THE USES AND TRIUMPHS OF MATHEMATICS

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The Uses and Triumphs of Mathematics

ITS BEAUTIES AND ATTRACTIONS POPULARLY TREATED IN THE LANGUAGE OF EVERYDAY LIFE

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'For by known principles you understand the most difficult subjects much more easily.'—*Cicero.*

Without Mathematics, expressed or implied, our knowledge of Physics would be friable in the extreme.'—*Tyndall.*

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AND SYDNEY.

DEDICATION.

———o———

To that long series of Illustrious Men who by their labours and discoveries have rendered the powers of Nature the servants of man, in contradiction to E*MPIRICISM*, *which subjects man to their service, this little Book is respectfully dedicated by*

THE AUTHOR.

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PREFACE.

'ETHEL,' I remember once hearing a lady say to her daughter, 'did Mr Antony say he intended to bring a friend with him this evening?' 'Yes, mamma,—a Mr Bertrand, a mathematician.' 'Oh dear!' replied her mother, 'what a wet blanket he will be! I hope he will not be trying to "Square the Circle," don't they call it, or something of that sort?'

Is not the opinion which this lady held with regard to Mathematicians and Mathematics the one very often entertained, namely, that Mathematics renders a man unsociable, unpoetical, calculating, and without faith in anything which will not admit of a rigid demonstration; and that the Science of Mathematics is a dreadful subject,—a compound of fearful words and symbols which only the initiated are able to appreciate or perceive any use in?

I shall endeavour to show, in a non-mathematical manner, in the following pages, that this Science can be put to every possible kind of practical use, and also to refute the statement that the Science of Mathematics is a dreadfully dry subject, without any beauty or poetry in it,—simply a conglomeration of straight lines and circles, a's and b's, x's and y's, sines and cosines, etc. etc., but that it is one of the most profound and the most fascinating, and, in some respects, the most beautiful, of all the Arts and Sciences.

The object of this Essay is an attempt to create a desire for the study of Mathematics, by showing its intimate and important connection with so many branches of Science (man's greatest helpmate), by stating and explaining a few of its uses and triumphs, and by attempting to prove that it is a subject possessed of a beauty and attraction entirely its own.

Much has been done at various times to popularise the subject of Mathematics, but much more remains to be done before this Science will be divested of that perhaps quite natural aversion which a non-mathematical person always experiences on opening any book treating on any of its various branches. Some knowledge of Mathematics (or what is commonly called such) is now required in every public examination, and therefore the subject is too often taught and learned *for* these examinations, and not on account of those uses, etc. (excepting arithmetic), which the subject possesses in itself, and as an aid to Science.[1] The majority of students, having acquired the necessary degree of skilfulness in the manipulation of certain symbols, or a mnemonic acquaintance with certain geometrical demonstrations, pass the examination, and then drop the subject at once, it having been to them only a means to an end,—an abominably dry subject to be avoided as soon as possible.

And this is not surprising, considering the uninteresting way in which this subject is usually presented to the student, ignorant alike as he is of its wonderful and

[1] 'The distinctive feature of mathematical instruction in England is, that an appeal is there made rather to the memory than to the intelligence of the pupil.'—Messrs Demogeot and Montucci—Report on English Education.

varied uses; of the attractions which this subject possesses on account of its intimate connection with so many departments of Science and Art; of its historical interest; and of what a vast though indirect influence this Science has had on the progress of the human race.

The contents of this little book is also an attempt to meet in some measure this want, being intended mostly for the non-mathematical; no mathematical knowledge is required for its perusal; and I have endeavoured, as far as I have been able, to make it as interesting as possible.

For much matter and many ideas I am indebted to 'The Orbs of Heaven' (a book, though antiquated, well worthy of a place in the library of everyone), the writings of R. W. Emerson, Professor Tyndall, Justus Liebeg, Dr Whewell, Lord Bacon, R. A.

Proctor, C. Flammarion, and the articles on Mathematics and Geometry in the 'National Encyclopædia,' and various other writers, to whom reference will be made in due course. Whatever well expressed thought or idea I have found (no matter where) illustrating my subject, I have taken, and, when possible, I have acknowledged it. I have *created* but a small portion of the materials of this work. The only claim to originality that I make, is the giving them proportion, place, design, and the shaping them to a new utility. Thus I am rather a compiler than a composer; and I say with Montaigne, 'I have gathered a nosegay of flowers, in which there is nothing of my own but the string which ties them.'

But in this case the string which ties them is entirely my own.

'They (the Egyptian Priests) applied themselves much to the study of Geometry and Arithmetic. The Nile, which annually changed the aspect of the country, gave rise to numerous lawsuits amongst neighbours, with regard to the boundaries of their possessions. These lawsuits would have been interminable without the intervention of the Science of Geometry. Their Arithmetic was useful in the administration of private affairs and in geometrical speculations.'—DIODORUS.

THE USES AND TRIUMPHS OF MATHEMATICS.

CHAPTER I.

INTRODUCTION.

'*The oldest of the Sciences.*'

THERE is a certain Science which may be compared to a series of mighty rivers, several of whose sources — tiny springs in some vast mountain—are inaccessibly profound,—to a series of mighty rivers, separate at first, and flowing in such diverse directions that you would never guess of any confluences ever occurring, but which, suddenly bending round, join themselves to one another, forming thereby a stream which, ever adding as it is new tributaries, —a stream (to pass from Nature to the Intellect) that under the name of MATHEMATICS has given to the mind of man a power, force, and rapidity in the investigation of Nature, increasing manifold its capacities.

It is this that the history of Mathematics teaches us. For from whence come our first principles of this science—the oldest in the world, and cœval, to some extent, with the existence of man—we know not. And then from what small beginnings has each department of the science arisen: a Definition, an Axiom, an Experiment. For, like the other sciences, it existed, in observation and practical applications, long before it was established or reduced to the form of a science by abstract reasoning.[1]

[1] Pythagoras sacrificed a hecatomb when he discovered the *proof* of Euc. I. 47.

Herodotus says: 'I was informed by the Priests of Thebes, that King Sesostris made a distribution of the territory of Egypt among all his subjects, assigning to each an equal portion of land, in the form of a quadrangle, and that from these allotments he used to derive his revenue by exacting every year a certain tax. In cases, however, where a part of the land had been washed away by the annual inundations of the Nile, the proprietor was permitted to present himself before the king and signify what had happened. The king used then to send proper officers to examine and ascertain by exact admeasurement how much of the land had been washed away, in order that the amount of the tax to be paid for the future might be proportional to the land which remained. From this circumstance I am of opinion that Geometry (the keystone of Mathemetics) derived its origin, and from thence it was transmitted into Greece.'

And Plato also says, 'I have heard it said that in the neighbourhood of Naucratis, a town of Egypt, there had existed one of the most ancient gods of this country, who was named Theuth, and who had invented the numbers, ciphering, geometry, astronomy, the games of chess and of dice, and writing.'

These were, of course, not all invented by one man, nor yet at one period, but this passage serves, however, to show their vast antiquity. But it is not only from such vague sources as the above that we derive our knowledge of the Egyptian Mathematics, for if we see a people possessing no mathematical knowledge before they

Note.—The British Museum preserves, under the name of *Papyrus de Rhind*, the only treatise on Geometry that Egypt has left us. This document dates from the XIXth Dynasty, but it is, according to M. Birch, the copy of an original which traces its origin back even to Cheops (B.C. 3091-67), *i.e.* about 5000 years ago. It is a very elementary manual, containing a series of rules for the measurement of surfaces and solids, presenting, at the same time, problems for solution.—'Les Premières Civilisations Egyptiennes.'

have had relations with the Egyptians, and possessing it very soon after these relations have been established, it is assuredly safe to infer that the first has borrowed its knowledge from the second. Our first principles of Geometry we can practically trace back through the Greeks to the Egyptians, our knowledge of Algebra (Arithmetic and Algebra were originally one, and, in fact, until the end of the 16th century Algebra was little more than a convenient shorthand for solving problems in Arithmetic) we derive from the Arabs, who transmitted it into Europe in the 10th or 13th century, and the Arabians were pupils of the Hindu mathematicians,

Note A.—The Jesuit missionaries found very little knowledge of Geometry amongst the Chinese. The Hindoos possess a much larger amount of knowledge, but it is of very uncertain date. No trace of any knowledge of Geometry is found in the writings of the Jews.

Note B.—Leonardo Bonacci (13th century), a Pisan, whose father was employed in the Custom House of Bugia, in Barbary, acquired from the Arabs a knowledge of arithmetic after the manner of the Indians.

who were, in their turn, most probably pupils of the Egyptians.

Thus we see that we owe the first principles of this science (as indeed many others) in all probability to the Egyptians.

Nevertheless, it is the Greeks whom we have to credit with the real foundation of the science of Geometry. Proclus says: 'That Pythagoras,' (about 600 B.C.) 'was the first who gave Geometry the form of a science.'

The science was then greatly advanced by the philosopher Plato, and the illustrious Euclid, whose 'Elements' has been the *principal text book* for beginners during a period of more than 2000 years. Ptolemy Lagus was one of his pupils, and it was he to whom he made the celebrated reply, when asked if there was no shorter way to Geometry than by studying his 'Elements,'—'No, sire, there is no *royal road* to Geometry.'

In the year 1619 Descartes, *at the age of twenty-three*, by one of those extraordinary strokes of genius, occurring once only in any age, fastened the irresistible power of Algebra upon Geometry, thereby giving to the mind a force and rapidity in mathematical investigations quadrupling its capacity. About this time also was invented another analytical branch of Mathematics, known as the Calculus or the Infinitesimal Analysis.[1] To explain the nature of this analysis is not my object; its power and capacity are all I wish to mention. Between the two methods—the geometrical and the analytical—the following comparison has been drawn:—

'Geometry had invigorated the reason, as exercise toughens and strengthens the

Note.—Most of Sir Isaac Newton's great discoveries were made before the age of twenty-seven.

[1] Newton's 'Fluxionary Calculus,' 1666; Leibnitz's 'Differential Calculus,' 1667.

muscles of the human frame. But it had given to the mind no mechanical power wherewith to conquer the difficulties which rose superior to its natural strength. Archimedes wanted but a place whereon to stand, and with his potent lever he would lift the world. The student of Physics demanded an analogous mental machinery. What the human mind demands and resolves to find out, it never fails to discover. The Infinitesimal Analysis was invented, its principles developed, and its resistless power compelled into the service of human knowledge. So great is the power of this analysis, that once having seized on a wandering planet it never relaxes its hold; no matter how complicated its movements, how various the influences to which it may be subjected, how numerous its revolutions, no escape is possible. This subtle analysis clings to its object, tracing its path and fixing its place with equal ease, at the beginning, middle, or close of a thousand revolutions, though each of them should require a century for its accomplishment.'[1]

It must be added, however, that the close and grasping character of the ancient reasoning was lost; but the time came when Monge (the inventor of descriptive Geometry) showed how to return to geometrical construction with means in many cases superior to those of analysis in many practical matters. The method of Monge recalled the attention of geometricians to the properties of Projection in general; and from the time of Monge to the present day this subject (Geometry), has been cultivated with a vigour which has produced remarkable results (notably in Electricity), and promises still greater.

[1] 'The Orbs of Heaven.'

Note.—The oldest work on Algebra now extant is that of Diaphantos of Alexandria, in the 4th century after Christ.

The history of the Applied Mathematics, or Natural Philosophy, is equally interesting. It had also its foundation in experiment and practical application before it existed as a science properly so-called.

The famous Archimedes was the great founder of the sciences of MECHANICS and HYDROSTATICS. Then Galileo, about the latter end of the 16th century, greatly advanced the science of MECHANICS, and his pupil Toricelli that of HYDROSTATICS; and then the illustrious Newton, with his immortal discoveries in so many branches of the science of Natural Philosophy.

ELECTRICITY was first observed by Thales (600 B.C.), who noticed the property by which yellow amber, on being subjected to friction, attracted light bodies. But nothing more was known of this science until the close of the 16th century. The celebrated mathematician Gauss (born 1777) invented the magnetometer, and made MAGNETISM an exact science; and, in fact, may be regarded as the founder of the truly scientific study of magnetism.

Descartes (about 1620) brought the science of OPTICS under command of Mathematics by the discovery of the laws of refraction through transparent bodies.

