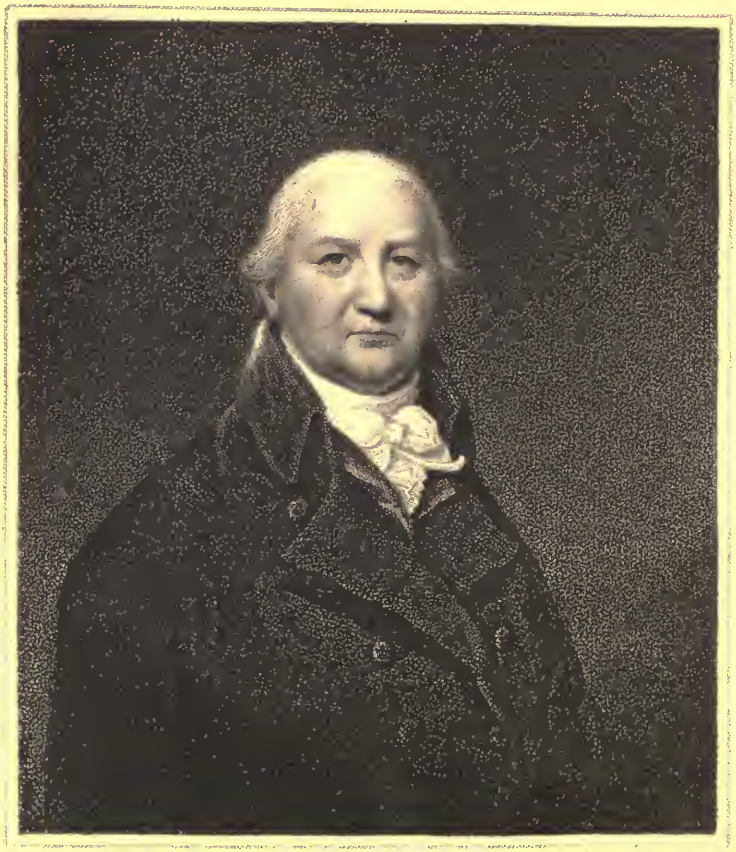

TRACTS

ON

MATHEMATICAL

AND

PHILOSOPHICAL SUBJECTS;

COMPRISING,

AMONG NUMEROUS IMPORTANT ARTICLES,

THE THEORY OF BRIDGES,

WITH SEVERAL PLANS OF RECENT IMPROVEMENT.

ALSO

THE RESULTS OF NUMEROUS EXPERIMENTS ON

THE FORCE OF GUNPOWDER,

WITH APPLICATIONS TO

THE MODERN PRACTICE OF ARTILLERY.

IN THREE VOLUMES.

BY CHARLES HUTTON, LL.D. AND F.R.S. &c.

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VOL. I.

LONDON:

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1812.

T. DAVISON, Lombard-Street,
Whitefriars, London.

PREFACE.

HAVING been, for a long series of years, in the constant habit of preserving original Tracts and dissertations on scientific subjects; and now enjoying, at a very advanced period of life, some degree of leisure, in consequence of my retirement from the laborious duties of the Royal Military Academy; I have anxiously embraced the opportunity of selecting, and revising, such of those papers as were likely to be most useful, and of presenting them to the public.

Some few parts of these Tracts have been already printed in the Philosophical Transactions, and in other works; but most of them are quite new; and such as are not so, having been recast and greatly improved, may be also considered in some measure as original compositions. These papers, being necessarily of a miscellaneous nature, are here arranged nearly according to the order of time in which they were composed; and the description of them, is briefly as follows.

VOLUME I.

TRACT I, is on the Principles of Bridges.—The original of this paper was a small pamphlet on the same subject, first published by me on a particular occasion at Newcastle, in the year 1772. It was also republished at London in 1801, nearly in the same state. But it has been now recomposed, and greatly enlarged with many additional propositions, as also numerous observations, both practical and scientific.

An Appendix is also added, containing my report to the Committee of Parliament on the project for a new iron bridge, of only one arch, proposed to be thrown over the river Thames at London; with several other appropriate articles, as below.

Tract ii, exhibits some curious queries concerning London Bridge, proposed in the year 1746 by the magistrates of the city; with the ingenious answers given to the same, by Mr. George Dance, surveyor-general of the city works, being the result of that gentleman's examination concerning the state of the bridge at that time.

Tract iii contains experiments and observations to be made on the state of London bridge; being the report of a committee of the members of the Royal Society, addressed to the common council of the city of London.

Tract iv treats of the effects which might be produced on the tides in the river Thames, in consequence of erecting a bridge at Blackfriars. This was an ingenious report, drawn up by the late Mr. John Robertson, at the request of the city of London.

Tract v consists of answers, given by me, to questions proposed by the Select Committee of Parliament, relative to a proposal, made by Messrs. Telford and Douglas, for erecting a new iron bridge, of a single arch only, over the river Thames, instead of the present London bridge.

Tract vi exhibits a brief history of the original invention, and subsequent improvements of iron bridges, as practised of late years in this country.

Tract vii is a dissertation on the nature and value of infinite series; explaining the properties of several forms of such series, as converging, diverging, and neutral.

Tract viii is a new method for the valuation of numeral infinite series, that have their terms alternately plus and minus; which is performed by taking continual arithmetical means between the successive sums, and between the means; a method by which the value or sum of any such series is very easily and quickly obtained.

TRACT IX is a method of summing the series $a+bx+cx^2+dx^3+ex^4+\&c$, in the case when it converges very slowly, namely, when x is nearly equal to 1, and the coefficients a, b, c, d, &c, decrease very slowly; the signs of all the terms being plus or positive :—a method which has been considered a great desideratum in infinite series.

TRACT X contains the investigation of certain easy and general rules, for extracting any root out of a given number; exhibiting a general and very easy formula, to serve for all roots whatever.

TRACT XI is a new method of finding, in general and finite terms, near values of the roots of equations of this form, $x^n - px^{n-1} + qx^{n-2} - \&c = 0$; namely, having the terms alternately plus and minus: being one method more to be added to the many we are already possessed of, for determining the roots of the higher orders of equations.

TRACT XII treats of the binomial theorem; exhibiting a demonstration of the truth of it in the general case of fractional exponents. The demonstration is of this nature, that it proves the law of the whole series in a formula of one single term only: thus, P, Q, R, denoting any three successive terms of the series, expanded from the given binomial $(1+x)^{\frac{1}{n}}$, and if $\frac{g}{h}\text{P} = \text{Q}$, then is $\frac{g-n}{h+n}\text{Q} = \text{R}$, which denotes the general law of the series, being a new mode of proving the law of the coefficients of this celebrated theorem. But, besides this law of the coefficients, the very form of the series is, for the first time, here demonstrated, viz, that the form of the series for the developement of the binomial $(1+x)^{\frac{1}{n}}$, with respect to the exponents, will be $1 + ax + bx^2 + cx^3 + dx^4 + \&c$, a form which has heretofore been assumed without proof.

TRACT XIII treats on the common sections of the sphere and cone: with the demonstration of some other new properties of the sphere, which are similar to certain known properties of the circle. The few propositions which form part of this tract, is a small specimen of the analogy, and even identity, of some of the more remarkable properties of the circle, with those of the sphere. To which are added some properties of the lines of section, and of contact, between the sphere and cone: both of which can be further extended as occasions may offer.

Tract xiv, on the geometrical division of circles and ellipses into any number of parts having equal perimeters, and areas either all equal or in any proposed ratios to each other: constructions which were never before given by any author, but which, on the contrary, had been accounted impossible to be effected.

Tract xv contains an approximate geometrical division of the circumference of the circle.

Tract xvi treats on plane trigonometry, without the use of the common tables of sines, tangents, and secants: resolving all the cases in numbers, by means of certain algebraical formulæ only.

Tract xvii is on Machin's quadrature of the circle; being an investigation of that learned gentleman's very simple and easy series for that purpose, by help of the tangent of the arc of 45 degrees; which series the author had given without any proof or investigation.

Tract xviii, a new and general method of finding simple and quickly-converging series; by which the proportion of the diameter of a circle to its circumference may easily be computed to a great many places of figures. By this method are found, not only Machin's series, noticed in the last Tract, but also several others that are much more simple and easy than his.

Tract xix, the history of trigonometrical tables, &c: being a critical description of all the writings on trigonometry made before the invention of logarithms.

Tract xx, the history of logarithms; giving an account of the inventions and descriptions by several authors on the different kinds of logarithms.

Tract xxi, on the construction of logarithms; exhibiting the various and peculiar methods employed by all the different authors, in their several computations of these very useful numbers.

Tract xxii, treats on the powers of numbers; chiefly relating to curious properties of the squares, and the cubes, and other powers of numbers.

Tract xxiii, is a new and easy method of extracting the square roots of numbers; very useful in practice.

Tract xxiv, shows how to construct tables of the square-roots, and cube-roots, and the reciprocals of the series of the natural numbers; being a general method, by means of the law of the differences of such roots and reciprocals of numbers.

Tract xxv, is an extensive table of roots and reciprocals, constructed in the above manner, accompanied also with the series of the squares and cubes of the same numbers.

VOLUME II.

Tract xxvi, an account of the calculations made from the survey and measures taken at mount Shichallin, in order to ascertain the mean density of the earth: being the result of a laborious calculation, the first ever made to ascertain that density; by which it is shown to be nearly equal to 5 times the density of water, or almost double the density of the rocks at the surface of the earth, and that consequently the interior of the earth must consist of immense quantities of metals or metallic ores.

Tract xxvii, consists of calculations to determine at what point, on the side of a hill, its attraction will be the greatest. This is inserted as an appendix to the preceding tract, and intended to direct operations of any future attempt to ascertain such density, or to corroborate the foregoing statement; and, by this determination, it is shown that the best situation is generally at about $\frac{1}{4}$ of the altitude of the hill.

TRACT XXVIII, is an extensive treatise on cubic equations and infinite series: showing their nature, prop rties, and solutions, both in finite formulas and by expressions in infinite series.

TRACT XXIX contains a curious project for a new division of the quadrantal arc of the circle, with a view to trigonometrical and other purposes: being intended for the novel design of constructing tables of the sines, tangents, and secants of arcs, to equal parts of the radius of the circle; or expressing all these lines, as well as the arcs themselves, in such parts.

TRACT XXX, on the sections of spheroids and conoids: showing that all such plane sections are the same as conic sections; and that all the parallel sections, in each of these solids, are like and similar figures.

TRACT XXXI, on the comparison of curves of the same species; showing their mutual relations.

TRACT XXXII contains a theorem for the cube-root of an algebraic binomial, one of the terms being a quadratic radical; useful in the solution of certain cubic equations by Cardan's rule.

TRACT XXXIII, is a complete history of algebra; tracing its origin and practice among the ancient Greeks, the Indians, Persians, and Arabians; with particular details of the various peculiarities and improvements, made among different people, and by several eminent individuals, especially among the European authors, namely, the Italians, Spaniards, French, Germans, and the English; in which all the discoveries and improvements are ascribed to the proper authors.

TRACT XXXIV, exhibits the results of new experiments in Artillery, for determining the force of fired gunpowder, the initial velocity of cannon balls, the ranges of projectiles at different elevations, the resistance of the air to their motions; the effect of different lengths of guns, and of different quantitities of powder, &c, &c: giving a complete detail of all the circumstances attending these very numerous and accurate experiments, with many useful philosophical and practical inferences deduced from them; the whole forming as it were a new era in the progress of this curious and important branch of knowledge.

VOLUME III.

TRACT XXXV, on a new Gunpowder Eprouvette; showing its construction and use, by means of which the strength and quality of gunpowder may be proved and evinced, in a way far more exact and easy than by any other machine.

TRACT XXXVI, on the Resistance of the Air to bodies in motion, as determined by the Whirling Machine: showing the exact quantity of the air's resistance to all forms of bodies, moved through it with slow and moderate motions; the effects of which, combined with those of the very high motions of cannon and musket shot, furnish us with a complete and uniform series of resistances to all degrees of velocity, from the very slowest perceptible motions, to those of the highest and most violent.

TRACT XXXVII, on the Theory and Practice of Gunnery, as dependent on the Resistance of the Air. This tract is employed in stating the deductions abstracted from all the preceding experiments, and applying them in many problems, to the important purposes of Artillery and projectiles. Here are given complete tables of the quantity of resistance to balls moving with every degree of velocity; with correct rules for ascertaining those that are proper to all other sizes of balls. Here are also given general rules and algebraic formulæ, for expressing the resistance to any size of ball in terms of the velocity; with a great variety of problems for determining the motions of balls in all directions, upwards, downwards, or obliquely, touching their velocities and times in motion, with the ranges of projectiles in the air, and practical applications to the cases of gunnery, in a great variety of useful instances.

TRACT XXXVIII, being the last, contains a miscellaneous collection of practical questions, illustrating several of the principles in the preceding Tracts, with the solutions at large.

Such are the outlines of a work, which is the result of many years assiduous study and persevering research; and which it is presumed will be found to contain several new articles, on civil and military science, that may be deemed of national importance.

It is, in all probability, the last original work that I may ever be able to offer to the notice of the Public, and I am therefore the more anxious that it should be found worthy of their acceptance and regard. To their kind indulgence, indeed, is due whatever success I may have experienced, both as an Author and Teacher for more than half a century: and it is no small satisfaction to reflect, that my humble endeavours, during that period, have not been wholly unsuccessful in the diffusion of useful knowledge.

To the same liberal encouragement of the Public must likewise be ascribed, in a great measure, the means of the comfortable retirement which I now enjoy, towards the close of a long and laborious life: and for which I have every reason to be truly thankful.

CHA. HUTTON.

London,
July, 1812.

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CONTENTS.

CORRECTIONS.

VOLUME I.

Note, *b* denotes counted from the bottom.

Page	Line	
8	9*b*	$Et = Ds$.
12	1*b*	horizontal line CH.
19		in the cut, at the bottom of the lines appended from the points *b*, *c*, *d*, *e*, *f*, set the letters *i*, *k*, *l*, *m*, *n*.
—	19	$DH = dk$,
57	1*b*	supposing $\dot{y}$.
89	13*b*	channel.
264	5*b*	$2 . 4 . 5r^5$.
267	13*b*	$\frac{120}{119}$.
430	14*b*	$\frac{1}{5}t^5$.
266	1	$4a^2b^2$.

VOLUME II.

153	4	for *Hill Street*, read *Portland Place*.
311	12*b*	and elsewhere, for *Bloomfield*, read *Blomefield*.
379	13*b*	for *pendulum*, read *gun*.

VOLUME III.

46	11*b*	dele *of*.
—	10*b*	for *such*, read *some*.
101	2	for 15, read 18.
303	9	for 128, read 1280.

TRACT I.

THE

PRINCIPLES OF BRIDGES:

CONTAINING

THE MATHEMATICAL DEMONSTRATION OF THE PROPERTIES OF THE ARCHES, THE THICKNESS OF THE PIERS, THE FORCE OF THE WATER AGAINST THEM, &c. WITH PRACTICAL OBSERVATIONS AND DIRECTIONS DRAWN FROM THE WHOLE.

THIS Tract, on bridges, originated from the circumstance of the fall of Newcastle bridge, in the year 1771; which, with other particulars relative to the Tract, are noticed in the Preface to that Edition of it; which was as follows:

THE ORIGINAL PREFACE.

A large and elegant bridge, forming a way over a broad and rapid river, is justly esteemed one of the noblest pieces of mechanism that man is capable of performing. And the usefulness of an art which, at the same time that it connects distant shores by a way over the deep and rapid waters, also allows those waters and their navigation to pass smooth and uninterrupted, renders all probable attempts to advance the theory or practice of it, highly deserving the encouragement of the public.

This little book is offered as an attempt towards the improvement of the theory of this art, in which the more essential properties, dimensions, proportions, and other relations of the various parts of a bridge, are strictly demonstrated, and clearly illustrated by various examples. It is divided into five sections: the 1st treats on the projects of bridges, containing a regular detail of the various circumstances and considerations that are cognizable in such projects. The 2d treats on arches, demonstrating their various properties, with the relations between their intrados and extrados, and clearly distinguishing the most preferable curves to be used in a bridge; the first two or three propositions being instituted after the manner of two or three done by Mr. Emerson in his Fluxions and Mechanics. The 3d section treats on the piers, demonstrating their thickness necessary for supporting any kind of an arch, springing at any height, both when part of the pier is supposed to be immersed in water, and when otherwise. The 4th demonstrates the force of the water against the end or face of the pier, considered as of different forms; with the best form for dividing the stream, &c. and to it is added a table, showing the several heights of the fall of the water under the arches, arising from its velocity and the obstruction of the piers; as it was composed by Tho. Wright, Esq. of Auckland, in the county of Durham, who informs me it is part of a work on which he has spent much time, and with which he intends to favour the public. And the 5th and last section contains a Dictionary of the most material terms relating to the subject: in which many practical observations and directions are given, which could not be so regularly nor properly introduced into the former sections. The whole, it is presumed, containing full directions for constituting and adapting to one another, the several essential parts of a bridge, so as to make it the strongest, and the most convenient, both for the passage over and under it, which the situation and other circumstances will admit: not indeed for the actual methods of disposing the stones, making of mortar, or the external ornaments, &c. those things are not here attempted, but are left to the discretion of the practical architect, as being no part of the plan of this undertaking; and for the same reason also here are not given any views of bridges, but only prints of such parts or figures as are necessary in explaining the elementary parts of the subject.

As my profession is not that of an architect, very probably I should never have turned my thoughts to this subject, so as to address the public upon it, had it not been for the occasion of an accident in that part of the country in which I reside, viz. the fall of Newcastle and other bridges on the river Tyne, on the 17th of November, 1771, occasioned by a high flood, which rose about 9 feet higher at Newcastle than the usual spring tides do. This occasion having furnished me with many opportunities of hearing and seeing very absurd notions advanced on the subject in general, I thought the demonstrations of the relations of the essential parts of a bridge, would not be unacceptable to those architects and others, who may be capable of perceiving their force and effects.

Newcastle, 1772.

The original edition, of 1772, being out of print, and the book being much asked for, a new edition was printed in 1801, at a time when the project of a cast-iron bridge of one arch, proposed to be built over the Thames at London, by Messrs. Telford and Douglass, was the subject of much conversation: on which occasion the following addition was made to the Preface; viz.

This little work, which was hastily composed on a particular occasion, having been long out of print, is now as suddenly reprinted in the same form, on the present occasion, of the report of a new bridge proposed to be thrown across the Thames, at London: reserving the long intended edition, on a much larger and more improved plan, till a more convenient opportunity.

Royal Military Academy, Jan. 12, 1801.

It may here be added, that the whole tract has been now quite re-cast and composed, and greatly enlarged with more propositions, and numerous observations, both practical and scientific. To the end is also added an Appendix, being the author's report to the Committee of Parliament, on the project for a new cast-iron bridge, of one arch, over the river at London; and several other appropriate appendages.

SECTION I.

ON THE PROJECTS OF BRIDGES; WITH THE DESIGN, THE ESTIMATE, &c.

When a bridge is deemed necessary to be built over a river, the first consideration is the place of it; or what particular situation will contain a maximum of the advantages over the disadvantages. In agitating this important question, every circumstance, certain and probable, attending or likely to attend the bridge, should be separately, minutely, and impartially stated and examined; and the advantage or disadvantage of it rated at a value proportioned to it; then the difference between the whole advantages and disadvantages, will be the net value of that particular situation for which the calculation is made. And by doing the same for other situations, all their net values will be found, and of consequence the most preferable situation among them.—Or, in a competition between two places, if each one's advantage over the other be estimated or valued in every circumstance attending them, the sums of their advantages will show which of them is the better. And the same being done for this and a third, and so on, the best situation of all will be obtained.

In this estimation, a great number of particulars must be included; nothing being omitted that can be found to make a part of the consideration. Among these, the situation of the town or place, for the convenience of which the bridge is chiefly to be made, will naturally produce an article of the first consequence; and a great many others, if necessary, ought to be sacrificed to it. If possible, the bridge should be placed where there can conveniently be opened and made passages or streets from the end of it in every direction, and especially one as nearly in the direction of the bridge itself as possible, tending towards the body of the town, without narrows or crooked windings, and easily communicating with the chief streets, thoroughfares, &c.—And here every person, in judging of this, should divest himself of all partial regards or attachments whatever; think and determine for the good of the whole only, and for posterity as well as for the present.

The banks or declivities towards the river are also of particular concern, as they affect the conveniency of the passage to and from the bridge, or determine the height of it, on which in a great measure depends the expense, as well as the convenience of passage. The breadth of the river, the navigation upon it, and the quantity of water to be passed, or the velocity and depth of the stream, form also considerations of great moment; as they determine the bridge to be higher or lower, longer or shorter. However, in most cases, a wide part of the river ought rather to be chosen than a narrow one, especially if it is subject to great tides or floods: for, the increased velocity of the stream in the narrow part, being again augmented by the further contraction of the breadth by the piers of the bridge, will both incommode the navigation through the arches, and undermine the piers and endanger the whole bridge. The nature of the bed of the river is also of great concern, it having a great influence on the expense; as upon it, and the depth and velocity of the stream, depend the manner of laying the foundations, and building the piers. These are the chief and capital articles of consideration, which will branch themselves out into other dependent ones, and so lead to the required estimate of the whole.

Having resolved on the place, the next considerations are, the form, the estimate of the expense, and the manner of execution. With respect to the form; strength, utility, and beauty ought to be regarded and united; the chief part of which lies in the arches. The form of the arches will depend on their height and span; and the height on that of the water, the navigation, and the adjacent banks. They ought to be made so high, as that they may easily transmit the water at its greatest height, either from tides or floods; and their height and figure ought also to be such as will easily allow of a convenient passage of the craft through them. This, and the disposition of the bridge above, so as to render the passage over it also convenient, make up its utility.—Having fixed the heights of the arches, their spans are still necessary for determining their figure. Their spans will be known by dividing the whole breadth of the river into a convenient number of arches and piers, allowing at least the necessary thickness of the piers out of the whole. In fixing on the number of arches, let an odd number always be taken; and few and large ones, rather than many and smaller, if convenient: For thus we shall have not only fewer foundations and piers to make, but fewer arches and centres, which will produce great savings in the expense; and besides, the arches themselves will also require much less materials and workmanship, and allow of more and better passage for the water and craft through them; and will appear at the same time more noble and graceful, especially if constructed in elliptical, or in cycloidal forms; for the truth of which, it may be sufficient to refer to that noble and elegant bridge lately built at Blackfriars, London, by Mr. Mylne; which might perhaps be accounted incomparable, at least in England, if the piers were of equal excellence: but these are too thick, and clumsy, and their appearance is made still less graceful by the double columns placed before them. So that Blackfriar's arches and the Westminster's piers united, would be preferable to either bridge separately.

If the top of the bridge be a straight horizontal line, the arches may be made all of a size; if it be a little lower at the ends than the middle, the arches must proportionally decrease from the middle towards the ends; but if higher at the ends than the middle, which can seldom happen, they may then increase towards the ends. A choice of the most convenient arches is to be made from some of the following propositions, where their several properties and effects are demonstrated and pointed out: Among these, the elliptic, cycloidal, and equilibrate arch, will generally claim the preference, as well on account of the strength, and beauty, as cheapness or saving in materials and labour: Other particulars also concerning them may be seen under the word ARCH in the Dictionary in the last section.

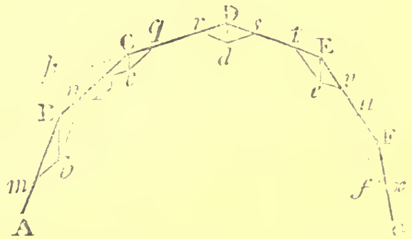
Next find what thickness at the keystone or top will be necessary for the arches. For which see the word KEYSTONE in the Dictionary in the 5th section.—Having thus obtained all the parts of the arches, with the height of the piers, the necessary thickness of the piers themselves are next to be computed. This done, the chief and material requisites are found; the elevation and plans of the design can then be drawn, and the calculations of the expense thence made, including the foundations, with such ornamental or accidental appendages as shall be thought fit; which, being no part of the plan of this undertaking, is left to the fancy of the Architect and Builder, together with the practical methods of carrying the design into execution. I shall however, in the Dictionary, in the last section, not only describe the terms, parts, machines, &c, but also speak of their dimensions, properties, and any thing else material belonging to them; and to which therefore I from hence refer for more explicit information in each particular article, as well as to these immediately following propositions, in which the theory of the arches, piers, &c, are fully and strictly demonstrated.

SECTION II.

OF THE ARCHES.

PROPOSITION I.

Let there be any number of lines AB, BC, CD, DE, &c. *all in the same vertical plane, connected together and moveable about the joints or angles* A, B, C, D, E, F; *the two extreme points* A *and* G *being fixed: It is required to determine the proportion of the weights to be laid upon the angles* B, C, D, &c. *so that the whole may remain in equilibrio.*



Solution.—From the several angles, having drawn the lines Bb, Cc, Dd, &c. perpendicular to the horizon; about them, as diagonals, constitute parallelograms such, that those sides of every two that are at the opposite ends of the given lines, may be equal to each other; viz, having made one parallelogram mn, take Cp = Bn, and form the parallelogram pq; then take Dr = Cq, and make the parallelogram rs; and take Er = Ds, and form the parallelogram tv; and so on: Then the said vertical diagonals Bb, Cc, Dd, Ee, &c, of those parallelograms, will be proportional to the weights, as required.

Demonstration.—By the resolution of forces, each of the weights or forces Bb, Cc, Dd, &c, in the diagonals of the parallelograms, is equal to, and may be resolved into, two forces, expressed by two adjacent sides of the parallelogram; viz. the force Bb may be resolved into the two forces Bm, Bn.

and in those directions; the force C*c*, into the two forces C*p*, C*q*, and in those directions; the force D*d*, into the forces D*r*, D*s*, and in those directions; and so on. Then, since two forces that are equal, and in opposite directions, do mutually balance each other; therefore the several pairs of forces B*n* and C*p*, C*q* and D*r*, D*s* and E*t*, &c, being equal and opposite, by the construction, mutually destroy or balance each other; and the extreme forces B*m*, E*v*, are balanced by the opposite resistances of the fixed points A, G. There is no force therefore to change the position of any one of the lines, and consequently they will all remain in equilibrio.

Corollary.—Hence, if one of the weights and the positions of all the lines be given, all the other weights may thence be found, as well as all the oblique forces in the direction of the bars or lines. And the weight which is given, may either be that at the lower extremity, as B*b*, or it may be that at the vertex D*d*, or it may be any of the intermediate ones, as C*c*: for, whichever of these is given, it will serve, as a diagonal, to form the parallelogram about it; then the sides of this parallelogram will give the sides of the two next parallelograms, on each side of the former; and so on through the whole collection of the bars. Thus, if the uppermost vertical weight, or diagonal D*d*, be the given one: Then draw *dr* parallel to DE, and *ds* to DC, so forming the parallelogram *r*D*sd*: then make C*q* = D*r*, and E*t* = D*s*: and, having drawn the several indefinite vertical lines B*b*, C*c*, E*e*, at the angles, form the parallelograms *pq* and *tv*, by drawing *qc* parallel to EC, and *cp* to CD, and *te* to EF, and *ev* to DE.—Lastly, take B*n* = *p*C, and make the parallelogram *mn*, by drawing *nb* parallel to AB, and *bm* parallel to BC. And so on through the whole.

PROP. II.

If any number of lines, that are connected together and moveable about the points of connection, be kept in equilibrio by weights laid on the angles, as in the last proposition: Then will the weight on any angle C *be universally proportional to* $\frac{\text{sine of the } \angle \text{ BCD}}{\text{s. } \angle \text{ BCc} \times \text{s. } \angle \text{ cCD}}$; *that is, directly as the sine of that angle, and reciprocally as the sines of the two parts or angles into which that angle is divided by a line drawn through it perpendicular to the horizon.* See the former figure.

Demonstration.—By the last proposition the weights are as Bb, Cc, Dd, &c, where Bn = pC, Cq = rD, Ds = tE, &c. But, since the angle ABb is = the angle Bbn, and the angle BCc = the angle cCq, &c, these being always the alternate angles made by a line cutting two other parallel lines; also the sine of the ∠ ABC = s. ∠ Bnb, and s. ∠ BCD = s. ∠ Cqc, these being supplements to each other; by plane trigonometry we shall have,

$$(\text{B}n =)\ \frac{\text{B}b \times \text{s. } \angle \text{ AB}b}{\text{s. } \angle \text{ ABC}} = (\text{C}p =)\ \frac{\text{C}c \times \text{s. } \angle\ c\text{CD}}{\text{s. } \angle \text{ BCD}},$$

$$(\text{C}q =)\ \frac{\text{C}c \times \text{s. } \angle \text{ BC}c}{\text{s. } \angle \text{ BCD}} = (\text{D}r =)\ \frac{\text{D}d \times \text{s. } \angle\ d\text{DE}}{\text{s. } \angle \text{ CDE}},$$

$$(\text{D}s =)\ \frac{\text{D}d \times \text{s. } \angle \text{ CD}d}{\text{s. } \angle \text{ CDE}} = (\text{E}t =)\ \frac{\text{E}e \times \text{s. } \angle\ e\text{EF}}{\text{s. } \angle \text{ DEF}},$$

and so on. Hence,

$$\text{B}b : \text{C}c :: \frac{\text{s. } \angle \text{ ABC}}{\text{s. } \angle \text{ AB}b} : \frac{\text{s. } \angle \text{ BCD}}{\text{s. } \angle\ c\text{CD}},$$

$$\text{C}c : \text{D}d :: \frac{\text{s. } \angle \text{ BCD}}{\text{s. } \angle \text{ BC}c} : \frac{\text{s. } \angle \text{ CDE}}{\text{s. } \angle\ d\text{DE}},$$

$$\text{D}d : \text{E}e :: \frac{\text{s. } \angle \text{ CDE}}{\text{s. } \angle \text{ CD}d} : \frac{\text{s. } \angle \text{ DEF}}{\text{s. } \angle\ e\text{EF}}, \text{ \&c.}$$

Or, by dividing the latter terms of the first of these proportions each by s. ∠ bBC, and then compounding together two of the proportions, then three of them, &c, striking out the common factors, and observing that the s. ∠ bBC is =

s. $\angle$ BCc, the s. $\angle$ cCD $=$ s. CDd, &c, we shall have the following proportions; viz,

$$\text{B}b : \text{C}c :: \frac{\text{s. } \angle \text{ ABC}}{\text{s. } \angle \text{ AB}b \times \text{s. } \angle\, b\text{BC}} : \frac{\text{s. } \angle \text{ BCD}}{\text{s. } \angle \text{ BC}c \times \text{s. } \angle\, c\text{CD}},$$

$$\text{B}b : \text{D}d :: \frac{\text{s. } \angle \text{ ABC}}{\text{s. } \angle \text{ AB}b \times \text{s. } \angle\, b\text{BC}} : \frac{\text{s. } \angle \text{ CDE}}{\text{s. } \angle \text{ CD}d \times \text{s. } \angle\, d\text{DE}},$$

$$\text{B}b : \text{E}e :: \frac{\text{s. } \angle \text{ ABC}}{\text{s. } \angle \text{ AB}b \times \text{s. } \angle\, b\text{BC}} : \frac{\text{s. } \angle \text{ DEF}}{\text{s. } \angle \text{ DE}e \times \text{s. } \angle\, e\text{EF}},$$

and so on.

Otherwise.

Since Cp or Bn : Bm or nb :: s. $\angle$ Bbn,

or s. $\angle$ ABb : s. $\angle$ bBC or s. $\angle$ BCc :: $\dfrac{1}{\text{s. } \angle \text{ BC}c} : \dfrac{1}{\text{s. } \angle \text{ AB}b}$;

and Cp or qC : Cq or Dr :: s. $\angle$ cCq or s. $\angle$ CDd : s. $\angle$ cCq or s. $\angle$ BCc :: $\dfrac{1}{\text{s. } \angle \text{ BC}c} : \dfrac{1}{\text{s. } \angle \text{ CD}d}$;

it is clear that Cp is as $\dfrac{1}{\text{s. } \angle \text{ BC}c}$; that is, the forces mB, pC, rD, &c. are always reciprocally as the sines of the angles which they make with the vertical line.

And since Cc $= \dfrac{\text{C}p \times \text{s. } \angle\, \text{C}p c}{\text{s. } \angle\, c\text{C}p} = \dfrac{\text{C}p \times \text{s. } \angle \text{ BCD}}{\text{s. } \angle\, c\text{CD}}$;

therefore any force or weight Cc is as $\dfrac{\text{s. } \angle \text{ BCD}}{\text{s. } \angle\, c\text{CB} \times \text{s. } \angle\, c\text{C}d}$.

And this is the same as the property in corol. 4 to the 3d proposition following.

