot or a dash, t the time for a space between dots and dashes, $3t$ between letters, and $7t$ between words. Then the average number of elements per letter, allowing for spaces between letters but not between words, has been found* to be 7·16, or the time to send a letter is $7{\cdot}16t$. Hence the time for a word of N letters, allowing for the space between it and the next, is $7{\cdot}16Nt+4t$ and the number of words per minute, t being measured in seconds, is

$$\frac{60}{(7{\cdot}16N+4)t}.$$

If $N=5$, the number of words is $1{\cdot}51/t$. Thus, if t be taken as 0·08 second, the number of five-letter words per minute is 18·9. Suppose that signals are transmitted through the San Francisco–Honolulu cable at this rate, and that the cable is used simplex with condensers each of 87·5 mfds. Then the arrival curve is that shown in Fig. 90, A, and to build up the message the curve must be taken positively or negatively at intervals of 0·08 second. The ordinates are then to be added together. This is done in Fig. 112, A, which shows the form on arrival of the word "Silvertown" when sent at the above speed. In Fig. 112, B, the same word is shown now sent at a higher speed, which may be taken as an upper limit to the actual speed in practice, just as the lower speed is a lower limit. In this case $t=0{\cdot}06$ second, corresponding to a speed of 25·2 five-letter words a minute. In a similar way Fig. 112, C, is obtained, which shows the influence of a leak on the signals sent at the higher speed. To build it up, Fig. 107, A, was used, which is the arrival curve when a leak of 1,244 ohms is placed at the centre of the cable. In order to secure sufficient

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* E. Raymond-Barker, "Electrical Review," XL., p. 516, 1897.

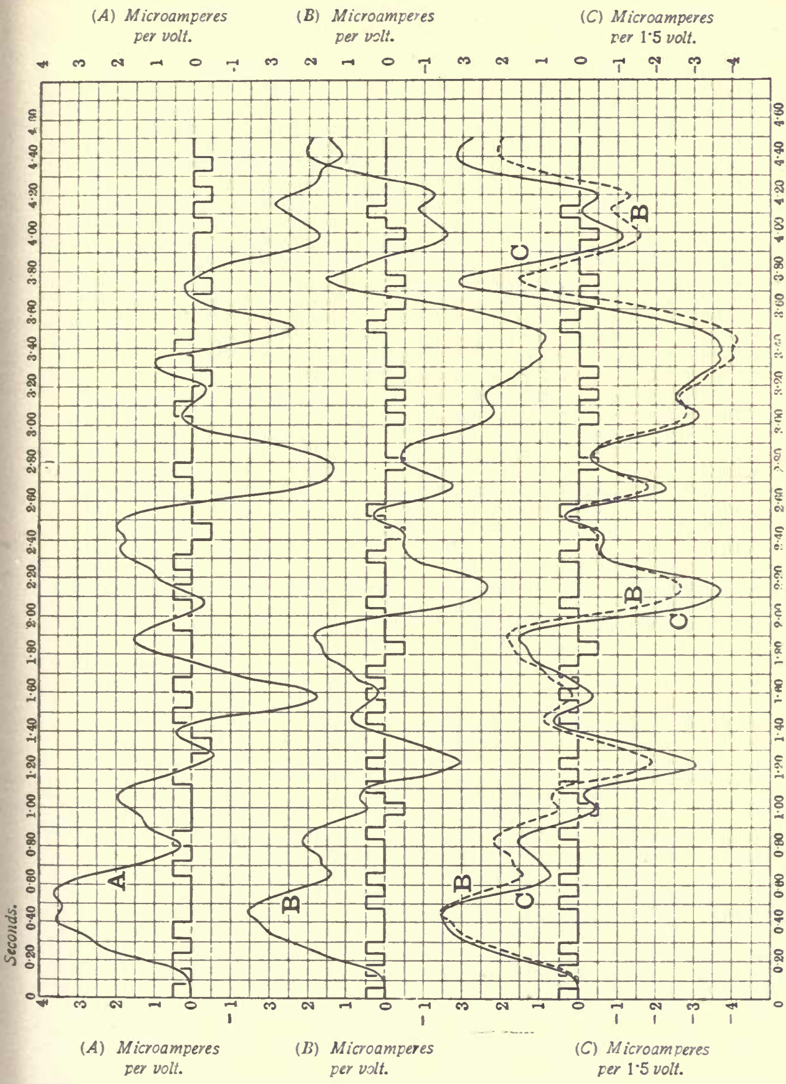

Fig. 112.—San Francisco-Honolulu Cable.

Curve A.—With sending and receiving condensers $K_s = K_r = \frac{Kl}{10} = 87{\cdot}5$ mfds. Time of elementary signal = 0·08 sec.

,, B.—Condensers as before. Time of elementary signal = 0·06 sec.

,, C.—Condensers as before and leak at middle $R_m = \frac{Kl}{4} = 1{,}244$ ohms. Time of elementary signal = 0·06 sec.

accuracy in the shape of the signals the points on the arrival curves, Fig. 90, A, and Fig. 107, A, were calculated for every 0·02 second, and the points on the curves of Fig. 112 are the result of combining these arrival curves also at intervals of 0·02 second.

By proceeding in this way, and constructing a variety of such cases, it is now possible, given the arrival curve, to predict accurately the speed which may be obtained from a projected cable, or to compare two proposed types of cable, or to examine the influence of change of apparatus on the maximum speed at which the cable may be worked.

CONCLUSION.

The endeavour of the foregoing treatment of the subject has been to place the theory of the telegraph cable on a sound basis, and to establish a systematic method of attacking by calculation all the problems that arise in connection with submarine telegraphy. It has been shown how, by the application of this method, the answer may be obtained to many questions which have hitherto been without solution. Such applications are, for example, to find the arrival and the sending currents and voltages with various arrangements of apparatus, to derive the theory of the inductive shunt, the theory of leaks, of inductive leaks, and of the cable with continuously distributed leakance, and to extend the "KR" law to include the signalling apparatus. The treatment is merely illustrative, and, owing to the infinite variety of cases that may arise, by no means exhaustive; but it is hoped that enough has been said to render easy its application in detail to any particular case. The difficulties in the way of high-speed, long-distance telegraphy seem by no means to be insuperable, and it is probable that the near future may see the speed of telegraphy greatly increased. But to bring this about, theory and experiment must go hand in hand. In this connection it is well to remember from the history of telephony that "all attempts to increase the inductance of a wave conductor by the introduction of inductance coils at periodically recurring points failed, because they had no mathematical thory to guide them, so as to avoid the difficulties of wave reflection."* As evidence of the extent of the experimental work that requires to be done, it may be pointed out that there is no available record of the observation in ordinary units of even the simplest arrival curve, although this curve has been known for half a century. When the problem of the high-speed submarine telegraph cable is attacked with the modern resources and the scientific knowledge that have won success in other branches of electrical seicnce, notably in telephony and in radio-telegraphy, a great transformation may be anticipated.

* M. I. Pupin, Am. I.E.E., 17, p. 450, 1900.

PART IV.—THE METHODS OF TELEGRAPHIC TRANSMISSION.

CHAPTER XI.

SIGNALLING BY INVERSE CURRENTS.

Introduction—Elementary Arrival Curve—Siphon Recorder Code—Ordinary Morse Code—The Gott Form of Morse—The Picard Form of Morse—Discharge of a Cable Insulated at the Sending End—Modification of the Picard Method—Retransmission by Gott Method—Higher Speed—Cross Letters—First Modification—Second Modification—Recent Improvements—Conclusion.

INTRODUCTION.

From time to time inventions have been made with the object of facilitating through working, in which the offices at the ends of a chain of communications are put in electrical connection, thus obviating the loss of time and the risk of error which are entailed when messages are received and retransmitted at intermediate stations.

If two submarine cables of equal lengths and of the same type are connected in series they behave as a single cable, the KRl^2 of which is four times that of either component, and therefore the speed of the combination, or the words which can be transmitted in a given time, is only one quarter that of one of the shorter cables. If, now, a perfect relay were inserted at the junction to repeat or retransmit the messages automatically, the combination could be worked with the speed of either of its components, but owing to the manner in which the signals spread out, and are rounded off in transmission, it is not easy to ensure that a relay shall work with certainty at the end of a long cable when the speed of sending is more than a moderate one. An instance of such through working is afforded by the cables along the West Coast of South America, which are linked up on the Muirhead system, so that at certain times New York is placed in direct communication with Buenos Ayres through four relays.

In order to test the merits of any proposal of modifications in the form of transmission, perhaps the best way to proceed is as follows : A typical word is supposed sent through a representative cable in the proposed conditions, and the form of the word on reception is calculated. The conditions are then varied, and the new shape of the word is obtained. This method, whereby the complex shape of the word is synthesised out of the elementary arrival curves, has been fully developed in Part III. It presupposes a knowledge of the elementary arrival curve or fundamental transient, which can be found by means of the general theorem of Chapter IX.

In what follows the San Francisco–Honolulu cable is taken as an example, as the data regarding it have already been tabulated. Should another type of cable be preferred, rules for conversion* allow this to be done. As forming a typical word the letters SO are chosen—three dots followed by three dashes—because they show the cumulative effect of two sets of three similar signals in succession, with a change midway from the one kind of signal to the other.

Sine-wave transmission does not come within the restricted scope of this chapter, but is dealt with separately in Chapter XII.

Elementary Arrival Curve.

Suppose that the cable is worked simplex with sending and receiving condensers, each equal to one-tenth part of the cable capacity. The shape of the arrival curve of current in these circumstances, for a single prolonged contact at the sending end with the positive pole of the battery, is given in Fig. 113, curve A. The current rises to a maximum in 0·53 second, and then declines slowly to zero, after which both condensers are fully charged, and the cable forms a connecting link between them, and is charged to one-twelfth the potential of the battery.

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* Chapter X., p. 326.

Now, suppose that at the end of a contact lasting 0·08 second the cable is put to earth at the sending end. This may be regarded as equivalent to the insertion of an equal negative E.M.F. (−E) to neutralise the first (+E) at the end of 0·08 second. The arrival curve for this second E.M.F., taken alone, would be similar to curve A in Fig. 113, but displaced to the right through 0·08 second, as is curve A′, and drawn downwards. The combined effect of the two operations may be obtained by adding algebraically the arrival curves which they

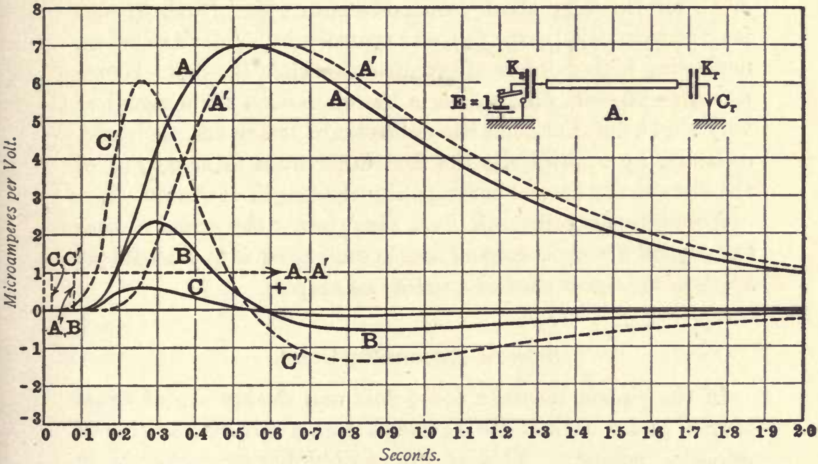

Fig. 113.—Arrival Current, San Francisco-Honolulu Cable, with Sending and Receiving Condensers, $K_s = K_r = Kl/10$. $l = 2{,}276$ n. ms. $Rl = 4{,}975$ ohms. $Kl = 874{\cdot}6$ mfds. $KRl^2 = 4{\cdot}352$ seconds.

Curve A—Single prolonged contact. Curve A′—Single prolonged contact beginning at time 0·08 second. Curve B.—Dot signal lasting 0·08 second. Curve C—Dot signal lasting 0·02 second. Curve C′ is curve C with the ordinates multiplied by 10.

produce—*i.e.*, by subtracting the ordinates of curve A′ from those of curve A. Curve B is obtained in this way. It reaches its maximum in 0·29 second, and becomes negative at 0·58 second, after which it approaches zero very slowly.

Curve C, Fig. 113, is obtained in a similar manner, contact lasting in this case for 0·02 second, and the turning value is reached in 0·26 second. For clearness the ordinates of curve

C′ are drawn to 10 times the scale of those of curve C, as though the voltage of the sending battery were increased tenfold, or receiving apparatus of 10 times the sensitivity were used, to compensate for the decrease in current arising from the shortness of contact.

In like manner the effect of any succession of signals may be synthesised.*

If $f(t)$ denote the elementary arrival curve A, then $f(t-dt)$ will denote curve A′, when the time of contact dt is very small, and $f(t)-f(t-dt)$ will represent the combined curve, as in B or C, for the very short period of contact dt. Now, $f(t-dt)$ may be written $f(t)-dt\,.\,f'(t)$, on expansion by Taylor's theorem, neglecting higher orders of infinitesimals than the first. Hence $f(t)-f(t-dt)=dt\,.\,f'(t)$, which is the expression for a signal of very slight duration, and the ordinates of which are, therefore, obtained by multiplying the first differential or slope $f'(t)$ of the elementary arrival curve $f(t)$ by the length of contact dt.

No advantage can result from diminishing the period of contact beyond a certain amount, as the only effect is to decrease the height of the signal without affecting its shape.

Siphon Recorder Code.

In the siphon recorder code, dots and dashes are of equal length, and a dash is distinguished from a dot by its being of opposite polarity. This is represented diagrammatically in Fig. 114 by the dotted rectangular line A. The letter S is supposed to be formed by three contacts, each lasting 0·08 second, with the positive pole of the battery, and the letter O by three contacts for the same length of time with the negative pole. The space between dots and dashes is also 0·08 second, and between the two letters $0{\cdot}08\times3=0{\cdot}24$ second.†

The resultant effect at the receiving end is shown in curve A. Just as the effect of the first dot is beginning to subside, the second one makes its appearance, running like a wave on top of the first, closely followed by the third. If no further signals

* For other examples, *see* Chapter X., p. 330.

† E. Raymond-Barker, "Electrical Review," Nov. 22, 1901.

were made the current would pass through zero and become negative, afterwards slowly declining to zero. But on the arrival of the first dash there is superimposed an additional negative current, and the same process is repeated as before, but in the opposite direction.

In actual practice these curves would be modified to some extent by various factors, such as the inertia of the siphon recorder, and the influence of the inductance shunt; they should not, therefore, be used for other than *comparative* purposes, unless these influences are taken into account.

The current in curve A, Fig. 114, starts up exactly as in curve A, Fig. 113, and if a train of dots were sent the current

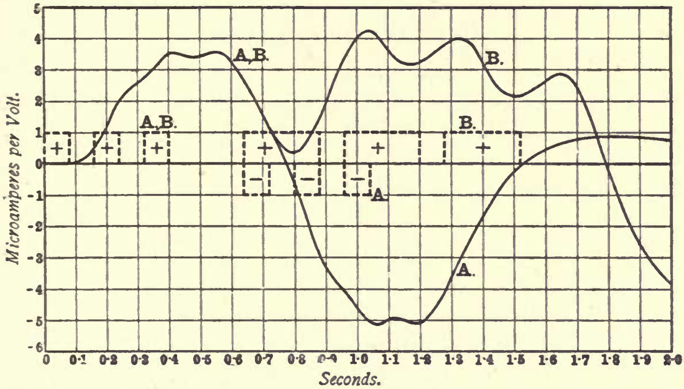

FIG. 114.—FORM ON ARRIVAL OF LETTERS SO. LENGTH OF ELEMENT =0·08 SECOND.

Curve A—Siphon recorder code. Curve B—Ordinary Morse code.

would settle down into a state of steady oscillation about the zero line, the amplitude of the oscillations depending on the frequency with which the dots were sent.*

Suppose that the sending battery has an E.M.F. of 50 volts. Then the ordinates in Fig. 114, which represent microamperes per volt, must be multiplied by 50. Again, let the relay through which the received current passes be of such a sensitivity that

* Chapter VIII., p. 249.

50 microamperes are required to actuate it. Then when the current curve A reaches the height of unit ordinate in Fig. 114, which it does at 0·19 second, the relay will be actuated, and will continue to retransmit into the second cable until the current decreases to 50 microamperes again, which it does not do until time 0·73 second. The three dots of the S would be sent into the second cable as a very long dot, and, similarly, the three dashes would be retransmitted as a very long dash.

A relay of fixed zero is here understood. One with a variable zero, or one provided with a local correction circuit, would follow the general shape of the curve and be actuated only by the ripples, behaving somewhat like the inertia governor fitted to marine engines to prevent racing.

Ordinary Morse Code.

In the Morse code the three dashes are represented by the dotted line B in Fig. 114, as contacts made with the same pole of the battery, and lasting for three times as long as a dot. The shape of the current-arrival curve is given in curve B, Fig. 114. It coincides until nearly 0·72 second with curve A for the siphon recorder code.

The shape of the dashes is well marked, as might naturally be expected, since they may be looked upon as dots sent so slowly that the signals have time to form.

The siphon recorder message is completely sent, and the cable is ready to receive a fresh one, in time $1{\cdot}04+0{\cdot}24$ (space before next letter) $=1{\cdot}28$ second, whereas the same message in Morse occupies $1{\cdot}52+0{\cdot}24=1{\cdot}76$ second.

With a fixed zero instrument actuated by 50 microamperes, curve B would be retransmitted as two long dashes. The use of the siphon recorder code, it is clear, has the advantage over the Morse code that the current passes through zero between the two letters. The change from the one letter to the other in curve A is strongly marked, while with faster sending the dip between the two letters in curve B would disappear and the received signal would be retransmitted as one long dash.

In high-speed working the siphon recorder clerk finds SS difficult, whereas SO is easy to recognise, because of the way in which the current crosses the zero line midway between the two letters.

The letters of the alphabet, as is well known, may be divided into two classes, cross letters such as *c* (dash dot dash dot) and non-cross letters such as *h* (four dots). Non-cross letters are difficult to read, while cross letters are easy, and the speed of transmission is limited by the degree of legibility of the non-cross letters alone.*

Now, the Morse code has no cross letters, and for this reason, as well as that the dashes require more time than the dots, Morse transmission is slower than siphon recorder transmission on long cables. But if messages are to be sent through a junction of cable and land lines—on which the Morse code is exclusively employed—the same code must be used on the cable also.

It may seem accordingly but a short step to modify the Morse code by making the even contacts of negative polarity, so that the received current may pass to and fro across the zero line, as though a succession of cross letters were being sent in the siphon recorder code.

The Gott Form of Morse.

This is the principle of the method with which the name of the late Mr. John Gott became associated early in the year 1913, although the field had been previously covered by other inventors.† In Fig. 115, the letters SO are supposed sent in this way. The apparatus by which the reversals of current are brought about may take a variety of forms, and is of subsidiary importance compared with the principle involved. At the receiving end a non-polarised relay or its suspended-coil equivalent, in which the two contacts are joined together, may be used, actuating a Morse inker, which will write down the message, Fig. 115, B, without respect to sign of current, as though it were couched in ordinary Morse.

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* *See* the Discussion on "The Sine Wave Transmitter," A. C. Crehore and G. O. Squier, Am.I.E.E., XVII., 1900, p. 385.

† *See* A. Fraser, The Electrician, LXX., 1913, p. 1018; P. O'Neil, "Electrical Review," LXXII., 1913, p. 377; C. Bright, "Submarine Telegraphs," 1898, p. 676.

Fig. 115 should be compared with Fig. 114. With a relay of the same sensitivity (50 microamperes) the message is now quite legible (Fig. 115, B). The dashes come through better than the dots, as they have more time to form. The second dot is rather short, and may be crowded out. The space between dashes is very short, but quite enough to enable them to be read, and the space between the letters is not much greater than between dot and dot.

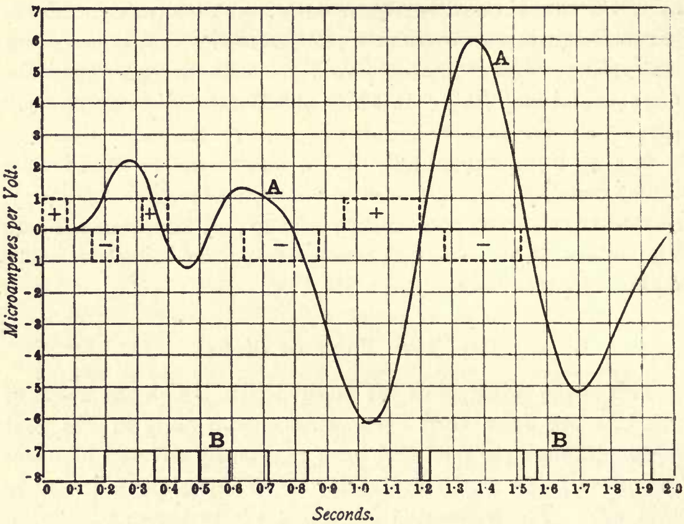

Fig. 115.—Form on Arrival of Letters SO, using Gott Method. Length of Element=0·08 second.

Curve A—As shown on siphon recorder. Curve B—As shown on Morse inker.

The Picard Form of Morse.

In an article* dealing with the Gott method there is recalled to notice a method of signalling which appears to have escaped the attention it deserves. Originally introduced to enable Baudot working to be carried out on the Marseilles–Algiers cable (900 km.), which was accomplished with great success

* By E. Raymond-Barker, "Electrical Review," Feb. 28, 1913.

between Paris and Algiers, the principle was intended by its inventor to be applied to submarine cables of greater length.

In this method the cable is normally free at the sending end. A dot is sent by making contact for an instant with the positive pole of the battery, closely followed, after an interval of insulation, by a contact of the same length with the negative pole. A dash is distinguished by having its two contacts further apart. There is thus a regular alternation of equal contacts of opposite sign, as represented imperfectly by the dotted rectangular line in Fig. 117.*

When contact is made with the battery the cable begins to charge up, and current begins to flow out at the receiving end. If the cable were then earthed at the sending end, current would flow out at both ends of the cable. But when the cable is freed at the sending end, current can flow out only at the receiving end.

The case of a cable insulated at the sending end is a new one, and requires separate treatment.

Discharge of a Cable Insulated at the Sending End.

When contact is made at the sending end, without apparatus, the voltage along the cable rises to a steady value as represented in diagram A, Fig. 116. If the insulation of the cable is perfect, the graph of voltage is a straight line falling from $+E$ at the sending end to zero at the receiving end. The current rises at the same time to a steady value ($=E/Rl$), and the growth of current is given by the Kelvin formula (cable alone, earthed at both ends)—

$$C_t = \frac{E}{Rl}\left[1 + 2\sum_1^{\infty} {}^m e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi\right] \quad . \quad . \quad . \quad . \quad (1)$$

—which is plotted in Fig. 116, curve A. When the battery is

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* This form is the one of which a description is given by A. Tobler, "Journ. Télég.," Dec., 1903. In another arrangement, described by A. Carletti, "Journ. Télég.," XXX., 1906, p. 270, and devised for Hughes working, the apparatus used is the same as in the Gott system with transformer, but Picard preferred to insulate the back stop of the sending key.

removed and the cable put to earth, the current dies away, as in Fig. 116, curve C, which is curve A drawn downwards. Now, when the cable is insulated at the sending end the current must be zero there, throughout the whole of the discharge. To take account of this constraint the following artifice may be used. Let two cables be joined in parallel, as in diagram B, Fig. 116. When the battery is applied each will charge up as before, and when the battery is removed the whole will behave as one cable discharging with ends to earth as in the former case, the initial distribution of voltage alone being altered.

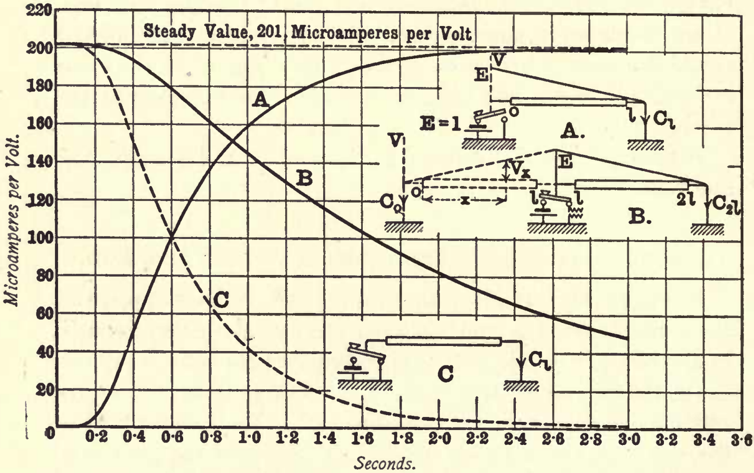

FIG. 116.—ARRIVAL CURRENT, CABLE WITHOUT APPARATUS, RECEIVING END TO EARTH.

Curve A—Current growth, sending end to earth through battery. Curve B—Current decay, sending end insulated. Curve C—Current decay, sending end to earth.

The graph of voltage in diagram B, Fig. 116, rising from zero at $x=0$ to $+E$ at $x=l$, and falling to zero again at $x=2l$, can be represented by a Fourier's series.

Thus

$$V_{\text{steady}}=\frac{8E}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2l}-\frac{1}{3^2}\sin\frac{3\pi x}{2l}+\ldots\right] \quad . \quad . \quad (2)$$

represents the initial state of voltage between $x=0$ and $x=2l$, at time $t=0$. At any subsequent time the voltage must satisfy the differential equation :—

$$\frac{d^2V}{dx^2}=KR\frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A particular solution of the equation suggests itself, of the form

$$\frac{8E}{\pi^2}\frac{\cos m\pi}{\overline{2m+1}^2} \cdot e^{-\frac{\overline{2m+1}^2\pi^2 t}{4KRl^2}} \cdot \sin\frac{\overline{2m+1}\pi x}{2l}, \quad . \quad . \quad . \quad . \quad (4)$$

for it satisfies the differential equation, is zero when $t=\infty$, and is the mth term of V (steady) when $t=0$. Each sine component decays according to the same law, the higher harmonics disappearing first. Hence—

$$V=\frac{8E}{\pi^2}\sum_0^\infty \frac{\cos m\pi}{\overline{2m+1}^2} \cdot e^{-\frac{\overline{2m+1}^2\pi^2 t}{4KRl^2}} \cdot \sin\frac{\overline{2m+1}\pi x}{2l}. \quad . \quad (5)$$

The current may be obtained for this by means of the equation

$$C_x=-\frac{1}{R}\frac{dV}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \quad C_x=-\frac{8E}{R\pi^2}\sum_0^\infty \frac{\pi\cos m\pi}{2l\,(2m+1)} \cdot e^{-\frac{\overline{2m+1}^2\pi^2 t}{3KRl^2}} \cdot \cos\frac{\overline{2m+1}\pi x}{l}. \quad (7)$$

When $x=l$,

$$C_l=\frac{4E}{\pi\,.\,Rl}\sum_0^\infty \frac{e^{-\frac{\overline{2m+1}^2\pi^2 t}{4KRl^2}}}{2m+1}\cos m\pi. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The series (8) is the same as that occurring in the formula for the arrival voltage when the receiving end is insulated, and its values have already been tabulated.* It is plotted in Fig. 116, curve B.

The current is considerably greater when the sending end is freed, since all the charge in the cable must flow out at the receiving end. The area between curve B and the time axis represents the whole charge in the cable, and equals $Kl\,.\,E/2$.

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* Chapter IX., p. 283.

The area between curves B and C is the quantity which flows out at the sending end when it is put to earth after charging, and the area between curve C and the time axis is the quantity which flows out at the receiving end. This last area is the same as the area between curve A and its steady value, and may be obtained from (1)—

Thus $$Q_{\smile} = -\frac{2E}{Rl}\int_0^\infty \Sigma_1^\infty e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi . dt,$$

$$= \frac{2E . Kl}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right] = \frac{E . Kl}{6}.$$

Hence, one-third of the charge flows out at the receiving end, and two-thirds at the sending end.

The rate at which the current decays is much lower than when the cable is earthed at both ends. When there is a condenser in the circuit, the cable retains its charge, when insulated at the sending end after full charging. In ordinary telegraphy, where signals of like polarity follow each other, it is therefore necessary to earth the cable at the sending end after every signal.

Modification of the Picard Method.

From a comparison between formula (8) and formula (1) it is seen that the laws governing the charge and the discharge of the cable are different, series (1) containing all the harmonics, and series (8) only the odd ones. When the cable is discharged before being fully charged, a highly complex state of affairs arises, and it is not permissible to superimpose B on A. Curve C, Fig. 113, may nevertheless be used in the ordinary manner, and this is carried out in Fig. 117, curve A.*

Although the reproduction by a fixed zero instrument, as shown in Fig. 117, B, would be useless, curve A is a great improvement on curve B, Fig. 114, and would be easily decipher-

* This procedure finds some justification in the fact that, according to A. Carletti, *l.c.*, p. 270, in certain cases Picard earthed the sending end for part of the time.

able, and readily recorded by an instrument with variable zero. It should be observed that the scale of ordinates is altered in this case, and reference should be made to the remarks regarding Fig. 113, curve C′.

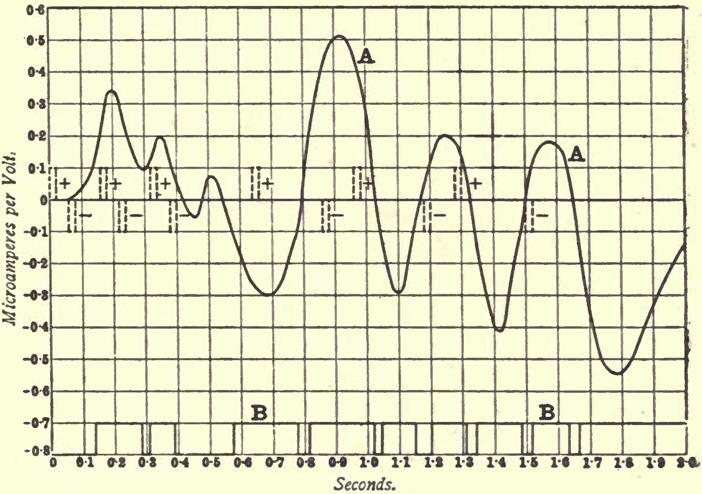

FIG. 117.—FORM ON ARRIVAL OF LETTERS SO, USING MODIFIED PICARD. LENGTH OF ELEMENT=0·02 SECOND.

Curve A—As shown on siphon recorder. Curve B—As shown on Morse inker.

RE-TRANSMISSION BY GOTT METHOD.

In Fig. 118 the letters received in Fig. 115, B, are supposed re-transmitted through a cable equal in length to the first, and of the same type. The dotted rectangular line represents the voltage (Fig. 115, B) as sent by the receiving relay. If a receiving instrument of the same sensitivity as formerly supposed were used, the second dot would fail to make a record. If the sensitivity were increased so that the instrument would be actuated by $50 \times 0{\cdot}2 = 10$ microamperes, the received signal would be as shown in Fig. 118, B.

The second dash is shorter than the first dot, and probably the message would provide great difficulty in interpretation.

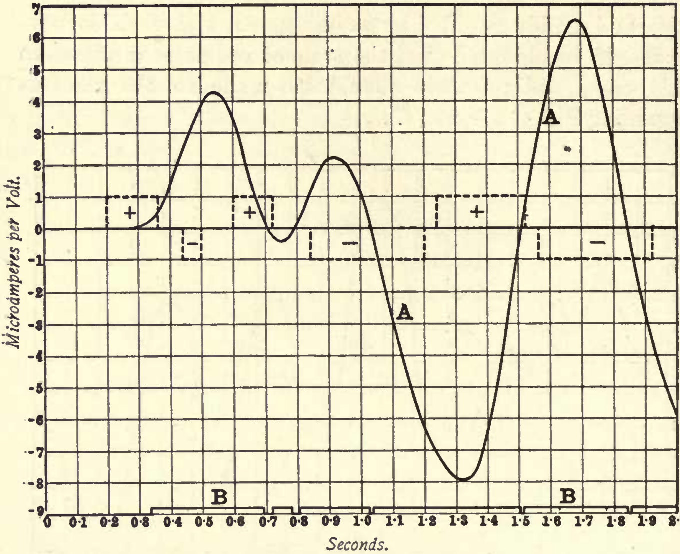

FIG. 118.—FORM ON ARRIVAL OF LETTERS SO, RE-TRANSMITTED BY GOTT METHOD.

Curve A—As shown on siphon recorder. Curve B—As shown on Morse inker.

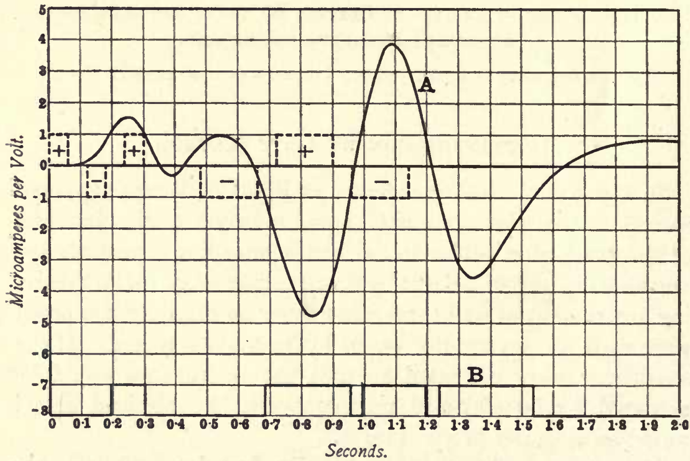

FIG. 119.—FORM ON ARRIVAL OF LETTERS SO, USING GOTT METHOD. HIGHER SPEED. LENGTH OF ELEMENT=0·06 SECOND.

Curve A—As shown on siphon recorder. Curve B—As shown on Morse inker.

It may be argued that the test is a severe one, in transmitting through two such long cables, but a still more severe test would have to be satisfied before direct communication could be established, say, between London and Sydney.

Higher Speed.

If the speed of transmission be increased, everything else being left unaltered, the result is shown in Fig. 119, where the length of an element is now 0·06 second instead of 0·08 second. The second and third dots have now disappeared, although they could be restored by increasing the sensitiveness of the receiving relay, but the lengths of dots and dashes would be out of proportion and cause difficulty in interpretation.

Cross Letters.

Hitherto only non-cross or cumulative letters have been considered. In Fig. 120 two cross letters are shown. They are K (dash dot dash) and R (dot dash dot).

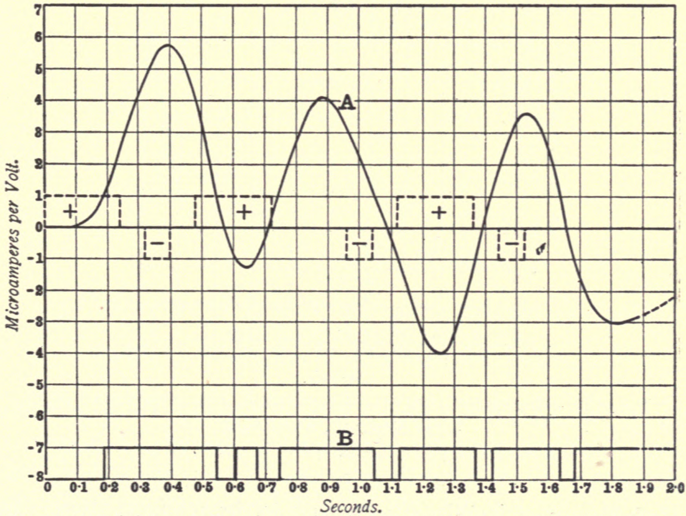

Fig. 120.—Form on Arrival of Letters KR, using Gott Method. Length of Element=0·08 second.

Curve A—As shown on siphon recorder. Curve B—As shown on Morse inker.

The dot between the two dashes in the letter K has shrunk in transmission to very small proportions, but it is still decipherable. On the other hand, the dash in the centre of the R is shorter than the preceding dot. The tail of the second dot in the R would be cut off by the beginning of the next letter. In siphon recorder code the letter R would be sent as is the letter S in Fig. 115, and received as in curve B, Fig. 115, except that the central signal would be drawn downwards. The letter K would be like the R turned upside down.

For the transmission of cross letters the new method would appear to be inferior to siphon recorder code.

First Modification.

The assumptions which have been made regarding the relative lengths of dots, dashes, and spaces are quite arbitrary,

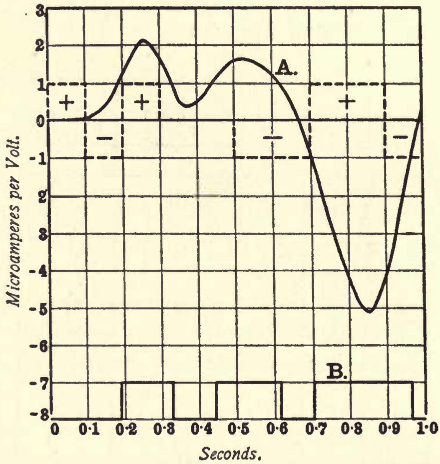

Fig. 121.—Form on Arrival of Letter S, and Beginning of O. Gott Method, re-spaced.
Curve A—As received on S.R. Curve B—As received on Morse inker.

and it may be found advantageous to depart from them. In Fig. 121 the dotted rectangular line shows the letter S, and the beginning of the letter O sent according to a different arrangement. The length of a dot is increased to 0·10 second, instead of 0·08 second. The space between the dots has been cut out, and a dash as well as the space between the letters is only twice as long as a dot, the net effect being to shorten the time required to send the message.

From curve A it is seen that the result is not a success, although the dots are given longer time, since the second one drops out and would not be received on a fixed zero instrument.

SECOND MODIFICATION.

Another and more promising way is to alter the relative length of the dots. This is carried out in Fig. 122. The first

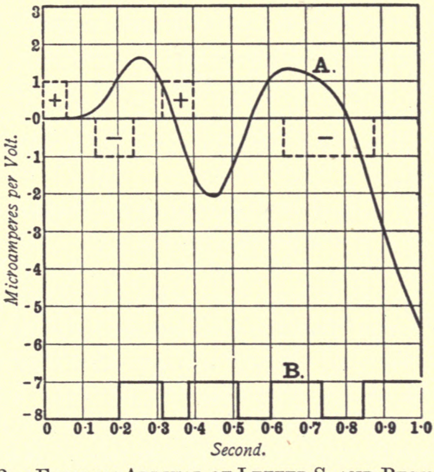

FIG. 122.—FORM ON ARRIVAL OF LETTER S, AND BEGINNING OF O. GOTT METHOD, RE-SPACED.

Curve A—As shown on S.R. Curve B—As shown on Morse inker.

dot is reduced from 0·08 to 0·06 second, and the second dot in increased by the same amount, the third dot and the spaces being left unaltered. The result is shown in Fig. 122, B, which should be compared with Fig. 115, B. The dots are now very nearly of equal length, and the second dot is somewhat longer than the first, which would be an advantage in retransmission, or by a proper adjustment they could be made exactly equal. With a slight curbing current the tail of the third dot could be cut off, and its length sharply defined, so as to equal that of the other two.

In like manner the forms of the cross letters K and R in Fig. 120 could be corrected so as to free the issuing signals from distortion.*

Recent Improvements.

Instead of distinguishing dash from dot by prolonging the duration of the sending E.M.F., an alternative plan is to use an augmented E.M.F. to form the dashes.† Consecutive signals are, as before, of opposite polarity. Now, it has been shown ‡ that a dash differs from a dot at the receiving end only in height, provided that the period of contact is very short. The effect of using increased E.M.F. for a dash instead of increased duration is therefore seen to be equivalent to an alteration in the spacing, through which the dashes are brought closer together.

Curve A, Fig. 123, shows the letters SO transmitted in this way. It should be compared with Fig. 115. The dashes are supposed to be formed by double the sending E.M.F. It is seen that at the one microampere per volt height the first dash is not much longer than the first dot, and the second dash is considerably shorter. The signals are not, therefore, to be separated out at the receiving end by their length. The alternative is to distinguish them by the greater height of a dash, which involves the use of a special relay. It is clear from curve A, since the second dash is of not quite the same height as the first dot, that either the dash E.M.F. must be increased

* Various other modifications, which will occur to the reader, may be treated in a similar manner. In the Kitsee form of Morse (B.P. 26,043 of 1904) all impulses, as in the Picard form, are of like duration and intensity, and it resembles both Picard and Gott forms in securing a regular alternation in the polarity of the signals. A dot is formed by a single contact, and two in quick succession constitute a dash.

† Gott's British Patent No. 22,364, of 1912.

‡ *See* p. 338.

or, equivalently, the dot E.M.F. must be reduced, if this distinction is to hold good in the present instance without necessitating a reduction in the speed of sending. Accordingly, in curve B the dash E.M.F. is trebled. On covering up the lower part and using the dotted curves A′ and B′ the signal is seen as it would affect a non-polarised receiver.

Fig. 124 shows apparatus specially devised for the reception of such two-power signals, as well as their rectification and

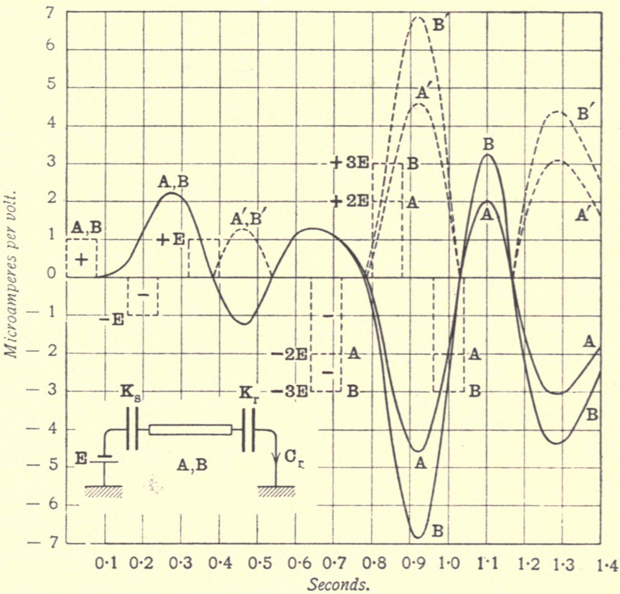

FIG. 123.—FORM ON ARRIVAL OF LETTERS SO USING INCREASED E.M.F. FOR DASHES.

Curve A. Dash E.M.F. doubled. Curve B. Dash E.M.F. trebled.
Curves A′ and B′ are mirror images of the negative portions of A and B.

retransmission.* The jet relay† is actuated by the incoming signals from cable C_1, and closes the contact K_1K_2 for dots. The circuit of S_1 is closed thereby through the local battery,

* Due to E. Raymond-Barker, THE ELECTRICIAN, July 9, 1915.
† *See* Chapter XIV.

and the sounder S_3 sends into cable C_2. On the arrival of a dash the contacts K_3K_4, which lies outside K_1K_2, is closed, the circuit of S_1 is broken and the circuit of S_2 closed. Sounder S_4 then transmits dashes into cable C_2. The relays PR_1 and PR_2 serve to obtain the required alternation in polarity of the retransmitted signals, the tongues of the relays being moved over by the cable discharge after every signal. The rectified signals are recorded at the same time on the local instrument LR.

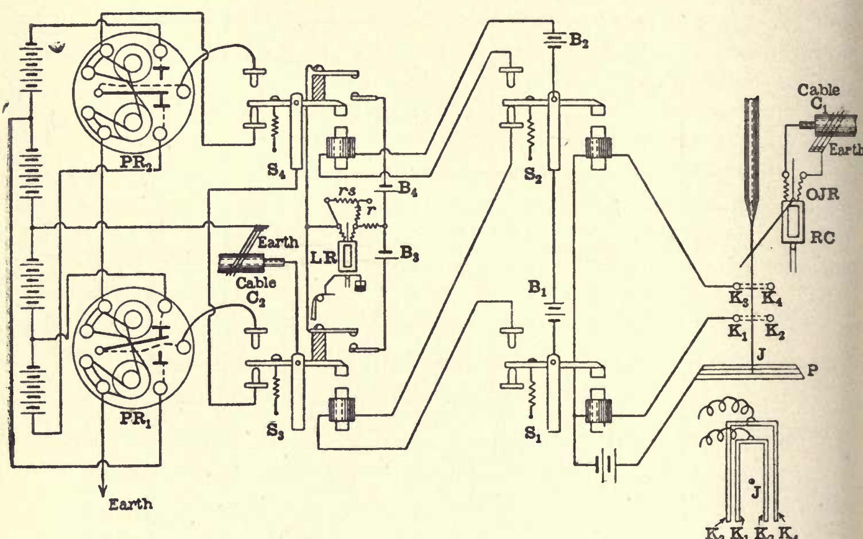

Fig. 124.—Raymond-Barker Method of Reception of Two-Power Signals.

Conclusion.

In the foregoing analysis it is evident that the Gott method is superior to ordinary Morse, and to siphon recorder code for the non-cross letters. In the case of the cross letters it offers no advantage over, or is even inferior to, siphon recorder code, and inasmuch as a dash is of greater duration than a dot, it is necessarily slower than siphon recorder code when read by an instrument of variable zero, whether relay or recorder. This objection does not apply to the form just described, although there, if the maximum voltage which may be applied to the cable is fixed, the height of the dashes cannot be increased and the height of the dots must necessarily undergo reduction.

It appears, moreover, that the method could be improved if each letter were given its theoretically best shape. Thus, in the case of three successive like signals the first would be slightly shortened and the second lengthened, so as to compensate for the distortion introduced in transmission and make the issuing signals of equal length and of equal spacing. Alternatively the sending E.M.F. would be proportioned in a similar way. Automatic transmission would probably be necessitated, the letters being punched out to their proper shape on the tape, and sent through a form of transmitter in which the contacts are under control. These and other modifications which suggest themselves naturally are applicable also to the other methods of signalling, and a great service will have been done cable telegraphy if the interest which has been aroused by the proposals described should culminate in a systematic and exhaustive investigation with the object of discovering the best sending conditions for transmission.

CHAPTER XII.

SINE WAVE TRANSMISSION.

Introduction—Resolution of a Semi-oscillation into Two Sustained Oscillations—Effect of Switching on a Sinusoidal E.M.F. to a Cable —Discussion of the Formula: The Periodic Terms—Discussion Continued: The Transient Terms—The Transient Component—The Periodic Component—Train of Battery Reversals—General Theorem Regarding Mean Value of Sending E.M.F.—Influence of Apparatus: Transient Component—Influence of Apparatus: Periodic Component—Effect of a Single Oscillation and of a Single Semi-oscillation—Battery Signals—Sending-end Phenomena—Stress on the Cable—Attenuation of Voltage throughout the Cable—Voltage Growth along Cable: Battery Contact—Deformation of Battery Signal in Transmission—Semi-period Signal at the Sending End—Deformation of Semi-period Signal in Transmission—Recent Improvements—Conclusion.

Hitherto the sending E.M.F. has invariably been considered to be derived from a battery. In a Paper read before the American Institute of Electrical Engineers some years ago,* a new form of telegraphic transmission was described by its inventors. In this form the battery ordinarily employed is replaced by an alternator giving a sinusoidal E.M.F. of low frequency. The machine is of the two-pole type, with drum-wound armature, and narrow air-gap, in order to give sufficient output for the low speed of rotation entailed.

Signals are formed in the siphon-recorder code by using impulses of sine shape, each one semi-period in length. For example, the letters *a*, *b*, *c*, *d* would be sent as in Fig. 125. Except when sending, the cable is to earth at the sending end. When sending, the alternator is inserted between the cable end

* A. C. Crehore and G. O. Squier, "Trans." Am.I.E.E., XVII., 1900, p. 385.

and earth. This insertion and the signalling are effected by a paper tape with two lines of holes as usual, one for the dots and another for the dashes. An additional row of perforations is engaged by a toothed wheel driven from the alternator shaft, so that the tape moves forward in strict synchronism with the rotating armature. Two levers provided with small rollers bear at one end on the dot and dash lines of perforations respectively, and close a contact at the other end when a hole in

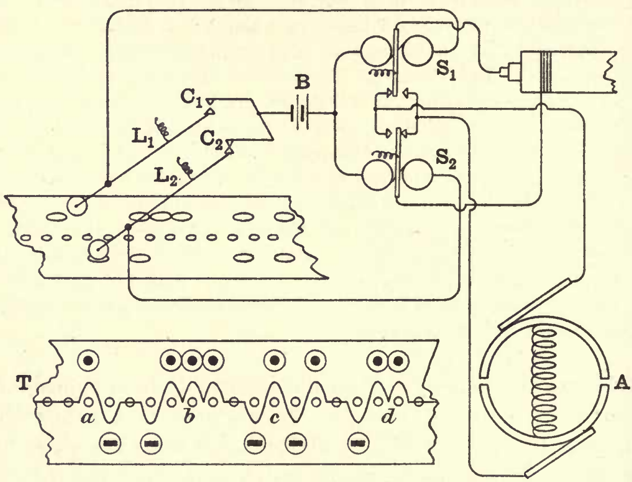

FIG. 125.—TRANSMITTING APPARATUS.

T—Tape, showing letters *a, b, c* and *d*. L_1—Dot lever operating contact C_1. L_2—Dash lever operating contact C_2. S_1—Dot sounder. S_2—Dash sounder. A—Alternator. B—Local battery.

the paper passes beneath. A local battery acting through the closed contact operates the ordinary sounder transmitter shown in Fig. 125, which is otherwise self-explanatory.* The contact levers can be moved and adjusted in position along the tape by means of a micrometer screw, the best position being arrived at by experiment.

* These particulars of the latest form of transmitter have been kindly furnished by Lieut.-Col. Squier.

In view of the claims which this system appears to possess towards furnishing an ideal source of transmitting E.M.F., the cable telegraphist cannot afford to pass it by without a critical examination. The value of such an investigation will be increased by the light which it throws on the phenomena accompanying the switching on and off of alternating currents in long cables—a matter of great practical importance in other ways.

Resolution of a Semi-Oscillation into Two Sustained Oscillations.

A signal of sine shape one semi-period in length, ending abruptly on the zero line, as drawn in Fig. 126, C, is not by any

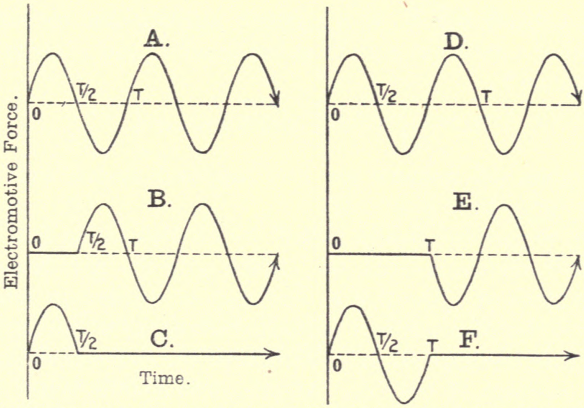

FIG. 126.—Superposition of One Sustained Oscillation on Another to Form an Oscillation One Semi-period or One Whole Period in Length. A+B=C. D+E=F.

means a sine-wave; neither is one a whole period in length, ending abruptly on the zero line, as in Fig. 126, F, although if the signal were repeated a number of times the phenomena in the cable would eventually become sinusoidal in consequence. Nevertheless a signal such as Fig. 126, C can be represented as the sum of two prolonged trains of sinusoidal oscillations, one beginning at time $t=0$, and the other at time $t=\mathrm{T}/2=1/2n=\pi/p$. Thus when Fig. 126, B is superposed on Fig. 126, A the result is Fig. 126, C. Similarly, two trains of oscillations indefinitely continued, the second beginning at time $t=T=1/n=2\pi/p$, and reversed in sign, as in Fig. 126, D, E, when combined give Fig. 126, F.

The problem is therefore reduced to finding what transient current is produced when an alternating current is switched on to a long cable and sustained. This transient current, when found, may be combined with a similar current displaced through the proper interval of time, and the resultant current will be that which is produced by a single semi-oscillation or whole oscillation, as the case may be.

Effect of Switching on a Sinusoidal E.M.F. to a Cable.

When a steady E.M.F., E, is suddenly introduced at the sending end of a cable the current at the receiving end is given by the formula * :—

$$C_r=\frac{E}{Rl}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2 t}{KRl^2}}\cos m\pi\right]. \quad . \quad . \quad (1)$$

Inductance and leakance are here supposed absent, and the cable is to earth at both ends.

Now, replace the steady E.M.F. by an alternating E.M.F., say, $Ee^{ipt}=E(\cos pt+i\sin pt)$. Later, only the part $E\sin pt$ will be retained. The E.M.F. Ee^{ipt} may be divided up into a great number of elementary portions, each equal to $ip\,.\,Ee^{ipt}dt$, which may be supposed to be introduced successively, at equal small intervals of time dt. At the instant $t=0$, when the E.M.F. is introduced, Ee^{ipt} has the value E.

This produces a current at the sending end which at time t_1 later has the value

$$\frac{E}{Rl}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2 t_1}{KRl^2}}\cos m\pi\right]. \quad . \quad . \quad . \quad . \quad (2)$$

An instant later the E.M.F. has increased by $ip\,.\,Ee^{ipdt}\,.\,dt$, which produces an additional current at the receiving end of

$$\frac{ip\,.\,Ee^{ipdt}\,.\,dt}{Rl}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2(t_1-dt)}{KRl^2}}\,.\cos m\pi\right].$$

* Chapter VIII., p. 233.

An instant later again, a still further additional current,

$$\frac{ip\,.\,\mathrm{E}e^{ip\,.\,2dt}dt}{\mathrm{R}l}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2(t_1-2dt)}{\mathrm{K}\mathrm{R}l^2}}\,.\cos m\pi\right]$$

is formed, and so on.

The last will be

$$\frac{ip\,.\,\mathrm{E}e^{ip(t_1-dt)}\,.\,dt}{\mathrm{R}l}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2dt}{\mathrm{K}\mathrm{R}l^2}}\,.\cos m\pi\right],$$

because the received current at time t_1 depends only on what has happened up to time t_1. When these elementary currents are all added up the result is :—

$$\frac{ip\,.\,\mathrm{E}}{\mathrm{R}l}\int_0^{t_1}\left[e^{ipt}dt+2e^{ipt}\,.\,dt\sum_1^\infty e^{-\frac{m^2\pi^2(t_1-t)}{\mathrm{K}\mathrm{R}l}}\,.\cos m\pi\right]$$

$$=\frac{\mathrm{E}}{\mathrm{R}l}\left[\int_0^{t_1}e^{ipt}\,.\,ip\,.\,dt+2ip\sum_1^\infty e^{-\frac{m^2\pi^2t_1}{\mathrm{K}\mathrm{R}l^2}}\,.\cos m\pi\int_0^{t_1}e^{\left(ip+\frac{m^2\pi^2}{\mathrm{K}\mathrm{R}l^2}\right)t}\,.\,dt\right]$$

$$=\frac{\mathrm{E}}{\mathrm{R}l}\left[e^{ipt_1}-1+2ip\sum_1^\infty e^{-\frac{m^2\pi^2t_1}{\mathrm{K}\mathrm{R}l^2}}\,.\cos m\pi\,.\,\frac{e^{\left(ip+\frac{m^2\pi^2}{\mathrm{K}\mathrm{R}l^2}\right)t_1}-1}{ip+\frac{m^2\pi^2}{\mathrm{K}\mathrm{R}l^2}}\right].\quad(3)$$

To (3) add (2), and the total current at time t, dropping the suffix to t—no longer necessary now that the integration has been performed—is

$$\mathrm{C}_r=\frac{\mathrm{E}}{\mathrm{R}l}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2t}{\mathrm{K}\mathrm{R}l^2}}\cos m\pi\right]$$

$$+\frac{\mathrm{E}}{\mathrm{R}l}\left[e^{ipt}-1+2ip\sum_1^\infty\frac{e^{ipt}-e^{-\frac{m^2\pi^2t}{\mathrm{K}\mathrm{R}l^2}}}{ip+\frac{m^2\pi^2}{\mathrm{K}\mathrm{R}l^2}}\cos m\pi\right]$$

$$=\frac{2\mathrm{E}}{\mathrm{R}l}\sum_1^\infty e^{-\frac{m^2\pi^2t}{\mathrm{K}\mathrm{R}l^2}}\cos m\pi+\frac{\mathrm{E}e^{ipt}}{\mathrm{R}l}+\frac{2ip\mathrm{E}}{\mathrm{R}l}\sum_1^\infty\frac{e^{ipt}-e^{-\frac{m^2\pi^2t}{\mathrm{K}\mathrm{R}l^2}}}{ip+\frac{m^2\pi^2}{\mathrm{K}\mathrm{R}l^2}}\cos m\pi.\quad(4)$$

Discussion of the Formula. The Periodic Terms.