ASTRONOMY[1] has always from the earliest ages been intimately connected with the science of Mathematics, and has always presented problems to the mathematician not only equal to all he could perform, but passing beyond the limits of his greatest intellectual power, and the solutions of many of these problems are, indeed, wonderful illustrations of the triumphs of mind over matter.

In the year 1788, Lagrange showed how to apply *mathematical analysis* to mechanical problems, thus making the science purely

[1] The first *triumph* of Mathematics would evidently be the *prediction* of an eclipse; the *first* prediction made by whom, alas, unknown.

analytical. And since then the art of applying Mathematics to the sciences of Electricity, Heat, Magnetism, Hydromechanics, Optics, etc., and, in fact, to every branch of Physical Science, and the introduction of mechanical principles into the theories of physical phenomena in general, has been most rapidly and extensively cultivated, until at the present day Mathematics and Physical Science have become in reality one. 'For,' says Professor Tyndall, 'no matter how subtle a phenomena may be, whether we observe it in the region of sense, or follow it into that of the imagination, it is in the end reducible to mathematical laws.'

Thus have I endeavoured, in a few words, not to present you with a short history of the science, but only to place before you a few of its most striking and salient points; hoping by so doing to excite your interest to pursue the subject further.[1]

[1] See Montucla's 'Histoire des Mathématiques.'

Its first principles, under the head of Arithmetic, are lost in the dim vista of the past; its origin rests on legendary ideas alone. All we know is that numeration, or the art of numbering, must have been to some extent coeval with the existence of man. Once transmitted into Europe, its rapid progress and marvellous growth are easily traceable. By the genius of different mathematicians the various branches of the science have been united, thereby giving to the mind a power enabling it to pursue its investigations with a force and rapidity increasing tenfold its capacity. And as the various branches of the Pure Mathematics have been invented and combined, so has it been applied, both analytically and geometrically, more and more to the different branches of science and art, and within the last half century has attained such a state of perfection as to enable a mathematician to determine almost immediately whether a problem can be solved by such means as he possesses or not,—no small advantage, when it is considered how much time was wasted in attempts to attain impossible solutions.

Applied to Engineering, it has enabled man to bridge rivers and tunnel mountains; under the head of Electricity, it has enabled him to 'flash his words from the far land, and girdle the earth with a spell;' and under the head of that sublime subject, Astronomy, its power is so great that should a star commence to revolve around some grand centre, moving so slowly that millions of years must roll away before it can complete one circuit, not even a single year shall pass before its motion be detected (by observation), in ten years its velocity shall be calculated, and in the lifetime of a single observer its period shall become known. In a word, the astronomer, by observation and calculation, writes out its history with perfect accuracy for a million years.

These and other marvels not less wonderful perhaps justify the lines :—

> ‘*Some by it learnt the mysteries of the sphere,*
> *The paths of comets, the movements of the stars,*
> *The distance of the sun: to some it gave*
> *To prove th' existence of the self-same power*
> *O'er time and space; and law's unbroken reign,*
> *And never varying energy: to others*
> *But the arithmetician's lessen'd power,*
> *Whilst others by its aid were led*
> *'Neath mountains and o'er rivers.*’

The further contents of this little book is an account of a few of its universal applications and triumphs, and also a few words about the inherent beauty and poetry of the subject. We stand, so to speak, on the verge of boundless possibilities; what new truths may be discovered, what new branches of Mathematics may be invented, as Nature becomes more and more disclosed, no man can say. This further revelation of Nature (God's work, and therefore His Word, if it can be rightly interpreted) is man's highest and noblest ambition.

The triumphs of mind over matter are indeed most wonderful, and, at times, even to the initiated, appear almost beyond man's power. But man is born to aspiration as the sparks fly upwards, and there is no nobler or loftier ambition than

'*To build in matter home for mind.*'[1]

But let us always carefully remember the words of two of the greatest mathematicians that the world has hitherto seen—Newton and Laplace; the first of whom compared himself to a child who had picked up here a bright pebble and there a shining shell on the shore, while the ocean of truth lay all unexplored before him; and the second of whom said,—'What we know is a very little (*peu de chose*), what we know not, is immense.' And the illustrious mathema-

[1] Emerson.

tician and philosopher, Pascal, wrote that memorable sentence :—'The highest perfection of human understanding is to know that there is an infinity of truth beyond its reach.'

And the lesson of modern science has been, in one sense, a negative one, for it has revealed to man his utter insignificance in the infinities by which he is surrounded, and has taught us that first lesson we all should learn—that of humble humility :—

> . . . '*Here*
> *In this interminable wilderness*
> *Of worlds, at whose immensity*
> *Even soaring fancy staggers.*'—SHELLEY.

'The Mathematics are either pure or mixed. To the pure Mathematics belong those sciences which handle *quantity determinate*,[1] merely severed from any axioms of natural philosophy. . . . The mixed Mathematics hath for subject some axioms or parts of natural philosophy, and considers *quantity determined*, as it is auxiliary and incident unto them; for many parts of Nature can neither be discovered with sufficient subtility, nor explained with sufficient perspicuity, nor accommodated unto practice with sufficient dexterity, without the aid and intervention of the Mathematics; of which sort are Perspective, Music, Astronomy, Cosmography, Architecture, Engineering, and divers others.

'In the Mathematics I can discover no deficiency, except that men do not sufficiently understand the excellent use of the pure Mathematics, in that they do remedy and cure many defects in wits and faculties intellectual. For if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it; so that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye, and a body ready to put itself into all postures, so in the Mathematics that use which is collateral and intervenient, is no less worthy than that which is principle and intended. And as for the mixed Mathematics, I may only make this prediction, *that there cannot fail to be more kinds of them as Nature grows further disclosed.*'—BACON.

[1] Viz., Arithmetic, Algebra, Geometry, Trigonometry, the Calculus, Logarithms, Probabilities, etc., etc.

Note.—How wonderfully has his prediction been fulfilled, not only with regard to the mixed Mathematics, but the pure also. Under the head of the mixed Mathematics is included Mechanics, Optics, Electricity, Heat, Astronomy, Pneumatics, Magnetism, and also those continually calling in the aid of pure Mathematics, as Geology, Geography, Geodosy, Land-Surveying, Navigation, Civil, Practical, and Military Engineering, etc., etc.

CHAPTER II.

THE USES OF MATHEMATICS.

'Histories make men wise; poets witty, the Mathematics subtle, natural philosophy deep.'

BACON.

A SPECIAL interest that the history of the science of Mathematics (hitherto unmentioned) possesses, is the fact 'that mathematical truths have always been referred to by each successive generation of thoughtful and cultivated men as examples of truth and demonstration, and have thus become standard points of reference among cultivated men, whenever they speak of truth, knowledge, or proof.'[1]

I now pass on to the consideration of the

[1] Dr Whewell.

subject-matter concerning which this Essay was more especially written.

The utility of the science of Mathematics in itself, and as a discipline of the mind, lies in its strengthening the power of the reasoning faculties by frequent examples, which are the best lessons all can read, and in its enabling anyone to distinguish between reasonings founded on only probable premises and on certain ones, and in forming that habit known as concentration—of which it has been said:—'The one evil in life is dissipation, the one prudence concentration;' and by means of which the greatest difficulties are overcome, and victory certain, if only the right means be used; and in cautioning anyone against receiving anything which may appear at first probable enough and based on sound reasoning, but which when examined and analysed is seen to be founded on false premises; and in giving to us a true and correct estimate of the powers of the mind, by showing the really wonderful and varied consequences which are able to be developed out of a few of its most inherent notions; and in giving to us the pleasure of possessing a science in which men of different nations, creeds, and habits might *a priori* be expected to agree. A knowledge of the pure Mathematics enables anyone to have all his knowledge systematised and arranged. What others have in confusion he will have in order. The elements of knowledge are more or less known to all, but in their most perfect, communicable, and usable state they are known only to a person possessing some knowledge of Mathematics, if not of its doctrines, at any rate of its methods. What training is to the soldier, Mathematics is to the thinker. Mathematics has conquered

Note.—The illustrious Newton when asked how he had been able to achieve all his wonderful discoveries, replied,—'By always intending my mind.'

contingency and verisimilitude, and shown the fallacy of Chance.[1]

'Chance,' says J. S. Mill, 'is usually spoken of in direct antithesis to Law; whatever (it is supposed) cannot be ascribed to any law is attributed to chance. It is, however, certain that whatever happens is the result of some law; it is an effect of causes, and could have been predicted from a knowledge of the existence of these causes, and from their laws. If I turn up a particular card, that is a consequence of its place in the pack. Its place in the pack was a consequence of the manner in which the cards were shuffled, or of the order in which they were played in the last game; which, again, were the effects of prior causes. At every stage, if we possessed an accurate knowledge of the causes in existence, it would have been abstractedly possible to foretell the event.'

[1] *Vide* 'Chance and Luck,' by R. A. Proctor.

Further on, in his work on Logic, he also says, ‘Every event in itself is certain, not probable, and if we knew all, we should either know positively that it will happen or positively that it will not. But the *probability* to us means the degree of expectation of its occurrence which we are warranted in entertaining by our present evidence.’

It is this kind of chance or probability with which Mathematics concerns itself. Chance, then, as connected with Mathematics, has no connection with the ordinary meaning of the word.

Mathematical Probability has shown the fallacy of what is known as common consent, and is a powerful auxiliary in the investigation or the discovery of some new law of

Note.—By means of the science of Mathematics is every *indirect* measurement made. The utility of the science from this point of view can hardly be over-estimated, because there are many measurements which *must* be effected indirectly, such as determining the distance and weight of the moon, sun, planets, the velocity of light, electricity, the weight of the earth, the distance of the stars, etc., etc.

Nature. Life in the aggregate is but a Mathematical problem.

'Man,' said Jules Sandeau, 'has been called the plaything of chance, but there is no logic more close or inflexible than that of human life; all is entwined together, and for him who is able to disentangle the premises and patiently await the conclusion, it is the most correct of syllogisms.'

But this is a digression. To attempt to enumerate the different uses of each of the various departments of the science of Mathematics would be an impossible task a task as wearisome to the general reader as myself.

The uses of ARITHMETIC are known to everyone: of them it would be absurd to attempt an account. A man *may* be successful in business and be ignorant of Greek or Latin, French or German, but unless he have some knowledge, at any rate, of Arithmetic, he certainly cannot be so. For some years of boyhood there ought to be a daily appropriation to the task of thoroughly acquiring a perfect knowledge of the manipulation of this all-powerful instrument, which by the new method (the unitary method) is much less mechanical, and requires more exercise of the reasoning faculties than formerly.

[1] The uses of GEOMETRY have always been admitted, from the time of the Egyptians, who settled their lawsuits by means of it, and Plato, who placed the following inscription over the door of his house: 'Whoso knows not Geometry, let him not enter here,' down to the present day, when the Parisian dressmakers are taught in the professional schools of the city of Paris not only sewing but Euclid or Geometry and Drawing. Geometry strengthens and invi-

[1] *Vide* Chalmers' 'Graphical Determination of Forces in Engineering Structures,' Sir W. Thompson's Papers on 'Electrostatics and Magnetism,' and Clerk Maxwell's 'Electricity.'

gorates the reason, as exercise toughens and strengthens the muscles of the body ; and Lord Bacon's statement, 'that if a man's mind be wandering, let him study the Mathematics,' applies to no branch of that science more than to Geometry (notably 'The Elements of Euclid'). Once lose the chain of reasoning, it is no use; you must go back to the beginning of the proposition, and begin again, if you wish to understand what you are reading.'

And pure Geometry must ever remain the most perfect type of the deductive method in general. 'And the recollections of the truths of pure Geometry has, in all ages, given a meaning and a reality to the best attempts to explain man's power of arriving at truth.'

The practical uses of Trigonometry (a combination of Algebra and Geometry), under the head of land surveying, geodosy, etc., are well known. And what is known as Spherical Trigonometry is of great use in Nautical Astronomy, to which navigation, and therefore commerce, owes so much.

I now pass on to the uses of ALGEBRA, LOGARITHMS, the CALCULUS, PROBABILITIES, *and the higher branches of the pure Mathematics.* The uses of these are not so well known, because, with the exception of a slight knowledge of Algebra, by far the greater portion of mankind have no knowledge of these subjects, and their uses, vast though they be, are known only to a few.