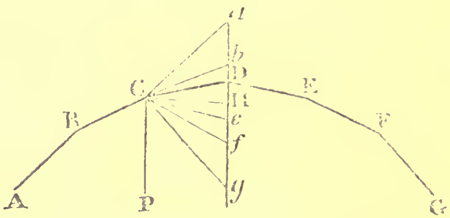
Corol. If DC be produced to h; then, the sine of the angle hCB being equal to the sine of its supplement BCD, the same weight or force Cc will be always proportional to $\dfrac{\text{s. } \angle\, h\text{CB}}{\text{s. } \angle \text{ BC}c \times \text{s. } \angle \text{ DC}c}$; which three angles together make up two right angles.

Properties similar to the foregoing are otherwise determined in the following propositions.

PROP. III.

Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, *&c. all in the same vertical plane, connected together and freely moveable about the joints or angles* A, B, C, D, E, *&c, and kept in equilibrio by their own weights, or by weights only laid on the angles: It is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.*

Solution.—Through any point, as D, draw a vertical line *a*DH*g*, &c: to which, from any point, as C, draw lines in the direction



of, or parallel to, the given lines or beams, viz, C*a* parallel to AB, and C*b* parallel to BC, and C*e* to DE, and C*f* to EF, and C*g* to FG, &c; also CH parallel to the horizon, or perpendicular to the vertical line *a*D*g*, in which also all these parallels terminate.

Then will all these lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz. vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars . . . . AB, BC, CD, DE, EF, FG, are proportional to their parallels . . C*a*, C*b*, CD, C*e*, C*f*, C*g*; and the vertical weights on the angles B, C, D, E, F, &c, are as the parts of the vertical . . . . *ab*, *b*D, D*e*, *ef*, *fg*, and the weight of the whole frame ABCDEFG, . . . is proportional to the sum of all the verticals, or to *ag*; also the horizontal thrust, at every angle, is every where the same constant quantity, and is expressed by the constant ho-

Demonstration.—All these proportions of the forces derive and follow immediately from the general well known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three forces, viz, the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But C*a* is parallel to AB, and C*b* to BC, and *ab* to the vertical weight; those three forces are therefore proportional to the three lines *ab*, C*a*, C*b*.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines *b*D, C*b*, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines D*e*, CD, C*e*.—And the three forces balancing the angle E, are proportional to their three parallel lines *ef*, C*e*, C*f*.—And the three forces balancing the angle F, are proportional to their three parallel lines *fg*, C*f*, C*g*. And so on continually, the oblique forces or thrusts in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz, the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force *cb*, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *ac*, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force *ce*, in direction DE, resolves into the two CH, H*e*, being the same horizontal force with the vertical H*e*; and the oblique force *cf*, in direction EF, resolves into the two CH, H*f*; and the oblique force *cg*, in direction FG, resolves into the two CH, H*g*; and the oblique force *cg*, in direction FG, resolves into the two CH, H*g*; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals, viz, *a*H the whole vertical pressure at B, from the weights on the angles B, C, D: and *b*H the whole pressure on C from the weights on C and D; and DH the part of the weight on D causing the oblique force DC; and H*e* the other part of the weight on D causing the oblique pressure DE; and H*f* the whole vertical pressure at E from the weights on D and E; and H*g* the whole vertical pressure on F arising from the weights laid on D, E and F. And so on.

So that, on the whole,

*a*H denotes the whole weight on the points from D to A;
and H*g* the whole weight on the points from D to G;
and *ag* the whole weight on all the points on both sides;

while *ab*, *b*D, D*e*, *ef*, *fg* express the several particular weights laid on the angles B, C, D, E, F.

Also, the horizontal thrust is every where the same constant quantity, and is denoted by the line CH.

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, C*a*, C*b*, CD, C*e*, C*f*, C*g*.

Corollary 1. It is obvious, and remarkable, that the lengths of the bars AB, BC, &c, do not affect or alter the proportions of any of these loads or thrusts; since all the lines C*a*, C*b*, *ab*, &c, remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, D*b*, *ba*, then C*b*, C*a* give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius; then it is visible that H*a* is the tangent, and C*a* the secant of the elevation of C*a* or AB above the horizon; also H*b* is the tangent and C*b* the secant of the elevation of C*b* or CB; also HD and CD the tangent and secant of the elevation of CD; also H*e* and C*e* the tangent and secant of the elevation of C*e* or DE; also H*f* and C*f* the tangent and secant of the elevation of EF; and so on; also the parts of the vertical *ab*, *b*D, *ef*, *fg*, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, a, b, D, e, f, g, &c, made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical ag, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant, than the above, and are therefore omitted; the following only excepted, which are inserted here on account of their usefulness.

Corollary 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle bCD, in which the side bD is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : Cb : : sin. bCD : bD; that is, bD is as $\frac{\text{sin. } b\text{CD}}{\text{sin. D}} \times \text{C}b$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore bD is as sin. bCD × sec. DCH × Cb; and Cb being as the secant of the angle bCH of the elevation of bC or BC above the horizon, therefore bD is as sin. bCD × sec. bCH × sec. DCH; and the sine of bCD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as bD, is as the sin. BCD

$\times$ sec. DCH $\times$ sec. bCH, that is, as the continual product of the sine of that angle and the secants of the elevations of its two sides above the horizon.

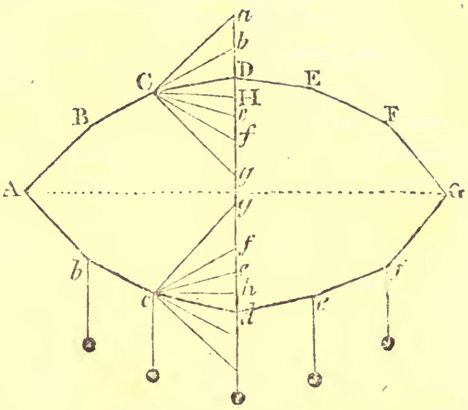
Corol. 4.—Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or sines of their complements: but BCP $=$ cbH, is the complement of the elevation bCH, and DCP is the complement of the elevation DCH; therefore the secant of bCH $\times$ secant of DCH is reciprocally as the sin. BCP $\times$ sin. DCP; also the sine of bCD is $=$ the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. bCD $\times$ sec. bCH $\times$ sec. DCH, is also proportional to $\frac{\text{sin. BCD}}{\text{sin. BCP} \times \text{sin. DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

Scholium.—The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures by, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz, that an assemblage of bars or beams, being connected together by joints at their extremities, and freely moveable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a perfect balance. It may be further observed, that the conclusions here derived, in this proposition and its corollaries, exactly agree with those derived in a very different way, in the former editions of the principles of bridges, viz, in props. 1 and 2, and their corollaries; and which have been here repeated, in the foregoing prop. 2.

PROP. IV.

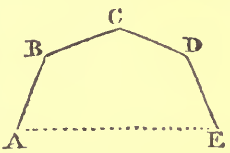
If the whole figure in the third proposition be inverted, or turned round the horizontal line AG *as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle* D, *d*, *&c, being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles* B, C, D, E, F, *of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; then will those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.*



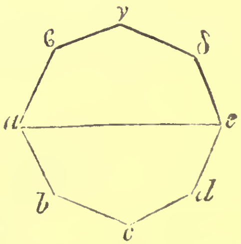
This necessarily happens, from the equality of the weights, and the similarity of the positions and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c, in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c, in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, *ab* = *ab*, *b*D = *bd*, *dh* = DH, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c, which being parallel to the beams A*b*, *bc*, &c, will denote the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium.—The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances, A*b*, *bc*, &c, on a thread, or cord, or chain, &c, having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So that, whatever properties are inferred in the corollaries to the 3d prop. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied. Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt



but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts, which may require the aid of ties or stays, to keep the frame in its position. Here the mechanic has nothing to do, but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them loosely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins, *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and

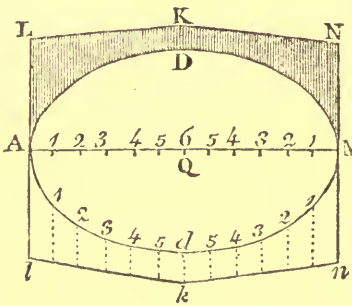


frame of a curb roof $a\beta\gamma\delta e$, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.

Indeed this principle is very fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method, in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem; some examples of which will be shown hereafter.

PROP. V.

To form mechanically a balanced Festoon arch, on the principles of the last proposition; having a given pitch or height and span, and also a given height and form of wall or roadway over it.



Let AM be the given or proposed span of the arch, DQ its pitch or greatest height, DK the thickness at the crown, and ALKNM the given anterior form of the wall: in order to determine the form of the curve ADM which shall put that wall in equilibrio.

Invert the whole figure ALKNM, as in the opposite position A*lkn*M, or construct this latter figure, on the lower side of AM, exactly equal and similar to the proposed upper one; the point *d* answering to the point D, and the point *k* to the point K, &c. Let a very fine and thin, but strong line, such as a fine silken cord, or a bricklayer's working line, or perhaps a very fine and slender chain of small links, be suspended from the extreme points A and M, and of such a length, that its middle point may hang at the point *d*, or a little below it. Divide the given span or width AM into a number of equal parts, the more the better, as at the points 1, 2, 3, 4, 5, &c; from which draw vertical lines, cutting the festoon chain or cord in the corresponding points 1, 2, 3, 4, 5, &c. Then take short pieces of another chain, and suspend them by these points of the festoon 1, 2, 3, &c, as represented by the dotted verticals in the lower part of the figure. This will somewhat alter the form of the curve. If now the new curve should correspond with the point *d*, and all the bottoms of the vertical pieces of appended chain also coincide with the given line of roadway *l k n*, the business is done. But if both those coincidences do not take place, then alterations must be made, by trials and by judgment, in lengthening or shortening either the festoon A*d*M, or the appended vertical pieces of chain, or in both, till such time as those coincidences are accomplished, namely, the bottom of the arch with the point *d*, and the bottom of the appended pieces with the boundary *l k n*. Then re-invert the whole figure, or otherwise trace out the upper curve ADM exactly like or the same as the lower one A*d*M, and there will be obtained an arch sustaining the wall above in perfect equilibrium.

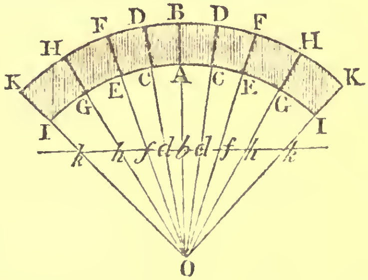
Scholium.—Thus then, as explained by professor Robison, we have an easy and practical way, by which any common intelligent workman may readily construct for himself the form of a real balanced arch, to any proposed design for a bridge. In this method, the thinner and lighter the festoon line is, so as to bear but a small proportion to the weight of the appended pieces of chain, so much the more exact will the conclusion be obtained, when the superincumbent wall is of uniform weight of masonry. But as the festoon line represents the line of voussoirs or arch stones, in the constructed arch, if these only are solid, and the rest of the wall or matter above them be looser and lighter, then there ought to be an equality of proportion between the weights of the festoon chain and the string or rib of arch stones, and between the superior wall and the appended pieces of chain; a circumstance of equality to be obtained by mutual accommodations and calculations adapted to the real circumstances of the case.

The chief objection to the curve found in this way is a want of elegance, and perhaps too of convenience and of economy, because it does not spring or rise at right angles to the horizontal line, but at a much smaller angle; and which indeed is the case with all curves of equilibration. However, this is a circumstance which can be very safely and profitably remedied; for in the part of the flanks near the piers, it may be cut away to hollow the arch out to any form we please, so as, for instance, to resemble the elliptical arch, which is one of the most graceful of all; because the masonry is so solid and strong in that part. And this will be not only more agreeable to the eye, but will also leave more room for water and boats to pass, and will be a saving in the expence of masonry. To accomplish this end with more regularity and method, instead of dividing the horizontal line into equal parts at the points 1, 2, 3, &c, if the festoon chain itself be so divided, viz, into equal parts, and the pieces of chain be appended at these, in the manner before mentioned, then the greater number of these pieces being thus near the extremities, they will draw the arch more down in that part, and thus hollow it out there in a more regular and uniform manner, making the shape more pleasing and commodious, and yet leaving it sufficiently near a true balance.

The following proposition is here added, to determine the figure of a balanced arch, on the supposition that the voussoirs are at liberty to slide on each other. A principle indeed having no real foundation in fact, though it has been much insisted on by some persons.

PROP. VI.

It is proposed to determine the nature and properties of a balanced arch, as derived from the property of the wedge, or by considering the voussoirs or arch-stones as frustrums of wedges.



Let ACEGI &c, be the inner or lower curve of an arch, formed of the voussoirs, or wedge pieces, the vertical sections of which are the quadrilaterals AD, CF, EH, GK, &c, considered as so many elementary parts of the arch, the upper sides of them forming the exterior or outer curve BDFHK, and their butting sides making the joints AB, CD, EF, GH, IK, &c, which joints produced, meet in the point O, of the vertical line OAB. Through any point b, in that line, draw the horizontal line $bdfhk$, or perpendicular to the vertical line OAB, and cutting the directions of the joints in the respective or corresponding points b, d, f, h, k, &c.

Now every wedge in the balanced arch, supposing its sides polished, must be kept in equilibrio, in its place, by the mutual action of three forces, viz, by its own weight acting in a direction perpendicular to the horizon, and by the thrust or pressure of the two adjacent wedges, one on each side, in directions perpendicular to their sides, or to the joints: So, for instance, the wedge AD is balanced, or kept in equilibrio, by its own weight acting in the vertical direction BO, and by two forces acting perpendicularly to AB and CD; and the stone CF, by its weight in the vertical direction, and by two forces perpendicular to CD and EF; also the stone EH, by its weight acting vertically, and by two forces perpendicular to EF and GH; also the stone GK, by its weight vertically, and by two forces perpendicular to GH and IK; and so on, the weights all acting in the vertical direction parallel to BAO.

But, whenever three forces balance one another, they have then to each other the same ratios as the sides of a triangle drawn perpendicular to the directions of the forces. Therefore the three forces balancing the wedge AD, are proportional to the three sides of the triangle *obd*, these sides being respectively perpendicular to those forces, viz, the side *bd* perpendicular to the vertical direction of gravity, also *ob* perpendicular to the force against the joint AB, and *od* perpendicular to the force against the joint CD. For the same reason the wedge CF is balanced by three forces proportional to the three sides *df*, *od*, *of*, of the triangle *odf*; and the wedge EH by forces proportional to the three sides *fh*, *of*, *oh*, of the triangle *ofh*; and the wedge GK by forces proportional to the three sides *hk*, *oh*, *ok*, of the triangle *ohk*; and so on. So that, in all these cases, the weights of the wedges, and their oblique push perpendicular to the joints, will have these following ratios, viz,

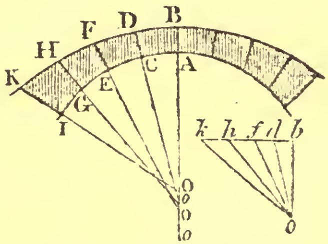
the weights of the wedges - - - AD, CF, EH, GK, &c,
as the parts of the horizontal - - *bd*, *df*, *fh*, *hk*, &c,
and the push at the joints as - - *ob*, *od*, *of*, *oh*, &c,
also the sums of the wedges, or the parts, AD, AF, AH, AK,
are proportional to the perpendiculars *bd*, *bf*, *bh*, *bk*,
which are the tangents of the angles BOD, BOF, BOH, BOK, &c,
of which the oblique thrusts *od*, *of*, *oh*, *ok*, are the secants, to the radius *ob*, which denotes the constant push in the horizontal direction at every wedge, or every point of the arch. Which, on the whole, amounts to this, viz, that the weights of any part of the balanced arch, or set of wedges, commencing from the vertex, are directly proportional to the tangents of the angles which the joints make with the vertical line or direction, while the oblique thrusts, in the directions of the arch at the extremity, or perpendicular to the joints, are proportional to the secants of the same angles; the constant horizontal push, at every point, being proportional to the radius.

And this property comes to the very same thing as the properties in the foregoing propositions, because the angles of elevation of the curve at every point, or of the direction of the tangents there, or of the curve itself, are equal to the angles in this proposition, which the joints form with the vertical direction. So that, all the three theories in these four propositions are all one and the same in effect, amounting to the very same thing, and yielding the same conclusions. And therefore, whatever consequences may further be drawn from any one of them, may be understood as deduced from the whole.

Scholium.—In the practice of bridge-building, the key piece, or wedge at the crown, is a solid, having its magnitude and weight half on each side of the middle vertical line; whereas, in this proposition, it has been supposed that this wedge is divided and actually separated in two by that line AB: this however will cause no difference in the theory, nor yet in the practice; for, in any calculations that may be required, it is only necessary to suppose the key piece divided exactly in the middle, then taking half its weight for the weight of the piece AD, and computing all the other weights and angles from the middle line AB.

It has also been supposed, in all the three theories that have been contemplated, that the constituent parts are formed of materials perfectly smooth and polished, and put together without cement, and without all kinds of ties or bars, so as to leave them quite at liberty to slide over each other, the parts being kept in a perfect balance by means of their shape, weight, and disposition only. This, it must be acknowledged, is not the case in real practice; as here all the materials are quite rough, which very much prevents them from sliding by each other, even when their abutting surfaces are laid at a considerable slope or angle. But this circumstance however, so far from being a disadvantage, by thus deviating from the theory, is on that very account of great use and benefit. For, the equilibrium among the constituent parts of the arch, established by the foregoing theories, is of that nice and critical nature, that the whole hangs in a kind of tottering state of balance, from the perfect polish of the parts, so that any the least accidental extraneous force or pressure, on any particular part, would destroy the equilibrium, and cause the whole to fall down, except for the length of the joints and stones. The theory also supposes the parts, constituting the fabric, to be exceedingly small, and may be even round, small, polished globules. But because of the shape and roughness and magnitude of the parts, of which an arch is constituted, it comes to pass, that a moderate degree of imperfection in the structure, or any accidental shocks or pressure from external objects, has no sensible effect in displacing or deranging the materials: for the wedge-like form prevents any piece from easily dropping out by itself; and the roughness of the sides prevents the wedges from sliding; also the considerable magnitude of the stones, or other matter, while it enables them to bear the weight and pressure of the whole fabric, without being crushed to pieces, admits of a small displacing of materials, or deviation from a perfect balance, as prescribed by theory, without suffering any sensible inconvenience.

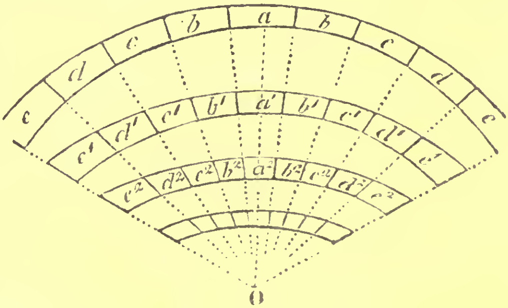
It has been supposed in this proposition, that the directions of the joints, CD, EF, GH, &c, when produced, all meet in the same point O, of the vertical line OAB. This however is not necessary in the theory; as the directions of the joints may meet the vertical in so many different points O, O, O, &c, as in this fig. and yet all the parts and their affections have still the same properties. This will be made evident by constituting the small triangles, *obd*, *obf*, &c, apart, as in



this figure, by drawing, from one point o, the lines ob, od, of, &c, still parallel to the joints AB, CD, EF, &c, meeting the horizontal line in the points b, d, f, &c: for, because these lines are perpendicular to the actions of the forces, of pressure and push, of the arch pieces, the same proportions among these, as before deduced, still take place, and hold good; viz. that the weights are in proportion as the parts of the line *b d f h k*, and the oblique push as the corresponding lines *ob*, *od*, *of*, &c, of which *ob* is as the horizontal thrust.

It has also been supposed, that the joints are cut or drawn perpendicular to the inner curve at every point, or that all the angles at it, C, E, &c, are right-angles. But neither is this necessary in the theory; for the system of balancing will be still the same, whatever those angles may be, whether all alike or all various, as these differences will only cause an alteration in the weight or length of the arch-pieces, which still will be represented in their proportions by the parts of the line *bdfhk*. And indeed we often see this kind of oblique joints employed in the small arches in the common practice of architecture and building, as over windows, doors, gateways, &c. But yet such a practice is not to be admitted into the larger kind of arches, employed in bridges, &c, as being both ungraceful and troublesome, as well as weakening the fabrick.

It is manifest, from all the theories, that the balancing of the arch is not restricted to any particular kind of curve or shape, for either the under or upper curve; as the arch may be balanced with any particular curves we please. It also follows very evidently, that the same angles or directions of the joints may be employed to balance a great variety of arches, and indeed any sort of an arch whatever; as in this fig.; where, if the wedges *a*, *b*, *c*, *d*, &c, form a balanced arch, by being taken in the required proportion to each other, viz, as the differences of the



tangents of the angles formed by their sides with the vertical line; then, if the under curve of any of the other lower arches be assumed of any shape at pleasure, the upper curve of them will be found, by taking their corresponding wedges, *a*1, *b*1, *c*1, &c, or *a*2, *b*2, *c*2, &c, or *a*3, *b*3, *c*3, &c, in the same proportions to each other as the wedges *a*, *b*, *c*, &c, are in the uppermost arch; and all the sets of wedges will form balanced arches.

EXAMPLE.

The theory laid down in the preceding propositions, which give, all of them, the same conclusions, will serve as a foundation on which to establish a method for constructing arches of equilibration, on any proposed curve whatever. The method however will require some further preparation, to render the application to practice easy and convenient. We may here, however, in the mean time, just take one example, in order to show the facility of the mode of calculation from the theory, so far as it has now been laid down. In this example, we shall suppose that the intrados curve is a circular arc, which is formed by the under sides of the wedge pieces, the joints between which are all perpendicular to that curve, as the only proper position, or all directed exactly to the centre of the curve. We shall also suppose the wedge pieces to form equal parts of that arc, of the quantity of 5° each, that is, each wedge subtending at the centre an angle of 5 degrees, the key, or middle wedge at the crown, therefore, extending 2 degrees and a half on each side of the vertical line passing through the centre; and have 17 other wedges, of equal angle (5°) on each side of the key, making in all 35 wedges, which, at 5 degrees each, will form an entire arch of 175 degrees. In this case, the angle which the sides of the middle wedge forms with the middle vertical line, will be that of half the breadth of the wedge, or $2\frac{1}{2}$ degrees; and the angles which the sides of the other wedges, on each hand of the crown or key wedge, form with the vertical direction, will be found by adding continually the breadth of each wedge (5 degrees), to the said $2\frac{1}{2}$ degrees; by which it will be found that the angles at the centre, formed with the vertical, by the said lower edges of the arch pieces, in order after the key, will be as follows, viz, that of the 2d wedge $7\frac{1}{2}$ degrees; that of the 3d, $12\frac{1}{2}$ degrees; that of the 4th, $17\frac{1}{2}$ degrees; and so on to the 17th or last on each side the key, which will have its lower edge making an angle of $87\frac{1}{2}$ degrees with the vertical direction: all which angles, of inclination to the vertical, are ranged in the 2d column of the following tablet, the first, or half the middle wedge, making an angle of $2\frac{1}{2}$ degrees. We shall also suppose the weight of the middle wedge at the crown to be a certain given quantity, represented by unity or 1, and express the several other weights and pressures, as in the other columns of the said tablet, in terms of that unit: so that all these proportional numbers for the other weights and pressures, will require to be multiplied by any other weight of middle wedge which may happen to occur in any other case.

Now, in regard to the rule for computing all the other weights and pressures, according to the conclusions from the preceding theory, it is very easy and simple indeed, viz, that the weight of any part of the arch, counted from the vertex or crown downward, is always proportional to the tangent of the angle of inclination of the lower wedge to the vertical, while the oblique push or pressure, in direction of the curve, is proportional to the secant of the same angle, and the constant horizontal thrust is proportional to the radius. For which reason it is, as formerly observed, that the constant horizontal thrust is a proper radical measuring unit, by means of which to compute the two other circumstances, namely, the weight of the arch, and the oblique push or pressure in the direction of the curve: for, the horizontal thrust being taken for radius, then the weight of the semiarch will be the tangent of the angle with the vertex, and the oblique pressure the secant of the same angle, to that radius Consequently, if the constant horizontal push be called h, then the weight of the semiarch will be $h \times t$, or h

multiplied by the tangent of the side's inclination to the vertical, and the oblique pressure of the arch will be $h \times s$, or h multiplied by the secant of the same angle. So that, in calculating the said several weights and oblique pushes of the arches, we have nothing to do but to take out, from a trigonometrical table, the tangents and secants of the several angles of inclination to the vertical, as contained in the 2d column of the tablet, and multiply all the tangents and secants by the number expressing the constant horizontal thrust, for all the values of the several weights and pressures, as arranged in the 3d and 4th columns of the tablet; the products of the tangents being the several weights of the half arches, in the 4th column, and the products of the secants being the oblique pressures of the same in the arch's direction, as in the 3d column. This calculation will be rendered still easier by using the log. tangents and secants; for there will then be nothing to do, but to take out all the log. tangents and secants; then to each of them add the constant log. of the horizontal thrust; lastly, take out the natural numbers answering to these sums, and they will be the required weights and pressures.

As to the uniform horizontal thrust, which is the constant multiplier, its value is easily found thus: It has been shown that this horizontal thrust is every where in the same proportion to the weight of half the middle or key wedge, as radius is to the tangent of half the angle of that wedge; that is, as $t : 1 :: \frac{1}{2}w : \frac{1}{2}w \div t = h$ the horizontal thrust, putting w for the weight of the key piece, and t for the tangent of half its angle; or, if we put its weight $w = 1$, then this will become $\frac{1}{2} \div t = h$ the horizontal thrust. Now, in the example, the angle subtended by the key is 5 degrees, the half of which is $2\frac{1}{2}$ degrees, and the tangent of this is $\cdot0436609$; then $\frac{1}{2}$ or $\cdot5 \div \cdot0436609 = 11\cdot451883 = h$ the constant horizontal thrust, that is, 11 times the weight of the key piece and nearly one half more; or, the same may be easier found from the cotangent of the same angle $2\frac{1}{2}$ degrees, which is $22\cdot903766$, the cotangent of any angle being equal to the reciprocal of its tangent, to the radius 1;

therefore, in general, $\frac{1}{2} \div$ tang. $= \frac{1}{2}$ the cotang. is $= h$ the horizontal thrust, and in the present instance the half of the cotangent 22·903766 is 11·451883 the same value of the horizontal thrust as before.

Hence then the constant number 11·451883 is to be multiplied by the tangents of all the vertical angles, to give the weights of the semiarch, in the 4th column, and by the secants of the same angles, to give their oblique pressures, as in the 3d column; or else, to work by the logarithms, the log. of the constant number 11·451883, which is 1·0588769, is to be added to all the log. secants and tangents of the said angles, then the corresponding natural numbers taken, and ranged in the 3d and 4th columns of the table.

The differences of the numbers in the 4th column are taken, and ranged in the 5th or last column, for the weights of the single wedge pieces taken separately, making the whole of the first or key wedge equal to 1.—The table is as follows.

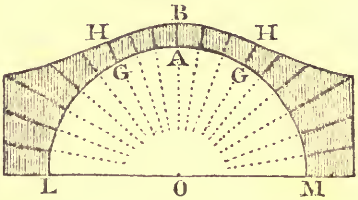
No. of sections.	Vertical angles of the joints, or $\angle$ s o.	Oblique pressures, $= h \times$ sec. $\angle$ o.	Wts. of half arches, $= h \times$ tan. $\angle$ o.	Wts. of the sections or wedges.
	degrees.			
1	$2\frac{1}{2}$	11·46279	0·5	1·
2	$7\frac{1}{2}$	11·55070	1·50767	1·00767
3	$12\frac{1}{2}$	11·72993	2·53882	1·03115
4	$17\frac{1}{2}$	12·00763	3·61076	1·07194
5	$22\frac{1}{2}$	12·39543	4·74352	1·13276
6	$27\frac{1}{2}$	12·91065	5·96147	1·21795
7	$32\frac{1}{2}$	13·57837	7·29565	1·33418
8	$37\frac{1}{2}$	14·43478	8·78734	1·49169
9	$42\frac{1}{2}$	15·53267	10 49372	1·70638
10	$47\frac{1}{2}$	16·95094	12·49753	2·00381
11	$52\frac{1}{2}$	18·81177	14·92439	2·42686
12	$57\frac{1}{2}$	21·31377	17·97585	3·05146
13	$62\frac{1}{2}$	24·80112	21·99886	4·02301
14	$67\frac{1}{2}$	29·92521	27·64727	5·64841
15	$72\frac{1}{2}$	38·08334	36·32073	8·67346
16	$77\frac{1}{2}$	52·91028	51·65611	15·33538
17	$82\frac{1}{2}$	87·73628	86·98568	35·32957
18	$87\frac{1}{2}$	262·54113	262·29125	175·30557

From this calculation, as well as from the theorems by which it is made, it is manifest how greatly the weight and the pressure of the semiarch increase towards the bottom or the extremity, where the position of the joint approaches towards the horizontal direction, or the angle it makes with the vertical approaches towards a right-angle; and when that angle actually becomes a right-angle, or the joint quite horizontal, then the weight and pressure become equal and infinite, which must naturally be expected, both because the tangent and secant of the angle (being a right one) are then infinite, and also because it must require an infinite weight or pressure to balance there the constant given horizontal thrust, which is perpendicular to the former.

We may here, by the way, stop to examine a little in what manner the preceding calculation of the weights of the voussoirs may be employed to give a familiar and easy mechanical construction, that may approach very near to a true balanced arch. In order to this, we are to consider, that since the bases, or extents of the under sides, of all the voussoirs, are equal, it will thence happen that their weights will have to each other nearly the same ratios as their lengths, from the under to the upper side of them, or taken in the direction of the radius, that is perpendicular to the under curve or intrados, at least when the breadth or angle of these wedges is very small, which is the case in real practice, the approach to equality being the nearer indeed as their breadth is the smaller. And though the angle of 5 degrees, employed in the preceding calculation, be not such a small breadth as to render the equality and the construction perfect, it will yet serve to show the manner of proceeding in such a way of forming the arch, and will besides approach tolerably near to the truth.

As it is most proper that the joints between the wedges, in the arch of a bridge, should be in directions perpendicular to the under curve of the arch, we shall only exemplify the method in cases of that sort. For this purpose then, let us suppose the intrados or under curve to be divided into a number of equal parts, answering to a breadth of 5 degrees each, or such that the angle formed by every two adjacent joints, when produced, shall be an angle of 5 degrees. Let us then draw a line through the middle point of every one of these breadths, bisecting them, and in a direction perpendicular to the curve at every point. Then, by setting off, upon these lines, from the curve upwards, by a proper scale, lengths which shall have the same ratios to each other as the weights of the corresponding wedges through which these lines pass, or proportional to the numbers in the last column of the foregoing table; then will the lengths of these lines be the extent of the several voussoirs nearly, and therefore, their upper extremities or points being connected, by drawing short lines from one to another, they will limit or form the extrados, or the upper curve or side of the arch, when built of uniform materials, so as to be very nearly in equilibrio.

As it is manifest that the theorems and the calculation have no peculiar restricted reference to any particular curve for the intrados, or under side of the arch, we are therefore at liberty to assume that curve of any form at pleasure; therefore the form of it being so assumed, by then applying the numbers of the foregoing table to it, in the manner above mentioned, we shall have a balanced arch as required. And thus by assuming any different shapes of curve for the intrados, the same numbers in the table will give as many balanced arches as we please. Assuming then, for the inner curve, a semicircle, as in the next fig. having its span or diameter LM 84 feet, consequently its pitch or height OA 42 feet. We shall also assume AB the thickness of the crown or key-piece, equal to 6 feet, or the 14th part of the span, being nearly the proportion employed by good engineers. Dividing each half arc AL, AM, into 9 equal parts, of 10 degrees each, which will be sufficiently small to show the nature and form of the extrados, containing each an extent of two wedges or voussoirs; then from the centre O drawing radii through all the points of division, these, when continued, passing through the middle of every second wedge, the first OAB passing through the middle of the key-piece. Then, on these radii produced, set off, from the arc of the semicircle, AB, GH, &c, every second number in the last column of the table, when multiplied by 6, the assumed length of AB; then, drawing with the hand a curved line through the extremities of all the exterior lines, it will be the extrados required, exhibiting the form and limit of the wall built of uniform materials, above the circular soffit, so as to constitute an arch of equilibration nearly as in the annexed fig.