When t is infinitely great, the terms in (4) which have a real exponential factor disappear, and there is left the series

$$\frac{\mathrm{E}e^{ipt}}{\mathrm{R}l}\left[1+2ip\sum_{1}^{\infty}\frac{\cos m\pi}{ip+\frac{m^2\pi^2}{\mathrm{KR}l^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This series may be summed as follows :—

Expand $\cosh \mu x$ by Fourier's theorem in a series of cosines—

$$\cosh \mu x=\frac{2\sinh \mu\pi}{\pi}\left[\frac{1}{2\mu}+\sum_{1}^{\infty}\frac{\mu\cos m\pi\cos m\pi}{\mu^2+m^2}\right] \quad . \quad . \quad (6)$$

and put $x=0$. Then

$$\frac{\mu\pi}{\sinh \mu\pi}=1+2\sum_{1}^{\infty}\frac{\mu^2\cos m\pi}{\mu^2+m^2}.$$

For $\mu\pi$ write $\mathrm{P}l$, where P is the propagation constant, and, in the cable under consideration—where inductance and leakance are zero, and $\mathrm{P}=\sqrt{\mathrm{R}\,.\,ip\mathrm{K}}$—it follows that

$$\frac{\mathrm{P}l}{\sinh \mathrm{P}l}=1+2ip\sum_{1}^{\infty}\frac{\cos m\pi}{ip+\frac{m^2\pi^2}{\mathrm{KR}l^2}}. \quad . \quad . \quad . \quad . \quad (7)$$

The series (5) is, therefore, equal to

$$\frac{\mathrm{E}e^{ipt}}{\sqrt{\frac{\mathrm{R}}{ip\mathrm{K}}}\sinh \mathrm{P}l}=\frac{\mathrm{E}e^{ipt}}{\mathrm{Z}_0\sinh \mathrm{P}l}. \quad . \quad . \quad . \quad . \quad (8)$$

where Z_0 is the characteristic impedance. Hence it follows, that when t is infinitely great—*i.e.*, when the alternating E.M.F. has been kept on for a very long time—the general expression (4) for the received current reduces to the ordinary periodic form, as it should, because then all the transient terms have died away.

Discussion Continued. The Transient Terms.

The remaining terms, those with a real exponent, are the transient terms, viz. :—

$$\frac{2\mathrm{E}}{\mathrm{R}l}\sum_{1}^{\infty}e^{-\frac{m^2\pi^2t}{\mathrm{KR}l^2}}\,.\cos m\pi-\frac{2ip\mathrm{E}}{\mathrm{R}l}\sum_{1}^{\infty}\frac{e^{-\frac{m^2\pi^2t}{\mathrm{KR}l^2}}\cos m\pi}{ip+\frac{m^2\pi^2}{\mathrm{KR}l^2}},$$

which may be simplified to

$$\frac{2E}{Rl}\sum_1^\infty \frac{\frac{m^2\pi^2}{KRl^2}e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi}{ip+\frac{m^2\pi^2}{KRl^2}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Multiply above and below by $-ip+\frac{m^2\pi^2}{KRl^2}$, and (9) becomes

$$\frac{2E}{Rl}\sum_1^\infty \frac{\frac{m^2\pi^2}{KRl^2}\left(-ip+\frac{m^2\pi^2}{KRl^2}\right)e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi}{p^2+\left(\frac{m^2\pi^2}{KRl^2}\right)^2}.$$

$$=\frac{2E}{Rl}\sum_1^\infty \frac{e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi}{p^2+\left(\frac{m^2\pi^2}{KRl^2}\right)^2}\left[\left(\frac{m^2\pi^2}{KRl^2}\right)^2-\frac{m^2\pi^2}{KRl^2} . ip\right]. \quad . \quad (10)$$

If the alternating E.M.F. is E sin pt, instead of Ee^{ipt}, the received current, taking the imaginary part of (10), is

$$C_{r\ (\text{transient})}=\frac{2E}{Rl}\sum_1^\infty \frac{e^{-\frac{m^2\pi^2 t}{KRl^2}}\cos \overline{m-1}\pi}{p . \frac{KRl^2}{m^2\pi^2}+\frac{m^2\pi^2}{p . KRl^2}}. \quad . \quad . \quad (11)$$

The Transient Component.

Take $p=2\pi\times 10=62{\cdot}832$, and let formula (11) be applied as before, so as to facilitate comparison with previous results, to the San Francisco–Honolulu section of the Commercial Pacific Cable, for which $Rl=4{,}975$ ohms, $Kl=875$ mfd. and $KRl^2=4{\cdot}352$ seconds. The corresponding values of the denominator of (11) are contained in Table LXXXI., and the values

Table LXXXI.—*Transient Factor.*

m.	$\frac{m^2\pi^2}{KRl^2}$	$\frac{m^2\pi^2}{pKRl^2}$	$\frac{p . KRl^2}{m^2\pi^2}$	$\frac{pKRl^2}{m^2\pi^2}+\frac{m^2\pi^2}{pKRl^2}$.
1	2·2678	0·03609	27·708	27·744
2	9·0704	0·14437	6·9260	7·0700
3	20·413	0·32490	3·0779	3·4028
4	36·283	0·57745	1·7318	2·3092
5	56·690	0·90221	1·1084	2·0106
6	81·636	1·2994	0·7694	2·0688

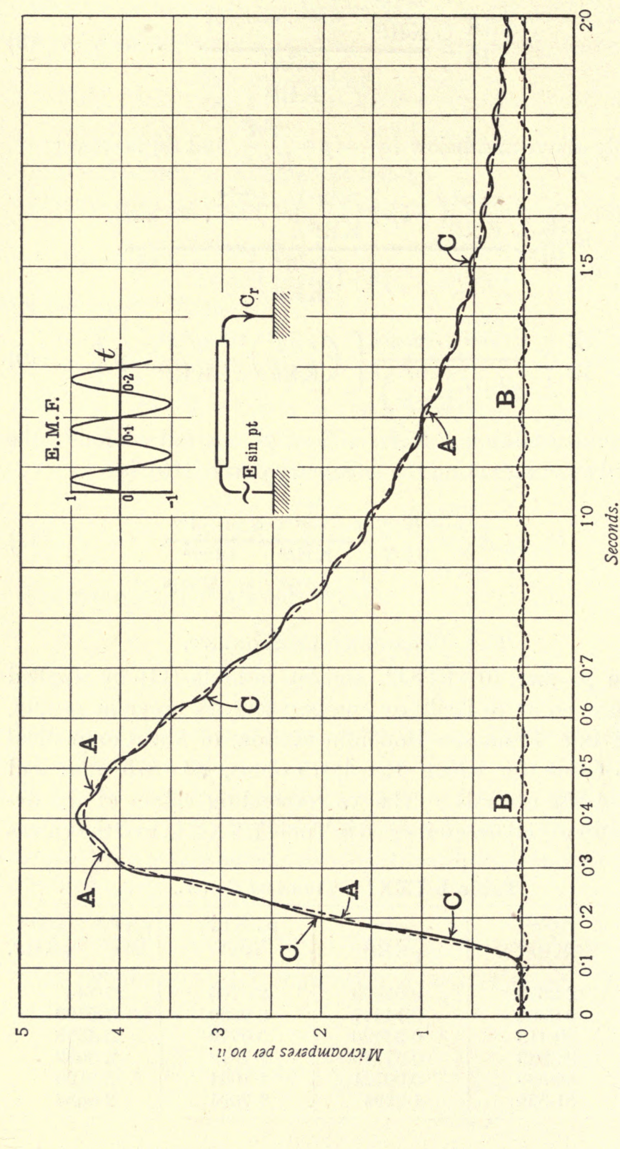

FIG. 127.—ARRIVAL CURRENT, ALTERNATING E.M.F. $n=10$.

Curve A (broken)—Transient component. Curve B (broken)—Periodic component. Curve C—Total current as actually received. C=A+B.

of the numerator have already been tabulated.* From the latter and Table LXXXI., Table LXXXII. is constructed. Curve A, Fig. 127, is plotted from Table LXXXII.

Table LXXXII.—*Fig.* 127, *Curve A.*

t (secs.)	0·1	0·2	0·3	0·4	0·5	0·07	1·0	1·5	2·0	3·0
$m=$ 1 +	11·548	9·205	7·337	5·848	4·662	2·962	1·500	0·483	0·155	0·016
2 −	22·948	9·263	3·740	1·509	0·610	0·100	0·007	...	...	...
3 +	15·338	1·993	0·259	0·032	0·003	...	...	...	...	...
4 −	4·621	0·121	0·004	......	...	...	...	...	...	...
5 +	0·692	...	...	...	...	...	...	...	...	...
6 −	0·054	...	...	...	...	...	...	...	...	...
C_r†	−0·045	+1·814	3·852	4·371	4·055	2·862	1·493	0·483	0·155	0·016

The transient component rises to a maximum value of about 4·4 microamperes per volt in 0·4 second, afterwards slowly declining to zero. In shape it is not unlike the arrival curve from a battery E.M.F. with condensers at the ends of the cable,‡ but it must be oscillatory at the beginning, because when combined with the periodic current to give the total current, as actually received, the result should be practically zero up to about 0·1 second, as in the case of the application of a constant E.M.F. from a battery.

The Periodic Component.

In order to obtain the actual current the periodic component must be added to the transient component. The periodic current is given as in (8) by the formula,

$$C_{r\,(\text{periodic})}=\frac{\mathrm{E}e^{ipt}}{\mathrm{Z}_0 \sinh \mathrm{P}l}.$$

This reduces,§ on substitution for Z_0 and P, to

$$C_r=\frac{\mathrm{E}\sin\left(pt-al+\frac{\pi}{4}\right)}{\sqrt{\frac{\mathrm{R}}{p\mathrm{K}}}\sinh l\sqrt{\frac{1}{2}p\mathrm{KR}}}.$$

* Chapter X., p. 302, Table LVIII.

† Microamperes per volt.

‡ Chapter IX., p. 273.

§ Chapter VIII., p. 243.

When $p=2\pi\times 10$, as before, the formula becomes

C_r (microamperes per volt) $=0\cdot0555 \sin\left(62\cdot832t-11\cdot693+\frac{\pi}{4}\right)$.

so that when $t=0,\ 0\cdot1,\ 0\cdot2\ \ldots$

$$C_r \text{ (microamperes per volt)} = 0\cdot0555 \sin 1\cdot482 = 0\cdot0553.$$

C_r (periodic) is plotted in Fig. 127, curve B, and curves B and A combined give curve C, the current as actually received.

The transient component constitutes by far the greater part of the received current.

Train of Battery Reversals.

In order to see how the transient component is formed, it is instructive to compare the current in Fig. 127 with that which would be produced by a train of battery reversals. This is done in Fig. 128, curve H, and the dotted curves show how it is built up, stage by stage. Curve A is the beginning of the ordinary arrival curve, cable without apparatus, and is calculated from the formula,

$$C_r = 2E\sqrt{\frac{K}{R\pi t}}\,e^{-\frac{KRl^2}{4t}} \text{ up to } t=0\cdot5 \text{ second.}$$

It rises rapidly at first, and then more slowly, to a maximum of 201 microamperes per volt.* If at the end of 0·05 second the sending end is put to earth, as in diagram B, curve A must be displaced through 0·05 second to the right and subtracted from itself. In this way curve B is obtained.† It rises eventually to a maximum of about 13·5 microamperes per volt, and then declines slowly to zero. In curve C, instead of earthing the sending end, the battery is reversed. Curve C falls rapidly to a steady value of -201 microamperes per volt. In curve D, which slowly sinks to zero, the cable is earthed at the end of 0·01 second as in diagram D. When the process here outlined is repeated indefinitely curve H is obtained.‡ It represents the effect of a long-continued train of reversals, and sinks eventually to the zero line, where all that is left of it is

* Chapter VIII., Fig. 76. † Chapter VIII., Fig. 77, Curve B.
‡ Chapter VIII., Fig. 82.

a pure sine oscillation, of amplitude 0·0555 microampere per volt. Up to 0·12 second curve H coincides with curve A; up to 0·17 second its direction is that of curve C; up to 0·22 second it coincides with E, and up to 0·27 second it coincides with G.

At frequency 10 a square-topped *periodic* E.M.F. produces eventually only a pure sine current at the receiving end of the cable.* The periodic current into which curve H eventually sinks is, therefore, the same as the periodic current in Fig. 127,

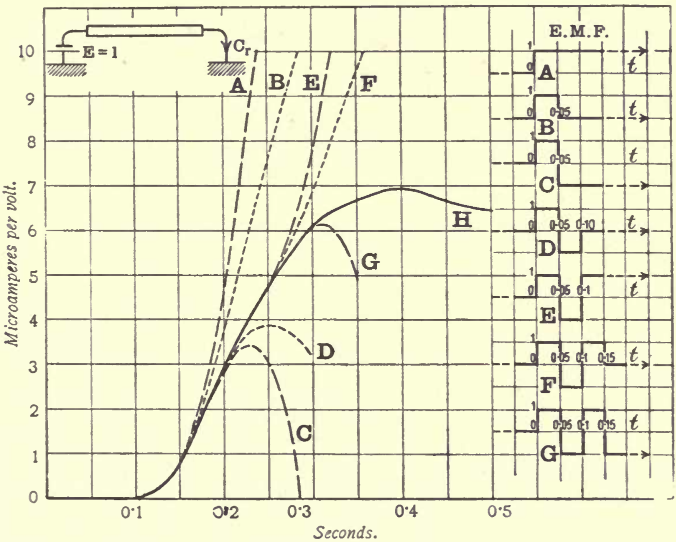

FIG. 128.

Curve A—Sustained battery contact. Curve B—Dot contact. Curve D—Dot-dash. Curve F—Dot-dash-dot. Curve H—Train of reversals.

where a pure sine E.M.F. is used at the sending end. On the other hand, the transient component of the arrival current produced by the square-topped train is much greater than that for the pure sine train. Whereas the ratio of the amplitude of the fundamental of the square-topped wave of E.M.F. to the

* Chapter VIII., p. 248.

amplitude of the pure sine wave is $4/\pi : 1$, or $1{\cdot}27 : 1$, the ratio of the height of curve H, Fig. 128, to that of curve C, Fig. 127, is considerably greater—approximately $1{\cdot}57 : 1$. *Although the periodic currents due to the harmonics vanish at the receiving end the transient components persist.*

Another way of regarding the matter is to write down the square-topped wave of sending E.M.F. as the Fourier series.

$$V_s = \frac{4E}{\pi}[\sin pt + \tfrac{1}{3}\sin 3pt + \tfrac{1}{5}\sin 5pt + \ldots], \quad . \quad (13)$$

and apply (11) to every term in the series.

Since $\frac{p\,.\,KRl^2}{m^2\pi^2}$ is fairly large compared with $\frac{m^2\pi^2}{p\,.\,KRl^2}$, (11) may be written approximately

$$C_r \text{ (transient, due to E } \sin pt) = -\frac{2E}{Rl}\cdot\frac{1}{p}\sum_1^\infty \frac{e^{-\frac{m^2\pi^2 t}{KRl^2}}\,.\cos m\pi}{KRl^2/m^2\pi^2},$$

so that

$$C_r \text{ (transient, due to } V_s) = \frac{4}{\pi}\times C_r \text{ (transient, due to E } \sin pt)$$
$$\times\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]. \quad (14)$$

The ratio C_r (due to V_s) to C_r (due to E $\sin pt$) is, therefore, approximately

$$\frac{4}{\pi}\sum_0^\infty\left(\frac{1}{2m+1}\right)^2 = \frac{\pi}{2},$$

and this formula becomes the more nearly true the greater the value of p.

General Theorem Regarding Mean Value of Sending E.M.F.

The ratio $\pi : 2$ is the same as that of the area of a semi-oscillation of a square-topped signal to that of a sine curve of the same height. In other words, the transient component of the received signal is directly proportional to the mean value in point of time of the applied E.M.F. This theorem may be generalised as follows: Let $F(t)$ be an alternating E.M.F. of any shape, provided only that the semi-oscillation is symmetrical in its rise and fall—*i.e.*, that the curve of the E.M.F. in the second half of the time of application (fall) is the mirror image of that in the first (rise). $F(t)$ can then be represented by the Fourier series—

$$F(t)=A_1 \sin pt+A_3 \sin 3pt+A_5 \sin 5pt+\ldots.$$

This produces, according to the approximate form of (11), when p is large, a transient current at the receiving end.

$$C_r \text{ (transient)}=\frac{2}{\pi . E}\left[A_1+\frac{A_3}{3}+\frac{A_5}{5}+\ldots\right]$$
$$\times[C_r \text{ transient, due to E}].$$

Now, the mean value of $F(t)$ is $\frac{2}{T}\int_0^{T/2} F(t)dt$,

$$=\frac{p}{\pi}\int_0^{\pi/p}[A_1 \sin pt+A_3 \sin 3pt+\ldots]dt,$$

$$=\frac{2}{\pi}[A_1+\tfrac{1}{3}A_3+\tfrac{1}{5}A_5+\ldots],$$

so that C_r (transient, due to $F(t)$) $=\dfrac{\text{mean value of } F(t)}{E}\times C_r$ (transient, due to E).

The periodic current produced would be proportional to the amplitude of the fundamental only, whatever the shape of the sending E.M.F. It will appear later that when a semi-oscillation of a sinusoidal E.M.F. is employed at the sending-end the transient arrival current is alone instrumental in building up the received signals. Hence whatever the *symmetrical* shape of the sending E.M.F., whether square-topped, circular, sinusoidal or triangular, the effect is the same at the receiving end, provided the mean value of the E.M.F. is the same in all cases. *If the time of contact be short the received signal is independent of the shape of the sending (symmetrical) E.M.F. and dependent only on its mean value.*

If the frequency of the E.M.F. is low this would be only approximately true. The criterion is that $p . KRl^2/m^2\pi^2$ should be large compared with $m^2\pi^2/p . KRl^2$. Now if T/2 is the time of the signal (semi-period of alternating E.M.F.), since $T=\frac{1}{n}=\frac{2\pi}{p}$, it follows by substitution for p, that $2KRl^2/m^2\pi T$ must be large compared with $m^2\pi T/2\,.\,KRl^2$. Hence T must be small compared with $2KRl^2/m^2\pi$. In the present instance $T=0{\cdot}05$ second, and $2KRl^2/\pi=2{\cdot}77$ seconds, so that in the sixth term, say, of the series, when $m=6$, the condition no longer holds, as is evident from Table LXXXI., but $e^{-\frac{m^2\pi^2t}{KRl^2}}$ is then very small except for small values of t, so that only at the beginning will the approximate curve differ appreciably from the true.

Influence of Apparatus. Transient Component.

It is necessary now to take account of the apparatus which is used in conjunction with the cable, as it may have a great influence on the phenomena. In particular the effect of the condensers universally employed at the ends of the cable must be considered.

The periodic solution for apparatus Z_s and Z_r at the sending and receiving ends respectively is

$$C_r=\frac{V_s}{(Z_s+Z_r)\cosh Pl+\left(\frac{Z_sZ_r}{Z_0}+Z_0\right)\sinh Pl}, \quad . \quad . \quad (15)$$

and when Z_s and Z_r are condensers, each K_r, the corresponding arrival current, arising from the sudden insertion of a battery E.M.F., E, is

$$C_r{}^{*}=\sum_x \frac{Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x}+\frac{1}{x}\frac{Kl}{K_r}\left(\frac{1}{\sin x}-\frac{\cos x}{x}\right)\right]} \quad . \quad . \quad . \quad (16)$$

where $\tan x=\dfrac{2Kl/K_r}{x-\frac{1}{x}\left(\frac{Kl}{K_r}\right)^2}$.

By the application of the same reasoning as in the case of the cable without apparatus, it follows at once, x taking the place

* Chapter IX., p. 273.

of $m\pi$, that the transient component of the arrival current corresponding to E sin pt is

C_r (transient) =

$$\sum_x \frac{Ee^{-\frac{x^2 l}{KRl^2}}}{Rl\left[\frac{\sin x}{x}+\frac{1}{x}\frac{Kl}{K_r}\left(\frac{1}{\sin x}-\frac{\cos x}{x}\right)\right]} \cdot \frac{1}{\frac{p\,.\,KRl^2}{x^2}+\frac{x^2}{pKRl^2}}. \quad (17)$$

Take $K_s=K_r=\frac{Kl}{10}$, and $p=2\pi\times 10=62{\cdot}832$, as before.

The roots of $\tan x=\frac{20}{x-\frac{100}{x}}$ are given in Table XVI. (Chapter IX., p. 274), and two more are given in Table LXXXIII.

Table LXXXIII.—*Roots of* $\tan x=\frac{20}{x-\frac{100}{x}}$

x	$\frac{x^2}{p\,.\,KRl^2}$.	$\frac{p\,.\,KRl^2}{x^2}$.	Sum.
2·6272	0·0252	39·619	39·644
5·307	0·1030	9·7102	9·8132
8·067	0·2380	4·2012	4·4392
10·909	0·4352	2·2980	2·7332
13·819	0·6983	1·4320	2·1303
16·782	1·0298	0·9709	2·0007
19·786	1·4316	0·6985	2·1301
22·817	1·9039	0·5252	2·4291

Curve A, Fig. 129, is drawn from Table LXXXIV., which is constructed from Table XVIII. (Chapter IX.) by the help of Table LXXXIII.

By comparing curve A with Fig. 127, curve A, it is seen that the greatest value of the transient current is reduced to about one-tenth of the value which it would have without condensers, but this value is reached in time 0·25 sec. instead of in 0·4 second.

At 0·53 second the current is zero, and it is then followed by a long negative tail.

Table LXXXIV.—*Fig.* 129, *Curve A.*

t sec	0·1	0·15	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1-	0·471	0·435	0·402	0·343	0·292	0·250	0·182	0·113	0·051	0·023	0·005
x_2+	4·075	2·948	2·133	1·117	0·585	0·306	0·084	0·012	0·001	...	...
x_3-	7·141	3·382	1·601	0·359	0·081	0·013	0·001	...	...	...	...
x_4+	4·754	1·209	0·308	0·020	0·001	...	...	...	...	...	...
x_5-	1·439	0·160	0·018	...	...	...	...	...	...	...	...
x_6+	0·218	0·009	...	...	...	...	...	...	...	...	...
x_7-	0·018	...	...	...	...	...	...	...	...	...	...
x_8+	0·001	...	...	...	...	...	...	...	...	...	...
*C_r.	−0·021	+0·189	0·421	0·435	0·213	0·039	−0·099	−0·101	−0·051	−0·023	−0·005

* Microamperes per volt.

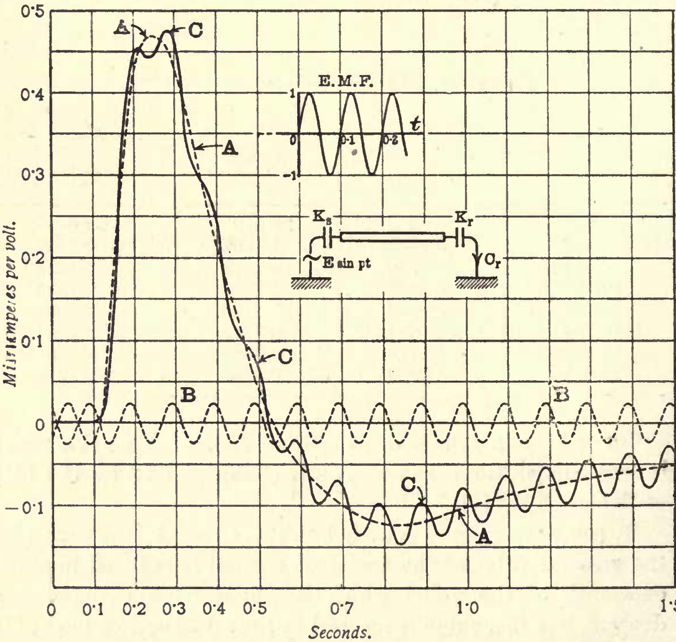

Fig. 129.—Arrival Current, with Condensers. $n=10$.

Curve A—Transient component. Curve B—Periodic component. Curve C—Total current. C=A+B.

Influence of Apparatus. The Periodic Component.

The periodic current is to be obtained from (15). Since Pl is large, sinh Pl=cosh Pl, and the formula becomes

$$C_r \text{ (periodic)} = \frac{\mathrm{E}e^{ipt}}{\left(2\mathrm{Z}_r + \frac{\mathrm{Z}_r^2}{\mathrm{Z}_0} + \mathrm{Z}_0\right) \sinh \mathrm{P}l} = \frac{\mathrm{E}e^{ipt} \div \sinh \mathrm{P}l}{\frac{2}{ip\mathrm{K}_r} - \frac{1}{p^2\mathrm{K}_r^2}\sqrt{\frac{ip\mathrm{K}}{\mathrm{R}}} + \sqrt{\frac{\mathrm{R}}{ip\mathrm{K}}}}$$

Taking the same values for p and K_r as before,

$$\begin{aligned} C_r \text{ (periodic)} &= -0{\cdot}0250 \times 10^{-6} \times \mathrm{E} \sin (pt - 51°37) \quad . \quad . \quad (18) \\ &= -0{\cdot}0250 \times \sin p(t - 0{\cdot}0143) \text{ microamperes per volt.} \end{aligned}$$

When $t=0$, 0·1, 0·2..., $C_r=0{\cdot}0196$.

Curve B, Fig. 129, is plotted from (18). Curve C is formed by adding together curves A and B, and represents, therefore, the current actually received.

Effect of a Single Oscillation and of a Single Semi-Oscillation.

The arrival curves for a continued sine-wave train being known, it is now possible to combine them by superposition and thus to obtain the arrival curve for one whole oscillation of the alternating E.M.F., or for a single semi-oscillation, as explained in Fig. 126. This is carried out in Fig. 130, for the cable with condensers, the curves of Fig. 129 being used for the purpose. Now it is evident that when curve C, Fig. 129, is moved to the right for a half period and then added to itself—as in the plan adopted in Fig. 126, A, B, C—its periodic component will cancel out except for the first semi-period, and the current throughout this first semi-period will cancel with the ripple at the beginning of the transient component, and will therefore be zero. In the construction of the curves, therefore, the transient component will alone be used, the current during the first 0·05 second being nevertheless zero, as just explained. Curve C is reproduced from Fig. 129, curve A, and represents the transient component of the arrival current when the alternating E.M.F. is introduced as it is passing through its zero value, and afterwards sustained. Curve A is formed by adding to curve C itself displaced through 0·05 second. Curve B is formed by subtracting from curve C itself displaced through 0·1 second. Curve A reaches its maximum value of 0·94 microampere per volt in 0·28 second, is zero at 0·56 second, and is followed by a long negative train. Curve B reaches its

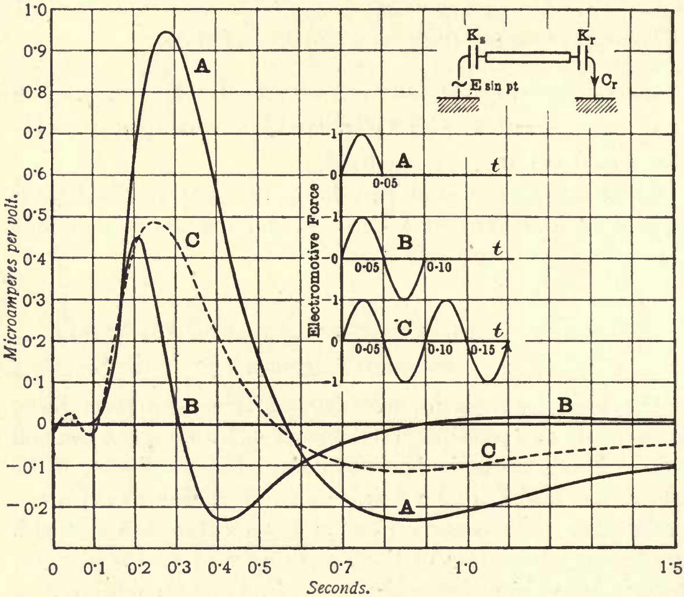

FIG. 130.—ARRIVAL CURRENT, WITH CONDENSERS. SEMI-PERIOD AND WHOLE PERIOD SIGNALS.

Curve A—Semi-period 0·05 second. Curve B—Whole period 0·10 second. Curve C—Continued train of oscillations, transient component. A=C+C′ (C displaced through 0·05 second.) B=C−C″ (C displaced through 0·10 second).

maximum of 0·45 at 0·21 second, but there the effect of the negative impulse begins to tell on the preceding positive one, and the current is zero at 0·3 second, and reaches a minimum value of −0·23 microampere per volt at 0·42 second. Curve C may be looked upon as arising from the continued repetition of curve A, each time displaced through 0·05 second further to the right and with the proper sign.

Curve B reaches its maximum sooner than curve A, and on this score it is better for signalling purposes than curve A. But it is only half the height, and the hollow at 0·42 second would cause the signals to be over-curbed. If the signals are sent as in Fig. 125, without space between the constituents of a letter, diagram B must be sent in half the time (in 0·05 second) if it is to take the place of diagram A. Curve B would then be about one-quarter of its present height, and its shape would not be greatly altered.

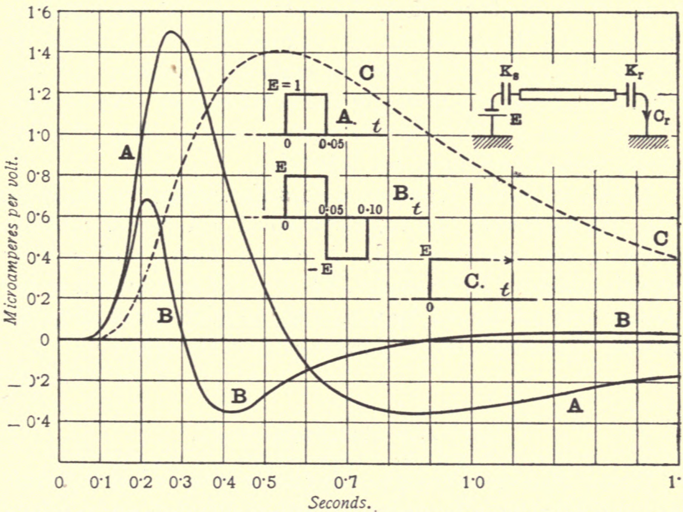

FIG. 131.—ARRIVAL CURRENT, WITH CONDENSERS. BATTERY CURVES.

Curve A—Single 0·05 dot. Curve B—Dot-dash, each lasting 0·05 second. Curve C—Prolonged contact. Curve A=C—C′ (C displaced through 0·05 second). Curve B=A—A′ (A displaced through 0·05 second). Curve C is to scale one-fifth.

BATTERY SIGNALS.

The corresponding signals as they would be formed in the ordinary way by a direct-current source of E.M.F. are drawn in Fig. 131. Curve C is the arrival current due to a single sustained contact with one pole of the battery.* From curve C, curves A and B are constructed. Curve A is found by subtracting from curve C, curve C itself displaced through 0·05 second, and curve B is formed in like manner from curve A.

Allowing for slight inaccuracies in drawing, curves A and B, Fig. 131, are identical in shape with the corresponding curves of Fig. 130, although greater in height and depth. Their greatest depth is, estimating by eye, about 0·352, compared with 0·229 for curves A and B, Fig. 6. The ratio 0·352 : 0·229 is 1·54, which is very nearly the value already found for $p = \infty$.

Battery and alternator produce signals of the same shape, but the battery signals are greater than the alternator signals in the ratio π *to* 2. This result is independent of the size of the signalling condensers. By increasing the voltage of the alternator by 57 per cent. it could be used to replace the battery in every respect as far as the receiving end is concerned. No account is taken here of the armature impedance, the effect of which would probably be slight, but beneficial.

If the sinusoidal impulses were sent without space, as in Fig. 125, successive like impulses would run together before reaching the receiving end. A certain amount of curbing would, therefore, be necessary, which could be adjusted to the precise amount desired by means of the micrometer screw for moving the contact levers along the line of the tape.

Sending-end Phenomena.

It now remains to examine the effects produced at the sending end when an alternating current is switched on to a long cable. The most important of these is the transient rise in voltage at that end. When there is no apparatus in front of the cable the voltage at the sending end is the same at every instant as that of the alternator. When a condenser is interposed the voltage is less, except at the moment of switching on, when the unchanged condenser behaves as though it were short-circuited.

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* Chapter IX., p. 275.

The voltage on the cable, V_0, with sending condenser, for a single battery contact has already been obtained,* and is represented by curve F in Fig. 132. At the instant of making contact V_0 is the same as the battery voltage. It then drops very quickly, and after about 1 second it begins either to approach a steady value or to fall to zero, according to whether the distant end is free or to earth. In the present instance the phenomena just after switching on are of most importance, and the later portion of the curve may here be ignored. It will suffice, therefore, to suppose that the cable has a condenser in front of it, and that it is put to earth at the receiving end.

The periodic voltage V_0 is then given by

$$V_0\,(\text{periodic}) = \frac{Ee^{ipt}}{1+\dfrac{Z_s}{Z_0}\coth Pl} \quad . \quad . \quad . \quad . \quad (19)$$

and the effect of a single battery current is

$$V_0 = \sum_x \frac{Ee^{-\frac{x^2t}{KRl^2}}}{1+\dfrac{x}{\sin x \cos x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Hence, as in (17)—

$$V_0\,(\text{transient}) = \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{1+\dfrac{x}{\sin x \cos x}} \cdot \frac{-1}{p\,.\,\dfrac{KRl^2}{x^2}+\dfrac{x^2}{p\,.\,KRl^2}}, \quad (21)$$

where $$\cot x = x\frac{K_s}{Kl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Take $K_s = Kl/10$. Then (22) becomes $\cot x = x/10$, the roots of which are given in Table XXXVIII.† Fig. 132, curve A, is plotted from (21) and Tables LXXXV. and LXXXVI.

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* Chapter X., p. 311. † Chapter IX., p. 271.

Table LXXXV.—*Transient Factor.*

x	$\frac{pKRl^2}{x^2}$.	$\frac{x^2}{pKRl^2}$.	$\frac{1}{\frac{pKRl}{x^2}+\frac{x^2}{pKRl^2}}$.
1·429	133·93	0·0075	0·00747
4·306	14·750	0·0678	0·06749
7·228	5·2342	0·1911	0·18432
10·200	2·6283	0·3805	0·3324
13·214	1·5662	0·6385	0·4536
16·260	1·0343	0·9669	0·4997

Table LXXXVI.—(*Fig.* 132, *Curve A.*).

t	0·05	0·1	0·2	0·3	0·4	0·5	0·7	1·0
x_1—	1·303	1·272	1·214	1·158	1·105	1·054	0·960	0·834
x_2—	8·487	6·859	4·479	2·925	1·910	1·248	0·532	0·148
x_3—	12·468	6·840	2·059	0·620	0·187	0·056	0·005	...
x_4—	9·400	2·845	0·260	0·024	0·002	...	...	...
x_5—	4·292	0·578	0·010	...	...	...	...	...
x_6—	1·280	0·061	...	...	...	...	...	...
*C_r	−37·230	−18·455	−8·023	−4·727	−3·204	−2·358	−1·497	−0·982

* Millivolts per volt.

The periodic current is obtained from (19), which, when the necessary substitutions have been made, reduces to

$$V_0=\frac{E\sin\left(pt+\tan^{-1}\frac{Kl}{Kl+K_s\sqrt{2pKRl^2}}\right)}{\sqrt{\left(1+\frac{Kl}{K_s\sqrt{2pKRl^2}}\right)^2+\frac{(Kl/K_s)^2}{2pKRl^2}}}, \quad . \quad . \quad (23)$$

from which

$$V_0=671{\cdot}0\sin(62{\cdot}832t+16^\circ\ 40{\cdot}3)\text{ millivolts per volt.} \quad (24)$$

When $t=0$, 0·1, 0·2..., $V_0=192{\cdot}5$ millivolts per volt.

Fig. 132, curve B, is plotted from (24). Curve A is drawn with its zero value equal and opposite to that of curve B, so that the voltage on the cable may be zero when $t=0$. Curve C is found by adding curve A to curve B. Curve F is the voltage curve for a single prolonged contact with one pole of a battery. Curve A resembles curve F in appearance, but is very much smaller, and curve A is small also in comparison with the periodic voltage, curve B. The total voltage, curve C, is almost entirely constituted by the periodic component. *When*

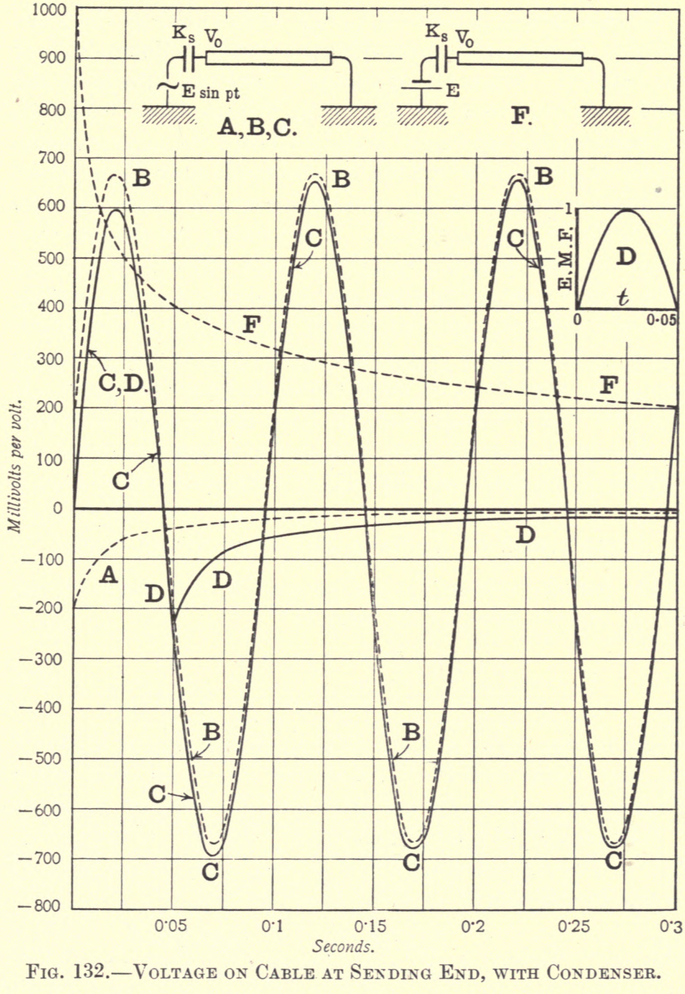

FIG. 132.—VOLTAGE ON CABLE AT SENDING END, WITH CONDENSER.

Curve A—Transient voltage. Curve B—Alternating voltage. Curve C—Total voltage C=A+B. Curve D—Semi-oscillation signal. Curve F—Battery contact sustained.

a sinusoidal E.M.F. is introduced at the sending end of a long cable the voltage on the cable is very nearly purely periodic from the start.

Stress on the Cable.

When a battery is used the voltage on the cable at the sending end, at the moment of making contact, is the full voltage of the battery. When an alternator is used the voltage on the cable rises from zero to a maximum, in this case of 0·60 volt. If, therefore, the alternator is to be allowed the same extreme values as the battery voltage, its amplitude may be increased roughly by two-thirds, the precise amount depending on the size of the sending condenser. If the size of the sending condenser be decreased the received signals will lose in size but gain in definition. The stress on the cable due to the battery will be unaltered, but that due to the alternator will be reduced. The smaller the condenser the greater may the voltage of the alternator be, for the same stress on the cable. If the alternating voltage were $\pi E/2$ the received signals would be the same in both cases, and in the present instance the stresses at the sending-end would be very nearly equal.

Attenuation of Voltage Throughout the Cable.

The periodic voltage $V_{l'}$, at distance l' from the sending-end of the cable is given by the formula

$$V_{l'} = \frac{V_s\,(\cosh P\overline{l-l'} + Z_0/Z_r \sinh P\overline{l-l'}),}{(1+Z_s/Z_r)\cosh Pl + (Z_s/Z_0 + Z_0/Z_r)\sinh Pl} \quad . \quad (25)$$

which may be obtained from (77)* by multiplying as in (114)† by the factor $\cosh P\overline{l-l'} + Z_0/Z_r \sinh P\overline{l-l'}$, so that when $l'=l$, $V_{l'}$ becomes V_l (77) and when $l'=0$, $V_{l'}$ becomes V_0, (114).

When $n=10$, Pl is large, and up to $l'=l/2.$, *i.e.*, as far as, say, half-way along the cable, $P\overline{l-l'}$ is also large, hence $\cosh Pl = \sinh Pl$ and $\cosh P\overline{l-l'} = \sinh P\overline{l-l'}$, so that

$$V_{l'} = \frac{V_s(1+Z_0/Z_r)}{1+Z_s/Z_r+Z_s/Z_0+Z_0/Z_r} \cdot \frac{\sinh P\overline{l-l'}}{\sinh Pl} \quad . \quad . \quad . \quad (26)$$

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* Chapter IX., p. 282. † Chapter X., p. 310.

Take $Z_s = Z_r = \frac{1}{ipK_r}$ as before; $Z_0 = \sqrt{\frac{R}{ipK}}$, and $P = \sqrt{RipK}$.

Then

$$V_{l'} = \frac{V_s\left(1 + ipK_r\sqrt{\frac{R}{ipK}}\right)}{2 + \frac{1}{ipK_r}\sqrt{\frac{ipK}{R}} + ipK_r\sqrt{\frac{R}{ipK}}} \cdot \frac{\sinh \overline{l-l'}\sqrt{RipK}}{\sinh l\sqrt{RipK}}$$

$$= \frac{0{\cdot}671 E \sin\left(pt - 11{\cdot}693\frac{l'}{l} + 16°40{\cdot}3\right) \sinh 11{\cdot}693\left(1 - \frac{l'}{l}\right)}{59860} . \quad (27)$$

Table LXXXVII. is calculated from (27), and curves A, B and C, Fig. 133, are plotted from (27) and Table LXXXVII.

Curve A shows the way in which the amplitude of the sinusoidal oscillation falls off as the distance from the sending-end increases. Curve B shows the phase of the oscillation, relative to the sending end, at every point along the cable. The two curves are combined in curve C, which represents the voltage for the moment when the sending E.M.F. is passing through its zero value. The wave-length of the oscillation is given by $11{\cdot}693 \times \frac{l'}{l} = 2\pi$, or $l' = 0{\cdot}5374 \times l = 1123{\cdot}4$ n.ms., *i.e.*, slightly greater than half the length of the cable.

The corresponding current is to be obtained from the formula

$$C_{l'} = \frac{V_s\,(Z_0 \cosh P\overline{l-l'} + Z_r \sinh P\overline{l-l'})}{Z_0\left[(Z_s + Z_r)\cosh Pl + \left(Z_0 + \frac{Z_r Z_s}{Z_0}\right)\sinh Pl\right]} . \quad . \quad (28)$$

which is related in a similar manner to (21)* and (89),† to which it reduces when $l' = l$, or $l' = 0$ respectively. As before, since the cable is a long one, so that Pl and $P\overline{l-l'}$ are big,

$$C_{l'} = \frac{V_{l'}}{Z_0} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

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* Chapter VIII., p. 241. † Chapter X., p. 297.

$C_{l'}$ is therefore to be obtained from $V_{l'}$ by dividing by Z_0 or $\sqrt{\frac{R}{pK}}$, or by multiplying by $\sqrt{\frac{pK}{R}\Big/\frac{\pi}{4}}$.

The values of C_l' are also contained in Table LXXXVII., from which and from (29) curves A′, B′, C′, Fig. 133, are plotted. Curve A′ shows how the amplitude falls off, curve B′ gives the phase, and curve C′ the current in the cable at the moment when the sending E.M.F. is passing through zero. The current at any point is 45 deg. in advance of the voltage there, and this is always true in a long cable, for the first half of the cable, before the effect of the receiving end apparatus makes itself felt, whatever apparatus may be interposed at the sending end.

In Fig. 133 the alternating E.M.F. is supposed to have been kept on for a very long time, so that all the transient effects have died away. In the corresponding direct E.M.F. case the distribution of voltage would be a straight line at height 83·3 millivolts per volt, or one-twelfth of the applied voltage. The corresponding current would be zero. In actual telegraphic practice the distributions represented by Fig. 133 would never be attained, because the sending E.M.F. would not be maintained sufficiently long.

Table LXXXVII.—*Fig.* 133, *Curves A and A′.*

l'/l	n.ms.	$\sinh 11{\cdot}693\left(1-\frac{l}{l'}\right)$	$V_{l'}$ (millivolts per volt).	$C_{l'}$ (milliamperes per volt).
0	0	59,860	671	2·231
0·01	22·8	53,252	597	1·984
0·02	45·5	47,367	531	1·765
0·03	68·3	42,143	472	1·570
0·04	91·1	37,484	420	1·397
0·05	113·8	33,349	374	1·245
0·07	159·3	26,392	296	0·983
0·10	227·6	18,598	208	0·693
0·15	341·5	10,363	116	0·386
0·20	455·3	5,772·5	64·7	0·215
0·25	569·1	3,217·3	36·06	0·120
0·30	682·9	1,792·5	20·09	0·0678
0·40	910·6	556·1	6·23	0·0207
0·50	1138·2	173·0	1·94	0·0065

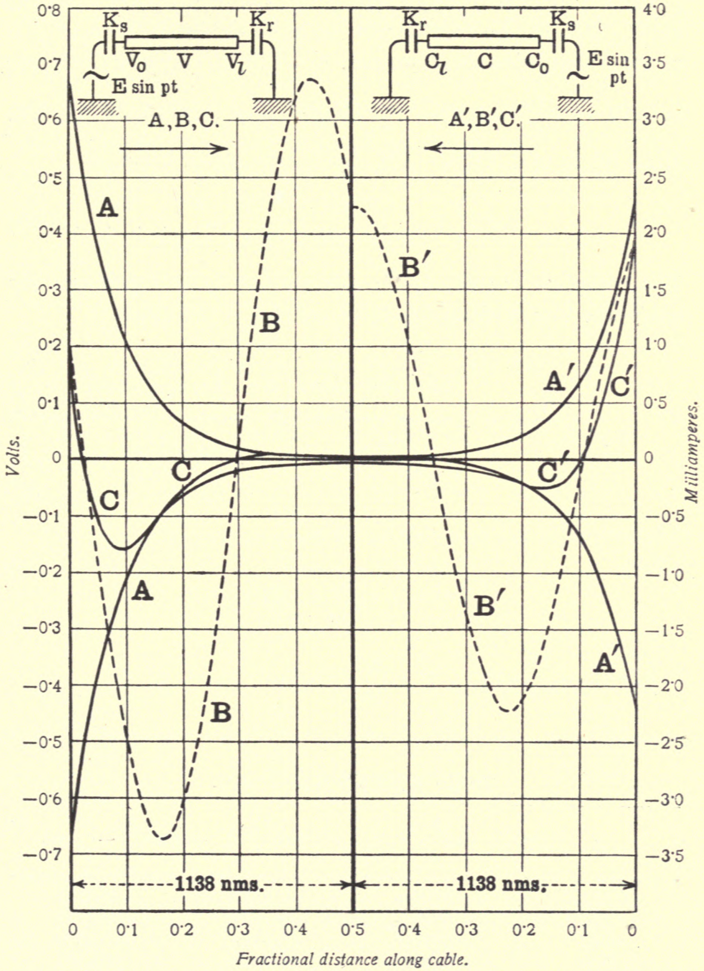

Fig. 133.—Attenuation of Alternating Voltage and Current in Long Cable. $n=10$. E=1.

Curve A—Attenuation of voltage. Curve B—Phase of voltage. Curve C—Voltage at instant when sending E.M.F. is passing through zero. Curves A′, B′, C′—Current curves.

Voltage Growth along Cable. Battery Contact.

In order to obtain the growth of voltage at any point along the cable, (25) must be transformed, so as to express the result of a single prolonged battery contact. Now (25) is (77)*with the factor $\cosh P\overline{l-l'}+\frac{Z_0}{Z_r}\sinh P\overline{l-l'}$. Substituting, according to rule, Rl/xi for Z_0 and $KRl^2/-x^2K_r$ for Z_r, the factor becomes

$$\cos x\left(1-\frac{l'}{l}\right)-\frac{xK_r}{Kl}\sin x\left(1-\frac{l'}{l}\right). \quad . \quad . \quad . \quad (30)$$

The complete expression for the voltage at any point of the cable due to a sustained battery contact is therefore

$$V_{l'}=\frac{E\,.\,K_s}{K_s+K_r+Kl}-\sum_x\frac{2Ee^{-\frac{x^2t}{KRl^2}}\left[\cos x\left(1-\frac{l'}{l}\right)-x\frac{K_r}{Kl}\sin x\left(1-\frac{l'}{l}\right)\right]}{\frac{x^2K_r}{Kl}\left(\frac{1}{\cos x}+\frac{\sin x}{x}\right)+\frac{Kl}{K_s}\left(\frac{\sin x}{x}+\frac{1}{\cos x}\right)} \quad (31)$$

where

$$\tan x=\frac{1+\frac{K_r}{Kl}}{x\frac{K_r}{Kl}-\frac{1}{x}\frac{Kl}{K_s}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Let $l'=l/4$, and $K_s=K_r=\frac{Kl}{10}$ as before.

Table LXXXVIII.—$l'=l/4$.

x	$\cos\frac{3x}{4}$	$\sin\frac{3x}{4}$	$\cos\frac{3x}{4}-\frac{x}{10}\sin\frac{3x}{4}$
2·6272	−0·3891	+0·9212	−0·6311
5·307	−0·6683	−0·7438	−0·2735
8·067	+0·9731	−0·2304	+1·1590
10·909	−0·3215	+0·9469	−1·3546
13·819	−0·5897	−0·8076	+0·5265
16·782	+0·9997	+0·026	+0·9561
19·786	−0·6472	+0·7623	−2·1554
22·817	−0·1640	−0·9864	+2·0869

* Chapter IX., p. 28?.

The values of the factor (30) are contained in Table LXXXVIII. Table LXXXIX. is found by multiplying the data of Table LII.* by the corresponding factor from Table LXXXVIII.

The values of $V_{l/4}$ from the first two columns of Table LXXXIX., where t is small, are to be regarded as only approximate, and more terms of the series would be required to obtain the exact values. Curve B, Fig. 134, is plotted from

Table LXXXIX.—*Fig.* 134, *Curve B.* $l'=l/4$.

t seconds		0·025	0·05	0·1	0·2	0·3	0·4	0·5	0·7	1·0
x_1	+	95·59	91·87	84·87	72·41	61·80	52·74	45·00	32·77	20·36
x_2	−	31·41	26·72	19·32	10·12	5·30	2·77	1·45	0·40	0·03
x_3	−	86·24	59·34	28·09	6·29	1·41	0·31	0·07	...	...
x_4	−	57·25	28·89	7·36	0·48	0·03	...	...	...	...
x_5	−	11·30	3·77	0·42	0·01	...	...	...	...	...
x_6	+	9·45	1·88	0·08	...	...	...	...	...	...
x_7	+	8·89	0·94	...	...	...	...	...	...	...
x_8	+	3·27	0·16	...	...	...	...	...	...	...
Steady	+	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33
†$V_{l/4}$		14·33	59·46	113·09	138·84	138·39	132·99	126·81	115·70	103·63

Table LXL.—$l=l'/2$.

x	$\cos\frac{x}{2}$	$\sin\frac{x}{2}$	$\cos\frac{x}{2}-\frac{x}{10}\sin\frac{x}{2}$
5·307	−0·8833	+0·4689	−1·1321
10·909	+0·6759	−0·7369	+1·4798
16·782	−0·5116	+0·8592	−1·9536
22·817	+0·4012	−0·9160	+2·4913

Table LXLI.—*Fig.* 134, *Curve C.* $l'=l/2$.

t seconds		0·025	0·05	0·1	0·2	0·3	0·4	0·5	0·7	1·0
x_2	−	130·00	110·57	79·96	41·88	21·93	11·48	6·01	1·65	0·24
x_4	+	62·54	31·57	8·04	0·52	0·30	...	...	...	...
x_6	−	19·31	3·83	0·16	...	...	...	...	...	...
x_8	+	3·91	0·20	...	...	...	...	...	...	...
Steady	+	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33	83·33
†$V_{l/2}$		0·47	0·70	11·25	41·97	61·70	71·85	77·32	81·68	83·09

* Chapter IX., p. 288. † Millivolts per volt.

Table LXXXIX. Curves A and D of the same figure are reproduced from Fig. 104* and Fig. 96, curve B.† Curve C of Fig. 134 is plotted from Table LXLI., constructed in a similar way from Table LXL. It represents the growth in voltage for a sustained battery contact at a point half way along the

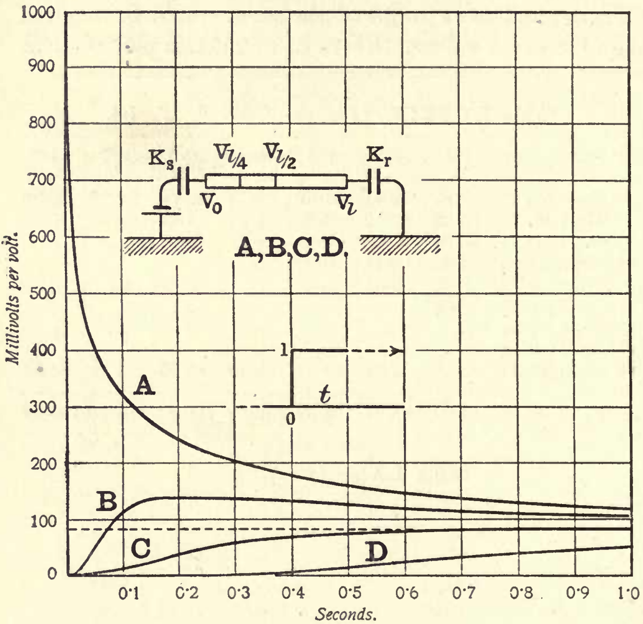

FIG. 134.—VOLTAGE GROWTH AT DIFFERENT POINTS ALONG CABLE. SUSTAINED BATTERY CONTACT.

Curve A—Voltage at sending end. Curve B—Voltage one-quarter way along. Curve C—Voltage half-way. Curve D—Voltage at receiving end.

cable. Table LXLI. has only half as many rows as Table LXXXIX. because the odd roots of (32) make (30) vanish.

Curve B rises quickly from zero, passes beyond the final or steady value at 0·07 second, reaches a turning value of 150 millivolts per volt at about 0·2 second, and thereafter declines

* Chapter X., p. 311. † Chapter IX., p. 288.

slowly to the final value of 83·3 millivolts per volt, one-twelfth of the battery voltage. Curve C rises slowly to the steady value, and the slope is positive throughout. In curve D the rate of growth is still smaller.

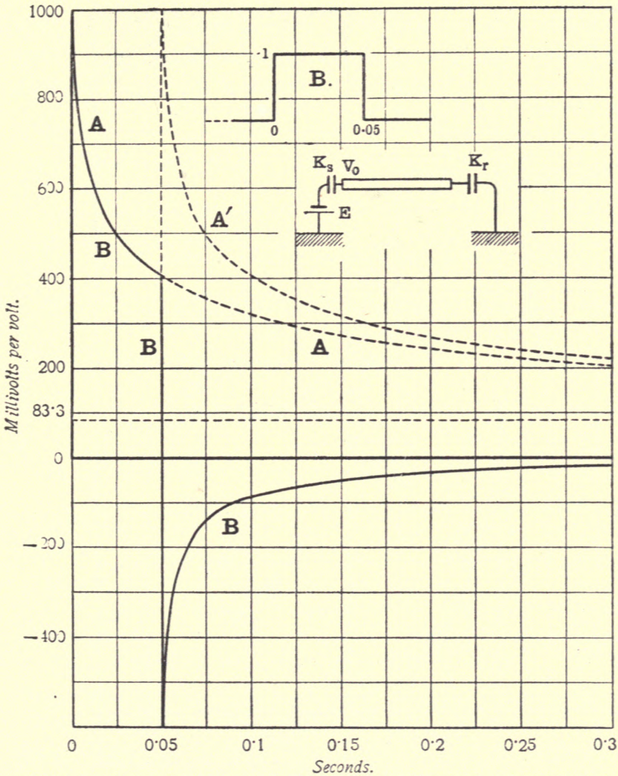

FIG. 135.—DOT SIGNAL LASTING 0·05 SECOND. VOLTAGE AT SENDING END.

Curve A—Sustained contact. Curve B—Difference of A and A′. Curve A′—Curve A displaced through 0·05 second.

DEFORMATION OF BATTERY SIGNAL IN TRANSMISSION.

The growth of voltage at various points along the cable being now known the curves can be combined so as to give the shape of the signal at these points. This is carried out for the sending-end in Fig. 135. Curve A is the voltage-curve corresponding to a sustained contact, as in Fig. 134, A. Curve A′ is the same curve displaced through 0·05 second. Curve B is the

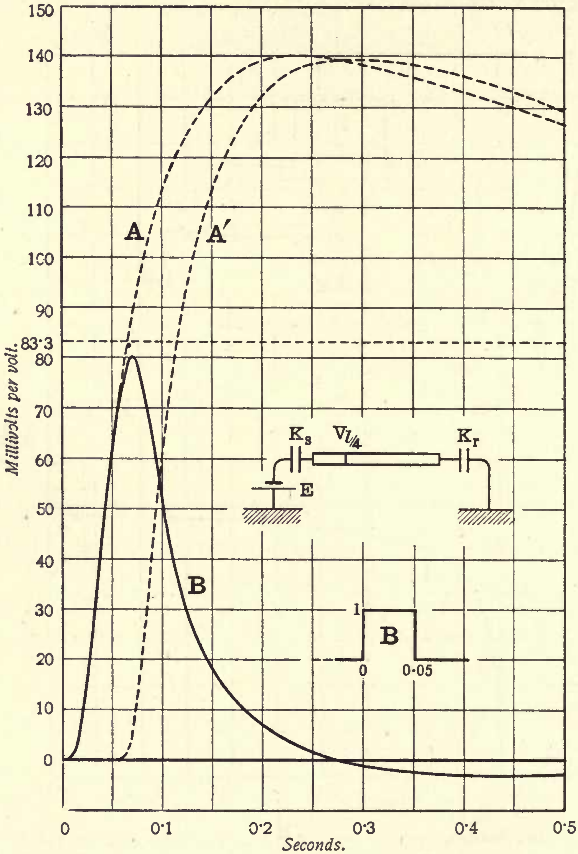

FIG. 136.—DOT SIGNAL LASTING 0·05 SECOND. VOLTAGE ONE-QUARTER-WAY ALONG CABLE.