One of their many uses is an *immense saving of time and labour;* problems which by more elementary means would require sheets of paper and days of labour, are solvable in a few lines and in a few minutes. And as this holds for even the elementary portions of Algebra over ordinary arithmetical calculations, so does it hold proportionately with regard to the Calculus over Algebra and Geometry. Problems which, without the Calculus, require much thought and labour, and often a great deal of ingenuity for their solution, can be solved systematically by the Calculus, without any need of ingenuity, so long as the proper rules are followed. By the higher Mathematics are solved of course problems *not* solvable by elementary means, however much ingenuity be used; and many of the results obtained by the Calculus have their practical applications in the rules used in Mensuration, for instance. But it is not these uses to which I wish particularly to call your attention. Mathematics has other uses besides the determining of heights and distances, the finding the volumes of solids, and the like; or

> '*By geometric scale*
> *To take the size of pots of ale.*'[1]

[1] Thomas Carlyle.

The Real Practical Use of the science of Mathematics lies in its *application* to the sciences of Mechanics, of Optics, of Acoustics, of Hydro-Mechanics, of Astronomy, of Electricity, of Magnetism, of Heat, of Chemistry, of Geology, of Biology, of Engineering, of Music, of Architecture, of Painting, of Pneumatics, of Navigation, and, in fact, directly or indirectly, to the whole domain of Science and Art; 'from investigations relating to the infinitely great and the infinitely little to the study of the most familiar objects of every-day life.'[1]

And what are the uses of Science? Faraday answered this question by demanding: 'What was the use of a baby?' But, apart from this, the use of Science, is this—mankind has at last realised the fact that *nothing happens by accident, and that there is no such thing as chance.* Thus, nothing happening by accident but by *Law*, it be-

[1] R. A. Proctor.

hoves us to become acquainted with these laws, in order that we may guide our practical conduct by them. This it is the aim of Science to achieve. We owe all our knowledge of Nature to Science. Hence its use. It was not for this—as an instrument in the discovery of the laws of Nature—that the science of Mathematics was valued by many of the ancients. They valued it only 'as leading men to the knowledge of abstract, essential truth.'[1] Archytas framed machines of extraordinary power by mathematical principles. Plato remonstrated with him, and declared that this was to degrade a noble intellectual exercise into a low craft, fit only for carpenters and wheelwrights. The office of Geometry was to discipline the mind, not to minister to the base wants of the body. This interference was successful, and from that time, the science of Mechanics was

[1] Plato's 'Republic,' Bk. 7.

considered as unworthy of the attention of a philosopher. And even Archimedes was not free from the prevailing notion that Geometry was degraded by being employed to produce anything useful. It was with difficulty that he was induced to stoop from speculation to practice. He was half ashamed of those inventions which were the wonder of hostile nations, and always spoke of them slightingly as mere amusements, as trifles in which a mathematician might be supposed to relax his mind after intense application to the higher parts of the science.[1]

With increased knowledge has come increased wisdom, and we now value Mathematics as the handmaid of Science. There is no doubt beauty in the idea that 'The soul, considered in relation to its Creator, is like one of those mathematical lines that may draw nearer to another for all eternity, without a possibility of touching it;' or in

[1] See Lord Macaulay's 'Essays,' Lord Bacon.

comparing 'the directrix or axis of a curve which extends both ways to infinity, without ever deviating to the one side or the other, to the infinite and unbending rectitude, truth, and justice of the great Creator.' But it is not for illustrations such as these that we desire to become acquainted with the conic-sections.[1]

The real use of the Science of Mathematics lies in its applications, and they may be said to be universal. From determining the stability or non-stability of a ship, to determining the stability or non-stability of the planetary system; from calculating the path of a projectile, to calculating the path of a comet; from finding the cubical contents of an ordinary wall, to finding the cubical contents of the sun; from computing the distance of an object a mile off, to computing the distance of a star at a distance of billions of miles;

[1] *Vide* Chap. V.

from estimating the weight of a few tons, to estimating the weight of the whole solar system; from calculating the velocity of a railway train, to calculating the velocity of light, with a velocity of over 180,000 miles per second; in a word, from the ordinary money or business transactions of everyday life, up to the most elaborate calculations of the student of Physics, the Science of Mathematics is of the greatest possible kind of use.

But, as Professor Tyndall so aptly puts it: 'The circle of human nature is not complete without the arc of feeling and emotion. And here the dead languages, which are sure to be beaten by science in a purely intellectual fight, have an irresistible charm. They supplement the work of Mathematics, by exalting and refining the æsthetic faculty, and must be cherished by all who desire to see human culture complete.'

To omit one is to leave a man half-educated. To cram into a man a certain amount of knowledge concerning the manipulation of certain symbols is not to educate him at all; in order that we may share in what men are doing in the world, we must share in what they have done. Thus arises the importance of history. As refining the æsthetic faculty, the Classics and Fine Arts are invaluable, and their claims must not be set aside.

And manifold and varied though the uses of Mathematics be, we must not let those uses become abuses; for, in our hours of pleasure and enjoyment, we are quite right to say with the poet Shelley,—

'As to nerves,
With cones, and parallelograms, and curves,
I've sworn to strangle them if once they dare
To bother me—when you are with me there,'

and, above all, let us take care of our

health; it is better to cultivate the body at the expense of the mind than the mind at the expense of the body, for

'*Health is the first wealth.*'

'Those long chains of reasoning, all simple and easy, by which Geometers used to arrive at their most difficult demonstrations, suggested to me that all things which come within human knowledge must follow each other in a similar chain, and that, provided we abstain from admitting anything as true which is not so, and that we always preserve in them the order necessary to deduce the one from the other, there can be none so remote to which we may not finally attain, nor so obscure but that we may discover them.'—DESCARTES.

CHAPTER III.

THE TRIUMPHS OF MATHEMATICS.[1]

'Mathematicians' art is ever able
To endow with truth mere fable.'

I.

UNIVERSAL GRAVITATION.

'Nature and Nature's law lay hid in night,
God said, "Let Newton be," and all was light.'

In giving a short account of a few—I may say a very few—of the Triumphs of Mathematics, I will first mention the discovery, or rather the demonstration, of the law of Universal Gravitation, as given by the illustrious Newton in that wonderful book 'The Principia,'—of that grand law, that mighty power, that mysterious hand, so to

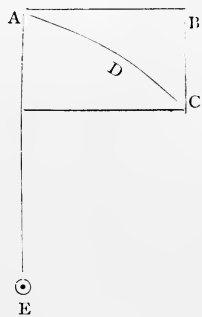
[1] *Vide* 'Mécanique Celéste' (Laplace), 'Mecanique Analytique' (Lagrange), and Euler's 'Scientific Papers.'

speak, which causes bodies when unsupported to fall to the ground, the moon to describe its orbit about the earth, the tides to perform their daily ebb and flow, the planets to revolve around the sun, floating isolated in space, and the whole solar system to revolve around some grand centre, describing an immense but purely ideal curve, existing on in theory and in the decree of eternal laws, — of that law which, as far as we know, exists everywhere, which has created order out of chaos, and which holds the universe together.

That a law existed had been suspected long before the time of Newton; and it had even been conjectured 'that it varied inversely as the square of the distance.'

Note.—The distinct part of Newton's great discovery was not the motion of attraction, which had occurred to many of the ancients,—not the law which had been suggested by Kepler and Bouillard, but the *proof* that the mechanical deductions from this law of attraction did really represent observed phenomena.

The manner in which Newton established his theory of Universal Gravitation was as follows. He first considered the moon as a body falling,[1] in one sense, towards the earth,—that is, seeing if the rate at which it fell towards the earth agreed with the laws of falling bodies as propounded by Galileo. Newton first computed, from the known velocity of the moon in its orbit, and from the radius of that orbit, the distance



[1] What is meant by the moon falling towards the earth, is supposing the curved line A D C to represent the path of the moon from A to C. The point E being the situation of the earth, A that of the moon, then were it not for the attraction of the earth the moon would proceed along the straight line A B, and traverse A B in the same time that it would have taken to go from A to C along the curved line A D C under the earth's attraction; thus supposing A D C to be the distance the moon goes in one second of time, B C is the distance, so to speak, through which it has fallen in that period, due to the earth's attraction.

through which the moon actually fell towards the earth in one second of time. He next computed the distance through which a heavy body would fall towards the earth's surface, if removed to the distance of the moon from the earth's surface. Now if these two quantities were equal, then the truth of his demonstration (as far as the moon and earth are concerned) was complete, because, if so, the moon did fall through that distance required by the assumed law, and therefore this law (for earth and moon) was a law of Nature. This, after the greatest labour and calculation, he found to be the case. Having thus established his great theory in the case of the moon, he next proceeded to establish it for the planets and solar system generally. This he accomplished by means of certain propositions, demonstrated in his 'Principia,' and three famous laws with regard to the sun and planets, known as Kepler's Laws, from the name of the discoverer, who discovered them after years of continuous observations of the sun and planets, and laborious mathematical calculations.

Having thus *demonstrated* his law for the Solar System generally, Newton was led on to *infer* his grand theory of Universal Gravitation, which is as follows:—Every particle in the universe attracts every other particle with a force of attraction in the line joining them, proportional directly to their mass; and proportional inversely to the square of the distance between them.

Newton was only able to *demonstrate* his law with regard to the Solar System, but his *inference* modern science has fully confirmed, and has but infinitely increased our ideas of the marvellous wonders and powers of this mysterious law. The illustrious French mathematician Laplace has shown that the velocity of the action of this law, or of gravity, must be several, if not many, millions of times greater than the velocity of light, and the velocity of light is over 180,000 miles per second. And as its rate of propagation is infinitely great, so is the distance through which it acts. It annihilates both space and time. This mystery of the universal action of Gravity is the greatest of all modern or ancient scientific marvels; and the deeper we go into it the deeper grows the mystery. If Light, Electricity, etc., be but modifications of the action of Gravity, that renders nothing more simple, but only infinitely more wonderful. It is the mystery of mysteries, and seems to be almost in some sense associated with the great First Cause.

Attempting its unravelment :—

'*Charmed and compelled thou climb'st from height to height,*
And round thy path the world shines wondrous bright,
Time, Space, and Size, and Distance cease to be,
And every step is fresh Infinity.'[1]

[1] Goethe.

II.

RE-DISCOVERY OF THE ASTEROID CERES.

'Oh thou small fragment of a world once
Beautiful and bright and fair as this is, till
Wreck't in some convulsion. Oh float
Into our azure sky once more.'

THE next triumph I will relate is the re-discovery of the Asteroid Ceres after it had become lost in the rays of the sun, owing to its having been discovered by Piazzi in such a position that he was able, on account of illness, only to make a few observations of it prior to its being lost in the rays of the sun. What a hopeless task it seemed, its re-discovery, as the telescope would have to grope its way around the heavens, slowly and carefully, in that region known as the Ecliptic, comparing every star with its place in the chart or catalogue of stars. What was to be done? In this dilemma mathematical analysis attempted to create an orbit for this lost planet by means of the data afforded by the few observations which Piazzi had been able to make. What were these data? It had been observed during its passage over an arc of 4° out of 360°, approximately taken the orbit as circular. What an absurd attempt, do you say? You know not the powers of this wonderful analysis. The genius of the great mathematician Gauss, then quite young, succeeded in this herculean task, and when the telescope was pointed to the heavens in the exact spot indicated by this daring computer, there, in the field of view of the telescope, shone the delicate and beautiful light of the long-lost planet. This was indeed a wonderful triumph of analytical skill and reasoning, and another verification of the saying, 'That fiction can never be more wonderful or superior to truth;' the latter is indeed a source of inspiration to us, richer and more enduring than the former.

III.

THE DISCOVERY OF NEPTUNE.

'*Hence the view is profound,*
It floats between the world
And the depths of the sky.'—GOETHE.