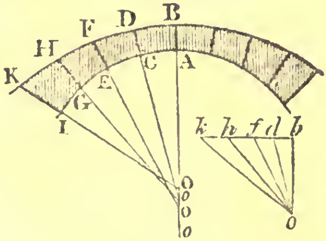


Where it is seen that the extrados follows nearly a course parallel to the intrados for about 30 degrees on each side of the vertex; after which, it begins to bend the contrary way, having there a contrary flexure during the rest of its course, going off to an infinite distance on each side parallel to the base, making the voussoirs at last of an infinite length, and composing all together a form of arch very unfit for adoption in practice.

We shall now show, in the next proposition, that, by another very strict and genuine construction, an exterior curve is derived exactly similar to the curve here obtained: in the determination of which, some part of the mode of reasoning in the demonstration of the last prop. is here again necessarily repeated.

PROP. VII.

If ACEGI *&c, be an arch, supporting a wall* ABKI, *formed of the voussoirs or arch stones* AD, CF, *&c, lying aslope, on smooth surfaces, and having the joints* AB, CD, *&c, every where perpendicular to the curve of the arch* ACE *&c. It is required to find the lengths of these arch stones, so that the whole fabric may be balanced, or kept in equilibrio.*



Let A be the vertex of the inner curve of the proposed arch; AB the given thickness of the wall at the crown, or length of the archstone there; also BAO, DCO, &c, the joints produced, making AO the radius of curvature at A, and CO at C, and EO at E, &c; the bases of the stones AC, CE, EG, GI, &c, being so many elements or small parts of the arch; and the vertical sections of the stones, or the areas of the quadrilaterals AD, CF, EH, GK, being proportional to the weights of them.

Now every stone in the balanced arch will be kept in equilibrio by three forces, viz, by its own weight acting perpendicular to the horizon, and by the pressures of the two adjacent stones, in directions perpendicular to their sides, or to the two adjacent joints: So, for instance, the stone AD is balanced, or kept in equilibrio, by its own weight, and by two forces acting perpendicularly to AB and CD; and the stone CF, by its weight, and by the two forces perpendicular to CD and EF; also the stone EH, by its weight, and by the two forces perpendicular to EF and GH; also the stone GK, by its weight, and by the two forces perpendicular to GH and IK; and so on; all these weights acting in the vertical direction BAO.

But whenever three forces balance one another, they have then the same ratios as the sides of a triangle drawn perpendicular to their directions. Therefore, if there be constructed another figure *obdfhk*, having *bk* horizontal, or perpendicular to a given vertical line *ob*; and having *od* parallel to OD, and *of* to OF, and *oh* to OH, and *ok* to OK, &c: then the three forces balancing the stone AD are proportional to the three sides of the triangle *obd*, these sides being respectively perpendicular to those forces; for the same reason, the stone CF is balanced by the three forces *df*, *od*, *of*; also the stone EH by the three *fh*, *of*, *oh*; and the stone GK by the three *hk*, *oh*, *ok*; and so on; in all these cases the weights of the stones being proportional to the bases *bd*, *df*, *fh*, *hk*, of the triangles *obd*, *odf*, *ofh*, *ohk*. But as these triangles have all the same common altitude *ob*, they have the same ratios as their bases *bd*, *df*, &c, which bases, it has been shown, are proportional to the weights of the stones, which have also been found proportional to the quadrilateral areas AD, CF, &c; therefore the quadrilaterals AD, CF, EH, GK, are respectively proportional to the triangles *obd*, *odf*, *ofh*, *ohk*.

But, as these small triangles have their angles respectively equal to the angles of the corresponding sectors, because their sides are parallel by the construction; that is, the angle bod = the angle BOD, &c; their areas are therefore proportional to the squares of their corresponding sides;

viz. the sectors OBD, OAC, *obd*,
proportional to OB^2, OA^2, ob^2;
and the sectors ODF, OCE, *odf*,
proportional to OD^2, OC^2, od^2; and so on.

Therefore, by taking the differences,

$$AD : obd :: OB^2 - OA^2 : ob^2,$$
$$\text{and } CF : odf :: OD^2 - OC^2 : od^2,$$
$$\text{and } EH : ofh :: OF^2 - OE^2 : of^2,$$
$$\text{and } GK : ohk :: OH^2 - OG^2 : oh^2, \text{ \&c.}$$

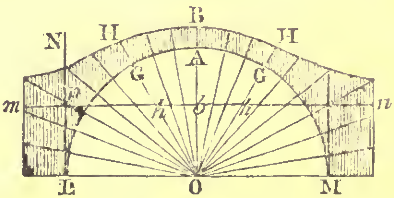
Hence, if ob^2 be taken $= OB^2 - OA^2$,
then od^2 is $= OD^2 - OC^2$,
and of^2 is $= OF^2 - OE^2$,
and oh^2 is $= OH^2 - OG^2$, &c.

Or, by transposing, $OB^2 = OA^2 + ob^2$,
and $OD^2 = OC^2 + od^2$,
and $OF^2 = OE^2 + of^2$,
and $OH^2 = OG^2 + oh^2$, &c.

Which gives us the following geometrical construction, viz, Produce the joints till OA, OC, OE, OG, &c, be equal to the several radii of curvature at the corresponding points, A, C, E, &c; to which also draw the parallels *ob*, *od*, *of*, &c. Then take $ob = \sqrt{OB^2 - OA^2}$, and draw *bdfhk* perpendicular to ob. Lastly, make OD $= \sqrt{OC^2 + od^2}$, and OF $= \sqrt{OE^2 + of^2}$, and OH $= \sqrt{OG^2 + oh^2}$, &c; then shall the line or curve drawn through all the points B, D, F, H, K, &c, be the top of the wall, so as the whole fabric may be balanced, or kept in equilibrio, by the mutual weights and pressures of the stones, having smooth or polished sides, and at liberty to descend along them.

Note.—When the given interior curve ACE &c, is a circle, all the radii of curvature will be equal to each other, and will all have the same centre O. But in other curves, having various degrees of curvature, the radii and centres of curvature will be all different.

EXAMPLE.



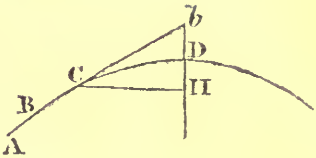
Suppose the interior curve to be a Semicircle. And suppose the span or diameter LM to be 84 feet, the height or pitch OA 42 feet, and the thickness at the crown AB 6 feet, which is the 14th part of the span. Then take Ob so, that Ob^2 be equal to $OB^2 - OA^2$, or O$b = \sqrt{OB^2 - OA^2}$ $= 23{\cdot}2379$, and through b draw mbn parallel to the base LM; from the centre O draw a number of radii OhGH &c, cutting the circle in as many points G, and the line mn in as many points h; on the perpendicular LN set off all the distances Lp equal to the several distances Oh, cut on the radii by the directrix mn; then transfer the distances Op to the same radii produced to H, namely taking OH $=$ Op; then shall the points H be so many points of the exterior curve, through all which points the bounding line being drawn with a steady hand, it will be as is seen in the figure to this example, which is accurately constructed and drawn by a scale to the dimensions above given, and which will extend infinitely along the directrix mn, this line being indeed an asymptote to the said curve.

The calculation in numbers is also equally easy and obvious. Thus, taking any given angle AOG, ob being $= \sqrt{OB^2 - OA^2}$, then $Lp = oh = ob \times$ sec. AOG, and hence $OH = Op = \sqrt{OL^2 + Lp^2} = \sqrt{OA^2 + Lp^2}$, which gives a point H in the curve. And the curve thus constructed gives the very same as the fig. p. 35, formed on the principles of prop. 6, as might be expected.

Examples of other curves, besides the circle, might be here taken, but the above case may suffice, as none of them are of a nature to be suitable for, or to hold good, in the construction of arches, at least for the ordinary purpose of bridges. Because, that in such arches, the parts do not endeavour to slide down in the oblique direction of the joints, both on account of the roughness or friction there, and because, when the parts are cemented together by the morter, or keyed together by pieces within side, the weights then all act perpendicular to the horizon, being each fixed to the other parts of the arch, after the manner supposed in the 9th and 10th propositions; and according to the examples to the latter of these, it will therefore be expedient to make such calculations as may occur in cases of real practice.

PROP. VIII.

When a curve is kept in equilibrio, in a vertical position, by loads or weights bearing on every point of it: then the load or vertical pressure on every point, is directly proportional to the product of the curvature at that point, and the square of the secant of the elevation above the horizon of the tangent to the curve at the same point, the radius being 1. *That is, the load or vertical pressure on any point* C, *is directly as the curvature at* C, *and as the square of the secant of the angle* bCH, *made by the tangent* bC *and the horizontal line* CH.



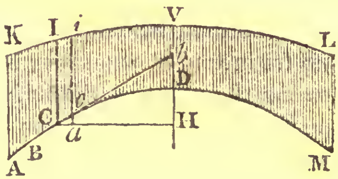
This property will be deduced as a corollary from the properties in the 2d and 3d propositions, according to the idea mentioned in the conclusion of the scholium there, by conceiving the bars or lines kept in equilibrio to become indefinitely small; for, by this means, those bars will form a continued curve line, after the manner of the arch stones in a bridge, constituting an arch of equilibration, by weights pressing vertically on every small or elementary part of the arch.

Now the consequence of the above idea, namely, of the bars becoming very small, and forming a continued curve, is, that the angle *b*CD becomes the angle of contact of the curve and tangent, and the angles *b*CH, DCH become equal to each other; consequently, the vertical load on the point C, which, in the 3d corol. prop. 3, was proportional to the sin. *b*CD × sec. *b*CH × sec. DCH, will be here proportional to the sin. *b*CD × sec^2. *b*CH, or as the angle *b*CD × sec^2. *b*CH, since a small angle (*b*CD) has the same proportion as its sine. But the angle of contact *b*CD, in any curve, is the measure of the curvature there; therefore, lastly, the vertical load or pressure, at any point C, in the curve of equilibration, is proportional to the curvature multiplied by the sec^2. of *b*CH; that is, proportional to the curvature at that point, and also to the square of the secant of the elevation of the curve or tangent above the horizon.

Corol.—Because the curvature at any point in a curve, is reciprocally proportional to the radius of curvature at that point; it follows, therefore, that the vertical load or weight on any point C, is as $\frac{\sec^2. \, b\text{CH}}{r}$, where r denotes the radius of curvature at the point C; that is, directly proportional to the square of the secant of elevation, and inversely proportional to the radius of curvature to the same point.

PROP. IX.

When an upright wall, bounded by a curve beneath, is kept in equilibrio by the mutual weight and pressure of its parts and materials; then the height of the wall above every point of the curve, is directly proportional to the cube of the secant of elevation of the tangent to the curve there, and also directly proportional to the curvature at the same point, or else, which is the same thing, inversely proportional to the radius of curvature there.

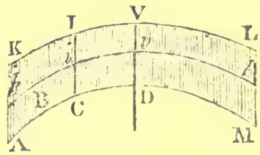


By the last proposition, the load or pressure on every elementary or small portion, cc, of the curve, is proportional to $\frac{\sec^2. b\text{CH}}{r}$. Now this load on every such small equal part of the arch, as cc, is a mass of solid matter $\text{C}\text{I}ic$, incumbent on that part of the curve, and pressing it vertically; and which may be considered as made up of a number of equal heavy lines standing vertically on it; the number of which lines may be expressed by the breadth $\text{C}a$ of the said pillar $\text{C}i$ of heavy materials: but the breadth $\text{C}a$ is $=$ $\frac{cc}{\sec. cca} = \frac{cc}{\sec. b\text{CH}}$, or as $\frac{1}{\sec. b\text{CH}}$, because the element cc is supposed given, or always of the same length, that is, $\text{C}a$ is reciprocally as the secant of the angle of elevation. Hence then the vertical load, or $\text{C}i$, or $\frac{\text{CI}}{\sec. b\text{CH}}$, is as $\frac{\sec^2. b\text{CH}}{r}$; consequently the altitude CI of the wall AKLM, at the point C, is as $\frac{\sec^3. b\text{CH}}{r}$, or as $\sec^3. b\text{CH} \times$ curvature there. That is, the height of the wall above every part of the arch of equilibration, is directly proportional to the cube of the secant of the curve's elevation at that part, also directly proportional to the degree of curvature there, or else inversely as the radius of curvature at the same part.

Corollary 1.—Hence, if the form of the arch, or the nature of the inner curve ABCDM, be given; then the form or nature of the outer line KIL, bounding the top of the wall, or forming what is therefore called the extrados, may be found, so as that the intrados ABCDM shall be an arch of equilibration, or be in equilibrio in all its parts, by the weight or pressure of the superincumbent wall. For, since the arch or nature of the curve is given, by the supposition, the radius of curvature and position of the tangent, at every point of it, will be given, and thence also the proportions of the verticals CI, &c. So that, by assuming one of them, as the middle one VD for instance, or making it equal to an assigned length, the rest of the verticals will be found from it, and will be in proportion as it is greater or less; and then the extrados line KIVL may be drawn through all their extremities.

Or, on the other hand, if the extrados KIVL, or line bounding the top of the wall, be given; then the nature of the correspondent curve of equilibration ABCDM may be found out. And the manner of the practical derivation of both these curves, mutually the one from the other, will be shown in the following propositions.

Corollary 2.—If the intrados curve ABCD should be a circle; then the radius of curvature will be a constant quantity, and equal to the semidiameter of that circle; also the angle *b*CH will be always measured by the arc DC, from the vertex D of the curve; and then the height CI of the wall, will be every where proportional to the cube of the secant of the arch DC.



Corollary 3.—Hence also it follows, that if between the intrados and extrados curves, an intermediate curve *kivl*, be drawn through the middle of the wall, bisecting all the verticals DV, CI, &c, or indeed dividing them in any ratio whatever, so as that it may be every where DV : Dv : : CI : Ci; then if ACDM be an arch of equilibration to the wall AKVLM, it will be an arch of equilibration to the inner wall AkvlM also.

PROP. X.

Having given the Intrados or Soffit, of a Balanced Arch; to find the Extrados. That is, having given the nature or form of an arch; from thence to find the nature of the line forming the top of the seperincumbent wall, by the pressure of which the arch is kept in equilibrio.

The solution of this problem is to be made out generally from the last proposition and its corollaries, by adopting general values of the lines there employed, which belong to all curves whatever: or otherwise by making use of the peculiar values proper to any individual curve, for the solution of particular cases.

For the general solution, in fig. pa. 41, KVL represents the extrados, the form of which is required, and ABCDM the given intrados or soffit of the arch, the vertex of which is D, and DV the height or thickness of the wall there, which is commonly a dimension that is known from the particular circumstances of the case. Now if we make the arch DC $= z$, its element Cc $= \dot{z}$, the absciss DH $= x$, its element $ca = \dot{x}$, the ordinate CH $= y$, its element C$a = \dot{y}$, the height or thickness of wall at the vertex DV $= a$, and the radius of curvature at any point C $= r$, that at the vertex D being $=$ R.

Then, because the height CI, at any point C, is as $\frac{\sec^3 . b\text{CH or of } cCa}{r}$, by the last proposition, and because the secant of cCa is $= \frac{\text{C}c}{\text{C}a} = \frac{\dot{z}}{\dot{y}}$, the radius being 1, therefore CI is as $\frac{\dot{z}^3}{r\dot{y}^3}$, or as $\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{r\dot{y}^3}$, because $\dot{z} = \text{C}c = \sqrt{ca^2 + \text{C}a^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$ or $(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}$.

Or, the general value of CI is $\frac{\dot{z}^3}{\dot{y}^3} \times \frac{Q}{r} = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}^3} \times \frac{Q}{r}$; where Q denotes a certain given or constant quantity, the value of which may be determined by making the general expression equal to a or DV, the height at the crown of the arch.

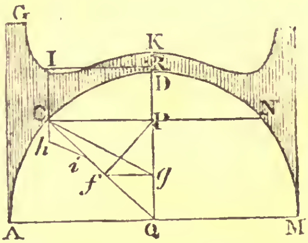
Corollary 1.—Because, at the vertex of the curve D, the angle of elevation is nothing, or its secant $\frac{cc}{ca} = \frac{\dot{z}}{\dot{y}} = 1$ the radius, and the radius of the curvature there being R; therefore the general expression for the height, becomes there DV $= a = \frac{Q}{R}$; consequently Q $= a$R, which is the general value of Q for all curves whatever, expressed in terms of the height a at the crown, and R the radius of curvature at the same point. Hence then, substituting this value of Q instead of it, the general expression or value of CI becomes $\frac{\dot{z}^3}{\dot{y}^3} \times \frac{a\text{R}}{r} = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}^3} \times \frac{a\text{R}}{r}$.

Corol. 2.—Because, in all curves that are referred to an axis, the general value of the radius of curvature r, is $= \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$; therefore, by substituting this value for r in the last expression, the general value of the height CI then becomes $\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^3} \times a\text{R} = \frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^3} \times \text{Q}$, or $= \frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times \text{Q}$ when $\dot{x}$ is constant.

For, as either x or y may be supposed to flow uniformly, and when, consequently, either of their second fluxions may be taken equal to nothing, which will cause one of the terms in the numerator of the above value of CI to vanish; therefore, by striking out either of those terms, and then exterminating either of the unknown quantities by means of the equation to the curve, the particular value of the height CI will be obtained: as is done in the following examples, except in some certain cases, where more peculiar methods may suit them better, as when the radius of curvature is a known quantity, &c.

EXAMPLE 1.

To find the Extrados of a Circular Arch.



That is, ACDM being a circular arc, of which AQ or QD is the semidiameter, Q the centre, and D the vertex of the given circular arch; also K the vertex of the extrados KIG, and the other lines as in the figure.

Making $a =$ DK, $r =$ AQ $=$ QD the radius of the circle, which is also equal to the radius of curvature throughout, or $r = \text{R}$; also $x =$ DP, and $y =$ PC $=$ RI, and $z =$ the arch DC. Then, because $\frac{\text{R}}{r} = \frac{r}{r} = 1$, and $\frac{\dot{z}^3}{\dot{y}^3} =$ the cube of the secant of elevation at C, which is $= \frac{\text{CQ}^3 \text{ or DQ}^3}{\text{PQ}^3}$; therefore the general value of CI, in corol. 1, becomes $\text{CI} = \frac{\text{DK} \times \text{DQ}^3}{\text{PQ}^3} = (\frac{r}{r-x})^3 \times a = (\frac{r}{\sqrt{r^2-y^2}})^3 a$; or, as $\text{PQ}^3 : \text{DQ}^3$ or $\text{CQ}^3 :: \text{DK} : \text{CI}$.

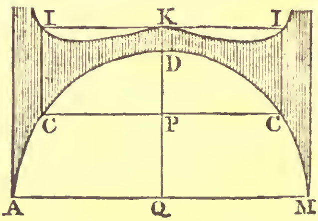
Corol. 1.—This expression for the value of CI, affords a very simple mode of calculation for the case of a circular arch; viz, to the constant logarithm of the height a, add triple the logarithm secant of the elevation, or of the arc DC; then the natural number answering to the sum will be the value of the height CI, at every point C.

Corol. 2.—It gives also a very simple construction by scale and compasses, which is as follows:—Join QC; draw Pf perpendicular to QC, and fg perpendicular to QP; then shall $Qg : QC :: QP^3 : QC^3$; because, by similar triangles, $Qg : Qf :: Qf : QP$ and $:: QP : QC$, or Qg, Qf, QP, QC are four terms in continued proportion, in which case the first Qg is to the fourth QC, as QP^3 to QC^3, the cube of the third to the cube of the fourth. Hence, if CI be taken a fourth proportional to Qg, QC, DK, it will be the length of the vertical line sought. And this fourth proportional will be easily determined in the following manner: viz, Join Cg, and in the vertical line IC downward take $Ch = DK$, and draw hi parallel to Cg, so shall CI be equal to Ci the fourth proportional to Qg, QC, DK, or to QP^5, QC^3, DK, as required.

Corol. 3.—The extrados line in this figure is accurately drawn according to the above construction and calculation, when the thickness DK at the crown is the exact 15th part of the span AM. It falls more and more below the horizontal line, from the crown all the way till the arch be between 30 and 40 degrees, where it takes a contrary flexure, tending upwards, passing the point I very obliquely, and thence rising very rapidly to an unlimited height, in an infinite curve, to which the vertical line AG is an asymptote; a circumstance which must always be the case with every curve, which, like AC, springs perpendicularly from the horizontal line AQM.

This curve cuts the horizontal line nearly over the point of 50 degrees. If DK were taken greater than the 15th part of AM, all the other vertical lines CI would be greater in the same proportion, and the curve KIG would cut the horizontal line drawn through K in some point still nearer to K; but the reverse, or farther off, if DK were taken less than the 15th part. Hence it appears, that a circular arch cannot be put in equilibrio by building on it up to a horizontal line, whatever its span may be, or whatever be the thickness at the crown. And consequently it may generally be inferred, that the circle is not a curve well suited to the purposes of a bridge which requires an outline quite horizontal, but may answer tolerably well when that line bends a little downwards, from the crown toward the extremities; and then a great variety of proportions between the thickness at the crown and the span of the arch might be assigned, which would put the circular arch in equilibrio, nearly.

Now these cases will happen in general when KR vanishes, or is of no length, and then CI must be equal to PK, or nearly so; with which general condition many particular cases may be found to agree nearly. But it may be proper here first to make out a general rule for such cases, which may be done in the following manner:



By the premises, the general value of CI being DK $\times$ $\sec^3$. DC, or as 1 : $\sec^3$ DC :: DK : CI; then, by taking CI $=$ PK, in order to cause the outer curve KI to cross the horizontal line KI at the point I, that proportion becomes

1 : $\sec^3$ DC :: DK : PK or DK + DP,

or $\sec^3$ DC $-$ 1 : 1 :: DP : DK $= \dfrac{\text{DP}}{\sec^3 \text{DC} - 1}$, the radius being 1.

Now, by taking the arch DC of various magnitudes, from DA or 90°, to O or nothing at D, the several thicknesses DK, at the crown, will be found by this theorem, corresponding to the several heights DP, or span CC, as here following, so as to make CDC a balanced arch very nearly. Thus,

1st. If DC be taken $=$ DA or 90°: then its height is DQ $= r$, its span AM $= 2r$, and its secant is infinite; consequently DK $= \dfrac{\text{DQ}}{\text{infin.}} = 0$. That is, the thickness at the crown comes out equal to nothing in this extreme case.

2d. If DC be taken $= 75°$: then its height DP $= \cdot 74118r$, the span CC $= 1\cdot 93185r$, and the sec. DC $= 3\cdot 8637$; therefore DK $= \frac{\text{DP}}{\sec^3. - 1} = \cdot01308r = \frac{\text{cc}}{148}$. That is, the thickness at the crown would be the 148th part of the span, being also much too small for common practice.

3d. If DC be taken $= 60°$: then its height DP $= \frac{1}{2}r$, the span cc $= r\sqrt{3}$, the sec. DC $= 2$; therefore DK $= \frac{\text{DP}}{2^3 - 1}$ $= \frac{1}{7}\text{DP} = \frac{1}{14}r = \frac{\text{cc}}{14\sqrt{3}} = \frac{\text{cc}}{24\cdot2487} = \frac{\text{cc}}{24\frac{1}{4}}$ nearly. That is, the thickness at the crown would be rather less than the 24th part of the span: which is still too small in ordinary bridges.

4. If DC be taken $= 54°$: then its height DP $= \cdot4122r$, the span cc $= 1\cdot618r$, and the sec. DC $= 1\cdot7013$; therefore DK $= \frac{\text{DP}}{\sec^3. - 1} = \cdot10504r = \frac{\text{cc}}{15\cdot41}$. That is, the thickness at the crown would be between the 15th and 16th part of the span; which is nearly the proportion allowed in common bridges.

5. If DC be taken $= 45°$: then its height DP $= r - \frac{1}{2}r\sqrt{2}$, the span cc $= r\sqrt{2}$, the sec. DC $= \sqrt{2}$; therefore DK $=$ $\frac{\text{DP}}{\sec^3. - 1} = \frac{1 - \frac{1}{2}\sqrt{2}}{2\sqrt{2} - 1}r = \frac{r}{2 + 3\sqrt{2}} = \frac{\text{cc}}{6 + 2\sqrt{2}} = \frac{\text{cc}}{8\cdot8284} =$ $\frac{1}{9}$cc nearly. That is, the thickness at the crown would be more than the 9th part of the span: which in common cases is too much.

6. If DC be taken $= 30°$: then its height DP $= r - \frac{1}{2}r\sqrt{3}$, the span cc $= r$, the sec. DC $= \frac{2}{\sqrt{3}}$; therefore DK $=$ $\frac{\text{DP}}{\sec^3. - 1} = \frac{r - \frac{1}{2}r\sqrt{3}}{\frac{8}{3\sqrt{3}} - 1} = \frac{6\sqrt{3} - 9}{16 - 6\sqrt{3}}r = \frac{6\sqrt{3} - 9}{16 - 6\sqrt{3}}\text{cc} = \frac{\text{cc}}{4\cdot03} =$ $\frac{1}{4}$cc nearly. That is, the thickness at the crown would then be almost the 4th part of the span.

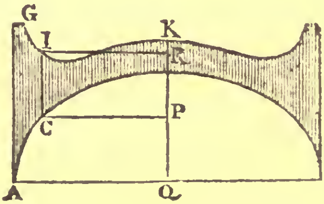
7. If DC be taken $= 15°$: then its height DP $= \cdot03407r$,

the span CC $=$ $\cdot 5176 r$, and the sec. DC $=$ $1\cdot 0353$; therefore $DK = \frac{DP}{\sec^3. - 1} = \frac{DP}{\cdot 11} = 9DP = \cdot 3r = \frac{4}{7}CC$ nearly, or $\frac{4}{7}$ of the span.

From all which it appears, that a whole arch CDC of about 108 or 110 degrees, is the part of the circle which may be used for most bridges with the least impropriety, the thickness at the crown being nearly the 16th part of the span, with a horizontal straight line at top.

EXAMPLE 2.

To determine the Extrados of an Elliptical Arch of Equilibration.



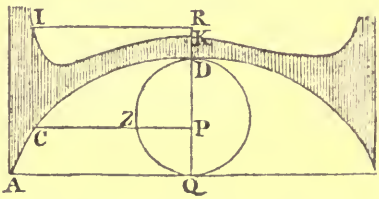
Suppose the curve in this figure to be a semiellipse, with either the longer or shorter axe horizontal: putting h to denote the horizontal semiaxe AQ, and r the vertical one DQ, also $x =$ DP, $y =$ PC, and $a =$ DK, as usual.

Then, by the nature of the ellipse, $r : h :: \sqrt{2rx - xx} : y$; therefore $y = \frac{h}{r}\sqrt{2rx - xx}$, and $\dot{y} = \frac{h\dot{x}}{r} \times \frac{r - x}{\sqrt{(2r - xx)}}$, also $\ddot{y} = \frac{-hr\dot{x}^2}{\sqrt{(2rx - xx)}}$ by making $\dot{x}$ constant. Hence the general value of CI, viz, $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times Q$, becomes $\frac{hr\dot{x}^3 Q}{(2rx - xx)^{\frac{1}{2}}} \times \frac{r^3}{h^3\dot{x}^3} \times \frac{(2rx - xx)^{\frac{3}{2}}}{(r - x)^3} = \frac{r^4 Q}{h^2(r - x)^3}$. But at the vertex of the curve D, where x is $= o$, this expression becomes only $\frac{rQ}{hh}$, which must be $=$ DK or a; therefore the value of Q is $= \frac{ahh}{r}$, which being substituted for it in the above general value of CI, this becomes $\text{CI} = \frac{ar^3}{(r-x)^3} = \frac{\text{DK} \times \text{DQ}^3}{\text{PQ}^3}$, which is the very same expression as the value of CI in the case of the circle in the former example, and which belongs equally to the ellipse in both positions, that is, both with the longer axe vertical, and with the shorter one vertical, as it is in the figure to this example.

Hence it appears, that the flat ellipse is more nearly balanced by a straight horizontal back or wall at top, than the circle is; but the circle more nearly than the sharp ellipse: the want of balance being least in the flat ellipse, but most in the sharp one, and in the circle a medium between the two.

EXAMPLE 3.

To determine the Extrados of a Cycloidal Arch of Equilibration.



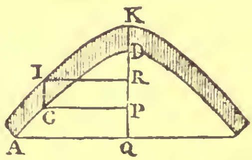
Let DZQ be the circle from which the cycloid ACD is generated; and the other lines as before.

Put a = DK, x = DP, and y = CP = IR, as usual; also put r = DQ the diameter of the circle, and z = the circular arc DZ. Then, by the nature of the cycloid, CZ is always equal to DZ $= z$; and, by the nature of the circle, PZ is $= \sqrt{rx - xx}$; therefore PC or y (= CZ + PZ) is $= z + \sqrt{rx - xx}$. Hence $\dot{y} = \dot{z} + \frac{\frac{1}{2}r - x}{\sqrt{(rx - xx)}} \times \dot{x}$; but $\dot{z}$ is $= \frac{\frac{1}{2}r\dot{x}}{\sqrt{(rx - xx)}}$ by the nature of the circle; therefore $\dot{y}$ is $= \frac{r - x}{\sqrt{(rx - xx)}} \times \dot{x} = \dot{x}\sqrt{\frac{r - x}{x}}$; then $\ddot{y} = \frac{-r\dot{x}^2}{2x\sqrt{(rx - xx)}}$, making $\dot{x}$ constant. Hence CI is $= \frac{-\dot{x}\dot{y}\text{Q}}{\dot{y}^3} = \frac{\frac{1}{2}r\text{Q}}{(r - x)^2}$. But at the vertex D, $x = 0$, and

$\text{CI} = \frac{\text{Q}}{2r} = a$; therefore $\text{Q} = 2ar$; consequently the general value of CI is $(\frac{r}{r-x})^2 \times a = (\frac{\text{DQ}}{\text{PQ}})^2 \times \text{DK}$; a formula which expresses the nature of the curve KI, for the extrados or back of a cycloidal curve of equilibration; a curve much resembling that for the circle and ellipse, in the two foregoing examples, as evidently appears by comparing the figures together, each of them being here accurately contracted. But this last figure, for the cycloid, seems to be rather better than either of those other two, as the extrados deviates rather less from a right line, and extends farther along before it bends upwards; and besides, the cycloidal arch is not deficient in either use or gracefulness.

EXAMPLE 4.

To determine the figure of the Extrados of a Parabolic Arch of Equilibration.

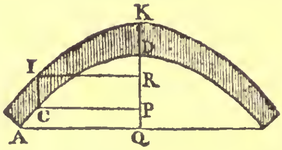


Putting, as before, $a = \text{KD}$, $r = \text{DQ}$, $h = \text{AQ}$, $x = \text{DP}$, and $y = \text{PC} = \text{RI}$. Then, by the nature of the curve, $hh : yy :: r : x = \frac{ryy}{hh}$; hence $\dot{x} = \frac{2ry\dot{y}}{hh}$, and $\ddot{x} = \frac{2r\dot{y}^2}{hh}$, by making $\dot{y}$ constant. Then $\text{CI} = \frac{\ddot{x}}{\dot{y}^2} \times \text{Q}$ is $= \frac{2r\text{Q}}{hh} =$ a constant quantity $= a$; that is, CI is every where equal to KD.

Consequently KR is $=$ DP; and since RI is $=$ PC, it is evident that KI is the same parabolic curve with DC, and may be placed any height above it, always producing an arch of equilibration.

EXAMPLE 5.

To find the figure of the Extrados for an Hyperbolic Arch of Equilibration.

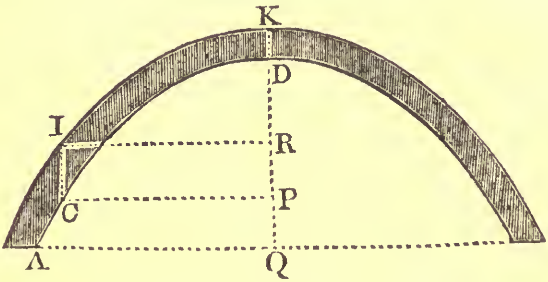


Here putting, as before, $a =$ KD, $r =$ the semi-transverse, and $h =$ the horizontal or semi-conjugate axe, also $x =$ DP, and $y =$ PC $=$ RI. Then, by the nature of the hyperbola, $y = \frac{h}{r}\sqrt{2rx + xx}$; hence $\dot{y} = \frac{h\dot{x}}{r} \times \frac{r+x}{\sqrt{(2rx+xx)}}$, and, by making $\dot{x}$ constant, $\ddot{y} = \frac{-hr\dot{x}^2}{(2rx+xx)^{\frac{3}{2}}}$. Therefore CI or $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times$ Q is $= \frac{r^4 \text{Q}}{h^2 \times (r+x)^3}$. But in the vertex D, where $x = 0$, this expression becomes $\frac{r\text{Q}}{hh} = a$; hence Q $= \frac{ahh}{r}$, and consequently CI or $\frac{r^4\text{Q}}{h^2 \times (r+x)^3}$ is $= \frac{ar^3}{(r+x)^3} = (\frac{r}{r+x})^3 \times a$, which is exactly similar to the formula for the circle and ellipse, only having $r + x$ in the denominator, instead of $r - x$, which causes the value of CI to become always less and less, as the point C is taken farther from the vertex D.