Curve A—Sustained contact. Curve A′—Curve A displaced through 0·05 second. Curve B—Difference between A and A′.

difference between curves A and A′, and represents the voltage at the sending end corresponding to an elementary, or dot, signal lasting $\frac{1}{20}$ second.

The voltage at the moment of making contact, when the sending-end condenser is without charge, is that of the battery. As the sending-end condenser charges up the voltage on the cable falls rapidly. At $t=0{\cdot}05$ second, the sending-end is put to earth, and the voltage on the cable drops to -590 millivolts per volt, afterwards slowly returning to zero. The positive part of the signal occupies 0·05 second and the negative tail about 0·15 second, a total of 0·20 second.

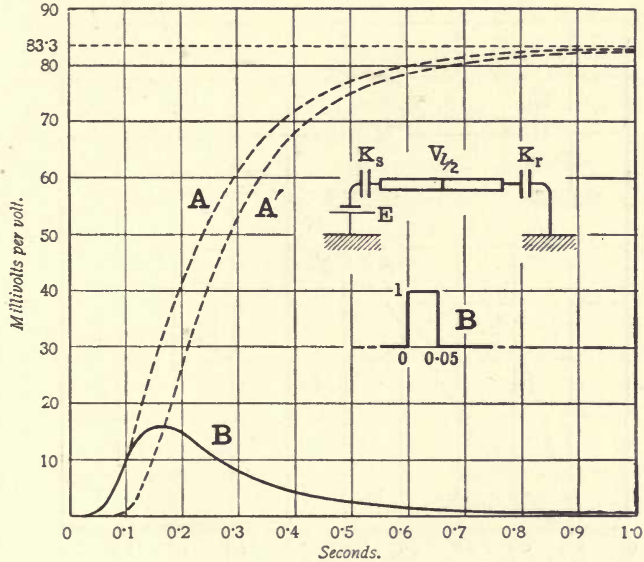

FIG. 137.—DOT SIGNAL LASTING 0·05 SECOND. VOLTAGE HALF-WAY ALONG CABLE.

Curve A—Sustained contact. Curve A′—Curve A displaced through 0·05 second. Curve B—Difference between A and A′.

The same procedure is followed in Fig. 136. Curves A and A′ are reproductions of curve B, Fig. 134. Curve B, found by subtracting curve A′ from curve A, gives the shape of the signal when it has travelled one quarter of its way in its journey through the cable.

The voltage now starts from zero and rises steeply in about 0·075 second to a maximum of 80 millivolts per volt. It then declines, and is zero at 0·28 second, on which there follows a long negative tail before the voltage is finally zero. The signal still occupies about 0·2 second, but its significant portion is practically all positive. Already at one quarter of the journey the sharpness at make and break of contact is completely gone from the signal.

In Fig. 137 curves A and A′ are reproductions of curve C, Fig. 134, and curve B is obtained by subtracting A′ from A.

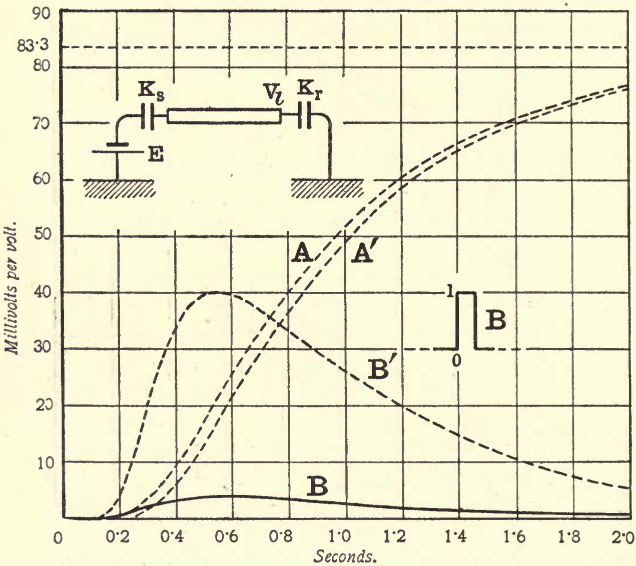

FIG. 138.—DOT SIGNAL LASTING 0·05 SECOND. VOLTAGE ON RECEIVING CONDENSER.

Curve A—Sustained contact. Curve A′—Curve A displaced through 0·05 second. Curve B—Difference between A and A′. Curve B′=Curve B to scale 10 times A and A′.

The negative tail is now completely gone, and the signal is entirely positive. The maximum of 16 millivolts per volt is not reached until 0·17 second, and the whole signal occupies more than ½ second.

Curves A and A′, Fig. 138, are reproduced from Fig. 134, curve D, and curve B is their difference, as before. Curve B′

is curve B to 10 times the scale. Curves A and A′ are too close together for an accurate estimation of their difference, and they have been calculated at more frequent intervals (every 0·05 second) to enable curve B′ to be drawn with sufficient exactitude.

The signal reaches its maximum of 4 millivolts per volt in 0·54 second, and the whole signal requires nearly 2 seconds before it can be said to approach completion.

Semi-period Signal at the Sending End.

Before the shape can be calculated which a single semi-oscillation, from a sinusoidal source of E.M.F., takes as it passes along the cable, it is necessary, as explained above, first to obtain the curve representing the way in which a sustained sinusoidal oscillation establishes itself. For the sending end this has already been done in Fig. 132, curve C. The voltage curve at the sending end of the cable, produced by a single semi-oscillation of a sinusoidal E.M.F. is therefore to be obtained by adding to curve C itself displaced through 0·05 second. Now curve B, the periodic component of curve C, cancels out with itself after 0·05 second, and up to 0·05 second it unites with curve A to form curve C. From $t=0{\cdot}05$ second onwards the required curve D is to be obtained by adding curve A to itself displaced to the right through 0·05 second.

Curve D rises to a maximum of 600 millivolts in 0·02 second, which is followed by a negative dip to 230 millivolts, and a long tail. Curve D is to be compared with curve B, Fig. 135. The extreme range in the latter case is from $+1$ to $-0{\cdot}59$ volt. In curve D, if the sending amplitude were increased to 1·57 volt ($=\pi/2$) the range would be from $+0{\cdot}94$ to $-0{\cdot}36$, or 1·30 volt.

It is to be observed that the sharp peak which marks the beginning of the battery signal in Fig. 135, curve B, is absent from Fig. 132, curve D. Instead, the voltage rises smoothly to its maximum. In Fig. 135, curve B, the same sudden change in the voltage as before, now from 0·41 to $-0{\cdot}59$ volt, is repeated when the cable is put to earth to conclude the signal. In Fig. 132, curve D, the shock is not so great. The voltage falls smoothly to $-0{\cdot}36$ volt ($0{\cdot}23\times1{\cdot}57$), and then declines to zero. The dip in this case is neither so great nor so steep as when the battery is used.

From curve B, Fig. 136, it is seen that the sharp corners of the signal are rounded off when it has travelled a little way into the cable. But it is precisely at the sending end that they are likely to do most harm, in upsetting the duplex balance. The more gradual the rise in the voltage at the sending end the more easy will it be to obtain a perfect balance, so that more sensitive receiving instruments can be used without their being liable to disturbance from the signals sent out at their own end.

Deformation of Semi-period Signal in Transmission.

To obtain the shape of the signal at other points along the cable, the curve representing the growth, at the point in question, of a sustained alternating voltage must first be found. The transient component of this is formed as before by multiplying the corresponding expression for a sustained battery-contact by the appropriate factor

$$\frac{-1}{p\,.\,\frac{\mathrm{KR}l^2}{x^2}+\frac{x^2}{p\,.\,\mathrm{KR}l^2}}$$

so that the complete expression becomes

$$\mathrm{V}_{l'}\,(\text{transient})=\sum_x \frac{2\mathrm{E}e^{-\frac{x^2t}{\mathrm{KR}l^2}}\left[\cos x\left(1-\frac{l'}{l}\right)-\frac{x\mathrm{K}_r}{\mathrm{K}_s}\sin x\left(1-\frac{l'}{l}\right)\right]}{\frac{x^2\mathrm{K}_r}{\mathrm{K}l}\left(\frac{1}{\cos x}+\frac{\sin x}{x}\right)+\frac{\mathrm{K}l}{\mathrm{K}_s}\left(\frac{\sin x}{x}-\frac{1}{\cos x}\right)}$$

$$\times\frac{1}{p\frac{\mathrm{KR}l^2}{x^2}+\frac{x^2}{p\mathrm{KR}l^2}}\quad .\quad (33)$$

The first part of this expression, for $l'=l/4$, is contained in Table LXXXIX., and Table LXLII. is formed from Table LXXXIX. by multiplying its terms by the appropriate values of the factor contained in the last column.

Table LXLII.—(*Fig.* 139, *Curve A*). $l'=l/4$.

t sec	0·025	0·05	0·10	0·20	0·30	0·40	0·50	0·70	1·00	$\frac{1}{\frac{pKRl^2}{x^2}+\frac{x^2}{pKRl^2}}$
x_1-	2·412	2·318	2·141	1·827	1·559	1·331	1·135	0·827	0·514	0·02523
x_2+	3·200	2·722	1·969	1·031	0·540	0·283	0·148	0·041	0·006	0·10191
x_3+	19·428	13·369	6·329	1·418	0·319	0·071	0·016	...	...	0·22528
x_4+	20·943	10·568	2·690	0·175	0·010	...	...	...	...	0·3658
x_5+	5·306	1·771	0·198	0·002	...	...	...	...	...	0·4694
x_6-	4·724	0·937	0·038	...	...	...	...	...	...	0·4998
x_7-	4·176	0·441	...	...	...	...	...	...	...	0·4695
x_8-	1·347	0·068	...	...	...	...	...	...	...	0·4117
*$V_{l/4}$	36·218	24·666	9·007	0·799	−0·690	−0·977	−0·971	−0·786	−0·508	...

* Transient component, millivolts per volt.

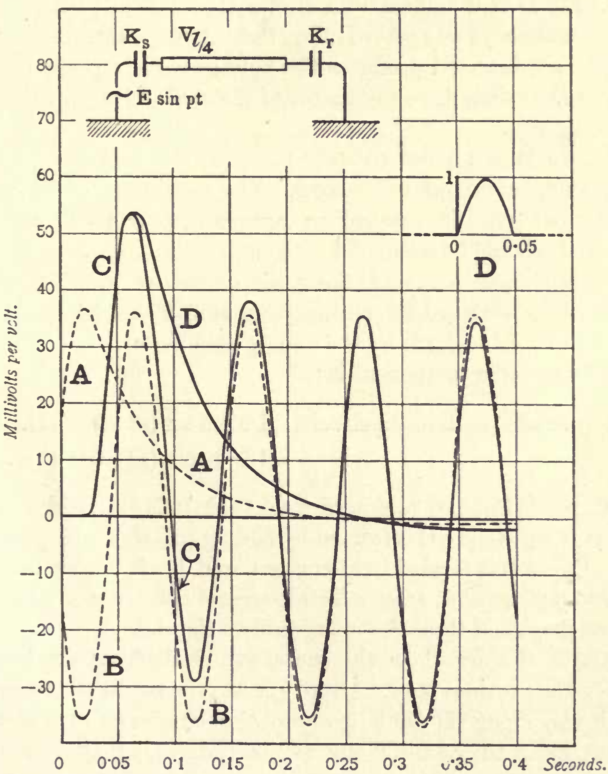

FIG. 139.—ALTERNATING VOLTAGE ONE-QUARTER-WAY ALONG CABLE. $n=10$.

Curve A—Transient component. Curve B—Periodic component. Curve C—Sum of A and B, sustained alternating voltage. Curve D—Dot signal.

The periodic component one-quarter way along is, from Table LXXXVII.,

$$V_{l/4}\,\begin{matrix}\text{(periodic, milli-}\\ \text{volts per volt)}\end{matrix} = 36{\cdot}06 \sin\left(pt - \frac{11{\cdot}693}{4} + 0{\cdot}2910\right)$$

$$= -36{\cdot}06 \sin p(t + 0{\cdot}0081). \quad . \quad . \quad (34)$$

When $t=0$, $V_{l/4}$ (periodic) $= -17{\cdot}59$ millivolts per volt.

Curve B, Fig. 139, is plotted from (34), and curve C is formed by adding curves A and B together. The first semi-oscillation in the positive direction is now much the biggest, and there is a general rise at the beginning of the curve during the first few oscillations which was absent from curve C, Fig. 132.

Curve D is obtained from curves C and A in precisely the same manner as was curve D, Fig. 132. A single semi-oscillation at the sending end produces the voltage-wave represented by curve D at distance one-quarter of the cable's length from the sending-end.

Curve D is almost entirely positive, the negative portion beginning at about 0·3 second. The maximum of 54 millivolts per volt is reached in approximately 0·065 second. Curve D should be compared with curve B, Fig. 136.

At distance half-way along the cable the transient component is obtained by multiplying, as in Table LXLIII., the terms of Table LXLI. by the appropriate factor.

The periodic component is :—

$$V_{l/2}\,\text{(periodic, millivolts per volt)} = 1{\cdot}9395 \sin(pt + 0{\cdot}7277),$$
$$= 1{\cdot}9395 \sin(pt + 0{\cdot}01158). \; (35)$$

Curve A, Fig. 140, is plotted from Table LXLIII., and curve B from (35). Curve C is formed by adding together curves A and B. Curve D is formed from curves C and A. It follows curve C up to 0·05 second, after which it consists of curve A added to itself displaced through 0·05 second to the right.

Curve C shows how the successive oscillations are heaped upon the previous ones. Already, when the signal has travelled half-way along the cable, it approaches as a wave with a steep front rising to 7·3 millivolts per volt, declining afterwards to

Table LXIII.—(*Fig.* 140, *Curve A*). $l' = l/2$.

t sec	0·025	0·05	0·10	0·20	0·30	0·40	0·50	0·70	1·00	$\dfrac{1}{\dfrac{pKRl^2}{x^2}+\dfrac{x^2}{pKRl^2}}$
x_2+	13·246	11·267	8·147	4·268	2·234	1·170	0·613	0·168	0·024	0·1019
x_4-	22·877	11·547	2·940	0·191	0·108	...	...	...	...	0·3658
x_6+	9·651	1·915	0·078	...	...	...	...	...	...	0·4998
x_8-	1·608	0·081	...	...	...	...	...	...	...	0·4117
*$V_{l/2}$	−1·588	+1·554	5·285	4·077	2·126	1·170	0·613	0·168	0·024	...

* Millivolts per volt.

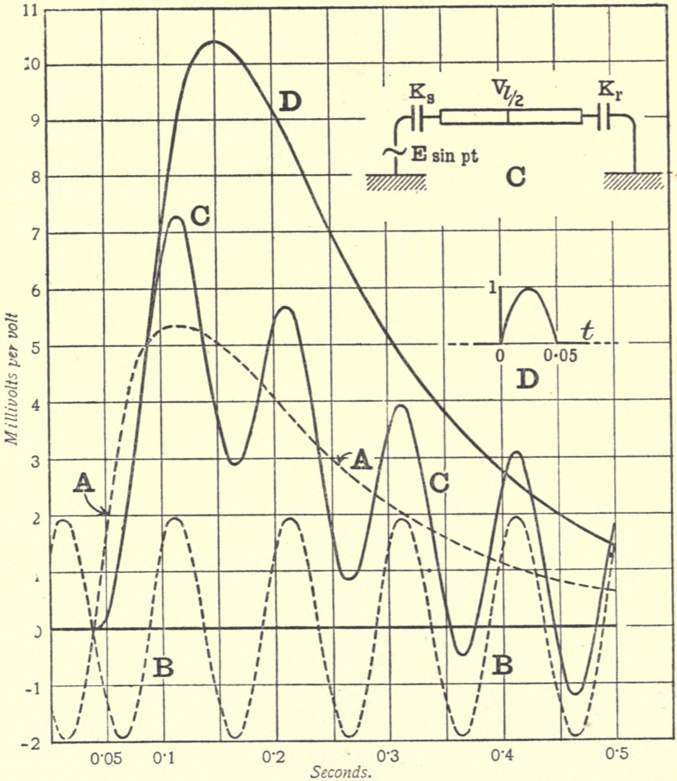

FIG. 140.—ALTERNATING VOLTAGE HALF-WAY ALONG CABLE. $n=10$.

Curve A—Transient component. Curve B—Periodic component. Curve C—Sum of A and B, sustained alternating voltage. Curve D—Semi-oscillation signal.

an oscillation of ± 2 millivolts per volt. The first oscillatiod shrinks back, as it were, into the following ones, which crown upon it and are piled up on top of one another in the wave front.

Curve D has a maximum of 10·3 millivolts per volt, which is reached in 0·15 second. Curve D should be compared with curve B, Fig. 137.

At the end of the cable the transient component is obtained by multiplying as in Table LXLIV. the terms of Table LII.,* giving the arrival voltage by the appropriate factor.

The periodic component at the end of the cable is :—

$$V_l \text{ (periodic, millivolts per volt)} = 0{\cdot}00455 \sin (pt + 0{\cdot}6700),$$

$$= 0{\cdot}00455 \sin p(t + 0{\cdot}0107). \quad (36)$$

Fig. 141, curve A, is plotted from Table LXLIV. The periodic component, as given by (36), is too small to show. Curve D is obtained by adding curve A to itself displaced through 0·05 second. The transient component of the arrival current may be obtained from curve A by multiplying the slope by $87{\cdot}5 \times 10^{-6}$ according to the equation

$$C_r = K_r \frac{dV_l}{dt}.$$

If C_r is in microamperes per volt, and V_l in millivolts per volt,

then $$C_r \text{ (transient)} = 0{\cdot}0875 \frac{dV_l}{dt}. \quad . \quad . \quad . \quad . \quad (37)$$

C_r (periodic) is to be obtained by multiplying V_l (periodic) by $K_r p \Big/ \frac{\pi}{2}$, so that, with the same units as before,

$$C_r \text{ (periodic)} = 5{\cdot}20 \times V_l \Big/ \frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad (38)$$

In this way curves A and B, Fig. 129, might be obtained from curve A, Fig. 141, and from (38), and from them curve C, Fig. 129, which is the slope of curve C, Fig. 141 (curve A with the periodic ripple superimposed) might be constructed.

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* Chapter IX., p. 288.

Table LXLIV.—(*Fig.* 141, *Curve A*). $l'=l$.

t sec	0·025	0·05	0·10	0·20	0·30	0·40	0·50	0·70	1·0	1·5	2·0	3·0
x_1+	3·82	3·67	3·39	2·90	2·47	2·11	1·80	1·310	0·81	0·37	0·17	0·03
x_2-	11·70	9·95	7·20	3·77	1·97	1·03	0·54	0·15	0·02	...	...	...
x_3+	16·76	11·54	5·46	1·22	0·27	0·06	0·01	...	...	...	...	...
x_4-	15·46	7·80	1·99	0·13	0·01	...	...	...	...	...	...	...
x_5+	10·08	3·36	0·38	0·01	...	...	...	...	...	...	...	...
x_6-	4·94	0·98	0·01	...	...	...	...	...	...	...	...	...
x_7+	1·94	0·20	...	...	...	...	...	...	...	...	...	...
x_8-	0·65	0·03	...	...	...	...	...	...	...	...	...	...
*V_l	−0·15	+0·01	0·01	0·22	0·77	1·14	1·27	1·16	0·79	0·37	0·17	0·03

* Transient, millivolts per volt.

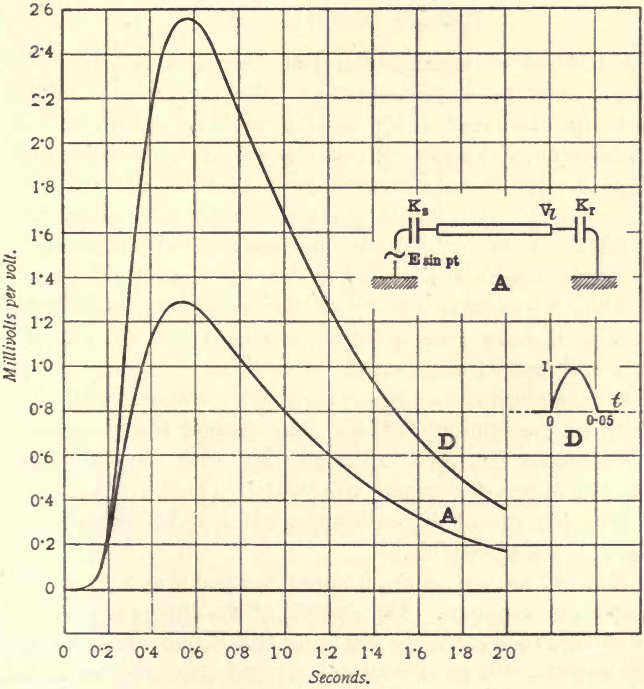

FIG. 141.—ALTERNATING VOLTAGE AT END OF CABLE. $n=10$.

Curve A—Transient component. Curve D—Signal formed by semi-oscillation of sending E.M.F.

The arrival voltage on the receiving condenser due to an alternating E.M.F. at the sending end is, as shown by Fig. 141, curve A, almost purely transient in nature, the alternating ripple being too small to show. The ripple is much more pronounced in the current arrival curve Fig. 129, curve C, because the receiving condenser admits more readily the periodic component of the current, according to (38), than it does the transient component, as given by (37), which is governed by the gradual rise and fall of the arrival voltage.

Curve D should be compared with curve B, Fig. 138 (obtained in an entirely different way), with which it will be found to coincide if its ordinates are multiplied by 1·55.

Recent Improvements.

The method of transmission just described has recently undergone several improvements.* The most important of these is that the circuit at the sending end is never broken, and alternating current flows without cessation through cable and apparatus. It is to the extreme fluctuations in the sending-end current which the breaking of the circuit produces that the chief disturbances in the duplex balance are to be attributed. Accordingly signals are formed merely by cutting out partly or in whole a resistance inserted at the sending end, so that the signals shall have greater amplitude than the spaces. As signals and spaces are formed by utilising successive semicycles of the alternating current there is a continual reversal of polarity in the sending E.M.F. The method thus combines the advantages described in Chapter XI. with the additional recommendations of a smooth rise and fall in E.M.F., free moreover from the violent discontinuities which would accompany the breaking of the circuit.

In Fig. 142 are shown the letters *a*, *b*, c and *d* as they would be sent on this system. The current at the spaces is just not great enough to actuate the receiving apparatus. The current at dot and the still greater current at dash are adjusted to be just sufficient to actuate the receiving mechanism, the under-

* G. O. Squier, "Proc." Phys. Soc., 27, 1915, pp. 540-563.

lying principle being to preserve as far as possible the smooth alternating current normally flowing into the cable in the absence of a message. The tape t is driven in synchronism with the generator, and the perforations in it are of such a size as to correspond to a semi-cycle of the alternating current. When a perforation occurs the tongue moves over and makes contact with S′, thus short-circuiting R and sending a dash. Duplicate mechanism short-circuits part of the resistance for dots through the connection l. Micrometer adjustments ensure that all operations take place at zero current. At the receiving end a gold wire relay* makes contact for dots with the stops f_2 which are adjustable in position and are joined together. The

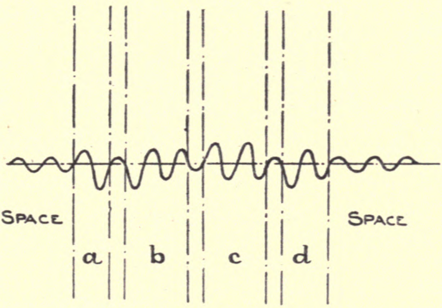

FIG. 142.—CURRENT WAVES FOR TRANSMITTING THE LETTERS *a, b, c* AND *d* BY THE ALTERNATING CURRENT.

greater current of a dash bends the wire round the stops and causes it to make contact with f_3. F′ is a local wire relay which behaves in a similar way and actuates the electro-magnets hh' which control the printing levers gg'. For dots g marks in the centre line of the tape, and for dashes g' also

* In the Muirhead Gold Wire Relay the gold wire is stretched between platinum contacts, one end being attached to the siphon tube of a recorder or similar instrument, and the other end to a vibrator. When the recorder coil is actuated by the arrival currents it deflects the siphon tube and brings the gold wire up against one or other of the platinum posts. The sawing action of the vibrator ensures good contact.

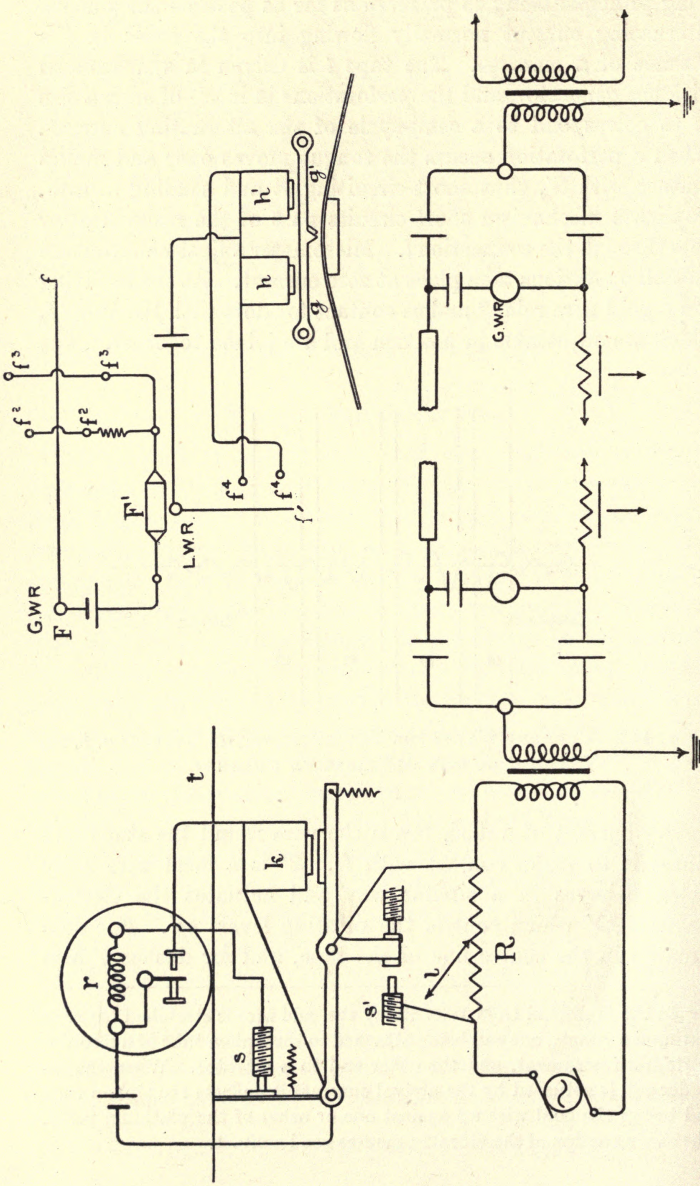

FIG. 143.—ONE FORM OF DUPLEX CIRCUIT ARRANGEMENT FOR TRANSMITTING AND RECEIVING MESSAGES BY THE ALTERNATING CURRENT.

marks simultaneously on both sides of the central mark. The whole forms a delicate receiver which interprets the fluctuations in amplitude of the arriving message and prints it in Morse characters.

In Fig. 142 the spaces between the constituents of a letter are

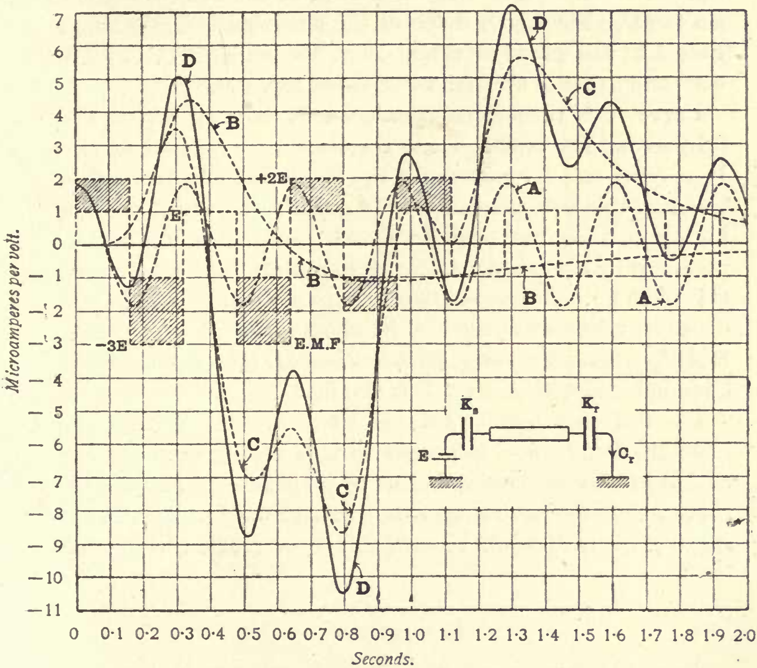

Fig. 144.—Form on Arrival of Letters A B.

Curve A. Continuous sine current constituting spaces; T = 0·32 second.
,, B. Single dot lasting 0·16 second.
,, C. Additional current due to letters. Curve D. Sum of A and C.

omitted and the space between letters is reduced to one semi-cycle, instead of being three times the length of a dot or dash, as assumed previously. Hence instead of 34 elements for the four letters only 17 semi-cycles are needed. Instead of taking the element at 0·08 second as in Chapter XI., we may increase the semi-period to 0·16 second without loss of sending speed, and make the periodic time 0·32 second, or the frequency $n=3{\cdot}125$ and $p=19{\cdot}64$.

In Fig. 144 the letters *a* and *b* are supposed to be transmitted on the system just described, except that battery contacts are used instead of the sine wave. Approximately the same signals would be given by an E.M.F. of the form $V=(\pi E/2) \sin 19{\cdot}64t$. The exact shape of the sine signals could be obtained by the use of an extension of Table LXXXIV., used to calculate in detail the arrival curves of Fig. 130.

Curve A is the current produced by battery reversals at 0·16 second intervals. These reversals are supposed to have been going on long enough for the steady state to be reached. Even at this low frequency it is a pure sine wave, the first harmonic being too small to show. Curve B is the current produced by a single dot lasting 0·16 second, from which curve C is built up. It represents the current which would be produced by that part of the E.M.F. which is shaded. The added E.M.F. is taken here as E for a dot and 2E for a dash. When C is combined with A curve D is obtained.

The first four signals are well marked. At the fifth and sixth the influence of the negative tails arising from the two dashes in close succession begins to preponderate and lift the curve above the horizontal axis. A slight adjustment in the shape of curve B would be sufficient to overcome this defect.*

Conclusion.

From the foregoing investigation it appears that a sine-wave of E.M.F., with equal conditions of maximum stress on the cable, will give the same signals at the receiving end as the square-topped wave which is at present employed. At the sending end, on the other hand, the rise in voltage due to the sinusoidal E.M.F. is a more gradual one, tending to facilitate the preservation of the duplex balance. Moreover the sine-wave transmitter has certain characteristics which are wanting

* *See* Chapter XIV., Part V., for fuller information on this subject.

in the battery. Thus it possesses great adaptability, inasmuch as it contains within itself both generator and transmitter. Phase-relationship and wave-shape are perfectly under control, rendering it for experimental purposes invaluable. By its aid the effective values of the cable constants, particularly of the inductance, could be determined, numerical data for which are long overdue. In telephony the change has been from primary battery to secondary cell, and more recently to dynamo. It is not unreasonable to suppose that the same change may follow in cable telegraphy.

Part V.—The Future Progress of Cable Telegraphy.

CHAPTER XIII.

DISTORTION AND HOW IT IS PRODUCED.

Introduction—Single Mesh Circuit—Circuit of Two Meshes—Three Meshes—Four Meshes—Five Meshes—Any Number of Meshes—Passage to the Distributed Case—Criterion for Uniform Subdivision—Sending Currents—The "KR" Law and the Cable Core.

Introduction.

Nearly half a century has elapsed since the first Atlantic cable was successfully laid.* During that time the arts of making and of laying submarine telegraph cables have been brought to a great state of perfection. The construction of a long cable no longer excites comment, and to such an extent has the element of chance been eliminated that in the specification of a new cable it is usual to stipulate under guarantee the precise date at which it shall be open for traffic. At the same time the signalling apparatus used in conjunction with the telegraph cable has been continually improved, and as a consequence the working speed of cables has been greatly increased. Indeed, so far has this process been carried that the consensus of telegraphic opinion appears to be that finality has almost been attained. The future, it is believed, holds out the promise only of refinements in existing practice and of modifications in detail.

In what follows the attempt is made to ascertain how far this standpoint is justified. It is proposed to pass in review the various means by which the commercial value of a cable—

* In 1866. Previous attempts in 1857, 1858 and 1865.

as measured by its carrying capacity—may be increased, and to expose them all to a critical and, as far as possible, quantitative examination. The present position of cable telegraphy may be gathered from certain descriptions which have recently appeared.* But so far no endeavour, however imperfect, seems to have been made to forecast the lines on which the future of cable telegraphy will run. For such an endeavour to be of value it is obvious that mere speculation must be avoided, and that no deduction may be made for which adequate justification does not exist.

In entering upon an undertaking of so formidable a nature it is fortunate that we have ready to hand a powerful weapon in the mathematical theory of cable phenomena. It is the crucial test of a scientific theory that not only should it explain satisfactorily known phenomena, but it should also be capable of pointing the way to new developments. Although it may be unavoidable in the early stages that theory should be unsupported by experimental evidence, nevertheless when that experimental confirmation shall be forthcoming the principles which it involves will be on the eve of incorporation in the practice of the day.

The working speed of a land telegraph is that of the end apparatus. If the speed at which the message can be impressed on the line, and still more if the speed at which the receiving apparatus works can be increased, the line itself will offer little objection. In the case of cable telegraphy an attempt to increase the speed of transmission may result only in rendering the messages indecipherable at the far end.

In order that a clear idea may be obtained of the way in which the distributed capacity and the resistance of a cable combine to retard and distort the signals which are transmitted through it, it is proposed to begin by the study of a few simple cases by way of illustration, gradually leading up to those more complicated. The advantage of this method of

* H. Larose, "Rapport sur l'Etat Actuel de la Télégraphie Sousmarine," "Lum. Elect.," August 7, 1909. M. Roscher, "60 Jahre technischer Entwicklung der Unterseetelegraphie," "E.T.Z.," XXXIII., p. 664 p. 714, p. 740, 1912.

procedure will be evident when we come to consider in what manner inductance can be regarded as acting counter to capacity to reduce distortion.

SINGLE MESH CIRCUIT.

Suppose that, in the first place, the capacity of the cable is concentrated, as in Fig. 145, in a single condenser, of total capacity Kl, where K is the capacity of the cable per unit length and

Table XCV.—*Fig.* 145.

t(sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
ϵ^{-t/KRl^2}	0·977	0·955	0·933	0·912	0·891	0·851	0·795	0·709	0·632	0·502
C(mm.a. per volt)	196·5	192·0	187·6	183·4	179·2	171·1	159·7	142·4	127·0	100·9

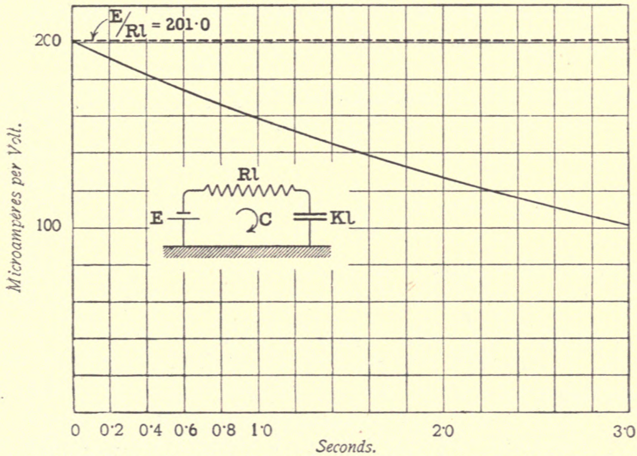

FIG. 145.—CHARGING A CONDENSER THROUGH A RESISTANCE.
Kl=875 mfd. Rl=4,975 ohms. KRl^2=4·352 seconds.

l the length, and let this be joined in series with a coil of resistance Rl, where R is the resistance per unit length of the cable. Let there be a periodic E.M.F., say, $E\epsilon^{ipt}$ in the circuit, and suppose it to have been maintained for so long a time that all the initial effects have subsided and the phenomena are all periodic and of the same period as the impressed E.M.F. The current C is then given by the symbolic extension of Ohm's law in the form

$$C=\frac{Ee^{ipt}}{Rl+\dfrac{1}{ipKl}}=\frac{E/pt-\tan(1/pKRl^2)}{\sqrt{R^2l^2+1/p^2K^2l^2}}\quad . \quad . \quad . \quad . \quad (1)$$

The periodic solution, although of vital importance in telephony, is of comparatively small interest to the telegraphist. His signalling is carried on by sudden and irregular changes of an E.M.F., which is inserted in and withdrawn from the circuit at the sending-end.

Instead of the periodic current, we require to know the transient current which is produced by a constant E.M.F. E when it is suddenly inserted in the circuit. The method of obtaining the transient solution from the periodic solution has already been fully explained.* Following the rule, in equation (1), write y for ip and it becomes

$$C=\frac{Ee^{yt}}{Rl+\dfrac{1}{yKl}}=\frac{Ee^{yt}}{\varphi(y)}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here $\qquad \varphi(y)=0$ gives $Rl+\dfrac{1}{yKl}=0$, or $y=-\dfrac{1}{KRl^2}$.

$$\varphi'(y)=-\frac{1}{y^2Kl}.$$

$$\therefore\ y\varphi'(y)=-\frac{1}{yKl}=Rl. \quad \text{Also } \varphi(0)=\infty.$$

Hence, finally,

$$C_{(\text{transient})}=E\left[\frac{1}{\varphi(0)}+\frac{e^{yt}}{y\varphi'(y)}\right]=\frac{Ee^{-t/KRl^2}}{Rl}.\quad . \quad . \quad (3)$$

This familiar result might also have been obtained from the differential equation of the circuit by trial integration and determination of the constants in the usual manner. But as the circuits grow more complicated this alternative method

* Chap. IX., p. 264.

soon becomes unworkable, whereas the method just illustrated —by transformation from the periodic case—follows closely the physical process, and gives the desired result in every case in a straightforward and simple manner. When a little experience has been gained in its use it soon becomes easy to handle, and one would not think of returning to the older method even in the simple cases in which alone that is an alternative.

The significance of formula (3) may be illustrated by applying it to some typical cable. In order that the results now to be obtained may be directly comparable with previous results, let Rl be 4,975 ohms and Kl be 874·6 mfd., as in the San Francisco-Honolulu section of the Commercial Pacific Cable. If it be desired to apply the results to any other cable conversion formulæ may be used.* Table XCV. is constructed from (3), and Fig. 145 is drawn from (3) and Table XCV.

The current at the moment of closing the circuit is given by the equation $C=\frac{E}{Rl}$, just as though there were no condenser there, but it at once begins to fall off as the condenser charges up, and the decline continues until the current is zero. Even at the end of 3 seconds the current has lost only one half of its initial value.

Circuit of Two Meshes.

Let there now be two meshes as shown in the diagram of Fig. 146. Here the resistance is split up into two coils, and the capacity is joined up between the junction of the two coils and earth. Let C_1 and C_2 be the periodic currents in the two meshes; C_1-C_2 is then the current through the condenser.

Write down the cyclic equations for the two circuits. From the first circuit

$$\frac{Rl}{2}\cdot C_1+\frac{1}{ipKl}(C_1-C_2)=Ee^{ipt}$$

and from the second

$$\frac{1}{ipKl}(C_2-C_1)+\frac{Rl}{2}C_2=0 \qquad \ldots\ldots (4)$$

* Chap. X., p. 326.

or, on re-arrangement,

$$\left.\begin{aligned} &\left(\frac{Rl}{2}+\frac{1}{ipKl}\right)C_1-\frac{C_2}{ipKl}=Ee^{ipt} \\ \text{and} \quad &\frac{-C_1}{ipKl}+\left(\frac{Rl}{2}+\frac{1}{ipKl}\right)C_2=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

The simultaneous equations (5) in C_1 and C_2 may be solved by substitution, or by the aid of determinants :—

$$\text{Thus} \quad C_{2(\text{periodic})}=\begin{vmatrix} \frac{Rl}{2}+\frac{1}{ipKl} & Ee^{ipt} \\ -\frac{1}{ipKl} & 0 \end{vmatrix} \div \begin{vmatrix} \frac{Rl}{2}+\frac{1}{ipKl} & -\frac{1}{ipKl} \\ -\frac{1}{ipKl} & \frac{Rl}{2}+\frac{1}{ipKl} \end{vmatrix}$$

$$=\frac{Ee^{ipt}}{Rl\left[1+\frac{ipKRl^2}{4}\right]}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

on reduction.

The periodic solution being found, the first step is now complete. The next step is to transform it so as to obtain from it the transient solution.

Following the rule,

$$\varphi(y)=Rl\left[1+\frac{KRl^2}{4}\,.\,y\right].$$

$$\text{from which} \qquad \varphi'(y)=Rl\,.\,\frac{KRl^2}{4}.$$

Also $\varphi(y)=0$, gives $y=-\frac{4}{KRl^2}$ $\therefore$ $y\varphi'(y)=-Rl.$

and $\varphi(0)=Rl.$

Hence, finally,

$$C_{2(\text{transient})}=\frac{E}{Rl}\left[1-e^{-\frac{4t}{KRl^2}}\right]. \quad . \quad . \quad . \quad . \quad (7)$$

Table XCVI. is constructed from (7), and Fig. 146, curve A, is plotted from (7) and Table XCVI.

The current now starts from zero, because at the moment of making contact the uncharged condenser acts as a shortcircuit to the second mesh. As the condenser charges up, less current flows through it to earth, and more through the second mesh, until the steady value E/Rl is reached. The condenser is then charged to half the potential of the battery instead of

Table XCVI.—*Fig.* 146, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
e^{-4t/KRl^2}	0·912	0·832	0·759	0·692	0·632	0·526	0·399	0·252	0·159	0·063
C_2(mm.a. per volt)	17·7	33·8	48·4	61·8	74·1	95·4	120·8	150·4	169·0	188·3
C_1(mm.a. per volt)	384·4	368·3	353·6	340·2	327·9	306·6	281·2	251·6	233·0	213·8

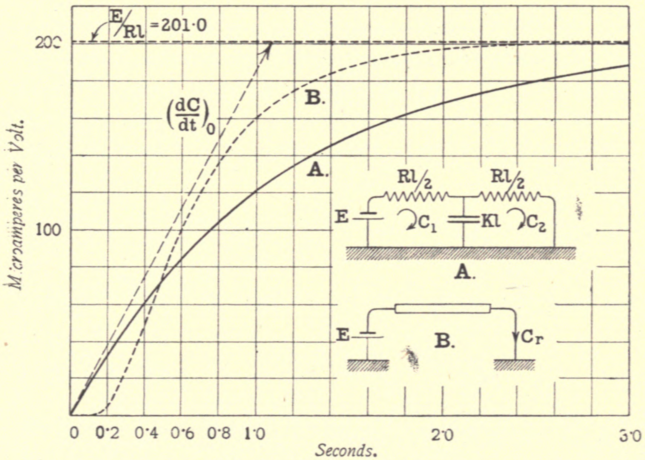

Fig. 146.—Two Meshes.

Curve A.—Current in the second mesh.

,, B.—Current in case of continuous distribution.

the whole potential, as in the former case of a single mesh. In three seconds the current has now nearly attained its maximum.

Differentiating (7) with respect to time,

$$\frac{dC_2}{dt}=\frac{E}{Rl}\cdot\frac{4}{KRl^2}\cdot e^{-\frac{4t}{KRl^2}}$$

so that when $t=0$, $\left(\frac{dC_2}{dt}\right)_{t=0}=\frac{4E}{Rl \, . \, KRl^2}=184{\cdot}74$ microamperes/seconds in the present instance. This is the direction of the tangent to the curve at the origin, represented by the straight line in Fig. 146. Curve B, Fig. 146, is the Kelvin arrival curve, in which case the capacity, instead of being contained in a single condenser, is distributed uniformly along the conductor. The tangent to this curve at the origin is horizontal, and the current does not start up at once as in curve A.

Accordingly we must go a stage further before we can obtain a first approximation to the distributed case.

Three Meshes.

In Fig. 147 the conductor resistance is represented by three coils each $Rl/3$, and the capacity by two condensers, each $Kl/2$. The periodic current may be found exactly as before. From the three mesh equations the expression for the current in the third mesh is written down as the quotient of two determinants,

$$C_3=\begin{vmatrix} \frac{Rl}{3}+\frac{2}{yKl} & -\frac{2}{yKl} & Ee^{yt} \\ -\frac{2}{yKl} & \frac{Rl}{3}+\frac{4}{yKl} & 0 \\ 0 & -\frac{2}{yKl} & 0 \end{vmatrix} \div \begin{vmatrix} \frac{Rl}{3}+\frac{2}{yKl} & -\frac{2}{yKl} & 0 \\ -\frac{2}{yKl} & \frac{Rl}{3}+\frac{4}{yKl} & -\frac{2}{yKl} \\ 0 & -\frac{2}{yKl} & \frac{Rl}{3}+\frac{2}{yKl} \end{vmatrix}$$

which are then evaluated. In this way the periodic current is found to be

$$C_{3(\text{periodic})}=\frac{108\times Ee^{ipt}}{Rl[(ip \, . \, KRl^2)^2+24ip \, . \, KRl^2+108]}. \quad (8)$$

The transient current is now to be obtained from (8). Here

$$\varphi(y)=0 \text{ gives } y \, . \, KRl^2=-6 \text{ or } -18.$$

Again, $\quad \varphi'(y)=Rl \, . \, KRl^2[2y \, . \, KRl^2+24]$.

from which $y\varphi'(y)=-72Rl$, or $216Rl$.

Also $\quad \varphi(0)=108Rl$.

Table CXVII.—*Fig.* 147, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
e^{-6t/KRl^2}	0·871	0·759	0·661	0·576	0·502	0·381	0·252	0·126	0·063	0·016
e^{-18t/KRl^2}	0·661	0·437	0·289	0·191	0·126	0·055	0·016	0·002	...	...
C (mm.a. per volt).	4·8	16·1	30·7	46·5	62·4	91·7	126·7	163·1	181·9	196·2
$_3C_1$ (mm. a.p'r v'lt)	530·2	473·8	429·4	393·9	365·0	321·4	278·6	239·3	220·2	205·8

Table CXVIII.—*Fig.* 147, *Curve B.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$e^{-\frac{24+12\sqrt{2}}{KRl^2}t}$	0·851	0·724	0·616	0·524	0·446	0·323	0·199	0·089	0·040	0·008
$e^{-\frac{24}{KRl^2}t}$	0·576	0·332	0·191	0·110	0·063	0·021	0·004	...	...	...
$e^{-\frac{24+12\sqrt{2}}{KRl^2}t}$	0·390	0·152	0·059	0·023	0·009	0·001	...	...	...	...
C_4 (mm.a. per volt)	1·9	10·4	24·6	42·0	60·2	94·4	133·6	170·6	187·4	198·3
$_4C_1$ (mm.a. per volt)	631·7	525·1	454·3	404·3	367·3	316·1	270·0	231·5	214·6	203 7

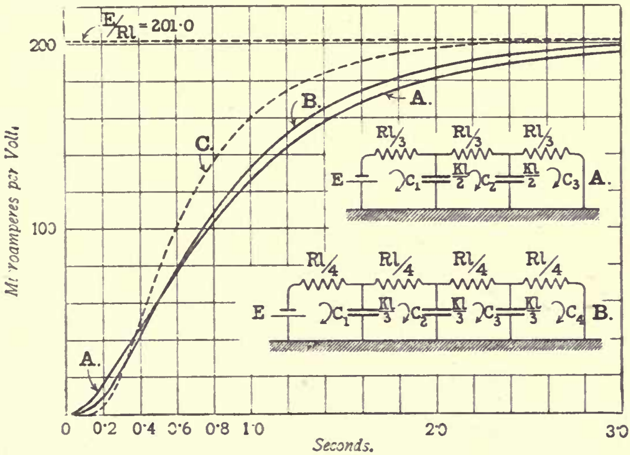

Fig. 147.—Received Currents.

Curve A.—Current in third mesh.
,, B.—Current in fourth mesh.
,, C.—Current in case of continuous distribution.

Hence, finally,

$$C_{3(\text{transient})}=\frac{E}{Rl}\left[1-\frac{3e^{-\frac{6t}{KRl^2}}}{2}+\frac{e^{-\frac{18t}{KRl^2}}}{2}\right] \quad . \quad . \quad . \quad (9)$$

Table XCVII. is constructed from (9), and curve A, Fig. 147, is plotted from (9) and Table XCVII.

Differentiating (9) with respect to t, and putting t zero, it follows that $\left(\frac{dC_3}{dt}\right)_{t=0}=0$. The tangent to curve A is horizontal at the origin, and the curve coincides initially with the horizontal axis. Three meshes, as in Fig. 147, may be said, therefore, to constitute the first approximation to the continuously distributed case.

Four Meshes.

When there are four meshes, arranged as in the diagram on Fig. 147, the current in the fourth mesh, when the E.M.F. E is suddenly inserted, at time $t=0$, in the first mesh, is

$$C_4=\frac{E}{Rl}\left[1-\frac{e^{-\frac{24+12\sqrt{2}t}{KRl^2}}}{2-\sqrt{2}}+e^{-\frac{24}{KRl^2}t}-\frac{e^{-\frac{24-12\sqrt{2}t}{KRl^2}}}{2+\sqrt{2}}\right] \quad . \quad . \quad (10)$$

Table XCVIII. is calculated from (10), and plotted in Fig. 147, curve B.

It is seen that C_4 overtakes C_3 at time 0·57 sec. Up to this point C_4 is less than C_3; after crossing, C_4 is greater than C_3. The effect of increasing the sub-division into meshes is to steepen the current arrival curve, by cutting away from it initially, and adding to it subsequently, so as to make it a nearer approach to the case of infinite sub-division, curve C.

Five Meshes.

When there are five meshes the current in the fifth mesh is given by the formula—

$$C_5=\frac{E}{Rl}\left[1-\frac{e^{-\frac{10t}{KRl^2}(3-\sqrt{5})}}{1-\frac{1}{\sqrt{5}}}+\frac{e^{-\frac{10t}{KRl^2}(5-\sqrt{5})}}{3-\sqrt{5}}-\frac{e^{-\frac{10t}{KRl^2}(3+\sqrt{5})}}{1+\frac{1}{\sqrt{5}}}+\frac{e^{-\frac{10t}{RKl^2}(5+\sqrt{5})}}{3+\sqrt{5}}\right] \quad (11)$$

Table XCIX.—*Fig.* 148, *Curve A.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$e^{-\frac{10t(3-\sqrt{5})}{KRl^2}}$	0·839	0·704	0·591	0·496	0·416	0·293	0·173	0·072	0·030	0·00 5
$e^{-\frac{10t}{KRl^2}(5-\sqrt{5})}$	0·530	0·281	0·149	0·079	0·042	0·012	0·002	...	...	...
$e^{-\frac{10t}{KRl^2}(3+\sqrt{5})}$	0·300	0·090	0·027	0·008	0·002	...	...	...	...	...
$e^{-\frac{10t}{KRl^2}(5+\sqrt{5})}$	0·190	0·036	0·007	0·001	...	...	...	...	...	...
C_5 (mm.a. per volt)	0·3	7·7	21·9	40·5	60·5	97·7	138·6	174·9	190·2	199·1
${}_5C_1$(mm.a. per volt)	693·8	544·6	458·9	403·1	363·5	310·5	264·3	227·2	211·9	202·9

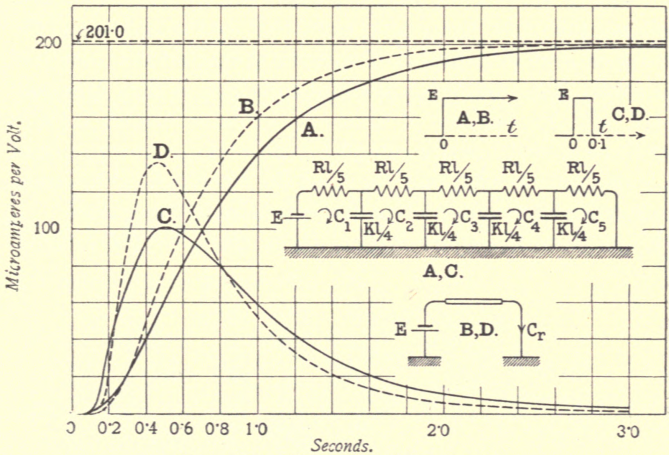

FIG. 148.—RECEIVED CURRENTS.

Curve A.—Current in fifth mesh.
,, B.—Current in case of continuous distribution.
Drawn to scale five times. { ,, C.—Tenth of second dot corresponding to A.
,, D.—Tenth of second dot corresponding to B.

Table XCIX. is calculated from (11), and plotted in Fig. 148, curve A. Curve B is the current in the distributed case. As the number of meshes is increased the point at which the two curves cross approaches the origin. The arrival curve, in the distributed case, is steeper than the current curve for a finite number of meshes, and permits, therefore, of a higher speed of signalling. This is further illustrated in curves C and D, representing the shape on arrival of a dot signal lasting $\frac{1}{10}$th second. The turning value of curve D is not only greater than that of curve C, but it also anticipates it in point of time. Curve D is, moreover, more acute than curve C; for all of which three reasons curve D would give a higher speed than curve C.

The speed of signalling is greatest when the capacity is continuously distributed.

Any Number of Meshes.

It is evident that if the process of sub-division be continued indefinitely the arrival current, when the number of meshes becomes infinitely great, must eventually be the same as in the case of uniform distribution of capacity throughout the length of the cable. The number of circuital equations is then infinitely great, and the number of terms in the determinants is $(\infty)^2$. Yet these arrival currents must be finite, and it is now proposed to investigate to what limiting value the formula for the current tends when the number of meshes is indefinitely increased.

When there are three meshes, C_3 is given by the formula

$$C_3 = Ee^{yt}\left(\frac{-2}{yKl}\right)^2 \div \begin{vmatrix} \frac{Rl}{3}+\frac{2}{yKl} & \frac{-2}{yKl} & 0 \\ \frac{-2}{yKl} & \frac{Rl}{3}+\frac{4}{yKl} & \frac{-2}{yKl} \\ 0 & \frac{-2}{yKl} & \frac{Rl}{3}+\frac{2}{yKl} \end{vmatrix}$$

$$= Ee^{yt} \div \frac{2}{yKl} \begin{vmatrix} \frac{y\,.\,KRl^2}{2\cdot3}+1 & -1 & 0 \\ -1 & \frac{yKRl^2}{2\cdot3}+2 & -1 \\ 0 & -1 & \frac{yKRl^2}{2\cdot3}+1 \end{vmatrix}$$

when numerator and denominator are multiplied by $\left(\frac{yKl}{2}\right)^3$.

Hence, when there are n meshes it follows by analogy that—

$$C_n = \frac{Ee^{yt}}{\varphi(y)}$$

where

$$\varphi(y) = \frac{n-1}{y\,.\,Kl}\begin{vmatrix} \frac{y\,.\,KRl^2}{n(n-1)}+1 & -1 & - & - & - & - \\ -1 & \frac{y\,.\,KRl^2}{n(n-1)}+2 & -1 & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & -1 & \frac{y\,.\,KRl^2}{n(n-1)}+1 \end{vmatrix}$$

In the determinant write $\cos\theta$ for $1+\frac{y\,.\,KRl^2}{2n(n-1)}$. . . . (12)

Then,

$$\varphi(y) = \frac{n-1}{y\,.\,Kl}\begin{vmatrix} 2\cos\theta-1 & -1 & - & - & - & - \\ -1 & 2\cos\theta & -1 & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & -1 & 2\cos\theta-1 \end{vmatrix}$$

This determinant has been shown* to have the value,

$$\frac{2\sin n\theta}{\sin\theta}(\cos\theta-1).$$

Hence $\quad \varphi(y) = \frac{n-1}{y\,.\,Kl}\;\frac{2\sin n\theta}{\sin\theta}(\cos\theta-1) = \frac{Rl\sin n\theta}{n\sin\theta},$

on substituting for y from (12).

Now, $\quad \varphi'(y) = \frac{d\theta}{dy}\;\frac{d}{d\theta}[\varphi(y)] = \frac{-KRl^2\;\;Rl\cos n\theta}{2n(n-1)\;\sin^2\theta},$

so that $\quad y\varphi'(y) = \frac{(\cos\theta-1).2n(n-1)}{KRl^2}\,.\,\varphi'(y) = \frac{Rl\cos n\theta}{1+\cos\theta}.$

Again, $\varphi(y)=0$ gives $\theta=\frac{m\pi}{n}$ and $y=\frac{2n(n-1)}{KRl^2}\left(\cos\frac{m\pi}{n}-1\right)$.

Hence, finally,

$$C_n = \frac{E}{Rl}\left[1+\sum_{m=1}^{m=n-1} e^{-\frac{2n(n-1)}{KRl^2}\left(1-\cos\frac{m\pi}{n}\right)t}\,.\,\left(1+\cos\frac{m\pi}{n}\right)\cos m\pi\right]. \quad (13)$$

---

* Chap. I., p. 26.

Passage to the Distributed Case.