THE astronomer M. Bouvard of Paris, in the year 1820, prepared tables by means of which the motions of three great planets, Jupiter, Saturn, and Uranus, might be predicted. The positions of the two planets, Jupiter and Saturn, were found to agree with those predicted, and their motions with the theory of Gravitation. But not so in the case of the planet Uranus. In a few years Uranus began to deviate from the places indicated, and in the year 1844 the error amounted to four minutes, or about one-seventh the diameter of the moon,—a very small quantity, from a non-astronomical point of view, but unable to be overlooked by this the exactest of the sciences. Nor could it have been overlooked if it had been an eighth part of that amount. Analogy suggested that these discrepancies must be due to the attraction of some *unknown* planet. This planet evidently could not be between Saturn and Uranus, for then Saturn would have been affected as well as Uranus. Thus the orbit of this unknown planet must be outside that of Uranus. Two *young* mathematicians, the one English, and the other French, whose names were respectively Adams and Le Verrier, *independently and quite unknown to each other*, undertook this apparently superhuman task of discovering the new planet, being given the perturbations of Uranus. And there is this difference between this task and the discovery of the Asteroid Ceres, in the case of Ceres the planet was known to exist, and had even been observed, although only for a very short time, but in the case of the discovery of the planet Neptune, no eye had ever beheld it, *i.e.*, as a planet, and about any of its elements or data nothing was of course known; these it was the business of the two mathematicians to discover, by means of Newton's theory of Gravitation, and Mathematical Analysis.

Mr Adams began his calculations and investigations in 1843, and in October 1845 he communicated the results of his calculations to the Astronomer-Royal, and to Professor Challis of Cambridge University in August 1846. Professor Challis found the planet, but, under pressure of other business, did not recognise it. In the meantime, M. Le Verrier, at the instigation of M. Arago, had investigated the problem, and communicated his results to the French Institute in November 1845, June 1846, and August 1846; and on 25th of September 1846, Dr Galle, assistant to Professor Encke, of Berlin, discovered the new planet, from a communication which he had received from Le Verrier. What is known as the Heliocentric position of the planet as found by—

Dr Galle was 326° 12′.

As compared by Mr Adams, 329° 19′.

As computed by M. Le Verrier, 326° 0′. Thus the real discovery of the planet was due to M. Le Verrier and Dr Galle, though this does not, of course, in any way detract from the fame and merit due to Mr Adams in the undertaking and so successfully solving so grand a problem. From the above we see that the computation of M. Le Verrier was rather more accurate than that of Mr Adams, but Mr Adams was of course quite correct enough for all practical purposes. And this double calculation must demonstrate that the *position* of the planet as assigned by the two computers was not one of mere chance, but that it was one determined by means of pre-eminent ability and skill, based on sound principles, and approached by an accurate and logical process. And in this respect these two great mathematicians are not rivals, but vindicators of each other, IV.

PREDICTIONS OF THE RETURN OF COMETS.

'*That mysterious visitant whose beauteous light*
Among the wandering stars so strangely gleams!
Like a proud banner in the train of night,
Th' emblazon'd flag of Deity it streams—
Infinity is written on thy beams:
And thought in vain would through the pathless sky
Explore thy secret course. Thy orbit seems
Too vast for Time to grasp. Oh, can that eye
Which numbers hosts like thee, this atom Earth descry?'

'Les Merveilles Célestes.'

Two thousand years ago Seneca wrote:—'A day will come when the course of these bodies (comets) will be known, and submitted to rules like those of the planets.' The prophecy of the philosopher has been fulfilled. Thought 'has explored their secret courses; their orbits are not too vast for man to grasp.' The comets, like the planets, obey Newton's law of Universal Gravitation, and are subject to all its varied influences. The first prediction of the return of a comet was made by an Englishman, viz., the illustrious Halley, which return he knew he himself would never be able to behold. This comet (Halley's) appeared in 1682, and Halley studied it with great care and attention; and after great labour he computed the elements of its orbit, and found it to be moving in an ellipse of great elongation, *i.e.*, greatly extended or flattened out, so to speak, and that it receded from the sun to a distance of 3,400,000,000 miles. And he predicted its return about the close of 1758 or the beginning of 1759. The first glimpse caught of it was by G. Pabtch, an amateur peasant astronomer, on December 25th, 1758, returned to crown with glory the English mathematician and astronomer who had predicted its return after an absence of seventy-six years.[1]

[1] The return of this comet was computed to within a period

The next return of this comet was computed to within nine days of its actual occurrence—a most remarkable calculation, since it never escapes from the attractive influence of the planet Neptune, even when at its furthest distance from the sun.

You may say, but all the 'Triumphs' which you have related, so far, are taken from the subject of Astronomy. Has Mathematics achieved no triumphs in any other department of science? It has, in every department of Physical Science. It has triumphs, no less nobly achieved, in the sciences of Electricity, Mechanics, Optics, and, in fact, in the whole range of Physical Science.[2] I will take one from

of nineteen days of its actual occurrence by two French mathematicians, Lalande and Clairvaut, assisted by Madame Lalande, they allowing themselves thirty days either way, on account of their neglecting small irregularities. The disturbing influence of Neptune was of course then unknown.

[2] *Vide* 'The Cambridge and Dublin Mathematical Journal,' 'The Philosophical Magazine,' etc.

the science of Optics, both recent and important, viz. :—

V.

THE DISCOVERY OF CONICAL REFRACTION.

'*First, mathematician's skill,*
And after, keen optician's gaze
Explored the doctrine *of those rays.*'

THE mathematician Fresnel had calculated the mathematical expression for the wave surface in crystals possessing two optic axes, but he did not seem to have any idea of refraction in such, except a refraction known as double refraction. Sir William Hamilton, of Dublin, the inventor of a mathematical method known as Quaternions, and a most profound mathematician, took the subject up at this point, and proved that the theory known as the Undulatory Theory of Light pointed to the conclusion that at four special points of the wave surface the ray of light was divided not into *two* but *into an infinite number of parts*, forming, therefore, at those four points, a continuous *conical envelope* or hollow cone, instead of two images, as had been hitherto supposed. No human eye had ever seen this conical envelope when Sir William Hamilton said it existed, any more than any eye had ever beheld the planet Neptune until Mr Adams and M. Le Verrier demonstrated its existence; both were previously ideas or theories in the minds of mathematicians. Dr Lloyd took a crystal of a mineral known as Arragonite, and following with scrupulous exactness the indications of Sir William Hamilton's theory, he discovered this wonderful envelope.[1]

You may say these triumphs are indeed most wonderful, almost incredible, and truly

[1] See 'Notes on Light,' by J. Tyndall, F.R.S.

prove 'the potent power of mind o'er matter;' but they, or, at any rate, some of them, scarcely appear to be of much practical use.

A few words, then, as to their uses. For this is the chief reason why I chose the above.

Newton's discovery of the law of Universal Gravitation remodelled and vastly improved the whole science of Mechanics; it gave us the true theory of the movements of the heavenly bodies, and became the parent of innumerable other discoveries.

Navigation, and, therefore, Commerce and Industry, immediately felt its influence, and every individual of our species has derived, and will continue to derive, as long as mankind exists, incalculable benefits therefrom, both intellectual and material.

The discovery of the planets Neptune and Ceres was a consummate verification of the law of Universal Gravitation, just as the discovery of Conical Refraction was a consummate verification of what is known as the Undulatory Theory of Light; these discoveries amounting to almost absolute proofs of two of the grandest and most useful theories ever propounded.

Comets were considered by the ancients, and in the Middle Ages, as objects of terror, —miraculous apparitions, forerunners of awful calamities, burning symbols of Divine wrath. But now, thanks to the labours of mathematicians and observers, these bodies (as we have seen) are regulated by and subject to the same laws as the planets. They have been robbed of their terrors, and are regarded by Schiaperelli and others as

[1] In Roman history there is a remarkable story of a Roman nobleman, an astronomer and mathematician, who, when he was serving against the Macedonians, under Julius Æmilius, *foretold* to the Roman soldiers an eclipse, and explained its causes, and thereby preventing the consternation they otherwise would have fallen into, and which, seizing their enemies, they were easily routed by the Romans.—*Guithric.*

analogous to meteors,—bodies fleeing from world to world, scattering in their course in the neighbourhood of the stellar systems the dust of the elements of which they are composed—carbon, and perhaps hydrogen; carbon, which is such an important factor in life, thus preserving perhaps life on the surface of those planets on which it falls.

And with regard to the practical use of certain other triumphs, it must be remembered that there are many great discoveries which, though they may appear at first sight of little use in themselves, have given birth to others of the greatest utility. When these results can be practically used in the increase of the mass of general knowledge and wealth, then is their use at once perceived. But of greater use (though perhaps unperceived) are those prior discoveries which led up to them; for without the first the second could have had no existence. By the existence and assistance of the higher branches of pure

Mathematics are those sciences to which they are applied rendered more powerful, perfect, and of greater service to man ; and by the aid of Electricity, Navigation, Engineering, Geodesy, etc., is commerce and industry vastly improved ; and thus is every individual incalculably benefited—indirectly, it may be, but none the less so on that account. And, moreover, these and other triumphs in every department of science 'have carried us to sublime generalisations—*have affected an imaginative race like poetic inspirations.* They have taught us to tread familiarly on giddy heights of thought, and to wont ourselves to daring conjectures.' And they have also revealed to us the infinities by which we are surrounded. They have taught us to see a system in every star, but they have

Note.—By the recent elaborate mathematical investigations of Sir William Thompson, Clerk Maxwell, etc., has the science of Electricity been entirely changed, and calculations can now be made with regard to electrical phenomena with as much certainty as calculations in dynamics.—See 'Electricity in the Service of Man.'

also taught us to behold a world in every atom, the one teaching us the insignificance of the world we tread on, the other redeeming it from every insignificance. Eager and ever curious, man presses onward for the accomplishment of further triumphs, the solution of grander problems. What we know is as nothing to what we know not. The solution of these problems is man's highest and noblest ambition; and there is no truer truth than that—

'*Nature when she adds difficulty adds brain.*'

EMERSON.

'It is an error to ascribe discoveries to Mathematics. It happens with this, as with a thousand other things, that the effect is confounded with the cause. Thus effects which have been ascribed to the steam-engine, belong properly to fire, to coals, or to the human mind. The true discoveries in Mathematics are successive steps towards the perfection of the instrument, by which it is rendered capable of innumerable useful applications, but Mathematics alone makes no discoveries in Nature.' —JUSTUS LIEBIG, 'Letters on Chemistry.'

CHAPTER IV.

THE LIMITS OF MATHEMATICS.

'Mathematical Science is the handmaid of Natural Philosophy.'—Bacon.

It is perfectly true that Mathematics of itself makes no discoveries in Nature, and that, besides Mathematics, a high degree of imagination, acuteness, and talent for observation are required to make discoveries in Physical Science. The imagination has always been a powerful factor in the discovery of Nature, and without observation we can know nothing. 'Observation of Nature is the only source of truth.' 'Experiment is invented observation.' It is the duty of the philosopher to explain and illustrate the facts of Nature by experiments. 'No single isolated phenomena, taken by itself, can furnish us with its own explanation; it is by tracing its consequences, by studying and arranging its antecedents and consequents, and well observing their several links, that we attain to a comprehension of it, and an understanding of its true cause. For we must never forget *that every phenomenon has its reason, every effect its cause.*'

And at this point Logic and Mathematics take up the subject, the one (Logic) to verify that which the imagination, with its far darting glance, has seen, for from the moment the imagination is allowed to solve questions left undecided by researches, investigation ceases, truth is unascertainable, and in error is created a MONSTER—envious, malignant, and obstinate—which, when at length truth endeavours to make its way, crosses its path, combats, and strives to annihilate it; and the other (Mathematics) to reduce the phenomena to mathematical laws for future use, and as a verification of his experiments; for if his calculations agree not with his experiments, then the conditions of an accurate and logical process are not satisfied, and no discovery has been made.