In this hyperbolic arch then, it is evident that the extrados KI continually approaches nearer and nearer to the intrados; whereas in the circular and elliptic arches, it goes off continually farther and farther from it; while in the parabola, the two curves keep always at the same distance. Observing, however, that, by the distance between the two curves, in all these cases, is meant their distance in the vertical direction.

EXAMPLE 6.

To find the Extrados for a Catenarian Arch of Equilibration.



Let $a =$ KD, $x =$ DP, and $y =$ PC $=$ RI, as before; also let c denote the constant tension of the curve at the vertex. Then, by the nature of the catenary, y is $= c \times$ hyp. log. of $\frac{c + x + \sqrt{2cx + xx}}{c}$; hence, taking the fluxions, we have $\dot{y} = \frac{c\dot{x}}{\sqrt{(2cx + xx)}}$, and $\ddot{y} = -c\dot{x}^2 \times \frac{c + x}{(2cx + xx)^{\frac{3}{2}}}$, by making $\dot{x}$ constant. Therefore CI, or $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times Q$, is $= \frac{c+x}{cc} \times Q$. But at the vertex x is $= 0$, and CI $= a = \frac{Q}{c}$; consequently Q is $= ac$. This being written for it, there results CI $= \frac{c+x}{c} \times a = a + \frac{ax}{c}$. And the same formula comes out for the logarithmic curve. Hence, for the nature of the curve KI, we have KR $= (a + x -$ CI $=) x - \frac{ax}{c} = \frac{c-a}{c} \times x$.

Corol.—And hence the abscissa DP, of the inner or soffit curve, is to the abscissa KR, of the exterior one, always in the constant proportion of c to $c - a$. So that, when a is less than c, R and the curve KI lie below the horizontal line; but when a is greater than c, they lie above it; and when a is equal to c, KR is always equal to nothing, and KI, or the extrados, coincides with the horizontal line. As a diminishes, the line KI approaches always nearer to DC in all its parts, till, when a entirely vanishes, or is so small in respect of c as to be omitted in the expression $\frac{c-a}{c} \times x =$ KR, the two curves quite coincide throughout.

Scholium.—As it has been found above, that the extrados will be a straight horizontal line when a is equal to c, a calculation may here be instituted to determine, in that case, the value of c, and consequently of a with respect to x and y, or a given span and height of an arch of equilibration in that case. Now the equation to the curve expressed in terms of c, x, and y, is $y = c \times$ hyp. log. of $\frac{c + x + \sqrt{2cx + xx}}{c}$; and when x and y are given, the value of c may be found from this equation, by the method of trial and error. But as the process would be at best but a tedious one, and perhaps the method not easy in this case to be practised by every person, we may here investigate a series for finding the value of c from those of x and y in a direct manner. Since then $y = c \times$ hyp. log. of $\frac{c + x + \sqrt{2cx + xx}}{c}$, by taking the fluxion of this equation, we have

$$\dot{y} = \frac{c\dot{x}}{\sqrt{(2cx + xx)}} = \frac{\frac{1}{2}d\dot{x}}{\sqrt{(dx + xx)}},$$ by writing d for $2c$; and by expanding this expression into a series, it becomes

$$\dot{y} = \tfrac{1}{2}\dot{x}\sqrt{\frac{d}{x}} \times \left(1 - \frac{x}{2d} + \frac{1.3x^2}{2.4d^2} - \frac{1.3.5x^3}{2.4.6d^3}\ \&c\right);$$ and, by taking the fluents, we have $y = \sqrt{dx} \times \left(1 - \frac{x}{2.3d} + \frac{1.3x^2}{2.4.5d^2} - \frac{1.3.5x^3}{2.4.6.7d^3} + \frac{1.3.5.7x^4}{2.4.6.8.9d^4}\ \&c\right)$; hence, dividing by x, we have $\frac{y}{x} = \sqrt{\frac{d}{x}} \times \left(1 - \frac{x}{2.3d} + \frac{1.3x^2}{2.4.5d^2} - \frac{1.3.5x^3}{2.4.6.7d^3} + \right.$

$\frac{1.3.5.7x^4}{2.4.6.8.9d^4}$ &c); or, by writing v for $\frac{y}{x}$, and w for $\sqrt{\frac{d}{x}}$, it is $v = w - \frac{1}{2.3w} + \frac{1.3}{2.4.5w^3} - \frac{1.3.5}{2.4.6.7w^5} + \frac{1.3.5.7}{2.4.6.8.9w^7}$ &c.

Then, by reverting this series, we have $w = v + \frac{1}{6v} - \frac{37}{360v^3} + \frac{547}{5040v^5} - \frac{337}{5600v^7}$ &c. Hence, by squaring, &c, and restoring the original letters, it is ($\frac{1}{2}d = \frac{1}{2}xw^2 =$) $c = \frac{1}{2}x \times (\frac{y^2}{x^2} + \frac{1}{3} - \frac{8x^2}{45y^2} + \frac{691x^4}{3780y^4} - \frac{23851x^6}{453600y^6}$ &c), where a few of the first terms are sufficient to determine the value of c pretty nearly.

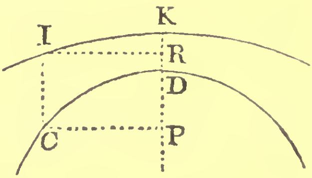
Now, for an example in numbers, suppose the height of the arch to be 40 feet, and its span 100, which are nearly the dimensions of the middle arch of Blackfriars Bridge at London. Then $x = 40$, and $y = 50$; which being substituted for them in this series, it gives $c = 36\cdot88$ feet nearly. So that, to have made that arch a catenarian one, with a straight line above, the top of the arch must have been almost of the immense thickness of 37 feet, to have kept it in equilibrio. But if the height and span be 40 and 100 feet, as above, and the thickness of the arch at top be assumed equal to 6 feet, then the extrados will not be a right line, but as it is drawn in the figure to this example, which figure is accurately constructed according to these dimensions.

It may be further remarked, that the curves in these last three examples, viz, the parabola, hyperbola, and catenary, are all very improper for the arches of a bridge consisting of several arches; because it is evident from their figures, which are all constructed from a scale, that all the building or filling up of the flanks of the arches will tend to destroy the equilibrium of them. But in a bridge of one single arch, whose extrados or back rises pretty much from the spring to the top, one of these figures will answer better than any of the former ones.—Other examples of known curves might be given; but those that have been here noticed, seem to be the fittest for real practice; and there is a sufficient variety among them, to suit the various circumstances of convenience, strength, and beauty, that may be desired.

We may now proceed to another general problem, which is the reverse of the last, and is, to determine the figure of the intrados for any given figure of the extrados, so that the arch may be in equilibrio in all its parts. This is a more difficult problem than the former, and the more useful one also. Here commonly, that the roadway may be of easy and regular ascent, we are confined to an outline nearly horizontal, to which the curve of the soffit or inner arch must be adapted.

PROP. XI.

Having the Extrados given; to find the Intrados. That is, having given the nature or form of a line, bounding the top of a wall above an arch; to determine the figure of the arch, so that, by the pressure of the superincumbent wall, the whole may remain in equilibrio.



Putting a = DK the thickness of the arch at top, x = DP the absciss of the required intrados arch DC, u = KR the corresponding absciss of the given extrados KI, and y = PC = RI their equal ordinates.

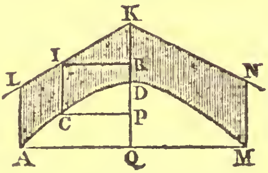
Then, by the last prop. CI is $= \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times$ Q; but CI is also evidently equal to $a + x - u$; therefore $a + x - u$ is $= \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times Q = \frac{Q}{\dot{y}} \times$ the fluxion of $\frac{\dot{x}}{\dot{y}}$; where Q is a constant quantity, as used in the last proposition, and is always to be determined from the nature or conditions of each particular case, commonly indeed by taking the real value of CI, viz. DK or a at the vertex of the curve.

Hence then, by substituting, in this equation, the given value of u instead of it, as expressed in terms of y, the resulting equation will then involve only x and y, together with their first and second fluxions, besides constant quantities. And from it the relation between x and y themselves may be found, by the application of such methods as may seem to be best adapted to the particular form of the given equation to the extrados. In general, a proper series for the value of x in terms of y is to be assumed with indeterminate coefficients; which series being put into fluxions, striking out of every term the fluxion of y; and the result put into fluxions again, striking out from every term of this also the fluxion of y; the last expression drawn into Q being equated to $a + x - u$, there will be produced an equation, from which may be found the values of the coefficients of the terms in the assumed value of x.

Fortunately however, the process is more simple and easy in the most common and useful cases, than might at first be expected from this general method, viz, when the extrados is a straight line, even when it is oblique, and still more when it is horizontal; two cases to which we shall now proceed to apply the general method, in the following examples.

EXAMPLE 1.

To find an Arch of Equilibration when the Extrados is a straight line, oblique or inclined.



In this case, the extrados will have a resemblance to the sloping roof of a house, as in the annexed figure, and is often used in the case of gunpowder magazines. Here employing the notation as in the proposition, the general equation there is CI, or w, $= a + x - u = \text{Q} \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= \text{Q} \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{x}$

a constant quantity. But KR or u is $= ty$, if t be put to denote the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.

But the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the second fluxion is $\ddot{w} = \ddot{x}$; therefore the general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w} = \frac{Q\dot{w}\ddot{w}}{\dot{y}^2}$, the fluent of which gives $w^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, because the curve at D is parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 = \frac{Q\dot{w}^2}{w^2 - a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{w^2 - a^2}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w + \sqrt{w^2 - a^2}}{a}$.

Now, when the vertical line CI is at the position AL, then $w =$ CI becomes AL $=$ the given quantity c suppose, and y $=$ AQ $= h$, in which case the last equation becomes $h = \sqrt{Q} \times$ hyp. log. of $\frac{c + \sqrt{c^2 - a^2}}{a}$; hence it is found, that the value of the constant quantity $\sqrt{Q}$ is $\frac{h}{\text{h. l. of} \frac{c + \sqrt{c^2 - a^2}}{a}}$; which being substituted for it in the above general value of y, that value becomes $y = h \times \frac{\text{log. of} \frac{w + \sqrt{w^2 - a^2}}{a}}{\text{log. of} \frac{c + \sqrt{c^2 - a^2}}{a}}$; from which equation the value of the ordinate CP may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner: Put A = the log. of a, and C $= \frac{1}{h} \times$ log. of

$\frac{c + \sqrt{c^2 - a^2}}{a}$; then the above equation gives $cy + A =$ co log. of $(w + \sqrt{w^2 - a^2})$; again, put $n =$ the number whose log. is $cy + A$; then $n = w + \sqrt{w^2 - a^2}$; and hence $w = \frac{a^2 + n^2}{2n} =$ CI.

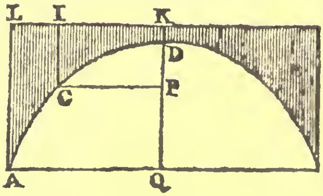
This example is more peculiarly adapted to the use of magazines for gunpowder, which are usually made in the manner represented in the figure above, that is in regard to their roof, for the inner curve itself has commonly been made a semicircle. But it is a constant observation, that after the centering of semicircular arches is struck, they settle at the crown, and rise up at the flanks, even with a straight horizontal extrados, and still much more so in powder magazines, where the outside at top is formed, like the roof of a house, by two inclined planes joining in an angle, or ridge, over the top of the arch, to give a proper descent to the rain; which effects are exactly what might be expected from a contemplation of the true theory of arches. Now this shrinking of the arches must be attended with very bad consequences, by breaking the texture of the cement, after it has in some degree been dried, and also by opening the joints of the vousoirs at one end; consequently the application of the formula above investigated must be accompanied with beneficial effects. It may be useful therefore to give here an example in numbers in a real case of that nature. If the foregoing figure then represent a transverse vertical section of a balanced arch in all its parts, in which the span AM is 20 feet, the pitch or height DQ 10 feet, the thickness DK at the crown 7 feet, and the angle of the ridge LKN 112° 37′, or the half of it LKD = 56° 18′$\frac{1}{2}$, the complement of which, or the elevation KIR, is 33° 41′$\frac{1}{2}$, the tangent of which is $= \frac{2}{3}$, which will therefore be the value of t in the investigation above. The values of the other letters will be as follows, viz, DK $= a = 7$; AQ $= h = 10$; DQ $= r = 10$; AL $= c = \frac{31}{3} = 10\frac{1}{3}$;

A = log. of 7 = ·8450980 ; $c = \frac{1}{h} \times$ log. of $\frac{c + \sqrt{c^2 - a^2}}{a} = \frac{1}{10}$ log. of $\frac{31 + \sqrt{520}}{21} = \frac{1}{10}$ log. of 2·56207 = ·0408591 ; cy + A = ·0408591y + ·8450980 = the log. of n. From the general equation then, viz, CI $= w = \frac{a^2 + n^2}{2n}$, by assuming y successively equal to 1, 2, 3, 4, &c, and thence finding the corresponding values of cy + A or ·0408591 + ·8450980, and to these, as common logs, taking out the corresponding natural numbers, or values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by finding so many points in it.

Val. of y or CP.	Val. of w or CI
1	7·0310
2	7·1243
3	7·2806
4	7·5015
5	7·7838
6	8·1452
7	8·5737
8	9·0781
9	9·6628
10	10·3333

EXAMPLE 2.

To find an Arch of Equilibration whose Extrados shall be a Horizontal line.



The process for this case differs in nothing from that in the former example, but in substituting the horizontal line of extrados KI, instead of the oblique one, by which the angle DKI becomes a right angle, therefore the angle KIR, in the former example, vanishes, and consequently its tangent also, that is, the value of t, in the last example, becomes nothing in this: all the other letters and the formula being the very same.

For an example therefore in numbers, let us suppose the span of the arch to be 100 feet, the pitch or height 40 feet, and thickness at the crown 6 feet, which are nearly the dimensions of the centre arch in Blackfriars bridge: then the values of the several letters will be as follows, viz, AQ $= h = 50$; DQ $= r = 40$; DK $= a = 6$; AL $= c = 46$. Hence the hyp. log. of $\frac{c + \sqrt{c^2 - a^2}}{a}$ = hyp. log. of $\frac{46 + 4\sqrt{130}}{6}$ = hyp. log. of 15·26784 = 2·7257487; by which dividing h or 50, the quotient is 18·343584. So that the ordinate y will be constantly, in that case, equal to 18·343584 × hyp. log. of $\frac{w + \sqrt{w^2 - a^2}}{a}$. Also $\frac{1}{18·343584}$ = ·05451497 is = C, and A = hyp. log. of 6 = 1·7917594; therefore n is = the number whose hyp. log. is Cy + A or ·05451497y + 1·7917594. Hence, by assuming several values of the letter y, which is = CP or IK, the corresponding values of n will be found as above, and then those of w or CI from the final general equation $w = \frac{a^2 + n^2}{2a} = \frac{36 + n^2}{12} = 3 + \frac{1}{12}n^2$. And in this manner were calculated the numbers in the following table; from which the curve being constructed, it will be as appears in the figure to the example.

And thus we have an arch in equilibrium in all its parts, and its top a straight line, as is generally required in most bridges; or at least they are so near a horizontal line, that their difference from it will cause little or no sensible ill consequence. It is also both of a graceful figure, and of a convenient form for the passage through it. So that no reasonable objection can be offered against its adoption in works of consequence, on account of its mechanical excellency.

The Table for Constructing the Curve in this Example.

Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC
0	6·000	15	8·120	24	11·911	33	18·627	42	29·919
2	6·035	16	8·430	25	12·489	34	19·617	43	31·563
4	6·144	17	8·766	26	13·106	35	20·665	44	33·299
6	6·324	18	9·168	27	13·761	36	21·774	45	35·135
8	6·580	19	9·517	28	14·457	37	22·948	46	37·075
10	6·914	20	9·934	29	15·196	38	24·196	47	39·126
12	7·330	21	10·381	30	15·980	39	25·505	48	41·293
13	7·571	22	10·858	31	16·811	40	26·894	49	43·581
14	7·834	23	11·368	32	17·693	41	28·364	50	46·000

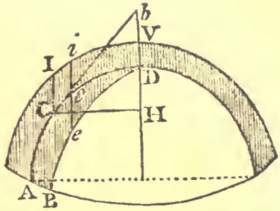
The above numbers may either be feet, or any other lengths, of which DQ is 40 and QA is 50. But when DQ is to QA in any other proportion than that of 4 to 5, or when DK is not to DQ as 6 to 40 or 3 to 20; then the above numbers will not answer; but others must be found by the same rule, to construct the curve by. In the beginning of the table, as far as 12, the value of KI is made to differ by 2, because the value of CI in that part increases so very slowly. Afterwards they differ by units or 1.

Other examples of given extrados might be taken; but as there can scarcely ever be any real occasion for them, and as the trouble of calculation would be, in most cases, very great, they are omitted.

As the theory for arch vaults, before laid down, will so easily apply to the arches for domes or cupolas also, a proposition or two may be here added for that purpose, as follows.

PROP. XII.

When a regular Concave Surface Dome, or Vault, formed by the rotation of a curve turned about its axis, is kept in equilibrio by the pressure of a solid wall built on every part of it; then the Height of the wall over any part, is directly proportional to the cube of the secant of elevation there, and inversely proportional to the radius of curvature, and to the diameter or width of the dome at the same part.



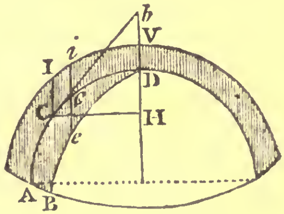
That is, VI being the form of the exterior surface of a balanced shell, the interior surface of which is formed by the rotation of the curve DCA about its axis DH; the elevation of any part C being the angle *b*CH, and CH the ordinate or semidiameter of the dome at the point C, also *r* the radius of curvature to the same point: then the height or vertical thickness of the shell over the point C, or CI, is proportional to $\frac{\sec^3 b\text{CH}}{r \times \text{CH}}$.

Let ACD*e*B be a small part of the inner surface, like a curved sector or gore, DCA and D*e*B being two near positions of the generating curve. Now the vertical load on any part C of a balanced arch, in a shell or dome, in the present case, is a solid pillar, C*i*, whose height is CI, its breadth C*a*, and thickness C*e*, and consequently is $= \text{CI} \times \text{C}a \times \text{C}e$. But C*a* is as $\frac{\text{CH}}{\text{C}b}$ or as $\frac{1}{\sec.\, b\text{CH}}$; and C*e* is always in the same proportion as CH; therefore the pillar C*i*, or $\text{CI} \times \text{C}a \times \text{C}e$ is as $\frac{\text{CI} \times \text{CH}}{\sec.\, b\text{CH}}$; which load, by the 8th prop. is also proportional to $\frac{\sec^2.\, b\text{CH}}{r}$; therefore $\frac{\text{CI} \times \text{CH}}{\sec.\, b\text{CH}}$ is as $\frac{\sec^2.\, b\text{CH}}{r}$; consequently the height CI is as $\frac{\sec^3.\, b\text{CH}}{r \times \text{CH}}$. That is, the vertical height of the wall over every part of a balanced shell, or dome, or vault, is directly as the cube of the secant of the curve's elevation at that part, and inversely as the radius of curvature, and also inversely as the width of the dome at the same place.

And here may be also understood several corollaries and observations exactly similar to those to the 3d and the 9th propositions, and which therefore need not be repeated in this place.

PROP. XIII.

Having given the form of the Inner Surface of a balanced Shell or Dome; to determine that of the Exterior or Outer Surface. That is, having given the nature or form of an inner shell; thence to find the nature of the outer or bounding surface of the superincumbent wall, by the pressure of which the shell is kept in equilibrio.



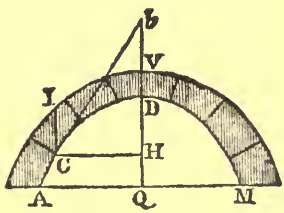
By reasoning here exactly as in the 10th proposition, it will be found that the general value of the height CI of the wall, will be proportional to the following forms or quantities, viz,

CI is either as $\frac{\sec^3. b\text{CH}}{r \times \text{CH}}$, or as $\frac{\dot{z}^3}{ry\dot{y}^3}$, or as $\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{ry\dot{y}^3}$, or as $\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{ry\dot{y}^3}$, or as $\frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$ when $\dot{x}$ is considered as invariable, or as $\frac{\ddot{x}}{y\dot{y}^2}$ when $\dot{y}$ is invariable: in which the letters have the usual values, namely, $x =$ DH the abscisss, $y =$ CH the ordinate, and $z =$ DC the curve, also r the radius of curvature at the point C. Or the general value of CI will be equal to any of these forms multiplied by a certain constant quantity Q, the particular value of which is always to be determined by putting the general value of CI equal to the given thickness of the shell, either at the crown, or at some other particular place, where that value may happen to be known or given.

Corol.—From this, and the foregoing prop. we may infer this general observation, namely, that no curve can produce the figure of a true or exact balanced dome or cupola, unless that curve be of such a nature as to have its radius of curvature at the vertex of an infinite length, or the curvature at the vertex nothing; which is the case with some curves; or unless the thickness at the crown be infinite. For, at the vertex, the angle of elevation bCH is nothing, and the secant $= 1$; the ordinate CH is there nothing also; therefore the general expression, $\text{CI} = \frac{\sec^3. b\text{CH}}{r \times \text{CH}}$, becomes, at the vertex, $\text{DV} = \frac{1}{r \times 0} = \frac{1}{0} =$ infinite, that is DV must be infinite, if r be a finite quantity.

Or, if DV be finite, as suppose $=$ A; then $a = \frac{1}{r \times 0}$, or $r = \frac{1}{a \times 0} = \frac{1}{0} =$ infinite, when a is finite. That is, the radius of curvature at the vertex must be infinite when the height there is finite or given; or, on the other hand, the height or pressure at the vertex must be infinite, when the radius of curvature there is a finite or given quantity, to have the shell truly balanced. Of this nature there are several curves, of the parabolic kind in particular, of a form both convenient and graceful, such as the cubical parabola in the following example.

EXAMPLE.



Taking, for an example, the curve DC of the cubical parabola, so called because its abscisses are proportional to the cubes of their ordinates. Thus, putting $x =$ DH the absciss, $y =$ CH the ordinate, and a the parameter, or a given quantity; then the equation to the curve is $ax = y^3$. Hence, taking the fluxions, we obtain $\dot{x} = \frac{3y^2\dot{y}}{a}$, and $\ddot{x} = \frac{6y\dot{y}^2}{a}$, when $\dot{y}$ is considered as invariable This value of $\ddot{x}$ being substituted for it in the general value of the height CI, viz, $\frac{\ddot{x}}{y\dot{y}^2}$, this becomes $\text{CI} = \frac{6y\dot{y}^2}{ay\dot{y}^2} = \frac{6}{a}$; that is, any given or constant quantity. Consequently the outer curve is the same as the inner, but placed in a higher position, as they appear in the figure to this example, where the curves are accurately constructed to a particular scale, when the greatest width AM is 80 feet, and the height DQ is 64 feet.

The foregoing principle for balancing dome vaults, it must be understood, is quite independent of the aid it receives from the circular or other form of its contour, in which indeed consists its great strength and stability. For, from this shape it happens, that the inside or outer one, in the vertical section, may take any form whatever, either convex outwards, as is usual in rotund domes, or a straight side, as in the cone of tile kilns or the pyramidal spire, or even concave outwards and convex inwards. For, by making all the coursing joints of masonry, quite around, not flat or horizontal, but everywhere perpendicular to the face, and all the vertical joints tending or pointing to the axis, all the stones or bricks, &c, will act as wedges in a round curb, and cannot possibly come down, or fall inwards, unless the component parts could be crushed to powder, or the bottom circular course burst outwards. To prevent this from happening, a strong hoop of iron may be passed round the bottom, and in other parts also, in works of consequence, which effectually secures the fabric from bursting open, or flying outwards, while the round form, like a curb, as securely prevents it from falling inwards. Hence too it happens, that considerable openings may be cut in the sides, or it may be left open, as if incomplete, at top, and over the opening may be erected any other figure, whether lantern or spire, &c, either for use or ornament.

GENERAL SCHOLIUM.

In the foregoing propositions have been delivered the chief variety of ways for constructing the arches of bridges, so as they may be in equilibrio or balanced in themselves. There are three of these different methods; first, that which is derived from the consideration of the equilibrium produced by the mutual thrusts, weights and pressures of the arch stones, supposing them prevented from sliding on each other at the oblique joints, either by their roughness and friction, or by the cement, or stone locks, or iron bars let into every adjacent pair of stones; which give the arch the effect of one compacted frame, pressed on vertically by the weight of the superincumbent load of wall above it: which seems to be the true and genuine way of considering the action of that load on the arch.

The second method, is that in which the balanced arch is computed on the supposition that the arch stones have their butting sides perfectly smooth, and at free liberty to slide on each other. A method which is but little insisted on, as it is founded on a supposition which is neither in nature nor art, and which can never take place in any real construction of an arch.

The third method, is that which has for its principle the catenarian or festoon arch, formed by the suspension of a slack chain or cord, by its two ends, and afterwards inverted. This idea it seems was first proposed by Dr. Hooke, near the latter part of the 17th century, when the Newtonian mathematics prepared the way to true mechanical science. This is a strictly just and useful principle, and may be most easily extended to every case that can happen in practice. At first indeed the idea had nothing more in view than the balancing of the single or thin arch, formed by the voussoirs only, as the catenarian curve, formed by a simple chain or cord, can aim at nothing further than the balancing of that simple string of arch stones, without any other wall to fill up the flanks, &c. This principle was also neatly treated of by Delahire, in prop. 123, 124, 125 of his Traité de Mecanique, published in 1695. But the same principle has been lately acted on, and extended much further, by professor Robison of Edinburgh, namely, by making thus a festoon arch balancing, not only the simple string of voussoirs, but also the whole load of the superincumbent wall, of any proposed form whatever. This method, so easy in its practical operation, depends on, and is easily deduced from the first, or that which balances the arch by the mutual thrusts and pressures of the parts; by showing that these forces, of mutual pressure of the parts, are exactly equal and opposite to those by which they pull or draw each other in the case of suspension.

It is true that the equilibrium which any theory establishes, is of so delicate a nature, by supposing the parts to touch only in single points, that it may be called a tottering equilibrium, since any other weight or force added at any part would press the arch out of its true balanced form, and, by shifting the points of contact of the parts, bring the whole down to the ground, if it were not that the arch stones have some considerable length, by which a stability is ensured, as the altered figure will find new points of contact, where the action of the parts will principally bear, and through all which points a new curve line may be conceived to pass, as the catenary or festoon balanced arch. And hence it follows, that the longer the butting joints or arch stones are, the more stable and secure the whole fabric will be; since this circumstance will allow of the more change either in the figure of the arch, or the true catenarian points of bearing or thrust, and yet have a competent substance of solid stone to sustain the great force of such actions. It is therefore of the greatest importance to have the arch stones made as long as may be, consistent with economy, and the other circumstances of the fabric. And this was the great use of the ribs that were employed in the old English architecture, the great projections of which augmented considerably the stiffness of the whole, and enabled the architects to make use of comparatively very small stones in the other parts of the work. This contrivance we find has been used in constructing roofs, as well as in bridges; the few old remaining ones of these we see have been constructed and strengthened by these ribs of long and large stones. It would therefore be perhaps the safest and firmest way, to give the whole masonry of the wall, over the arch stones, the same position of joints as these stones themselves have, namely, not in horizontal courses, but everywhere the joints in the direction perpendicular to the curve of the arch, quite up to the top or road way; as we see indeed has been practised in the face of the masonry at Westminster bridge. For, by this means, the whole has the effect of arch stones, considered as extended the whole length, from the soffit of the arch, all the distance up to the road way: thus ensuring a strength and safety so complete, as to render even considerable deviations from the theory of a balanced arch of no material bad effect whatever.

SECTION III.

OF THE PIERS.

When an arch is supposed to stand alone, and well balanced, it is necessary that its piers or abutments should be at least sufficiently firm and massive to resist completely the shoot, drift, or horizontal push of the arch. For should the pier yield in the least to this drift, and be pushed aside, the arch must infallibly fall down. It is therefore essential that every arch should have its abutments properly adapted to resist effectually its shoot. And the same precaution ought also to be employed in a string or series of arches, such as an arcade, or a long bridge composed of several openings: for though, in these cases, the arches may be supposed to sustain mutually each other's thrust, while they are all standing, and to require only a slender pier between every adjacent pair of arches, to serve as a thin plane between their mutual pushes, like the ridge board between the butting ends of the rafters in the roof of a house; yet provision should be made against any possible accident that may happen to any one of the arches in the string, so as that any of them may be supposed cut open, or to fall down, and yet not affect the adjacent ones, but leave them standing firm and independent, sustained by their own piers alone. For otherwise, should the arches be made in a string as it were, all dependent on each other for support, then on an accident befalling any one arch, the entire series of arches must follow it, and the whole fabric come down.

Prudent architects therefore take care to employ various means of constructing their piers to be, as they expect, sufficiently stable and firm, to sustain the shoot of the arches; without however being always certain of the just and adequate effect. For this reason it sometimes happens, that their piers are made too slender for perfect safety, and sometimes indeed, erring on the other hand, they are made unnecessarily thick and massive; a mistake which, to say nothing of the ungraceful appearance, both enhances the expence, and also impedes the free and easy passage of the water and navigation, by occupying too much of the breadth of the river, by such loads of solid masonry. It is therefore intended, in this section, to give rules and examples for computing nearly the proper thickness and weight of a pier, so as to be an exact balance to the shoot of the arch; that by then giving it a very little more thickness in practice, a security is provided against any accidental and extraneous effort.

But this equilibrium is not easily or certainly to be effected: it is by all authors attempted, though not always justly, by determining the thickness of the piers such, that the resistance of its weight to being overset, may be at least equal to the force of the shoot or drift of the arch against it. This principle is obvious enough; but then all authors have not agreed in the method of estimating the value of this last force in particular. Some have determined this point on supposition that the wedges or arch stones are perfectly smooth and unconnected with each other; while others have supposed them so firmly connected, as to form the arch into a solid mass, acting like one rigid body only. It is true, and it has been proved in the beginning of this work, that in an arch of equilibration, formed of parts properly disposed, whether of wedges, or of vertical pieces, the horizontal push or shoot is constantly the same quantity in every part of the arch; being to the weight of the arch above that part, as radius to the tangent of the elevation of that part of the arch above the horizontal line: from which circumstance some persons have imagined that, by computing the shoot or drift for any small given part, as at the key stone for instance, which can easily be done, that will be a sufficient measure or value of the whole; then by applying it at some particular part of the pier, as a force or action tending to overturn it, an equilibrium is established between them. But this method will not do; because it is founded on the supposition that the constituent parts of the arch are perfectly polished, and at liberty to slide freely on each other. Whereas, on the contrary, the parts that compose the arch are completely hindered from sliding on each other, partly by their roughness and friction, and partly by the cement employed between them, and still more by the ties and fastenings placed within, to bind them together. By these means it happens, that all the parts are firmly compacted and united, so as to form the whole arch in some measure, into one rigid and solid mass; and besides that many of the voussoirs, in the lower parts of the arch, are built and bonded into the very body of the pier itself, and forming a part of its very mass.

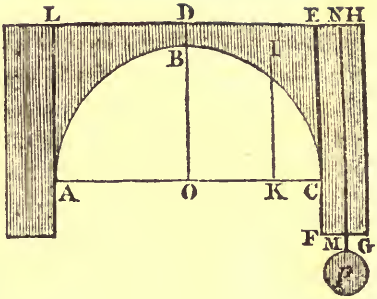
The same principle also, of the constant and determinate magnitude of the horizontal push, is founded on the supposition, that the arch is a true and real arch of equilibration; which perhaps can never be justly said to be the case. Besides, if it were such an arch, and the quantity of the constant horizontal push duly found, it would still be doubtful at what point of the pier to apply it, in making the calculation of its effect, on account of the circumstance that the arch has a bearing and oblique thrust, not against one point only, but in a different degree at all the points in that part of the pier extending from the impost, or foot of the arch, upward to the very top or roadway over the bridge.

On all these accounts then, and perhaps others, not here adverted to, it would seem that there is not, and perhaps cannot be, any true and perfect mathematical calculation made, of the exact balance between the push of an arch and the stability of the piers. Hence it has happened that various methods have been employed for this purpose, by different authors, with more or less show of reason or grounds of propriety: and hence also many practical engineers, neglecting all such calculations as unsatisfactory, have depended on practice and experience only, taking care, as they think, to err on the safe side, by making the piers much too massive, rather than risk the hazard of a failure by the chance of the contrary case. In this uncertainty, after several trials and examinations, two of the most promising, among the various ways of solving this problem, have been selected and delivered in the following prop. as affording probably a near approach to a true conclusion.