In the formula just obtained let n be infinitely great. Then $n(n-1)\left(1-\cos\frac{m\pi}{n}\right)=\frac{m^2\pi^2}{2}$, and the formula becomes

$$\underset{n=\infty}{\mathrm{C}}=\frac{\mathrm{E}}{\mathrm{R}l}\left[1+2\sum_{m=1}^{\infty}e^{-\frac{m^2\pi^2 t}{\mathrm{KR}l^2}}\cos m\pi\right]. \quad . \quad . \quad . \quad (14)$$

This expression, here deduced as a particular case of a more general formula which was obtained from first principles, is usually derived from the telegraphic equation, as on page 232. It is represented by the curve which is plotted in Fig. 146, curve B; Fig. 147, curve C; and Fig. 148, curve B. Well known as the "Kelvin Arrival Curve," it is one of the simplest of the infinite number of such curves by means of which the properties of the telegraph cable and its adjuncts can be interpreted.

The method by which it is here approached has certain advantages. First, it shows to what degree of approximation a system of condensers and resistance coils (artificial cable) can be taken to represent a cable of evenly distributed constants. This criterion is obtained in the following paragraph. Second, as will appear later, it allows us to adopt the same plan in more complicated cases, such as when inductance and leakance are present, and thus obtain from elementary considerations a clear idea of the way in which these factors act.

Criterion for Uniform Distribution.

The criterion as to whether the subdivision has been carried sufficiently far—a criterion corresponding to the Pupin criterion in telephony for uniform distribution—may be obtained as follows :—

From Figs. 147 and 148 it is seen that, even when the number of meshes is small, the mesh curve and the cable curve lie closely together at the origin, the divergence being most pronounced for the greater values of t. Now, from the corresponding tables, Tables XCVII., XCVIII. and XCIX., it is evident that in the later stages the curves are represented with sufficient accuracy by the first exponential term subtracted from unity. Hence, when n is great, C_n becomes, for the greater values of t, ($t > 0{\cdot}5$ sec., say), with sufficient exactitude

$$C_n = \frac{E}{Rl}\left[1 - e^{-\frac{2n(n-1)}{KRl^2}\left(1-\cos\frac{\pi}{n}\right)t} \cdot \left(1+\cos\frac{\pi}{n}\right)\right]. \quad . \quad (15)$$

In the continuous case the expression corresponding is *

$$C_r = \frac{E}{Rl}\left[1 - 2e^{-\frac{\pi^2 t}{KRl^2}}\right]. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When these two formulæ are compared the following, easily verifiable, deduction can be made :—

The constants of the artificial line may be regarded as uniformly distributed when the number of meshes (n) *is so great that* $\cos \pi/n$ *is sensibly equal to* $1 - \frac{\pi^2}{2n^2}$.

The amount by which the mesh current falls short of the cable current is obtained by subtracting (16) from (15).

Thus

$$C_r - C_n = \frac{E}{Rl}\left[e^{-\frac{2n(n-1)}{KRl^2}\left(1-\cos\frac{\pi}{n}\right)t} \times \left(1+\cos\frac{\pi}{n}\right) - 2e^{-\frac{\pi^2 t}{KRl^2}}\right]. \quad (17)$$

Table C. is calculated from (13) for $n=100$, and in the same table are given also the values of C_r ($n=\infty$).†

Table C.—100 *Meshes.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady...	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0
$m=1,-$	321·1	256·5	204·9	163·7	130·8	83·5	42·6	13·9	4·5	0·5
$m=2,+$	163·7	66·7	27·2	11·1	4·5	0·7	...	...	...	...
$m=3,-$	53·2	7·1	0·9	0·1	...	...	...	...	...	...
$m=4,+$	11·1	0·3	...	...	...	...	...	...	...	...
$m=5,-$	1·5	...	...	...	...	...	...	...	...	...
C_{100}‡	...	4·4	22·4	48·3	74·7	118·2	158·4	187·1	196·5	200·5
C_r‡	0·03	4·6	23·0	49·3	75·9	119·5	159·4	187·6	196·7	200·6
${}_{100}C_1$‡......	751·6	531·6	434·0	375·9	336·3	285·2	243·6	214·9	205·5	201·5
C_s‡	748·1	529·0	431·9	374·1	334·6	283·9	242·7	214·4	205·3	201·5

* Chapter VIII, p. 237.

† From Table XXVIII., Chap. VIII., p. 236

‡ Microamperes per volt.

If C_{100} were plotted on Fig. 148 the curves representing it would almost coincide with curves B and D.

The question of the degree of approximation of an artificial cable to a real cable is of more importance in the consideration of the phenomena which occur at the sending end.

SENDING CURRENTS.

The sending currents may be obtained in the same way as the arrival currents, by writing down the periodic equations, solving them by determinants, and developing the transient current from the periodic. Thus, when there are two meshes, as in Fig. 146, A, the current in the first mesh is given by

$$ {}_2C_1 = \frac{E}{Rl}\left[1 + e^{-\frac{4t}{KRl^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (18) $$

This resembles (7), but has the positive sign before the exponential term. In general, the series giving the sending currents are exactly the same at those for the arrival currents, except that all the terms of the series are positive.*

The values of ${}_2C_1$ as a function of t are contained in Table XCVI., and are plotted in Fig. 149. The current begins at 402 microamperes per volt, the uncharged condenser acting as a shortcircuit to the second mesh, and falls off to a stationary value of 201 microamperes per volt.

When there are three meshes the current in the first mesh is given by

$$ {}_3C_1 = \frac{E}{Rl}\left[1 + \frac{3}{2}e^{-\frac{6t}{KRl^2}} + \frac{e^{-\frac{18}{KRl^2}}}{2}\right] \quad . \quad . \quad . \quad . \quad (19) $$

The values of ${}_3C_1$ are contained in Table XCVII., and are plotted also in Fig. 149. The current starts at 603 microamperes and falls off to the same steady value as before. The two curves cross at 0·94 second.

At the moment of making contact, owing to the short-circuiting action of the first condenser, the current is entirely

* Chap. X., p. 302.

confined to the first mesh, and is to be obtained by dividing the E.M.F. of the battery by the resistance in the mesh. Thus, when there are n meshes, the instantaneous current is $\frac{E}{Rl/n} = \frac{nE}{Rl}$, and therefore when the number of meshes is infinite the instantaneous current would be infinitely great, if there were no impedance in front of the cable, and no inductance in the cable itself.

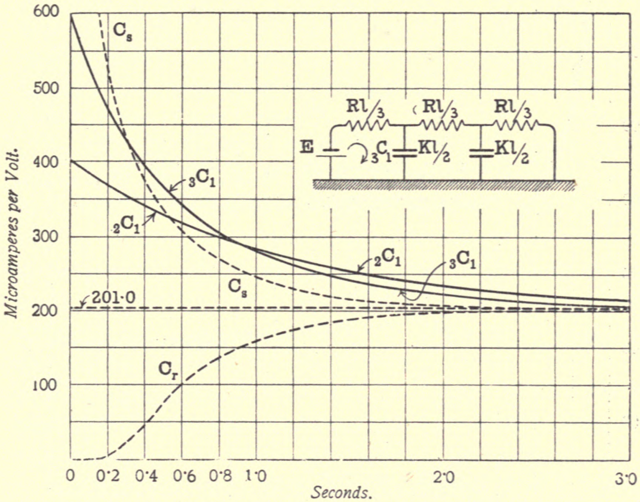

FIG. 149.—SENDING CURRENTS.

Curve $_2C_1$.—Current in first of two meshes.
,, $_3C_1$.—Current in first of three meshes.
,, C_s.—Cable sending current.
,, C_r.—Cable received current.

The curve for the current, C_s, entering the cable with uniformly distributed constants, is also shown in Fig. 149.* As the number of meshes is increased, the curve representing the current in the first mesh is bent upwards at the beginning and downwards later, until it coincides with the curve C_s.

* Reproduced from Fig. 27, A, Chap. X., p. 299.

Fig. 150 shows the current ${}_4C_1$ in the first of four meshes, whichis plotted from Table XCVIII., and calculated from the expression

$${}_4C_1=\frac{E}{Rl}\left[1+\frac{e^{\frac{-24+12\sqrt{2}}{KRl^2}t}}{2-\sqrt{2}}+e^{-\frac{24t}{KRl^2}}+\frac{e^{\frac{-24-12\sqrt{2}}{KRl^2}t}}{2+\sqrt{2}}\right], \qquad (20)$$

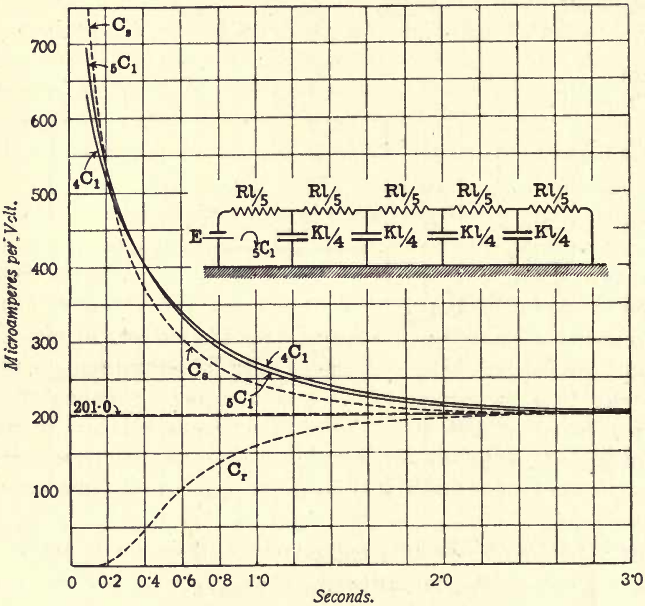

FIG. 150.—SENDING CURRENTS.

Curve ${}_4C_1$.—Current in first of four meshes.
" ${}_5C_1$.—Current in first of five meshes.
" C_s.—Cable sending current.
" C_r.—Cable received current.

as well as the current in the first of five meshes, which is plotted from Table XCIX. and is calculated from

$${}_5C_1=\frac{E}{Rl}\left[1+\frac{e^{-\frac{10(3-\sqrt{5})}{KRl^2}t}}{1-\frac{1}{\sqrt{5}}}+\frac{e^{-\frac{10(5-\sqrt{5})}{KRl^2}t}}{3-\sqrt{5}}+\frac{e^{-\frac{10(3+\sqrt{5})}{KRl^2}t}}{1+\frac{1}{\sqrt{5}}}+\frac{e^{-\frac{10(5+\sqrt{5})}{KRl^2}t}}{3+\sqrt{5}}\right] \qquad (21)$$

In general, the current entering the first of n meshes is, from (13), given by the expression

$$ {}_nC_1=\frac{E}{Rl}\left[1+\sum_{m=1}^{m=n-1} e^{\frac{-2n(n-1)}{KRl^2}\left(1-\cos\frac{m\pi}{n}\right)t}\times\left(1+\cos\frac{m\pi}{n}\right)\right]. \quad (22) $$

When n is infinite this expression becomes the same as that for C_s.* Table C. contains the values of (22) for $n=100$. As the number of meshes increases, the point at which the mesh curve cuts the cable curve moves up to the left, as is clear from Fig. 150. Initially the mesh current is less than the cable current; after the curves cross it is greater. By taking a sufficiently great number of terms of (22), the sending current could be calculated to any required degree of exactitude, even for the initial stages where t is small.

In duplexing a short cable, where the instruments employed are of a robust character, only a few subdivisions are necessary in the artificial cable to produce a sufficiently close approximation to the real; but in a long cable, where the receiving instruments are extremely delicate, and are tending to become still more delicate, the subdivisions must be carried to a greater extent if the instruments are not to be upset by inequality in the balance. Given the number of divisions which it is proposed to adopt, by the aid of (22) the nature of the shock on the instrument to be expected from this form of inequality can be calculated.

It is not necessary that the distribution should be uniform for the whole of the artificial cable. Certain devices used in conjunction with the recording instrument, such as the series condenser and the inductive shunt, tend to facilitate the passage through it of rapidly varying currents, and render it insensitive to slow changes. It is, therefore, just after switching on, when the changes of voltage at the sending end are most rapid, that closeness of approximation is most to be desired. Now, it has been shown that the influence of the state of the distant end of a cable does not make itself felt at the sending end until a certain characteristic time has elapsed.† In

* Chap. X., p. 302. † Chap. X., p. 303.

the case of the present cable this may vary from 0·5 second to one second, according to the conditions. But if the cable were cut down to one-tenth of its length, by the law of reduction* the time of indifference would be at least $0{\cdot}5 \div (10)^2 = 0{\cdot}005$ second. If the first tenth of the artificial cable were an exact equivalent of the real cable the two curves would coincide during the first five-thousandths of a second, after which the falling off in the steepness of the curves would render the constitution of the remainder of the artificial cable of less importance. With increase of distance from the sending end the subdivisions can be made greater.

The "KR" Law and the Cable Core.

The conclusions of the electrostatic theory of propagation are summed up in the "KR" law, according to which the time of any electrical operation, and therefore the rate of signalling, are inversely proportional to the product of the total capacity and the total resistance of the cable. The actual size of the signals varies inversely as the total resistance of the cable, but by an alteration of the sending E.M.F. or of the sensitivity of the receiving apparatus the height may be adjusted to what is most convenient for reading on the recorder strip.

As the "KR" law made its appearance in the very earliest days of cable telegraphy, its influence was great, and endeavours were at once set on foot to reduce both resistance and capacity. The gain of speed by such a change is immediately apparent, since it is directly proportional to the reduction in either quantity. These efforts have, therefore, been continued to the present day, We shall consider these two ways of reduction in turn.

1. The resistance of the conductor may be decreased by increasing the weight of copper. For high-speed working a very heavy conductor may be used, weighing as much as 700 lb. per nautical mile. The drawback is that the amount of gutta-percha required to cover the conductor must also be

* Chap. X., p. 329.

increased if the insulating coating is to have sufficient mechanical strength and if the capacity is to remain the same. Such a core is necessarily a very expensive one.

The capacity of a cable core may be written in the form,

$$K = \frac{k}{\log_{10} D/d},$$

where D is the outer diameter of the core, d the diameter of the conductor, and k is a constant depending on the insulating material. Putting $R = r/d^2$, where r is the resistance of a conductor of unit diameter, it follows that

$$KR = \frac{kr}{d^2 \log_{10} D/d}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now, by volume gutta-percha and copper are approximately of equal cost.* Hence, if the cost of the core is fixed the overall diameter D is fixed. Since R varies inversely as d^2 and K only inversely as log D/d, it is of advantage to make d large.

Differentiating (31) with respect to d, and equating to zero, the minimum value of KR is found to occur when $d = 0{\cdot}6065$ D, or $D = 1{\cdot}649\, d$.† This proportion of d to D is much greater than what is customarily regarded as being safe mechanically. Although the proportion which the weight of conductor bears to that of the core has been gradually increased in course of time, it still falls short of its most economical value. In Fig. 151, curve A, is plotted the diameter of a stranded copper conductor against the weight of the conductor, and curve B is drawn as a mean curve to represent as closely as possible the corresponding overall diameters of the core according to current practice. Curve B′ is a true parabola, so drawn as to pass below the smaller values of D and above the greater. Curve A′ is drawn to height 0·6065 of curve B′, and represents, therefore, the diameter of conductor to give the maximum speed for a core of outer diameter fixed by curve B′. It is seen that the actual diameters of curve A could be increased with electrical advantage by 60 per cent., or the weight of conductor made $2\frac{1}{2}$ times its usual value. Thus a core of 200/180 might with advantage,

* For a more general examination *see* THE ELECTRICIAN, Vol. LXXIV., Dec. 11, 1914, p. 329.

† W. A. Price, "Electrical Review," 41, 1897, p. 190.

from the electrical point of view, be replaced by one for which $d=0{\cdot}190''$, instead of $0{\cdot}114''$, and, therefore, $\log_{10} D/d = \log_{10}\frac{0{\cdot}312}{0{\cdot}190}=0{\cdot}2154$, instead of $\log_{10}\frac{0{\cdot}312}{0{\cdot}114}=0{\cdot}4373$, so that $d^2 \log_{10} D/d$ would be $0{\cdot}00778$, instead of $0{\cdot}00568$, and the KR would be reduced in the proportion 1 : 1·37, giving an increase of 37 per cent. in speed for the same cost.

Probably the ratio D/d could be brought down to some extent without undue risk to the core through lack of mechanical

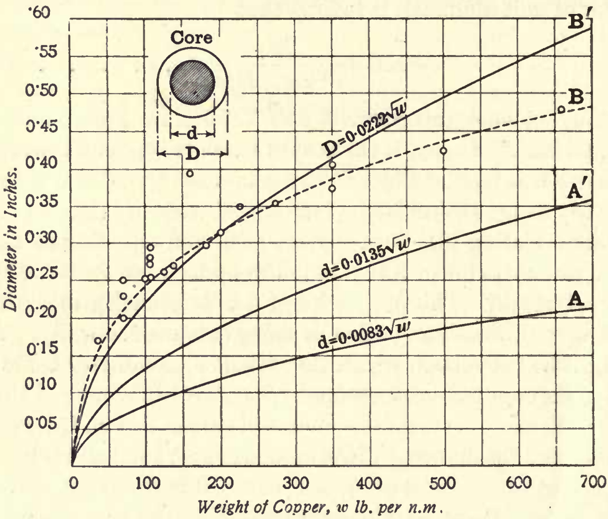

FIG. 151.—DIAMETERS OF CONDUCTOR AND OF CORE.

Curve A.—Diameter of conductor d.
„ B.—Diameter of core D for typical cables.
„ B′.—$D=\sqrt{w}/45$.
„ A′.—Best value of d corresponding to B′.

stability. The greater part of cable breakdowns seem to be attributable to external influences, against which a few extra mils of gutta-percha offer no protection.*

2. There is much greater scope for improvement in the constant k. With a very few exceptions, the only insulating

* On this point *see* R. M. Sayers, THE ELECTRICIAN, Vol. LXXIV, Nov. 6, 1914, p. 155.

material used in submarine cables is gutta-percha. For this substance the constant k may be taken as 0·140 for average present-day cores, when K is in microfarads per nautical mile. Any improvement in k is felt at once in a directly proportional increase in speed. Improvement in this direction, though continuous, has been slow for two reasons. On the one hand the cable manufacturer is restricted by stipulations as, for example, to dielectric resistance and electrification, and, on the other hand, the cable user is naturally reluctant to depart from well-tried materials, and may not realise the possibilities in this direction. Nevertheless, the way has recently been thrown open in the case of telephone cables. The essential properties of an insulating material are (*a*) permanency of insulating qualities, (*b*) low capacity, and (*c*), in the case of telephone cables, low leakance. A specification may militate against these by insistence on other non-essential qualities.

It is possible that a substitute for gutta-percha will be found in rubber, the low price of which is in its favour, provided it can be made to retain its insulating qualities after long immersion.* The advantages to be gained by its use would probably not be electrical, but possibly mechanical, and the lower cost would enable cores of heavier type, and, therefore, of greater speed to be laid for the same expenditure.

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* Hooper's Core, *see* Charles Bright, "Submarine Telegraphs." "Gutta Gentzsch" *see* Archiv, f. Post u. Teleg., 1902, p. 717; 1904, p. 89.

CHAPTER XIV.

PRESENT-DAY METHODS OF OVERCOMING DISTORTION.

Introduction—Signalling without Apparatus—Curbed Signalling—Signalling with Condensers—Limiting Shape of Condenser Curve: Best Size of Condensers—Theory of the Shunted Condenser—Limiting Shape of Curve: Condenser Zero—Inductive Shunt—Train of Damped Oscillations—Damped Train Continued: Influence of Signalling Condensers—Method of Compensation—Ideal Signal—The Use of Magnifiers—The Heurtley Magnifier—The Orling Jet Relay—Conclusions.

Introduction.

The problem which transmission presents in telegraphy is more difficult in one respect than the corresponding problem in telephony. In telephony it is sufficient, for the present at any rate, if the current at the receiving end of the line is powerful enough to affect the diagram of the telephone receiver. In submarine telegraphy, both the strength of the current and also the shape of the current arrival curve must be taken into account.

The principle underlying those methods of overcoming distortion which have grown up in the past history of cable telegraphy and are in use, and in course of further development, at the present day may be stated as follows: *The shape of the current curve is of greater importance than its height.* By height is meant here the absolute length of the ordinates forming the curve. "Shape" may be understood to take account of the rate of growth and decay of the curve.

The application of this principle in practice is illustrated below in a variety of typical and important cases, and it will be shown that the progressive increase in the speed of signalling which has been brought about in recent years is based upon it.

SIGNALLING WITHOUT APPARATUS.

In Fig. 152 the arrival curve due to a single prolonged contact is shown in part in curve A. The cable is supposed to be used

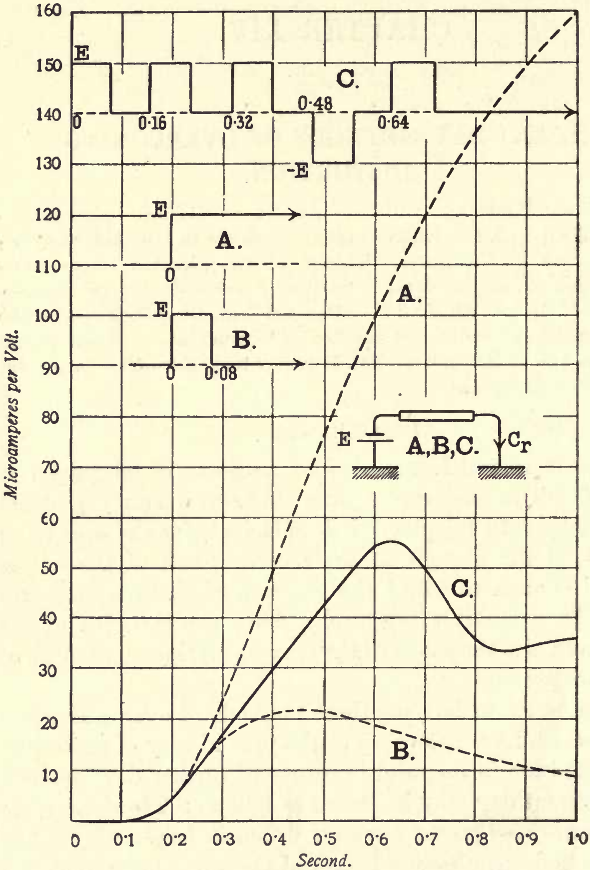

FIG. 152.—CABLE WITHOUT APPARATUS.

Curve A.—Arrival curve.
„ B.—Elementary dot curve, contact lasting 0·08 second.
„ C.—Signal " Understand."

without apparatus of sensible impedance, with both its ends to earth. Curve B represents the elementary dot signal when the time of contact is taken as 0·08 second. In curve C the signal "understand" (- - - — -) is supposed sent by mirror code, the length of a space being taken equal to that of a dot or dash, in this case 0·08 second.

From curve C it is seen that successive dots are scarcely distinguishable in the received signal. A trained eye might read such a message in plain English, but not with the necessary degree of certainty in code.

Curbed Signalling.

At an early stage in cable history it was recognised that if the signal in sending was followed immediately by an impulse of opposite sign, or curb, the effect was to sharpen the signals at the receiving end. The action of the opposing impulse following close on the heels of the first impulse was in a measure to clear the cable more quickly and bring it to a neutral condition. Such a signal is usually called a *curbed* signal.

The result of curbing on the signal of Fig. 152 is represented in Fig. 153. Curve A is the elementary dot signal for a time of contact 0·04 second. It does not differ appreciably in shape from curve B of Fig. 152, but it is only half the height.* Curve B is the corresponding curbed signal, in which the two contacts are of equal length. Curve C, built up from curve B, represents, as before, the signal "understand." Comparing curve C, Fig. 153, with curve C, Fig. 152, the improvement produced is striking. Although the greatest height in the second case is only 3 microamperes, as against 55 microamperes in the other, the definition of the signals has been greatly improved. The dots are well marked and clearly separated; the dash crosses the zero line, and the curve tends to run parallel to the zero line, instead of climbing up as in Fig. 152. Shape has been gained at the expense of size.

The received current is still great enough to be well within the range of sensitivity of the siphon recorder when the cable is worked with, say, 50 volts. By taking different forms of

* Chap. XI., p. 338.

elementary signal B, in which the proportions of dot, dash and space are altered from zero upwards, the influence of various factors may be determined.

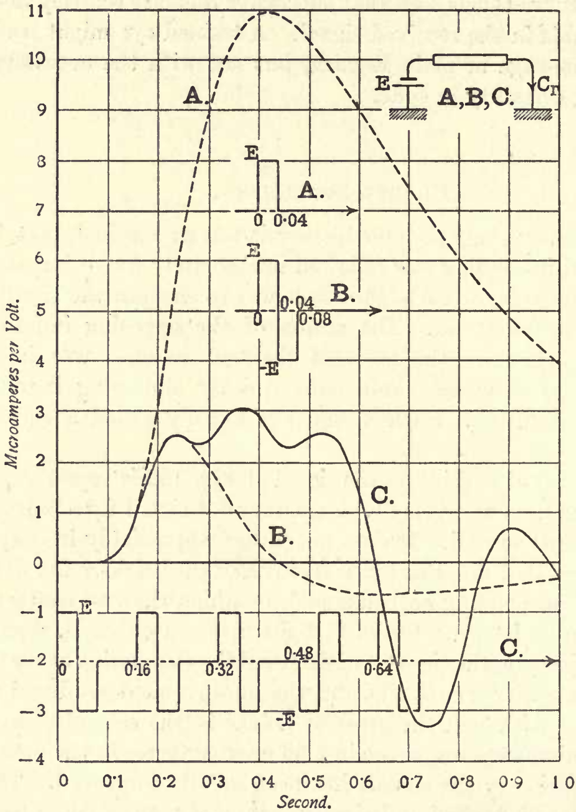

FIG. 153.—CURBED SIGNALLING. CABLE WITHOUT APPARATUS.

Curve A.—Dot lasting 0·04 second.
„ B.—Dot-dash, 0·04+0·04 second.
„ C.—Signal " Understand."

SIGNALLING WITH CONDENSERS.

Signalling condensers were first introduced to prevent the passage through the receiving apparatus of parasitic earth currents. These currents may be regarded as alternating currents of low periodicity, to which a condenser offers only low conductance. It was not long before it was discovered that the employment of a condenser in this way led to a gain in the working speed of the cable, which was further increased when another condenser was inserted at the sending end. The use

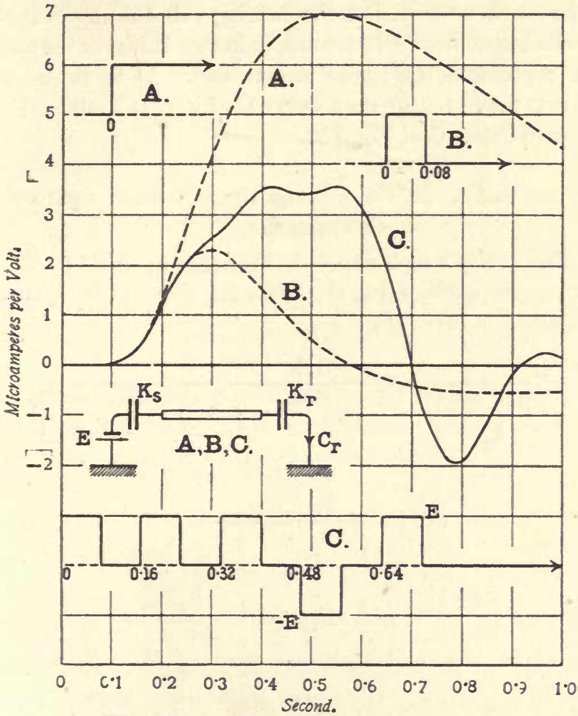

Fig. 154.—Signalling with Condensers.

Curve A.—Single prolonged contact.
" B.—Dot lasting 0·08 second.
" C.—Signal "Understand."

of sending and receiving condensers became universal thenceforth for long cables.

To understand the manner in which this increase of speed arises need present no difficulty in the light of what has been stated in the preceding paragraphs. In Fig. 154, curve A, is drawn the current arrival curve for the cable with condensers which are each one-tenth of the total cable capacity.* Curve B represents the elementary dot signal for a contact lasting 0·08 second. Comparing curve B with curve B of Fig. 152, it is seen that, although the height of the dot has been reduced from 22 microamperes per volt to 2·3 microamperes per volt, or to about one-tenth, nevertheless the turning value is now reached in 0·3 second instead of 0·45 second, or in two-thirds of the time. Curve C represents the signal understand. As is to be expected, it is much better than curve C of Fig. 152, although it is inferior to curve C of Fig. 153.

Limiting Shape of Condenser Curve—Best Size of Condensers.

The relationship which curve A, Fig. 154, bears to curve A, Fig. 152, may be exhibited in the following way. The equation to the condenser curve is†

$$C_r = \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x}+\frac{1}{\cos x}+\left(\frac{\sin x}{x}-\frac{1}{\cos x}\right)\frac{1}{x^2}\cdot\frac{Kl}{K_s}\cdot\frac{Kl}{K_r}\right]},$$

where $$\tan x = \frac{\frac{Kl}{K_s}+\frac{Kl}{K_r}}{x-\frac{1}{x}\cdot\frac{Kl}{K_s}\cdot\frac{Kl}{K_r}}.$$

Suppose that $\frac{Kl}{K_s}=\frac{Kl}{K_r}$, and that both ratios are great—*i.e.*, that the signalling condensers are very small. Then $\tan x = -2x\cdot\frac{K_s}{Kl}$ approximately, in the limit when K_s/Kl is very small. To the same degree of approximation $\sin x = -2x\cdot\frac{K_s}{Kl}$ and $\cos x = \cos m\pi$, since when $K=0$, $\tan x=0$ and $x=m\pi$. When these values are substituted in C_r, it becomes, small quantities being rejected,

$$\underset{\text{(with small condensers)}}{C_r} = -m^2\pi^2\left(\frac{K_s}{Kl}\right)^2\sum_{m=1}^{\infty}\frac{2E\,.\,e^{-\frac{m^2\pi^2t}{KRl^2}}\cos m\pi}{Rl}\quad . \quad . \quad (24)$$

* Chap. IX., p. 275.   † Chap. IX., p. 273.

Now the Kelvin arrival curve, Fig. 152, A (cable earthed without apparatus), has as equation

$$\underset{\text{(no apparatus)}}{C_r} = \frac{E}{Rl}\left[1+2\sum_{m=1}^{\infty} e^{-\frac{m^2\pi^2 t}{KRl^2}} . \cos m\pi\right]$$

from which, on differentiation,

$$\underset{\text{(no apparatus)}}{\frac{dC_r}{dt}} = -\frac{m^2\pi^2}{KRl^2}\sum_1^{\infty}\frac{2E . e^{-\frac{m^2\pi^2 t}{KRl^2}}\cos m\pi}{Rl} \quad . \quad . \quad . \quad (25)$$

Hence, comparing (24) with (25), it is seen that

$$\underset{\text{(with small condensers)}}{C_r} = \left(\frac{K_s}{Kl}\right)^2 . KRl^2 . \frac{dC_r}{dt}_{\text{(cable without apparatus)}} \qquad (26)$$

In the limit, therefore, when the condensers are very small, the shape of the condenser arrival curve, Fig. 154, A, becomes merely the slope of the arrival curve without apparatus. Now the slope of Fig. 152, curve A, is approximately given by curve B, Fig. 152, and if the interval of contact is smaller, as in curve A, Fig. 153, the approximation is still closer. Hence in the limit when the condensers in the one case and the time of contact in the other case are infinitely small, the condenser arrival curve (single prolonged contact) has the same shape as the dot signal (make and break) of the cable without apparatus, which shape is the slope of the Kelvin arrival curve.

From (26) it follows that the size of the condenser arrival curve is proportional to $(K_s/Kl)^2$, when K_s/Kl is small. Hence if the signalling condensers are halved the size of the received signal is reduced to one-quarter. Further reduction in the size of the condensers when these are already small has, therefore, the effect of greatly reducing the height of the arrival curve without appreciably affecting its shape, which merely tends to approximate somewhat more closely to the slope of the arrival curve without apparatus.

From (26) the greatest height of the condenser arrival curve occurs when $\frac{dC_r}{dt}$ as given by (25) is a maximum. To find the maximum value of this series would be difficult. But the expression *—

$$C_{r\,(\text{no apparatus})} = 2E\sqrt{\frac{K}{R\pi t}}\,e^{-\frac{KRl^2}{4t}}$$

represents the arrival curve with sufficient exactitude initially and beyond the point of greatest slope. On differentiating this expression twice with regard to t,

$$\frac{dC_r}{dt} = \frac{C_r}{2t}\left[\frac{KRl^2}{2t} - 1\right], \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

and

$$\frac{d^2C_r}{dt^2} = \frac{C_r}{16t^4}[(KRl^2)^2 - 12KRl^2 \,.\, t + 12t^2].$$

Hence, $\frac{d^2C_r}{dt^2}$ is zero when $t = 0{\cdot}09175 \times KRl^2$, and then $\frac{dC_r}{dt}$ has its maximum value. Substituting this value of t in (27), and from (27) in (26), we obtain finally the following simple formula for the approximate height H of the condenser arrival curve, for small condensers—

$$H = \frac{6E}{Rl}\left(\frac{K_s}{Kl}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

The formula becomes exact only when the condensers are extremely small. As the size of the condensers diminishes the turning point of the arrival curve moves further to the left towards the limiting value of t, $(0{\cdot}09175 \times KRl^2)$, in this case 0·4 second.

Curve A, Fig. 155, is the slope of curve A, Fig. 152, and is calculated from (27). When $t = 0{\cdot}4$ second, $\frac{dC_r}{dt} = 273{\cdot}5$ mm.a. per volt per second, which is its maximum value. In Fig. 155 the height of curve A is arbitrarily chosen as 7 microamperes per volt in order that it may compare with curve A, Fig. 154. In reality, since $dC_{r(\text{maximum})} = 273{\cdot}5 \times dt$, it is clear that when dt is zero the curve vanishes. Nevertheless, curve A of Fig. 155 will represent closely the effect of a contact of $7 \div 273{\cdot}5 = 0{\cdot}026$ second duration. Comparing the corresponding curves A of Figs. 154 and 155 with each other, it is seen that although the

* Chap. VIII., p. 236.

two curves have the same height, curve A of Fig. 155 is the better, since its turning value is reached in 0·4 second instead of 0·54 second, as in curve A, Fig. 154. Curve B, Fig. 155, is the

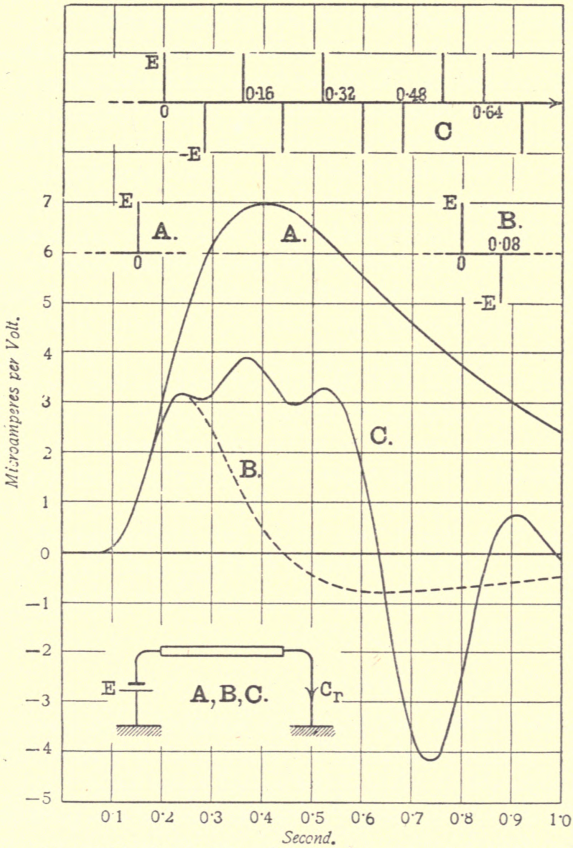

FIG. 155.—INFINITELY SHORT CONTACT. NO APPARATUS.

Curve A.—Single make-and-break.
„ B.—Dot-dash.
„ C.—Signal "Understand."

elementary 0·08 dot signal, and curve C the signal understand. As was to be expected from the superiority of the arrival curve, curve C, Fig. 155, is better than curve C, Fig. 154.

At the same time, it has practically the same shape as curve C, Fig. 153, from which it differs appreciably only in height, as was to be anticipated since the arrival curves A of Figs. 153 and 155 differ only slightly in shape, the maximum of curve A, Fig. 153 being reached in 0·42 second instead of 0·4.

The foregoing simple considerations lead to the following important results : (1) The time of contact in the elementary dot signal is so short compared with the KRl^2 of the cable that the shape of the elementary dot signal may be regarded as identical with the slope of the arrival curve when the cable is used without apparatus ; (2) in signalling with condensers, the height of the elementary dot signal diminishes much more rapidly than the proportion which the condensers bear to the total capacity of the cable ; (3) with the smallest condensers that will give permissible size of signal, the shape is still inferior to that of the slope of the arrival curve, cable without apparatus, to which it tends in the limit ; (4) the abandonment of signalling condensers would, therefore, result in increased speed, and their retention is to be regarded as due to their twofold function in stopping earth currents and in forming the bridge arms in duplex telegraphy. The best size for the condensers is that which just gives full deflections on the recorder strip ; greater condensers would give excess current, and smaller condensers fail to yield a proportionate improvement in the wave shape.

Theory of the Shunted Condenser.

It has been shown that the employment of condensers is a first approximation to the use of the slope of the arrival curve without apparatus in place of the arrival curve itself. Now, if the condensers are shunted by resistances so that two paths are offered to the current—one through the resistances and the other through the condensers, both having the cable in common—we may expect that the resultant arrival curve will in some degree be a combination of the arrival curves A of Figs. 152 and 154.

The arrival curve with one-tenth condensers, Fig. 154, curve A, rises to a maximum of 7 microamperes per volt, and then returns to zero. If there were no condensers and the ends of the cable were to earth through the battery the current would rise to a steady value of 201 microamperes per volt. By the insertion of resistance coils this steady value could be brought down also to 7 microamperes per volt. When both condensers and resistance coils are used the arrival curve may be anticipated to possess the quick initial rise of the condenser curve combined with the later steady value of the resistance curve, thus affording a closer approximation than either alone to the ideal instantaneous rise to steady current.*

Take the case of a single condenser at the receiving end shunted by a resistance, and let the battery be applied directly to the cable at the sending end.† The equation to the arrival curve for the single receiving condenser is

$$C_r = \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x} + \frac{1}{\cos x}\right]}, \quad . \quad . \quad . \quad . \quad (29)$$

where x is given by

$$\tan x = \frac{Kl}{K_r} \cdot \frac{1}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Let $K_r = Kl/10$. Curve A, Fig. 156, is plotted from (29) and Table XL.‡ The curve rises to a maximum of about 31 microamperes per volt, which is reached in 0·85 second.

The equation to the arrival curve with resistance alone is

$$C_r = \frac{E}{Rl + R_r} + \sum_x \frac{2Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{1}{\cos x} - \frac{\sin x}{x}\right]}, \quad . \quad . \quad . \quad (31)$$

where $$\tan x = -\frac{x \cdot R_r}{Rl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The steady value to which the current rises is $E/(Rl + R_r)$. If this is to be 31 microamperes per volt, R_r must equal 27,283

* This principle forms the basis of patent 5,589 of 1901, granted to Dr. Alexander Muirhead. For earlier applications of the idea *see* THE ELECTRICIAN, Vol. XXVI., 1890, p. 241.

† Stated *l.c.* to give the best result.

‡ Chap. IX., p. 272.

Table CI.—*Roots of* $\tan x = -5{\cdot}484 \times x$.

x.		Sin x.	Cos x.
1·6790	$\pi-83°\ 48$	0·99415	−0·10800
4·7508	$2\pi-87°\ 48$	−0·99926	0·03839
7·8770	$3\pi-88°\ 41$	0·99973	−0·02315
11·0122	$4\pi-89°\ 3$	−0·99986	0·01657

Table CII.—*Fig.* 156, *Curve B.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0
x_1-	38·248	35·844	33·602	31·490	29·516	25·928	21·350	15·440	11·169	5·843
x_2+	9·114	5·425	3·230	1·922	1·144	0·405	0·086	0·006	...	...
x_3-	2·230	0·536	0·129	0·031	0·008	...	...	...	...	...
x_4+	0·410	0·051	0·015	...	...	...	...	...	...	...
C_r(mm.a. per volt).	0·046	0·096	0·515	1·402	2·621	5·478	9·736	15·566	19·831	25·157

Table CIII.—*Roots of* $\tan x = \dfrac{10}{x - 1{\cdot}8235/x}$.

x.		Sin x.	Cos x.
1·5360	$88°\ 02·$	0·99939	0·03485
4·3389	$\pi+68°\ 36$	−0·93106	−0·36488
7·2437	$2\pi+55°\ 2$	0·11315	0·57310
10·2087	$3\pi+44°\ 55$	−0·70608	−0·70813
13·2192	$4\pi+37°\ 24$	0·60738	0·79441
16·2624	$5\pi+31°\ 46$	−0·52646	−0·85020

Table CIV.—*Fig.* 156, *Curve C.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0	31·0
x_1+	11·263	10·669	10·106	9·571	9·066	8·134	6·913	5·272	4·020	2·337
x_2-	86·903	56·378	36·580	23·731	15·397	6·481	1·770	0·204	0·023	...
x_3+	64·514	19·321	5·785	1·732	0·519	0·047	...	...	...	...
x_4-	24·704	2·253	0·187	0·019	...	...	...	...	...	...
x_5+	5·574	0·101	...	...	...	...	...	...	...	...
x_6-	0·764	...	...	...	...	...	...	...	...	...
C_r(mm.a. per volt).	−0·020	2·460	10·124	18·553	25·188	32·700	36·143	36·068	34·997	33·337

ohms, and therefore $\tan x = -x \times 5{\cdot}484$. The roots of this equation are contained in Table CI. Curve B, Fig. 156, is plotted from Table CII. The curve approaches its steady value of 31 microamperes per volt only very gradually. When the ordinates of curves A and B are added, curve C′ is obtained. This curve may be regarded as a first approximation to the arrival curve when condenser and shunt are combined. To obtain the true curve, first write down the formula for the periodic current in the form,

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl},$$

where Z_r in this case stands for $\dfrac{R_r}{1 + ipK_r \,.\, R_r}$.

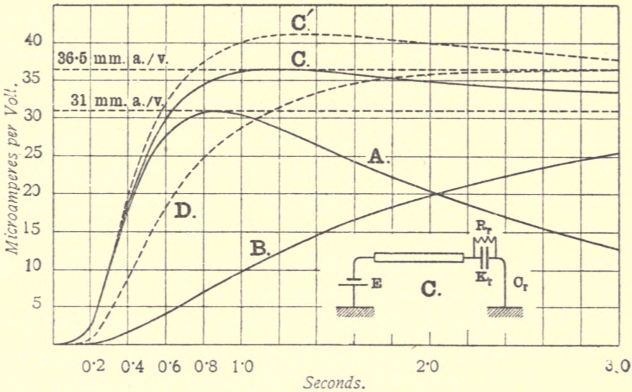

FIG. 156.—SHUNTED CONDENSER.

Curve A.—Condenser alone, $K_r = Kl/10$.
,, B.—Resistance alone, $Rl + R_r = 10^6/31$.
,, C′ = A + B.
,, C.—Condenser and resistance.
,, D.—No apparatus, reduced in proportion 36·5 : 201.

Transforming this expression in the usual manner, the current due to a single prolonged battery contact is

$$C_r = \frac{E}{Rl + R_r} + \sum_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{Rl\left[\dfrac{1}{\cos x} - \dfrac{\sin x}{x} + \dfrac{2K_r}{Kl} \sin x \tan x\right]}, \quad (33)$$

where $$\tan x = \frac{-x}{\mathrm{R}l/\mathrm{R}_r - x^2 \,.\, \mathrm{K}_r/\mathrm{K}l} \quad . \quad . \quad . \quad . \quad (34)$$

When R_r is infinite or K_r is zero, (33) reduces as it should to one of the formulæ already given and used in calculating curves A and B, Fig. 156.

The roots of (34) are contained in Table CIII., and curve C, Fig. 156, is plotted from (33) and Table CIV.

The arrival current for the combination of resistance and condenser is less in height, as was to be expected, than the sum of the two independent currents, but it resembles the sum of the two in shape. The maximum value of 36·5 mm.a./v. is reached in 1·2 second, after which the curve descends very gradually to its steady value of 31 microamperes per volt. Curve D is the arrival curve corresponding when the cable is to earth at the receiving end. The ordinates of curve D have been reduced in the proportion 36·5/201 in order that its steady value may be the same as the maximum value of curve C. It is evident that for signalling purposes the shape of curve C is much superior to that of curve D, embodying, as it does, a steeper rise to the steady value, which is practically attained in 1·2 second, since the slow decline which follows would produce only a gradual shifting of zero without affecting the legibility of the signals to any extent.

In Fig. 157, curves A and B, are shown the elementary dot signals for contacts lasting 0·04 second and 0·08 second respectively. They are obtained in the usual manner by displacement of curve C, Fig. 156. Comparing them with curve B, Fig. 152, and curve A, Fig. 153, it is seen that curve A, Fig. 157, although only one-third of the height of the curve for a contact of the same length without apparatus, reaches its turning value in 0·35 second, instead of 0·45 second, and that curve B, Fig. 157, although also reduced in roughly the same proportion, reaches its turning value in 0·32 second in place of 0·42 second; both curves are, therefore, considerably superior in shape to the corresponding curves without apparatus.

In Fig. 158, curve A is a curbed signal formed from curve B, Fig. 157, in which the contacts are both 0·04 second in duration.

Curve B, Fig. 158, is the signal "understand." It is constructed from the ordinates of curve A in the usual manner, which necessitates the calculation of curve A at short intervals. In this case 0·04 second has been considered sufficiently close. Curve B should be compared with curve C, Fig. 152, to which it is greatly superior in shape, and with curve C, Fig. 153, to which it is inferior both in height and in shape. It is evident that the curbing in this instance has been overdone. The negative dip

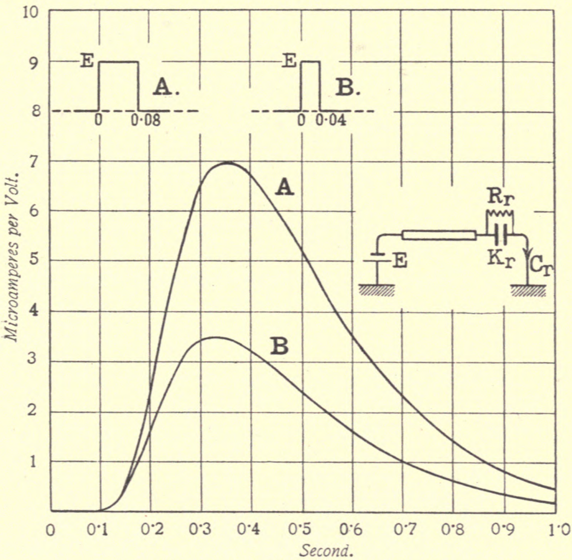

FIG. 157.—SHUNTED CONDENSER.

Curve A.—Dot signal, lasting 0·08 second.
,, B.—Dot signal, lasting 0·04 second.

in curve A at 0·5 second is too great, and gives the signal in curve B a downward tendency. Successive dots grow smaller, and the first dash after the three dots is exaggerated.

When curve A, Fig. 157, is compared with curve B, Fig. 154, it is seen to be greater in height but inferior in shape, its turning value being attained at 0·35 second against 0·3 second. The fact that the curves of Fig. 157 do not cross the zero line, whereas curve B of Fig. 154 does, renders the use of the shunted condenser of importance in certain methods of "through working." *

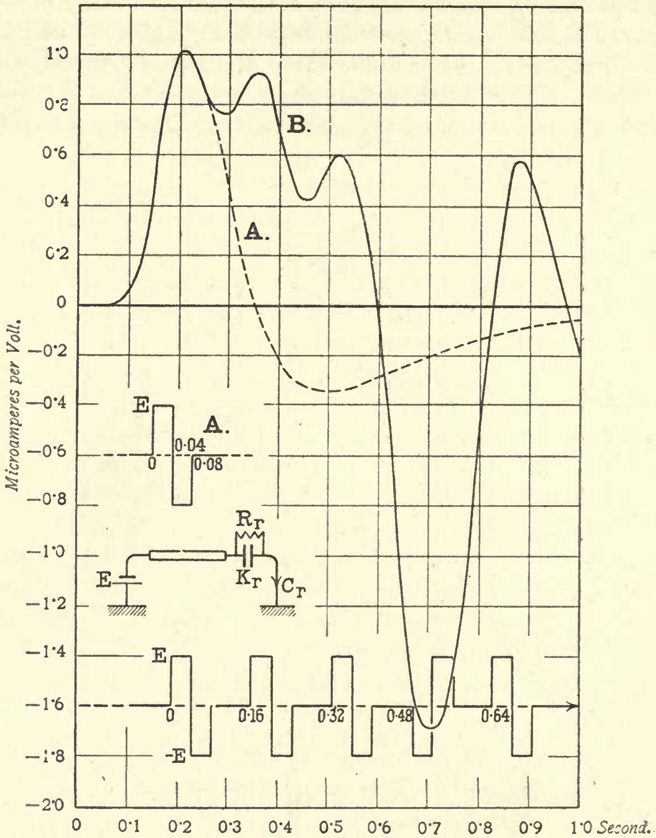

Fig. 158.—Shunted Condenser.

Curve A.—Curbed signal lasting 0·04+0·04 second.
" B.—Signal " Understand."

Limiting Shape of Curve—Condenser Zero.

It is interesting to trace what happens when the size of the condenser is reduced without limit. Take first the case of the con-

* Muirhead, *l.c.*

denser alone, without shunt, and in (30) let K_r be extremely small. Then $\tan x = \infty$, and $x = \frac{2m+1}{2}\pi$ ($m=0, 1, 2, \ldots$). At the same time, $\cos x = \cot x \times \sin x = \frac{K_r}{Kl} \cdot \frac{(2m+1)\pi}{2} \times \sin \frac{2m+1}{2}\pi$. Substituting in (29),

$$C_r = \sum_m \frac{(2m+1)\pi E e^{-\frac{(2m+1)^2\pi^2 t}{4KRl^2}}}{Rl} \cdot \frac{K_r}{Kl} \quad . \quad . \quad . \quad (35)$$

Now, the curve of arrival voltage when the receiving end is free is given by *

$$V_r = E - \sum_0^\infty \frac{4Ee^{-\frac{(2m+1)^2\pi^2 t}{4KRl^2}}}{(2m+1)\pi \sin \frac{2m+1}{2}\pi} \quad . \quad . \quad . \quad . \quad (36)$$

Hence, on differentiation of V_r, it is seen that

$$\underset{(K_r=0)}{C_r} = K_r \frac{dV_r}{dt} \text{ (end free)}. \quad . \quad . \quad . \quad . \quad (37)$$

as might have been anticipated.

In like manner from (31) on following out the same line of reasoning it is found that

$$\underset{(R_r=\infty)}{C_r} = \frac{V_r}{R_r} \text{ (end free)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Hence, for the combination,

$$\underset{(K_r=0,\ R_r=\infty)}{C_r} = \frac{V_r}{R_r} + K_r \frac{dV_r}{dt} \text{ (end free)}. \quad . \quad . \quad . \quad . \quad (39)$$

It should now be clear why it is advantageous to use signalling condensers at both ends of the cable instead of merely at one. For it has been shown that in the former case the limiting shape of the arrival curve, when the condensers are both zero, is the slope of the current arrival curve with receiving end to earth. Now, the current arrival curve with receiving end to earth rises much more quickly to its steady value than does the voltage arrival curve with receiving end free, the slope of which is the limiting or best shape of the arrival curve for a single condenser which is diminished indefinitely.

* Chap. IX., p. 283.

Again, from (37) it is seen that the height of the arrival current with a single small condenser is in the limit directly proportional to the size of the condenser, instead of as in (26), when there are two condensers, proportional to the square of the size of the condensers. Halving the single condenser will halve the arrival current, instead of reducing it to one-quarter, as when there are two condensers. The improvement in shape in the latter case is, therefore, gained at great expense in height.

Inductive Shunt.

The equation (39)

$$C_r = \frac{V_r}{R_r} + K_r \cdot \frac{dV_r}{dt}.$$

reciprocates into

$$V_r = R_r \cdot C_r + L_r \frac{dC_r}{dt}. \quad . \quad . \quad . \quad . \qquad (40)$$

The reciprocal of a condenser shunted by a resistance is, therefore, an inductive coil.* We may anticipate that the voltage arrival curve on an inductive coil will resemble in shape the current arrival curve through a shunted condenser.

The theory of the action of the inductive coil which is used as a shunt to the recorder has already been obtained.† Reference should be made to the curves given there, and they should be compared with the curves of Fig. 156.

Curbing with Condensers.

In Fig. 159 are brought together diagrammatically the various forms of signal which have been considered up to the present. By $f(t)$ is here understood the Kelvin or current arrival curve for the cable without apparatus. It has been shown that the shape to which the elementary dot signal approaches when the time of contact is made infinitely small is the slope of $f(t)$, or $dt \cdot f'(t)$; and, moreover, that the same shape is the limit of the

* Russell, "Alternating Currents," Vol. I., p. 384.

† Chap. IX., p. 289.

current due to a prolonged contact when signalling condensers are used and these are made infinitely small. Proceeding a stage further, a very short dot-dash, without apparatus, is $dt \,.\, f'(t) - dt \,.\, f'(t-dt)$, or $dt \,.\, f'(t) - dt[f'(t) - dt \,.\, f''(t)] = (dt)^2 f''(t)$. Short dot-dash without apparatus and short dot with small condensers have therefore in the limit the same shape. Finally, short dot-dash with small condensers has the shape $(dt)^3 f'''(t)$, and is the slope of the elementary dot curve.

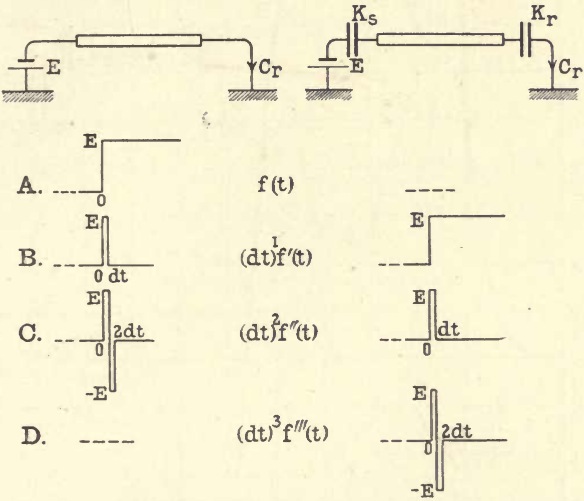

FIG. 159.—RECEIVED SIGNALS OF SAME SHAPE, WITH AND WITHOUT CONDENSERS.

Curve A.—Single prolonged contact; no apparatus.
„ B.—Infinitely short dot contact; or prolonged contact with Ks and Kr small.
„ C.—Short dot-dash; or short dot, condensers small.
„ D.—Short dot-dash, condensers small.

Since each stage in the derivation yields a curve which is of better shape than the foregoing, it may be expected that the last, $(dt)^3 f'''(t)$, will prove the best of all. To show the effect of curbing on the signals of Fig. 154 the curves in Fig. 160 have been constructed. Here the dot contact has been assumed to last for 0·06 second, and the subsequent dash for 0·02 second; each elementary dot-dash signal being separated from the next by a space of 0·08 second.

Fig. 160 should be compared with Fig. 154. Curve A, Fig. 160, is better than curve B, Fig. 154, in having its turning value at 0·26 second instead of 0·3 second, although the height is only

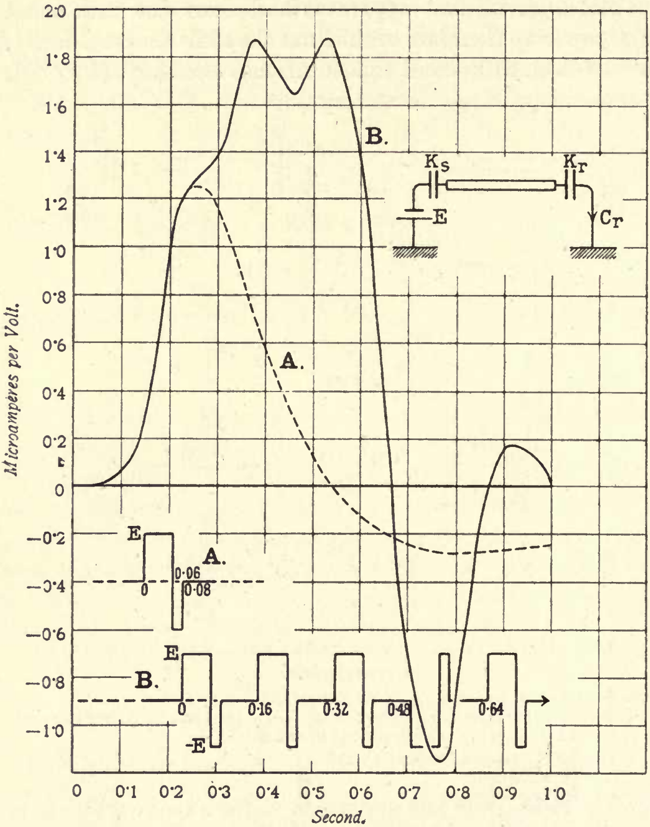

FIG. 160.—SLIGHT CURB, WITH CONDENSERS.