Thus Mathematics, though the *last*, is by no means the *least*, factor in the discovery. And it must be remembered that all the sagacity, acuteness, and talent in the world would be useless without the instrument. The instrument might be created, you say. Exactly what has been done, but this detracts not either from its power or use. A steam-engine is none the less important because it is only a steam-engine.[1]

An erroneous and rather curious idea often entertained with regard to the higher

[1] The order indicated above, of course, is not always followed. In the discovery of *Conical Refraction*, for instance, Mathematics came first, Observation afterwards. The above is the most unfavourable case, so to speak, Mathematics being not so much the discoverer as verifier, though one is useless without the other.

branches of Elementary Mathematics is that it is possible to prove all manner of incongruities by means of them,—that two is equal to four, and many such like absurdities. This, of course, is not so. This and other illustrations are simply examples of what absurd results may be arrived at if our data be incorrect, our reasoning false, or some element or factor left out in the calculations. It was proved by the Fluxionary Calculus that steamships could never get across the Atlantic. But in spite of the Fluxionary, or any other Calculus, this has been done. But this does not of necessity denote, as some would suppose, the falsity or weakness of the Fluxionary Calculus, but points to false data, incorrect reasoning, or unknown elements left therefore out of consideration. Another erroneous idea is that the results of the calculations of the higher Mathematics are only approximations, and not exact. With regard to this, the illustrious Carnot has said,—'The important—one may say the sublime—value of the Infinitesimal Analysis is in joining with the facility of the process of a simple approximate calculation, the exactitude of the results of ordinary analysis. . . . The objections made against it (the Infinitesimal Analysis) all rest on the *false* supposition that the errors committed in the course of the calculation, in neglecting the infinitely small quantities, remain in the results of that calculation, however small one may suppose them to be. But this is not so: the elimination takes them all away.' Thus we see, as every student knows, that the results of the higher Mathematics are not approximations but exact.

But the science of Mathematics deals with the physical and not with the ideal. It is an instrument constructed by man for the use of man, and of necessity, therefore, like all of man's creations, is far from perfect, and has its narrow limits. I believe I am right in saying that no mathematician is able to calculate the exact curve which a tossed penny describes, the influences to which it is subjected being so numerous and intermixed in such a complex manner with one another. And there are problems whose solution, even supposing the laws of the influences to which they are subjected being known, the computation of their aggregate effect appears to be beyond the powers of the Mathematical Analysis as it is or is ever likely to be. Thus this instrument, powerful as it is, has its limits.[1]

And it has also its limits in another way, namely, the limits of its applicability to the improvement of the other sciences. It is not difficult to conceive how chimerical would be the hope of applying mathematical *principles* to some of the complex inquiries of such subjects as physiology, society, govern-

[1] See Comte's 'Positive Philosophy,' vol. iii.

ment, etc. But the failure of the science even here is only partial. Its *principles* may fail but its *methods* be still applicable. 'The value of mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, etc.) consists in the applicability, not of its doctrines, but of its methods.

'Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of Mathematics to the simpler branches of physics furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex.

'These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding, with Plato, one who is *ageometrê-*

tos as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy.'[1]

[1] J. S. Mill, 'Logic,' vol. ii. p. 180.

‘*Beauty chased he everywhere*
In flame, in storm, in clouds of air.
He smote the lake to feed his eye
With the beryl beam of the broken wave;
He flung in pebbles well to hear
The moment's music which they gave.
Oft pealed for him a lofty tone
From nodding pole and belting zone.
He heard a voice none else could hear
From centred and from errant sphere.
The quaking earth did quake in rhyme,
Seas ebbed and flowed in epic chime.’—EMERSON.

CHAPTER V.

THE BEAUTY OF MATHEMATICS.

'There can be no Beauty where Chaos reigns.'

'BEAUTY,' said a philosopher, 'possesses that which is simple; which has no superfluous parts; which exactly answers its end; which stands related to all things; which is the mean of many extremes.'

This definition applies equally well to the science of Mathematics. In Mathematics all our knowledge is in the number of primitive data or conclusions which can be drawn therefrom. Around these first principles as around a standard everything associates; no matter how remote it may be, it unites itself to that to which it has been well attached. Every theorem in each department of the science has a kindred connection with every other theorem. What has been so exquisitely sung of the associations of childhood, is true (altering the connection only) of the associations of the different departments, etc., of Mathematics. For in Mathematics—

> '*Up springs at every step, to claim a place,*
> *Some little* Axiom *making sure the 'pace';*
> *And not a* Problem—*but what truly teems*
> *With golden visions and romantic dreams.*'

The language of Mathematics has its own peculiar beauty. Symbolical as it be, it is also symbolical in another sense.

As the artist has to employ symbols to give us either the spirit or the splendour of Nature, and to convey his enlarged sense to his fellow-men, and to open men's eyes to the mysteries of eternal art, so the mathematician is compelled to use symbols to clothe *his* art in a language enabling him to best convey *his* enlarged sense with brevity and clearness to others, and to enable him also to open men's eyes to the powers of the human mind, and to grapple so successfully with those problems pressing for solution on every hand. It is the wonderful simplicity, power, and utility of these symbols which constitute their beauty.

The beauty of Mathematics is not that of a pageant or ballet. There are many beauties,—moral beauty, beauty of manners, of the human face and form, of the intellect, of Nature, and of that which enables us to understand Nature, and discover her laws. It is this latter beauty that the science of Mathematics possesses.

There is an ascending scale of the perception of beauty, from the joy which some grand spectacle affords the eye, to the perception that symmetry of any form is beauty up to perception of the Goethe, that the beautiful is but a manifestation of the secret laws of Nature, which, but for this appearance, had been for ever concealed from us.

'I do not wonder,' says Emerson, 'that Newton, with an attention habitually engaged on the paths of planets and suns, should wonder what the Earl of Pembroke found to admire in stone dolls.'

Each department of Mathematics has its own peculiar beauty. The especial beauty of *Geometry* consists in its being the most perfect type of the Deductive Method in general. In Geometry, too, there is no difference of style; in a geometrical demonstration we are unable to distinguish by internal evidence whether it is Euclid's, Archimedes', or Apollonius'. In this severe necessity of form, Geometry is unique.

In Geometry, each link of the chain of reasoning hangs to the preceding, without any insecurity in the whole. In Geometry, we tread every step of the ground ourselves, at every step feeling ourselves firm, directing our steps to the required end.

The beauty of *Analysis*, of the analytical method, is exactly the opposite. Here we no longer tread the ground ourselves,—we are carried along as it were in a railway train, entering in at one station and coming out at the other without having any choice in our progress in the intermediate space. In geometrical reasoning we reason concerning things as they are; in analysis, the contrary is the case. The analyst represents everything—lines, angles, forces, mass, etc.—by letters of the alphabet. All curves are represented by what are known as co-ordinates. His reasonings are merely operations upon symbols. He obtains his required results equally well if he has forgotten, or even does not know, what he is reasoning about.

This, of course, arises from the perfection of the analysis,—from the entire generality of its symbols and its rules. It is not possible, in any other subject than analytical Mathematics, to do this; that is, to express things by symbols, once for all, and then go on with our reasonings, forgetting all their peculiarities. Any attempt to do this (for such attempts have not been wanting) lead to the most extravagant and inapplicable conclusions.

That department of Mathematics known as the conic sections, has an especial beauty, from the fact that those curves with which it is concerned are the curves in which the planets, comets, etc., move around the Sun, and in which, also, the moons or satellites move around their planets.

But as the principal use of the science of Mathematics lies in its applications to the Sciences and Arts, so there, as I have said, also lies its chief beauty. The pleasure which a temple, or a palace, or a bridge gives the eye, is that an order and method has been communicated to stone and iron, so that they speak and geometrize, becoming tender or sublime with expression. What in a great measure gives to Architecture those beautiful curves and angles which we admire so much?—what but the labours and discoveries of Geometers? What figure is more often repeated than that of the circle?—that curve which meets the eye often enough as we go about our daily task. It is brought before us in the wheels of every vehicle we meet; we behold it in the plates and dishes from which we eat, in the cups and glasses from which we drink. It is the most beautiful, the most perfect, the most useful, and yet the simplest of all curves or forms. Under the head of a snake holding its tail in its mouth, the ancients adopted it as the emblem of eternity, which has no beginning, and which has no end. It is, as a writer has said,—'The highest emblem in the cipher of the world. Throughout Nature this primary figure is repeated without end. We are all our lifetime reading the copiousness of this first of forms.' St Augustine defined the nature of God as a circle whose centre was everywhere, and circumference nowhere. It has certainly something Divine about it, being without beginning and without end, perfect in form, in beauty, and in power. Surely, then, a science which will unfold its manifold uses and beauties, is not without use and beauty too.

Architecture and Geometry have always been intimately connected, and, indeed, it is probable that to Geometry Architecture owes its origin and rise. For there is a manifest and oftentimes perfect resemblance between the tombs and temples of the ancients and the forms and figures of Geometry. And, moreover, the principles, at any rate, of Geometry must have been known before Architecture was possible.

Mathematics and Music have also from the earliest times been closely united. For Euclid, besides his famous 'Elements,' was also the author of two books entitled 'The Divisions of the Scale,' and 'An Introduction to Harmony,' at that time held to be a part of Mathematics. Pythagoras and Plato were also writers on Music.

And the eminent mathematicians, Descartes, Euler, and D'Alembert, were writers on the subjects of Harmony and Counterpoint. Of them D'Alembert said,—'It is solely by closely observing facts, by reconciling one with the other, *and by making them all*, if possible, depend upon some single fact, or at most upon a very few principal ones, that they can succeed in giving to Music a correct, lucid, and unexceptionable theory.'

The illustrious astronomer Sir William Herschel was originally a poor organist. Desiring to study the theory of Music, he applied himself to the perusal of a treatise on Harmony. Finding that for a complete comprehension of the work he required some knowledge of Mathematics, he applied himself to this new study, and having mastered

Geometry and Algebra, the science so fascinated him that it came to occupy the first place in his mind. And he often, after a fatiguing day's work of *fourteen* or *sixteen* hours with pupils, repaired for *recreation* to what many would deem these severer exercises. He would, I think, hardly have done this had he not perceived Beauty as well as Use in the science.

In Painting, too—a subject in which all educated persons feel a lively interest—where would those grand effects of distance, of solidity, etc., be, without Perspective?—a branch of Geometry and Optics.

Thus, not with science alone, but also with the Fine Arts, is Mathematics intimately connected. Of the intrinsic beauties which the science of Mathematics possesses, I have made mention. But, as I have said, its chief beauty is owing to its being *a* sometimes *the* most powerful factor in the discovery of a new Law of Physics, and to its intimate connection with the Fine Arts. For that which is associated with, and is an aid to, what is beautiful, cannot fail in itself to possess beauty too. For whatever sympathises, is of precisely the same nature as *that with which it sympathises.*

It is the office of Art to embellish, to beautify, wherever or whenever opportunity offers.

Of Mathematics, therefore, as applied to and part of the Fine Arts, as well as to the demonstration of the existence everywhere in Nature of fixed, eternal, and immutable Laws, may it not be said,—

'*Beauty chased he everywhere.*'

'In our lecture-room we teach the letters of the alphabet; in our laboratory their use. . . . . As soon as these signs, letters, and words have become formed into an intellectual language, there is no longer any danger of their being lost, or obliterated from his mind. With a knowledge of this language he may explore unknown regions, gather information, and make discoveries wherever its signs are current. This language enables him to understand the manners, customs, and wants prevailing in those regions. He may, indeed, without this knowledge, cross the frontiers of the known, and pass into the unknown territory, but he exposes himself to innumerable misunderstandings and errors. He asks for bread and he receives a stone.'—JUSTUS LIEBIG, 'Letters on Chemistry.'

'It may be laid down as a general rule for *Electrical* students, that he who has not a quantitative (*i.e.* mathematical) knowledge of the principles of Electrical Science will only waste his time in making original experiments.' —JOHN PERRY, 'Electricity in the Service of Man.'

CHAPTER VI.

THE ATTRACTIONS OF MATHEMATICS.

'*Nature has made it delightful to man to know, disquieting to him to know imperfectly, while anything remains in his power that can make his knowledge more accurate or comprehensive.*'

Dr Thomas Brown.

One of the principal attractions of the Science of Mathematics lies in the fact of its explaining the *why* and *wherefore* of so much, and in rendering us capable of reading and understanding ourselves many great discoveries which we must remain ignorant of or take for granted.

The celebrated Locke,[1] who was incapable of understanding the 'Principia' from his want of mathematical knowledge,

[1] Life of Newton.

inquired of Huygens if all the mathematical propositions in that work were true. When he was assured that he might depend upon their certainty, he took them for granted, and carefully examined the reasonings and corollaries deduced from them. In this way he acquired a knowledge of the *physical* truths of the 'Principia,' and became a firm believer in the discoveries it contained. Of his annoyance experienced through his want of mathematical knowledge, or the pleasure he would have experienced from a *complete* perusal of the 'Principia,' I need make no mention.