PROP. XIV.

To find the thickness of the piers of an arch, necessary to keep the arch in equilibrio, or to resist its drift or shoot, independent of any other arches.

First Solution.—Let BDEC be the half arch, and EFGH the pier necessary to balance and support it, considered as moveable about the extreme point G of the base.



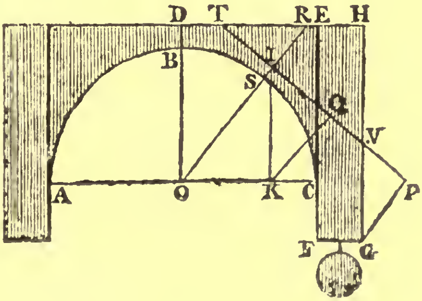
Through the centre of gravity I, of the arch BDEC, let IK be drawn perp. to the span AOKC. Now the semiarch BDEC is supported against the part of the pier EC, but chiefly on the impost or lowest point C, which sustains its weight, and by the horizontal thrust of the other semiarch ALDB, acting against it in the line of meeting BD. If both of these pressures be taken at their lowest points B, C, the arch may be considered as supported at these two points after the manner of a solid beam. But when such a body is supported in this way, it is well known, from the principles of mechanics, that the weight of the body downward, is in proportion to the horizontal push at its foot, as the vertical line IK is to the horizontal line KC; therefore the weight of the semiarch BDEC, is to its shoot against the pier at C, as IK is to KC: this force or push therefore will be expressed by $\frac{KC}{KI} \times a$, where a denotes the arch BDEC, or its weight or its area: and if this force be drawn into the length of the lever CF, the product $\frac{KC.CF}{IK} \times a$ will express the efficacious force tending to overturn the pier, by causing it to turn back about the point G, supposing the pier to be firmly compacted into one mass.

Now, to oppose and balance this force to overset the pier, arising from the push of the arch, we have the resistance depending on the weight of the pier itself. This weight may be supposed to be collected into its middle vertical line MN, or it may be represented by an equal weight p suspended from its middle point M; p, acting by the lever MG, and denoting the weight of the pier, or its area EF.FG.

Therefore the resistance of the pier will be expressed by $EF \cdot FG \cdot \frac{1}{2}FG$ or $\frac{1}{2}EF \cdot FG^2$.

Then, by making this opposing force of the pier equal to the efficacious force of the arch, both as expressed above, that there may be a just balance between them, they will form an equation, from which will easily be determined the unknown quantity, or thickness of the pier, so as to produce the desired equilibrium. And, by adding a little more to it, for better security, the stability is considered as sufficiently obtained. Thus then, having made the equation $\frac{1}{2}EF \cdot FG^2 = \frac{KC \cdot CF}{IK} \cdot a$, its resolution gives us $FG = \sqrt{\frac{KC \cdot CF}{IK \cdot EF} \cdot 2a}$, which is the first theorem or rule for the thickness of the pier; but which will probably be too small, by having taken the whole push of the arch as acting at the lowest point C.

Second Solution.—In the second mode of solving this problem, though the arch stones are supposed to be laid in mortar, and so cemented or locked together as to prevent them from easily sliding on one another, yet the whole not considered so firm or hard as to form as it were one solid stone; but the mortar or connection being only so firm, that if the piers were not sufficiently strong, the arch would break in the weakest part, and overturn the piers. In this method too let all the matter in the arch BDEC be supposed collected into its centre of gravity I, through which draw OI from the centre O, and through the joint SR of the arch in which the centre of gravity is situated: perpendicular to the joint SR draw IQP, the direction in which the



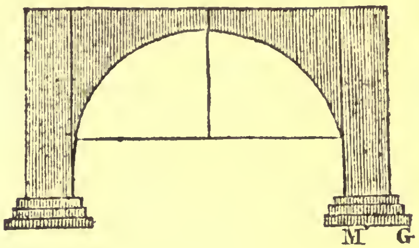
joint SR resists and supports the action of the arch at I: draw IK perpendicular to AC, or in the direction of gravity, also GP and KQ perpendicular to IP or parallel to OIR. Then if IK represent the weight of the arch BDEC in the direction of gravity, this will resolve into IQ the force acting against the pier perpendicular to the joint SR, and QK the part of the force parallel to the same: the line IQ is the only force acting perpendicular on the arm GP, of the crooked lever FGP, to turn the pier about the point G; consequently IQ × GP will express the efficacious force of the arch to overturn the pier, and which must be equal to the force of the pier itself, denoted by the area $EG \times \frac{1}{2}FG$ as before; that is $\frac{IQ}{IK} . a . GP = EF . FG . \frac{1}{2}FG = \frac{1}{2}EF . FG^2$, a denoting the area of the section BDEC of the arch, as EF . FG denotes the section EFGH of the pier. And this equation, after substituting for GP its value, will be a 2d theorem for the thickness of the pier, and which may probably be rather above the just quantity.

Schol.—As the centre of gravity is employed in both the preceding methods, it will be necessary to employ a few lines on the manner of finding the place I of that centre, together with the various other lines in the figure dependent on and connected with it. Now the centre of gravity I may be known either by mathematical calculation, or by mechanical and geometrical measurement. The best way of performing the first method seems to be on this principle, viz. 'That the content of the solid described by any plane surface, either in moving parallel to itself, or in revolving about a line as an axis, is always equal to the product of the generating plane, and the line described by its centre of gravity.' Hence, if the whole figure ODEC be first revolved about the axis OC, the rectangle ODEC will describe a cylinder, and the space OBSC, of a given figure, will describe a solid of a known magnitude; the difference of these two solids will give the content of the solid described by the mixed space BDECSB;

this solid content divided by the area of its said generating figure, gives the circumference of the circle described by the centre of gravity I, which circumference divided by the number 6·2832, or by $\frac{44}{7}$, will be the length of the radius IK. Next, by conceiving the same figure to revolve about the axis OD, and proceeding in the same way, there will be found the line OK, or the distance of the centre of gravity I from the axis OD. The point I being thus determined, there will hence be known all the lines KC, OI, RS, IQ, IT, TE, &c. Then, by denoting the unknown breadth of the pier, EH or FG, by any letter, as z, in terms of it will be expressed the perpendicular GP: thus, by similar triangles, as IK : OK :: TH : HV; hence GH − HV gives GV, and OI : IK :: GV : GP expresses the unknown line GP. Lastly, the value of GP substituted in the foregoing equation $\frac{1}{2}EF \cdot FG^2 = \frac{IQ \cdot GP}{IK} \cdot a$, it will be in the form of a quadratic, the solution of which will give the value of FG, the thickness of the pier sought, very near the truth.

The mechanical way of finding the centre of gravity I, and the geometrical measurement, is thus performed: On card-paper or pasteboard, or any other thin plate, construct the given figure BDECB very correctly, of a pretty large size, from a scale: then cut it out very neatly by the extreme edges, and lay it so as just to balance itself over the straight edge of a table, the line CE parallel to the edge, and close by the edge of the table draw a line on the paper, which will be the line IK; next balance the same figure in like manner with the line DE parallel to the edge of the table, close by which draw another line, crossing the former line in the point I, which will be the centre of gravity of the figure, determined sufficiently near the truth. This done, lay this point I down on another general construction of the figure, having the representation of the pier annexed, on which also draw all the other lines before mentioned, measuring their lengths by the scale of construction, and noting them down. Then with these, together with the thickness of the pier EH or FG, denoted by the unknown letter z, compute the value of GP, which, with z the value of FG, substitute in the equation $\frac{1}{2}\text{EF} \cdot \text{FG}^2 = \frac{\text{IQ} \cdot \text{GP}}{\text{IK}} \cdot a$, which reduce and solve as above mentioned, to determine the value of z or FG the thickness of the pier; which may thus be easily determined in all cases, and with a sufficient degree of accuracy.—The same methods of determining the centre of gravity, and the lines IK, KC, in the fig. to the 2d example following may be employed, to substitute in the expression $\text{FG} = \sqrt{\frac{\text{KC} \cdot \text{CF}}{\text{IK} \cdot \text{EF}}} \cdot 2a$, for determining the thickness of the pier by the first rule.

In the foregoing solutions, it appears that, besides having given all the measures or dimensions of the arch and height of the pier, it is necessary to know the areas of their vertical transverse sections, or at least that of the superstructure BDEC: and this is easily to be found, when the figure of the arch BC and the exterior DE are known, viz, by deducting the area of the space or vacuity OBSC from that of the whole figure ODEC.—The foregoing solutions may also be considered as taking place either when the pier is all dry, or when it stands partly in water, which can penetrate its foundation or the joints of the masonry: and whether this last circumstance takes place or not, can probably be well judged of and ascertained by the experienced builder: if it do take place, which is perhaps commonly the case, then in the calculation the weight of the part in water must be reduced in the proportion of 5 to 3, as stone loses 2 parts in 5 of its weight when immersed in water.—In the foregoing solution it has also been supposed that the pier is made every where straight alike, or equally thick down to the very bottom, as represented in the two preceding figures. But, instead of that, it is very common to enlarge the pier towards the bottom, both to give it a broader base to stand on, without increasing the weight or dimensions above, and to make the lever WK longer at the



base, to oppose a greater resistance to its oversetting or turning about the point G, and without any sensible increase to the weight of the pier. On the contrary, as the thickness, and consequently the weight of the pier, may be diminished above, in proportion as it is enlarged at the foundation, without diminishing its force of resistance and stability, the experienced architect will avail himself of the circumstance, to reduce in a considerable degree the size of the pier, and the expense of the work.

In the investigation of this proposition, the sections of the arch and pier are used for their solidities, as being evidently in the same proportion, or in that of their weights, since they are of the same length, viz, the breadth of the bridge. By the above rules then, the necessary thickness of a pier may be found, so that it shall *just* balance the spread or shoot of the arch, independent of any other arch on the side of the pier. But the weight of the pier ought a little to preponderate against, or exceed in effect, the shoot of the arch: and therefore the thickness ought to be taken a little more than what will be found by these rules; unless it be supposed that the pointed projections of the piers against the stream, beyond the common breadth of the bridge, will be a sufficient addition to the pier, to give it the necessary preponderancy. We may now take some examples of the calculation in numbers, to show the manner of operation, and in them also to point out the easiest methods of calculation.

EXAMPLE 1.

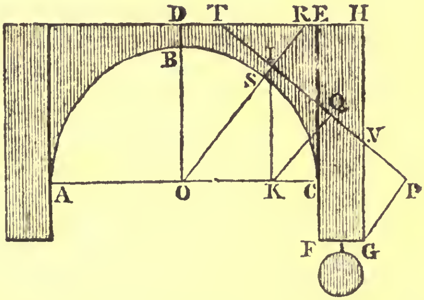
Supposing the arch in the figure to be a semicircle, whose height or pitch is 45 feet, and consequently its span 90 feet;

also supposing the thickness DB at top to be 7 feet, and the height CF to the springing 20; let it be required to find the thickness FG of the pier, necessary to resist the shoot of the arch; the roadway being a horizontal right line.

Now in this example we have OB or OC $= 45$, BD $= 7$, (fig. p. 74) OD or CE $= 52$, CF $= 20$, and EF $= 72$. Hence, the rectangle ODEC $=$ OD $\times$ OC $= 52 \times 45 = 2340$, and the circular quadrant OBC $= 45^2 \times \frac{11}{14} = 1590$ nearly, the difference of these gives $750 = a$, the area of the arch BDEC. Again, the content of the cylinder generated by the rotation of the rectangle ODEC, about the axis OD, is $4\text{OC}^2 \times \frac{11}{14} \times \text{OD}$; and the content of the semisphere, generated by the rotation of the quadrant OBC, about the axis OB, is $4\text{OC}^2 \times \frac{11}{14} \times \frac{2}{3}\text{OB}$; therefore the difference of these gives $4\text{OC}^2 \times \frac{11}{14} \times (\text{OD} - \frac{2}{3}\text{OB}) = 8100 \times \frac{11}{14} \times (52 - 30) = 8100 \times \frac{11}{14} \times 22 = 8100 \times \frac{11}{7} \times 11 = 140000$, for the content of the solid generated by the area BDEC (750) about the axis BD. Hence $140000 \div 750 = 186\frac{2}{3}$ the circumference or path described by the centre of gravity I about OD; consequently $186\frac{2}{3} \times \frac{7}{44} = 29{\cdot}7 =$ OK, the radius of that circle. Hence OC $-$ OK $= 45 - 29{\cdot}7 = 15{\cdot}3 =$ KC.

Again, the content of the cylinder generated by the rotation of the rectangle OBEC, about the axis OC, is $4\text{OD}^2 \times \frac{11}{14} \times \text{OC}$; and the content of the semisphere, as above, is $4\text{OB}^2 \times \frac{11}{14} \times \frac{2}{3}\text{OC}$; therefore the difference of these two $(\text{OD}^2 - \frac{2}{3}\text{OB}^2) \times \frac{44}{14} \times \text{OC}$, gives $(52^2 - \frac{2}{3} . 45^2) \times \frac{22}{7} \times 45 = 1354 \times \frac{11}{7} \times 90 = 191494$, for the content of the solid generated by the area BDEC (750) about the axis OC. Hence $191494 \div 750 = 255{\cdot}325$ the circumference or path described by the centre of gravity I about OC; conseq. $255{\cdot}325 \times \frac{7}{44} = 40{\cdot}6 =$ IK, the radius of that circle. Lastly, the 1st theorem $\sqrt{\frac{\text{KC} . \text{CF} . 2a}{\text{IK} . \text{EF}}}$ gives $\sqrt{\frac{15{\cdot}3 \times 20 \times 1500}{40{\cdot}6 \times 72}} = \sqrt{\frac{510000}{3248}} = 12\frac{1}{2}$ feet $=$ FG, for the required thickness of the pier; but which is probably below the truth, and perhaps below what a practical engineer would fully trust to.

It may be added, that the method of determining the place of the centre of gravity I, by balancing the figure BDEC, gave, within a small fraction, the same values of the two lines IK, KC, viz, 40 +, and 15 +, which were above calculated to be 40·6 and 15·3.



Secondly, to apply our example to the 2d theorem, $\frac{1}{2}\text{EF}\cdot\text{FG}^2 = \frac{\text{IQ}\cdot\text{GP}}{\text{IK}}\cdot a$, the same methods of determining the position of the centre of gravity I may be employed. If the mechanical method of balancing and measurement on a scale be used, we may then measure, not only the lines IK, OK, KC, but all the other lines also depending on it, as OI, OR, TI, TR, TE, KQ, IQ, &c, excepting only such lines as depend on the unknown breadth FG of the pier. But, instead of that, we shall calculate the accurate value of all the lines wanted by strict mathematical principles, as follows. In the example are given OB = OC = DE = 45, OD = CE = 52, CF = 20, EF = 72; and just above we have found by calculation OK = 29·7, KC = 15·3, IK = 40·6, and the area BDEC or a = 750; and we have to compute IQ and GP. Now OI = $\sqrt{(\text{OK}^2 + \text{IK}^2)} = \sqrt{(29{\cdot}7^2 + 40{\cdot}6^2)} = 50{\cdot}3$; then by similar triangles OI : OK : : IK : IQ = 23·97.

Again, to get an expression for GP, put the required thickness of the pier EH or FG = z; then, because by similar triangles, IK : OK : : OD : DR = 38·04,
and IK : IO : : OD : OR = 64·42, hence OR − OI = 14·12 = IR,
and OK : OI : : IR : TR = 23·91, also DE − DR = 6·96 = RE,
hence TR + RE = 30·87 = TE, and TH = TE + EH = TE + z,
then IK : OK : : TH : HV = 22·58 + 0·7315z,
and GH − HV = GV = 49·42 − ·7315z,
lastly OI : IK : : GV : GP = 39·89 − ·5904z.
These values being now substituted in the 2d theorem $\frac{1}{2}$EF .

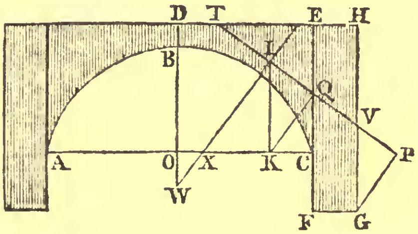
$FG^2 = \frac{IQ \cdot GP}{IK} \cdot a$, give $36z^2 = 17664{\cdot}9 - 261{\cdot}5z$, or $z^2 + 7{\cdot}26z = 490{\cdot}69$; the root of which quadratic equation gives $z = 18{\cdot}82 =$ EH or FG, the thickness of the pier sought.

It may be presumed that this theorem brings out the thickness of the piers very near the truth, and very near what would be allowed in practice by the best practical engineers, as may be gathered from a comparison of the two cases of Westminster and Blackfriars bridges, in the former of which the centre arch is a semicircle of 76 feet span, and 17 feet thickness of piers, and in the latter it is a semiellipse, of 100 feet span, 40 feet in height, and 19 feet thickness of piers.

EXAMPLE 2.

Suppose the span to be 100 feet, the height 40 feet, the thickness at top 6 feet, and the height of the pier to the springer 20 feet, as before.

Here the figure either is, or may be considered as, a scheme arch, or the segment of a circle, in which the versed sine OB is $= 40$, and the right sine OA or OC $= 50$; also DB $= 6$, CF $= 20$, and EF $= 66$. Now, by the nature of the circle, whose centre is W, the radius WB or WC $= \frac{OB^2 + OC^2}{2OB} = \frac{40^2 + 50^2}{80} = 51\frac{1}{4}$; hence OW $= 51\frac{1}{4} - 40 = 11\frac{1}{4}$; and the area of the semisegment OBC is found to be 1491;



which being taken from the rectangle ODEC $=$ OD $\times$ OC $= 50 \times 46 = 2300$, there remains $809 = a$, the area of the space BDECB. Hence, by the method of balancing this space, and measuring the lines, there will be found, KC $= 18$, IK $= 34{\cdot}6$, IX $= 42$, KX $= 24$, OX $= 8$, IQ $= 19{\cdot}4$, TE $= 35{\cdot}6$, and TH $= 35{\cdot}6 + z$, putting $z =$ EH, the breadth of the pier, as before. Then IK : KX : : TH : HV $= 24\cdot7 + 0\cdot7z$; hence GH $-$ HV $= 41\cdot3 - 0\cdot7z =$ GV, and IX : IK : : GV : GP $= 34\cdot02 - 0\cdot58z$. These values being now substituted in the theorem $\frac{1}{2}\text{EF} \cdot \text{FG}^2 = \frac{\text{IQ} \cdot \text{GP} \cdot a}{\text{IK}}$, give $33z^2 = 15431\cdot47 - 263\cdot09z$, or $z^2 + 8z = 467\cdot62$, the root of which quadratic equation gives $z = 18 =$ EH or FG the breadth of the pier, and which it may be presumed is sufficiently near the truth.

These two cases it may be expected are sufficient to exemplify this method of determining the proper dimension of the piers; a method, the propriety of which is thus confirmed by conclusions that are so conformable to the practice of the best engineers. In all cases it appears to be the easiest course, and sufficiently correct, to construct accurately the semiarch and superstructure above it; then find its centre of gravity by the method of balancing it in two positions perpendicular to each other, viz. in lines parallel and perpendicular to the base AC; next through that centre I draw a line IW perpendicular to the curve of the arch, or in the direction of the arch joints there, and meeting the base line in the point X; next, through I draw TVP perpendicular to IX, and IK perpendicular to AC, and KQ perpendicular to TP. Then measure by the scale as many of these lines as are necessary in the intended calculation, and as are used in working the 2d example above, viz. the lines IK, KX, TE, IQ, and compute the area EDEC $= a$, which may be sufficiently done in a mechanical manner, and to an approximate degree, whatever may be the figure of the curve, and shape of that area. After this, continue to complete the rest of the calculation as in the example above.

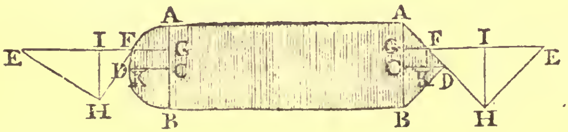
SECTION IV.

THE FORCE AND FALL OF THE WATER, &c.

PROP. XV.

To determine the Form of the Ends of a Pier, so as to make the Least Resistance, or be the Least subject, to the Force of the Stream of Water.

LET the following figure represent a horizontal section of the pier, AB its breadth, CD the given length or projection of the end, and ADB the line required, whether right or curved; also let EF represent the force of a particle of water acting on AD at the point F, in the direction parallel to the axis CD; produce EF to meet AB in G, and draw the tangent FH; also draw EH perpendicular to FH, HI perpendicular to EF, and FK perpendicular to DC.



Now the absolute force EF of the particle of water may be resolved into the two forces EH, HF, and in those directions; of these, the latter one, acting parallel to the face at F, is of no effect; and the former EH is resolved into the two EI, IH; so that EI is the only efficacious force of the particle to move the pier in the direction of its axis or length: That is, the absolute force is to the efficacious force, as EF is to EI. Then, since EF is the diameter of a semicircle passing through H, by the nature of the circle it will be, as EF : EI : : EF^2 : EH^2 : : (by similar triangles) HF^2 : HI^2 and : : the square of the fluxion of the curve or line : the square of the fluxion of the ordinate FK, because HF, HI are parallel to the line and ordinate.

Therefore, putting the abscissa DK $= x$, the ordinate KF $= y$, and the line DF $= z$, it will be, as $\dot{z}^2 : \dot{y}^2 :: 1$ (the force EF) : $\frac{\dot{y}^2}{\dot{z}^2} =$ the force of the particle at F to move the pier in the direction EFG. But the number of particles striking against the indefinitely small part of the line, is as $\dot{y}$; this drawn into the above found force of each, we have $\frac{\dot{y}^3}{\dot{z}^2} = \frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ for the fluxion of the force, or the force acting against the small part z' of the line.

But, by the proposition, the whole force on DFA must be a minimum, or the fluent of $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ must be a minimum, when that of $\dot{x}$ becomes equal to the constant quantity DC; in which case it is known that $\frac{\dot{x}\dot{y}^3}{(\dot{x}^2 + \dot{y}^2)^2}$ must be always equal to some constant quantity q; and hence $\dot{x}\dot{y}^3 = q \times (\dot{x}^2 + \dot{y}^2)^2$.

Now, in this equation, it is evident that $\dot{x}$ is to $\dot{y}$ in a constant ratio; but when two fluxions are always in a constant ratio, their fluents x, y, it is known, are also in a constant ratio, which is the property of a right line. Therefore DFA is a right line, and the end ADB of the pier must be a right-lined triangle, that the force of the water upon it may be the least possible.

PROP. XVI.

To determine the Quantity of the Resistance of the End of a Pier against the Stream of Water.

USING here the same figure and notation as in the last proposition, by the same it is found, that the fluxion of the force of the stream against the face DF, s $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$; and since the fluxion of the force against the base is $\dot{y}$, it follows, that the force of the stream against the base AB, is to the force against the face ADB, as (y) the fluent of $\dot{y}$, is to the fluent of $\frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2}$. That is, the absolute force of the stream, is to the efficacious force against the face of the pier, as its breadth is to double the fluent of $\frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2}$, when y is equal to half the breadth.

Corollary 1.—If the face ADB be rectilineal.

Putting DC $= a$, AC $= b$, and AD $= \sqrt{(aa+bb)} = c$; then, as $a : b :: x : y$ by similar triangles; hence $x = \frac{ay}{b}$, and $\dot{x} = \frac{a\dot{y}}{b}$; this being written for it in the general expression above, it becomes $\frac{bb\dot{y}}{aa+bb} = \frac{bb\dot{y}}{cc}$, for the fluxion of the force on AD; the fluent of which, or $\frac{bby}{cc}$, is the force itself. Consequently the force on the flat base AB, is to that on the triangular end ADB, as y to $\frac{bby}{cc}$, or as cc to bb, that is, as AD^2 to AC^2.

And if AC be equal to CD, or ADB a right angle, which is generally the case, then $AD^2 = 2AC^2$, and the force on the base will be to that on the face, as 2 to 1. Moreover, as the force on ADB, when ADB is a right angle, is only half of the absolute force, so it is evident that the force will be more than one-half when ADB is greater than a right angle, and less when it is less; and also, that the longer AD is, the less the force is, it being always inversely as the square of AD.

Corollary 2.—If ADB be a semicircle.

The radius AC $=$ CD $= a$; then $2ax - xx = yy$, or $x = a - \sqrt{(aa-yy)}$, and $\dot{x} = \frac{y\dot{y}}{(\sqrt{aa-yy})}$; hence $\frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2}$ becomes

$\frac{aa - yy}{aa} \times \dot{y}$, the fluent of which is $\frac{aa - \frac{1}{3}yy}{aa} \times y$; and therefore the force on the base is to the force on the circular end, as y is to $\frac{aa - \frac{1}{3}yy}{aa} \times y$, or as aa to $aa - \frac{1}{3}yy$, or as $3aa$ to $3aa - yy$. And when $y = a =$ AC, the proportion becomes that of 3 to 2. So that, only one-third of the absolute force is taken off by making the end a semicircle.

Corollary 3.—When the face ADB is a parabola.

Then, the notation being as before, viz, DC $= a$, and AC $= b$, it is $a : x :: bb : yy$; hence $x = \frac{ayy}{bb}$, and $\dot{x} = \frac{2ay\dot{y}}{bb}$; which being written in the general expression, the fluent of it becomes the circular arc whose radius is $\frac{bb}{2a}$ and tangent y, or $= \frac{bb}{2a} \times$ arc whose radius is 1 and tangent $\frac{2ay}{bb}$; so that the absolute force is to the force on the parabolic end, as y is to the arc whose tangent is y and radius $\frac{bb}{2a}$; that is, as the tangent of an arc is to the arc itself, the radius being to the tangent, as 1 to $\frac{2ay}{bb}$, or as 2 to $\frac{ay}{bb}$. And when $y = b$, the ratio of the tangent to radius, is that of 2 to $\frac{b}{a}$; or that of 2 to 1 when DC $=$ CA. In which case, the whole force is to the force on the parabolic end, as the tangent, which is double the radius, is to the corresponding arc; that is, as the tangent of 63° 26′ 4″ to the arc of the same, or as 2 to 1·10714; which is a less force than on the circle, but greater than on the triangle. And so on for other curves; in which it will be found, that the nearer they approach to right lines, the less the force will be, and that it is least of all in the triangle, in which it is one-half of the whole absolute force when right-angled.

It must be noted, however, that in determining the best form of the end of the pier to be a right-lined triangle, the water is supposed to strike every part of it with the same velocity: had the variably increased velocity been used, the form of the ends would come out a little curved; but as the increase of the velocity in the best bridges is very small, the difference in them is quite imperceptible.

PROP. XVII.

To determine the Fall of the Water in the Arches.

HAVING, in the foregoing propositions, treated of the resistance made by the piers to the current of water, it will now be proper to contemplate the effects of that resistance, and of the contraction of the passage they produce in the water-way. These effects are, a fall, or sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction; being somewhat observable at the place of all bridges, even where the arches are very large and the piers small, but in a high and extraordinary degree at London bridge, and some others, where the piers, and the sterlings, are so very large, in proportion to the arches. Now, in an open canal or river, an equal quantity of water passing in every part, in the same time, if in any part the passage be narrower, there, the bottom continuing the same, the velocity of the stream must be so much the greater, and a correspondent rise in the surface must also take place, to produce that increased celerity. Similar effects also occur in a river when any obstacles, as the piers of a bridge, are placed in the way of a stream. This is resisted and obstructed by the piers; of course the water rises against them, and consequently the stream from thence descends the more rapidly. And this is the case, not only in such canals or rivers where the stream runs always the same way, but in tide rivers also, both upward and downward. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the conclusion of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb again, the resistance of the piers against the stream, and the contraction of the water-way, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall below the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little above the bridge, becomes known. From the same velocity also will be found that of the increased stream in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz. by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the water-fall in the arches; allowing however, in the calculation, for the contraction of the stream, in the narrowed passage, at the rate as observed by Sir I. Newton. Such then are the elements and principles on which the solution of the problem is to be made out; and which it is now easy for any one to perform.

But, as it may be desirable to exhibit the manner of the solution of this curious problem, by some former noted authors, in this instance I shall give the solution from some manuscripts that have now been many years in my possession: viz, one solution by the celebrated Wm. Jones, Esq. the friend of Sir I. Newton, and father of the late Sir Wm. Jones; which is in Mr. Jones's own hand writing, and which I had from the late Mr. John Robertson, many years clerk and librarian to the Royal Society, who had the paper from Mr. Jones himself. Another solution is by the same Mr. Robertson himself, from a paper found among a great number of other manuscripts which I purchased at the sale of his books, after his death in the year 1776; and among which papers there are also other solutions that have never been published. The solutions here inserted, are given in the same words and peculiar manner as in those authors, in order to show their different forms and modes of stating and working. And first the solution by Mr. Jones, done in his usual manner, which was always remarkably concise, neat, and accurate.

The Solution of Wm. Jones, Esq.

"*Lemma.* In a chanel, whose stream runs with such an uniform velocity, in any given time, as is acquired by falling from a certain hight (h); if an obstacle should contract the passage of the water, in any place, the water above the obstacle will rise to such a hight (H) as to acquire a velocity that will discharge the stream as it comes; but will occasion a fall at the obstacle: and the difference ($H - h$) between these hights, is the measure of that fall.

"In a chanel of running water, whose breadth (b feet), and the velocity of its stream (v feet in 1″), being given: To determine the quantity of the fall, occasioned by an obstacle that takes up p feet of the breadth of the chanel.

"Let the hight fallen (near the surface of the earth) in 1″

of time, be (a feet); and the contraction of streams, in the water-way, be as r to 1. Put $c = \frac{b}{b-p}$; $d = rrcc$: Then the quantity of the fall is $\overline{d-1} \times vv \times \frac{1}{4a}$ feet.

" For, the water-way takes up w $(\frac{b-p}{b})$ part of the breadth of the chanel. But streams are found to be contracted in the water-way, in the proportion of r to 1. Therefore the water-way contracted will be $(\frac{w}{r} =) \frac{1}{rc}$ $(= m)$. But the current above the obstacle moves v feet in 1″ of time; and the velocities of water through different passages, of the same hight, are as the reciprocals of the breadth of those passages. Therefore the current, in the true water-way, must move $(\frac{v}{m} = \frac{1}{m}v =)$ nv feet in 1″ of time.

" Now, since (a) feet is the hight fallen in 1″ of time to acquire a velocity to move uniformly the length $2a$ in that time: Let x and z feet be the hights fallen to acquire a velocity to move uniformly the lengths v and nv feet in 1″ of time: and because hights fallen are as the squares of their velocities; therefore $\frac{\overline{2a}^2}{vv} = \frac{a}{x}$, and $\frac{\overline{2a}^2}{nnvv} = \frac{a}{z}$: consequently $x = \frac{vv}{4a}$, and $z = \frac{nnvv}{4a}$. That is, $\frac{vv}{4a}$ feet is the hight of water necessary to produce, in the chanel, a current that moves v feet in 1″ of time. And $\frac{nnvv}{4a}$ feet, is the hight of water necessary to produce, in the water-way, a current that moves nv in that time. Then the difference $\overline{nn-1} \times \frac{vv}{4a}$ of these hights, is the fall in feet. But $n = (\frac{1}{m} =) rc$, therefore $nn = rrcc = d$ per supposition. Therefore $\overline{d-1} \times vv \times \frac{1}{4a}$ feet, is the quantity of the fall. Q. E. D.

" Hence, putting $A = L.\frac{1}{4a}$, $B = L.r$, $C = L.c$, $D = 2 \times \overline{B + C} = L.d$: Then $L.\overline{d - 1} + 2L.v + A =$ Log. of the quantity of the fall, in feet*.

" Now, if the length of a pendulum vibrating seconds, is 39·126 inches, then will $a = 16·0899$ feet; and, according to Newton, $r = \frac{25}{21}$: consequently $A = \bar{2}.1913861$; and $B = 0.0757207$."

Such is the solution of this problem as given by Mr. Jones. And as there is contained in the same paper with this, a short solution of another kindred problem, it is here inserted, as follows.

" The length, p inches, of a pendulum that performs one vibration in 1″ of time, at a given place, being known; the altitude (a) fallen from, in 1″ of time, will be $\frac{1}{2}p\pi\pi$ inches, or $\frac{1}{24}p\pi\pi$ feet, at that place.

" For $\left(\frac{\text{time of } 1''}{\text{time in } \frac{1}{2}p} =\right) \frac{T}{t} = \frac{c}{d} = \frac{\pi\pi}{1}$; therefore $\left(\frac{TT}{tt} =\right) \frac{a}{\frac{1}{2}p} = \left(\frac{cc}{dd} =\right) \frac{\pi\pi}{1}$.