Curve A.—0·06 dot+0·02 dash.
„ B.—Signal "Understand."

half as great. In curve B, Fig. 160, the hollows are more distinct than in curve C, Fig. 154, although the curve has not lost its tendency to run upwards.

In Fig. 161 the curb is increased, the dot contact now lasting for only 0·04 second, and the dash for an equal length of time. The spacing is as before. Curve A now reaches its maximum

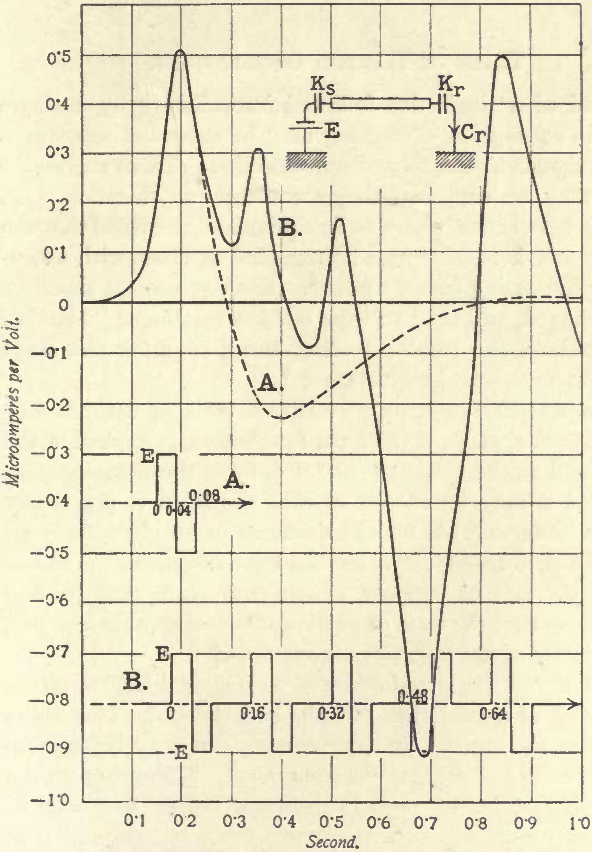

FIG. 161.—INCREASED CURB, WITH CONDENSERS.

Curve A.—Dot-dash, 0·04+0·04 second.
„ B.—Signal "Understand."

in 0·2 second. In curve B, dots and dashes are well marked and clearly separated, but the curve now runs downwards. The curbing in this case appears to be overdone.

The effect of further variations of length of contacts and spaces may be studied in the same manner. An interesting case would be one intermediate between those illustrated in Figs. 160 and 161.

TRAIN OF DAMPED OSCILLATIONS.

Instead of a single dot followed immediately by a single dash, the conception of curbing may be extended to cover a train of signals alternately of opposite sign. These successive signals may be supposed to grow shorter in duration; or, what has been shown above to be equivalent, provided that the contacts are short, they can be regarded as made with a continually decreasing battery power. Each successive signal, it may be hoped, will tend to wipe out the residue or "tail" of the preceding, the resultant effect being to bring the cable quickly to a neutral electrical state.*

To examine this case, suppose that a train of damped sine oscillations of E.M.F., of the form $Ee^{-at} \sin pt$, is applied at the sending end at the instant $t=0$. Here a is the damping constant, and it may be chosen so that the train is practically zero in a time suitable for the duration of an elementary (e) signal. Thus, if $a=40$, e^{-at} is less than one-fiftieth at the end of 0·1 second. Let us suppose in the first place that $p=2\pi n = 2\pi \times 10$, so that the train of oscillations consists, as in Fig. 162, B, practically of the first two semiperiods.†

The theory of the introduction of a sinusoidal E.M.F. at the sending-end of a cable has already been given.‡ It is there shown that the current at the receiving end may be split up into a periodic and a transient component. Following up the same reasoning in the present instance, the sending E.M.F.

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* This was the idea underlying the inventions of Sir Charles Bright and Sir William Thomson. *See* Charles Bright, "Submarine Telegraphs," p. 537. It forms also the basis of patent 21,939A of 1903, granted to Sir O. J. Lodge and Dr. Muirhead. Mr. C. F. Varley distinguishes this infinite series from the simple dot-dash by the term "true" curbing, although five or six contacts may be sufficient in practice.

† Lodge and Muirhead, *l.c.*

‡ Chap. XII., p. 360.

Table CV.—*Fig.* 162, *Curve B.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$m=$ 1+	8·498	6·774	5·400	4·304	3·431	2·180	1·104	0·355	0·114	0·012
2−	18·856	7·611	3·073	1·240	0·501	0·081	0·006	...	...	...
3+	15·454	2·008	0·261	0·033	0·003	...	...	...	...	...
4−	6·139	0·161	0·006	...	...	...	...	...	...	...
5+	1·171	...	...	...	...	...	...	...	...	...
6−	0·099	...	...	...	...	...	...	...	...	...
C_r(mm.a. per volt)	0·029	1·010	2·582	3·097	2·933	2·099	1·098	0·355	0·114	0·012

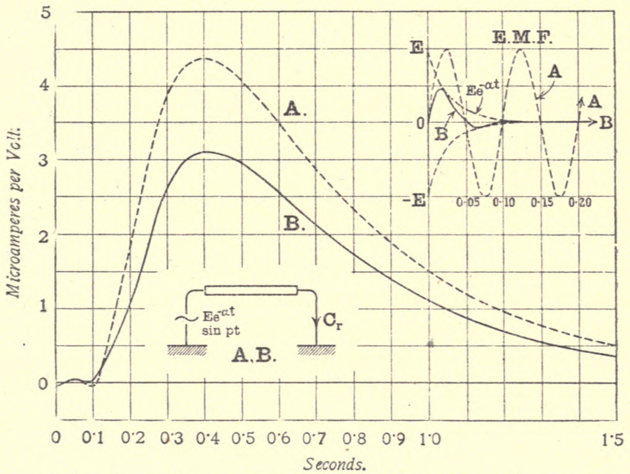

Fig. 162.—Transient Component of Damped Sine Wave; Ee^{-at} sin pt. No Apparatus.

Curve A.—$\alpha=0$. $p=2\pi\times 10$.
,, B.—$\alpha=40$. $p=2\pi\times 10$.

may be taken as $\mathrm{E}e^{-at}\,.\,e^{ipt}$. It is necessary, therefore, only to replace ip in the purely periodic reasoning by $ip-\alpha$ in the damped case.

This leads at once to the expression for the periodic component,

$$C_{r\,(\text{periodic})}=\frac{\mathrm{E}e^{-at}\,.\,e^{ipt}}{\sqrt{\mathrm{R}l(ip-\alpha)\mathrm{K}}\times\sinh l\sqrt{\mathrm{R}(ip-\alpha)\mathrm{K}}},$$

which decays rapidly, and may be neglected after the first tenth of a second, or complete oscillation. At the same time the transient component is found to be

$$C_{r\,(\text{transient})} = \frac{2Ee^{-at}}{Rl} \sum_1^\infty \frac{e^{-\frac{m^2\pi^2 t}{KRl^2}} \cdot \frac{m^2\pi^2}{KRl^2} \cdot \cos m\pi}{ip - a + \frac{m^2\pi^2}{KRl^2}}.$$

Multiplying above and below by $-ip-a+\frac{m^2\pi^2}{KRl^2}$, and neglecting the real part of the resultant expression, the transient component for an E.M.F. $Ee^{-at}\sin pt$ is found to be

$$C_{r\,(\text{transient})} = \frac{2Ee^{-at}}{Rl} \sum_1^\infty \frac{e^{-\frac{m^2\pi^2 t}{KRl^2}} \cos (m-1)\pi}{p \cdot \frac{KRl^2}{m^2\pi^2} + \frac{m^2\pi^2}{p \cdot KRl^2}\left(1 - \frac{aKRl^2}{m^2\pi^2}\right)^2}. \qquad (41)$$

The transient component in the damped case differs from the component in the case of continuous oscillations only in a single term in the denominator. It may, therefore, be calculated readily by making the necessary alterations to Tables LXXXI. and LXXXII.* In this way Table CV. is obtained.

Curve B, Fig. 162, is plotted from (41) and Table CV. Curve A is the corresponding transient component for an undamped periodic E.M.F. Curves A and B do not differ appreciably in shape, and curve B is only a little over two-thirds of the height of curve A.

Damped Train Continued—Influence of Signalling Condensers.

The arrival current for a damped train of oscillations when there are signalling condensers present may be obtained in a similar way by modification of the formula for continuous oscillations. Proceeding in this way, we arrive at the formula

* Chap. XII., pp. 363, 365.

$$C_{r\,(\text{transient})} = \sum_x \frac{Ee^{-\frac{x^2t}{KRl^2}}}{Rl\left[\frac{\sin x}{x}+\frac{1}{x}\frac{Kl}{K_r}\left(\frac{1}{\sin x}-\frac{\cos x}{x}\right)\right]}$$

$$\times\frac{-1}{\frac{p\,.\,KRl^2}{x^2}+\frac{x^2}{p\,.\,KRl^2}\left(1-\frac{a\,.\,KRl^2}{x^2}\right)^2}, \quad (42)$$

where $\tan x = \dfrac{2Kl/K_r}{x-\frac{1}{x}\left(\frac{Kl}{K_r}\right)^2}$.

Table CVI. is obtained from Table LXXXIV.,* and from it curve B, Fig. 163 is plotted.

Curve A is the corresponding curve for the undamped oscillations. Curve B is less in height than curve A, and does not offer any appreciable advantage over it in shape.

Table CVI.—*Fig.* 163, *Curve B.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
x_1-	0·343	0·292	0·250	0·213	0·182	0·132	0·082	0·037	0·017	0·003
x_2+	3·205	1·678	0·878	0·460	0·241	0·066	0·009	...	...	...
x_3-	6·511	1·460	0·327	0·073	0·016	0·001	...	...	...	...
x_4+	5·434	0·353	0·023	0·001	...	...	...	...	...	...
x_5-	2·132	0·026	...	...	...	...	...	...	...	...
x_6+	0·390	...	...	...	...	...	...	...	...	...
x_7-	0·033	...	...	...	...	...	...	...	...	...
x_8+	0·002	...	...	...	...	...	...	...	...	...
C_r(mm.a. per volt)	0·012	0·252	0·324	0·175	0·043	−0·067	−0·073	−0·037	−0·017	−0·003

Table CVII.—$p=2\pi\times 20$.

x.	$\frac{x^2}{p\,.\,KRl^2}$.	$\frac{pKRl^2}{x^2}$.	$\frac{x^2}{pKRl^2}\left(1-\frac{aKRl^2}{x^2}\right)^2$.	Sum.
2·6272	0·0126	79·238	7·406	86·644
5·307	0·0515	19·420	1·383	20·803
8·067	0·1190	8·402	0·334	8·736
10·909	0·2176	4·596	0·047	4·643
13·819	0·3492	2·864	0·003	2·867
16·782	0·5149	1·942	0·075	2·017
19·786	0·7158	1·397	0·221	1·618
22·817	0·9520	1·050	0·422	1·472

* Chap. XII., p. 372.

Table CVIII.—*Fig.* 163, *Curve C.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0
x_1 −	0·215	0·184	0·157	0·134	0·114	0·083	0·052	0·023	0·011
x_2 +	1·922	1·006	0·527	0·276	0·144	0·040	0·006	...	...
x_3 −	3·629	0·813	0·182	0·041	0·009	...	...	...	...
x_4 +	2·799	0·182	0·012	0·001	...	...	...	...	...
x_5 −	1·069	0·013	...	...	...	...	...	...	...
x_6 +	0·217	...	...	...	...	...	...	...	...
x_7 −	0·024	...	...	...	...	...	...	...	...
x_8 +	0·001	...	...	...	...	...	...	...	...
C_r(mm.a. per volt)	0·003	0·177	0·199	0·102	0·021	−0·044	−0·046	−0·023	−0·011

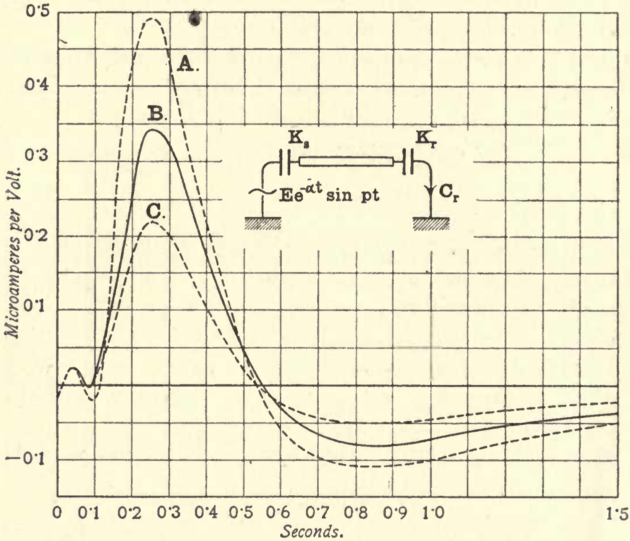

Fig. 163.—Transient Component of Damped Sine Wave with Condensers. $Ee^{-\alpha t} \sin pt$. $K = K_r = Kl/10$.

Curve A.—$\alpha=0$. $p=2\pi\times 10$.
,, B.—$\alpha=40$. $p=2\pi\times 10$.
,, C.—$\alpha=40$. $p=2\pi\times 20$.

To examine the influence of frequency, p may be increased to $2\pi\times 20$. There are then two complete oscillations in one-tenth of a second, by which time the E.M.F. has fallen as before, the damping being unaltered, to less than one-fiftieth of its initial value. Curve C, Fig. 163, is plotted from Tables CVII. and CVIII.

Its height is less again than that of curve B, and the improvement in shape, if any, is only slight. It is now clear why this method of signalling has not met with success, although advocated on high authority and supported by great inventive ingenuity, because even although the complications of the apparatus could be overcome, it would not lead to increased speed of working.

By differentiation it is easily found that the greatest value of $Ee^{-at}\sin pt$ occurs when $\tan pt = p/a$. Then $\sin pt = p/\sqrt{p^2+a^2}$ and the greatest value which the sending voltage attains is

$$V_{s\,(\text{max.})} = \frac{pE}{\sqrt{p^2+a^2}} e^{-\frac{a}{p}\tan^{-1}(p/a)} \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

Taking $a=40$, and $p=20\pi$, $V_{s\,(\text{max.})}$ is 0·445 volt when E is unity, and this value is reached at $t=0{\cdot}016$ second. If, therefore, the same maximum voltage is to be applied in the damped wave case as for the continuous wave, the amplitude of the damped wave may be increased initially in the proportion 1 : 0·445, or 2·25 : 1.

This magnification is carried out in Fig. 164, curve D, which is curve B of Fig. 163, with the ordinates increased in the above proportion. The signals produced by various forms of E.M.F. are brought together in this figure. Curve A is the result of a single semi-period lasting 0·05 second. Curve B shows the effect of a whole period, and curve C the effect of a continuous train, which may be regarded as built up out of curves A and B.

Of all four curves, curve B, although it is less than half the height of curve A, has the best shape, since it rises most quickly to its turning value. Nevertheless, its excellence in this respect is spoiled to a great extent by the deep trough by which the crest is followed at 0·4 second. It is necessary, therefore, to seek some means of removing this imperfection.

Method of Compensation.

In Fig. 164, curve A′ is curve A moved to the right through 0·10 second, and reduced in height to one-fifth. It is seen that its shape is not unlike that of the trough of curve B. When the ordinates of these two curves are added curve E is the result. Now, curve E has the same crest as curve B, but the depth of the trough is greatly reduced. Curve E is, therefore, much superior to curve B for signalling purposes.

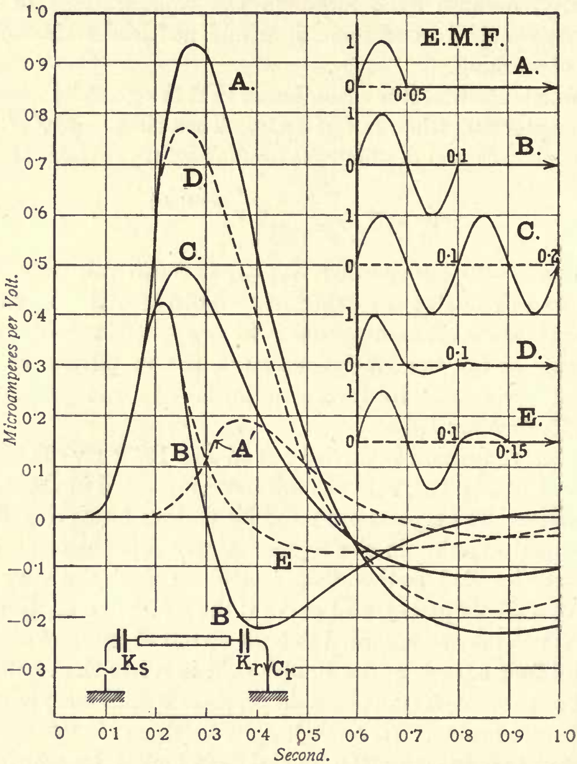

FIG. 164.—ELEMENTARY SIGNAL FOR VARIOUS E.M.F.S.

Curve A.—Single semi-period. $n=10$.
,, B.—Whole period.
,, C.—Continued train of oscillations.
,, D.—Damped train, $\alpha=40$, initial voltage 2·25.
,, E.—Compensated signal $E=B+A'$ ($A'=A/5$ displaced through 0·10 second).

The same method of compensation is illustrated for battery curves in Fig. 165. Here curve A is reproduced from Fig. 161. Curve B is a single 0·08 dot signal, as shown in Fig. 154, moved to the right through 0·16 second, and reduced in height to one-fifth. Curve C is found by adding curves A and B together.

Curve C preserves the steep descent of curve A, but the trough is nearly gone, and it is followed quickly by a gradual

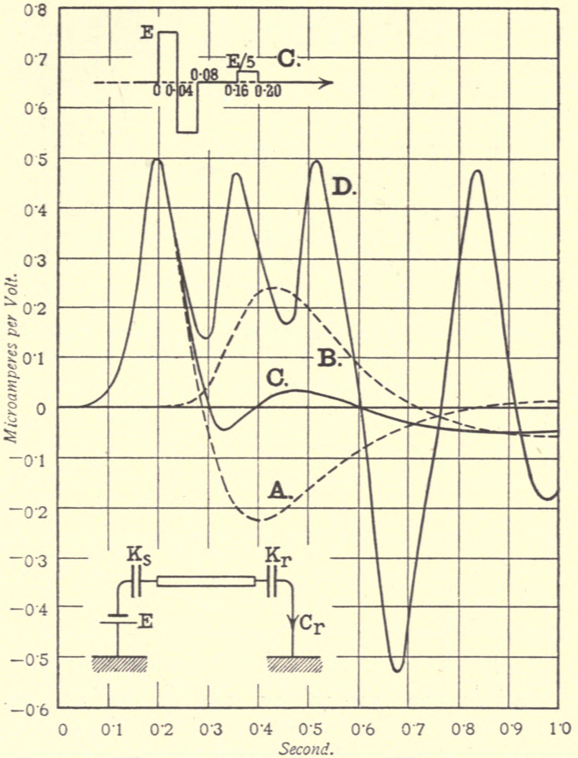

FIG. 165.—COMPENSATED CURVE WITH CONDENSERS.

Curve A.—Dot-dash, 0·04+0·04 second.
,, B.—0·2×single 0·04 dot, displaced to right through 0·16 second.
,, C.—Sum of A and B.
,, D.—Signal "Understand," using Curve C.

rise and fall. Its shape is excellent for signalling purposes. Curve D is formed from it in the usual manner. Any tendency to wander from the zero line is now reduced to a minimum; the successive dots are clearly marked and are well separated;

the dash is of nearly the same height as the dots. The signal is clear to read, and would operate a relay of fixed zero. For all practical purposes distortion has been entirely removed.

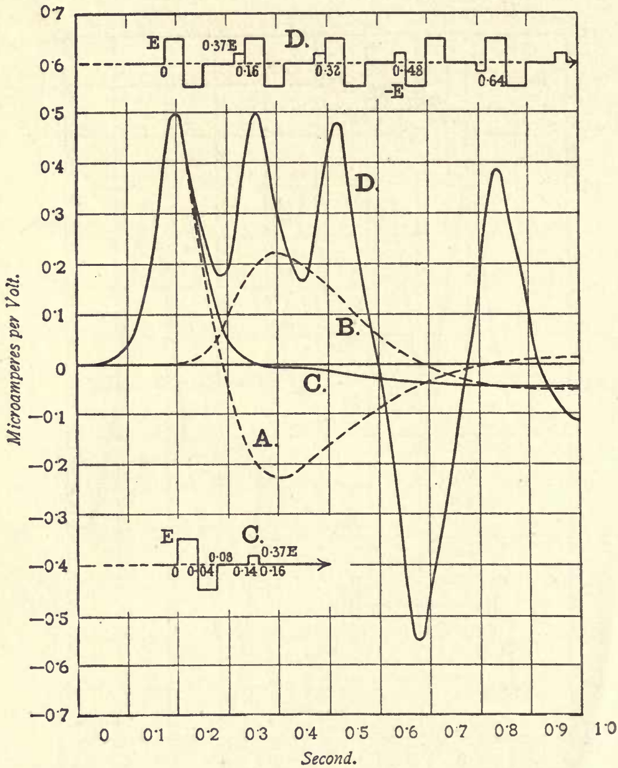

FIG. 166.—COMPENSATED CURVE.

Curve A.—Dot-dash, 0·04+0·04 second.
„ B.—0·3685 × single 0·02 dot, displaced through 0·14 second.
„ C.—Sum of A and B.
„ D.—Signal " Understand," using Curve C.

Fig. 165 should be compared with Figs. 161, 160, 154, 153 and 152, when the improvement which may be effected in this way is rendered striking. Comparing curve B of Fig. 154 with curve C of Fig. 165, it is seen that the elimination of distortion has been accomplished at the expense of height, which has been reduced from 22 microamperes per volt to 0·5 microampere per volt, or to less than one-fortieth.

It may be urged in objection to the way in which the compensation is effected in Fig. 165 that the elementary signals overlap, which would necessitate special arrangements. Accordingly, in Fig. 166 compensation is carried out by means of a dot signal of less duration but slightly greater height. The maximum of the first dot-dash signal composing curve D is reached in 0·2 second, and therefore the maximum of the second signal, which follows at 0·16 second interval, should occur at 0·36 second. If the second maximum is to be of the same height as the first, curve C, the compensated curve, must cross the horizontal axis at 0·36 second. Now at 0·36 second curve A has the value $-0·199$ microampere per volt, and if the 0·02 second compensating dot signal is to be introduced at 0·14 second, so that it may just be completed in the allotted time, the compensating curve must have the value $+0·199$ microampere per volt at $0·36—0·14=0·22$ second. The actual ordinate is 0·540, and accordingly the ordinates of curve B are those of the 0·02 dot signal reduced in the proportion 0·540 : 0·199, or 1 : 0·3685.

Curve C, found in this way, is not so good as curve C of Fig. 165, because it lacks the dip at 0·33 second, and, therefore, the hollows which reveal the spaces are not so deep. Moreover, the rise which follows in curve C, Fig. 165, at 0·48 second is wanting in Fig. 166, and consequently the dash following the dots is deepened, and the effect of the accumulated negative tails is still more apparent in the cutting away of the fourth dot.

In order to increase the speed of transmission we may cut out the spaces separating the elementary signals. In this way distortion would be re-introduced, and to counteract it the curbing may be increased. In Fig. 167 the dash or curbing signal is of the same length as the dot, but the E.M.F. which produces it is doubled. In consequence the crest of curve A is much reduced and the trough greatly increased. In fact, if curve A were turned upside down it would resemble curve A of Fig. 160. The curbing is overdone, and the message would appear inverted. Curve B is a single 0·04 second dot, displaced to the right through 0·06 second, and slightly increased in height. Curve C is the sum of curves A and B, and curve D is formed from curve C. Curve D is not so good as curves D of

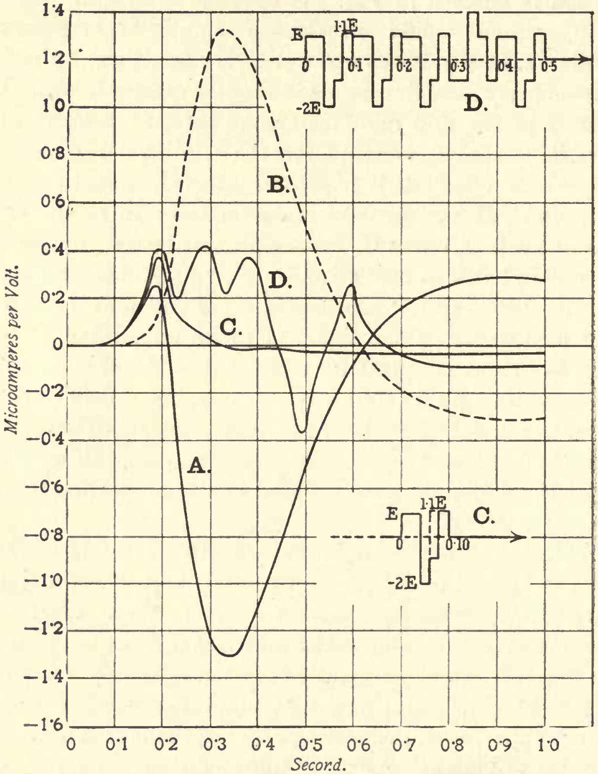

Fig. 167.—Compensated Curve; Greater Curbing.

Curve A.—Dot-dash, 0·04 + 2 × 0·04 second.
,, B.—1·11 × 0·04 dot signal displaced through 0·06 second.
,, C.—Sum of A and B.
,, D.—Signal "Understand."

Figs. 166 and 165, but it would still be satisfactory for recorder reading, or for operating a movable zero relay. It is slightly less in height than the two preceding curves, but the gain in speed is 60 per cent.

Ideal Signal.

It is unnecessary to pursue this subject into further detail. Sufficient has been said if the principle has been established which was enunciated at the beginning. The importance of curbing has been brought out and shown to lie in the fact that it enables the slope of the current curve to be used instead of the current curve itself. In the foregoing demonstration this procedure has been carried to the third stage, and it may obviously be carried indefinitely further by the aid of transformers. At each stage the crest of the current curve moves further to the left and the signals are sharpened. Moreover, it has been shown that the drawbacks to this procedure may be overcome by the introduction of a suitable compensating current. The success of the method depends on the correct choice of position and magnitude or duration for the compensating signal, and hence the importance of an exact knowledge of the arrival curve and its derivatives is accentuated. This knowledge would be best acquired from experimental study in the actual working conditions.

The Use of Magnifiers.

Since all the methods that have been considered up to the present for improving the definitions of the signals lead to a loss in strength—whether by direct curbing, or by indirect curbing such as the use of signalling condensers, transformers and inductance shunts entails—it is clear that if the process be carried too far it may happen that the strength of the signals is finally insufficient to produce a legible reading on the siphon recorder or other receiver. To increase the sensitivity without affecting the period or the robustness of the recorder is difficult, and an increase in the battery strength would have undesirable consequences at the sending end. There remains the possibility of the introduction of a relay.

Two such relays have recently come into prominence for telegraphic purposes—the Heurtley magnifier and the Orling water jet. Both instruments are characterised by (1) extreme sensitiveness and (2) high periodicity, produced by a strained suspension. It follows that the amplitude of vibration is in both cases very slight.

The explanation which is given of the working of these magnifiers is that since the motion of the coil is slight it does not deviate sensibly from the region in which the magnetic field is constant, and therefore distortion, which accompanies the wider angle through which the recorder coil moves, is eliminated. But any distortion which is introduced by the siphon recorder in this way is probably insignificant. In the light of the preceding study of arrival curves no such explanation is necessary. Turning to such a curve as that plotted in Fig. 166, curve D, it is seen that the essential characteristics of a receiver which is to follow the variations of current are (*a*) quickness in action and (*b*) sensitivity—quickness of movement in order that the crispness of the signal may be preserved, and sensitivity that full advantage may be taken of all curbing devices.

It is not necessary that the relay should give an accurate reproduction of the current actuating it. On the contrary, it may introduce a beneficial distortion. If its movement is underdamped it will overshoot the mark, and lead to the exaggeration of crests and hollows. To secure the best effect it is obvious that the semi-period of the coil movement should coincide with the length of the elementary dot signal, and as the speed of signalling is increased so must the periodic time of the coil be reduced in proportion.

One of the earliest forms of magnifier was that invented by Mr. Charles Cuttriss.* In this instrument the receiving-coil, actuated by the arrival currents, opens and closes two carbon helices which form the arms of a Wheatstone bridge. In consequence their resistances are affected and the balance of the bridge, in one diagonal of which is the local recorder, is upset. The helices were filled up with powdered carbon, so that they acted in a similar way to the carbons (under varying pressures) in Edison's carbon relay. In an improve-

* THE ELECTRICIAN, XXIX., 1892, p. 147. "El. Rev.," XXXI, 1892, p. 8. *See also* E. Raymond-Barker, "El. Rev.," XXXI., 1892, p. 94.

ment* on the Raymond-Barker double differential water relay the movable coil carries light conducting arms which dip each into a short narrow channel containing conducting liquid, along which is a fall of potential from a local battery. The local receiving or translating apparatus is joined up across the conducting arms.

Why these instruments, although establishing the principle that speed could be gained by the use of a magnifier, met with only partial success compared with that which has been obtained with the later forms, may be explained as follows. In order to secure rapidity of motion the suspension must be stiff (bifilar). Hence the deflection produced in the receiving coil by the feeble arrival currents is microscopic. In consequence the motion of the coil must be critical—*i.e.*, the whole change of resistance or potential must be concentrated into the smallest possible space. This requirement is fulfilled in two different ways in the magnifiers described below.

THE HEURTLEY MAGNIFIER.†

In the magnifier invented by Mr. E. S. Heurtley, of which the form adopted in practice is shown diagrammatically in Fig. 168, the motion of the receiving coil is communicated through silk fibres to an aluminium cradle. This cradle carries two fine platinum wires (0·5 mil in diameter), which are heated by current from a local battery, and form two arms of a Wheatstone bridge. The tubes T_1 and T_2 communicate with some source of cold air; they are slit down their entire length where they are adjacent to the wires W_1 and W_2. The positions of the tube slits are so adjusted that a deflection of the coil in one direction brings W_1 into the cold blast, W_2 remaining out of the cold blast—that is, remaining hot—whilst a deflection in the opposite direction brings W_2 into the cold blast, W_1 remaining out. In this manner reversal currents received from the cable are caused to alter the bridge balance in one direction

* British patent No. 15,853 of 1908. *See also* W. Judd, THE ELECTRICIAN, XXIX., 1892, p. 637.

† British patent No. 17,555 of 1909. *See also* Nos. 13,337 of 1910 and 8,397 of 1911.

or the other, thereby producing reversal signals on the local recorder. Similar results may be obtained by the coil being caused to move screens over one slit or the other. To compensate for the time lag in the heating and cooling of the wires, inductances—or capacities—may be inserted in the bridge arms

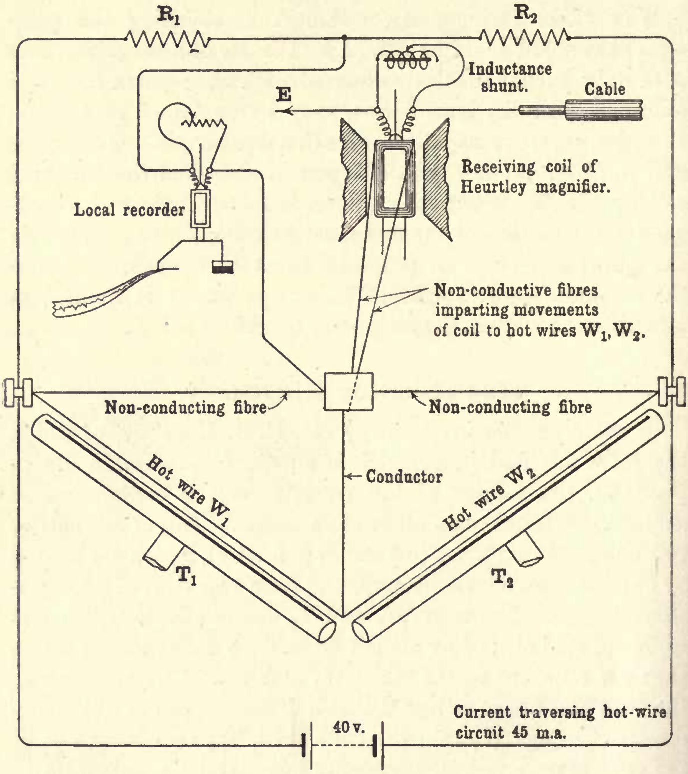

FIG. 168.—CONNECTIONS OF THE HEURTLEY MAGNIFIER.

to square up the signals. This instrument is said to be capable of receiving at the rate of 75 words a minute with an actuating current of 0·7 mfd.* It is in use continuously on the long

* THE ELECTRICIAN, LXXII., 1913, p. 433.

cables of no fewer than five companies. On a cable worked duplex it increased the speed from 135 to 190 letters per minute, or by 40 per cent. in both directions.*

The Orling Jet Relay.†

In the magnifier invented by Mr. Axel Orling the receiving coil is caused to deflect a jet of water, through the instrumentality of a quartz fibre. In the latest form of this instrument, shown in Fig. 169, the jet of conducting water impinges

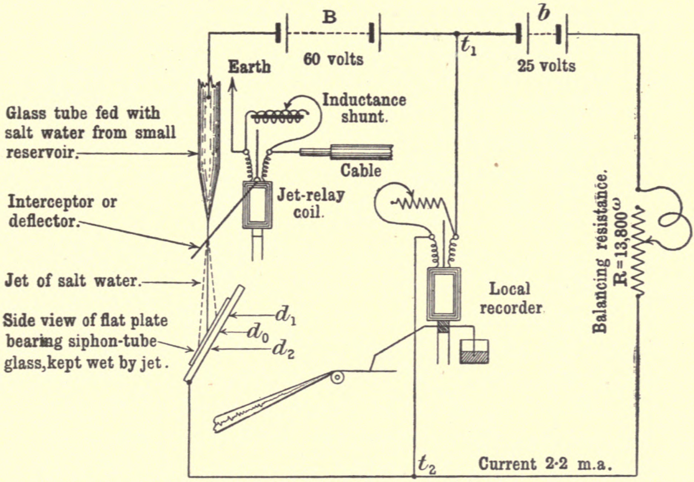

Fig. 169.—Varying-resistance Type of Orling Jet Relay.

Good results are also obtained with B=45 volts; b=18 volts; R=5,300 ohms c=3·4 m.a.

on an inclined plane which carries a number of fine glass tubes. In series with the jet is a local battery, and the current from it through the local recorder is balanced by a second smaller

* "Archiv. f. Post u. Teleg.," 1912, p. 65.

† British patent No. 18,001 of 1911. For the particulars given here of this instrument, as well as of the Heurtley magnifier, the writer is indebted to original notes and diagrams kindly supplied by Mr. E. Raymond-Barker.

battery through an adjustable resistance. Alternatively a second winding may be used on the recorder coil. When the jet is moved by the quartz fibre from its normal position d_0 to make contact with the inclined plate at d_1, the length of the jet is reduced and its resistance decreases; when the jet is moved to d_2 the resistance increases. The current fluctuations produced by these changes of resistance actuate the local recorder. It is a peculiar and characteristic property of the instrument that a microscopic deflection of the coil and quartz fibre suffices to produce a large deflection of the jet. The sensi-

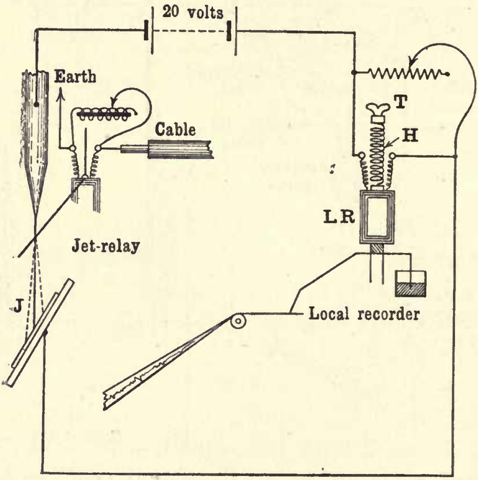

FIG. 170.—ORLING JET RELAY—IMPROVED ARRANGEMENT OF LOCAL CIRCUIT.

tivity of the instrument may be varied in different ways. Thus, conveniently, the angle of incidence of the jet on the plate may be adjusted by altering the angle of inclination of the plate to the vertical. Again, a fixed deflector may be placed between the moving deflector and the plate, and this process may be repeated without limit, and without the moving coil being called upon to perform any extra work. The instrument may obviously be used as a make-and-break relay, in which all contact trouble has been eliminated. Employed in this way it is stated to require a current of only 0·01 microampere to give good signals. This instrument appears to combine in a high degree the opposing requisites of high sensitivity and high periodicity, and the bifilar suspension under tension—which may amount to as much as 7 lb.—is sufficient to ensure stability. Remarkable results are claimed for its use on long cables; increases of speed amounting to as much as 125 per cent. have been obtained, and a speed constant of 25 letters per KRl^2.

A somewhat different form of connections is shown in Fig. 170. Here the mechanical bias afforded by the helical suspension H, and adjustable by the thumb-screw T, counteracts the coil deflection due to the local battery. The effective current in the local circuit is about 3 milliamperes.

Conclusions.

The introduction of the magnifier, although it permits the extension of the application of methods for securing increased definition, brings with it attendant disadvantages. At the sending end of the cable in duplex working the sensitive instrument is exposed to the shocks which arise should any inequality exist in the duplex balance. The smaller the ratio which the received current bears to the sent current, and, therefore, the greater the sensitivity of the receiving instrument, the greater must be the care exercised to secure an exact reproduction of the cable in the artificial line. These demands have grown to such a pitch that they bar the way to much further advance in this direction.

The question thus arises: What can be done to remove this deadlock? The answer is not far to seek. Distortion, it has been shown, may be reduced indefinitely, but at the cost of increased attenuation. The next step must be to reduce the attenuation. Now, the only way of reducing attenuation, if the other cable constants are to remain the same, is to increase the inductance of the cable. We are thus led inevitably to the study of loading in the submarine telegraph cable.

CHAPTER XV.

THE DUPLEXED CABLE.

Periodic Received Current—Particular Cases—Cable Insulated at Receiving End—Resistance Arms—Condenser Arms—General Proof that the Recorder Current is Unaffected by Position of Receiving-end Key.

PERIODIC RECEIVED CURRENT.

Before leaving the subject of the preceding chapter, it is important to determine to what extent, if any, the conclusions there drawn must undergo modification when they are applied to the duplexed cable. In order that communication may be carried on in both directions at once along a telegraph cable, it is the practice, when the traffic is at all considerable, to work the cable duplex. The arrangement of apparatus may be according to the scheme of Fig. 171, Diagram A. Here K_s, K_s are the sending-end condensers, constituting the arms of a Wheatstone bridge, of which the siphon recorder (S.R.) with its inductive shunt (I.S.) forms one diagonal. A.C. is the artificial cable, which in the ideal case would be an exact duplicate of the real cable. If, now, the balance is in adjustment, equal currents flow from the sending-end battery (E_s) into the cable and its artificial equivalent; the fluctuations of potential at the ends of the recorder diagonal are simultaneous and equal, and no current passes through the recorder. At the receiving end the arrival current forks, flowing partly through the condensers (K_r) and partly through the receiving recorder, which is deflected accordingly.

Let a single contact be made and maintained with one pole of the battery at the sending end, and let the current through the receiving recorder be calculated. Instead of diagram A,

Fig. 171, use diagram B. Here the sending end connections are simplified by omitting one bridge arm and the artificial line, which is justifiable for the present purpose, since when the balance is exact their only effect, so far as the receiving end is concerned, is to produce a greater drain on the battery (of zero resistance), which is required to furnish a duplicate of

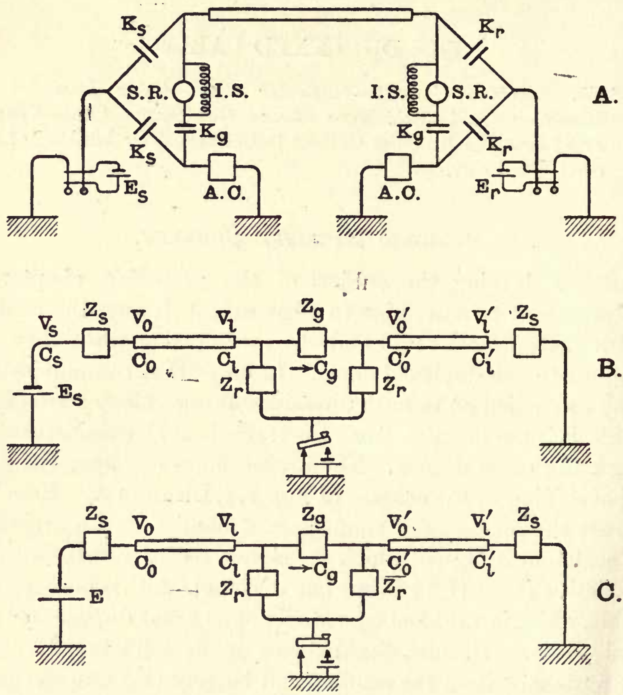

FIG. 171.—DUPLEX WORKING.

Diagram A. General Arrangement of Apparatus. Cable duplexed.
" B. Schematic representation of same; receiving end to earth.
" C. " " " " " insulated.

the current entering the real cable. V_0, C_0 and V_l, C_l are the voltage and current at the beginning and end of the real cable; V_0', C_0' and V_l' C_l' the voltages and currents for the receiving end artificial line. The cable is supposed to be to earth at the receiving end.

The following 10 equations then hold :—

For the cables,

$$\left.\begin{aligned} V_l &= V_0 \cosh Pl - Z_0 C_0 \sinh Pl, \\ C_l &= C_0 \cosh Pl - (V_0/Z_0) \sinh Pl, \\ V'_l &= V_0' \cosh Pl - Z_0 C_0' \sinh Pl, \\ C_l' &= C_0' \cosh Pl - (V_0'/Z_0) \sinh Pl, \end{aligned}\right\} \quad . \quad (44)$$

and for the apparatus,

$$\left.\begin{aligned} V_l - V_0' &= Z_g C_g, \\ V_l &= Z_r(C_l - C_g), \\ V_0' &= Z_r(C_g - C_0'), \\ V_s - V_0 &= Z_s C_s, \\ V_l' &= Z_s C_l', \\ C_s &= C_0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

From these 10 equations in 11 unknowns the ratio of any two can be found. In the present instance the ratio C_g/V_s is required. On substitution from one equation to another in turn the result is found to be

$$C_g = \frac{V_s}{(2Z_s + Z_g + Z_s Z_g/Z_r) \cosh Pl + (2Z_0 + Z_0 Z_g/Z_r + Z_s Z_g/Z_0) \sinh Pl}. \quad (46)$$

Particular Cases.

Some particular cases of (46) are of interest. Thus (*a*) let Z_g be very small, practically zero. Then

$$C_g = \frac{V_s/2}{Z_s \cosh Pl + Z_0 \sinh Pl}. \quad . \quad . \quad . \quad . \quad (47)$$

The recorder current is exactly half what it would be if the cable were worked simplex, through an impedance Z_s at the sending-end, and with the receiving end to earth. Again (*b*) let $Z_r = \infty$. Then

$$C_g = \frac{V_s/2}{(Z_s + Z_g/2) \cosh Pl + \left(Z_0 + \dfrac{Z_s Z_g/2}{Z_0}\right) \sinh Pl} \quad . \quad . \quad (48)$$

The current is half the simplex current with Z_s at the sending end, and $Z_g/2$ at the receiving end.

Finally, (*c*) let $Z_g=Z_r$. Then,

$$C_g=\frac{V_s/3}{(Z_s+Z_g/3)\cosh Pl+\left(Z_0+\frac{Z_sZ_g/3}{Z_0}\right)\sinh Pl}. \quad (49)$$

The current is now one-third of that which would actuate the recorder in simplex working, with apparatus Z_s at the sending end, and $Z_g/3=Z_r/3$ at the receiving end. If Z_r and Z_g consist chiefly of condensers this reduction in effective impedance is equivalent to a threefold increase in capacity. In order that the wave-shape may be as good as in simplex working the capacity of the condensers must be reduced, with simultaneous reduction in the height of signals. By this action Z_r would be affected but not Z_s, and transmission would not be symmetrical, but would depend on the direction in which traffic is moving, to obviate which a compromise may be effected by decreasing the capacity in the diagonal Z_g only, and leaving $Z_r=Z_s$. The height of the curve would then suffer, on account of the increased shunting action of the bridge arm condensers.

Cable Insulated at Receiving End.

Now, suppose that the key at the receiving end is depressed slightly—so that contact is broken through the back-stop to earth—but not sufficiently to make the circuit of the battery there, as in diagram C, Fig. 171. The total current issuing from the real cable and entering the artificial cable may be obtained from (46) by putting $Z_r=\infty$, and substituting $\frac{2Z_gZ_r}{Z_g+2Z_r}$, the impedance of the apparatus between the cables, for Z_g. Of this total current, the fraction $\frac{2Z_r}{2Z_r+Z_g}$ passes through the galvanometer.

Hence—

$$C_g=\frac{V_s\cdot\frac{2Z_r}{2Z_r+Z_g}}{\left(2Z_s+\frac{2Z_rZ_g}{2Z_r+Z_g}\right)\cosh Pl+\left(2Z_0+\frac{Z_s}{Z_0}\cdot\frac{2Z_gZ_r}{2Z_r+Z_g}\right)\sinh Pl},$$

$$=\frac{V_s}{\left(2Z_s+\frac{Z_sZ_g}{Z_r}+Z_g\right)\cosh Pl+\left(2Z_0+\frac{Z_0Z_g}{Z_r}+\frac{Z_sZ_g}{Z_0}\right)\sinh Pl},$$

which is the same as (46). The current through the receiving end recorder is independent of the position of the signalling key there.

Resistance Arms.

To illustrate the effect of the apparatus used in duplexing in modifying the curves obtained in simplex working, two important cases will be considered below. In the first place let $Z_r = Z_g = R_g$, where R_g is a pure resistance, and let $Z_s = 0$.

Then

$$C_g = \frac{V_s}{Z_g \cosh Pl + 3Z_0 \sinh Pl}. \quad . \quad . \quad . \quad (50)$$

Here,

$$f(x) = R_g \cos x + \frac{3Rl}{x} \sin x,$$

$$\therefore \tan x = -\frac{R_g \cdot x}{3Rl}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and finally

$$C_g = \frac{E}{3Rl + R_g} + \frac{2E}{3Rl} \sum_x \frac{e^{-\frac{x^2 t}{KRl^2}}}{\left(\frac{1}{\cos x} - \frac{\sin x}{x}\right)}. \quad . \quad . \quad (52)$$

Take R_g as $3Rl/10$. The roots of (51) are then the same as in the simplex case* when R_g is $Rl/10$. If R_g were the same in the duplex case as in the simplex its effect would be less than in simplex working. Hence it is seen that the influence of the duplex connections when the receiving arms and diagonal are equal resistances is (1) to improve the shape of the current arrival curve by lowering the impedance of the apparatus to one-third, but (2) at the expense of a reduction to one-third the height.

Condenser Arms.

Now let the receiving arms of the bridge be condensers K_r. From (46) as before

$$C_g = \frac{V_s}{Z_g \cosh Pl + \left(2Z_0 + \frac{Z_0 Z_g}{Z_r}\right) \sinh Pl},$$

where $$Z_g = R_g \text{ and } Z_r = 1/ipK_r.$$

* Chap. IX., p. 265.

Hence, $$f(x)=\mathrm{R}_g\cos x+\mathrm{R}l\left(\frac{2}{x}-\frac{\mathrm{R}_g\mathrm{K}_r'}{\mathrm{KR}l^2}x\right)\sin x,$$

where $$\tan x=\frac{-\mathrm{R}_g}{\mathrm{R}l\left(\frac{2}{x}-\frac{\mathrm{R}_g\mathrm{K}_r x}{\mathrm{KR}l^2}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and, finally,

$$\mathrm{C}_g=\frac{\mathrm{E}}{2\mathrm{R}l+\mathrm{R}_g}-\sum_x\frac{2\mathrm{E}e^{-\frac{x^2t}{\mathrm{KR}l^2}}}{\mathrm{R}_g\left(\frac{x}{\sin x}+\cos x\right)+\frac{4\mathrm{R}l}{x}\sin x} \quad . \quad . \quad (54)$$

Equation (53) is the same as (34), which gives the roots in the case of a shunted condenser in simplex working, if $\mathrm{R}_g/2$ be substituted for R_r.

Take R_g as $\mathrm{R}l/10$ and K_r as $\mathrm{K}l/10$. Then (53) becomes $\tan x=\dfrac{10}{x-\frac{200}{x}}$, the roots of which are contained in Table CIX.

Curve B, Fig. 172, is plotted from (54) and Table CX.

Table CIX.—*Roots of* $\tan x=\dfrac{10}{x-200/x}$.

x.		Sin x.	Cos x.
2·9866	$\pi-8°\ 53$	+0·15442	−0·98800
5·9373	$2\pi-19°\ 49$	−0·33901	+0·94078
8·8020	$3\pi-35°\ 41$	+0·58331	−0·81225
11·5235	$4\pi-59°\ 45$	−0·86384	+0·50377
16·8206	$5\pi+63°\ 45$	−0·89687	−0·44229

In Fig. 172 are shown also in curve A the arrival current simplex without apparatus, both ends to earth, and in curve C, the arrival current simplex with a resistance R_g at the receiving end. Both these curves are reduced in scale to be one-half of their actual height. Curve B lies between them, and if R_g were infinitely small it would coincide with curve A. If, on the other hand, K_r were diminished, curve B would approach curve C, but would stop short of reaching it, since when K_r is zero the arrival curve duplex is one-half of the simplex curve with a resistance $R_g/2$ at one end. This result follows at once as a particular case of the general proposition which was given above for any kind of apparatus, or it may be deduced readily from (53) and (54) by putting $K_r=0$.

Table CX.—*Fig.* 172, *Curve B.*

t (sec.).	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady	95·71	95·71	95·71	95·71	95·71	95·71	95·71	95·71	95·71	95·71
x_1—	160·38	130·65	106·45	86·71	70·64	46·88	25·35	9·10	3·27	0·42
x_2+	94·81	42·18	18·76	8·35	3·71	0·73	0·06	...	...	...
x_3—	40·04	6·75	1·14	0·19	0·03	...	...	...	...	...
x_4+	12·01	0·57	0·03	...	...	...	...	...	...	...
x_5+	0·28	...	...	...	...	...	...	...	...	...
C_r(mm. a. per volt)	2·39	1·06	6·91	17·16	28·75	49·56	70·42	86·62	92·45	95·29

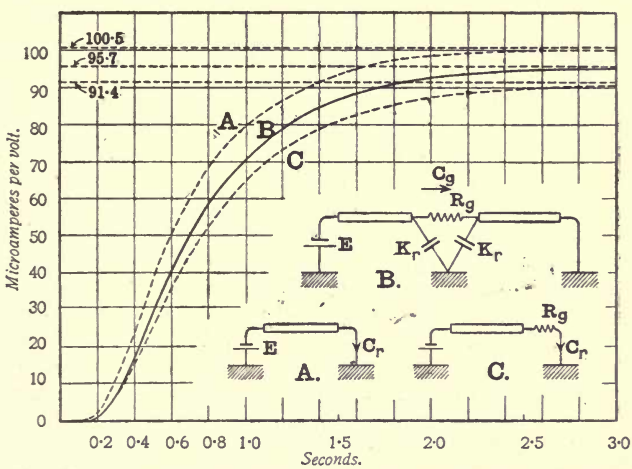

FIG. 172.—ARRIVAL CURRENTS: DUPLEX AND SIMPLEX COMPARED.

Curve A. Simplex; no apparatus; ordinates halved.
„ B. Duplex. $Rg=Rl/10$. $K_r=Kl/10$.
„ C. Simplex with $Rg=Rl/10$; ordinates halved.

If R_g were small, and if a condenser $K_g=K_r$ were inserted in the receiving diagonal so as to make its impedance approximately the same as that of the bridge arms, then, as was shown above generally, the arrival curve would be one-third of the simplex curve with a condenser $3K_g$ at one end. Thus, if the condensers were each equal to $Kl/30$, the arrival curve would be the same as curve A, Fig. 88,* but with the ordinates reduced one-third.

The conclusions to which we have just been led may be summarised as follows. *The shape of the arrival curve in duplex working may be slightly better or worse, depending upon the connections, than in simplex working, but its height is invariably reduced—in the most favourable case to one-half the simplex curve.*

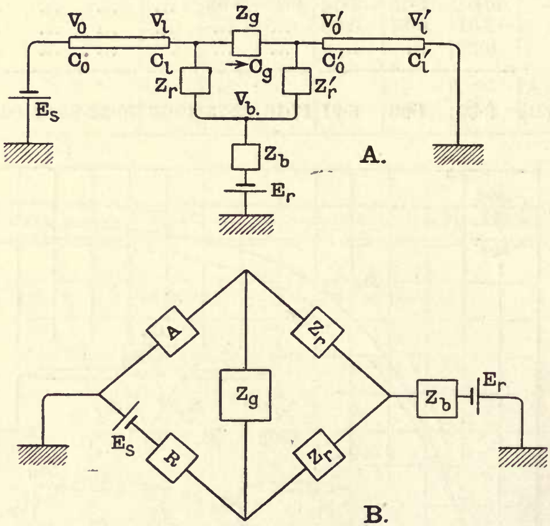

FIG. 173.—INFLUENCE OF RECEIVING-BATTERY RESISTANCE.

Diagram A. Impedance in Receiving Battery Circuit.
„ B. Equivalent Localised Wheatstone Bridge.

GENERAL PROOF THAT THE RECORDER CURRENT IS UNAFFECTED BY POSITION OF RECEIVING-END KEY.

A more general proof of this result may be obtained from the following considerations. In Fig. 173, diagram A, an impedance, Z_b, is inserted in the battery circuit at the receiving end: when Z_b is zero the receiving end is to earth, and when infinite,

* Chap. IX., p. 272.

the receiving end is insulated. The bridge arms Z_r and Z_r' are not necessarily equal.

Writing down the periodic equations as before, and solving them, after some elimination, finally,

$$C_g = \frac{V_0}{Z_g \cosh Pl + 2Z_0 \sinh Pl + \frac{Z_0 Z_g}{Z_r}\left[\frac{2Z_b + Z_r' + Z_0\left(\frac{Z_r' - Z_r}{Z_g} + 1\right)\tanh Pl}{Z_b + Z_r' + \frac{Z_b Z_r'}{Z_r} + Z_0 \tanh Pl}\right]\sinh Pl}. \quad (55)$$

When $Z_r' = Z_r$ the quantity in the square bracket becomes unity, and

$$C_g = \frac{V_0}{Z_g \cosh Pl + \left(2Z_0 + \frac{Z_0 Z_g}{Z_r}\right)\sinh Pl},$$

which is independent of Z_b. *Provided that the cable is perfectly balanced at the receiving end the received signals are independent of the impedance between the apex of the bridge and earth.* If the balance is not perfect a correcting factor which contains Z_b must be applied as in (55).

The equivalent theorem in the non-distributed case is illustrated in diagram B, where cable and artificial are shrunk to pieces of apparatus R and A. If the bridge is balanced, any charge in Z_b does not affect the current from E_s through Z_g.

CHAPTER XVI.

THE LOADED TELEGRAPH CABLE.

INTRODUCTION.

The employment of loading in cable and long-distance telephony has been attended with such phenomenal success that it is of the greatest importance to ascertain whether any corresponding improvement can be expected from loading in telegraphy. The question is of especial interest in submarine telegraphy where the line cannot be split up into sections and worked through repeaters as on land.

In order that the effect of artificial increase of inductance in a telegraph cable may be rendered evident, the cable is replaced, as before, by a meshwork, the number of meshes of which is increased until it ultimately coincides with a cable of uniformly distributed constants.

Single Inductive Mesh.

Taking the simplest case of all first, suppose that the capacity of the cable is concentrated in a single condenser Kl, as in Fig. 174, in series with a coil of resistance Rl and inductance Ll. The periodic current may be written down at once. It is, symbolically,

$$C=\frac{Ee^{ipt}}{Rl+ipLl+\frac{1}{ipKl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

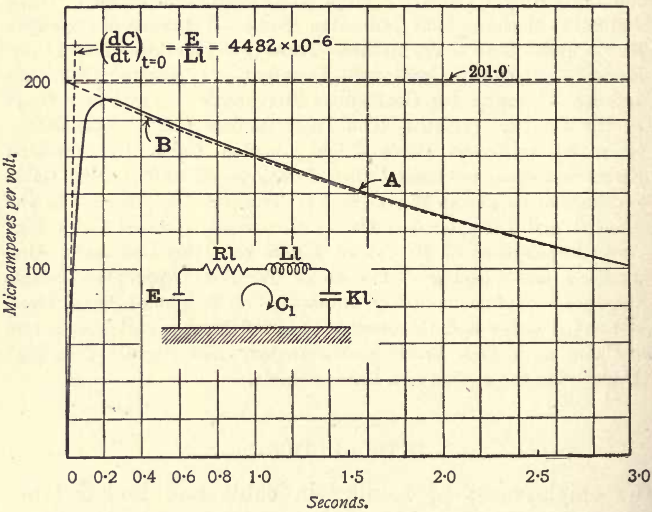

Fig. 174.—Charging a Condenser through an Inductive Coil.