The student of geometrical drawing knows the *how* of his subject, but (if he be unacquainted with the elements of geometry) the *why* is hidden from him. The practical part of his subject is known to him, the theoretical is not. But it is by theory that practical men are rendered of service to the world. In the science of Mathematics also, no one is able to pass through a course of Mathematics in a few sentences, and but few facilities are given for acquiring that superficial acquaintance with facts which enable their possessor to shine in society, without really enriching his understanding. In Mathematics, the very comprehension of the theorems to be demonstrated and problems to be solved implies an exercise of the faculties analogous to that by means of which every step of the demonstration is successfully traced out. And even genius can make no *royal* road to learning here. The greatest genius, perhaps, that ever lived, viz., Sir Isaac Newton, it is true, assumed the propositions proved in Euclid's 'Elements of Geometry' as self-evident truths; but in a letter to a friend he regretted that he had not studied the writings of Euclid with that thoroughness and attention which so excellent a writer deserved, before passing on to the works of Descartes and other algebraic writers.

The achievements and triumphs of Mathematics are essentially those of patient industry and study, even when treated with the most masterly skill. This may not seem an attraction at all to many, in fact, rather the reverse, but to a man possessed of fair ability, and of sufficient determination to enable him to stick to his work, this is a great advantage. Of course, the brilliant, if he be a worker, will always surpass the non-brilliant in whatever branches of Art, Science, or Business he may happen to be placed; but still, I think, brilliancy counts for less in this subject than in any other.

But the great secret of success in everything is *not* luck but work, and that faculty known as 'common sense.'

'Common sense,' said Guizot, 'is the genius of humanity.' Common sense is certainly the genius of Mathematics.

'I was informed,' said the celebrated mathematician Stone, 'that there was a science called Arithmetic. I purchased a book on Arithmetic, and learnt it. I was told there was another science called Geometry. I bought the books, and learnt Geometry. By reading, I found that there were good books on these two sciences in Latin. I bought a dictionary and learnt Latin. I understood that there were good books of the same kind in French. I bought a dictionary and learnt French, this, my lord, is what I have done. It seems to me that we may learn everything when we know the letters of the alphabet.'

The Science of Mathematics enables us to draw correct logical conclusions according to definite rules; it teaches us a peculiar language which, by the aid of signs and symbols, allows us to express such conclusions in the simplest manner possible, intelligent to every one of those who understands the language. Before we can comprehend the results, we must learn the language, and this is the part which is *not* attractive. But having learnt this language, the reward for our labour is most ample. We are enabled to become acquainted with discoveries and truths formerly obscure and unknown to us, or we may be enabled to make some original investigations.

For discoveries are *not* (as we are so often informed) the result of accident; the discovery of the aberration of light by Bradley was not the result of accident, nor of the orbits of the planets, nor of Universal Gravitation. 'Malus did *not*, by turning round and looking through a prism of calcareous spar, *accidentally* discover the polarisation of light by reflection, but by considering the position of the prism and the window; he repeated the experiment often, and by virtue of the eminently distinct conceptions of space which he possessed, he was able to resolve the phenomenon into its geometrical conditions.'[1]

Facts (no matter how noticed by the observer) can only become a part of exact knowledge when the discoverer's mind be already provided with precise and suitable conceptions by means of which he may analyse and connect them.

The fact that a beam of sunlight on passing through a prism throws upon the opposite wall a spectrum of different colours, has been noticed by hundreds without their ever inferring that which has helped to make the name of Sir Isaac Newton immortal.

Accidents are the theme of the spiritualist, not of the arithmetician. The most casual and extraordinary event—the data being large enough—is a matter of fixed calculation. 'Everything which pertains to the human species,' said Quetellet, 'con-

[1] See Dr Whewell, 'Phil. of Induct. Sciences,' vol. ii. 190-1.

sidered as a whole, belongs to the order of physical facts.'

The greatest 'attraction' of the science of Mathematics lies, of course, in its applications and uses, and its aid to science; and it is solely from a want of mathematical knowledge that a great number of people are deterred from the study of many scientific subjects. But the days have passed when the acquirement of knowledge is a matter of indifference to the general public. Is it not now rather a matter of universal emulation? And the knowledge of Nature is by no means exhausted. Many great discoveries yet await the student of Physical Science and Mathematics. And let it be remembered that any one who can discover any one new fact in any science, or assist others to do so, has thereby rendered the life of man more glad, and more productive of benefit and of good to others, than it has hitherto been in this world of ours.

'Rhyme soars and refines with the growth of the mind. The boy liked the drum, the people liked an overpowering Jew's-harp tune. Later they like to transfer that rhyme to life, and to detect a melody as prompt and perfect in their daily affairs. . . . By-and-by, when they apprehend real rhymes, namely, the correspondence of parts in nature—acid and alkali, body and mind, man and maid, character and history, action and reaction—they no longer value rattles and ding-dongs, or barbaric word-jingle. Astronomy, Botany, Chemistry, Hydraulics, and the elemental forces have their own periods and returns, their own grand strains of harmony not less exact. . . . . They furnish the poet with grander pairs and alternations, and will require an equal expansion in his metres.'—R. W. EMERSON.

CHAPTER VII.

THE POETRY OF MATHEMATICS.

'Poetry is the record of the best and happiest of moments of the best and happiest of minds.'—SHELLEY.

MATHEMATICS is not without poetry. This statement may be new to some of my readers. Of Mathematics as a cure for mind-wandering; of Mathematics as the most perfect type of the deductive method; of Mathematics as a great auxiliary to science, we have not unfrequently heard; but of Mathematics as a subject possessing poetry, I think little indeed has been said.

The poetry which the Science of Mathematics possesses may be a poetry quite its own, but which, I maintain, is poetry all the same. For what is Poetry? Not words, nor yet rhymes, for verse faultless in form may be utterly destitute of true poetry. Poetry is

> '*No smooth array of phrase,*
> *Artfully sought and ordered though it be,*
> *Which the cold rhymer lays*
> *Upon his page with languid industry.*'[1]

> '*But high and noble matter, such as flies*
> *From brain entranced, and filled with ecstasies.*'[2]

It is not the ear which tells us what is poetry and what is not, it is our innate feeling of truth and beauty. If, then, poetic genius can exist independent and in spite of phraseology, then many whom we have not been accustomed to call poets must be reckoned such; and much which we have hitherto regarded not only as not poetry, but as, perhaps, its very opposite be so called in the highest sense of the word.

Euclid's 'Elements of Geometry' is a

[1] Bryant. [2] Emerson.

book of poetry, one of the grandest the ancients have left us. In the simplicity of its first principles, the clearness and beauty of its demonstrations, the wonderful and regular concentration of its different parts, and the universality of its applications, it possesses a power and beauty such as no other subject can boast of. In most branches of Art and Science the moderns have far surpassed the ancients, but, after a lapse of more than two thousand years, this great composition of the ancients still maintains it original pre-eminence and grandeur, and has acquired additional celebrity from the fruitless attempts which have been made to create such another work. Does the 'Principia' of Newton possess no poetry? It was of this book that the illustrious Halley said, 'So near the gods, man cannot nearer go;' and Laplace, 'placed it

Note.—To found a superior system of Geometry upon and by the aid of Euclid is not to create such another work.

above all other productions of the human intellect.' And the great American philosopher Emerson says, 'Newton may be permitted to call Terence a playbook, and to wonder at the frivolous taste for rhymers; he only predicts, one would say, a grander poetry; he only shows that he is not yet reached—that the poetry which satisfies more youthful souls is not such to a mind like his, accustomed to grander harmonies; this being a child's whistle to his ear; that the music must rise to a loftier strain, up to Handel, up to Beethoven, up to the thorough bass of the sea-shore, up to the largeness of Astronomy.'

The 'Mécanique Céleste' of Laplace, and the 'Mécanique Analytique' of Lagrange, are grand volumes of poetry, for there *is* poetry in a mathematical demonstration when it is the emblem of some great difficulty solved, or some wonderful result simply arrived at. There is an ascending scale of poetry, from the poetry of Words to the poetry of Actions, and from the poetry of Actions to the poetry of Actions again—only not man's but Nature's.

'Presented rightly to the mind' (says Professor Tyndall), 'the discoveries and generalisations of modern science constitute a poem more sublime than has ever yet addressed the human imagination. The *natural philosopher* of to-day may dwell amid conceptions which beggar those of Milton. Look at the integrated energies of our world,—the stored power of our coal-fields; our winds and rivers; our fleets, armies, and guns. What are they? They are all generated by a portion of the sun's energy, which does not amount to $\frac{1}{2,300,000,000}$ of the whole. This is the entire fraction of the sun's force intercepted by the earth, and we convert but a small fraction of this fraction into mechanical energy. Multiplying all our powers by millions of millions, we do not reach the sun's expenditure. And still, notwithstanding this enormous drain, in the lapse of human history we are unable to detect a diminution of his store. Measured by our largest terrestrial standards, such a reservoir of power is infinite; but it is our privilege to rise above these standards, and to regard the sun himself as a speck in infinite extension,—a mere drop in the universal sea, We analyse the space in which he is immersed, and which is the vehicle of his power. We pass to other systems and other suns, each pouring forth energy like our own, but still without infringement of the law, which reveals immutability in the midst of change, —which recognises incessant transference or conversion, but neither final gain nor loss. This law generalises the aphorism of Solomon, that there is nothing new under the sun, by teaching us to detect everywhere, under its infinite variety of appearances, the same primeval force. The energy of Nature is a constant quantity, and the utmost man can do in the pursuit of physical truth, or in applications of physical knowledge, is to shift the constituents of the never-varying total, sacrificing one if he would produce another. The law of conservation rigidly excludes both creation and annihilation. Waves may change to ripples, and ripples to waves; magnitude may be substituted for number, and number for magnitude; asteroids may aggregate to suns, suns may invest their energy in florae and faunae; and florae and faunae may melt in air—the flux of power is eternally the same. It rolls in music through the ages, whilst the manifestation of physical life, as well as the display of physical phenomena, are but the modulation of its rhythm.' [1]

[1] 'Heat a Mode of Motion,' pp. 502-503.

It has been said 'Science does not know its debt to Imagination,' but (after the above passage) who will deny that the converse also holds—'Imagination does not know its debt to Science?' It has been very wisely said that 'the test of the poet is the power to take the passing day, with its news, its cares, its fears, as he shares them, and hold it up to a divine reason, till he sees it to have a purpose and beauty, and to be related to astronomy and history, and the eternal order of the world.' So it is, or will be, the test of the mathematician to take his Geometry and Calculus, with its uses, its beauties, and its triumphs, *as he shares them*, and beholding therein both Truth and Beauty, show its relation with every-day life, and bring it down to the minds and comprehension of the teeming millions.

If Mathematics be unpoetical it is false, and if poetry be illogical it is unreal; for the mathematician must not be devoid of poetic feeling, and the poet must be a true logician. 'Dante was free imagination, all wings, yet he wrote like Euclid.' Euclid had no wings, was all restrictions, yet he wrote like Dante.

'We think that,' said Macaulay, 'as civilisation advances poetry almost necessarily declines.' I think not. I think we shall have a grander poetry, with mightier strains of harmony, with loftier modulations, with mightier rhythms. The greatest of poets has said :—

> '*As the imagination bodies forth*
> *The forms of things unknown, the poet's pen*

Note.—By mathematician I do not mean a calculating machine,—that is, a man who, favoured by a good memory, may have rendered himself intimately acquainted with every theorem of mathematics, but who is totally unable to propose a problem for solution. When he possesses the capacity and talent of proposing a question to himself, and testing the truth of his calculations by experiment, he becomes qualified to investigate Nature. For from whence should he derive his problems if not from Nature?

Turns them to shapes, and gives to airy nothing
A local habitation and a name.'