" Consequently $a = \frac{1}{2}p\pi\pi$ inches $= \frac{1}{24}p\pi\pi$ feet.

" And putting $N = (L.\frac{1}{24}\pi\pi = 2L.\pi - L.24 = \bar{1}.6140885$; then $L.a = L.p + N$."

Proceed we now to Mr. Robertson's solution of the problem, which is on the principles, but more in detail, than Mr. Jones's. This solution was published by Mr. R. in the Philos. Trans. vol. 50, or in my new Abridgement, vol. 13, from which it is chiefly here extracted.

Mr. John Robertson's Solution of the Problem.

" Sometime before the year 1740, the problem about the fall of water, occasioned by bridges built across a river, was

* This is the theorem, adapted to working by logarithms, given by Mr. Jones to Mr. Gardiner, and printed in p. 12 of his Logarithms in 4to; the latter L denoting logarithm, in the theorem.

much spoken of at London, on account of the fall that was supposed would be at the new bridge to be built at Westminster. In Mr. Hawksmoor's and Mr. Labelye's pamphlets, the former published in 1736, and the latter in 1739, the result of Mr. Labelye's computations was given: but neither the investigation of the problem, nor any rules, were at that time published.

"In the year 1742 was published, Gardiner's edition of Vlacq's Tables; in which, among the examples there prefixed, to show some of the uses of those tables, drawn up by the late Wm. Jones, esq. there are two examples, one showing how to compute the fall of water at London-bridge, and the other applied to Westminster-bridge: but that excellent mathematician's investigation, by which those examples were wrought, was not printed, though he communicated copies of it to several of his friends. Since that time, it seems as if the problem had in general been forgotten, as it has not made its appearance, to my knowledge, in any of the subsequent publications. As it is a problem somewhat curious, though not difficult, and its solution not generally known, (having seen four different solutions, one of them very imperfect, extracted from the private books of an office in one of the departments of engineering in a neighbouring nation), I thought it might give some entertainment to the curious in these matters, if the whole process were published.

"PRINCIPLES.

"1. A heavy body, that in the first second of time has fallen the height of a feet, has acquired such a velocity, that, moving uniformly with it, will in the next second of time move the length of $2a$ feet.

"2. The spaces run through by falling bodies are proportional to one another as the squares of their last or acquired velocities.—These two principles are demonstrated by the writers on mechanics.

"3. Water forced out of a larger chanel, through one or more smaller passages, will have the streams through those passages contracted in the ratio of 25 to 21.—This is shown in the 36th prob. of the 2d book of Newton's Principia.

" 4. In any stream of water, the velocity is such, as would be acquired by the fall of a body from a height above the surface of that stream.—This is evident from the nature of motion.

" 5. The velocities of water through different passages of the same height, are reciprocally proportional to their breadths.—For, at some time, the water must be delivered as fast as it comes; otherwise the bounds would be overflowed. At that time, the same quantity, which in any time flows through a section in the open chanel, is delivered in equal time through the narrower passages; or the momentum in the narrow passages must be equal to the momentum in the open chanel; or the rectangle under the section of the narrow passages, by their mean velocity, must be equal to the rectangle under the section of the open chanel by its mean velocity. Therefore the velocity in the open chanel is to the velocity in the narrower passages, as the section of those passages is to the section of the open chanel. But, the heights in both sections being equal, the sections are directly as the breadths. Consequently the velocities are reciprocally as the breadths.

" 6. In a running stream, the water above any obstacles put therein, will rise to such a height, that by its fall the stream may be discharged as fast as it comes.—For the same body of water, which flowed in the open chanel, must pass through the passages made by the obstacles: and the narrower the passages, the swifter will be the velocity of the water: but the swifter the velocity of the water, the greater is the height, from which it has descended: consequently the obstacles, which contract the chanel, cause the water to rise against them. But the rise will cease, when the water can run off as fast as it comes: and this must happen when, by the fall between the obstacles, the water will acquire a velocity in a reciprccal proportion to that in the open chanel, as the breadth of the open chanel is to the breadth of the narrow passages.

" 7. The quantity of the fall, caused by an obstacle in a running stream, is measured by the difference between the heights fallen from, to acquire the velocities in the narrow passages and open chanel.—For, just above the fall the velocity of the stream is such, as would be acquired by a body falling from a height higher than the surface of the water: and at the fall, the velocity of the stream is such, as would be acquired by the fall of a heavy body from a height more elevated than the top of the falling stream; and consequently the real fall is less than this height. Now as the stream comes to the fall with a velocity belonging to a fall above its surface; consequently the height belonging to the velocity at the fall, must be diminished by the height belonging to the velocity with which the stream arrives at the fall.

" PROBLEM.

" In a chanel of running water, whose breadth is contracted by one or more obstacles; the breadth of the chanel, the mean velocity of the whole stream, and the breadth of the water-way between the obstacles, being given; to find the quantity of the fall occasioned by those obstacles.

" Let b = breadth of the chanel in feet;

v = mean velocity of the water in feet per second;

c = breadth of the water-way between the obstacles.

Now $25 : 21 :: c : \frac{21}{25}c$, the water-way contracted, by prin. 3.

And $\frac{21}{25}c : b :: v : \frac{25b}{21c}v$, the veloc. in the contr. way, prin. 5.

Also $(2a)^2 : vv :: a : \frac{vv}{4a}$, height fallen to gain the vel. v, 1 and 2.

And $(2a)^2 : (\frac{25b}{21c}v)^2 :: a : (\frac{25b}{21c})^2 \times \frac{vv}{4a}$, ditto for the velocity $\frac{25b}{21c}v$, by princ. 1 and 2.

Then $\frac{25b}{21c} \times \frac{vv}{4a} - \frac{vv}{4a}$ is the measure of the fall required, prin. 7.

Or $[(\frac{25b}{21c})^2 - 1] \times \frac{vv}{4a}$ is a rule for computing the fall.

Here $a = 16{,}0899$ feet; and $4a = 64{,}3596$.

" EXAMPLE 1. *For London-Bridge.*

" By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
Sum of water-ways at the time of low water is 236 feet;
Mean veloc. of stream just above bridge is $3\frac{1}{6}$ f. per sec.
Under almost all the arches there are great numbers of drip-shot piles, or piles driven into the bed of the water-way, to prevent it from being washed away by the fall. These drip-shot piles considerably contract the water-ways, at least $\frac{1}{6}$ of their measured breadth, or about $39\frac{1}{3}$ feet in the whole. So that the water-way will be reduced to $196\frac{2}{3}$ feet.

" Now $b = 926$; $c = 196\frac{2}{3}$; $v = 3\frac{1}{6}$; $4a = 64{,}3596$.

Then $\frac{25b}{21c} = \frac{23150}{4130} = 5{,}60532$; its square $= 31{,}4196$;

And $31{,}4196 - 1 = 30{,}4196 = (\frac{25b}{21c})^2 - 1$;

Also $vv = (\frac{19}{6})^2 = \frac{361}{36}$; And $\frac{vv}{4a} = \frac{361}{36 \times 64{,}3596} = 0{,}15581$.

Then $30{,}4196 \times 0{,}15581 = 4{,}739$ f. $= 4$ f. $8{,}868$ inc. the fall required.

" By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches."

" EXAMPLE 2. *For Westminster-Bridge.*

" Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to 820 feet: to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water just above the bridge, from many experiments, is not greater than $2\frac{1}{4}$ feet per second.

"Here $b = 994$; $c = 820$; $v = 2\frac{1}{4}$; $4a = 64{,}3596$.

Now $\frac{25b}{21c} = \frac{24850}{17220} = 1{,}443$; and its square $= 2{,}082$;

Hence $2{,}082 - 1 = 1{,}082 = (\frac{25b}{21c})^2 - 1$.

Also $vv = (\frac{9}{4})^2 = \frac{81}{16}$; and $\frac{vv}{4a} = \frac{81}{16 \times 64{,}3596} = 0{,}0786$.

Then $1{,}082 \times 0{,}0786 = 0{,}084$ f. $= 1$ inch, the fall required; and is about half an inch more than the greatest fall observed by Mr. Labelye."

Among the old papers of Mr. Robertson I find several other solutions of the same problem, by different persons, and on somewhat different principles. Several of the papers also, which are of a miscellaneous nature, relate to other branches of the subject of bridges; some of which, being curious, I shall avail myself of, by insertion in the appendix to this Tract.—The following table shows, at one view, the quantity of fall in the water under the arches, in consequence of its obstruction and contraction by the piers, according to several rates of velocity and quantity of obstacles; as computed on the foregoing principles.

A Table of the natural Rise of Water, in Proportion to the Resistance or Obstruction it meets with, in its Passage.

Construction of a modern bridge of 2 arches. Resistance 1-11th. Rise of Water.	Velocity of the Current in one Second.	OBSTRUCTIONS, OR RESISTANCES. 1-8th	1-4th	3-8ths	1-half	5-8ths	3-4ths	7-8ths	Stages of Accumulation in Floods.	Construction of an ancient bridge of 3 or more arches. Resistance 5-18ths. Rise of Water.
		Proportional Rise of Water, in Feet, Inches, and Parts.								
ft. in. pts.		ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.		ft. in. pts.
0 0 ·133	1 foot	0 0 ·158	0 0 ·283	0 0 ·49	0 0 ·87	0 1 ·69	0 4 ·041	1 4 ·728	Uniform Tenors.	0 0 ·320
0 0 ·533	2 feet	0 0 ·635	0 1 ·133	0 1 ·96	0 3 ·48	0 6 ·77	1 4 ·164	5 6 ·9		0 1 ·28
0 1 ·2	3 feet	0 1 ·428	0 2 ·549	0 4 ·41	0 7 ·835	1 3 ·234	3 0 ·368	12 6 ·53	Ordinary Floods.	0 2 ·881
0 2 ·133	4 feet	0 2 ·539	0 5 ·439	0 7 ·89	1 1 ·928	2 3 ·08	5 4 ·656	22 3 ·6		0 5 ·119
0 3 ·333	5 feet	0 3 ·967	0 7 ·083	1 0 ·25	1 9 ·763	3 6 ·316	8 5 ·024	34 10 ·31	Extra-ordinary Floods.	0 8 ·003
0 4 ·799	6 feet	0 5 ·713	0 10 ·199	1 5 ·64	2 7 ·339	5 0 ·934	12 1 ·476	50 2 ·112		0 11 ·525
Piers 12 / River 132	Velocities above seldom happen.	Piers 20 / Arches 140	Piers 40 / Arches 140	Piers 60 / Arches 160	Piers 80 / Arches 160	Piers 100 / Arches 160	Piers 120 / Arches 160	Piers 140 / Arches 160	Torrents above generally Inundations.	Piers 50 / River 180

This, next to one arch which has no resistance, without the flood encroaches on its crown, the most eligible mode.

N. B. These several numbers, respectively, show how high the water is constrained to rise above its natural level, or surface; which would otherwise carry it off, in a free and uninterrupted passage; therefore these numbers must everywhere be added to the depth of water, below the fall, to give the true height of the flood.—The seven predicaments above show the excellence or imperfection of bridges, of every construction, and in all states of a flood, either in its uniform or variable tenors; and by which appears the great advantage of bridges of a sufficient capacity, and the pernicious consequences of all such as are not so.—London-bridge is nearly in the 6th predicament of this table, and Westminster-bridge nearly in the 2d. At the 1st of these the Thames, with a velocity of about 3f. 2in. per second, rises to about 4f. 7in. and at the latter, with a velocity of 2f. 6in. per second, to only 2 inches and a half.

In this most common mode, seldom sufficient in a flood, the water soon encroaches on the arches, and changes the predicament.

SECTION V.

OF THE TERMS OR NAMES OF THE VARIOUS PARTS PECULIAR TO A BRIDGE, AND THE MACHINES, &c, USED ABOUT IT; DISPOSED IN ALPHABETICAL ORDER.

ABUTMENT, or BUTMENT, which see in its place below.

ARCH, an opening of a bridge, through or under which the water and vessels pass, and which is usually supported by piers or by butments. Arches are denominated circular, elliptical, cycloidal, catenarian, &c, according to the figure of the curve of them. There are also other denominations of circular arches, according to the different parts of a circle: So, a semicircular arch, is half the circle; a scheme or skeen arch, is a segment less than the semicircle; and arches of the third and fourth point, or gothic arches, consist of two circular arcs, excentric and meeting in an angle at top, each being 1-3d or 1-4th, &c, of the whole circle.

The chief properties of the most considerable arches, with regard to the extrados they require, &c, may be learned from the second section. It there appears, that none, but the arch of equilibration in the 2d example to prop. 5, can admit of a horizontal line at top: that this arch is not only of a graceful, but of a convenient form, as it may be made higher or lower at pleasure with the same opening: that, with a horizontal top, it can be equally strong in all its parts, and therefore ought to be used in all works of much consequence. All the other arches require tops that are curved, either upward or downward, some more and some less. Of these, the elliptical, or the cycloidal arch, seems to be the fittest to be substituted instead of the balanced one, with the least degree of impropriety: it is in general also the best form for most bridges, as it can be made of any height to the same span, or of any span to the same height, while at the same time its flanks are sufficiently elevated above the water, even when it is pretty flat at top; a property of which the other curves are not possessed in an equal degree: and this property is the more valuable, because it is remarked that, after any arch is built, and the centering struck, it settles more about the hanches than the other parts, by which other curves are reduced near to a straight line at the flanks. Elliptical arches also look bolder, are really stronger, and require less materials and labour than the others. Of the other curves, the cycloidal arch is next in quality to the elliptical one, for all the above properties. And, lastly, the circle. As to the others, the parabola, hyperbola, and catenary, they may not at all be admitted in bridges of several arches; but may in some cases be used for a bridge of one single arch, which is to rise very high, because then not much loaded at the flanks. We may hence also perceive the fallacy of those arguments which assert, that because the catenarian curve supports itself equally in all its parts, it will therefore best support any additional weight laid upon it: for the additional building made to raise the bridge to a horizontal line, or nearly such, by pressing more in one part than another, must force those parts down, and the whole must fall. Whereas, other curves will not support themselves at all, without some additional parts built above them, to balance them, or to reduce their parts to an equilibrium.

Archivolt, the curve or line formed by the upper sides of the voussoirs or arch stones. It is parallel to the intrados or underside of the arch when the voussoirs are all of the same length; otherwise not. By the archivolt is also sometimes understood the whole set of voussoirs.

Banquet, the raised foot path at the sides of the bridge next the parapet. This ought to be allowed in all bridges of any considerable size: it should be raised about a foot above the middle or horse passage, being made 3, 4, 5, 6, 7, &c, feet broad, according to the size of the bridge, and paved with large stones, of a length equal to the breadth of the walk.

Battardeau, or Coffer-dam, a case of piling, &c, without a bottom, fixed in the bed of the river, water-tight or nearly so, by which to lay the bottom dry for a space large enough to build the pier on. When it is fixed, its sides reaching above the level of the water, the water is pumped out of it, or drawn off by engines, till the included space be laid dry; and it is kept so, by the same means, if there are leaks which cannot be stopped, till the pier is built up in it; and then the materials of it are drawn up again.

Battardeaux are made in various manners, either by a single inclosure, or by a double one, with clay or chalk rammed in between the two, to prevent the water from coming through the sides. And these inclosures are also made, either with piles only, driven close by one another, and sometimes notched or dove-tailed into each other; or with piles, grooved in the sides, and driven in at a distance from one another, with boards let down between them in the grooves.

The method of building in battardeaux cannot well be used where the river is either deep or rapid. It also requires a very good natural bottom of solid earth or clay: for, though the sides be made water-tight, if the bottom or bed of the river be of a loose consistence, the water will ooze up through it, in too great abundance to be evacuated by the engines. It is almost needless to remark, that the sides must be made very strong, and well propt or braced on the inside, to prevent the ambient water from pressing the sides in, and forcing its way into the battardeaux.

Bridge, a work of carpentry, masonry, or iron, built over a river, canal, &c, for the conveniency of crossing the same. A bridge is an edifice forming a way over a river, &c, supported by one arch, or by several arches, and these again supported by proper piers or butments. A stately bridge, over a large river, is one of the most noble and striking pieces of human art. To behold huge and bold arches, composed of an immense quantity of small materials, as stones, bricks, &c, so disposed and united together, that they seem to form but one solid compact body, affording a safe passage for men and carriages over large waters, which with their navigation pass free and easy under them at the same time, is a sight truly surprizing and affecting.

To the absolutely necessary parts of a bridge, already mentioned, viz, the arches, piers, and abutments, may be added the paving at top, the parapet wall, either with or without a balustrade, &c; also the banquet, or raised foot way, on each side, leaving a sufficient breadth in the middle for horses and carriages. The breadth of a bridge for a great city should be such as to allow an easy passage for three carriages and two horsemen a-breast in the middle way, and for three foot passengers in the same manner on each banquet. And for other less bridges, a less breadth.

As a bridge is made for a way or passage over a river, &c, so it ought to be made of such a height, as will be quite convenient for that passage; but yet so as to be consistent with the interest and concerns of the river itself, easily admitting through its arches the craft that navigate on it, and all the water, even at high tides and floods. The neglect of this precept has been the ruin of many bridges, and particularly that at Newcastle, over the river Tyne, on the 17th of November 1771. So that, in determining its height, the conveniences both of the passage over it, and under it, should be considered, and the height made to answer the best for them both, observing to make the *convenient* give place to the *necessary*, when their interests are opposite.

Bridges are generally placed in a direction perpendicular to the stream in a direct line, to give free passage to the water, &c. But some think they should be made, not in a straight line, but convex towards the stream, the better to resist floods, &c. And some such bridges have been really made.—Again, a bridge should not be made in too narrow a part of a navigable river, or one subject to tides or floods: because the breadth being still more contracted by the piers, will increase the depth, velocity, and fall of the water under the arches, and endanger the whole bridge and navigation.

Bridges are usually made with an odd number of arches, as one, or three, or five, or seven, &c; either that the middle of the stream or chief current may flow freely without the interruption of a pier; or that the two halves of the bridge, by gradually rising from the ends to the middle, may there meet in the highest and largest arch; or else, for the sake of grace, that by being open in the middle, the eye, in viewing it, may look directly through there, as one always expects to do in looking at it, and without which opening we generally feel a disappointment in viewing it.

If the bridge be equally high throughout, the arches, being all of a height, are made all of a size; which causes a great saving of centring. If the bridge be higher in the middle than at the ends, the arches are made to decrease from the middle towards each end, but so, as that each half may have the arches exactly alike, and that they decrease in span, proportionally to their height, so as to be always the same kind of figure, and similar parts of that figure: thus, if one be a semicircle, the rest should be semicircles also, but proportionally less; if one be a segment of a circle, the rest should be similar segments of other circles; and so for other figures. The arches being equal at equal distances, on both sides of the middle, is not only for the strength and beauty of the bridge, but that the centring of the one half may serve for the other also. But if the bridge be higher at the ends than the middle, which is a very uncommon case, the arches ought to increase in span and pitch from the middle towards the ends. When the middle and ends are of different heights, their difference however ought not to be great in proportion to the length, that the ascent may be easy; and then also it is more beautiful to make the top one continued curve, like Blackfriars, than two inclined straight lines, from the ends towards the middle, like that of Westminster bridge.

Bridges should rather be of few and large arches, than of many and small ones, if the height and situation will allow of it; for this will leave more free passage for the water and navigation, and be a great saving in materials and labour, as there will be fewer piers and centres, and the arches themselves will require less materials. And, one large single arch only is best, when it can be executed. For the fabric of a bridge, and the proper estimate of the expence, &c, there are generally necessary three plans, three sections, and an elevation. The three plans, are so many horizontal sections, viz, the first a plan of the foundation under the piers, with the particular circumstances attending it, whether of gratings, planks, piles, &c: the second, is the plan of the piers and arches, &c: the third, is the plan of the superstructure, with the paved road and banquet. The three sections, are vertical ones: the first of them a longitudinal section, from end to end, and through the middle of the breadth: the second, a transverse one, or across it, and through the summit of an arch: and the third also across, but taken on a pier. The elevation, is an orthographic projection of one side or face of the bridge, or its appearance as viewed at a great distance, showing the exterior aspect of the materials, and the manner in which they are worked and decorated.—Other observations are to be seen in the first section.

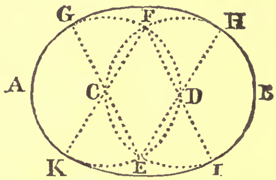
Butments, or Abutments, are the extremities of a bridge, by which it joins to, or abuts on, the land or sides of the river, &c. These must be made very secure, quite immovable, and more than barely sufficient to resist the drift of its adjacent arch. So that, if there are not rocks or very solid banks to raise them against, they must be well reinforced with proper walls or returns, &c. The thickness of them, which will be barely sufficient to resist the shoot of the arch, may be calculated as that of a pier by prop. xi.

When the foundation of a butment is raised against a sloping bank of rock, gravel, or good solid earth, it will produce a saving of materials and labour, to carry the work on by returns at different heights against it, like steps of stairs. And if the foundation, and all the courses, parallel to it, be laid, not horizontal, but rising backwards, so as to be perpendicular to the springing and pressure of the arch, it will be less liable to slide or be forced back by the push of the arch.

Caisson, a kind of Chest, or flat-bottomed boat, in which a pier is built, then sunk to the bed of the river, and the sides loosened and taken off from the bottom, by a contrivance for that purpose; the bottom of it being left under the pier as a foundation. It is evident therefore, that the bottoms of caissons must be made very strong, and fit for foundations of the piers. The caisson is kept afloat till the pier be built to about the height of low-water mark; and, for that purpose, its sides must either be made of more than that height at first, or else gradually raised to it as it sinks by the weight of the work, so as always to keep its top above water. And therefore the sides must be made very strong, and be kept asunder by cross timbers within, lest the great pressure of the ambient water should crush the sides in, and so not only endanger the work, but also drown the men who work within it. The caisson is made of the shape of the pier, but some feet wider on every side, to make room for the men to work: the whole of the sides are of two pieces, both joined to the bottom quite around, and to each other at the salient angles, so as to be disengaged from the bottom, and from each other, when the pier is raised to the desired height, and sunk. It is also convenient to have a small sluice made in the bottom, occasionally to open and shut, to sink the caisson and pier sometimes by, before it be finished, to try if it bottom level and rightly; for, by opening the sluice, the water will rush in and fill it to the height of the exterior water, and the weight of the work already built will sink it; then, by shutting the sluice, and pumping out the water, it will be made to float again, and the rest of the work may be completed: but it must not be sunk except when the sides are high enough to reach above the surface of the water, otherwise it cannot be raised and laid dry again. Mr. Labelye says, that the caissons in which he built some of the piers of Westminster bridge, contained above 150 load of fir timber, of 40 cubic feet each, and was of more tonnage, or capacity, than a 40 gun ship of war.

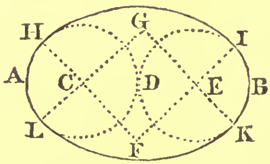
CENTRES, and CENTRING, or CENTERING, are the timber frames erected in the spaces of the arches, to turn them on, by building on them the voussoirs of the arches. As the centre serves as a foundation for the arch to be built on, when the arch is completed, that foundation is struck from under it, to make way for the water and navigation, and then the arch will stand of itself from its curved figure. A centre must therefore be constructed of the exact figure of the intended arch, convex as the arch is concave, to receive it on as a mould. If the form be circular, the curve is struck from a central point by a radius: if it be elliptical, it ought to be struck with a doubled cord, passing over two pins or nails fixed in the foci, as the mathematicians and gardeners describe their ellipses. Very often, in practice, an oval is employed, as made of three circular arcs. This very nearly resembles the true geometrical ellipsis, being formed of two equal arcs of small circles at the extremities, having between them a longer arch of a much larger circle, the ends of these arches being made to butt and join to each other, that they seem like the same curve only continued. As this mechanical oval will have nearly the same properties and effect as the true ellipsis, and can be more conveniently worked by the builders, as it requires the voussoirs to be cut only to two moulds, or for two centres, while those for the true ellipsis have them all different, we shall add in this place some of the most approved methods of describing these ovals. These methods indeed are, and must be, various, according as the length or span is required to be more or less, in proportion to the breadth or height. But in all of them, the centres of the large and small arcs must be so taken, that the right line passing through them, may also, when continued, pass through exactly the point where the ends of those arches butt and join together; for by this means they will have the same common tangent at that point, and consequently they will unite together, or run into each other, like parts of the same curve produced.

FIRST METHOD.—*When the Length and Breadth differ not very much.*



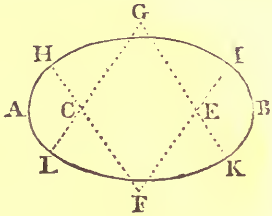
Divide the given length or span AB into three equal parts, at the points C and D. With one of those parts, CD, as a radius, and from the two centres C, D, describe two circles, intersecting each other in the two points E and F. Through these two points E, F, and the two centres C, D, draw four lines ECG, EDH, FDI, FCK, cutting the two circles in the four points G, H, I, K. Lastly, with one of these lines, as a radius, and from the two centres E, F, describe the two arches GH, KI, and they will complete the oval, forming a figure so much resembling a true ellipse, that the eye cannot perceive the difference between them. In this oval it is evident that the radius of the larger circular arch is just double of that of the smaller arches.

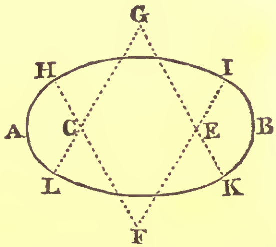
SECOND METHOD.—*For a Narrower Oval.*



Divide the length or span AB into four equal parts; then, with one of those parts as a radius, and from the three points of division, C, D, E, as centres, describe three circles. Find the uppermost and lowest points, F, G, of the middle circle; or through the middle point D draw a perpendicular to AB, which will give the points F, G, or construct the square CGEF, which will give the centres of the larger arch. Through these two points F, G, and the two C, E, draw four lines FH, FI, GK, GL; with any one of which as a radius, and the two centres F, G, describe the other two arcs HI, KL, to complete the oval; which does not rise so high as the former.

THIRD METHOD.





Other ovals may be made to the same length, or any other length, but rising still less in the crown, in any degree whatever, if, after having described the two smaller or end circles from the centres C and E, as in the second method, instead of forming the right angled triangles CGE, CFE, these be described with acute angles at F and G, by making the equal lines CF, CG, EF, EG, longer than before in any ratio at pleasure; these being then produced to the little circles at the four points H, I, K, L, from the centres F, G, describe the other two arches HI and KL, to complete the ovals, narrower and narrower at pleasure.

The little circles also at the ends, may have their radius taken smaller to any degree, or a less portion of the whole span; and indeed it is evident that its radius ought always to be less than the pitch or height of the arch.

There are other methods of making such ovals, but those above given are some of the best. The last method is general too, and will serve to accommodate an oval to any length and breadth whatever, at pleasure. Having thus described the half of such an oval to any span and pitch proposed, for any arch of a bridge, &c, the whole of the voussoirs may be cut by two mold boards only, viz, one for the voussoirs for the arch AH and IB, and the other for those in the arch HI.

But if the arch be of any other form, the several abscisses and ordinates ought to be calculated; then their corresponding lengths, transferred to the centring, will give so many points of the curve, and exactly by these points bending a bow of pliable matter, the curve may be drawn close by it.

The centres are constructed of beams, &c, of timber, firmly pinned and bound together, into one entire compact frame, covered smooth at top with planks or boards to place the voussoirs on, the whole supported by offsets in the sides of the piers, and by piles driven into the bed of the river, and capable of being raised and depressed by wedges, contrived for that purpose, and for taking them down when the arch is completed. They ought also to be constructed of a strength more than sufficient to bear the weight of the arch.

In taking down the centring; it is first let down a little, all in a piece, by easing some of the wedges; it is there let to rest a few hours, or days, to try if the arch make any efforts to fall, or any joints open, or stones crush or crack, &c, that the damage may be repaired before the centring is entirely removed, which is not to be done till the arch ceases to make any visible efforts.

In some bridges the centring makes a considerable part of the expence, and therefore all means of saving in this article ought to be closely attended to; such as making few arches, and as nearly alike or similar as possible, that the centring of one arch may serve for others, and at least that the same centre may be used for each pair of equal arches, on both sides of the middle.

Chest, the same as Caisson.

Cofferdam, the same as Batterdeau.

Drift, Shoot, or Thrust, of an arch, is the push or force which it exerts in the direction of the length of the bridge. This force arises from the perpendicular gravitation or weight of the stones of the arch, which, being kept from descending by the form of the arch and the resistance of the piers, exert their force in a lateral direction. This force is computed in prop. XI, where the thickness of the pier is determined which is necessary to resist it; and is the greater as the pitch is lower, *cæteris paribus*.

ELEVATION, the orthographic projection of the front of a bridge, on the vertical plane, parallel to its length. This is necessary to show the form and dimensions of the arches, and other parts, as to height and breadth, and therefore it has a plain scale annexed to it, to measure the parts by. It also shows the manner of working up and decorating the fronts of the bridge.

EXTRADOS, the exterior curvature or line of an arch. In the propositions of the second section, it is the outer or upper line of the wall above the arch; but it often means only the upper or exterior curve of the voussoirs.

FOUNDATIONS, the bottoms of the piers, &c, or the bases on which they are built. These bottoms are always to be made with projections, greater or less according to the spaces on which they are built. And according to the nature of the ground, the depth and velocity of water, &c, the foundations are laid, and the piers built, after different manners, either in caissons, in batterdeaux, or on stilts with sterlings, &c; for the particular methods of doing which, see each under its respective term.

The most obvious and simple method of laying the foundations, and raising the piers up to water-mark, is to turn the river out of its course above the place of the bridge, into a new channel, cut for it near the place where it makes an elbow or turn; then the piers are built on dry ground, and the water turned into its old course again, the new one being securely banked up. This is certainly the best method, when the new channel can be easily and conveniently made; but which however is very seldom the case.

Another method is, to lay only the space of each pier dry, till it be built, by surrounding it with piles and planks driven down into the bed of the river, so close together as to exclude the water from coming in; then the water is pumped out of the inclosed space, the pier built in it, and lastly the piles and planks drawn up. This is cofferdam work; but it evidently cannot be practised when the bottom is of a loose consistence, admitting the water to ooze and spring up through it.

When neither the whole nor part of the river can be easily laid dry, as above, other methods are to be used; such as, to build either in caissons or on stilts, both which methods are described under their proper words; or yet by another method, which hath, though seldom, been sometimes used, without laying the bottom dry, and which is thus: the pier is built upon strong rafts or gratings of timber, well bound together, and buoyed up on the surface of the water by strong cables, fixed to other floats or machines, till the pier is built; the whole is then gently let down to the bottom, which must be made level for the purpose. But of these methods, that of building in caissons is the best.

But before the pier can be built in any manner, the ground at the bottom must be well secured, and made quite good and safe, if it be not so naturally. The space must be bored into, to try the consistence of the ground; and if a good bottom of stone, or firm gravel, clay, &c, be met with, within a moderate depth below the bed of the river, the loose sand, &c, must be removed and digged out to it, and the foundation laid on the firm bottom, on a strong grating, or base of timber, made much broader every way than the pier, that there may be the greater base to press on, to prevent its being sunk. But if a solid bottom cannot be found at a convenient depth to dig to, the space must then be driven full of strong piles, the tops of which must be sawed off level, some feet below the bed of the water, the sand having been previously digged out for that purpose; and then the foundation, on a grating of timber, laid on their tops as before. Or, when the bottom is not good, if it be made level, and a strong grating of timber, two, three, or four times as large as the base of the pier, be made, it will form a good base to build on, its great size in a great measure, preventing it from sinking. In driving the piles, the method is, to begin at the middle, and proceed outwards, all the way to the borders or margin: the reason of which is, that if the outer piles were driven first, the earth of the inner space would be thereby so jammed together, as not to allow the inner piles to be driven at all. And besides the piles immediately under the piers, it is also very prudent to drive in a single, double, or triple row of them, around and close to the frame of the foundation, cutting them off a little above it, to secure it from slipping aside out of its place, and to bind the ground under the pier the firmer. For, as the safety of the whole bridge depends much on the foundations, too much care cannot be used to have the bottom made quite secure.

Jettee, the border made around the stilts under a pier; being the same with Sterling.

Impost, is the part of the pier on which the feet of the arches stand, or from which they spring.

Keystone, the middle voussoir, or the arch stone in the crown, or immediately over the centre of the arch. The length of the keystone, or thickness of the archivolt at top, is allowed to be about 1-15th or 1-16th of the span, by the best architects.

Orthography, the elevation of a bridge, or front view, as seen at a great distance.