Kl=875 mfd. Rl=4,975 ohms. KRl^2=4·352 sec. $L/R=KRl^2/100$.

Here $\quad \phi(y)=Rl+yLl+\frac{1}{yKl},$

from which $\quad y=-\frac{R}{2L}\pm\sqrt{1-\frac{4}{KRl^2}\cdot\frac{L}{R}}.$

Also $\quad y\phi'(y)=\pm Rl\sqrt{1-\frac{4}{KRl^2}\cdot\frac{L}{R}},$ and $\phi(o)=\infty.$

Table CXI.—*Fig.* 174, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$\frac{Rt}{2L}\sqrt{1-\frac{4L}{R.KRl^2}}$	1·126	2·252	3·378	4·503	5·629	7·880	11·259	16·888	22·517	33·777
Sinh do.	1·3793	4·6994	14·635	45·143	139·21	1321·9	38·741	$1·078\times10^{7}$	$3·002\times10^{9}$	$2·326\times10^{14}$
$e^{-\frac{Rt}{2L}}$	0·3170	0·1005	0·0319	0·0101	$3·201\times10^{-3}$	$3·215\times10^{-4}$	$1·025\times10^{-5}$	$3·285\times10^{-8}$	$1·050\times10^{-10}$	$1·072\times10^{-15}$
C (mm.a. per volt)	176·7	190·9	188·4	184·3	180·1	171·8	160·5	143·2	127·4	100·8

Table CXII.—*Fig.* 175, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
$\frac{Rt}{2L}\sqrt{1-\frac{16L}{KRl^2.R}}$	1·053	2·106	3·159	4·212	5·266	7·371	10·531	15·794	21·062	31·593
Sinh do.	1·259	4·048	11·756	33·745	96·77	794·9	$1·872\times10^{4}$	$3·624\times10^{6}$	$0·701\times10^{9}$	$2·625\times10^{13}$
$e^{-\frac{Rt}{2L}}$	0·317	0·100	0·032	0·010	$3·200\times10^{-3}$	$3·214\times10^{-4}$	$1·025\times10^{-5}$	$3·286\times10^{-8}$	$1·049\times10^{-10}$	$1·074\times10^{-15}$
$e^{-\frac{Rt}{L}}$	0·100	0·010	0·001	...	...	...	...	...	...	...
C_2 (mm.a. per volt)	5·79	20·6	36·6	51·6	65·2	88·9	116·9	148·8	168·7	188·7

Hence, finally,

$$C=\frac{Ee^{\left(-\frac{R}{2L}+\sqrt{1-\frac{4L}{KRl^2\,.\,R}}\right)t}}{Rl\sqrt{1-\frac{4L}{KRl^2\,.\,R}}}+\frac{Ee^{\left(-\frac{R}{2L}-\sqrt{1-\frac{4L}{KRl^2\,.\,R}}\right)t}}{-Rl\sqrt{1-\frac{4L}{KRl^2\,.\,R}}}$$

$$=\frac{2Ee^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4L}{KRl^2\,.\,R}}}{Rl\sqrt{1-\frac{4L}{KRl^2\,.\,R}}}\quad .\ .\ .\ .\quad (2)$$

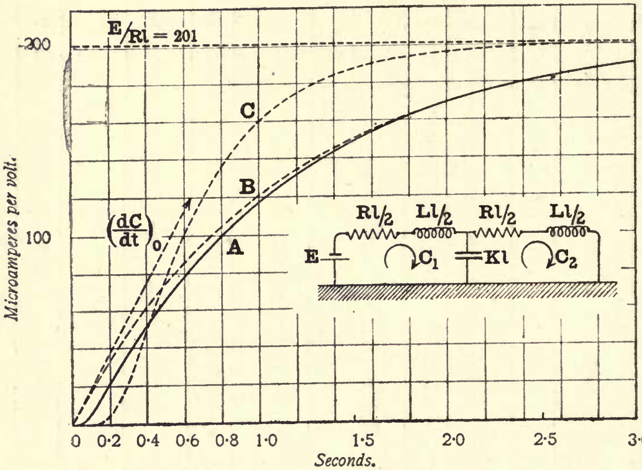

FIG. 175.—TWO INDUCTIVE MESHES.

Curve A. $L/R=KRl^2/100$. Current in second mesh.
" B. $L=0$.
" C. Infinite subdivision ; $L=0$.

Let $\frac{L}{R}=\frac{KRl^2}{100}$, so that when $KRl^2=4{\cdot}352$ seconds, $\frac{L}{R}=0{\cdot}04352$ second, and when R is 2·1856 ohms, $L=0{\cdot}0951$ henry per n.m. Then $4L/KRl^2\,.\,R$ is a small quantity $(=\frac{1}{25})$, and

$$\sqrt{1-\frac{4L}{KRl^2\,.\,R}}=0{\cdot}9798.$$

Curve A, Fig. 174, is plotted from (2) and Table CXI. Curve B is reproduced from Fig. 145. The current now starts from zero, instead of from the steady value which it would attain if the condenser were short-circuited, passes quickly to a turning-point which is not far short of that value, and then falls off slowly to zero. The rate of increase at the beginning may be obtained by differentiating (2), and putting $t=0$, from which $\left(\frac{dC}{dt}\right)_{t=0}=\frac{E}{Ll}=\frac{1}{223\cdot 1}=4482\times 10^{-6}$. It is indicated by the tangent line through the origin of co-ordinates. The steepness of the initial rise and the slowness of the subsequent fall are due to the smallness of the coil time-constant L/R compared with the condenser-resistance time-constant KRl^2, and by readjusting them the curve could be made to rise and fall as desired. In the present instance the effect of the inductance is practically zero at 0·2 second.

Two Inductive Meshes.

Now let there be two meshes as in Fig. 175. The periodic current C_2 in the second mesh is given by—

$$C_2=\frac{V\times\frac{1}{yKl}}{\begin{vmatrix}\frac{Rl+yLl}{2}+\frac{1}{yKl} & \frac{-1}{yKl}\\ \frac{-1}{yKl} & \frac{Rl+yLl}{2}+\frac{1}{yKl}\end{vmatrix}}$$

$$=\frac{V}{(Rl+yLl)\left[1+y\frac{Kl}{4}(Rl+yLl)\right]}.$$

Here $\varphi(o)=Rl$, and $\varphi(y)=0$ gives $y=-R/L$ or

$$y=-\frac{R}{2L}\pm\frac{R}{2L}\sqrt{1-\frac{16L}{KRl^2\,.\,R}}.$$

Also $y\varphi'(y)=-Rl$ or $\pm Rl\sqrt{1-\frac{16L}{KRl^2\,.\,R}}$.

Hence

$$C_2=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+\frac{e^{\left(-\frac{R}{2L}+\frac{R}{2L}\sqrt{1-\frac{16L}{KRl^2 . R}}\right)t}}{\sqrt{1-\frac{16L}{KRl^2 . R}}}+\frac{e^{\left(-\frac{R}{2L}-\frac{R}{2L}\sqrt{1-\frac{16L}{KRl^2 . R}}\right)t}}{-\sqrt{1-\frac{16L}{KRl^2 . R}}}\right]$$

$$=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}-\frac{2e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{16L}{KRl^2 . R}}}{\sqrt{1-\frac{16L}{KRl^2 . R}}}\right] \quad . \quad . \quad . \quad (3)$$

Curve A, Fig. 175 is plotted from (3) and Table CXII.

Curve B is reproduced from Fig. 146, for two meshes without inductance. The two curves lie closely together except at the beginning. The rate of growth of Curve A at the origin can be found by differentiating (3) and putting $t=0$. This gives $\left(\frac{dC_2}{dt}\right)_0=0$, and the current curve for the loaded meshes is horizontal at the start, whereas for the meshes without inductance it makes a finite angle with the horizontal.

Three Inductive Meshes.

When there are three meshes the current in the third mesh is found, as before, by writing down the determinant giving the periodic solution and applying the rules to obtain from it the telegraphic solution. In this case, as there are three meshes there are three terms with an exponential factor in the expression for the current

$$C_3=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{2L}}-\frac{3e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{24L}{KRl^2 . R}}}{\sqrt{1-\frac{24L}{KRl^2 . R}}}\right.$$

$$\left.+\frac{e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{72L}{KRl^2 . R}}}{\sqrt{1-\frac{72L}{KRl^2 . R}}}\right] . \quad (4)$$

Curve A, Fig. 176, is plotted from (4). Curve B is reproduced from Fig. 147 for the same meshes devoid of inductance. Curve C is the arrival current for the cable of the same constants uniformly distributed, without inductance. Curves A and C coincide initially, afterwards diverging widely on their way to the attainment of the same steady value. Curves A and B run together at about $1\frac{1}{2}$ second, after which Curve A is slightly the upper. The effect of inductance in converting Curve B into Curve A is evidently to cut away from the lower part of the curve, afterwards adding slightly to the upper part, thus rendering the current curve more steep.

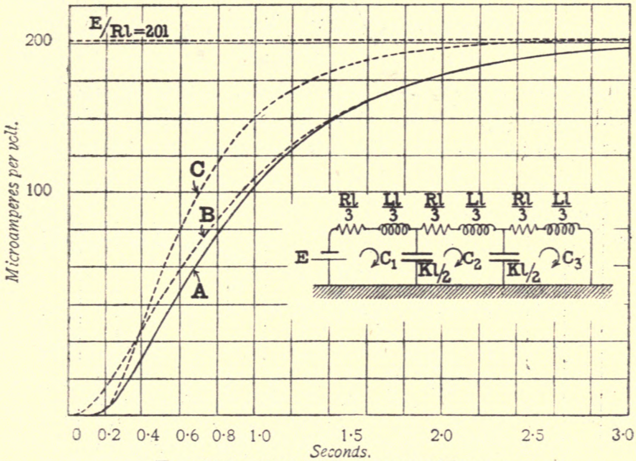

Fig. 176.—Three Inductive Meshes.

Curve A. $L/R = KRl^2/100$. Current in third mesh.
" B. $L=0$.
" C. Infinite subdivision ; $L=0$.

Four Inductive Meshes.

When there are four meshes, as in Fig. 177, the current in the fourth mesh is found as before to be

$$C_4 = \frac{E}{Rl}\left[1 - e^{-\frac{Rt}{L}} + \frac{2e^{-\frac{Rt}{2L}}\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{96L}{KRl^2 . R}}}{\sqrt{1-\dfrac{96L}{KRl^2 . R}}}\right.$$

$$\left. - \frac{2e^{-\frac{Rt}{2L}}\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{48(2-\sqrt{2})L}{KRl^2 . R}}}{(2-\sqrt{2})\sqrt{1-\dfrac{48(2-\sqrt{2})L}{KRl^2 . R}}} - \frac{2e^{-\frac{Rt}{2L}}\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{48(2+\sqrt{2})L}{KRl^2 . R}}}{(2+\sqrt{2})\sqrt{1-\dfrac{48(2+\sqrt{2})L}{KRl^2 . R}}}\right] . \quad (5)$$

Since $\dfrac{48(2+\sqrt{2})\mathrm{L}}{\mathrm{K}\mathrm{R}l^2 \,.\, \mathrm{R}} = \dfrac{48(2+\sqrt{2})}{100}$, when L/R is taken as $\mathrm{KR}l^2/100$, and is therefore greater than unity, it follows that the quantity under the root in the last term is negative. Turning it about, the term may be written in the form

$$\frac{2e^{-\frac{\mathrm{R}t}{2\mathrm{L}}}\sin\dfrac{\mathrm{R}t}{2\mathrm{L}}\sqrt{\dfrac{48(2+\sqrt{2})\mathrm{L}}{\mathrm{KR}l^2\,.\,\mathrm{R}}-1}}{(2+\sqrt{2})\sqrt{\dfrac{48(2+\sqrt{2})\mathrm{L}}{\mathrm{KR}l^2\,.\,\mathrm{R}}-1}},$$

which is that of a decaying oscillation. Curve A, Fig. 177, is plotted from (5). Curve B is for four meshes without inductance, and Curve C for the cable without inductance.

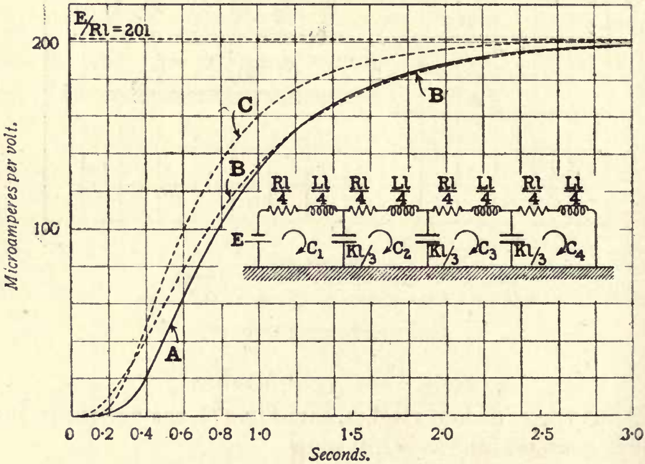

FIG. 177.—FOUR INDUCTIVE MESHES.

Curve A. $\mathrm{L/R}=\mathrm{KR}l^2/100$. Current in fourth mesh.
„ B. L=0.
„ C. Infinite subdivision. L=0.

This process, of increasing the number of meshes step by step, might be continued indefinitely. But it has been carried already far enough for the general character of the result to be evident. As the number of meshes is increased the number of terms in the series representing the current in any mesh increases, and since the quantities subtracted from unity under the root signs increase with the subdivision, it follows that the terms will all be of the transient oscillatory type, except perhaps a few at the beginning. Again, taking Figs. 175, 176 and 177 in turn, it is seen (*a*) that with increase of subdivision the point where Curves A and B, with and without loading, practically coincide moves further to the left, *i.e.*, occurs earlier; (*b*) that both curves in their later stages tend to approach Curve C for uniform distribution; and (*c*) that the initial difference between Curves A and B tends to increase. When the subdivision is infinitely fine, Curve B coincides with Curve C throughout, and we should expect, therefore, that Curve A would also coincide with Curve C in its later stages, but initially the divergence would increase, and the horizontal portion of Curve A would tend to increase. In general terms, therefore, the effect of inductance is to stop-back and steepen the arrival curve initially.

Influence of Leakage—Single Mesh.

The part which inductance plays being sufficiently clear, its action when combined with leakage may now be considered. In Fig. 178 is shown a single mesh in which a shunt is placed across the condenser of resistance $1/Gl$ or conductance Gl. The periodic solution is, symbolically,

$$C=\frac{Ee^{yt}}{Rl+yLl+\dfrac{1}{Gl+yKl}},$$

from which $\varphi(y)=0$ is $(Rl+yLl)(Gl+yKl)+1=0$,

or $$y=-\left(\frac{R}{2L}+\frac{G}{2K}\right)\pm\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{1}{LKl^2}}.$$

Let G be chosen for a reason which will appear later so that $G/K=R/L$. The quantity in brackets under the square root sign then vanishes, and

$$y=-\frac{R}{L}\pm\frac{i}{\sqrt{LKl^2}}.$$

Also $\varphi'(y)=2Ll$ and $\varphi(0)=Rl+1/GL$.

Table CXIII.—*Fig.* 178, *Curve A.*

(sec.)	0·05	0·10	0·15	0·2	0·3	0·4	0·5
$e^{-Rt/L}$	0·31703	0·10051	0·03186	0·01010	0·00102	0·00010	0·00001
$\frac{t}{\sqrt{KLl^2}}$	0·11489	0·22978	0·34467	0·45957	0·68934	0·91914	1·1489
$\tan^{-1}\frac{KRl^2}{\sqrt{KLl^2}} - \frac{t}{\sqrt{KLl^2}}$	1·3562	1·2413	1·1265	1·0116	0·7818	0·5520	0·3222
$\sin\left(\tan^{-1} - \frac{t}{\sqrt{KLl^2}}\right)$	0·9771	0·9462	0·9029	0·8476	0·7045	0·5244	0·3165
C (mm.a. per volt)	137·1	180·0	193·3	197·3	198·9	199·0	199·0

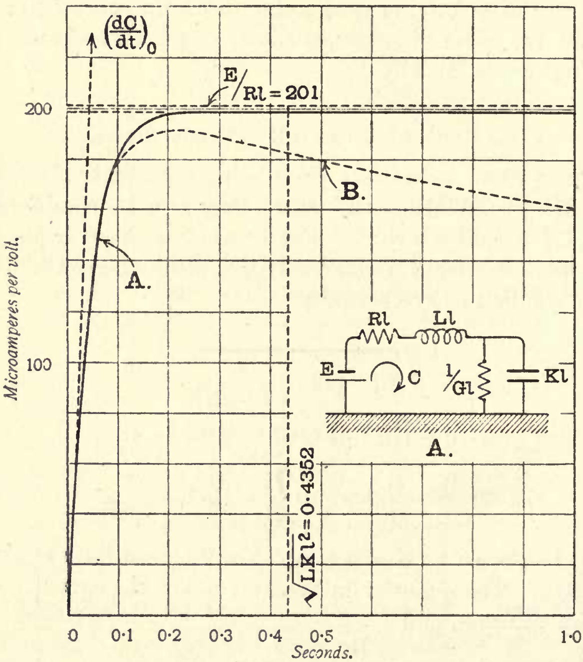

Fig. 178.—Single "Distortionless" Mesh.

Curve A. $L/R = KRl^2/100$; $Gl = 100/Rl$.
„ B. $G = 0$.

Hence, finally,

$$C=\frac{E}{Rl+\frac{1}{Gl}}+\frac{Ee^{-\frac{Rt}{L}}}{2Ll}\left[\frac{e^{\frac{it}{\sqrt{KLl^2}}}}{-\frac{R}{L}+\frac{i}{\sqrt{KLl^2}}}+\frac{e^{-\frac{it}{\sqrt{KLl^2}}}}{-\frac{R}{L}-\frac{i}{\sqrt{KLl^2}}}\right]$$

$$=\frac{E}{Rl\left(1+\frac{L}{R\,.\,KRl^2}\right)}+\frac{Ee^{-\frac{Rt}{L}}}{Rl\sqrt{1+\frac{L}{R\,.\,KRl^2}}}\sin\left(\frac{t}{\sqrt{KLl^2}}-\tan^{-1}\frac{KRl^2}{\sqrt{KLl^2}}\right) \quad (6)$$

Since Kl is of dimensions $(l^{-1}t^2)$ and Ll is of dimensions (l), KLl^2 is of dimensions (t^2), and $\sqrt{KLl^2}$ is a time and may be measured in seconds. The quotients $t/\sqrt{KLl^2}$ and $KRl^2/\sqrt{KLl^2}$ are, therefore, of zero dimensions—*i.e.*, they are numerical quantities.

Curve A, Fig. 178, is plotted from (6) and Table CXIII., taking L/R as before equal to $KRl^2/100$, and therefore $\sqrt{KLl^2}$ $=\frac{KRl^2}{10}=0{\cdot}04352$ second; also $Gl=Kl\times\frac{R}{L}=\frac{KRl^2}{L/R\times Rl}$ $=\frac{100}{Rl}=0{\cdot}0201$ mhos, *i.e.*, $1/Gl$ is $Rl/100$, or 49·75 ohms; in other words, the resistance of the leak is one-hundredth part of the total copper resistance.

The curve rises abruptly from the time axis. Differentiating (6) and putting $t=0$, it follows that $(dC/dt)_{t=0}=E/Ll=E\times 4{,}618\times10^{-6}$, which is represented by the tangent line at the origin. Curve B is reproduced from Fig. 174 for a single mesh without leak. Curve A reaches its maximum in 0·25 second and stays there, whereas curve B falls off to zero.

Two Leaky Meshes.

In Fig. 179 the cable constants are bunched together to form two meshes. Proceeding as before, the current in the second mesh is found to be

$$C_2=\frac{E}{Rl\left[1+\frac{GRl^2}{4}\right]}-\frac{Ee^{-\frac{Rt}{L}}}{Rl}-\frac{Ee^{-\frac{Rt}{L}}\sin\left(\frac{2t}{\sqrt{LKl^2}}-\tan^{-1}\frac{KRl^2}{2\sqrt{LKl^2}}\right)}{Rl\sqrt{1+\frac{4L}{R\,.\,KRl^2}}} \quad (7)$$

provided the condition $GL=KR$ holds good.

Curve A, Fig. 179, is plotted from (7), and curve B is a reproduction of curve A of Fig. 175, but is drawn to one-twentieth of its actual size. Both curves now run horizontally initially and rise to a steady value. The influence of the leak is seen (1) in reducing the steady value of 201 mm. a. per volt in the case of

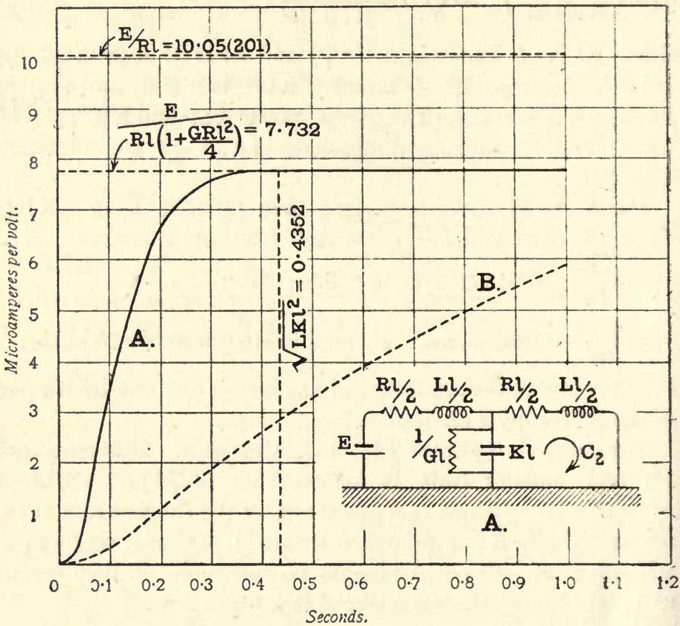

FIG. 179.—TWO "DISTORTIONLESS" MESHES.

Curve A. $L/R = KRl^2/100$; $Gl = 100/Rl$.

„ B. $G = 0$. Ordinates one-twentieth actual size.

curve B to 7·7 mm.a. per volt in curve A, and (2) in greatly reducing the time at which the steady value is attained—*i.e.*, in increasing the steepness of the curve. Improved shape has been gained at a great cost in height.

Three Leaky Meshes.

The current in the third mesh of the diagram in Fig. 180 is given by the formula

$$C_3 = \frac{E}{Rl\left(1+\frac{GRl^2}{6}\right)\left(1+\frac{GRl^2}{18}\right)} - \frac{Ee^{-\frac{Rt}{L}}}{Rl} - \frac{\frac{3}{2}Ee^{-\frac{Rt}{L}}\sin\left(t\sqrt{\frac{6}{LKl^2}} - \tan^{-1}\frac{KRl^2}{\sqrt{6LKl^2}}\right)}{Rl\sqrt{1+\frac{6L}{R\,.\,KRl^2}}}$$

$$+\frac{\frac{1}{2}Ee^{-\frac{Rt}{L}}\sin\left(t\sqrt{\frac{18}{LKl^2}} - \tan^{-1}\frac{KRl^2}{\sqrt{18LKl^2}}\right)}{Rl\sqrt{1+\frac{18L}{R\,.\,KRl^2}}}. \quad (8)$$

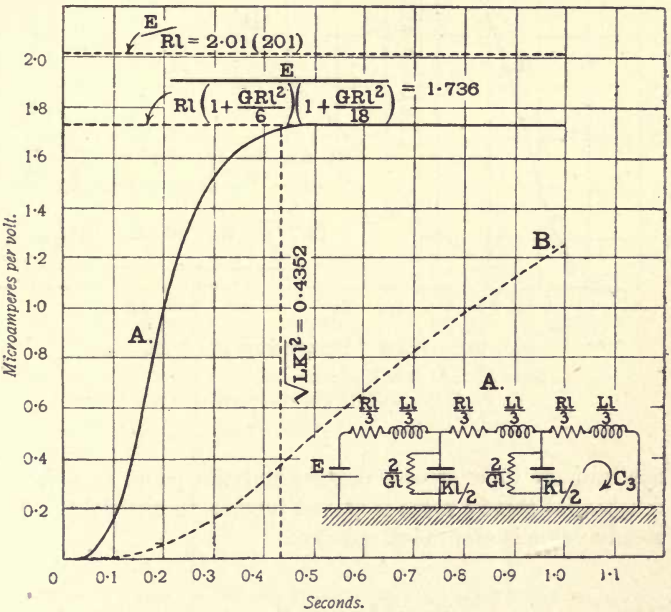

Fig. 180.—Three "Distortionless" Meshes.

Curve A. $L/R = KRl^2$; $Gl = 100/Rl$.

„ B. $G=0$. Ordinates one-hundredth actual size.

Curve A is plotted from (8) and curve B is taken from Fig. 176, where it is curve A. The effect of greater subdivision without alteration of the total constants is seen by comparing curve A, Fig. 180, with curve A of Fig. 179. The curve is now hollowed

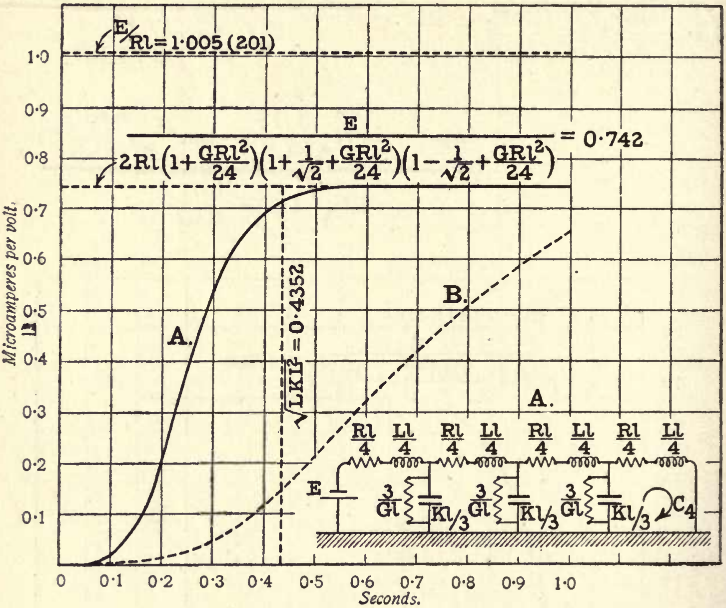

FIG. 181.—FOUR "DISTORTIONLESS" MESHES.

Curve A. $L/R = KRl^2/100$; $Gl = 100/Rl$.
" B. $G = 0$. Ordinates one-two-hundredth actual size.

out initially to a greater degree, and the point at which it touches its steady value is moved further to the right. The steady value is also much reduced.

FOUR LEAKY MESHES.

The current in the fourth mesh of the diagram in Fig. 181 is given by the formula,

$$C_4 = \frac{E}{2Rl\left(1+\frac{GRl^2}{24}\right)\left(1+\frac{1}{\sqrt{2}}+\frac{GRl^2}{24}\right)\left(1-\frac{1}{\sqrt{2}}+\frac{GRl^2}{24}\right)} - \frac{Ee^{-\frac{Rt}{L}}}{Rl} - \frac{\left(1+\frac{1}{\sqrt{2}}\right)Ee^{-\frac{Rt}{L}}\sin\left(t\sqrt{\frac{24\left(1-\frac{1}{\sqrt{2}}\right)}{LKl^2}}-\tan^{-}\frac{KRl^2}{\sqrt{24\left(1-\frac{1}{\sqrt{2}}\right)LKl^2}}\right)}{Rl\sqrt{1+24\left(1-\frac{1}{\sqrt{2}}\right)\frac{L}{R\,.\,KRl^2}}}$$

$$+ \frac{Ee^{-\frac{Rt}{L}}\sin\left(t\sqrt{\frac{24}{LKl^2}}-\tan^{-1}\frac{KRl^2}{\sqrt{24LKl^2}}\right)}{Rl\sqrt{1+\frac{24L}{R\,.\,KRl^2}}} - \frac{\left(1-\frac{1}{\sqrt{2}}\right)Ee^{-\frac{Rt}{L}}\sin\left(t\sqrt{\frac{24\left(1+\frac{1}{\sqrt{2}}\right)}{LKl^2}}-\tan^{-1}\frac{KRl^2}{\sqrt{24\left(1+\frac{1}{\sqrt{2}}\right)LKl^2}}\right)}{Rl\sqrt{1+24\left(1+\frac{1}{\sqrt{2}}\right)\frac{L}{R\,.\,KRl^2}}}, \quad (9)$$

assuming the same relationship among the constants as before. Curve A is plotted from (9), and curve B is taken from Fig. 177, where it is curve A. Comparing Fig. 181 with Fig. 180, the effect of the further subdivision is seen in (1) the greater flattening of the curve at the beginning, (2) the movement of the point of reaching the maximum to the right, and (3) the reduction in the height of the curve. The improvement which curve A presents in shape when compared with curve B has been gained at the expense of an enormous reduction in height.

Passage to Continuous Distribution—Number of Meshes Infinite.

The process of subdividing the cable step by step into an increasing number of meshes might be continued indefinitely, but it has already been carried sufficiently far for its general tendency to be rendered evident. We pass on, therefore, to consider the general case when there are n meshes present, with the ultimate intention of making n infinitely great.

The periodic solution for the current in the last of n inductive and leaky meshes may be written down at once. It is—

$$C_n=\frac{\mathrm{E}e^{yt}}{\varphi(y)}, \text{ where } \varphi(y)=\frac{n-1}{\mathrm{G}l+y\mathrm{K}l}\begin{vmatrix} \dfrac{(\mathrm{R}l+y\mathrm{L}l)(\mathrm{G}l+y\mathrm{K}l)}{n(n-1)}+1 & -1 & \text{—} & \text{—} \\ -1 & \dfrac{(\mathrm{R}l+y\mathrm{L}l)(\mathrm{G}l+y\mathrm{K}l)}{n(n-1)}+2 & -1 & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{vmatrix}$$

which may be obtained by a simple extension from the case already treated, where L and G are zero.

Write, as before, $\cos\theta$ for $\dfrac{(\mathrm{R}l+y\mathrm{L}l)(\mathrm{G}l+y\mathrm{K}l)}{2n(n-1)}+1$.

Then

$$\varphi(y)=\frac{n-1}{\mathrm{G}l+y\mathrm{K}l}\begin{vmatrix} 2\cos\theta-1 & -1 & \text{—} & \text{—} \\ -1 & 2\cos\theta & -1 & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{vmatrix}=\frac{\mathrm{R}l+y\mathrm{L}l}{n}\cdot\frac{\sin n\theta}{\sin\theta}.$$

Again,

$$\varphi'(y)=\frac{\mathrm{L}l}{n}\frac{\sin n\theta}{\sin\theta}+\frac{\mathrm{R}l+y\mathrm{L}l}{n}\cdot\frac{d\theta}{dy}\cdot\frac{n\cos n\theta\sin\theta-\sin n\theta\cos\theta}{\sin^2\theta},$$

where $-\sin\theta\cdot\frac{d\theta}{dy}=\frac{\mathrm{L}l(\mathrm{G}l+y\mathrm{K}l)+\mathrm{K}l(\mathrm{R}l+y\mathrm{L}l)}{2n(n-1)}$.

Also $\varphi(y)=0$ gives (1) $y_1=-\mathrm{R/L}$, $\cos\theta=1$ and $\theta=0$,

or (2) $\frac{\sin n\theta}{\sin\theta}=0$, from which $n\theta=m\pi$.

Substituting these values in $\varphi'(y)$, it becomes either

$$\frac{\mathrm{L}l}{n}\frac{\sin n\theta}{\sin\theta}=\mathrm{L}l,$$

or $$\frac{\mathrm{R}l+y_2\mathrm{L}l}{n}\cdot\frac{\mathrm{L}l(\mathrm{G}l+y_2\mathrm{K}l)+\mathrm{K}l(\mathrm{R}l+y_2\mathrm{L}l)}{-2n(n-1)\sin\theta}\cdot\frac{n\cos n\theta}{\sin\theta},$$

where $$y_2=-\left(\frac{\mathrm{R}}{2\mathrm{L}}+\frac{\mathrm{G}}{2\mathrm{K}}\right)\pm\sqrt{\left(\frac{\mathrm{R}}{2\mathrm{L}}-\frac{\mathrm{G}}{2\mathrm{K}}\right)^2-\frac{2n(n-1)(1-\cos\theta)}{\mathrm{LK}l^2}}$$

Hence, finally, on substitution of these values, the arrival current in the last mesh is given by

$$\mathrm{C}_n=\frac{\mathrm{E}}{\varphi(0)}+\mathrm{E}\Sigma\frac{e^{yt}}{y\varphi'(y)}$$

$$=\frac{\mathrm{E}}{\dfrac{\mathrm{R}l}{n}\dfrac{\sin n\cos^{-1}\left(1+\dfrac{\mathrm{GR}l^2}{2n(n-1)}\right)}{\sin\cos^{-1}\left(1+\dfrac{\mathrm{GR}l^2}{2n(n-1)}\right)}}-\frac{\mathrm{E}e^{-\frac{\mathrm{R}t}{\mathrm{L}}}}{\mathrm{R}l}$$

$$+\sum_{m=1}^{m=n-1}\frac{2n(n-1)\mathrm{E}e^{-\left(\frac{\mathrm{R}}{2\mathrm{L}}+\frac{\mathrm{G}}{2\mathrm{K}}\right)t}\left(\sin\dfrac{m\pi}{n}\right)^2\cos m\pi\sinh\left(t\sqrt{\left(\dfrac{\mathrm{R}}{2\mathrm{L}}-\dfrac{\mathrm{G}}{2\mathrm{K}}\right)^2-\dfrac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{\mathrm{LK}l^2}}-\psi\right)}{\mathrm{L}l\sqrt{\left(\dfrac{\mathrm{R}}{2\mathrm{L}}-\dfrac{\mathrm{G}}{2\mathrm{K}}\right)^2-\dfrac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{\mathrm{LK}l^2}}\sqrt{2n(n-1)\left(1-\cos\dfrac{m\pi}{n}\right)}\sqrt{2n(n-1)\left(1-\cos\dfrac{m\pi}{n}\right)+\mathrm{GR}l^2}}\quad(10)$$

$$\text{where } \tanh\psi = \frac{\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2 - \frac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{LKl^2}}}{\frac{R}{2L}-\frac{G}{2K}+\frac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{LGl^2}}. \quad (11)$$

Criterion for Continuous Distribution.

In the series (10), let n be very great. Then $\cos\frac{m\pi}{n}=1-\frac{m^2\pi^2}{2n^2}$, nearly, and $1-\cos\frac{m\pi}{n}=\frac{m^2\pi^2}{2n^2}$. Hence $2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)$ $=2n^2\left(1-\cos\frac{m\pi}{n}\right)$, nearly, $=m^2\pi^2$. Substituting in (10) it becomes

$$C_n=\frac{E}{\frac{Rl}{n}\cdot\frac{\sin i\sqrt{GRl^2}}{\sin\frac{i\sqrt{GRl^2}}{n}}}-\frac{Ee^{\frac{Rt}{L}}}{Rl}$$

$$+\sum_{m=1}^{m=n-1}\frac{2m^2\pi^2Ee^{-\left(\frac{R}{2L}+\frac{G}{2K}\right)t}\cos m\pi\sinh\left(t\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{m^2\pi^2}{LKl^2}}-\psi\right)}{Ll\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{m^2\pi^2}{LKl^2}}\cdot m\pi\sqrt{m^2\pi^2+GRl^2}}$$

and

$$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh\sqrt{GRl^2}}-\frac{Ee^{-\frac{Rt}{L}}}{Rl}$$

$$+\sum_{m=1}^{\infty}\frac{2Ee^{-\left(\frac{R}{2L}+\frac{G}{2K}\right)t}\cos m\pi\sinh\left(t\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{m^2\pi^2}{LKl^2}}-\psi\right)}{Ll\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{m^2\pi^2}{LKl^2}}\cdot\sqrt{1+\frac{GRl^2}{m^2\pi^2}}}. \quad (12)$$

where
$$\tanh \psi = \frac{\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2 - \frac{m^2\pi^2}{LKl^2}}}{\frac{R}{2L}-\frac{G}{2K}+\frac{m^2\pi^2}{LGl^2}} \quad . \quad . \quad . \quad (13)$$

This formula gives the arrival current in a cable which is connected to earth at the ends, and in which resistance, inductance, capacity and leakance are present. The criterion for uniform distribution is evidently that n should be so great that $2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)$ should practically be equal to $m^2\pi^2$, where m is the number of the highest term in the series (10), which requires to be taken into consideration. If we assume for the present that the mesh curve in the general case coincides initially with the distributed curve—as was shown to be true when L and G are zero—then it will suffice if the criterion hold for the first term of (10). Putting $m=1$, the criterion becomes, as before, that n should be so great that $\cos\left(\frac{\pi}{n}\right)$ practically equals $1-\frac{\pi^2}{2n^2}$.

Particular Cases of the General Formula.

In the mesh formula let G be zero. Then, from (11), the denominator of the expression for $\tanh \psi$ is infinite, and $\tanh \psi = 0$, from which $\psi = 0$. Substituting in (10), the formula reduces to

$$_n = \frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+2e^{-\frac{Rt}{2L}}\sum_{m=1}^{m=n-1}\frac{\left(1+\cos\frac{m\pi}{n}\right)\cos m\pi \sinh \frac{Rt}{2L}\sqrt{1-\frac{8n(n-1)\left(1-\cos\frac{m\pi}{n}\right)L}{R\,.\,KRl^2}}}{\sqrt{1-\frac{8n(n-1)\left(1-\cos\frac{m\pi}{n}\right)L}{R\,.\,KRl^2}}}\right], \quad (14)$$

which gives the current in the last of n meshes containing resistance, inductance and capacity only. If n be so great that $2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)=m^2\pi^2$, the formula further degenerates to

$$C_r=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+4e^{-\frac{Rt}{2L}}\sum_1^\infty\frac{\cos m\pi\sinh\frac{Rt}{2L}\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}}{\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}}\right], \quad (15)$$

which gives the arrival current in a loaded cable connected to earth at the ends.

Formula (15) was obtained by Vaschy* in 1876, who built it up by analogy from the Thomson formula as a solution to the differential equation with an inductance term. It is also to be found in Heaviside.† It is here obtained as a particular case of a much more general formula, and the method by which it was deduced permits of still further extension, if need be, which will be the case, for example, when we come to take account of the influence of the terminal apparatus.

Carrying the reduction a stage further, put $L=0$. Then in the limit, when L is very small,

$$\sinh\frac{Rt}{2L}\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}=\sinh\frac{Rt}{2L}\left(1-\frac{2m^2\pi^2L}{KRl^2\,.\,R}\right), \text{ and}$$

$$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\left(1-\frac{2m^2\pi^2L}{KRl^2\,.\,R}\right)=\tfrac{1}{2}e^{-\frac{Rt}{2L}}\left[e^{\frac{Rt}{2L}-\frac{m^2\pi^2t}{KRl^2}}-e^{-\frac{Rt}{2L}+\frac{m^2\pi^2t}{KRl^2}}\right]$$

$$=\tfrac{1}{2}e^{-\frac{m^2\pi^2t}{KRl^2}} \text{ when L is zero.}$$

Hence the series becomes, as it should,

$$C_r=\frac{E}{Rl}\left[1+2\sum_1^\infty e^{-\frac{m^2\pi^2t}{KRl^2}}\cos m\pi\right],$$

which is the formula for the Kelvin arrival curve, and dates back to 1855. The general formula (10) thus degenerates when n is made infinite, or L and G are made zero, to less comprehensive formulæ applicable in particular instances.

* Eclair Elect., 11, 1897.

† "Electromagnetic Theory," Vol. II., p. 381.

The Distortionless Cable.

Returning to the general mesh formula (10), it is seen that the series contains in numerator and denominator the expression,

$$\sqrt{\left(\frac{R}{2L}-\frac{G}{2K}\right)^2-\frac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{LKl^2}}.$$

The quantity under the square root sign may be negative, in which case the square root is purely imaginary. This will certainly be true whatever n and m may be if the relationship $R/L=G/K$ holds among the constants. The formula then simplifies considerably. The series becomes

$$\sum_{m=1}^{m=n-1}\frac{\left(1+\cos\frac{m\pi}{n}\right)Ee^{-\frac{Rt}{L}}\cos m\pi\sinh\left(it\sqrt{\frac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{LKl^2}}-\psi\right)}{\sqrt{L/K}\,.\,i\sqrt{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)+RGl^2}};$$

and the current in the nth mesh is given by

$$C_n=\frac{E}{\frac{Rl}{n}\,\frac{\sin n\cos^{-1}\left(1+\frac{GRl^2}{2n(n-1)}\right)}{\sin\cos^{-1}\left(1+\frac{GRl^2}{2n(n-1)}\right)}}$$

$$-\frac{Ee^{-\frac{Rt}{L}}}{Rl}\left[1-\sum_{m=1}^{m=\infty}\frac{\left(1+\cos\frac{m\pi}{n}\right)\cos m\pi\sin\left(t\sqrt{\frac{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{LKl^2}}-\tan^{-1}\frac{Rl}{\sqrt{2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)\frac{L}{K}}}\right)}{\sqrt{1+2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)\frac{L}{R\,.\,KRl^2}}}\right] \quad . \quad . \quad . \quad (16)$$

Of this formula those already obtained from first principles, (6), (7), (8), and (9), may be regarded as particular cases when n is 1, 2, 3 and 4 respectively.

Now, put $n=\infty$, and (16) becomes

$$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh\sqrt{GRl^2}}$$

$$-\frac{Ee^{-\frac{Rt}{L}}}{Rl}\left[1-\sum_{m-1}^{\infty}\frac{2\cos m\pi\sin\left(\frac{m\pi t}{\sqrt{LKl^2}}-\tan^{-1}\frac{KRl^2}{m\pi\sqrt{LKl^2}}\right)}{\sqrt{1+\frac{m^2\pi^2L}{R\,.\,KRl^2}}}\right], \quad (17)$$

which gives the arrival current in a cable for which the relationship $LG=KR$ holds among the constants.

This formula may be simplified greatly. It has been shown in Chap. I. that

$$e^{\mu x}=\frac{2\sinh\mu\pi}{\pi}\left[\frac{1}{2\mu}-\sum_{1}^{\infty}\frac{\sin\left(mx-\tan^{-1}\frac{\mu}{m}\right)}{\sqrt{\mu^2+m^2}}\cos m\pi\right], \quad . \quad . \quad . \quad (18)$$

where x lies between $-\pi$ and $+\pi$. As x increases and passes through the value π, the series shows a finite discontinuity, and drops abruptly from the value $e^{\mu x}$ to the value $e^{\mu(x-2\pi)}$.

Put $\mu=\frac{Rl}{\pi}\sqrt{\frac{K}{L}}$, and $x=\frac{\pi t}{\sqrt{LKl^2}}$, and substitute in (18).

Then $$e^{\frac{Rt}{L}}=\frac{\sinh Rl\sqrt{\frac{K}{L}}}{Rl\sqrt{\frac{K}{L}}}[1-\Sigma \text{ series of } (17)].$$

Hence $$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh\sqrt{GRl^2}}-\frac{Ee^{-\frac{Rt}{L}}}{Rl}\cdot\frac{Rl\sqrt{\frac{K}{L}}\cdot e^{\frac{Rt}{L}}}{\sinh Rl\sqrt{\frac{K}{L}}}=0.$$

This holds up to $x=\pi$ or $t=\sqrt{LKl^2}$, when the series becomes $e^{\frac{Rt}{L}-2Rl\sqrt{\frac{K}{L}}}$, and

$$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh\sqrt{GRl^2}}\left[1-e^{-2Rl\sqrt{\frac{K}{L}}}\right]=2E\sqrt{\frac{G}{R}}\cdot e^{-\sqrt{GRl^2}},$$

$$=2E\sqrt{\frac{K}{L}}e^{-Rl\sqrt{\frac{K}{L}}}. \quad (19)$$

It is evident, therefore, that the arrival current is zero initially and remains so until $t=\sqrt{LKl^2}$, when it jumps up abruptly to the value given by (19), which value it maintains until $t=3\sqrt{LKl^2}$, when there is a second discontinuity, and so on. These successive jumps contain the factors $e^{-3Rl\sqrt{\frac{K}{L}}}$, $e^{-5Rl\sqrt{\frac{K}{L}}}$, . . . and are infinitesimal in a long cable, where the first alone need be considered.

Attenuation in a Distortionless Cable.

The time $t=\sqrt{LKl^2}$ is sometimes called the *relaxation-time.** It is indicated in Figs. 178, 179, 180 and 181 by the vertical line at 0·4352 second. Provided the relationship LG=KR holds for the cable constants, the arrival curve for a single contact is square-topped, and all signals are received as an exact scale reproduction of the changes in the sending E.M.F. For this reason Heaviside called the above relationship the distortionless condition.†

Although distortion is elimated, enormous attenuation may yet be present. In Fig. 182, curve A, is shown the manner in which the arrival current in a distortionless cable depends on the inductance. It is plotted from (19) and Table CXIV. Curve A is drawn for $L=KRl^2/100$ or L=95 m.h. per n.m. and it forms the limit towards which the curves of Figs. 178, 179, 180 and 181 converge when the number of meshes is infinitely great. The other curves are for smaller values of the inductance, the leakance being adjusted in every case so that LG=KR. Curve

* A. G. Webster, "Theory of Electricity and Magnetism," p. 540.

† "Electromagnetic Theory," Vol. I., p. 366.

Table CXIV.—Fig. 182.

$KRl^2 \div \frac{L}{R}$.	L (m.h. per n.m.)	$\sqrt{KLl^2}$ (sec.)	$\alpha l = \frac{Rl}{2}\sqrt{\frac{K}{L}}$.	$10^6 \times e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$.	$10^6 \times e^{-Rl\sqrt{\frac{K}{L}}}$.	$10^6 \times 2\sqrt{\frac{K}{L}}$.	$2\sqrt{\frac{K}{L}}\,e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$ (mm.a. per v).	$2\sqrt{\frac{K}{L}}\,e^{-Rl\sqrt{\frac{K}{L}}}$ (mm.a. per v).
1000	9·5	0·138	15·8	0·136	$1·846 \times 10^{-8}$	12,713	$1·728 \times 10^{-3}$	$2·347 \times 10^{10}$
500	19·0	0·224	11·2	13·95	$1·945 \times 10^{-4}$	8,988	0·125	$1·748 \times 10^{-6}$
400	23·8	0·218	10·0	45·38	$2·060 \times 10^{-3}$	8,040	0·365	$1·657 \times 10^{-}$
300	31·7	0·251	8·7	173·3	$3·005 \times 10^{-2}$	6,963	1·207	$2·092 \times 10^{-4}$
250	38·0	0·275	7·9	368·8	0·136	6,356	2·344	$8·647 \times 10^{-4}$
200	47·6	0·308	7·1	848·8	0·722	5,685	4·826	$4·106 \times 10^{-3}$
170	55·9	0·334	6·5	1474	2·177	5,241	7·725	$1·141 \times 10^{-2}$
150	63·4	0·355	6·1	2189	4·797	4,923	10·78	0·0236
120	79·3	0·397	5·5	4179	17·47	4,404	18·41	0·0769
100	95·1	0·435	5·0	6736	45·41	4,020	27·08	0·1818

F is the beginning of the arrival curve without either L or G, and curve G is the same with its ordinates one-thousandth of their actual size.

In curve A, where $L/R = KRl^2/100$, the inductance is of the amount which it is customary to insert in coil-loaded telephone cables. Here $G = KR/L = 100K/KRl^2$, and $Gl = 100/Rl$ or $1/Gl = Rl/100$. The total insulation resistance of the distortionless cable is therefore only 49·75 ohms. If the inductance be reduced to one-tenth part or 9·5 millihenries per n.m.—which is of the order of the natural inductance of the cable, or of the same increased by continuous loading—the leakage resistance drops to 4·975 ohms, and the current falls, as is seen from the table, to an infinitesimal amount, much too small to be detected by the most sensitive measuring instrument.

From the foregoing considerations two weighty objections appear to the practical introduction of the distortionless cable. The first is, that the extremely low insulation resistance which is entailed by the distortionless condition, even with high values of the inductance, would render testing and maintenance of the cables a matter of extreme difficulty; the second is, that, unless the inductance is high, freedom from distortion

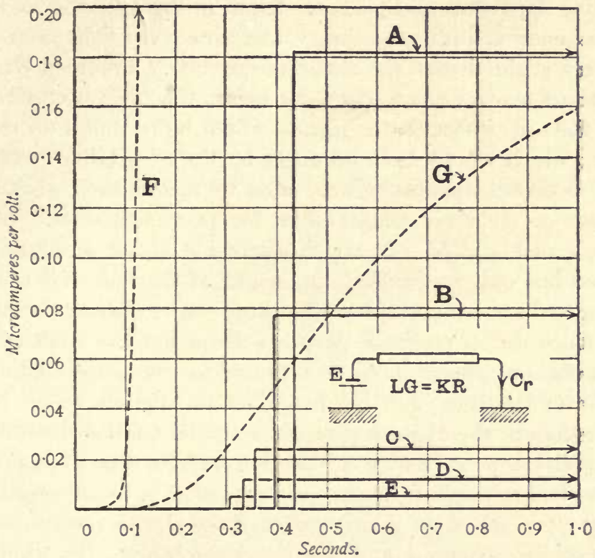

FIG. 182.—DISTORTIONLESS CABLE. INFLUENCE OF INDUCTANCE ON RECEIVED CURRENT.

Curve A. $L/R = KRl^2/100$; $L = 0{\cdot}095$ henry per n.m.
„ B. $L/R = KRl^2/120$; $L = 0{\cdot}079$.
„ C. $L/R = KRl^2/150$; $L = 0{\cdot}063$.
„ D. $L/R = KRl^2/170$; $L = 0{\cdot}056$.
„ E. $L/R = KRl^2/200$; $L = 0{\cdot}048$.
„ F. $L = 0 = G$.
„ G. Curve F $\div$ 1,000.

is purchased at the expense of enormous attenuation. Now, it has already been shown in Chap. XIV., that if attenuation be conceded, distortion can be removed to any desired extent by adjustment of the apparatus at the ends of the cable without the necessity for employing a distributed effect.

Although the distortionless cable would transmit signals undistorted at any speed—assuming that the cable constants do not vary appreciably with frequency—another limit is set to the speed of working by the recording apparatus, which must be sensitive as the currents actuating it are small. It will suffice therefore if the cable does not introduce sensible distortion at the highest speed of transmission of which the signalling apparatus will permit. Thus, in Fig. 182, the sharp corners of curve A are unnecessary, and the curve might assume in a very slight degree the shape of curve G. Provided that the current reaches its steady value before the cable is earthed again, the only effect on the signals would be to round off the corners, which would have no effect on their legibility—even if the receiving apparatus were able to record such abrupt changes—or on their adaptability for re-transmission. We may call such a cable—in which distortion is not absolutely removed but only so far that the height of the signals is constant up to the maximum speed of working—a *quasi-distortionless* cable, since for all practical purposes distortion has vanished. Such a state of affairs could be attained for the same loading by a lower leakance, and the height of the signals would be greater than in the distortionless cable. The quasi-distortionless condition is evidently a function of the rate of transmission, and it would be an interesting research to investigate for any given speed the relative sizes of the signals received in the quasi-distortionless and distortionless cables. It would take too long to enter upon that here, and moreover, when we come to consider loading alone the fatal objection to the distortionless condition will be found to lie in the fact that with high-speed working in the loaded cable the signals are quasi-distortionless even without leakance at all.

Lightly-loaded Cable.

Leaving the distortionless cable, let us now suppose that the insulation resistance is too high to have any influence on the signalling, and let us examine the effect of a small continuously distributed inductance. Take $L/R = KRl^2/1{,}000$, or in the present instance, $L = 9{\cdot}5$ millihenries per nautical mile, which may be regarded, in the present state of knowledge, as an amount to which the natural inductance of the cable may reasonably be brought up by loading on the continuous plan. The formula for the arrival current is (15), or

$$C_r=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+4e^{-\frac{Rt}{2L}}\sum_1^\infty\frac{\cos m\pi \sinh\frac{Rt}{2L}\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}}{\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}}\right].$$

Here $\frac{4\pi^2L}{KRl^2\,.\,R}=\frac{4\pi^2}{1{,}000}=0{\cdot}03948$, and the first five terms of the series, as shown in Table CXV., are sinh terms, after which the quantity under the square root sign is negative and the series becomes

$$\sum_1^\infty\frac{\cos m\pi \sin\frac{Rt}{2L}\sqrt{\frac{4m^2\pi^2L}{KRl^2\,.\,R}-1}}{\sqrt{\frac{4m^2\pi^2l}{KRl^2\,.\,R}-1}}$$

which converges very slowly. Nevertheless, on account of the factor $e^{-\frac{Rt}{2L}}$ by which they are multiplied, the sum of these terms is small, and it will suffice to neglect them, and take only the first five terms, without serious error except when t is small. The formula thus becomes, neglecting also $e^{-\frac{Rt}{L}}$,

$$\underset{(\text{L small})}{C_r}=\frac{E}{Rl}\left[1+\sum_{m=1}^{5}\frac{2\,e^{-\frac{Rt}{2L}\left(1-\sqrt{1-\frac{4m^2\pi^2L}{KRl^2.R}}\right)}.\cos m\pi}{\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}}\right]\quad.\qquad(20)$$

Table CXV.—R/2L=114·88.

$m.$	$\frac{4m^2\pi^2L}{KRl^2\,.\,R}$.	$1-\frac{4m^2\pi^2L}{KRl^2.R}$.	$\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}$	$\frac{R}{2L}\left(1-\sqrt{1-\frac{4m^2\pi^2L}{KRl^2\,.\,R}}\right)$
1	0·03948	0·96052	0·9801	2·26
2	0·15790	0·84210	0·9177	9·45
3	0·35528	0·64472	0·8030	22·63
4	0·63162	0·36838	0·6069	45·16
5	0·98690	0·01310	0·1145	101·73

Curve A, Fig. 183 is plotted from (20) and Table CXVI.

Curve B, Fig. 183, is the arrival curve without inductance. The effect of inductance is seen in making curve A slightly steeper than curve B, but soon after one second the two curves

Table CXVI.—*Fig.* 183, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady +	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0
m=1 −	327·2	260·9	208·2	166·1	132·5	84·3	42·9	13·8	4·5	0·5
2 +	170·3	66·2	25·7	10·0	3·9	0·6	...	...	...	...
3 −	52·1	5·4	0·6	0·1	...	...	...	...	...	...
4 +	7·2	0·1	...	...	...	...	...	...	...	...
5 −	0·1	...	...	...	...	...	...	...	...	...
C_r(mm.a. per volt)	−0·9	+0·9	18·0	44·9	72·4	117·3	158·2	187·2	196·5	200·5

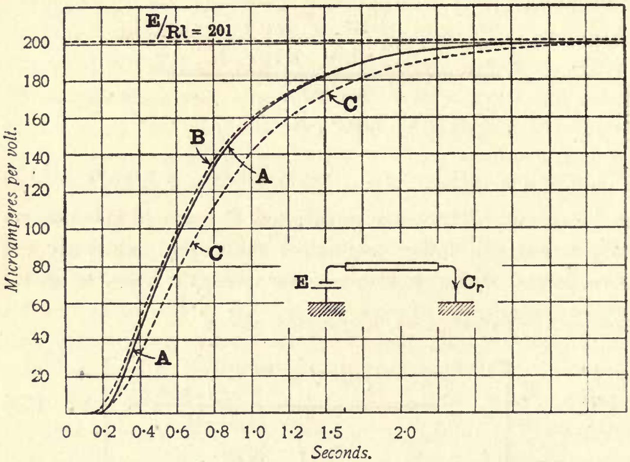

FIG. 183.—LIGHTLY-LOADED CABLE.

Curve A. $L/R = KRl^2/1000$; $L = 9{\cdot}5$ m.h. per n.m.

„ B. $L=0$.

„ C. $L=0$; $K'=1{\cdot}2\times K$.

coalesce. We have here the first instance of an improvement in shape which is not paid for by a loss in height.

The difference between the shapes of the curves is not great enough to exercise much influence on the speed of signalling. Moreover, when we come to consider the ways in which increase of inductance may be effected it will be found that in practice such an increase is invariably accompanied by an increase in resistance, or in capacity, or in both, if the cost of the cable is to remain unaltered. Thus, let us assume that the increased inductance is due to a single whipping of 8-mil iron wire, and that the copper conductor is reduced in diameter to admit of the iron layer. The diameter of a 500 lb. conductor would then be reduced from 0·182 in. to 0·166 in., and the resistance would increase in the proportion of 331 : 276, or by 20 per cent. Now, an increase of this amount in the *capacity* would increase the KRl^2 of the cable by 20 per cent. also, and would, therefore, increase the time abscissæ of Fig. 183 by 20 per cent., thus converting curve B into curve C. Alternatively, an increase of 20 per cent. in the resistance would, in addition, reduce the ordinates of curve B by 20 per cent. Hence, instead of comparing curves A and B we should compare curve B with a curve closely approximating in shape to curve C. The practical introduction of light loading would thus apparently lead to an actual loss in speed.