But the scientific conceptions of to-day surpass what the most daring of imaginations said but yesterday. 'The Comets,' said Kepler (speaking metaphorically), 'are as numerous in the sky as the fish in the ocean.' But extending the calculations of M. Arago, from the planet Neptune to the furthest limit of the sun's attractive action we arrive at the appalling minimum number of 74,000,000,000,000,000 of comets, that for one of their periods at least are subject to the empire of the sun.[1]

I close this chapter with an anecdote concerning the celebrated astronomer and mathematician Euler, as related by Arago to the Chambre des Députés, at a meeting on the 23d March 1837. I quote in full, illustrating so well as it does this portion of

[1] 'Les Comètes,' pp. 120-122.

my subject; it is an anecdote deserving to be far more widely known than it is.

'Euler, the great Euler, was very pious; one of his friends, a minister of one of the Berlin churches, came to him one day and said, "Religion is lost; faith has no longer any basis; the heart is no longer moved, even by the sight of beauties, and the wonders of Creation. Can you believe it? I have represented this Creation as everything that is beautiful, poetical, and wonderful; I have quoted ancient philosophers, and the Bible itself: half the audience did not listen to me, the other half went to sleep or left the church." "Make the experiment which truth points out to you," replied Euler. "Instead of giving the description of the world from the Greek philosophers or the Bible, take the astronomical world, unveil the world such as astronomical (*i.e.*, physical and mathematical) research constitute it. In the sermon which has been so little attended to, you have probably, according to Anaxagoras, made the sun equal to Peloponnesus. Very well! Say to your audience that, according to exact, incontestable (mathematical) measurements, our sun is 1,200,000 times larger than the earth. You have, doubtless, spoken of the fixed crystal heavens; say that they do not exist,—that comets break through them. In your explanation, planets were only distinguished from stars by movement; tell them they are worlds,—that Jupiter is 1400 times larger than the earth, and Saturn 900 times so; describe the wonders of the ring; speak of the multiple moons of these distant worlds. Arriving at the stars, their distances, do not state miles—the numbers will be too great, they will not appreciate them; take as a scale the velocity of light; say that it travels about 186,000 miles per second; afterwards add there is no star whose light reaches us under three years,—that there are some of them with respect to which no special means of observation has been used, and whose light does not reach us under thirty years. On passing from certain results to those which have only a great probability, show that, according to all appearance, certain stars would be visible several of millions of years after having been destroyed, for the light emitted by them takes many millions of years to traverse the space which separates them from the earth."

'This advice was followed; *instead of the world of fable, the minister preached the world of science.* Euler awaited the coming of his friend after the sermon with impatience. He arrived despondent, gloomy, and in a manner appearing to indicate despair. The geometer, very much astonished, cried out, "What has happened?" "Ah, Monsieur Euler," replied the minister, "I am very unhappy: they have forgotten the respect which they owed to the sacred temple, *they have applauded me*."'[1]

Of a truth there is a thousand times more poetry in the reality than in the fable :—

> '*For the world was built in order,*
> *And the atoms march in tune ;*
> *Rhyme, the pipe, and Time, the warder,*
> *Cannot forget the sun and moon.*' [2]

[1] See 'Les Merveilles Célestes.'

[2] Emerson.

'*If but one hero knew it*
The world would blush in flame;
The sage, till he hit the secret,
Would hang his head for shame.
But our brothers have not read it,
Not one has found the key;
And henceforth we are comforted,—
We are but such as they.'—R. W. EMERSON.

'Among all men, though all men be unfit (to penetrate within the temple wherein the Divine Mystery is enshrined), none can nearer attain fitness to approach the temple than those who contemplate the mysteries of Infinite Time and Infinite Space, of Infinite Might and Infinite Life, all ruled by Infinite and Eternal Law. They alone perceive what marvels of knowable truth lie within the infinite domain of the Unknowable.'

Knowledge, 'Science and Religion.'

CHAPTER VIII.

METAPHYSICAL OR SPIRITUALISTIC MATHEMATICS.

'*Never be deceived by words. Always try to penetrate to realities.*'—W. J. Fox.

'*No difficulty is unsurmountable if words be allowed to pass without meaning.*'—Lord Kames.

In writing about the uses of Mathematics, I made some mention of the difference between the ancient and modern ideas on that part of my subject, — how some of the ancients valued it, not for its practical or applied uses, but only as habituating the mind to the contemplation of truth, and raising man above the material universe, and in leading him to the knowledge of the essential, the eternal, the abstract truth,—as disciplining the mind (not in the sense now usually understood), and not as ministering to the base wants of the body.

But there are some now-a-days who, thinking they are creating a new era of thought (when in reality they are only going back some two or three thousand years; it is the old spirit under a new form), have resumed these old supposed uses of the science, and have been good enough to bestow on us, amongst other things, a 'Fourth Dimension.'

Of their manner of doing this I give here two instances, more being superfluous. The general character of their attempts is by juggling with the symbols of the pangeometers,—by appealing to the metageo—metrical vagaries of Lobatschewsky, Rieman, etc.—and by using very long words, and making up very learned-looking sentences, which are just as sensible very often read backwards as forwards. Why will the public be so taken in by words!

words! words! and sentences which they (or anyone else, as far as that goes) are totally unable to understand? one of these fourth dimensionists *actually assuming the identity of an algebraic multiple with a spatial magnitude!*

Mathematicians employ algebraic quantities of the first, second, and third degree to denote geometrical magnitudes of one, two, and three dimensions respectively, and the fourth dimensionists say (or must say, if they consistently adhere to their principles) therefore there must be a geometrical magnitude of a fourth dimension corresponding to an algebraic quantity of the fourth degree. But if this be so, there must be by analogy or common sense a geometrical magnitude of the fourth, fifth, sixth, . . . nth dimension corresponding to an algebraic quantity of the fourth, fifth, sixth, . . . nth degree, where n may be any number, ten or ten millions, or ten thousand billion millions, or any other number you like to write. For this is really all the evidence we possess of the fourth dimension, holding, you see, thus just as good for fifth, sixth, or sixth billionth millionth dimensional space; which is indeed a land of mist and shadows, a bourne from which no traveller hath, or is ever likely, to return. This is, nevertheless, one of the means whereby some would now attempt to 'demonstrate the existence of another world.' If it did really require such bolstering up as this, it would indeed be in a perilous state.

Another way is as follows:—It requires a certain amount of conceivability. But no matter.—It supposes you to imagine 'a direction which is *at one and the same time* perpendicular to what we know as height, breadth, and length,—that is, perpendicular to the sides of a box, and yet only in one direction. If it be possible to realise this, further illustration is valueless; geometrical four dimensional space is already understood, but, if not, further illustration is useless.' Quite so. If any of my readers possess the requisite imagination, he or she then understands four dimensional space. For my own part, I can only say *I* do not. But with regard to this I just wish to point out one thing:

If it be possible to conceive a line or direction at one and the same time perpendicular to the sides of a rectangular box, and yet only in one direction, it is just as possible that 'two straight lines should enclose a space,'—that 'the whole should be less than its part,' etc., etc.; the whole of Euclid falls to the ground, Science becomes of no value, and Chaos is once more triumphant in the world,—a world maybe—

> '*Where nothing is, and all things seem,*
> *And we are shadows of a dream.*'

Why have I inserted the above, does the reader ask? Not with any intention of being funny, nor yet as so much padding in order to fill up a certain number of pages, but because this is an age of empiricism; they flourish like the green bay tree, about which we have heard so much. But their days are numbered, for nothing can endure but what is genuine. They may be—nay, many we know are—self-delusionists, but the misfortune is that they not only delude themselves, but delude or attempt to delude hundreds or maybe thousands of others. Metaphysical Mathematics has always been a subject attended with danger and difficulty, and loss of both time and labour. Beware of those who inform you that you must neglect, must get beyond the vulgar uses of Mathematics, and attain to a science which is as independent of the actual subject considered as geometrical truth is independent of an ill-drawn diagram. Had Euclid, instead of explaining and demonstrating the properties of lines and curves, called upon men to reverence the mystery of Mathematics, we should have had a creed of Geometry, in the place of a Science, and our architects believing in tunnels and bridges instead of building them.

Metaphysics is a science too often resembling the ox of Prometheus, a sleek, well-shaped hide, stuffed with rubbish, goodly to look at, but containing nothing to eat. And does Philosophy possess that supreme pre-eminence generally assigned it? By its study we obtain a knowledge of the intellectual world, the laws of thought, of mental inquiry, and of the spiritual nature of man; but, nevertheless, philosophy has not been able to prevent people from being burnt for witchcraft, for when the illustrious Kepler went to Tubingen to save his mother from the stake, he succeeded only by proving that she possessed none of the characteristic signs essential to a witch. And the discovery by Descartes of algebraic geometry is, to my mind, a greater discovery than his *cogito ergo sum* (I think, therefore I am), and of more beneficial use to mankind at large. But then there is philosphy *and* philosophy.

But to return from this digression. The ancient philosopher Socrates, when speaking of the ancient metaphysical speculators, etc., of his day, demanded of such inquirers whether they had attained a perfect knowledge of human things since they searched into heavenly things, '*or if they could think themselves wise in neglecting that which concerned them, to employ themselves in that which was above their capacity to understand.*'

The spiritualists of to-day would do well to bear these words in mind, for they are full of wisdom; and wiser, perhaps, are the words of Emerson: 'Let us know what we know for certain. What we have, let it be solid, seasonable, and our own. A world in the hand is worth two in the bush. Let us have to do with real men and women, and not with skipping ghosts.'

There are problems in Metaphysics and Mathematics which may be demonstrated to be insolvable. To describe the limit of the human power with respect to these problems is not yet possible. Nevertheless the capacities of our understanding will probably one day be well considered, and the line drawn between what is and what is not comprehensible by us.

That such a thing as some Psychic Force exists, whose laws scientists are at present unable to explain, is both possible and conceivable. But we must remember that

Attraction or Gravity[1] was known to exist hundreds, nay thousands, of years before mankind was able to discover the law that it obeyed, so we should await the coming of that second Newton to explain this Psychic Force (if it exist), and not assign a supernatural cause to what science is certain in time to explain.

And be not deceived by the wonders that the spiritualists say man is able to perform, aided by this Psychic Force. These marvels are paltry as compared with those which man has been able to achieve by the assistance of Science. Of some of the wonders you have read in Chap. III., and there are others no less wonderful.—Man, *aided by Science*, can tell

[1] '*Gravity* is often incorrectly spoken of as being a Force. Gravity is a name for the general fact that any two material bodies, if free to move, approach each other with a gradually increasing swiftness; Force is the name which we give to the unknown cause of this fact.'—PROFESSOR HUXLEY.

It is now possible (by means of the spectroscope) to detect the presence of $\frac{1}{180000000}$th part of a grain of salt.

you the exact number of waves of light emitted from the sun per second, and their exact length; he can tell you what a star, billions of miles out there in space, is made of; he has surpassed the old miracles of Mythology, flying across the sea, and sending his messages under it; the artist (aided by Science) can display to your astonished gaze true and realistic pictures of what this earth was like ten thousand or ten hundred thousand or ten millions of years ago. Man is to-day able to speak, and his descendants, whose grandparents are yet unborn, shall hear his voice. The man of science is able to make a jet of gas twenty feet distant from him sing, and to continue its song for hours, loud enough to be heard by an assembly of a thousand people. The comparative anatomist, possessing but a small fragment of a bone, a tooth, is able to relate the whole history of this being belonging to a past world, describe its size and shape, point out the medium in which it lived and breathed, and demonstrate whether its nourishment consisted of animal or vegetable food, and its organs of motion. And the chemist is able, knowing the proportion in which any single substance unites with another substance, to assign the exact proportion in which the former will unite with all other bodies whatever. Such are a few of the marvels of Science.

From men who resort to pan-geometry and logomachy to prove the existence of another world, what may not be expected? what can we hope? The man of Science has achieved these triumphs, and may safely assert all of them as realities, because he has acquired a knowledge of natural phenomena, and an intimate acquaintance with natural laws, and because everything being subject to definite Laws, when these Laws are known, the rest follows from them.

Many of these triumphs we are, of course, able to verify for ourselves.

The difference between the *absolute* knowledge of the man of Science and the empiric or spiritualist is well illustrated in that anecdote of Socrates: 'Men call me wise. Certainly I know little; I will inquire.' He then questioned many people, and indeed found that they knew little, but that they thought that little much. 'In fact,' said Socrates, at length, 'though I know as little, yet in one sense I *am* their superior; I know how little that little is, whereas they are ignorant how ignorant they are.'