Parapet, the breast wall made on the top of a bridge, to prevent passengers from falling over. In good bridges, to build the parapet only a little part of its height close or solid, and on that a balustrade to above a man's height, has an elegant and useful effect.

Piers, are the walls built for the support of the arches, and from which they spring as their bases. These ought to be built of large blocks of stone, solid throughout, and cramped together with iron, or otherwise, which will make the whole like one solid stone. Their faces or ends, from the base up to high-water mark, ought to project sharp out with a salient angle, to divide the stream. Or perhaps the bottom of the pier should be built flat or square up to about half the height of low-water mark, to allow a lodgment against it for the sand or mud, to cover the foundation; lest, by being left bare, the water should in time undermine, and so ruin or injure it. The best form of the protection for dividing the stream, is the triangle; and the longer it is, or the more acute the salient angle, the better it will divide it, and the less will the force of the water be against the pier; but it may be sufficient to make that angle a right one, as it will make the work stronger, and in that case the perpendicular projection will be equal to half the breadth or thickness of the pier. In rivers on which large heavy craft návigate, and pass the arches, it may perhaps be better to make the ends semicircular; for though it does not divide the water so well as the triangle, it will both better turn off and bear the shock of the craft.

The thickness of the piers ought to be such, as will make them of weight or strength sufficient to support their interjacent arch, independent of any other arches. The thickness, in most cases of practice, may be made about $\frac{1}{8}$ of the span of the arch. And then, if the middle of the pier be run up to its full height, the centring may be struck, in order to be used in another arch, before the hanches are filled up. The whole theory of the piers may be seen in the third section. They ought to be made with a broad bottom on the foundation, and gradually diminished in thickness by offsets, up to low-water mark. The methods of laying their foundations, and building them up to the surface of the water, are given under the word FOUNDATION.

PILES, are timbers driven into the bed of the river for various purposes, and are either round, square, or flat like planks. They may be of any wood which will not rot under water, but elm, oak, and fir are mostly used, especially the latter, on account of its length, straightness, and cheapness. They are shod with a pointed iron at the bottom, the better to penetrate into the ground; and are bound with a strong iron band or ring at top, to prevent them from being split by the violent strokes of the ram by which they are driven down. It is said, that the stilts, or piles, under London-bridge, are of elm, which lasts a long time in the water.

Piles are either used to build the foundations on, or are driven about the pier as a border of defence, or to support the centres on; and in this case, when the centring is removed, they must either be drawn up, or sawed off very low under water; but it is perhaps better to saw them off, and leave them sticking in the bottom, lest the drawing of them out should loosen the ground about the foundation of the pier. Those to build on, are either such as are cut off by the bottom of the water, or rather a few feet within the bed of the river; or else such as are cut off at low-water mark, and then they are called stilts. Those to form borders of defence, are rows driven in close by the frame of a foundation, to keep it firm; or else they are to form a case or jettee about the stilts, to keep within it the stones that are thrown in to fill it up; in this case, the piles are grooved, driven at a small distance from each other, and plank piles let into the grooves between them, and driven down also, till the whole space is surrounded. Besides using this for stilts, it is also sometimes necessary to surround a stone pier with a sterling or jettee, and fill it up with stones to secure an injured pier from being still more damaged, and the whole bridge ruined. The piles to support the centres may also serve as a border of piling to secure the foundation, cutting them off low enough after the centre is removed.

PILE DRIVER, is an engine for driving down the piles. It consists of a large ram or square block of iron, sliding perpendicularly down between two guide posts; which being drawn up to the top of them, and there let fall from a great height, it comes down on the top of the pile with a violent blow. It is worked either by men or horses, and either with or without wheel work. That which was used at the building of Westminster-bridge, is perhaps one of the best kind.

PITCH, of an arch, is the perpendicular height from the spring, or impost, to the keystone.

PLAN, of any part, as of the foundations, or piers, or superstructure, is the orthographic projection of it on a plane parallel to the horizon.

PUSH, of an arch, the same as drift, shoot, or thrust.

SALIENT ANGLE, of a pier, is the projection of the end against the stream, to divide it. The right-lined angle best divides the stream, and the more acute the better for that purpose; but the right angle is generally used, as making the best masonry. A semicircular end, though it does not divide the stream so well, is sometimes better in large navigable rivers, as it carries the craft the better off, or bears their shocks the better.

SHOOT, of an arch, is the same as drift, thrust, &c.

SPAN, of an arch, is the extent or width at the bottom, or on the level at its springing.

SPANDRELS, or SPANDRILS, are the spaces about the flanks or haunches of the arch, above the curve or intrados.

SPRINGERS, are the first or lowest stones of an arch, being those at its feet, bearing immediately on the impost.

STERLINGS, or JETTEES, a kind of case, made of stilts, &c, about a pier, to secure it. It is particularly described under the next word STILTS.

STILTS, a set of piles driven into the space intended for the pier, whose tops being sawed level off about low-water mark, the pier is then raised on them. This method was formerly used, when the bottom of the river could not be laid dry; and these stilts were surrounded, at a few feet distance, by a row of piles and planks, &c, close to them like a coffer-dam, and called a sterling or jettee; after which, loose stones, &c, are thrown or poured down into the space, till it be filled up to the top, by that means forming a kind of pier of rubble or loose work, which is kept together by the sides of the sterlings: this is then paved level at the top, and the arches turned upon it. This method was formerly much used, most of the large old bridges in England being constructed in that way; such as London-bridge, Newcastle-bridge, Rochester-bridge, &c. But the inconveniencies attending it are so great, that it is now quite exploded and disused: for, because of the loose composition of the piers, they must be made very large or broad, otherwise the arch would push them over, and rush down as soon as the centre should be drawn: which great breadth of piers and sterlings so much contracts the passage of the water, as not only very much incommodes the navigation through the arch, from the fall and quick motion of the water, but from the same cause also the bridge itself is in much danger, especially in time of floods, when the quantity of water is too much for the passage. Add to this, that besides the danger there is of the pier bursting out the sterlings, they are also subject to much decay and damage by the rapidity of the water, and the craft passing through the arches.

THRUST, the same as drift, shoot, &c.

VOUSSOIRS, the stones which immediately form the arch, their under sides constituting the intrados or soffit. The middle one, or keystone, ought to be, in length, about $\frac{1}{15}$ or $\frac{1}{16}$ of the span, as has been observed; and the rest should increase in size all the way down to the impost; the more they increase the better, as they will the better bear the great weight which rests upon them, without being crushed, and also will bind the firmer together. Their joints should also be cut perpendicular to the curve of the intrados.

TRACT II.

QUERIES CONCERNING LONDON BRIDGE: WITH THE ANSWERS, BY GEORGE DANCE, ESQ.

AS an Appendix to the foregoing Tract, on the Principles of Bridges, a few smaller papers, on kindred subjects, are inserted in this and some of the Tracts immediately following. The present paper is one, among several of a curious nature, which I purchased at the sale of Mr. Robertson's books, in the year 1776, and appears to contain circumstances of too much importance to be kept private. It seems to have originated from enquiries formerly made, for improving the bridge and the port of London, in the year 1746. It consists of queries proposed by the magistrates of the city; and answers to those queries, by Mr. George Dance, the Surveyor General of all the works of the city of London, who was the father of that excellent architect the present City Surveyor. It seems also that the queries had been proposed to the public in general, to solicit answers from any ingenious engineers or architects; for the paper remarks that,

" The persons who are to answer these queries, may add to their answers what further remarks and observations they shall think proper, to the same purpose as these queries.—In the middle of every arch there are driven down piles, called dripshot piles, in order to prevent the waters from gullying away the ground.—I am of opinion, from the nature of the work, that the bridge was not so wide originally as it now is; and that the points of the piers have been much extended, in order to erect houses thereon.—I observe likewise, that in some of the piers, there are fresh casings of stone, before the original ashler.

" July the 9th, 1746. George Dance."

" QUERY 1. What are the shapes and dimensions of the stone piers, the sterlings, and the openings at high and low water? N. B. This will be best answered by figured sketches, or plans, correctly laid down from an exact mensuration by a scale, provided that scale be not smaller than 8 or 10 feet to an inch."

" *Answer*. I have described the shapes and dimensions of the stone piers, sterlings, and openings at high and low water, in a figured plan, which I delivered to Mr. Comptroller."

" QUERY 2. What are the depths of water, just above, under, and just below the arches, or locks, at a common low water? N. B. These depths may be marked on the plans or sketches."

" *Answer*. The depth of water, beginning at the south end of the bridge, is as follows: viz.

	On the west side.		Under the arch.		On the east side.	
	ft.	inc.	ft.	inc.	ft.	inc.
1st lock	16	0	5	9	8	10
2d	14	6	9	0	10	4
3d	22	3	3	0	14	0
4th	14	0	7	0	15	7
5th	18	9	10	3	18	7
6th	17	7	8	7	15	11
7th	18	1	8	10	15	11
8th	25	1	9	2	18	3
9th	17	8	5	9	18	6
10th	21	2	5	6	17	8
11th	18	11	3	5	12	8
12th	17	0	2	4	22	0
13th	24	6	8	9	20	0
14th	22	3	9	0	17	4
15th	23	9	6	9	20	7
16th	19	9	6	11	21	10
17th	20	3	4	6	21	10
18th	19	4	7	9	14	1
19th	10	10	4	0	13	10
20th	6	7	6	1	10	10

I have likewise described the dimensions in the plan aforesaid."

"Query 3. At what height, above low-water mark, and at what depth below the surface of the sterlings, is the underbed, or lower side of the first course of stones?"

"*Answer*. The height of the underbed of the first course of stones, is various: some being 2 feet 4 inches, some 1 ft. 11 inc., some 1 ft. 10 inc., some 1 ft. 3 inc., some 1ft. 1 inc. above low-water mark; and some are 6 feet, some 5 ft. 8 inc., some 4 ft. 6 inc., some 4 ft. 1 inc., and some 4 feet below the surface of the sterlings. These are the dimensions, as far as I am able to get them: there being no opportunity to make observations but when a breach happens to any of the piers."

"Query 4. What is there between the stones and the heads of the piles? Is it one row of planks only; or two rows, crosslaid; or timber: what wood are they made of, and what are their dimensions or scantlings?"

"*Answer*. In general I find nothing between the stones and piles, but sometimes pieces of plank, mostly of oak, and a little of elm, some of which is 6 inches and 4 inc. in thickness; which I apprehend were not originally placed there, but only when reparations have been made, on which account they were fixed, in order to wedge up tight to the stonework; it being impossible to make sound work in that case by any other method."

"Query 5. Are the piles which surrounded the foundations of the piers, before the sterlings were added, square or round, rough or hewn, driven as close as possible, or at a distance? If they touch one another, are they fastened together with a dovetail, or by any other contrivance of the same nature; and if they do not touch, at what distance are they at a mean?"

"*Answer*. These piles are round, rough, and unhewn: they are driven close, and touch one another: they do not seem to be fastened together by any contrivance, except that some have planks upon them, and some have none. But these observations I have made where breaches have happened, so that one might get 1, 2, or 3 feet within the surface of the piers: but how they are in the middle of the piers, is impossible to determine."

"Query 6. Are the heads of those surrounding piles fastened together by any kirb or capcile? If there be any, let it be described, and its dimensions, by a figured sketch."

"*Answer*. They are fastened by no kirb or capcile.— There are only planks upon some of them, as I mentioned in the former answer."

"Query 7. Are the inside piles, on which the foundations of the piers are laid, round or square, hewn or rough, very close, or at what distance at a mean; of what timber, and size; are they shod or not?"

"*Answer*. This query is very difficult to answer. I can only say, that I have had an opportunity to examine one pier, about 7 feet within. It is the south pier of the dam lock; a great part of which was undermined, by some of the sterlings being carried away, and leaving it defenceless there. I observe that the piles are round, rough, unhewn, and driven close together; and they are chiefly elm, of about one foot diameter. Some of these piles, being taken up, were shod with iron; and I think it is reasonable to suppose they are all so."

"Query 8. Whether the foundations of the piers, before the sterlings were added, extended beyond the naked line of the stone-work: and if so, as it is most likely, describe how much, at a mean, and the manner, by a figured sketch?"

"*Answer*. There is, to every pier, a setoff, or foundation, which extends about 7 inches beyond the naked line of the pier; and that setoff or foundation is of stone. But I am of opinion that sterlings were fixed at the first erecting of the bridge; because I think it impossible for the piers to stand long without some such defence. But whether they were so much extended, or in the same shape they are now, is not easy to determine."

"Query 9. Are the piles, that are under the foundations of the piers, much decayed and galled by the action of the currents of waters, before the sterlings were added?"

"*Answer*. All those piles under the foundations of the piers, which I ever saw, are very sound at heart. But about one inch of their surface hath been decayed: but these were piles which had been for some time exposed to the violence of the flood, by the breaches made in the sterlings. But I apprehend that cannot be the case with the piles which go farther under, or in the middle of the piers; because water cannot act upon them."

"Query 10. What is the inside of the stone piers made of? whether of the same sort of stone as the outside; cut and laid regular, or only common rubble stones, laid in very bad mortar, as it is in Rochester-bridge?"

"*Answer*. I have seen, in several breaches, the texture of the piers: and by them it appears to me, that the insides of the said piers are filled with rubble; and the external faces are formed with ashler laid in courses: but the rubble appears to be laid with good mortar.

"George Dance."

TRACT III.

EXPERIMENTS AND OBSERVATIONS TO BE MADE ABOUT LONDON BRIDGE.

THIS is another of the papers, relating to the state of London bridge, bought at the sale of the late Mr. John Robertson's books. It appears to be an answer given to certain queries, addressed to the Royal Society from the Committee of Common Council of the City of London. This answer is signed by the President, the Vice-Presidents, and several other respectable members of the Royal Society; viz. by Martin Folkes, esq. the president, and by Wm. Jones (father of the late Sir Wm. Jones), James Jurin, M. D., Geo. Lewis Scott, esq., Benj. Robins, esq., and John Ellicott, esq., all names highly respectable for their eminent scientific labours.—Their report is in the following words:

"In order to answer the queries proposed by the Committee, with regard to the alterations of London bridge, we apprehend it will be necessary,

"1st. To have an exact level taken, between some fixed point on the west side of London bridge, and another point on the east side of Westminster bridge; as also, to take the like level between some fixed point on the east side of London bridge, and another point at some convenient place about 2 miles below the bridge.

"2. To take the perpendicular height of each of those 4 points above the surface of the river at low-water, and likewise at every quarter of an hour before and after low-water; and to observe the time when the low-water happens at those places; and the same for high-water.

"3. To take the height of the fixed point on the west side of London bridge, above the surface of the river, at the low still water, and high still water under the drawbridge, with the time of each.

"4. To take the height of the same point, above the surface of the river, just above the sterling, at the time of low-water below bridge.

"5. To take the depth of the water in all the gullets, or at least in that under the drawbridge, at the time of low still water.

"6. To ascertain in how many of the arches the dripshot piles are driven; how close together; and how far the tops of them are below low still water mark.

"7. To know particularly at what time the sterlings are first intirely covered, and when first intirely uncovered.

"8. To know exactly the time of low and high water mark, and the height the water rises to, at the Nore, Gravesend, and Woolwich.

"9. That all the foregoing observations of the tides, be made at some one spring tide, and likewise at some one neap time. Was signed,

M. Folkes; Wm. Jones; Jas. Jurin; Geo. L. Scott; Benj. Robins; John Ellicott."

TRACT IV.

ON THE CONSEQUENCES TO THE TIDES IN THE RIVER THAMES, BY ERECTING A NEW BRIDGE AT LONDON. BY MR. JOHN ROBERTSON.

WHILE it was in contemplation to erect the new bridge over the river at Blackfriars, there was much public conversation and speculation on the probable effects of such erection, relative to the tides in the river, and other matters connected with it. On this occasion, the magistrates of the city of London consulted many scientific men and practical engineers, touching those points. Among others, they requested the advice and opinion of Mr. John Robertson, then master of the Royal Mathematical School in Christ's Hospital, by a special letter from the Town Clerk, as follows.

" To Mr. Robertson at Christ's Hospital.

" Sir,—The Committee of Common Council appointed to consider, whether the Navigation of the river Thames will in any and what manner be affected by a new Bridge, intend to meet at Guildhall, on Thursday the 12th instant, at 10 o'clock in the forenoon, and desire you will be so kind as to favour them with your company at that time, in order to give them your opinion and assistance therein. I am, Sir,

" Your most obedient, humble Servant,

" James Dobson."

Town Clerk's Office, Guildhall, 5 Dec. 1754.

Mr. Robertson's Answer.

" Before I deliver my opinion concerning the question proposed, I think it necessary to premise some few principles relating to the Tides, and particularly those which affect the river Thames; because a just solution to this question depends chiefly on the phenomena of the tides.

"1. It is now well known that the tides are regulated by the motion of the moon; and that this planet takes something less than 25 hours, between the times of its departing from any meridian, to its return to the same; in which time she causes two floods and two ebbs; so that in most parts of the earth there is a new time in every revolution of about 12 hours and a half.

"2. There is a flood tide which flows round the northern parts of Europe, and thence proceeds southward through the western ocean: a branch of this tide runs southward along the German sea, and makes high water to all the eastern coasts of Great Britain, in a successive order, in regard to the time the moon has passed the meridians of those places: this branch of the tide runs but a little to the southward of the mouth of the river Thames.

"3. While the said branch is running down the German sea, the grand body of the tide is marching southward along the western coasts of Ireland, and thence flowing partly southward, partly south-eastward; one branch runs up St. George's Channel, and another branch flows eastward, up the English Channel, and makes, in a successive order of time, the high waters upon all the southern coasts of England: this branch extends something to the northward of the mouth of the river Thames.

"4. The said tides, meeting near the mouth of the river Thames, contribute to send a powerful tide up that river; and so long as the said southern and northern branches continue to flow, so long will the waters continue to accumulate at the mouth of this river, and make their way up it, in order to restore the waters to a level.

"5. The flowing of the tide up the river Thames is greater or less, in proportion only to the accumulation of the waters at its mouth; and therefore, in the common course of things, there is, relative to the moon's age, a fixed quantity of tide which the river Thames is to receive; and therein to be disposed of in the best manner that its situation will admit.

"6. On account of the water being confined between the banks of the river, the tide must flow up higher, in proportion as the river becomes narrower, till the fixed quantity is received. But then it must be observed, that when the tide acts against the stream of a river, the tide up that river becomes progressively stronger and stronger, for a time, according as the velocity of the natural stream is checked; and in this manner the river waters themselves by degrees obtain a contrary direction, and run up with the tide, and so may be considered as waters coming in with the tide of flood, and part of the fixed quantity which that river is to receive.

"7. The return of the tide, or the time of ebbing, is not every where performed in the same time as it took to flow in. For, in the ebb tide there is to be discharged, not only the waters which were brought in by the tide, but also all the river water which has been retarded by it.

"8. Whatever obstacles are laid in the way of the tide, across any channel, the utmost rise, or the high-water mark, at different times, will be respectively the same: because the water will continue to rise till the fixed quantity of tide is disposed of, and no longer. And, in like manner, the low-water mark will not be affected by such obstacles. Indeed, between the limits of high and low-water marks, the water will be raised higher against those obstacles, both in the flood and ebb tides, than they would be in those places, were the obstacles removed. For, as the velocity of the current must, on both sides of the obstacle, be equal, in order for one part of the water to run away, as fast as the successive ones follow; therefore the waters must rise on that side of the obstacle which they run against, till they be so high, that by their fall they acquire a velocity sufficient to carry them off, as fast as they arise at the obstacle.

"These principles being premised, the solution to the question proposed naturally follows. And in order to this, let us for the present suppose, that between London and

Westminster bridges, another bridge were built; and to show what might be the consequences in the worst case, let us suppose it occasioned as great a fall as at London bridge.

"Consequences during the time of the Flood Tide.

"The flood tide, meeting with the obstruction of the new bridge, would accumulate on the eastern side thereof, much in the same manner as it does now at London bridge: this would cause the flood tide at first to run through London bridge with less velocity than it does at present. For, the new bridge, by penning up the water, would throw some of it back again, towards London bridge; and consequently the waters on the eastern side of London bridge, would rise higher than they now do, that they might run off with the same velocity, with which they came to the bridge.

"The tide would not run up the river so far as it now does; and consequently the tide of flood would be sooner spent, than at present: nevertheless the rise of the waters would not, at any place, be lessened beneath the present standard. For, the more obstacles any moving body has to encounter with, the sooner will its motion be destroyed. But the fixed quantity of the tide being in no wise diminished, the waters must necessarily rise as many feet high, either above or below the bridges, as they would, were there no bridges over the river.

"Consequences during the Tide of Ebb.

"The ebb tide would be obstructed, on the western side of the new bridge, in the same manner as it is now at London bridge; but the rise of the water at the new bridge would be highest*. For, as London bridge, by penning up the water,

* It is manifest that all this reasoning, by Mr. Robertson, we must remember, has been on the supposition, that the new bridge would be built with piers and sterlings, like London bridge, and so cause a similar obstruction to the currents.

would cause it, at the beginning of the ebb, to revert or fall back again towards the new bridge: consequently the waters on the western side of the new bridge must rise higher, on account of the pen below, that they might run away as fast as they were succeeded by the following water.

"The length of the tide of ebb would be greater than it is at present, by as much time as the tide of flood would be shortened. For though the same quantity of flood tide, being poured through London bridge, would spend its force sooner than at present, yet the time of the return of the aggregate of the flood tide, and the retarded land waters, would be greater; in proportion as the obstacles, they would have to pass by, were increased.

"From what has been said, I apprehend it is evident, that a new bridge, built between London and Westminster bridges, cannot alter the present high and low-water marks; even though this new bridge should be so constructed, as to occasion a fall of the waters, equal to what they have at London bridge.

"But experience has shown, how a bridge may be built, so as to cause no sensible fall: and were such a bridge substituted in the place of that we have before supposed, the consequences already remarked would become so inconsiderable, in respect to the tides, that I believe, and it is my opinion, that there would ensue no apparent alteration in the present state of the navigation of the river Thames, either above or below London bridge."

John Robertson.

TRACT V.

ANSWERS TO QUESTIONS, PROPOSED BY THE SELECT COMMITTEE OF PARLIAMENT, RELATIVE TO A PROPOSAL FOR ERECTING A NEW IRON BRIDGE, OF A SINGLE ARCH ONLY, OVER THE RIVER THAMES, AT LONDON, INSTEAD OF THE OLD LONDON BRIDGE.

AMONG the various means of improving the port of London, which have lately been devised, was one by removing the old inconvenient London bridge, and erecting another in its stead, which might be more commodious, and better according with the improved state of the port. Several projects were given in to the Committee of Parliament, appointed to consider those improvements, among which was one proposed by Messrs. Telford and Douglass, to be of a single arch, made of cast iron, which the Committee so far noticed, as to order engravings to be made of the design, and, for more safety, to issue a set of questions, concerning this extraordinary project, to be sent to several ingenious professional and literary men, requesting their answers to all or any of them, within a limited time.

The present tract contains my answers, which were delivered in, to those questions, and for which I was honoured with the thanks of the Committee; which answers are here given as a proper appendix, among other articles, to the essay on bridges in the first Tract.

The situation proposed for this new bridge, is about 200 yards above the old bridge, which brings it to run nearly in a line with the Royal Exchange, and with the wide part of the main street of the Borough of Southwark. This is the narrowest part of the river, being here but 900 feet over. It was also proposed to narrow the river still more in this part, by building strong abutments of masonry, running 150 feet into the river on each side, against which to abut the proposed arch of cast iron, which consequently was to be of 600 feet span, extending across the river at one stretch. The height of the arch at the crown or key piece, was to be 65 feet above high-water, to allow ships of considerable burden, with their top masts only struck, to sail through beneath it, up to Blackfriars bridge; to load or unload by the side of new wharfs, to be built into the river, on both sides of it, all the way up to Blackfriars. The width of the bridge, to be 45 feet in the middle, and from thence widening all the way, in a curved form, till it should become enlarged to 90 feet at the extremities.

The letter of the Committee is here given first, with the set of questions, followed by the answers as delivered in consequence of that requisition.

THE ORDER OF THE COMMITTEE.

"Lunæ 23 die Martii 1801,

"At the Committee for the further improvement of the Port of London;

"Charles Abbot, Esq. in the Chair:

"Ordered, That the Print, Drawings, and Estimates of an Iron Bridge, of a single arch, 600 feet in the Span, together with the annexed Queries, be sent to Dr. Hutton, requesting that he will, on or before the 25th of April next, transmit to Mr. Samuel Gunnell, the Clerk to this Committee, his opinion upon all of these queries, or such of them as he may be disposed to consider.

"Charles Abbot, Chairman.

"*To Dr. Hutton,*

"*Military Academy, Woolwich.*"

"*Estimate.*

	£.
" Getting out and securing the foundation of the two abutments - - - -	20,000
432,000 cubic feet of granite or other hard stone	86,400
20,029 cubic yards of brickwork, at 20s. - -	20,029
19,200 cubic feet of timber in tyes, at 3s. 6d.	3,360
6,500 tons of cast iron, including scaffolding and putting up, at 20l. - -	130,000
Making roadways and footpaths - - - -	2,500
	£262,289"

"*Questions respecting the Construction of the annexed Plate and Drawings of a Cast Iron Bridge of a Single Arch*, 600 *feet in the Span, and* 65 *feet Rise.*

" 1. What parts of the bridge should be considered as wedges, which act on each other by gravity and pressure, and what parts as weight, acting by gravity only, similar to the walls and other loading, usually erected upon the arches of stone bridges.—Or, does the whole act as one frame of iron, which can only be destroyed by crushing its parts?

" 2. Whether the strength of the arch is affected, and in what manner, by the proposed increase of its width towards the two extremities, or abutments; when considered vertically and horizontally. And if so, what form should the bridge gradually acquire?

" 3. In what proportions should the width be distributed from the centre to the abutments, to make the arch uniformly strong?

" 4. What pressure will each part of the bridge receive, supposing it divided into any given number of equal sections, the weight of the middle section being given. And on what parts, and with what force will the whole act upon the abutments?

" 5. What additional weight will the bridge sustain; and what will be the effect of a given weight placed upon any of the before mentioned sections?

" 6. Supposing the bridge executed in the best manner, what horizontal force will it require, when applied to any particular part, to overturn it, or press it out of the vertical plane?

" 7. Supposing the span of the arch to remain the same, and to spring ten feet lower, what additional strength would it give to the bridge.—Or, making the strength the same, what saving may be made in the materials.—Or, if instead of a circular arch, as in the plate and drawings, the bridge should be made in the form of an elliptical arch, what would be the difference in effect, as to strength, duration, convenience, and expences?

" 8. Is it necessary or adviseable, to have a model made of the proposed bridge, or any part of it, in cast iron. If so, what are the objects to which the experiments should be directed; to the equilibration only, or to the cohesion of the several parts, or to both united, as they will occur in the intended bridge?

" 9. Of what size ought the model to be made, and what relative proportions will experiments, made on the model, bear to the bridge, when executed?

" 10. By what means may ships be best directed in the middle stream, or prevented from driving to the side, and striking the arch, and what would be the consequence of such a stroke?

" 11. The weight and lateral pressure of the bridge being given, can abutments be made in the proposed situation for London bridge, to resist that pressure?

" 12. The weight and lateral pressure of the bridge being given, can a centre or scaffolding be erected over the river, sufficient to carry the arch, without obstructing the vessels which at present navigate that part?

" 13. Whether it would be most adviseable to make the bridge of cast and wrought iron combined, or of cast iron only. And if of the latter, whether of the hard white metal, or of the soft grey metal, or of gun metal?

"14. Of what dimensions ought the several members of the iron work to be, to give the bridge sufficient strength?

"15. Can frames of cast iron be made sufficiently correct, to compose an arch of the form and dimensions as shown in the drawings No. 1 and 2, so as to take an equal bearing as one frame; the several parts being connected by diagonal braces, and joined by an iron cement, or other substance?

N. B. The plate is considered as No. 1.

"16. Instead of casting the ribs in frames, of considerable length and breadth, as shown in the drawing, No. 1 and 2, would it be more adviseable to cast each member of the ribs in separate pieces of considerable lengths, connecting them together by diagonal braces, both horizontally and vertically, as in No. 3?

"17. Can an iron cement be made, which shall become hard and durable. Or can liquid iron be poured into the joints?

"18. Would lead be better to use in the whole or any part of the joints?

"19. Can any improvement be made in the plan, so as to render it more substantial and durable, and less expensive. And, if so, what are those improvements?

"20. Upon considering the whole circumstances of the case, and agreeable to the resolutions of the Committee, as stated at the conclusion of their third report: Is it your opinion, that an arch of 600 feet in the span, as expressed in the drawings produced by Messrs. Telford and Douglass, or the same plan, with any improvements you may be so good as to point out, is practicable and adviseable, and capable of being made a durable edifice?

"21. Does the estimate communicated herewith, according to your judgment, greatly exceed or fall short of the probable expence of executing the plan proposed, specifying the general grounds of your opinion?

"The Resolutions referred in No. 20, are as follow.

"1st. That it is the opinion of this Committee, that it is essential to the improvement and accommodation of the port of London, that London Bridge should be rebuilt, upon such a construction, as to permit a free passage at all times of the tide, for ships of such a tonnage, at least, as the depth of the river would admit of, at present, between London Bridge and Blackfriars Bridge.

"2d. That it is the opinion of this Committee, that an Iron Bridge, having its centre arch not less than 65 feet high in the clear, above high-water mark, will answer the intended purpose, and at the least expence.

"3d. That it is the opinion of this Committee, that the most convenient situation for the new bridge, will be immediately above St. Saviour's Church, and upon a line from thence to the Royal Exchange.

"Charles Abbot.

"*To Dr. Hutton, Woolwich.*"

The Answers to the foregoing Queries, were as follow; where each question is repeated immediately before its answer, to preserve the connection more close and immediate.

Answers to the Questions concerning the proposed New Iron Bridge, of one arch, 600 feet in the span, and 65 feet high.

Quest. 1. What parts of the bridge should be considered as wedges, which act on each other by gravity and pressure, and what parts as weight, acting by gravity only, similar to the walls and other loading usually erected upon the arches of stone bridges. Or, does the whole act as one frame of iron, which can only be destroyed by crushing its parts?

Answer. It is my opinion, that all the small frames or parts ought to be so connected together, at least vertically, as that the whole may act as one frame of iron, which can only be destroyed by crushing its parts.—For, by this means, the pressure and strain will be taken off from every particular arch or course of voussoirs, and from every single voussoir or frame, and distributed uniformly throughout the whole mass. Hence it will happen, that any particular part which may by chance be damaged, or be weaker than the rest, will be relieved, and prevented from a fracture, or, if broken, prevented from dropping out and drawing other parts after it, which may be next to it, either above or on the sides of it. By this means also, the effect of any partial or local pressure, or stroke, or shock, whether vertical or horizontal, will be distributed over or among a great number of the adjacent parts, and so the effect be broken and diverted from the immediate place of action. By this means also will be obviated, any dangerous effects arising from the continual expansion or contraction of the metal, by the varying temperature of the atmosphere, in consequence of which the bridge will, all together, in one mass, in a small and insensible degree, keep perpetually and silently rising or sinking, as the arch lengthens by the expansion, or shortens by the contraction of the metal.—This unity of mass will be accomplished, by connecting the several courses of arch pieces together vertically, or the lower courses to the next above them, and also by placing the pieces together in such a way as to break joint, after the manner of common or wall masonry, and that perhaps in the longitudinal and transverse joints, as well as the vertical ones.

QUEST. 2. Whether the strength of the arch is affected, and in what manner, by the proposed increase of its width towards the two extremities, or abutments; when considered vertically, and horizontally; and if so, what form should the bridge gradually acquire?

Answer. There can be no doubt but the bridge will be greatly strengthened by an increase of its width towards the two extremities, or abutments, especially if the courses or parts be connected together in the manner above mentioned, in the answer to the first question. For thus, the extent of the base of the arch at the impost being enlarged, the strength or resistance of the abutment will be increased in a much higher degree than the weight and thrust of the arch, and consequently will resist and support it more firmly. The arch itself will thus also acquire a great increase of strength and stability, both from the quantity and disposition of the materials, as well vertically as horizontally, by which, in the latter direction in particular, the arch will be better enabled to preserve its true vertical position, and to resist the force or shock of any thing striking against it in the horizontal direction. And, for the better security in these particulars, considering the immense stretch of the arch, it will perhaps be adviseable to enlarge the width in the middle to 50 feet, instead of 45, and at the extremities to 100 feet, instead of 90, as proposed in the design.—As to the form of this width or enlargement, the side of the arch might be bounded either by a circular arch, or by any curve that will look most graceful: perhaps a very excentric ellipse will answer as well as any other curve, or better.