This conclusion is borne out by the only experimental evidence which is available on the subject of loading in telegraph cables.* Experiments made at Electra House in 1904-5 on an artificial cable of KRl^2 varying from 1·54 seconds to 3·2 seconds, and loaded up to 0·030 henry per nautical mile, showed that the all-copper core was preferable to the composite core. Accordingly the distributed load was abandoned, and the plan of concentrating the inductance at the ends of the cable was adopted.

Concentrated Series Inductance.

To investigate the effect of concentrating the inductance at the ends of the cable, suppose that as in Fig. 184, a single inductance coil is placed at one end of the cable. The periodic current at the receiving end in that case is

$$C_r = \frac{V_s}{Z_r \cosh Pl + Z_0 \sinh Pl},$$

* B. Davies, "Jour." I.E.E., 46, p. 377, 1911.

where $Z_r = R_r + ipL_r$. The arrival current is, therefore, easily found to be

$$C_r = \frac{E}{Rl + R_r} + \Sigma_x \frac{2Ee^{-\frac{x^2 t}{KRl^2}}}{Rl\left[\frac{1}{\cos x} - \frac{3\sin x}{x} - \frac{2R_r}{Rl}\cos x\right]}, \quad (21)$$

where

$$\tan x = \frac{xR_r}{Rl}\left[\frac{L_r}{R_r}\cdot\frac{x^2}{KRl^2} - 1\right] \quad \ldots\ldots\ldots \quad (22)$$

Equation (22) has already been discussed when dealing with the arrival voltage on an inductance shunt,* and its roots are known. The most interesting case is when $L_r/R_r = KRl^2/\pi^2$, in which case (22) becomes, assuming $R_r = Rl/10$,

$$\tan x = \frac{x}{10}\left[\frac{x^2}{\pi^2} - 1\right], \quad \ldots\ldots \quad (23)$$

the roots of which are given in Table LIV. The complex root is $a + ib$, or $4{\cdot}669 + i \times 1{\cdot}988$. Writing the complex term in the series of (21) in the form $\frac{H + iJ}{M + iN}$, it follows that

$$H = e^{-\frac{(a^2 - b^2)t}{KRl^2}}\left[\cos a \cosh b\left(a\cos\frac{2abt}{KRl^2} + b\sin\frac{2abt}{KRl^2}\right)\right.$$
$$\left. - \sin a \sinh b\left(a\sin\frac{2abt}{KRl^2} - b\cos\frac{2abt}{KRl^2}\right)\right].$$
$$= e^{-4{\cdot}101t}\left[-7{\cdot}8668\cos\frac{2abt}{KRl^2} + 16{\cdot}391\sin\frac{2abt}{KRl^2}\right].$$

$$J = -e^{-\frac{(a^2 - b^2)t}{KRl^2}}\left[\cos a \cosh b\left(a\sin\frac{2abt}{KRl^2} - b\cos\frac{2abt}{KRl^2}\right)\right.$$
$$\left. + \sin a \sinh b\left(a\cos\frac{2abt}{KRl^2} + b\sin\frac{2abt}{KRl^2}\right)\right].$$
$$= e^{-4{\cdot}101t}\left[16{\cdot}391\cos\frac{2abt}{KRl^2} + 7{\cdot}8668\sin\frac{2abt}{KRl^2}\right].$$

$$M = \frac{9a}{10} - \frac{3}{2}\sin 2a\cosh 2b - \frac{a}{10}\cos 2a\cosh 2b - \frac{b}{10}\sin 2a\sinh 2b$$
$$= 12{\cdot}6842.$$

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* Part III., Chap. IX., p. 289.

$$N=\frac{9b}{10}-\frac{3}{2}\cos 2a \sinh 2b-\frac{b}{10}\cos 2a \cosh 2b.$$
$$=46{\cdot}8868.$$

The conjugate term is $\frac{H-iJ}{M-iN}$, and the two combined give $\frac{2(HM+JN)}{M^2+N^2}$. Substituting the values of H, J, M, and N, the term yielded by the complex roots is

$$300{\cdot}93\ e^{-4{\cdot}101t}[\sin(4{\cdot}2656t+49^\circ\ 13{\cdot}6)].$$

Table CXVII.—*Fig.* 184, *Curve A.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0	3·0
Steady......+	182·7	182·7	182·7	182·7	182·7	182·7	182·7	182·7	182·7	182·7
3·1416......−	400·6	319·3	254·5	202·8	161·7	102·7	52·0	16·7	5·4	0·6
7·5880......+	31·8	8·5	2·3	0·6	0·2	...	...	...	...	...
10·913......−	2·2	0·1	...	...	...	...	...	...	...	...
14·100......+	0·2	...	...	...	...	...	...	...	...	...
17·259......−	...	...	...	...	...	...	...	...	...	...
Oscillatory...	191·6	131·2	74·1	31·8	5·8	−11·0	−4·6	0·5	...	...
C_r (mm. a. per volt).........	3·5	3·0	4·6	12·3	27·0	69·0	126·2	166·5	177·4	182·2

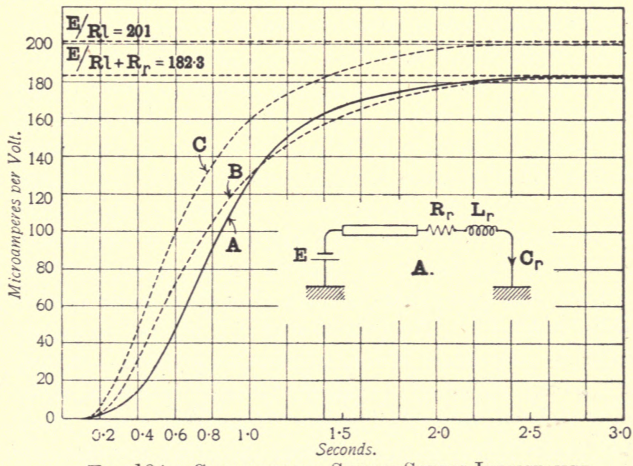

Fig. 184.—Cable with a Single Series Inductance.

Curve A. $\frac{L_r}{R_r}=\frac{KRl^2}{\pi^2}$; $R_r=\frac{Rl}{10}$.

Curve B. $L_r=0$; $R_r=\frac{Rl}{10}$.

Curve C. $L_r=0=R_r$.

Curve A, Fig. 184, is plotted from (21) and Table CXVII. The term given by $x=\pi$ is not zero, but is $\dfrac{2\mathrm{E}e^{-\frac{\pi^2 t}{\mathrm{KR}l^2}}}{2\mathrm{R}_r-\mathrm{R}l}$.

Curve B is the arrival current when the coil has zero inductance, and curve C is the arrival current for the cable alone. Curve A is much superior in shape to curve B, and the improvement produced by the inductance is marked. At the same time it is evident that curve B differs too much from curve C, and the resistance of the coil, $\mathrm{R}_r=\mathrm{R}l/10=497{\cdot}5$ ohms is much too high. To secure the best result the resistance of the coil should be as low as is possible and consistent with obtaining the required time-constant. If R_r were small, curve B would not differ appreciably from curve C, and the improvement in shape of curve A on both would then be clearly shown, free from the additional complication caused by the resistance.

It is clear from the discussion which has just been given of the effect of introducing inductance whether distributed or localised, that theory and experiment are here quite in agreement. If the problem had been attacked from the theoretical standpoint in the first instance instead of by experiment, the same conclusions would have been forced upon the investigator. We are justified, therefore, in feeling reasonably sure in the investigations which follow, where experimental evidence is entirely lacking, that the deductions of theory will be maintained when they come to be submitted to the test of experiment. Now, there are three great advantages which pure theory offers over experiment, and which render it the natural complement of experiment, so that no investigation in which both theory and experiment do not play their parts can be considered complete. The first is, that it is possible to isolate the influence of a single factor, whereas in practice it is very difficult, as the beginner in experimental research has painfully to learn, to keep from altering two things at a time. Thus, the early experiments on loaded telephone cables gave a negative result, because the coils being air-cored had too high resistance which swamped any benefit from the inductance itself. The second is, that it is possible in theoretical calculation to vary at will the magnitude of any quantity likely to influence the result, so that its effect, if any, can be pushed to extreme. In practice it is often impossible, except at great cost, to make such changes, and in general the apparatus is not sufficiently flexible. In the third place, it is often possible to obtain from theoretical considerations a *quantitative* estimate of the magnitude of the result which is anticipated. It is the reason that renders a preliminary calculation before the apparatus is brought together of inestimable value whenever possible.

In the next paragraph we shall, therefore, suppose that the inductance of the cable is substantially increased, without increase of resistance, and without considering for the time being the practicability of the change.

Heavily-loaded Cable.

Returning to formula (15) for the arrival current in a continuously loaded cable—

$$C_r=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+4e^{-\frac{Rt}{2L}}\sum_1^\infty\frac{\cos m\pi\sinh\frac{Rt}{2L}\sqrt{1-\frac{4m^2\pi^2L}{KRl^2.R}}}{\sqrt{1-\frac{4m^2\pi^2L}{KRl^2.R}}}\right]$$

—suppose that L/R now equals $KRl^2/100$, or that $L=0{\cdot}095$ henry per n.m. The quantity under the square-root sign is now negative except for the first term of the series, as is evident from Table CXVIII. The subsequent terms are of the nature

Table CXVIII.

m.	$\frac{4m^2\pi^2L}{KRl^2.R}$.	$\sqrt{\pm\left(1-\frac{4m^2\pi^2L}{KRl^2.R}\right)}$	$\frac{R}{2L}\sqrt{\pm}$	$\frac{4E}{Rl\sqrt{\pm}}$
1	0·3948	0·7780	8·938	1033·3
2	1·5790	0·7610	8·743	1056·5
3	3·5528	1·5978	18·357	503·2
4	6·3162	2·3057	26·489	348·7
5	9·869	2·9781	34·213	270·0
6	14·213	3·6350	41·761	221·2
7	19·346	4·2832	49·209	187·7
8	25·267	4·9262	56·594	163·2
9	31·979	5·5659	63·946	144·5
10	39·476	6·2030	71·262	112·8

of damped oscillations, and the series converges only slowly for small values of t. In its later stages, where t is greater than 0·5 second, the current is represented with sufficient accuracy by the formula :

$$C_r = \frac{E}{Rl}\left[1 - \frac{2e^{-\frac{Rt}{2L}\left(1-\sqrt{1-\frac{4m^2\pi^2}{KRl^2 \cdot R}}\right)}}{\sqrt{1-\frac{4m^2\pi^2 L}{KRl^2}}}\right] \quad . \quad . \quad . \quad . \quad (24)$$

Table CXIX. contains the steady term, and the first term of the series, according to (24). In Table CXX. the calculation of the series is extended to the first 10 terms for two particular values of the time.

Curve A, Fig. 185, is plotted from Tables CXIX. and CXX. Curve B is the ordinary arrival curve, without inductance. The dotted line through $t=0{\cdot}435$ second marks the time displace ment in the distortionless case, although the height (0·18) is too small to show on the diagram.

Curves A and B lie close together in their later stages. For smaller values of t curve A is considerably steeper, but the curve is given in Table CXIX. only as far back as $t=0{\cdot}5$ second. When t is very small the series becomes only very slowly convergent, and a great number of terms must be taken to obtain a close approximation to the current.

Another way of approaching the problem of the initial current values must be adopted. It has been shown * that the influence of leakance alone is to cause a quicker rise to a reduced steady value. On the other hand, as has been brought out in the foregoing study of inductive meshes, inductance leaves the steady value unaltered, but stops back the current initially. Now, in Fig. 182, curve A, the current in the distortionless case does not begin till 0·435 second, and, therefore, in the present instance where the constants are the same, except that leakance is absent, we may expect that the current will also not begin until the same time, and will then rise to a greater value than in the distortionless case. Accordingly, as a first approximation to the shape of the curve, the vertical line is extended upwards until it cuts the projection of curve A.

* Chap. X., p. 322.

Table CXIX.—*Fig.* 185 *Curve A.*

t (sec.)	0·5	0·7	1·0	1·5	2·0	3·0
Steady..................	201·0	201·0	201·0	201·0	201·0	201·0
1st term ....—	144·4	86·7	40·3	11·3	3·2	0·3
C_r (mm. a. per v.)	56·6	114·3	160·7	189·7	197·8	200·7

Table CXX.—*Fig.* 185, *Curve A.*

m.	Steady	$\dfrac{Ee^{-\frac{Rt}{L}}}{Rl}$	1	2	3	4	5	6	7	8	9	10	C_r (mm.a per v.)
=0·5 sec.	201·0	...	−144·3	−3·2	−0·4	+0·7	+0·9	+0·6	+0·3	−0·0	−0·2	−0·3	55·0
=0·6 sec.	201·0	...	−111·9	−0·9	+0·5	−0·1	−0·3	−0·0	+0·2	+0·1	−0·1	−0·1	88·4

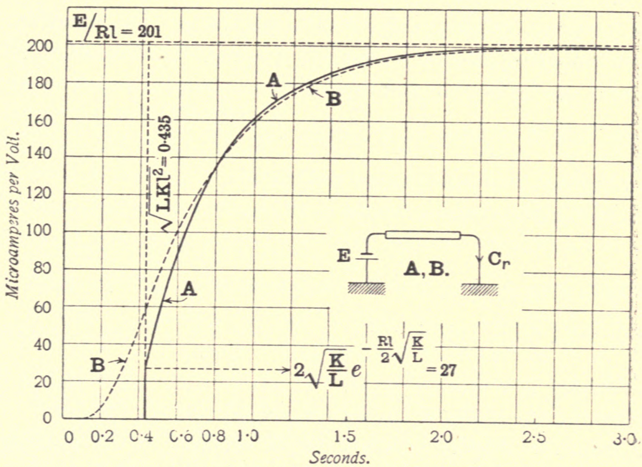

FIG. 185.—HEAVILY-LOADED CABLE.

Curve A.—$L/R=\dfrac{KRl^2}{100}$; L=0·095 h. per n.m. Curve B.—L=0

The exact height of the ordinate at $t=\sqrt{LKl^2}$ may be obtained as follows. The periodic received current after all transient effects have died away, is given by

$$C_{r\,(\text{periodic})}=\frac{Ee^{ipt}}{Z_0 \sinh Pl},$$

where $Z_0=\sqrt{\frac{R+ipL}{ipK}}=\sqrt{\frac{L}{K}}$ when p is great; and $P=\alpha+i\beta$. Hence,

$$C_{r\ (\text{periodic})}=2E\sqrt{\frac{K}{L}}e^{-\alpha l}\sin(pt-\beta l).$$

Now, also when p is great, $\alpha=\frac{R}{2}\sqrt{\frac{K}{L}}$ and $\beta=p\sqrt{LK}$.

Substituting these values,

$$C_{r\ (\text{periodic})}=2E\sqrt{\frac{K}{L}}e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}\sin p(t-\sqrt{LKl^2}). \quad . \quad (25)$$

The received current suffers, therefore, the same attenuation and the same phase displacement whatever the value of p may be. A square topped wave would appear at the receiving end undistorted, but moved through a time $\sqrt{KLl^2}$ to the right. To every contact at the sending end corresponds, therefore, an abrupt rise in current to a height $2\sqrt{\frac{K}{L}}e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$, at a time, $\sqrt{KLl^2}$, later, as is represented in Fig. 185.

The foregoing considerations have shown (a) that the wave-front in its passage through the loaded cable is vertical, (b) that it suffers attenuation, $e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$, in its journey, and ($c$) that it travels with finite velocity, $v=\frac{l}{\sqrt{KLl^2}}=\frac{1}{\sqrt{KL}}$. When L is zero the velocity is infinitely great. It has already been shown * that on the electrostatic theory of propagation, no interval of time whatever elapses between contact-making at the sending end and the beginning of the signal at the receiving-end.

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* Part III., Chap. VIII., p. 237. The belief in a definite "silent interval" in the simple cable, devoid of inductance, is of interest, because of its frequent occurrence. It is clearly enunciated in the writings of the early investigators, and from them has been copied, until the reader will meet it in almost every book and Paper devoted to cable telegraphy, the works of Vaschy and Heaviside naturally excepted. One would suppose that it is founded on a misunderstanding of the letter to Stokes, in which the arrival curve is first obtained, but it is strange that it should have passed uncontradicted by Sir William Thomson.

ATTENUATION OF THE WAVE-FRONT IN TRANSMISSION.

It is interesting to trace the attenuation which the wave-front suffers in its passage through the cable. The periodic current at distance x from the sending-end is given by

$$C_x = \frac{V_s \cosh P(l-x)}{Z_0 \sinh Pl} = \frac{Ee^{ipt} \cosh (l-x)(a+i\beta)}{Z_0 \sinh l(a+i\beta).}.$$

Now, when $l-x$ is great, $\cosh (l-x)a = \sinh (l-x)\, a$, and

$$C_x = E\sqrt{\frac{K}{L}} e^{-ax} \sin (pt-\beta x)$$

$$= E\sqrt{\frac{K}{L}} e^{-\frac{Rx}{2}\sqrt{\frac{K}{L}}} \sin p(t-x\sqrt{LK}). \quad . \quad . \quad . \quad . \quad (26)$$

The current produced by any periodic E.M.F. may be built up from (26) by giving p a succession of appropriate values according to the requirements of Fourier's theorem. Let us choose an extremely short square-topped dot signal repeated at intervals of 0·1 second. From (26) it follows that at time $t = x\sqrt{LK}$ the signal would pass through a point at distance x from the sending-end still as a square-topped signal, but with height reduced by the factor $e^{-\frac{Rx}{2}\sqrt{\frac{K}{L}}}$.

Table CXXI. is calculated from (26), and from it Fig. 186 is plotted. The zero of time is taken just as a signal is being formed at the sending-end. The signal preceding it by one-tenth second has already travelled 523 n.m., and has shrunk to less than one-third the height. The signal 0·4 second in advance has almost reached the end of the cable.

Suppose that the cable is put to earth at the receiving end. Comparing (25) with (26), it is seen that the current at the end is then exactly twice as great as it would be at the same distance in the infinite cable. This may be expressed by saying that at the closed end the current wave-front is reflected without change of sign, but with doubled amplitude. A signal 0·435 second in advance would just have reached the distant end and would be in process of rising to twice its height before starting to return through the cable to the sending-end. The smaller diagram shows the signals, at 0·5 second and 0·6

Table CXXI—*Fig.* 186.

$t.$	0·1	0·2	0·3	0·4	0·4352	0·5	0·6
$x.$	523	1,046	1,569	2,092	2,276	2,615	3,139
$\frac{Rx}{2}\sqrt{\frac{K}{L}}$......	1·149	2·298	3·447	4·596	5·0	5·745	6·893
$e^{-\frac{Rx}{2}\sqrt{\frac{K}{L}}}$ ...	0·3170	0·1005	0·0318	0·0101	0·0067	0·0032	0·0010
C_x (mm.a. per volt)	637	202	64·0	20·3	13·5	6·43	2·04

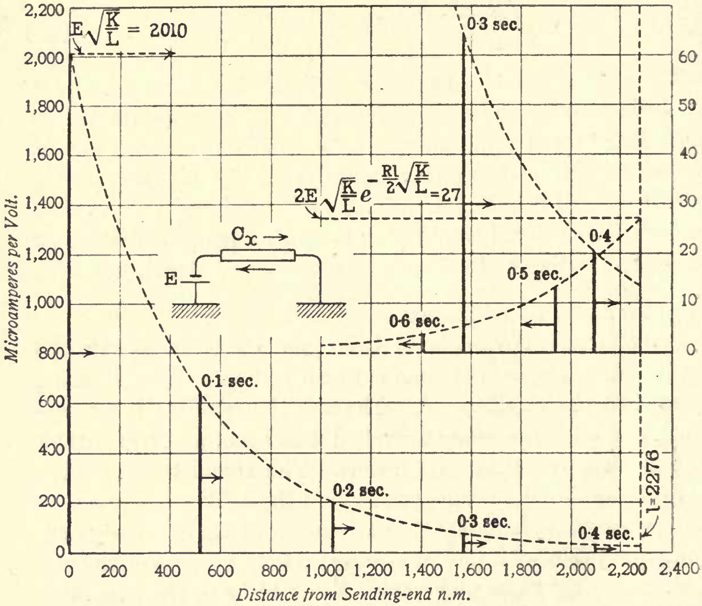

Fig. 186.—Attenuation of Wave-front in Transmission through Loaded Cable, at 0·1 Second Intervals.

second in advance, which have already suffered reflection. Before reaching the sending-end again the reflected signals would be infinitesimally small in comparison with the outgoing signals. In a long cable reflection need be considered as taking place only at the receiving-end.

Signals in the Loaded and Distortionless Cables Compared.

From (19) the arrival current in a distortionless cable is given by the formula

$$C_r = 2E\sqrt{\frac{K}{L}}e^{-Rl\sqrt{\frac{K}{L}}},$$

whereas from (25) the abrupt-rise which marks the beginning of the arrival curve in the loaded cable is given by

$$C_r = 2E\sqrt{\frac{K}{L}}e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}},$$

which differs in having $l/2$ in the exponent in place of l. In Fig. 185 the abrupt rise in current at time $l\sqrt{LK}$ is caused by the reflection of the *head* of the wave, and the subsequent slow increase to the steady value is due to the *tail*.*

Now, it is clear that if the time of contact in the elementary dot signal be decreased indefinitely then in the limit only the head of the arrival-curve will be appreciable in the formation of the received signal; the tail, which is of finite slope, being completely eliminated. The following important proposition may, therefore, be deduced :—

In the loaded cable the greater the speed of signalling the less is distortion. The attenuation is independent of the speed, and in the limit when the speed is infinitely great the signals approximate to those in a distortionless cable of half the length.

It is evident that a different state of affairs altogether is in question here from that to which we have been accustomed in the plain cable. It is assumed that the cable constants are true constants, and do not vary with the speed. Table CXIV. contains the heights of the signals in the distortionless and loaded cables for different values of the loading. It is seen that even when the loading is small the signals in the loaded cable are of appreciable height, whereas in the distortionless cable they are infinitesimal. Any attempt to approach the distortionless condition by the introduction of leakance in a loaded cable would cause an enormous loss in current with no appreciable improvement in the shape of the signals.

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* Heaviside, " Electromagnetic Theory," II., p. 422.

Loaded Cable Signals.

In Fig. 187, curves A and B, are shown the elementary dot signals for times of contact 0·08 and 0·02 second respectively. They are obtained graphically in the usual manner from curve A of Fig. 185. The head of curve A rises to a height of 59 mm. a. per volt, and the tail begins at 32 mm. a. per volt, or more than half the height of the head. In curve B the

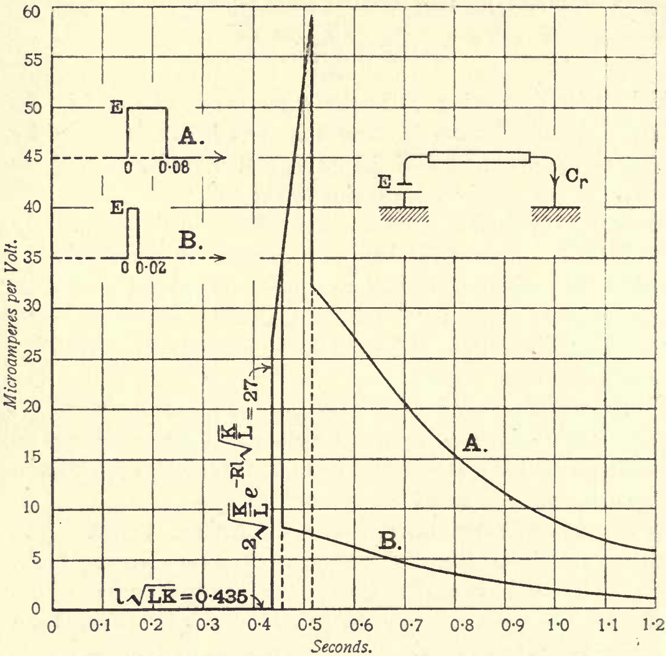

Fig. 187.—Elementary Dot Signal; Loaded Cable.

$L/R = \frac{KRl^2}{100}$; L=0·095 h. per n.m.

Curve A.—Time of Contact, 0·08 second. Curve B.—Time of Contact, 0·02 second.

heights are 35 and 8 mm. a. per volt, so that in this case the relative height of tail and head is less than one-quarter. If the period of contact were still further reduced, the tail would tend to disappear altogether, and the head to reach a limiting height of 27 mm. a. per volt.

In Fig. 188 the signal "understand" is built up from curve A of Fig. 187, taking the length of a space as equal to that of a dot. It should be compared with Fig. 152, for the plain cable. It is clear from Fig. 188 that the signals would be improved (*a*) by reducing the length of the space until it almost vanished entirely, and (*b*) by reducing the length of the contact—*i.e.*,

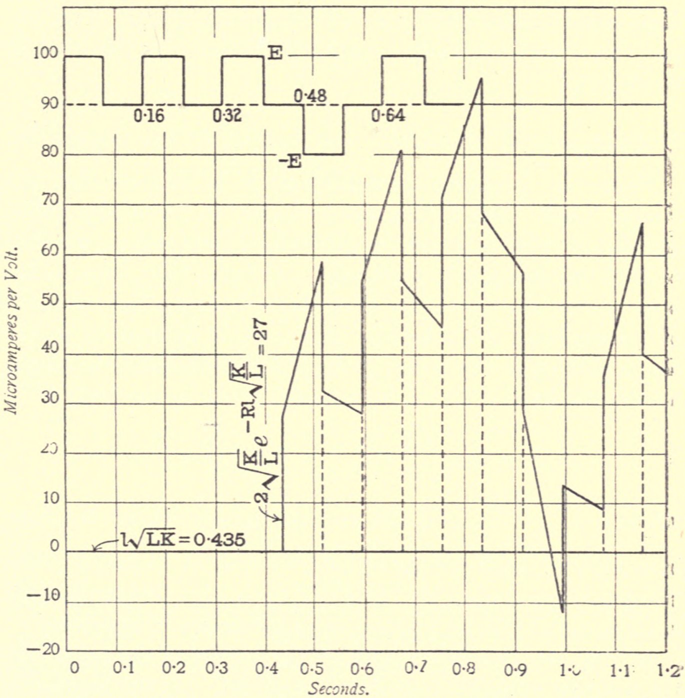

Fig. 188.—Loaded Cable: Signal "Understand."

$L/R = \frac{KRl^2}{100}$; L=0·095 h. per n.m.

employing curve B, say, of Fig. 187 as the elementary dot signal instead of curve A. Both these methods of improvement in shape call for increased speed of sending. The only limitations would arise from the receiving apparatus, and any increase which could be effected in the speed of response of the receiver would permit of a further increase in the speed of transmission. Assuming that, as on land, a speed of 500 words, or say 3,000 letters, a minute were attainable, the length of the elementary contact—taking 7·16 such contacts to form a letter—would be $\frac{60}{3{,}000 \times 7{\cdot}16} = 0{\cdot}00279$ second, or about one-tenth part of the time of contact in curve B, Fig. 187. It is clear, therefore, that with high-speed transmission the tail would disappear altogether and the signals would be constituted entirely by the head or wave-front.

Dependence of Height of Signals on Cable Constants.

From the foregoing considerations it is evident that the simple "KR" law breaks down entirely in the loaded cable. Instead of it we may use the generalised form, taking account of both leakance and inductance.* Alternatively, since, as has been shown, the signals are quasi-distortionless when the speed of transmission is sufficiently great, slight alterations in the shape of the tops of the signals may be neglected and the height alone considered. Now, the height is given by the formula

$$C_r = 2E\sqrt{\frac{K}{L}}\, e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}} = 2E\sqrt{\frac{K}{L}}\, e^{-al},$$

where $a = \frac{R}{2}\sqrt{\frac{K}{L}}$ and is the same as the attenuation constant in a highly-loaded telephone cable devoid of leakance. The problem of producing big signals in the loaded telegraph cable is the same as that of reducing the attenuation in telephony.

In Fig. 189 the dependence of the height of the signals on the amount of added inductance is shown. As the loading is reduced the height falls off rapidly. From Table CXIV. it is seen that a total inductance of 9·5 mh. per n.m. would give $1{\cdot}728 \times 10^{-3}$ mm. a. of received current. With 50 sending volts the received current would be 0·0864 microampere. Such a current would require magnification, either by a telephone relay or by a highly-sensitive telegraph relay, before it could be used to operate a local recording circuit. If the inductance

* Chap. VIII., p. 258.

could be brought to 19 mh. per n.m., the received current would be 6·27 microamperes for 50 volts. The lower figure for the inductance could be attained by continuous loading, and whether the higher figure could be reached by the same means or not would depend essentially on the allowable cost for the cable.

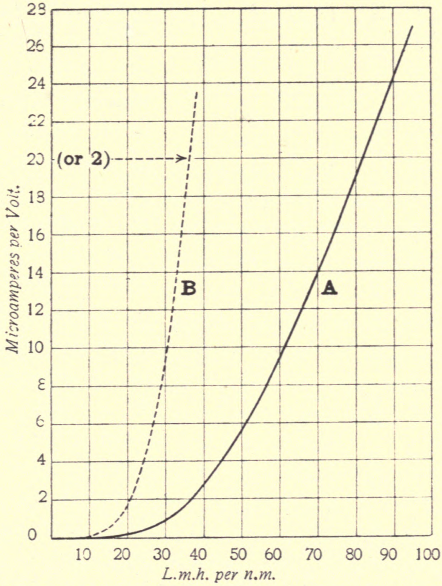

FIG. 189.—DEPENDENCE OF HEIGHT OF REFLECTED WAVE-FRONT ON AMOUNT OF LOADING.

Curve B is Curve A to 10 times the vertical scale.

COIL LOADING AND SPACING.

If the inductance of the cable is to be high the use of loading coils is necessitated. Now, as the wave-front travels along the cable and strikes an inductance coil it is partially reflected, and may undergo considerable loss before reaching the receiving end. It is, therefore, of primary importance to give attention to the spacing of the coils.

Table CXXII.

$m.$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$1+\cos\frac{m\pi}{20}$	1·951	1·891	1·809	1·707	1·588	1·454	1·309	1·156	1	0·844	0·691	0·546	0·412	0·293	0·191	0·109	0·049	0·012
$\frac{3040\mathrm{L}\left(1-\cos\frac{m\pi}{20}\right)}{\mathrm{R}\,.\,\mathrm{K}\mathrm{R}l^2}$	1·515	3·313	5·806	8·903	12·53	16·60	21·01	25·65	30·40	35·16	39·79	44·20	48·27	51·89	54·99	57·49	59·31	60·43
$\sqrt{\text{do.}-1}$........	0·718	1·521	2·192	2·811	3·396	3·950	4·473	4·964	5·422	5·844	6·228	6·573	6·876	7·133	7·348	7·516	7·636	7·709
$\frac{2\left(1+\cos\frac{m\pi}{20}\right)}{\sqrt{\ }}$	5·436	2·487	1·650	1·214	0·935	0·736	0·585	0·466	0·369	0·289	0·222	0·166	0·120	0·082	0·052	0·029	0·013	0·003

The fewer the coils and the more concentrated the loading the greater can their time-constant be made, and the higher the proportion which the added inductance bears to the added resistance; but at the same time their influence tends to become localised instead of distributed, and the wave-front tends to lose its steepness. Take as a possible compromise that the coils are placed at intervals of 100 miles instead of at every mile, as in telephone cables. In a cable 2,000 miles long there would, therefore, be 20 coils. If the San Francisco-Honolulu cable is to have 20 coils to bring its inductance up to 0·0951 henry per n.m., the inductance of each coil must be 10·8 henries.

In order to investigate the influence of the spacing, suppose that the cable is replaced by a system of 20 loaded meshes. The formula for the arrival current may then be obtained from the general formula (10) by putting $G=0$. This leads to

$$C_n=\frac{E}{Rl}\left[1-e^{-\frac{Rt}{L}}+2e^{-\frac{Rt}{2L}}\times\sum_{m=1}^{m=n-1}\frac{\left(1+\cos\frac{m\pi}{n}\right)\cos m\pi\sinh\frac{Rt}{2L}\sqrt{1-\frac{8Ln(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{R\,.\,KRl^2}}}{\sqrt{1-\frac{8Ln(n-1)\left(1-\cos\frac{m\pi}{n}\right)}{R\,.\,KRl^2}}}\right] \quad . \quad . \quad . \quad (27)$$

When $L/R=KRl^2/100$ and $n=20$, the first term of the series is a hyperbolic sine and the remaining 18 terms are sines. The approximate values of the principal constants are contained in Table CXXII., and Fig. 190, Curve A, is plotted from Table CXXIII.

Curve B, Fig. 190, is the arrival current when the loading is uniformly distributed. In order to trace the mesh curve at its steepest part, where it crosses the vertical through $t=l\sqrt{KL}$, or 0·435 sec., points have been calculated on it for the instants $t=0{\cdot}40$, 0·42, 0·44, 0·46, 0·48 and 0·50 second.

The effect of the sub-division into meshes is seen in Curve A in the rounding-off which the steep part of curve B has undergone. The slope of the curve is now everywhere finite, instead of being infinitely great, as in curve B, when $t=l\sqrt{LK}$. The height of the signals built up from curve B would therefore depend on the speed of signalling, on which would also depend whether the distortion that would be present to some extent would be harmful or not.

In Fig. 191 the arrival current for 20 loaded meshes is compared with the current for 20 plain meshes. The required formula may be obtained from (27) by making L vanishingly small. The formula then reduces to

$$C_n=\frac{E}{Rl}\left[1+\sum_{m=1}^{m=n-1}\left(1+\cos\frac{m\pi}{n}\right).\cos m\pi\,.\,e^{\frac{-2n(n-1)\left(1-\cos\frac{m\pi}{n}\right)t}{KRl^2}}\right] \quad (28)$$

which is the same as (13) of Chap. XIII.

Table CXXIII.—*Fig.* 190, *Curve A.*

t (sec.)	0·4	0·5	0·6	0·7	1·0	1·5	2·0
$e^{-Rt/L}$	0·000102	0·000010	...	...	...	...	...
$e^{-Rt/2L}$	0·010102	0·003203	0·001015	0·000322	0·0000	...	...
Sinh term	0·9614	0·7566	0·5952	0·4682	0·2279	0·0686	0·0207
First 3 terms (mm.a. per v.)	7·6	48·9	81·4	106·9	155·2	187·2	196·9

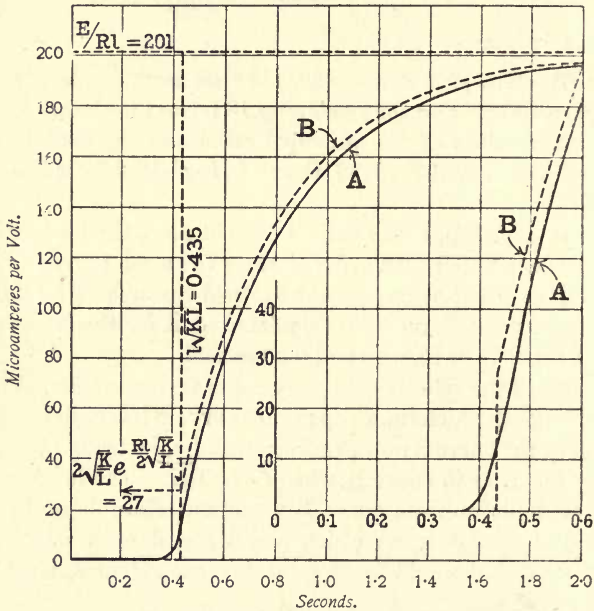

FIG. 190.—LOADED MESHES AND CABLE.

Curve A.—20 loaded meshes. $L/R = \frac{KRl^2}{100}$; $L = 0·095$ h. per n.m.

Curve B.—Loaded Cable. Inset: the discontinuity to a greater scale.

Curve A, Fig. 191, is plotted from (28) and Table CXXIV.

Curve B is the arrival current for the non-loaded cable. Curves A and B coincide initially; afterwards the cable current is greater than the mesh current. For purposes of comparison

Table CXXIV.—*Fig.* 191, *Curve A*.

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0
Steady	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201·0	201 0
$m=1$, ...−	322·2	259·9	209·6	169·1	136·4	88·7	46·6	15·9	5·4
$m=2$, ...+	166·8	71·0	30·2	12·8	5·5	1·0	0·1	...	...
$m=3$, ...−	56·7	8·4	1·3	0·2	...	...	...	...	...
$m=4$, ...+	13·0	0·5	...	...	...	...	...	...	...
$m=5$, ...−	2·1	...	...	...	...	...	...	...	...
C_r (mm. a. per v.)	0·1	4·1	20·4	44·6	70·1	113·3	154·5	185·1	195·6

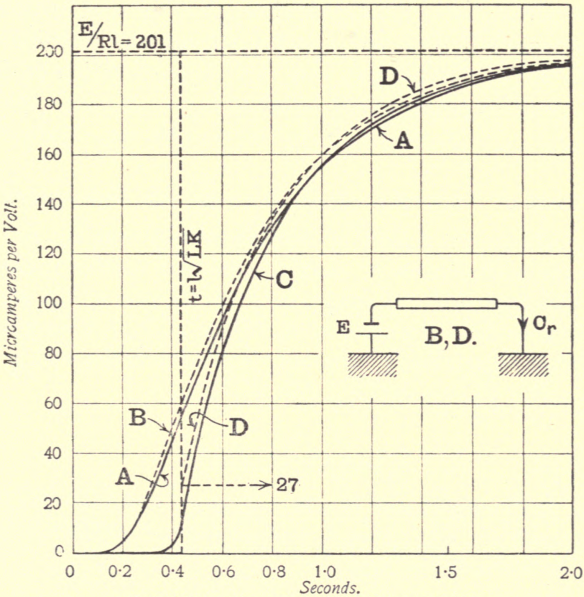

FIG. 191.—PLAIN AND LOADED MESHES.

Curve A.—20 plain meshes.
Curve C.—20 loaded meshes.
Curve B.—Non-loaded cable.
Curve D.—Loaded cable.

the curves of Fig. 190 are reproduced in curve C—for the loaded meshes—and curve D—for the loaded cable. Curve C in its later stages is less than curve D, in the same manner and to the same extent as curve A is less than curve B; the slight falling off in the tail is therefore to be attributed to the concentration of capacity and resistance. Again, curve D rises in its later stages above curve B, and similarly curve C rises above curve A so as nearly to coincide with curve B ; in other words the influence of loading is shown in a slight increase in the height of the tail, whether in the loaded cable or in loaded meshes.

It has been shown, however, that the tail is unimportant as it is practically eliminated at high speeds. Now, so far as the head is concerned sub-division into meshes does not cause much difference to exist between the plain curves initially. Hence it is not unreasonable to suppose that sub-division of capacity will not cause any great change when the cable is loaded, and that the loaded mesh curve is a close approximation to the cable curve with concentrated loading.

Influence of Terminal Apparatus—Single Condenser.

It is possible that signalling condensers may continue to be used in the loaded cable in order to eliminate earth currents. In any case some kind of apparatus must be used at the ends of the cable to form the arms of the bridge in duplex working. Suppose first that a single condenser is placed at one end of a loaded cable. The periodic solution is

$$C_r = \frac{V_s}{Z_0 \sinh Pl + Z_r \cosh Pl},$$

where Z_r is the impedance of the end apparatus. Let Z_r be the reciprocal of a conductance, and consist of a condenser K_r shunted by a leak G_r, so that

$$Z_r = \frac{1}{G_r + ipK_r}.$$

The denominator of C_r is

$$\phi(y) = \sqrt{\frac{R+yL}{G+yK}} \sinh l\sqrt{(R+yL)(G+yK)} + \frac{\cosh l\sqrt{(R+yL)(G+yK)}}{G_r + yK_r}.$$

Let $\frac{Gl}{G_r}=\frac{Kl}{K_r}=\frac{1}{\lambda}$, and put $l\sqrt{(R+yL)(G+yK)}=xi$. The roots of $\phi(y)=0$ are given by

$$\tanh l\sqrt{(R+yL)(G+yK)}=\frac{-1}{G_r+yK_r}\sqrt{\frac{G+yK}{R+yL}},$$

which reduces to $\tan x=\frac{l}{\lambda x}$. Also y is connected with x by the relationship

$$y=-\frac{(LG+KR)\pm\sqrt{(LG-KR)^2-4LKx^2/l^2}}{2LK}.$$

Further, $\quad \phi(0)$ is $\sqrt{\frac{R}{G}}\sinh l\sqrt{RG}+\frac{\cosh l\sqrt{RG}}{G_r}$.

Hence, finally, forming $y\phi'(y)$ and substituting for y in terms of x, after some manipulation the complete solution is found to be

$$C_r=\frac{E}{\sqrt{\frac{R}{G}}\sinh l\sqrt{RG}+\frac{\cosh l\sqrt{RG}}{G_r}}$$

$$+\frac{4Ee^{-\left(\frac{R}{2L}+\frac{G}{2K}\right)t}}{Rl}\sum_x\frac{\sinh\left(\frac{t\sqrt{(LG-KR)^2-4LKx^2/l^2}}{2LK}-\theta\right)}{\sqrt{1+\frac{GRl^2}{x^2}\sqrt{\left(1-\frac{LG}{KR}\right)^2-\frac{4Lx^2}{R\,.\,KRl^2}}\left[\frac{1}{\cos x}+\frac{\sin x}{x}\right]}}$$

$$\text{where } \tan x=\frac{Gl}{G_rx}=\frac{Kl}{K_rx} \text{ and } \tanh\theta=\frac{\sqrt{\left(1-\frac{LG}{KR}\right)^2-\frac{4Lx^2}{R\,.\,KRl^2}}}{1-\frac{LG}{KR}+\frac{2x^2}{GRl^2}} \quad . \quad . \quad (29)$$

When G and G_r are zero, $\tanh\theta$ and θ are zero, $\cosh\theta=1$ and $\sinh\theta=0$. The formula becomes

$$C_r=\frac{4Ee^{-\frac{Rt}{2L}}}{Rl}\sum_x\frac{\sinh\frac{Rt}{2L}\sqrt{1-\frac{4Lx^2}{R\,.\,KRl^2}}}{\sqrt{1-\frac{4Lx^2}{R\,.\,KRl^2}}\left[\frac{1}{\cos x}+\frac{\sin x}{x}\right]} \quad . \quad (30)$$

where $\qquad \tan x=Kl/K_rx$ . . . . (31)

If, in addition, L be zero, the formula reduces further to that for a single condenser with a plain cable.*

Let $K_r = Kl/10$, and let $L/R = KRl^2/100$, as before. The roots of (31) are contained in Table CXXV., and are taken from Table XXXVIII.†

Table CXXV.—$\tan x = 10/x$; $L/R = KRl^2/100$.

x.	$\frac{4Lx^2}{R \cdot KRl^2}$	$\frac{R}{2L}\sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}}$	$\frac{4}{Rl\sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}}\left(\frac{1}{\cos x}+\frac{\sin x}{x}\right)}$
1·429	0·08168	11·011	+108·13
4·306	0·74168	5·839	−577·12
7·228	2·0898	11·993	+423·42
10·200	4·1616	20·428	−307·87
13·214	6·9844	28·106	+247·13
16·260	10·5756	35·551	−215·45

The first and second terms of the series are hyperbolic; the subsequent terms are circular. Fig. 192, curve A, is plotted from (30) and Table CXXVI. The sine terms have been calculated for the times 0·50, 0·55 and 0·60 second, and the curve is shown dotted in its probable course to meet the ordinate through $t = l\sqrt{LK}$. Comparing Fig. 192 with Fig. 185, it is seen that the effect of the condenser is to produce a marked improvement in the shape of the curve. The tail is now much reduced, and the curve runs horizontally from the head for some distance before eventually sinking to zero. Curve B is the condenser arrival curve without inductance. The total area between the time axis and the curves is the same in both cases, being equal to the charge in the condenser. What is cut away from curve B by the vertical at $t = l\sqrt{LK}$ is added in its later stages to curve A. If Q_t be the quantity at time t by which the condenser is short of its full charge, then $Q_t = \int_t^\infty C\, dt$, which, from (30),

$$= 2E \cdot Kl \cdot e^{-\frac{Rt}{2L}} \sum_x \frac{\sinh \frac{Rt}{2L}\sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}} + \sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}} \cosh \frac{Rt}{2L}\sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}}}{x^2\sqrt{1-\frac{4Lx^2}{R \cdot KRl^2}}\left[\frac{1}{\cos x}+\frac{\sin x}{x}\right]}$$

* Formula (63), Chap. IX., p. 270. † Chap. IX., p. 271.

Table CXXVI.—*Fig.* 192, *Curve A.*

(sec.)	0·4	0·5	0·55	0·6	0·7	1·0	1·5	2·0
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{1-\frac{4Lx_1^2}{R.KRl^2}}$	0·1917	0·2396	0·2634	0·2875	0·3355	0·4792	0·7188	0·9584
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4Lx_1^2}{R.KRl^2}}$	0·8256	0·7869	0·3843	0·7501	0·7150	0·6193	0·4874	0·3835
1st term, $+^{ve}$	44·63	42·55	41·55	40·56	38·66	33·48	26·35	20·73
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{1-\frac{4Lx_2^2}{R.KRl^2}}$	...	...	3·108	...	...	...	...	...
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4Lx_2^2}{R.KRl^2}}$	0·0517	0·0296	0·0224	0·0168	0·0096	0·0018	0·0001	...
2nd term, $-^{ve}$	29·84	17·06	12·90	9·72	5·55	1·02	0·06	...
1st and 2nd	14·79	25·49	28·65	30·48	33·11	32·47	26·29	20·73

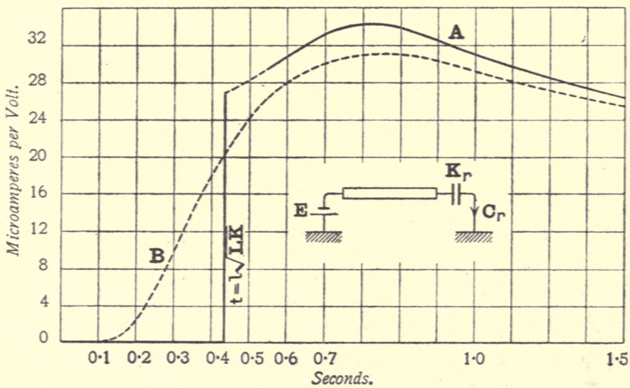

FIG. 192.—LOADED CABLE WITH SIGNALLING CONDENSER.

Curve A.—$K_r=\frac{Kl}{10}$; $L/R=\frac{KRl^2}{100}$. Curve B.—$K_r=\frac{Kl}{10}$; L=0.

This series, containing as it does the factor x^2 in the denominator, is much more rapidly convergent than the one from which it is derived, and it furnishes a useful check on the calculation.

Influence of End Apparatus on Height of Signal Head.

To find the height of the ordinate marking the head of the signal, return to the periodic case. The sinusoidal current through sending apparatus Z_s and receiving apparatus Z_r is

$$C_r = \frac{V_s}{(Z_s+Z_r)\cosh Pl + (Z_sZ_r/Z_0+Z_0)\sinh Pl},$$

$$= \frac{V_s}{(Z_s+Z_r+Z_sZ_r/Z_0+Z_0)\sinh Pl},$$

when Pl is big. Now, $Z_0 = \sqrt{\dfrac{R+ipL}{G+ipK}} = \sqrt{\dfrac{L}{K}}$ in the limit when p is great. Also, if Z_s and Z_r are condensers, $Z_s = 1/ipK_s$ and $Z_r = 1/ipK_r$, and both are zero when p is infinitely great. Hence, when the frequency is great,

$$C_r = \frac{V_s}{Z_0 \sinh Pl} = 2E\sqrt{\frac{K}{L}}\, e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}} \sin p(t - l\sqrt{LK}),$$

as before when there was no apparatus. A square-topped oscillation of high frequency is not affected by the sending and receiving condensers. *The wave-front is propagated unchanged through such sending and receiving apparatus as is initially of zero impedance.* The height to which the current jumps at the beginning of the signal in curve A, Fig. 192, is, therefore, the same as in Fig. 185, curve A.

When Z_s or Z_r is an impedance, of the form $R_s + ipL_s$, it is infinite when p is infinite, and therefore C_r is zero when p is infinite. The arrival curve in this case starts from zero, and the delay which the signal head would experience in rising to its full value would depend on the time constant of the coil. When Z_s and Z_r are pure resistances, it is easy to see that the height of the signal head is reduced in the proportion,

$$\sqrt{L/K} : \; R_s + R_r + R_sR_r\sqrt{K/L} + \sqrt{L/K},$$

or

$$1 \; : \; 1 + R_s \,.\, R_r \,.\, K/L + (R_s+R_r)\sqrt{\frac{K}{L}}.$$

If $R_s = R_r = Rl/m$, the proportion may be written

$$1 : 1 + \frac{KRl^2}{(L/R)m^2} + \frac{1}{m}\sqrt{\frac{KRl^2}{L/R}}.$$

When $L/R = KRl^2/100$, this simplifies to

$$1 : 1 + \frac{100}{m^2} + \frac{10}{m},$$

and if $m = 10$ the proportion is $1 : 3$. The head of the signal would be reduced to one-third the height by the end apparatus. From Fig. 92,* curves A and B, it is seen that the resistances would not affect greatly the subsequent course of the curve.

It is clear that the addition of apparatus of considerable impedance at the end of a cable in which the rise of the arrival curve is steep would be highly detrimental to the shape and size of the received signals. Moreover, even although reflection losses arising from discontinuity in loading—which will be considered immediately—were eliminated by the employment of continuous loading, it does not follow that the wave-front would be perpendicular. For that to be strictly true all the fundamental cable constants R, K, L, and G would require to be true or geometrical constants and quite independent of the frequency when measured with alternating current. It is the same difficulty as is met with in telephony, where it necessitates careful design if the attenuation at the higher frequencies is not to be prohibitive. Yet if equal care were given to the design of the telegraph cable we may conclude that the departure of the signal head from the rectangular form would not be so great as to reduce the speed of transmission to any serious degree.

Loaded Cable and Condensers.

Now, suppose that the cable is to earth through a condenser at both the sending and the receiving end. The periodic solution is

$$C_r = \frac{V_s}{(Z_s + Z_r)\cosh Pl + (Z_s Z_r/Z_0 + Z_0)\sinh Pl}.$$

Here $\varphi(y) = (Rl + yLl)\dfrac{\sin x}{x} + \dfrac{Kl}{yK_sK_r}\dfrac{\sin x}{x} + \dfrac{1}{y}\left(\dfrac{1}{K_s} + \dfrac{1}{K_r}\right)\cos x,$

where $xi = l\sqrt{(R + yL) \,.\, yK}$. Also $\dfrac{d}{dy} = \dfrac{dx}{dy} \cdot \dfrac{d}{dx},$

* Chap. IX., p. 278.

where $\dfrac{dx}{dy}=-\dfrac{2KLl^2y+KRl^2}{2x}=\mp\dfrac{KRl^2}{2x}\sqrt{1-\dfrac{4x^2L}{KRl^2 \,.\, R}}$.

Hence

$$y\varphi'(y)=(Rl+2yLl)\frac{\sin x}{x}+\frac{dx}{dy}\left[\left\{y(Rl+yLl)+\frac{Kl}{K_sK_r}\right\}\frac{x\cos x-\sin x}{x^2}-\left(\frac{1}{K_s}+\frac{1}{K_r}\right)\sin x\right],$$

which becomes on substitution for $\dfrac{dx}{dy}$,

$$y\varphi'(y)=\frac{\pm Rl\sqrt{1-\dfrac{4Lx^2}{KRl^2 \,.\, R}}}{2}\left[\frac{2\sin x}{x}+\frac{1}{x}\left(\frac{Kl}{K_s}+\frac{Kl}{K_r}\right)\left(\frac{1}{\sin x}-\frac{\cos x}{x}\right)\right].$$

Finally,

$$C_r=\frac{4Ee^{-\frac{R}{2L}}}{Rl}\sum_x\frac{\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{4Lx^2}{KRl^2 \,.\, R}}}{\sqrt{1-\dfrac{4Lx^2}{KRl^2 \,.\, R}}\left[\dfrac{2\sin x}{x}+\dfrac{1}{x}\left(\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}\right)\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{x}\right)\right]} \quad (32)$$

where $$\tan x=\frac{\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}}{x-\dfrac{1}{x}\cdot\dfrac{Kl}{K_r}\cdot\dfrac{Kl}{K_s}}.$$

When L is zero, (32) reduces to (66)*, the formula for a plain cable with signalling condensers.

Table CXXVII.

x	$\dfrac{4Lx^2}{R \,.\, KRl^2}$	$\dfrac{R}{2L}\sqrt{1-\dfrac{4Lx^2}{R \,.\, KRl^2}}$	$\dfrac{4}{Rl\sqrt{1-\dfrac{4Lx^2}{R \,.\, KRl^2}}\left[\dfrac{2\sin x}{x}+\dfrac{1}{x}\left(\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}\right)\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{x}\right)\right]}$
2·6272	0·2761	9·7752	+ 51·378
5·307	1·1266	4·0874	−429·32
8·067	2·6031	14·546	+223·41
10·909	4·7611	22·282	−206·32
13·819	7·6385	29·602	+191·63
16·782	11·265	36·811	−175·94
19·786	15·658	43·989	+160·66
22·817	20·826	51·157	−146·72

* Chap. IX., p. 273.

Table CXXVIII.—*Fig.* 193, *Curve A.*

t (sec.)	0·4	0·5	0·6	0·7	1·0	1·5	2·0
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{1-\frac{4Lx_1^2}{R\,.\,KRl^2}}$ ...	0·6854	0·8567	1·0278	1·1999	1·714	2·571	3·426
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{1-\frac{4Lx_1^2}{RKRl^2}}$	0·2518	0·2123	0·1789	0·1506	0·0901	0·0382	0·0163
1st term, + ...	12·94	10·91	9·19	7·74	4·63	1·96	0·84

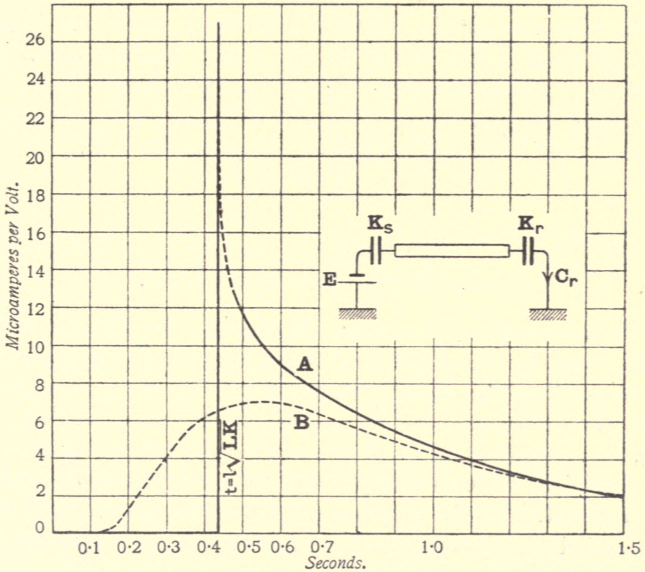

FIG. 193.—LOADED CABLE WITH SIGNALLING CONDENSERS.

Curve A.—$K_s=K_r=Kl/10$; $L/R=\frac{KRl^2}{100}$. Curve B.—$K_s=K_r=Kl/10$; $L=0$.

When $K_s=K_r=Kl/10$ and $L/R=KRl^2/100$ as before, the first term in the series is hyperbolic, and all the subsequent terms are circular. The values of x are contained in Table CXXVII. (taken from Table XLII.)* and from it Table CXXVIII. is calculated, and plotted in Fig. 193, curve A. Curve B is the current in the case of the plain cable.

* Chap. IX., p. 274.

The circular terms from x_1 to x_8 have been calculated also for $t=0{\cdot}45$, $0{\cdot}5$ and $0{\cdot}6$. The probable course of curve A for lower values of t is shown by the dotted part of the curve. The height at time $t=l\sqrt{\text{LK}}$ is the same as in curve A, Fig. 192 and curve A, Fig. 185.

Comparing curve A, Fig. 193, with curve A, Fig. 192, it is evident that the effect of the second condenser is to cut away the tail of the signal. This result might have been anticipated from a comparison of the corresponding curves in the non-loaded cable, Fig. 88, curve A, and Fig. 89, curve A.*

The area between the curves and the time axis is the same in both cases and is equal to the charge in the receiving condenser. Curve A may be regarded as formed from curve B by cutting away the area to the left of the vertical ordinate through $t=\sqrt{\text{LK}}$ and adding it on the right to curve B.

Loaded Meshes and Condensers.

It is now necessary to consider the effect on the signals which is produced by the discontinuities necessitated by coil loading in conjunction with signalling apparatus. Suppose, as before, that the cable is replaced by a series of n loaded meshes as in Fig. 195, connected to earth through condensers at the ends. To obtain the arrival current in this case return to the n-mesh determinant. To the first constituent of the determinant is now added Z_s, and to the last, Z_r. Substituting as before $\cos\theta$ for $1+\dfrac{y\,.\,\text{KR}l^2}{2n(n-1)}$, and expanding the determinant, it follows that

$$\varphi(y)=\frac{1}{\sin\theta}\left[(\text{R}l+y\text{L}l)\,\frac{\sin n\theta}{n}+\frac{y\,.\,\text{K}l}{n-1}\,\frac{1}{\text{K}_s\,.\,y}\,.\,\frac{1}{\text{K}_r\,.\,y}\,.\sin\overline{n-1}\theta\right.$$

$$\left.+\left(\frac{1}{\text{K}_s y}+\frac{1}{\text{K}_r y}\right)(\sin n\theta-\sin\overline{n-1}\theta)\right].$$

---

* Chap. IX., pp. 272, 274.