It is quite true that the man of Science (in one sense) does not know

> '*How the chemic atoms play,*
> *Pole to pole, and what they say.*'
>
> EMERSON.

Nor

> '. . . *What wove yon woodbird's nest,*
> *Of leaves and feathers from her breast?*

> *Or how the fish outbuilt her shell,*
> *Painting with morn each annual cell?*
> *Or how the sacred pine-tree adds*
> *To her old leaves new myriads?'*
>
> EMERSON.

Nor does the spiritualist, though he pretends to have solved the mystery of life. He knows no more than his fellows, only less, because his fellows know that they do not know; they also know that *he* does not know, whereas the spiritualist does not know that he does not know. We owe all our knowledge of Nature to Science, we owe nothing at all *directly*[1] to Spiritualism. The spiritualist allows effects to govern his will, whilst by a true insight into their hidden connections he might govern them.

Having thus warned my readers against Fourth Dimensionists, Metaphysicians, and

[1] During the Middle Ages, and amongst the ancients, the noble studies of Astronomy, Chemistry, etc., were often cultivated as subsidiary to those of Astrology, Alchemy, etc.; Alchemy chiefly from 13th to 17th century.

Spiritualists, I will close this chapter with a passage, and a few remarks on it, taken from S. Bailey's Essay on 'The Progress of Culture.' In it he says, 'It is unwise for any one to enter *very minutely* into the history of the science to which he devotes himself—more especially at the outset. Let him perfectly master the present state of the science, and he will be prepared to push it further while the vigour of his mind remains unbroken; but if he previously attempts to embrace all that has been written on the subject,—to make himself acquainted with all its exploded methods and obsolete doctrines, his mind will probably be too much entangled in their intricacies to make any original efforts; too wearied with tracing past achievements to carry the science to a further degree of excellence. When a man has to take a leap, he is materially assisted by stepping backwards a few paces and giving his body an impulse by a short run to the starting place; but if his precursory range is too extensive, he exhausts his forces before he comes to the principal effort.' But the whole question rests on those two words —*very minutely*. For, in order that we may share in what men are doing in the world, we must share in what they have done. 'For to know certain general symbolical results—that is to say, certain modern analytical methods, which are supposed to render all scientific history superfluous—is an accomplishment which can only be of little value in education; for a good education must connect us with the past as well as with the present, even if such mere generalities did supply the best mode of dealing with all future problems, which, in fact, they are very far from doing.' [1]

[1] See 'A Liberal Education,' by Dr Whewell.

The Future hides in it
Gladness and sorrow :
We press still thorow :
Nought that abides in it
Daunting us—Onward !

And solemn before us,
Veiled the dark Portal,
Goal of all Mortal.
Stars silent rest o'er us—
Graves under us, silent.

While earnest thou gazest,
Comes boding of terror,
Come phantasm and error ;
Perplexes the bravest
With doubt and misgiving.

But heard are the voices,
Heard are the Sages,
The Worlds and the Ages :
"*Choose well : your choice is*
Brief, and yet endless."

Here eyes do regard you
In Eternity's stillness ;
Here is all fulness,
Ye brave to reward you,
Work, and despair not.'—GOETHE.

First quoted by T. Carlyle in 'Past and Present.'

CHAPTER IX.

CONCLUSION.

'Any intelligent man may now, by resolutely applying himself for a few years to mathematics, learn more than the great Newton knew after half a century of study and meditation.'—LORD MACAULAY.

IF some of the statements made in Chapter III. should seem to be too wonderful to be credited, or if the nature and difficulties of some of the problems which the science of Mathematics has so successfully solved appear to overwhelm the mind, let it be remembered that the science of Mathematics has ever lived and never dies. One mathematician dies, but his works remain; another takes up the work where he left it, and the chain of reasoning is unbroken, the work carried on.

Commencing his calculations thousands of years ago amongst some of the nations of the East, in Babylon he toiled, and amongst the Egyptians he found a dwelling-place. Among the temples of India, the pagodas of China, the pyramids of Egypt, and the plains of Arabia he thought and studied.

When Science fled to Greece, his refuge was in the schools of her philosophers; and when darkness and bigotry covered the face of Europe for hundreds of years, he pursued his studies amidst the burning plains of Arabia. When Science returned again to Europe, the Mathematician was there, toiling in Leonardo Bonacci, suffering in Galileo, triumphing in Descartes, and triumphing still more in Leibnitz, and Newton, justly regarded as the greatest genius that ever lived.[1]

[1] And soon after in Laplace (born 1749), and Lagrange (born 1736), and Euler (1707), etc.

Standing on the lofty pinnacle of the Temple of Science of the present day, of which we are so justly proud, and which Mathematics has so powerfully and so effectually helped to raise, and looking around us, we become aware of the deep debt which the world owes to *original discoverers* in this Science, and we see what an important, though not openly apparent, part it has played in the history of the world.

What mathematicians will accomplish in the future remains to be seen, but one thing we know, the past and the present constitute one unbroken chain of reason, condensing all time to the mathematician into one mighty NOW. We shall not live to behold these anticipated triumphs of mind over matter, but who can doubt the final result? Look back to the ancient mathematicians; compare their power and knowledge with those of the modern, grasping in a few years of patient study far more than his predecessor was able to learn in a lifetime. Are the problems remaining to be solved more difficult, more inaccessible, than those which have been so successfully solved?

The results recorded by the ancient Mathematicians are of inestimable value in the solution of some of the most difficult problems of to-day; similarly the records made now shall descend to generations yet unborn, and aid them in the same manner as the records made thousands of years ago aid the mathematicians of the present day.

In conclusion, then, Science (in its broadest sense) is the great power of the day, and it is, as W. P. Fox says in his Lectures to the Working Classes, the friend of man; the history of its advance is the history of human progress; it sheds a light on the past, and by doing so, in some measure illuminates the coming future; it is in harmony with the being and well-being of all the inhabitants of this world of ours; and in proportion as it makes known to us the great principles and influences that pervade Creation, it makes us at one with Creation, and the recipients of its good and its blessings. Science is the friend of man, raising and dignifying man, and qualifying him more and more for the full possession of his rights, the exercise of his powers, and the accomplishment of whatever is good and great in this world, and of all that its various means and appliances are capable of rendering.'

Mathematics is one of—sometimes *the* greatest—auxiliary to Science; by Science are the inner works of Nature reverently uncovered. I commend, therefore, the study of Mathematics to you as worthy of all your acceptation, only bidding you remember that

> '*Industry is the* TRUE *philosopher's stone,*'

and that any one who is able to decipher any of the hieroglyphics of the volume of

Nature, or to carry any Science to a further degree of excellence, has not lived in vain, but has added something to the sum of human happiness and human knowledge.

FINIS.

APPENDIX.

APPENDIX.

THE SQUARING OF THE CIRCLE.[1]

THERE are four famous problems which from time immemorial almost have had a multitude of patient devotees. They are:—The discovery of perpetual motion; the trisection of *any* angle; the finding of two mean proportionals between two given straight lines (often referred to as the duplication of the cube); and the quadrature or 'Squaring' of the Circle.

With regard to the first, I make no further mention here than to suggest that before any one attempt its solution, he should read, mark, learn, and inwardly digest the principle of the conservation of energy, and he will then comprehend the absurdity of his attempt.

With regard to the second and third—The Tri-

[1] See 'Budget of Paradoxes,' by Professor De Morgan, being a series of papers in the *Athenæum* for 1863, and subsequent years.

section of *any* Angle, and the Duplication of the Cube, they are not unsolvable or impossible problems, but only so by means of elementary geometry, for by the postulates of ordinary geometry all constructions must be made by the aid of a circle and undivided ruler. Now straight lines intersect each other only in one point, and a straight line and a circle intersect each other only in two points. But the trisection of any angle, or the duplication of the cube, requires for its solution the intersection of a straight line and a curve,—of what is known as the third degree, or two conics ; but all of these are excluded by the postulates of ordinary geometry. If the postulates of elementary geometry allowed that a parabola or an ellipse could be described with what is known as a given focus and directrix, as they allow that a circle can be described with a given centre and radius, then these two problems are solvable by elementary geometry, their so-called insolvability being merely a restriction placed (by whom unknown, but prior to Euclid) upon the postulates of ordinary geometry.

Passing on now to the Quadrature, more generally known as the Squaring of the Circle, the question which arises is, What is the Squaring of the Circle? It is to make a circle containing *exactly* the same area as a square. And the solution of this problem depends on finding the precise or *exact* ratio which exists between the diameter and the circumference. This may appear at first sight absurd, for what ratio can possibly exist between two things so perfectly unlike? For the diameter of a circle is a straight line passing through the centre and terminated both ways by the circumference, while the circumference is, of course, a curved line. But supposing the circumference of the circle stretched out into a straight line to its full extent, similarly as a wire ring might be done, by cutting it through in one point, and then stretching it out into a straight piece of wire, what then is the proportion between the diameter and the circumference? Many persons think that it is an easy matter to determine this ratio—namely, by measuring. First measure the diameter, and then the stretched out circumference, and you have the required proportions, which, supposing the diameter to be 7 inches, you will probably find the circumference to be perhaps 22 inches. This must be right, you say, for we have measured it. Only it is not: the proportion is erroneous. The ratio of the diameter to the circumference is not *exactly* as 7 to 22. And this failure results simply from the nature of the thing. For it is impossible to compare the physical with the ideal,—the mechanical operations of the finest arts with the pure and simple abstractions of the mind. For even as in the Fine Arts there is an ideal beauty which no artist or connoisseur can or could ever attain, so in mathematics there is an accuracy of proportion, or ratio, which the finest instrument constructed or ever likely to be constructed by man can never attain. The above proportion, 7 to 22, is a very rough approximation; a nearer approximation is possible by measurement, but a rough approximation is only possible by instrumental means. Measurement and arithmetic this ratio surpasses, but ideas and mathematical expressions are able to reach it. By means of ordinary numbers we can only approximate to this ratio or proportion, for this proportion is simply what is known as an infinite series of which the law cannot be stated in ordinary terms of the decimal notation. Archimedes found this ratio to be between $3\frac{10}{70}$ and $3\frac{10}{71}$. By which he meant that the ratio of the circumference to the diameter lies between $3\frac{10}{70}$ and $3\frac{10}{71}$. And this is approximately as 22 to 7. But a small error in the circumference leads to a greater error in the surface or area of the circle, and to a still greater error in the solid or cubic contents of the sphere; for errors increase by multiplication.

Metius, a Dutch mathematician of the seventeenth century, found the ratio to be 355:113. The Hindoos, however, had obtained an expression nearly as accurate. From the time of Metius down to the present day, closer and closer approximation by different mathematical methods have been obtained.

Ludolph Von Coulen, by simple arithmetical processes, showed that this ratio was between

$$3.14159265358979323846264338327950288$$

and,

$$3.14159265358979323846264338327950289$$

a result so accurate, that, says Montucla, 'if there be supposed a circle whose radius is the distance of the nearest fixed star (250,000 times the earth's distance from the sun) the error in calculating its circumference is so excessively small a fraction of the diameter of a human hair as to be utterly invisible not only merely under the most powerful microscope yet made, but under any which future generations may be able to construct;' the above result being true to 36 significant figures, a result which will perhaps demonstrate to the general reader that the approximations of mathematics are hardly approximations in the ordinary sense of the word. In the year 1688 James Gregory gave a demonstration of the impossibility of effecting exactly the quadrature of the circle, now generally accepted. It can be expressed in the form of a Definite Integral, which it is perhaps possible may be expressed in finite terms containing irrational numbers, this being hailed, perhaps, as a solution of the grand problem. But be this so or not, the reader will at once see that the so-called approximations mentioned above are far more than sufficient for any practical applications ever required by man, even in those most delicate of all delicate calculations, the calculations of Astronomy. The solution of the problem mentioned just above, it is almost needless to say, is not the one attempted by the 'Squarers,' who, I am sorry to hear, grow more numerous every day. I can only suggest to them that when they have arrived at the conclusion that 3.1415 or 3.14159, etc., is the *exact* ratio, they should carefully bear in mind the result of Ludolph Von Coulen, stated above, and also meditate upon the fact that *by other methods its value is now known exact to* 600 *places of decimals.*

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