The equation $\varphi(y)=0$ gives the required θ roots. Also

$$y\varphi'(y)=\frac{\pm Rl\sqrt{1-\dfrac{8n(n-1)(1-\cos\theta)L}{R\,.\,KRl^2}}}{\sin\theta}\Bigg[\frac{\sin n\theta}{n}$$

$$-\frac{y\,.\,Kl}{2n(n-1)\sin\theta}\Bigg\{(Rl+yLl)\Big(\cos n\theta-\frac{n-1}{n}\sin n\theta\cot\overline{n-1}\theta\Big)$$

$$+\Big(\frac{1}{K_s y}+\frac{1}{K_r y}\Big)(n\cos n\theta-(n-1)\cot(n-1)\theta\sin n\theta)\Bigg\}\Bigg].$$

in which θ is to be substituted for y.

Hence, finally,

$$C_n=\frac{2Ee^{-\frac{Rt}{2L}}}{Rl}\sum_\theta\frac{n\sin\theta\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{8n(n-1)(1-\cos\theta)L}{KRl^2\,.\,R}}}{\sqrt{1-\dfrac{8n(n-1)(1-\cos\theta)L}{KRl^2\,.\,R}}\Bigg[\sin n\theta+\Bigg\{\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}}$$

$$\left.-2(n-1)(1-\cos\theta)\Bigg\}\times\frac{n\sin\theta-\sin n\theta\cos(n-1)\theta}{2(n-1)\sin\theta\sin(n-1)\theta}\Bigg]\right. \quad (33)$$

where

$$\frac{\tan n\theta}{\sin\theta}=\frac{\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}-\dfrac{1}{n-1}\cdot\dfrac{Kl}{K_s}\cdot\dfrac{Kl}{K_r}}{(\cos\theta-1)\Big(\dfrac{Kl}{K_s}+\dfrac{Kl}{K_r}-2(n-1)\Big)-\dfrac{\cos\theta}{n-1}\dfrac{Kl}{K_s}\cdot\dfrac{Kl}{K_r}}. \quad (34)$$

Certain particular cases of this formula are of interest.

1. When n is infinite the formula reduces to (32), the arrival current in a loaded cable with signalling condensers.

2. When $K_s=K_r=\infty$, the formula becomes the same as the series in (27), the arrival current in a set of loaded meshes without end apparatus.

3. When there is only one signalling condenser, and K_s or $K_r=\infty$, the formula reduces to

$$C_n=\frac{4Ee^{-\frac{Rt}{2L}}}{Rl}\sum_\theta\frac{\cos\dfrac{\theta}{2}\cos\dfrac{2n-1}{2}\theta\sinh\dfrac{Rt}{2L}\sqrt{1-\dfrac{8n(n-1)(1-\cos\theta)L}{R\,.\,KRl^2}}}{\Big(1+\dfrac{\sin 2n\theta}{2n\sin\theta}\Big)\sqrt{1-\dfrac{8n(n-1)(1-\cos\theta)L}{R\,.\,KRl^2}}}, \quad (35)$$

Table CXXIX.—Fig. 194. *Roots of* $\tan 20\theta = \dfrac{\sin\theta}{1{\cdot}2214 - 1{\cdot}5786\cos\theta}$.

—	θ.	20θ.	Sin θ.	Cos θ.	Sin 20θ.	Cos 19θ.	$\sqrt{1-\text{etc}}\cdot$	$\dfrac{2\text{E}}{\text{R}l}\cdot\dfrac{20\sin\theta\sinh\frac{\text{R}t}{2\text{L}}\sqrt{-}}{\sqrt{-}[\text{——}]}$
1	7°54·3	π−21°53·7	0·13753	0·99042	+0·37291	−0·86777	0·84188	+55·621
2	15°52·1	2π−42°38	0·27343	0·96189	−0·67730	+0·52247	0·34038	−501·90
3	23°55·9	3π−61°22	0·40564	0·91403	+0·87776	−0·08185	1·27028	+262·19
4	32°6·8	4π−77°44	0·53160	0·84710	−0·97714	−0·33942	1·9099	−258·20
5	40°24·9	4π+88°17	0·64829	0·76138	+0·99955	+0·67084	2·5008	+249·46
6	48°49·2	5π+76°25	0·75266	0·65841	−0·97203	−0·88626	3·0634	−232·71
7	57°19	6π+66°20	0·84167	0·54000	+0·91590	+0·98764	3·6033	+210·42
8	65°53·2	7π+57°43	0·91274	0·40856	−0·84542	−0·98985	4·1208	−185·68
9	74°30·9	8π+50°18	0·96370	0·26699	+0·76940	+0·91201	4·6133	+160·28
10	83°11·5	9π+43°50	0·99295	0·11855	−0·69256	−0·77320	5·0791	−135·66
11	π−88°5·3	10π+38°7	0·99944	−0·03336	+0·61726	+0·59068	5·5145	+112·49
12	π−79°21	11π+33°0	0·98277	−0·18481	−0·54464	−0·38026	5·9175	−91·254
13	π−70°34·9	12π+28°22	0·94312	−0·32246	+0·47511	+0·15554	6·2855	+72·114
14	π−61°47·2	13π+24°8	0·88120	−0·47276	−0·40886	+0·07115	6·6158	−55·149
15	π−52°59·5	14π+20°11	0·79855	−0·60193	+0·34503	−0·99876	6·9063	+38·390
16	π−44°10·5	15π+16°30	0·69685	−0·71721	−0·28402	−0·13912	7·1554	−28·094
17	π−35°21	16π+13°0	0·57857	−0·81563	+0·22495	−0·14950	7·3616	+17·964
18	π−26°31·1	17π+9°37	0·44650	−0·89478	−0·16706	−0·13493	7·5234	−10·092
19	π−17°41	18π+6°21	0·30376	−0·95275	+0·11060	−0·10102	7·6394	+4·482
20	π−8°50·5	19π+3°10	0·15370	−0·98812	−0·05524	−0·05403	7·7097	−1·121

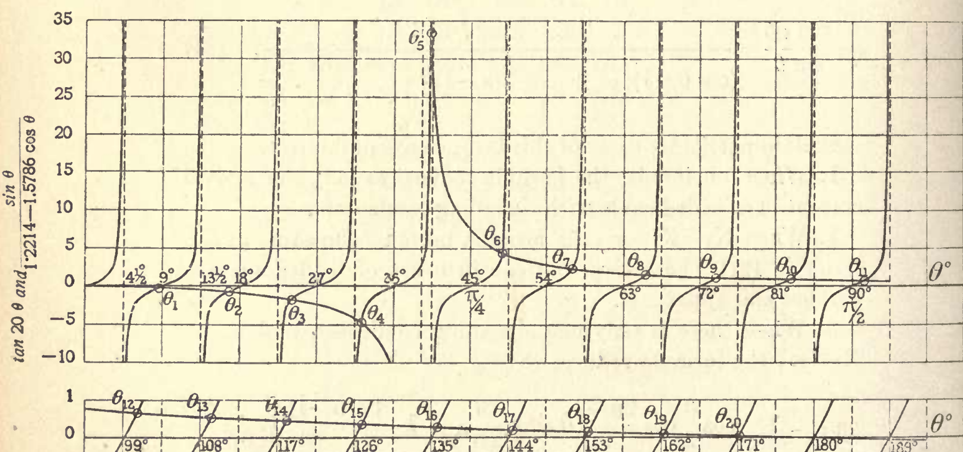

Fig. 194.

Roots of $\tan 20\,\theta = \dfrac{\sin\theta}{1{\cdot}2214 - 1{\cdot}5786\cos\theta}$.

where $$\tan n\theta = \frac{\cot\frac{\theta}{2}}{2(n-1)\frac{K_r}{Kl}-1}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In the general formula (33) let $n=20$, and let $K_s=K_r=\frac{Kl}{10}$, and $\frac{L}{R}=\frac{KRl^2}{100}$, as before. Then (34) becomes

$$\tan 20\theta = \frac{\sin\theta}{1{\cdot}2214-1{\cdot}5786\cos\theta}.$$

The situation of the roots of this equation is shown graphically in Fig. 194. At $\theta=39°\ 19'$, the fraction becomes $-\infty$, and reappears on the other side of the ordinate as $+\infty$. When $\theta=180$ deg., both curves cut in the horizontal axis, but on substituting $\theta=\pi$ in C_n the corresponding term is zero. After $0=180$ deg. the points of intersection are the same as those to the left, but in the inverse order and below the horizontal axis. They yield no new terms to the series. The 20 roots corresponding to the 20 meshes are indicated on the diagram, and their exact values are contained in Table CXXIX. When n is infinite the number of roots extends to infinity, and Fig. 194 changes into Fig. 89.*

The first root makes the expression under the square-root in the formula for C_n positive, and the first term in the series is, therefore, hyperbolic. All the remaining 19 terms are circular. The values of the first term are contained in Table CXXX.

Table CXXX.—Fig. 195, Curve A.

t (sec.)	0·3	0·4	0·42	0·5	0·6	0·7	1·0	1·5
$e^{-\frac{Rt}{2L}}$	0·03184	0·01010	0·00802	0·00320	0·00102	0·00032	0·00001	...
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{-}$ ...	0·5450	0·7266	0·7629	0·9079	1·0894	1·2718	1·817	2·724
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{-}$	0·2891	0·2417	0·2331	0·2017	0·1682	0·1402	0·0813	0·0328
1st term	16·08	13·44	12·97	11·22	9·36	7·80	4·52	1·82

* Chap. IX., p. 274.

When t is greater than 1 second $e^{-Rt/2L}$ is less than 0·00001, and the circular terms may be neglected. They are calculated here for t=0·30, 0·40, 0·42, 0·44, 0·46, 0·48, 0·50, 0·52 and 0·60 second. A single example is given in Table CXXXI. The complete curve is given in Fig. 195, curve A.

On comparing Fig. 195 with Fig. 193, it is seen that the mesh curve is also straight at the period of most rapid growth, but is inclined to the vertical. The influence of the subdivision is further shown in the rounding off of the base of the curve, and also of the tip, leading to a reduction in the height of the curve by nearly one-half.

Table CXXXI.—*Fig.* 195, *Curve A.* t=0·42 sec.

θ	2	3	4	5	6	7	8	9	10
$\frac{Rt}{2L}\sqrt{-}$	1·643	6·130	9·216	12·07	14·78	17·39	19·89	22·26	24·51
Sin do..	+0·997	−0·153	+0·207	−0·478	+0·799	−0·994	+0·860	−0·268	−0·584
Term...	−500·3	−40·1	−53·5	−119·2	−186·0	−209·2	−159·7	−42·9	+79·2

θ	11	12	13	14	15	16	17	18	19	20
$\frac{Rt}{2L}\sqrt{-}$	26·61	28·56	30·33	31·93	33·33	34·53	35·52	36·30	36·86	37·21
Sin do..	+0·996	−0·278	−0·883	+0·487	+0·943	+0·029	−0·823	−0·985	−0·742	−0·474
Term ...	+112·0	+25·4	−63·7	−26·9	+36·2	−0·8	−14·8	+9·9	−3·3	+0·5

Sum of Terms=−1,158; Sum $\times e^{-Rt/2L}$=−9·29; Total (with first term from Table CXXX.)=+3·67.

Table CXXXII.—*Fig.* 195, *Curve B.*

t (sec.)	0·1	0·2	0·3	0·4	0·5	0·7	1·0	1·5	2·0
θ_1+	19·80	16·75	14·17	11·99	10·14	7·26	4·39	1·90	0·82
θ_2-	43·91	22·26	11·60	5·96	3·06	0·81	0·11	...	...
θ_3+	37·10	8·26	1·84	0·41	0·09	...	...	...	...
θ_4-	17·07	1·18	0·08	...	...	...	...	...	...
θ_5+	4·83	0·07	...	...	...	...	...	...	...
θ_6-	0·91	...	...	...	...	...	...	...	...
C_{20} (mm.a. per v.)...	−0·16	+1·65	4·34	6·43	7·17	6·45	4·28	1·90	0·82

Curve B is for the same set of meshes and apparatus, without inductance. The formula for this case may be obtained directly, or from (33) by putting L zero. The arrival current is,

$$C_n=\frac{E}{Rl}\sum_{\theta}\frac{n\sin\theta\, e^{-\frac{2n(n-1)(1-\cos\theta)t}{KRl^2}}}{\sin n\theta+\left[\frac{Kl}{Ks}+\frac{Kl}{Kr}-2(n-1)(1-\cos\theta)\right]\left[\frac{n\sin\theta-\sin n\theta\cos(n-1)\theta}{2(n-1)\sin\theta\sin(n-1)\theta}\right]}\qquad \text{(3}$$

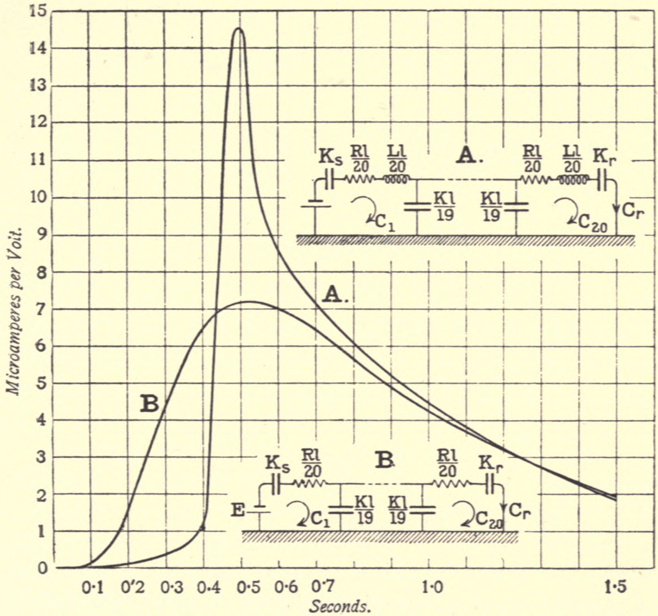

FIG. 195.—TWENTY LOADED AND TWENTY PLAIN MESHES WITH CONDENSERS.

Curve A.—$K_s=K_r=\frac{Kl}{10}$; $\frac{L}{R}=\frac{KRl^2}{100}$

Curve B.—$K_s=K_r=\frac{Kl}{10}$; $L=0$.

where θ has the same set of values as before. The values of C_{20} calculated from (37) are contained in Table CXXXII.

For the purpose of comparison all the four curves of Figs. 193 and 195 are brought together in Fig. 196.

As in the case of the cable without apparatus, it is clear from the closeness with which curves C and D approximate to each other that subdivision into 20 meshes produces little change in the curve of arrival current when the cable and meshes do not contain inductance. The same argument as before may therefore be adduced here with a certain degree of plausibility: that the effect on Curve A of redistributing the resistance and

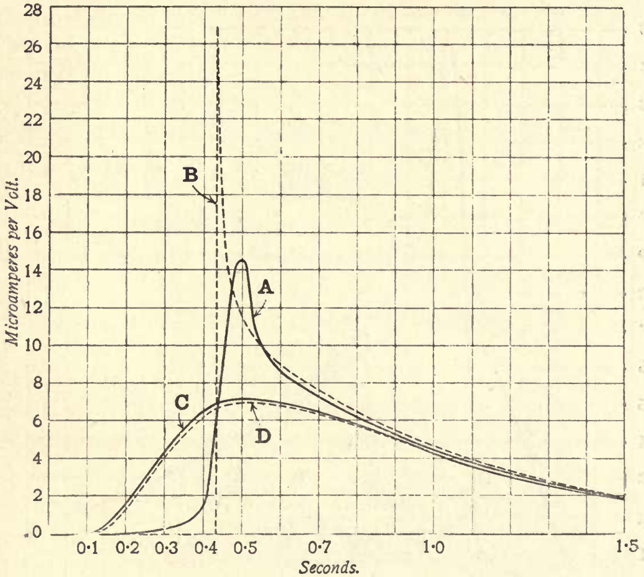

FIG. 196.—COMPARISON BETWEEN CABLE AND MESHES, WITH CONDENSERS.

Curve A.—20 loaded meshes. Curve B.—Loaded cable. Curve C.—20 plain meshes. Curve D.—Plain cable.

capacity—leaving only the inductance at equidistant intervals—would be only slight, and that Curve A is a close approximation to the true discontinuously-loaded cable curve, which may lie between it and curve B.

CABLE WITH LESS LOADING.

It is interesting to examine the effect of a reduction in the amount of loading on Curve A of Fig. 195. From Fig. 189 it is clear that the height of the head of the signal is much dependent on the inductance, and a substantial reduction in the inductance may cause the sharp peak to vanish altogether. Keeping the condensers of the same size as before, let $L/R = KRl^2/200$, and therefore $L = 47{\cdot}6$ m.h. Then $l\sqrt{LK} = 0{\cdot}308$ second, and the height of the signal head is 4·826 mm.a. per volt.

Table CXXXIII.—$R/2L = 22{\cdot}98$.

x.	$\frac{4x^2L}{KRl^2 . R}$	$\frac{R}{2L}\sqrt{1-\text{do.}}$	$\frac{4}{Rl\sqrt{-[\ \ ]}}$.
2·6272	0·13804	21·338	+ 47·084
5·307	0·56328	15·187	−231·14
8·067	1·30153	12·620	+515·00
10·909	2·38056	27·000	−340·57
13·819	3·81924	38·584	+294·06
16·782	5·6324	49·461	−261·89
19·786	7·8292	60·057	+235·35
22·817	10·413	70·500	−212·95

The first and second terms of the series (32) are now hyperbolic, as is seen from Table CXXXIII. They are contained in Table CXXXIV., from which curve A, Fig. 197, is plotted. The circular terms have been calculated for the instant $t = 0{\cdot}4$, and also for $t = 0{\cdot}5$, by which time they almost cease to count, owing to the rapid decay in $e^{-Rt/2L}$ due to the smaller value of L. Curve B is the same as Curve B of Fig. 193. The sharp peak is now gone and the curves closely resemble each other except that the front of Curve A is steep at 0·3 second, and resembles curve A of Fig. 192 for a single condenser with twice the loading. If the cables were replaced by meshes the discontinuities in the curvature of both these curves would disappear, and although steep they would not be absolutely perpendicular, but would be smoothed off as in curve A of Fig. 190.

To restore the sharpness to the head of curve A, Fig. 197, all that is necessary is to reduce the size of the condensers. The arrival current curve for the plain cable with condensers $K_s = K_r = Kl/20$, or half those of Fig. 197, is given in Fig. 90, curve B.* The general course of the loaded cable curve may

* Chap. IX., p. 275.

Table CXXXIV—*Fig.* 197, *Curve A.*

t (sec.)	0·3	0·4	0·5	0·7	1·0	1·5
$e^{-Rt/2L}$	0·001015	0·000102	0·000010	...	...	...
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{-}$	0·4926	0·6570	0·8210	1·1494	1·6420	2·4631
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{-}$	14·386	12·208	10·358	7·459	4·558	2·005
$\frac{Rt}{2L}-\frac{Rt}{2L}\sqrt{-}$	2·338	3·117	3·897	5·456	7·794	11·691
$e^{-\frac{Rt}{2L}}\sinh\frac{Rt}{2L}\sqrt{-}$	11·153	5·115	2·346	0·493	0·047	...
1st and 2nd terms (mm.a. per v.)	3·233	7·093	8·012	6·966	4·511	2·006

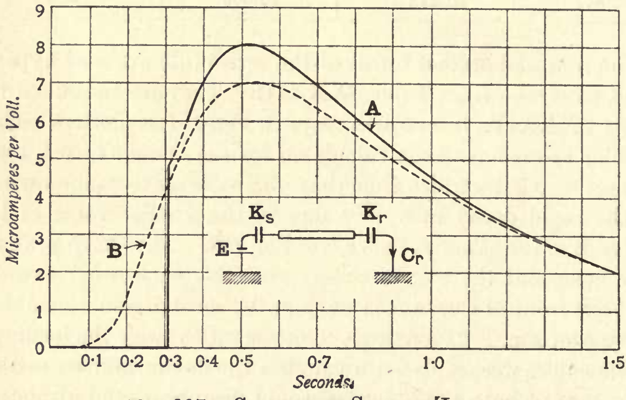

FIG. 197.—CABLE WITH SMALLER LOADING.

Curve A.—$K_s = K_r = \frac{Kl}{10}$; $\frac{L}{R} = \frac{KRl^2}{200}$

Curve B.—$K_s = K_r = \frac{Kl}{10}$; $L = 0$.

easily be drawn from it without calculation by combining with it the steep front of curve A, Fig. 197.

When t is only slightly greater than $l\sqrt{LK}$, the series (32) is only very slowly convergent. A more rapidly convergent series may be obtained by integrating it with respect to t.

The quantity of electricity still lacking in the condenser at time t after the steady E.M.F. has been applied at the sending end is

$$Q_t = \int_t^\infty C_r dt = 2EKle^{-\frac{Rt}{2L}}$$

$$\sum_x \frac{\sinh \frac{Rt}{2L}\sqrt{1-\frac{4x^2L}{R\,.\,KRl^2}} + \sqrt{1-\frac{4x^2L}{R\,.\,KRl^2}}\cosh\frac{Rt}{2L}\sqrt{1-\frac{4x^2L}{R\,.\,KRl^2}}}{x^2\sqrt{1-\frac{4x^2L}{R\,.\,KRl^2}}\left[\frac{2\sin x}{x}+\frac{1}{x}\left(\frac{Kl}{K_s}+\frac{Kl}{K_r}\right)\left(\frac{1}{\sin x}-\frac{\cos x}{x}\right)\right]} . \quad (38)$$

which contains the rapidly-increasing factor x^2 in the denominator. For all values of t which are not close to the critical it is sufficient to consider only the first two or hyperbolic terms of the series. Thus, when $t=0{\cdot}4$ second, Q_t calculated from (38) is 6·765 microcoulombs per volt. Now the total charge in the receiving condenser at infinite time is $Q_\infty = \frac{Kl}{10}\times\frac{E}{12}$ $=7{\cdot}289$ microcoulombs per volt. Hence the charge accumulating between time $t=l\sqrt{LK}$ and $t=0{\cdot}4$ second is the difference of these two, or 0·524 micro-coulomb per volt. Dividing this quantity by the time, $0{\cdot}4-0{\cdot}308$ second, the mean current is found to be 5·67 mm.a. per volt at $t=0{\cdot}35$ second, which affords a confirmation of the portion of curve A shown broken because uncertain.

Elementary Dot Signals.

The curves of Fig. 195 may be used to form their elementary dot signals in the usual way, and from them any message may be built up. In Fig. 198 the time of contact is supposed to be 0·04 second in both cases. Curve A is for 20 loaded meshes with condensers and curve B for 20 plain meshes with condensers. If the breadth of the signals at half their height be taken as a measure of the relative times required for their formation, the times required are in the proportion 27 to 7, or nearly 4 to 1. In other words, if the rate of signalling in the one case were 50 words a minute, in the other case the speed could be increased to 200 words a minute without sacrifice of legibility. The sharp dip in curve A could easily be removed by a curbing current of half the height or half the duration, put on at 0·08 second later. Moreover, the excess of height in curve A over curve B could be converted into increased sharpness of response, as illustrated fully in Chap. XIV., and used to bring about a further increase in speed of transmission.

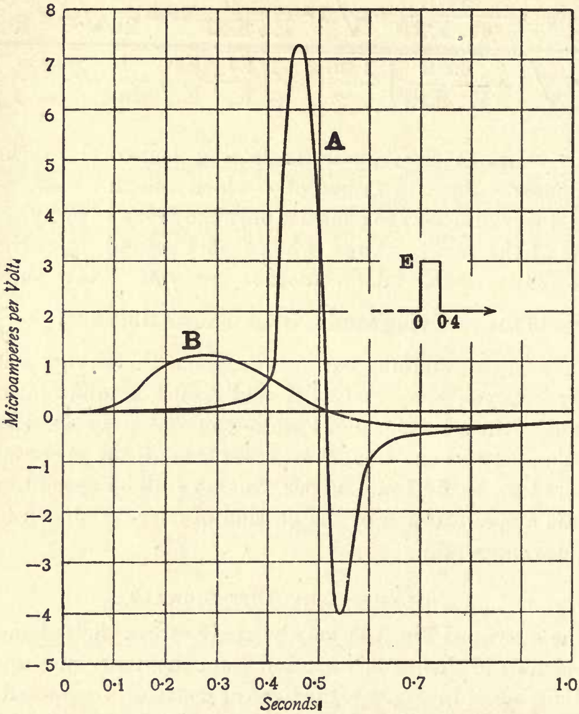

FIG. 198.—ELEMENTARY DOT SIGNAL. TIME OF CONTACT 0·04 SECOND.

Curve A.—20 loaded meshes with condensers; $K_s = K_r = Kl/10$; $L/R = \frac{KRl^2}{100}$
Curve B.—20 plain meshes with condensers: $K = K_s = K_r l/10$; $L = 0$.

PRINCIPLES UNDERLYING THE DESIGN OF A LOADED CABLE.

As the conditions which determine the speed of a telegraph cable when loaded differ entirely from those that have hitherto prevailed, it may be well to outline here the process of thought which may be followed by the telegraph engineer when he is called upon to design a loaded cable. Instead of the "KR" law we have the formula

$$C_r = 2E\sqrt{\frac{K}{L}}\,e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$$

for the height of the signal head, which requires correction as explained for the presence of any apparatus with resistance or impedance. The length of the cable and the speed at which it is to be worked are fixed by economic considerations. Now, instrumental limitations fix a certain minimum of current which the best available receiving apparatus requires if it is to work satisfactorily over long periods at the given speed. To make it more sensitive would lessen its reliability, and tests on the apparatus itself will determine this limiting current. Hence C_r is known; also E is fixed from considerations of safety of the dielectric. There remain only R, K and L. Of these three K can vary only within narrow limits, say from 0·3 to 0·4 mfd. per n.m. There are left, therefore, only R and L at disposal. The factor $\sqrt{K/L}$ in C_r is of small importance compared with the exponential $e^{-\frac{Rl}{2}\sqrt{\frac{K}{L}}}$; hence, for C_r to be great, it is necessary to make $Rl\sqrt{\frac{K}{L}}$ as small as possible; or $\sqrt{\frac{KRl^2}{L/R}}$ as small as possible—*i.e.*, the ratio of the conductor time-constant L/R as great as possible in comparison with the capacity time-constant KRl^2.

Since $\frac{Rl}{2}\sqrt{\frac{K}{L}}$ is directly proportional to R and inversely proportional only to the square-root of L, it follows that a change in R makes itself felt in much greater proportion than one in L. But the magnitude of R fixes the weight of copper, and therefore of dielectric, and with these the cost of the cable is fixed. A trial value for R may be assumed, and the mechanically best available coils with the lowest time-constant may be supposed placed in the cable at suitable distances. Substituting these values in the formula, C_r is determined. If not satisfactory the assumptions must be readjusted. Alternatively, if continuous loading is adopted, one, two or three layers of iron wire may be placed round the wire ; the increase in L which they produce is known from experiments on short sample lengths, and also the alteration to K. If these do not give a satisfactory value of C_r they must suffer readjustment—if necessary, R must be increased. Finally, the arrival curve is to be calculated, and corrected for the spacing if coil loading is used. By means of the arrival curve typical signals may be constructed at the required speed, and a comparison of these with similarly calculated signals and with those actually obtained in practice will enable the designer to state with confidence whether his plan will meet the specification or not

CONCLUSION.

From the foregoing discussion of the question of improved transmission it is evident that much may be expected from the present type of cable by advance along the lines which are indicated by theory for the improved formation of the signals at the sending end, and for their better detection at the receiving end. The principle of all these methods has been shown to consist in the sacrifice of the magnitude of the signal for the purpose of obtaining improvement in its shape. The natural corollary, and the complement to this process, is the restoration of size by reduction of attenuation, a result which is only to be attained by loading. In contradiction to the generally accepted opinion, it has been shown that the increase in inductance which can be attained in practice is far from being ineffective in the telegraph cable. But in order that the full significance of loading may be brought out the signalling conditions must be re-arranged. Hitherto signals have been formed exclusively by the extended " tail " of the arriving wave, the " head " being infinitesimal. By the use of loading the steep head may be given sufficient height to actuate the receiver, and the tail may be reduced to a convenient size by adjustment of the signalling apparatus. It is not merely a question of improved speed, but rather of a fundamental change in the method of transmission — a change which will bring submarine telegraphy into line with telegraphy on land and render available all the systems of signalling which are employed on land. It follows, as a first step, that if the design of a new cable is to make any pretence to scientific exactitude it is essential that all the cable constants should be specified and measured, for on the attenuation constant and not on the "K.R." of the cable will the speed of working—and, therefore, the value of the cable—increasingly depend. At least as great an advance may be hoped for as in telephony, where such an altogether disproportionate increase in efficiency has been attained without increase of cost, that it is now inconceivable for a new cable of any importance to be laid without loading of some kind.

It is probable that the loading of the first telegraph cable will be carried out on the continuous plan. In order to obtain from it the full speed of which it is capable it may be necessary at first to work the cable simplex. This objection would disappear if the cable were multiplexed, for then the allocation of the circuits could be arranged at will. Thus, it might be found convenient to work a type-printing multiplex system during the day, and high-speed automatic at night. Into the methods of multiplexing long cables it would be inadvisable to enter at this stage, but it may be predicted, in passing, that all will be found to depend for the degree of their success on the amount of inductance present in the cable.

Nevertheless, sooner or later the problem of coil loading must be faced. The difficulties of placing coils of fine wire within the sheathing of a cable, and of maintaining their insulation, are great, but within recent years they have all been overcome in telephony. The objection may still be raised that the problem of loading the telegraph cable is more difficult on account of the greater depth at which it must be laid. But it is probable that telephone coils could withstand the pressure unscathed, and if misgivings were felt on that score they could be specially constructed accordingly. Moreover, it may be pointed out that the telegraphic problem is more simple electrically and mechanically, because the coils need be (*a*) fewer in number and at greater distances apart, and (*b*) of a lower degree of electrical perfection, and, therefore, of greater robustness, since there is no requirement of balance, which must be accurately maintained, as there is in the twin cores of a telephone cable. Once laid there is no reason to apprehend more trouble at great depths than at present where the majority of faults occur in shallow water.

There is no difficulty, mechanical or electrical, in the way of laying a continuously-loaded cable. From both these standpoints the problem may be regarded as solved. At the same time it would be optimistic indeed to assume that telegraphic loading will soon become an accomplished fact. The student of cable history will recall the length of time that elapsed between the discovery by Oliver Heaviside* of the part played by inductance and its experimental verification abroad in the case of telephony and final acceptance here. But cable telegraphy is in a different category ; there is much prestige to lose ; its connections are international, and if events were allowed to follow the same course in cable telegraphy a blow would be struck at the supremacy which this country has long enjoyed.

There is another reason why the introduction of loading if long-deferred is in danger of coming too late. Interest in the problem has been quickened by the advance of radiotelegraphy. At present wireless acts as a feeder to cable telegraphy, but there is no reason why it should continue to do so indefinitely, and it is the manifest determination of its supporters that this state of affairs shall not continue. Unless its development should meet an unexpected check the indications point to the probability that it will ere long be fulfilling the functions of an economical high-speed transmitter of intelligence over long distances. The advantage of holding the field is on the side of cable telegraphy, and against wireless, which has to prove itself not only a success, but also preferable to the older method of communication. But no industry, however firmly established, can afford to wait until it begins to feel the stress of active competition. If this great advantage were lost by cable telegraphy the consequences could hardly fail to be

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* THE ELECTRICIAN, Nov. 17 1893; "Electromagnetic Theory," Vol. I., p. 445.

serious, for it is probable that had wireless been in existence half a century ago long-distance cable telegraphy would never have come into being at all.

It is customary in discussing the relative merits of the two methods of communication unconsciously to compare cable telegraphy as it now is with radiotelegraphy as it will be when it has attained maturity. In so doing it is tacitly assumed that cable telegraphy, with its many years of greater experience, is already in close approximation to its final shape. Nevertheless, it must not be forgotten that the resources of modern electrical research were not available in the early pioneering days, when telegraphic practice was being established, and that progress was consequently slow. On the other hand, the growth of radiotelegraphy has coincided with, and has reacted upon, the development of these methods of investigation and forms of apparatus.

In proportion as radiotelegraphy has gained in the universality of its appeal, cable telegraphy has suffered in the difficulty which it presents to the experimenter. The novelty and fascination of the problems of wireless have enabled it to draw upon the best ability of every country; and to the zeal with which the discoveries of the physicist have been assimilated and applied its rapid advance is in great measure due. On the other hand, the difficulties in the way of research in cable telegraphy to other than those actually engaged in the profession are almost insuperable. In consequence the history of cable telegraphy, while prolific in invention, has been relatively barren in discovery.

The possibilities in the future of wireless telegraphy are great and unquestioned in the field of communication between moving objects. For strategic reasons it may be of dominant importance; although such an occasion for its use fortunately arises but seldom, and the greatness of a modern nation is founded on its commercial prosperity built up in time of peace.* In high-speed transmission between fixed points the advantage of a metallic conductor, in guiding and localising the trans-

* Written before the War.

mitted energy, cannot be overestimated, *once the obstacles of distortion and attenuation have been removed.*

The possibilities in the future of cable telegraphy in its own sphere are as brilliant as are those of wireless telegraphy. Although the difficulties in the way of their realisation must not be underestimated, none appear to be insuperable. Their speedy translation into fact is a task worthy of the greatest efforts of the engineers to whom it is entrusted. But much depends on the manner in which they are supported. Given the spirit of indomitable perseverance, the commercial large-mindedness and the scientific intelligence which alone made Atlantic telegraphy possible, there can be but little question that cable telegraphy is about to enter upon a period in which all its past achievements will be excelled.

APPENDIX I.

USEFUL DATA.

Length.

1 nautical mile (n.m.) = 2,029 yards (yd.) = 1·1528 statute mile = 1·8553 kilometre (km.)

1 inch (in.) = 1,000 mils = 2·540 cm.

1 metre (m.) = 39·37 in. = 1·0936 yd.

1 foot = 30·48 cm.

1 statute mile (st. m.) = 1·6093 km.

1 fathom = 2 yards. = 1·829 m.

Volume.

1 litre = 61·03 cubic in.

1 cubic in. = 16·387 cubic centimetres (c.c.)

1 cubic ft. = 28,316 c.c.

Weight.

1 gramme (gm.) = 15·432 grains.

1 pound avoirdupois (lb.) = 453·59 gm

1 kilogramme (kg.) = 2·2046 lb.

1 ton = 20 cwt. = 2,240 lb. = 1,016 kg.

To convert square millimetres of copper to pounds per nautical mile :—(lb. per n.m) = (sq. mm.) × 36·41.

Number.

$e = 2{\cdot}71828$ $\log_{10} e = 0{\cdot}43429.$ $\pi = 3{\cdot}14159$

$\log_{10} \pi = 0{\cdot}49715$ $\pi^2 = 9{\cdot}8696$ $4\pi^2 = 39{\cdot}4784.$

To convert common into hyperbolic logarithms multiply by 2·3026.

To convert hyperbolic into common logarithms multiply by 0·43429.

Electrical.

1 volt $= 10^8(l^{3/2}m^{1/2}t^{-2})$. 1 ampere $= 10^{-1}(l^{1/2}m^{1/2}t^{-1})$.

1 ohm $= 10^9(lt^{-1})$. 1 henry $= 10^9(l)$. 1 microfarad $= 10^{-15}(l^{-1}t^2)$ $= 10^{-6}$ farad. 1 dyne $= (lmt^{-2})$. 1 erg $= (l^2mt^{-2})$.

1 watt $= 10^7(l^2mt^{-3})$.

1 B.A. ohm = 0·9866 ohm. 1 ohm = 1·0136 B.A. unit.

APPENDIX II.

Useful Formulæ.

$$\int e^{at} \cos bt = \frac{e^{at}(a \cos bt + b \sin bt)}{a^2+b^2}$$

$$\int e^{at} \sin bt = \frac{e^{at}(a \sin bt - b \cos bt)}{a^2+b^2}$$

$$\int e^{at} \cosh bt = \frac{e^{at}(a \cosh bt - b \sinh bt)}{a^2-b^2}$$

$$\int e^{at} \sinh bt = \frac{e^{at}(a \sinh bt - b \cosh bt)}{a^2-b^2}$$

Binomial theorem : $(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots$

When x is small, $(1+x)^n = 1+nx$ and $\frac{1}{(1+x)^n} = 1 - nx$ approximately.

Hence $\sqrt{x^2+(\delta x)^2} = x + \frac{(\delta x)^2}{2x}$ where δx is small compared with x.

Taylor's theorem : $f(x+\delta x) = f(x) + \delta x f'(x) + \frac{(\delta x)^2}{\lfloor 2} f''(x) + \ldots$

When x is small, $f(x+\delta x) = f(x) + \delta x \, . \, f'(x)$.

Maclaurin's theorem : Put $x=0$,

and $f(\delta x) = f(0) + \delta x \, . \, f'(0)$.

Weights of stranded conductor, 10 and 12 round one :—

10/1, Weight of single outer conductor = total weight ÷ 15·0

12/1, „ „ „ = „ ÷ 20·2

APPENDIX III.

Properties of Circular and Hyperbolic Functions.

Circular Functions.	Hyperbolic Functions.
$e^{ix}=1+ix-\frac{x^2}{\underline{\|2}}-\frac{ix^3}{\underline{\|3}}+\dots$	$e^{x}=1+x+\frac{x^2}{\underline{\|2}}+\frac{x^3}{\underline{\|3}}+\dots$
$\cos x=\frac{e^{ix}+e^{-ix}}{2}=1-\frac{x^2}{\underline{\|2}}+\frac{x^4}{\underline{\|4}}-\dots$	$\cosh x=\frac{e^{x}+e^{-x}}{2}=1+\frac{x^2}{\underline{\|2}}+\frac{x^4}{\underline{\|4}}+\dots$
$\sin x=\frac{e^{ix}-e^{-ix}}{2i}=x-\frac{x^3}{\underline{\|3}}+\frac{x^5}{\underline{\|5}}-\dots$	$\sinh x=\frac{e^{x}-e^{-x}}{2}=x+\frac{x^3}{\underline{\|3}}+\frac{x^5}{\underline{\|5}}+\dots$
$\cos x\pm i\sin x=e^{\pm ix}$	$\cosh x\pm\sinh x=e^{\pm x}$
$\cos^2 x+\sin^2 x=1$	$\cosh^2 x-\sinh^2 x=1$
$\cos^2 x-\sin^2 x=\cos 2x$	$\cosh^2 x+\sinh^2 x=\cosh 2x$
$=1-2\sin^2 x=2\cos^2 x-1$	$=1+2\sinh^2 x=2\cosh^2 x-1$
$\sin 2x=2\sin x\cos x$	$\sinh 2x=2\sinh x\cosh x$
$\cos x=\frac{1}{\sqrt{1+\tan^2 x}}$	$\cosh x=\frac{1}{\sqrt{1-\tanh^2 x}}$
$\sin x=\frac{\tan x}{\sqrt{1+\tan^2 x}}$	$\sinh x=\frac{\tanh x}{\sqrt{1-\tanh^2 x}}$
$\cos ix=\cosh x$	$\cosh ix=\cos x$
$\sin ix=i\sinh x$	$\sinh ix=i\sin x$
$\cos(x\pm iy)=\cos x\cosh y\mp i\sin x\sinh y$	$\cosh(x\pm iy)=\cosh x\cos y\pm i\sinh x\sin y$
$\sin(x\pm iy)=\sin x\cosh y\pm i\cos x\sinh y$	$\sinh(x\pm iy)=\sinh x\cos y\pm i\cosh x\sin y$
$\frac{d}{dx}(\cos x)=-\sin x$; $\frac{d}{dx}(\sin x)=\cos x$	$\frac{d}{dx}(\cosh x)=\sinh x$; $\frac{d}{dx}(\sinh x)=\cosh x$
$\cos(x+y)=\cos x\cos y\mp\sin x\sin y$	$\cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y$
$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$	$\sinh(x\pm y)=\sinh x\cosh y\pm\cosh x\sinh y$
$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$	$\cosh x+\cosh y=2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
$\cos x-\cos y=2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$	$\cosh x-\cosh y=2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$	$\sinh x+\sinh y=2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$
$\sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$	$\sinh x-\sinh y=2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$

APPENDIX IV.

TYPES OF CORE OF SOMEWHAT FREQUENT OCCURRENCE.

Copper lb. per n.m.	Dielectric lb. per n.m.	No. of wires.	Overall diameter d mils. (1)	Covered diameter D mils. (2)	Log_{10} (D/d).	$\frac{D-d}{2}$ mils.	$8{\cdot}7\sqrt{d}$ mils. (3)	R ohm per n.m. (4)	K mfd. per n.m. (5)	RK micro-second per n.m.
42¼	55 (6)	7	54	170	0·496	58	64	28·0	0·294	8·23
70	75	7	70	200	0·457	65	73	16·9	0·319	5·39
70	100	7	70	228	0·515	79	73	16·9	0·284	4·79
70	120	7	70	249	0·552	89	73	16·9	0·265	4·46
100	120	7	84	252	0·479	84	80	11·8	0·305	3·60
107	120	7	86	253	0·466	83	81	11·0	0·314	3·46
107	140	7	86	271	0·496	92	81	11·0	0·294	3·25
107	150 (6*a*)	7	86	280	0·510	97	81	11·0	0·286	3·16
107	166	7	86	293	0·531	103	81	11·0	0·275	3·04
130	130 (7)	7	95	264	0·444	85	85	9·08	0·329	2·99
140	140	7	99	274	0·443	88	86	8·44	0·329	2·78
160	300 (8)	7	106	392	0·569	143	89	7·38	0·256	1·89
160	150	7	106	285	0·431	90	89	7·38	0·339	2·50
180	160	7	112	295	0·421	92	92	6·56	0·347	2·28
200	180	11	114	313	0·439	99	93	5·91	0·333	1·97
220	180 (9)	11	121	314	0·416	97	96	5·37	0·351	1·88
225	225	7	125	348	0·444	111	97	5·25	0·329	1·73
275	225	7	138	352	0·405	107	103	4·29	0·361	1·55
350	250	11	151	374	0·395	112	107	3·37	0·370	1·25
350	300	11	151	405	0·429	127	107	3·37	0·340	1·15
500	315	11	180	423	0·372	122	117	2·36	0·393	0·93
650	375 (10)	13	203	465	0·360	131	124	1·82	0·406	0·74
661	397 (11)	13	205	477	0·367	136	125	1·79	0·398	0·71
700	360 (10*a*)	13	211	460	0·339	125	126	1·69	0·431	0·73

(1) These are nominal figures. No correction is made for stranding or stretch, nor for clearance between the wires.

(2) Calculated from the formula $D=\sqrt{\frac{W+w/8{\cdot}9}{2071}}$, taking the dielectric as of unit density.

(3) An empirical formula for normal radial thickness of dielectric covering. *See* THE ELECTRICIAN, Vol. LXXIV., 1914, p. 329.

(4) At 75°F., $R=1181\div w$.

(5) Approximate figures based on 0·146 mfd. per n.m. for a core in which $D=10d$.

(6) and (6*a*) Used by the G.P.O. in multiple telegraph cables.

(7) Extensively used for cables of medium length where the traffic does not warrant a heavier type.

(8) Exceptionally heavy dielectric employed on G.P.O. telephone cables, now obsolete, being superseded by 160/150. The choice of this type for the first Anglo-French cable was the outcome of KR law considerations. *See* THE ELECTRICIAN, Vol. XXV., 1890, p. 688.

(9) Pacific Cable Board.

(10) and (10*a*) Western Union.

(11) Compagnie Française des Câbles Télégraphiques and Anglo-American Atlantic.

APPENDIX V.*

Effective Resistance, Inductance and Impedance of Standard Apparatus at 1,000 p.p.s.

Apparatus.	R eff. ohms.	L henries.	Impedance.		Loss in milliwatts per 1 volt.
			Ohms.	Angle.	
Bells—					
1,000ω magneto	7,580	1·305	11,140	47° 9	0·061
Indicators—					
1,000ω tubular, ordinary	8,000	1·2	11,000	43°24	0·066
,, ,, differential	20,200	0·224	20,300	5° 0	0·049
600ω self-restoring	8,055	1·3	11,410	44°55	0·062
100ω+100ω eyeball signal, unoperated	3,900	0·512	4,035	14°45	0·240
,, ,, ,, operated	4,300	0·539	4,440	14° 3	0·219
Instruments—					
Local battery subscribers, battery key up	434	0·189	1,265	69°57	0·027
,, ,, ,, ,, down	563	0·182	1,275	63°48	0·035
Receivers—					
Double-pole bell (60ω central battery)	134	0·018	176	40°24	4·33
Relays—					
500ω double make and break (W.E.):					
Armature not attracted	7,160	1·157	10,210	44°54	0·069
Ditto ditto ,, attracted	7,960	1·238	11,150	44°24	0·064
1,000ω ditto ,, not attracted	9,910	1·543	13,845	44°18	0·052
Ditto ditto ,, attracted	9,970	1·617	14,230	45°30	0·049
Retards—					
100ω tubular	1,116	0·191	1,640	47° 6	0·414
200ω ,,	3,170	0·550	4,690	47°30	0·144
400ω ,,	4,700	0·664	6,280	41°30	0·119
600ω ,,	5,906	0·890	8,132	43°20	0·089
1,000ω ,, differential	19,100	0·538	19,400	10° 0	0·051
75ω+75ω W.E. pattern	1,827	1·367	8,770	77°58	0·024
200ω+200ω ,, toroidal	3,600	13.5	85,000	87°34	0·0005
No. 1 Central Battery Termination—					
(Repeater, supervisory relay, local line and subscribers' instrument)					
(*a*) Local line, 0ω	330	0·049	451	42°57	1·62
(*b*) ,, 300ω (ohmic)	630	0·068	760	83°54	1·09
(*c*) ,, 3 m. 20 lb. cable	680	0·049	746	23°51	1·22

Note.—To obtain loss in milliwatts at any voltage V, multiply figures in last column by V^2.

* From experiments carried out by Mr. B. S. Cohen in the Investigation Laboratory of the late National Telephone Company, *see* National Telephone "Journal," Vol. IV., 1909, p. 113. Currents of from 0·3 to 2 milliamperes were used in the measurements, which were made at a frequency of 1,000 p.p.s.

APPENDIX VI.*

Constants of some Typical Telephone Conductors.

All electrical figures are per mile loop at $p=5{,}000$.

Cables.

Weight lb. per mile.	R ohm.	L henry.	K micro-farad.	G micro-mho.	P	α	β	λ	Z_0
$6\frac{1}{2}$	272	Nil.	0·0639	5	0·295 /44°33	0·210	0·207	30·4	923 \44°33
10	176	0·001	0·0714	5	0·251 /45°25	0·176	0·179	35·2	702 \43°47
20	88	0·001	0·0714	5	0·177 /46°14	0·123	0·128	49·1	497 \42°59
40	44	0·001	0·0714	5	0·126 /47°51	0·0844	0·0932	67·4	352 \41°21
70	26	0·001	0·0714	5	0·0972 /50° 3	0·0624	0·0745	84·3	273 \39° 9
100	18	0·001	0·0714	5	0·0817 /52°22	0·0499	0·0647	97·2	229 \36°51
150	12	0·001	0·0714	5	0·0681 /55°55	0·0382	0·0564	111·4	191 \33°12
200	9	0·001	0·0714	5	0·0606 /59° 8	0·0311	0·0521	120·7	170 \30° 5
20 (standard)	88	0·001	0·054	5	0·154 /46° 6	0·107	0·111	56·5	571 \42°51

Open Wire Lines.

Weight lb. per mile.	R ohm.	L henry.	K micro-farad.	G micro-mho.	P	α	β	λ	Z_0
40 (bronze)	90·00	0·00420	0·00750	1	0·0589 /50°48	0·0373	0·0456	138	1570 \37°40
70 ,,	52·00	0·00400	0·00786	1	0·0468 /54°47	0·0270	0·0383	164	1191 \33°45
100 (copper)	18·00	0·00390	0·00810	1	0·0328 /67°56	0·0123	0·0304	207	810 \20°39
150 ,,	11·90	0·00376	0·00840	1	0·0306 /73° 9	0·00887	0·0293	215	728 \15°29
200 ,,	9·00	0·00366	0·00862	1	0·0297 /76°15	0·00705	0·0288	218	688 \12°25
300 ,,	5·86	0·00355	0·00893	1	0·0289 /80°13	0·00482	0·0285	221	647 \ 8°29
400 ,,	4·50	0·00344	0·00920	1	0·0286 /82° 3	0·00396	0·0283	222	622 \ 6°43
600 ,,	2·97	0·00331	0·00959	1	0·0284 /84°19	0·00281	0·0283	222	592 \ 4°29
800 ,,	2·25	0·00322	0·00987	1	0·0283 /85°26	0·00226	0·0282	223	574 \ 3°24

* Based on the values of R, K, L, G given by A. J. Aldridge, "Journal" I.E.E., Vol. LI., 1913, p. 426.

APPENDIX VII.

TELEPHONE TRANSMISSION FORMULÆ

$$P=\sqrt{(R+ipL)(G+ipK)}=a+i\beta$$

$$Z_0=\sqrt{\frac{R+ipL}{G+ipK}}$$

$$V_x=V_0 \cosh Px-Z_0C_0 \sinh Px$$

$$C_x=C_0 \cosh Px-(V_0/Z_0) \sinh Px$$

or

$$V_0=V_x \cosh Px+Z_0C_x \sinh Px$$

$$C_0=C_x \cosh Px-(V_x/Z_0) \sinh Px$$

$$2a^2=\sqrt{(R^2+p^2L^2)(G^2+p^2K^2)}+(RG-p^2LK)$$

$$2\beta^2=\sqrt{(R^2+p^2L^2)(G^2+p^2K^2)}-(RG-p^2LK)$$

$$\lambda=2\pi/\beta.$$

When L and G are zero, $a=\beta=\sqrt{\frac{1}{2}pKR}$; $Z_0=\sqrt{R/pK}\ \overline{\pi/4}$
When G is zero,

$$2a^2=pK(\sqrt{R^2+p^2L^2}-pL)$$

$$2\beta^2=pK(\sqrt{R^2+p^2L^2}+pL)$$

When pL/R and pK/G are large,

$$a=\frac{R}{2}\sqrt{\frac{K}{L}}+\frac{G}{2}\sqrt{\frac{L}{K}} \qquad \beta=p\sqrt{LK} \qquad Z_0=\sqrt{\frac{L}{K}}$$

APPENDIX VIII.

ARRIVAL CURRENT, S.F.–H. CABLE, ENDS EARTHED.

(Table XXVIII., p. 236 extended.)

Time.	Current.	Time.	Current.	Time.	Current.	Time.	Current.
0.10	0·03	0·33	30·47	0·56	90·59	0·79	134·31
0·11	0·07	0·34	33·08	0·57	92·90	0·80	135·77
0·12	0·16	0·35	35·71	0·58	95·20	0·81	137·22
0·13	0·30	0·36	38·41	0·59	97·43	0·82	138·63
0·14	0·53	0·37	41·09	0·60	99·64	0·83	140·02
0·15	0·86	0·38	43·82	0·61	101·80	0·84	141·38
0·16	1·32	0·39	46·54	0·62	103·92	0·85	142·71
0·17	1·91	0·40	49·28	0·63	106·02	0·86	143·98
0·18	2·64	0·41	52·02	0·64	108·07	0·87	145·26
0·19	3·54	0·42	54·73	0·65	110·05	0·88	146·46
0·20	4·59	0·43	57·47	0·66	112·00	0·89	147·72
0·21	5·80	0·44	60·15	0·67	113·96	0·90	148·92
0·22	7·18	0·45	62·87	0·68	115·84	0·91	150·06
0·23	8·70	0·46	65·51	0·69	117·72	0·92	151·21
0·24	10·38	0·47	68·18	0·70	119·53	0·93	152·32
0·25	12·19	0·48	70·80	0·71	121·31	0·94	153·39
0·26	14·13	0·49	73·39	0·72	123·04	0·95	154·46
0·27	16·18	0·50	75·95	0·73	124·76	0·96	155·46
0·28	18·36	0·51	78·46	0·74	126·45	0·97	156·50
0·29	20·62	0·52	80·96	0·75	128·07	0·98	157·49
0·30	22·98	0·53	83·42	0·76	129·68	0·99	158·46
0·31	25·42	0·54	85·85	0·77	131·27	1·00	159·42
0·32	27·91	0·55	88·26	0·78	132·81		

Calculated from the formula $C_r = \frac{473\cdot 1}{\sqrt{t}} e^{-\frac{1\cdot 088}{t}}$. To apply these figures to any other cable, multiply times by $\frac{KRl^2}{4\cdot 351}$ and currents by $\frac{4975}{Rl}$.
See Chapter X., p. 326.

APPENDIX IX.

ARRIVAL CURRENT, S.F.–H. CABLE, ENDS TO EARTH THROUGH $Kl/10$ CONDENSERS.

(Table XLIV., p. 275, extended.)

Time.	Current.	Time.	Current.	Time.	Current.	Time.	Current.
0·08	0·02	0·94	4·75	1·80	1·26	2·66	0·32
0·10	0·04	0·96	4·62	1·82	1·22	2·68	0·31
0·12	0·10	0·98	4·49	1·84	1·18	2·70	0·30
0·14	0·22	1·00	4·36	1·86	1·14	2·72	0·29
0·16	0·44	1·02	4·23	1·88	1·11	2·74	0·28
0·18	0·83	1·04	4·11	1·90	1·07	2·76	0·27
0·20	1·29	1·06	3·99	1·92	1·04	2·78	0·27
0·22	1·83	1·08	3·88	1·94	1·01	2·80	0·26
0·24	2·41	1·10	3·76	1·96	0·98	2·82	0·25
0·26	3·01	1·12	3·65	1·98	0·95	2·84	0·24
0·28	3·60	1·14	3·54	2·00	0·92	2·86	0·23
0·30	4·16	1·16	3·43	2·02	0·89	2·88	0·23
0·32	4·68	1·18	3·33	2·04	0·86	2·90	0·22
0·34	5·15	1·20	3·23	2·06	0·83	2·92	0·21
0·36	5·56	1·22	3·13	2·08	0·81	2·94	0·21
0·38	5·91	1·24	3·03	2·10	0·78	2·96	0·20
0·40	6·21	1·26	2·94	2·12	0·76	2·98	0·19
0·42	6·45	1·28	2·85	2·14	0·73	3·00	0·19
0·44	6·65	1·30	2·77	2·16	0·71	3·02	0·18
0·46	6·79	1·32	2·68	2·18	0·69	3·04	0·18
0·48	6·90	1·34	2·60	2·20	0·67	3·06	0·17
0·50	6·97	1·36	2·52	2·22	0·65	3·08	0·17
0·52	7·00	1·38	2·44	2·24	0·63	3·10	0·16
0·54	7·01	1·40	2·36	2·26	0·61	3·12	0·16
0·56	6·99	1·42	2·29	2·28	0·59	3·14	0·15
0·58	6·95	1·44	2·22	2·30	0·57	3·16	0·15
0·60	6·89	1·46	2·15	2·32	0·55	3·18	0·14
0·62	6·81	1·48	2·09	2·34	0·53	3·20	0·14
0·64	6·72	1·50	2·02	2·36	0·52	3·22	0·13
0·66	6·62	1·52	1·96	2·38	0·50	3·24	0·13
0·68	6·50	1·54	1·90	2·40	0·49	3·26	0·12
0·70	6·38	1·56	1·84	2·42	0·47	3·32	0·11
0·72	6·26	1·58	1·78	2·44	0·46	3·38	0·10
0·74	6·13	1·60	1·73	2·46	0·44	3·44	0·09
0·76	5·99	1·62	1·67	2·48	0·43	3·50	0·08
0·78	5·85	1·64	1·62	2·50	0·41	3·58	0·07
0·80	5·72	1·66	1·57	2·52	0·40	3·68	0·06
0·82	5·58	1·68	1·52	2·54	0·39	3·78	0·05
0·84	5·44	1·70	1·47	2·56	0·38	3·94	0·04
0·86	5·30	1·72	1·43	2·58	0·37	4·06	0·03
0·88	5·16	1·74	1·38	2·60	0·35	4·30	0·02
0·90	5·02	1·76	1·34	2·62	0·34	0·62	0·01
0·92	4·88	1·78	1·30	2·64	0·33		

These figures may be applied to any cable by the use of the same multipliers as in Appendix VIII.

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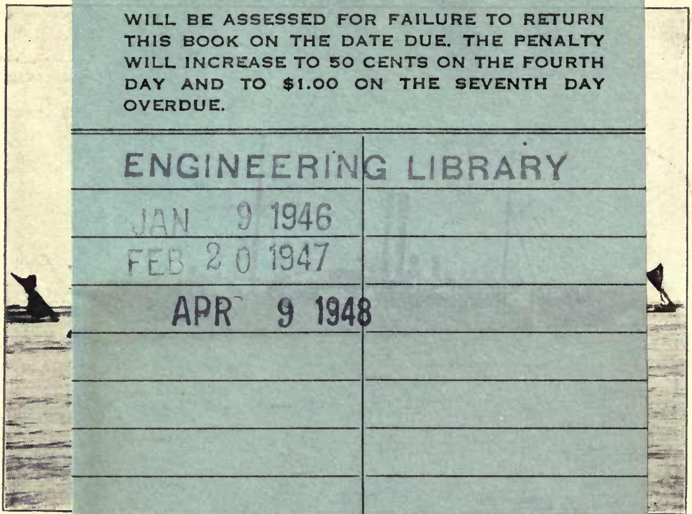

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