QUEST. 3. In what proportions should the weight be distributed from the centre to the abutments, to make the arch uniformly strong?

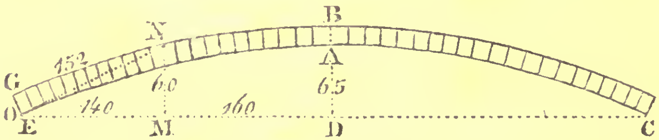
Answer. To make the arch uniformly strong throughout, it ought to be made an arch of equilibration, or so as to be equally balanced in every part of its extent.—When the materials of the arch are uniform and solid, then, to find the weight over every part of the curve, so as to put the arch in equilibrio, is the same thing as to find the vertical thickness of the arch in every part, or the height of the extrados, or back of the arch, over every point of the intrados or soffit of the under curve of the arch: the rule for determining and proportioning of which, is described at large in my Treatise on Bridges, particularly in prop. 4*, and the examples there given to the same. But in the case of the present proposed design for a bridge, a strict mathematical precision is not to be expected or attained by mere calculation, on account of the open frame work of iron, in parts of various shapes and sizes. We must therefore be content with a near approach to that point of perfection; which can be accomplished in a degree

* The same as prop. 10, tract 1, of this volume.

sufficient to answer all the purposes of safety and convenience. Now this can be conveniently done, by a comparison of the present design of a bridge, with the example of a similar intrados curve in the book above mentioned, and which is the case of the first example to the said 4th prop., being that with a circular soffit. By that example it appears, that the weight above every point in the soffit curve, should increase exactly in proportion as the cube of the secant of the number of degrees in the arch, from the centre or middle, to the several points in going toward the abutments. This proportion, though it require an infinite weight or thickness at the extremities of a whole semicircle, where the arch rises perpendicular to the horizon; yet for a small part of the circle near the vertex, the necessary increase of weight or thickness, toward the extremities, is in a degree very consistent with the convenient use and structure of such a bridge; as will be evident by a glance of the figure and curve to that example. For, as the whole extent of the soffit arch, in the present design for an iron bridge, is but about 48° 54′, or 24° 27′ on each side, from the middle point to the abutments, that is, little more than the fourth part of the arch in that example; therefore, by cutting out the fourth part of that arch, it will give us a tolerable idea of the requisite shape of the whole structure, and increase in the thickness where the materials are solid, or at least the increase in weight over every point in the soffit; that is, the figure exhibits a curve for the scale of such increase. Or, if we compute the numeral values of the weights or thickness, by the rule in that example, in the proportion of the cube of the secants, they will be as in the annexed tablet; which is computed for every degree in the arch,

Deg.	Wt. or height	Deg.	Wt. or height
0	10·000	13	10·810
1	10·000	14	10·947
2	10·018	15	11·096
3	10·041	16	11·258
4	10·073	17	11·434
5	10·115	18	11·625
6	10·166	19	11·831
7	10·227	20	12·052
8	10·298	21	12·290
9	10·379	22	12·546
10	10·470	23	12·821
11	10·572	24	13·116
12	10·685	$24\frac{1}{2}$	13·272

from the middle, supposing the middle thickness or weight to be 10. And the true representation of the figure, as constructed from these numbers, or the extrados curve determining the true scale of weight or thickness, over every such point in the soffit curve, is as is here exhibited below. Where the thickness or height in the middle being supposed 10, the vertical thickness or height of the outer curve, above the inner, at the extremities, is 13·272, or nearly $13\frac{1}{4}$, and the



other intermediate thicknesses, at every degree from the vertex, are as denoted by the numbers in the latter column of the table. If the thickness at top be supposed 7, or 8, or 12, or any other number, instead of 10, all the other numbers must be changed in the same proportion. Now the upper curve in this figure is constructed from these computed tabular numbers, and exhibits an exact scale of the increase of weight or thickness, so as to make the whole an arch of equilibration, or of uniform strength throughout, when the materials are of uniform shape and weight. And in this case the upper curve does not sensibly differ from a circular arc in any part of it. But, as the convenient passage over the bridge requires that the height or thickness at the extremities, or imposts, should be a great deal more than in proportion to these numbers denoting the equilibrium of weight, it therefore follows, that the frame work of the pieces above the arch, in the filling up of the flanks, ought to be lighter and lighter, or cast of a form more and more light and open, as in the engraved design, so as to bring the loading in those parts as near to the equilibrium weight, as the strength and stability of the iron frames will permit.

QUEST. 4. What pressure will each part of the bridge receive, supposing it divided into any given number of equal sections, the weight of the middle section being given; and on what part, and with what force, will the whole act upon the abutments?

Answer. By the equal sections, mentioned in this question, may be understood, either vertical sections of equal weight, or those perpendicular to the curve of equal weight, or of equal length; and whichever of these is intended, their thrust or pressure in direction of the curve may be easily computed, if wanted for the purpose of making experiments on the strength of the frames, to know whether they will bear those pressures, or what degree of pressure they *will* bear, without being crushed in pieces. But as it is evident that the frames next the abutments will suffer the greatest pressure of any, I shall here give a computation of the actual pressure there, which may be sufficient, since if the frames at the abutments are capable of sustaining that greatest pressure, we may safely conclude, that all the others, from thence to the vertex, will be more than capable of sustaining the lesser loads or pressures to which they are subject; and this computation will answer the latter and most essential part of the question, viz. "on what part, and with what force, will the whole act on the abutments." Now, from the nature of an arch, it appears that the whole pressure on the abutments, will be chiefly on the lower part of the impost, where the lower frame rests on it, and where we shall therefore, in our computation, suppose it to act. And in the calculation, the whole weight of the half arch AO must be supposed united in its centre of gravity N. Then, if a vertical line MN be drawn through the centre of gravity N, by computation it is found that DM is nearly equal to 160 feet, and consequently ME equal to 140 feet: also, if NO be perpendicular to the impost, or in the direction of the arch at OE; we shall have this proportion, viz, as MN (60), is to the weight of the half arch (3250 tons), so is NO (152), to the pressure on the impost in the direction of the arch at O, and so is ME (140), to the horizontal thrust or pressure in the direction ME; this gives 8233 tons for the pressure on the impost at O in direction of the arch, and 7583 tons for the horizontal thrust in direction ME; being the pressures at each end of the bridge. We may therefore estimate the greatest pressure on the last or abutment frame, at about 8 or 9 thousand tons.

QUEST. 5. What additional weight will the bridge sustain, and what will be the effect of a given weight placed upon any of the before mentioned sections?

Answer. It is perhaps not possible to pronounce exactly what additional weight the bridge will sustain, without breaking, as it depends on so many circumstances, some of which are not known. But, considering the great dimensions and strength of the arch frames, and of the whole fabric, we are authorized to conclude, that there is no possible weight which can pass over any part of the bridge, even heavy loaded waggons, whose pressure can be great enough to cause any danger to such strong and massy materials, and especially when it is considered that, by connecting all the frames together, by proper bond and otherwise, as mentioned in the answer to the first question, the local additional pressure will soon be distributed through the whole series of the iron framing.

QUEST. 6. Supposing the bridge executed in the best manner, what horizontal force will it require, when applied to any particular part, to overturn it, or press it out of the vertical plane?

Answer. This question will be much better answered by means of experiments, made on a proper model, than by theoretical calculations *a priori*. But when the bridge is executed in the best manner, with the frames properly bonded and connected together, it seems more likely that any violent horizontal shock, such as a ship driving against it, would break any particular frame, rather than overturn such a mass of bonded materials, or even move it sensibly out of the vertical position.

QUEST. 7. Supposing the span of the arch to remain the same, and to spring ten feet lower, what additional strength would it give to the bridge.—Or, making the strength the same, what saving may be made in the materials.—Or, if instead of a circular arch, as in the plate and drawings, the bridge should be made in the form of an elliptical arch, what would be the difference in effect, as to strength, duration, convenience and expence?

Answer. Should the arch spring ten feet lower than in the design, the bridge would be more stable, because the thrust or pressure on the abutments would be directed lower down, and more into the solid earth: and in general, the lower the springing of the arch, the more firm the abutments and stable the bridge, if the height of the crown above the springing of the bridge be the same.—But the greatest advantage would be, by making the bridge in the form of an elliptical arch, instead of the circular one, in all the articles of strength, duration, convenience, and expence. For, as the elliptical flanks require less filling up than the circular, this will produce a great saving in the iron frame work: and this same reduction of materials in the flanks, toward the abutments, is the very cause of greater strength, by reducing the weight there nearer to the case of equilibration; since that very extraordinary mass employed in the flanks of the circular arch destroys the equilibrium of the whole, by an overload in that part. The elliptical arch will be also much more convenient, as it will allow of a greater height of navigation way between the water and the soffit of the arch. The elliptical arch is also a much more graceful and beautiful form than the circular arch.

QUEST. 8. Is it necessary or adviseable, to have a model made of the proposed bridge, or any part of it, in cast iron. If so, what are the objects to which the experiments should be directed; to the equilibration only, or to the cohesion of the several parts, or to both united, as they will occur in the intended bridge?

Answer. It appears to be very adviseable, to have a model made of the whole of the proposed bridge, in cast iron, as well for the greater safety and satisfaction, as for the benefits and improvements to be derived from the experiments to be made with it, and from the experience and knowledge derived from the casting and making it.—The objects to which the experiments should be directed, might be, the equilibrium of the whole, the cohesion and fitting of the several parts, the effects of a vertical load on every part separately, and the effects of a horizontal blow or shock against every part in the side of the arch. Also what weight would be requisite to break or to crush the model frames.

Quest. 9. Of what size ought the model to be made, and what relative proportions will experiments, made upon the model, bear to the bridge, when executed?

Answer. The greater the size of the model, the more satisfactory the experiments and conclusions will be. For this purpose, it seems adviseable, that the model be not less than the 20th part or dimensions of the bridge, that is, of 30 feet in length. Now, as the solid contents of similar bodies are in the same proportion as the cubes of their linear dimensions, such a model would require only the 8 thousandth part of the weight or metal in the bridge, because the cube of 20 is 8000. So that, as it is estimated the bridge will require 6500 tons of metal, it follows, that about 3 quarters of a ton weight of metal will suffice for the model of 30 feet in length. As to the relative proportions of experiments made with the model: those relating to the equilibrium, will be in the same direct proportion with the masses of the model and bridge, as well as those relating to loads or shocks. But the strength of any particular bar or frame will be only as the square of the scantling, while the stress upon it will be barely in the same proportion as the length.

Quest. 10. By what means may ships be best directed in the middle stream, or prevented from driving to the side, and striking the arch; and what would be the consequence of such a stroke?

Answer. Some kind of fences might be placed in the river, to direct the navigation to the proper opening in the middle. The effect of the stroke or shock of a vessel, striking the side of the bridge, if very heavy, might endanger the breaking of the particular frame or bar so struck. But, the whole being well bonded and connected together, none of the others would probably be displaced.

QUEST. 11. The weight and lateral pressure of the bridge being given, can abutments be made in the proposed situation for London bridge, to resist that pressure?

Answer. No doubt of it; and especially if the courses of masonry have the joints directed towards the centre of the arch.

QUEST. 12. The weight and lateral pressure of the bridge being given, can a centre or scaffolding be erected over the river, sufficient to carry the arch, without obstructing the vessels which at present navigate that part?

Answer. I doubt not that the requisite centring or scaffolding can be erected, without obstructing the present navigation.

QUEST. 13. Whether it would be most adviseable to make the bridge of cast iron and wrought iron combined, or of cast iron only; and if of the latter, whether of the hard white metal, or of the soft grey metal, or of gun metal?

Answer. It appears most adviseable to make the bridge of cast iron only, and that of the soft grey metal, the bars and frames of which will be less liable to fracture by a blow or shock, than the hard metal.

The mixture of wrought iron with the cast metal, would be very improper, as the sorts are of unequal expansion and contraction by heat and cold, and as the several arch frames should not be tied or bolted together, but suffered to have a little play lengthways, in their butting grooves, so as that no one part be more confined than another.

QUEST. 14. Of what dimensions ought the several members of the iron work to be, to give the bridge sufficient strength?

Answer. This question will be best answered by experiments made on the metal.

QUEST. 15. Can frames of cast iron be made sufficiently correct, to compose an arch of the form and dimensions as shown in the drawings No. 1 and 2, so as to take an equal bearing as one frame, the several parts being connected by diagonal braces, and joined by an iron cement, or other substance?

N. B. The plate is considered as No. 1.

Answer. There can be no doubt that cast iron frames may be made sufficiently correct to compose an arch of any form whatever, and give them an equal bearing; because the wooden moulds, from which the metal is cast, can be made or cut to any shape desired.

QUEST. 16. Instead of casting the ribs in frames, of considerable length and breadth, as shown in the drawing No. 1 and 2, would it be more adviseable to cast each member of the ribs in separate pieces of considerable lengths, connecting them together by diagonal braces, both horizontally and vertically, as in No. 3?

Answer. It is, in my opinion, better to cast the ribs in frames, of considerable length and breadth.

QUEST. 17. Can an iron cement be made, which will become hard and durable, or can liquid iron be poured into the joints?

QUEST. 18. Would lead be better to use in the whole, or any part of the joints?

Answers to Questions 17 *and* 18. The joints might either be filled with an iron cement; or liquid iron might be poured into the joints, having a furnace near at hand for that purpose; or, melted lead may be run in, which will be best of all; because, being a soft metal, it will yield to, and accommodate itself to the inequalities of pressure or of shape, forming a sound and soft bond or bearing between frame and frame; and preventing their fracturing each other by a too hard and unequal bearing; in some respect performing the same office as the cartilages between the joints of the bones in the animal frame.

QUEST. 19. Can any improvement be made in the plan, so as to render it more substantial and durable, and less expensive. And if so, what are those improvements?

Answer. Although the plan appears to possess a very extraordinary degree of excellence, I am of opinion, that it is not incapable of some further improvements, so as to render it more substantial and durable, as well as less expensive. The circumstances which, it appears to me, would be improvements, are as follow:

1st. To make the vertical arch or curve of the bridge elliptical, instead of circular; which will be an improvement in stability, in convenience, in beauty, and in saving expence.

2d. To make the width of the bridge 50 feet in the middle, and 100 feet at the extremities: which will add greatly to its stability and security.

3d. To make the thickness of the arch at the crown, or the height of the middle or key frame there, to be not less than 10 or 12 feet, instead of 6 or 7 as proposed; because, in so extended and massy a fabric, *that* seems to be the least thickness that can afford a rational ground for security and stability.

4th. I would tie or connect every course of frames to those next above them, so as that the whole bridge may rise or settle together as one mass, by expansion or contraction. Yet I would not tie or bolt the frames together lengthways, but would simply make the edge, or the tenons, of the side of each frame, fit into the groove or the mortice holes of the next, going into each other two or three inches; by which means the arch frames will always sit or fit close together, in every degree of temperature, without straining or tearing asunder at the ties.

5thly. I would place the frames of the whole fabric so together, as to make a proper bond, in the manner of good masonry, by making them all to break joint both longitudinally and transversly: by which means, every shock or pressure on any part, would be broken and divided, or shared, among a great many, and any openings be prevented, which might arise from the manner of placing the frames with straight joints continued quite through.

QUEST. 20. Upon considering the whole circumstances of the case, and agreeable to the resolutions of the Select Committee, as stated at the conclusion of their Third Report, Is it your opinion that an Arch of 600 feet in the span, as expressed in the drawings produced by Messrs. Telford and Douglass, or the same plan, with any improvements you may be so good as to point out, is practicable and adviseable, and capable of being rendered a durable edifice?

Answer. On considering the whole circumstances of the case, It is my opinion, that an Arch of 600 feet in the span, as expressed in the drawings produced by Messrs. Telford and Douglass, especially when combined with the improvements above mentioned, is practicable and adviseable, and capable of being rendered a durable edifice.

Charles Hutton.

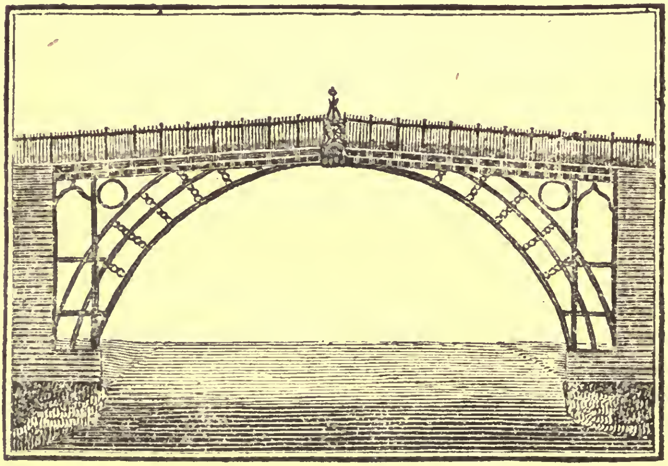
Woolwich, April 21, 1801.

TRACT VI.

HISTORY OF IRON BRIDGES.

A GENERAL History of all Arches and Bridges, both ancient and modern, and constituted of either wood, or stone, or iron, would be a very curious and important work. It should contain a particular account of every circumstance relating to them: such as their history, date, place, artificer, form, dimensions, nature, properties, &c. Such a work, in a chronological order, would make a considerable volume, and much too large to form a part of the present work. I confine my views, therefore, in the present Tract, to a short account of the novel invention of Iron Bridges, in several instances that have recently been executed or proposed; some few of which have been lately noticed in the new edition of Dr. Rees's Encyclopedia.

Bridges of cast iron appear to be the exclusive invention of British artists. The first that was executed on a large scale, is that on the river Severn, at Colebrook Dale, which was erected in the year 1779, by Mr. Abr. Darby, iron-master at that place. This bridge is composed of five ribs; and each rib of three concentric rings or circles, which are connected together by radiated pieces. The inner ring, of each rib, forms a complete semicircle: the others only segments, being terminated and cut off at the road-way. These rings pass through an upright frame of iron, which stands on the same plate as the ribs spring from; which not only acts



as a guide to the ribs, but also supports a part of the road-way. Between the inner upright of this frame and the outer ring of the ribs, in the haunches, is a circular ring of iron, of about 7 feet diameter; and between the outer upright of the frame, and the ribs, are two horizontal pieces, which act as abutments between the stonework and the ribs. There are also two diagonal stays, to keep the ribs upright. The roadway is covered with cast iron plates; and it has an iron railing on each side. The inner or under ring, of each rib, is cast in two pieces, each of which is about 78 feet in length, the arch being 100 feet 6 inches span: and the whole of the iron in it weighs $178\frac{1}{2}$ tons.

Whoever judiciously examines the construction of this bridge, will see, that its fame has arisen chiefly from the circumstance of its having been the first of the kind: for the construction is very bad. The cast iron indeed is in the best state of preservation: but the stone-work has cracked in several places. It is probable, therefore, that its duration will not be long; though not from any deficiency in the iron-work.

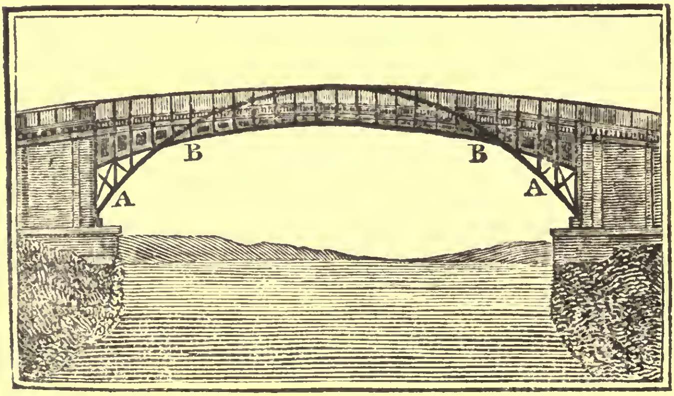
The second iron bridge which has come to my knowledge, is that which was designed by the noted Mr. Thomas Payne. This arch was set up in a bowling-green, at the public-house called the Yorkshire Stingo, at Lisson-Green, in the year 1790. This bridge was intended to be sent to America; but, owing to Mr. Payne's being unable to defray the expense, the arch was taken down by Messrs. Walker of Rotherham, the persons who made it, and some of the materials were afterwards employed in the bridge at Wearmouth and Sunderland, next following.

The third iron bridge that has come to our knowledge, was that executed on the river Wear, at Sunderland, by Rowland Burdon, Esq. M. P. for the county of Durham, by the assistance of Messrs. Walker the founders, Mr. Wilson, and several other persons: and for erecting bridges on similar principles, the first gentleman took out a patent in the year 1794. This bridge was begun in the year 1793, and completed in August 1796. The stone abutments are 70 feet high, above the ordinary surface of the low-water in Sunderland harbour, to the spring of the arch. The iron arch is 236 feet span; and the springing stones project about 2 feet beyond the face of the masonry: so that the whole span, from abutment to abutment, is 240 feet. The versed sine of the arch is 30 feet: its soffit is therefore 100 feet above the surface of low-water in Sunderland harbour.

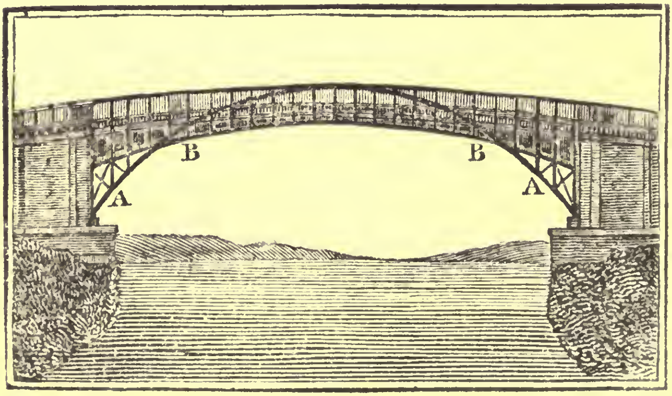
The arch is composed of 6 ribs; and each rib of 3 concentric rings, or segments of circles. Each ring is $5\frac{1}{4}$ inches deep, by $4\frac{1}{2}$ inches thick ; and these rings are connected by radii, $4\frac{1}{2}$ inches by $2\frac{1}{2}$; the rings being at such a distance from each other, as to make the whole depth of a rib 5 feet. The ribs are composed of pieces of about $2\frac{1}{2}$ feet long; and worked iron bars are let into grooves in the sides of the rings, and fastened by rivets. These ribs are connected transversely by hollow iron tubes, or pipes, with flanches on their ends, and fastened to the ribs by screw-bolts: there are also diagonal iron bars, to prevent the ribs from twisting. The haunches are filled with circular rings; and the top is covered with a frame of wood, and planked, to sustain the roadway. It has also an iron railing on each side.

The construction of this bridge is thought to be superior to that at Colebrook Dale; and its weight is much less, in proportion to the length, the whole being only 250 tons, of which 210 tons are cast iron, and 40 tons of worked iron. Yet it is considered in no small danger of falling, the arch having settled several inches, as well as twisted from a straight direction, and the whole vibrating and shaking in a remarkable manner in passing over it.

The fourth iron bridge that has been executed, is that over the river Severn at Buildwas, about 2 miles above Colebrook



Dale. It was begun in the year 1795, and finished in 1796, the iron work by the Colebrook Dale Company, under the direction of Mr. Thomas Telford. The arch is 130 feet span, with a versed sine or height of only 17 feet; and it is but 18 feet wide to the outside. This bridge seems to have been contructed on the principle of the famous wooden bridge at Schaufhausen. The ribs under the roadway are segments of a large circle, each cast in two pieces: but, on each side of the railing, there is a rib, cast in 3 pieces, which springs from a base, 10 feet lower, then crosses the others, and rises as high as the top of the railing: and from the upper part of these outer side ribs, the other ribs, which bear the covering plates, are suspended by king-posts: the covering plates, which are 46 in number, each extending quite across the bridge, have flanges 4 inches deep, and act as an arch. The outside ribs are 18 inches deep, and $2\frac{5}{8}$ inches thick; the middle ribs 15 inches deep, and $2\frac{1}{2}$ thick; and the whole weight of iron is about 174 tons.



Perhaps this may not be the most favourable construction that might be contrived: the tendency of the rib AA, when it expands, being to raise the ribs BB a little higher than they would by their own expansion, and to depress them lower when it contracts: which is not the case in a wooden bridge, this material not being so affected by heat and cold.

About the same time as the bridge at Buildwas was erected, an iron bridge was thrown over the river Tame in Herefordshire; but its parts were so slender, and so ill disposed, that no sooner was the wooden centring taken from under it, than the whole gave way, and tumbled into the river.

In the same year also as the Buildwas bridge was begun, another was erected by the Colebrook Dale Company, over the river Parret, at Bridgewater. The arch of this bridge is an ellipsis of 75 feet span, with 23 feet rise. The haunches are filled with circular rings of iron, and other fanciful figures: it is composed of ribs connected together by cross ties of iron; and the roadway is supported by plates. This bridge is very neat, and thought to be exceedingly firm and durable.

From the completion of the above bridge, few of any note were executed in this country, till about the year 1800, when the stone bridge erected over the Thames, at Staines, gave way. On this occasion the magistrates of the counties of Middlesex and Surrey came to a resolution to erect an iron bridge there, on the abutments of the stone bridge, the piers of which had failed; and Mr. Wilson, the agent of Mr. Burdon, was employed for this purpose. He accordingly undertook the construction of an iron arch of 181 feet span, with 16½ feet rise or versed sine; the arch being the segment of a circle. In this bridge the ribs were similar to those of Wearmouth: but instead of having the blocks, of which the ribs are composed, kept together by worked iron bars, let into grooves in their sides, the rings of the ribs were cast hollow, and a dowel was let into the hollow ring at each joint; so that the two adjacent blocks were fixed together by this dowel, and by keys passing through the rings. The ribs were also connected transversely by frames, instead of pipes as in the Sunderland bridge. The haunches were filled with iron rings, and the whole was covered with iron plates.

It is to be noted, that an iron arch, in small blocks, is not set up after the manner of a stone one, by beginning at the abutments, and building upwards; but is begun at the top, and continued downwards; it being easier to join the stone to the iron, than to cut the iron at the top, if it should not fit. It is somewhat remarkable, therefore, that when these ribs were put together, and before they joined the masonry, it was so nicely balanced, and its parts were so firmly locked together, that after all the supports were taken out, except those next the abutment, the whole was moved by a man, with a crowbar under the top, and it seemed to have little tendency to push the abutments asunder. This, however, turned out unfortunately not to be the case. The centring was taken away, and the bridge was opened for the use of the public, about the end of the year 1801, or beginning of 1802. At first it seemed to stand firm, and the public were much pleased with its light and elegant appearance. But in a short time it was found that the arch was sinking; and soon after it had gone so much, that it was obliged to be shut up, and the old bridge opened again. The sinking of the arch broke several of the transverse frames, and many of the radii at the haunches; which left no doubt that the abutments had given way. But on examination there appeared no visible sign of such failure: there was not a crack in the masonry, nor had they gone out of the upright.—After much investigation however, it appeared that the whole masonry of the abutments, to the very foundation, had slidden horizontally backwards, still preserving the perpendicular, or upright position. The failure took place in the south abutment, which was supposed to be owing to a cellar, that had been made in it. The inhabitants of Staines therefore, by the advice of an engineer whom they consulted, had this abutment strengthened: but no sooner was this done, than the north one failed: and they had intended to strengthen this also; but their funds being nearly exhausted, they came to the resolution to take the whole down, and erect a wooden bridge in its stead.

Before the completion of the iron bridge at Staines, another was begun of the same dimensions, and on the same principle, over the river Tees at Yarm. This bridge was completed also: but, instead of gradually yielding, as that at Staines had done, the whole suddenly tumbled into the river at once.

From the accidents above described, and from several others of less note, iron bridges have lost a good deal of their celebrity, but probably on no just grounds. Those failures that have happened, have not been through any intrinsic deficiency in the iron material, but from the injudicious manner in which they have been constructed. An opinion has gone forth, not only among the practical builders of iron bridges, but among some men of science, that the lateral pressure of iron bridges, in consequence of their parts being so firmly bound together, is comparatively small, to that of stone arches. But, on a due consideration of their principle, I believe it will be found quite different, and that an iron arch, of the same weight as one of stone, requires much stronger abutments, to resist its lateral pressure or push, than the stone arch does. And this we shall here endeavour to account for.

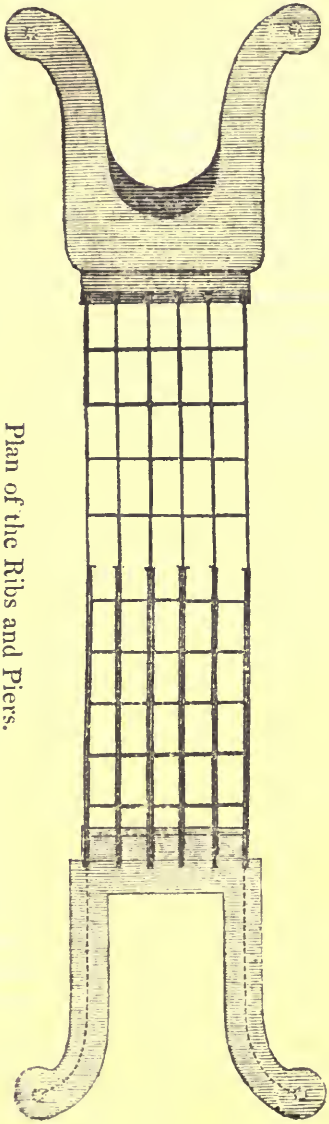
Stone may, in a great measure, be considered as an unelastic substance, being very little subject to expansion or contraction. When, therefore, an arch is composed of this material, and the abutments are sufficiently strong, to support it, when left to itself, there is little probability of its failure. No ordinary load upon it will excite a tremulous motion; nor will it change by heat or cold. The lateral pressure on the piers or abutments is therefore uniform.

But iron is an elastic substance, and is greatly affected by heat or cold, expanding with the one, and contracting by the other. When, therefore, a heavy load acts upon an iron bridge, such as a loaded waggon, the whole is put in motion, and the arch vibrates like the string of a violin, contracting and expanding while its parts are in the act of vibration. Thus at one part of the vibration it pulls the abutments together, and at the other it pushes them asunder, with a force compounded of the quantity of matter in motion, and the velocity with which it moves. When it expands, the whole weight of the arch is raised, and the pressure on the abutments is compounded of the matter and velocity of the weight raised. No such pressure, or rather impulsive momentum, takes place in a stone bridge: therefore the strength of the abutments of an iron bridge should be such, as not only to sustain the weight of the arch, but also the additional push arising from the causes above stated. The abutments of Staines bridge were only 14 feet thick; whereas they ought to have been at least 25 feet. There were also other causes which contributed to the failure of this bridge, such as the improper manner in which the foundations were made.—The abutments of Yarm bridge were made still weaker than those of Staines: no wonder, therefore, that its failure was more sudden.

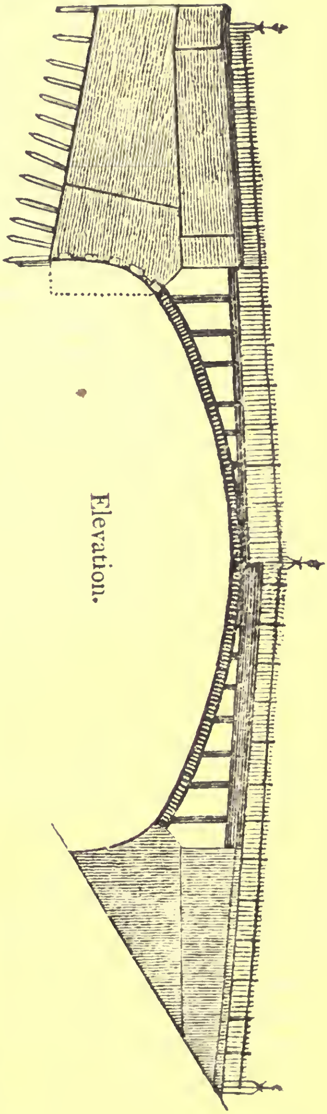
I am therefore most decidedly of opinion, from what has happened in the bridges above described, and in several others, that no part of the failure is attributable to the iron material, at least respecting its strength.—I do not however mean to say, that iron is generally to be preferred to stone:—on the contrary, I think a stone bridge is preferable to an iron one, when it can be executed with propriety and conveniency. But there are many cases where stone would not answer the purpose; in which cases therefore iron is most valuable.—The cases here chiefly alluded to, are when the foundations cannot be made within the width that a stone arch can with convenience be erected; or when the requisite rise would be very inconvenient for a stone bridge, or in places where stone cannot easily be procured. The bridge at Wearmouth is an example of the former, as stone piers would have very much obstructed the navigation of the river; and of the latter, as the arch is a segment of a circle of about 500 feet diameter.

The bridge at Boston, in Lincolnshire, is another example, though of less extent: the banks of the Witham are very low, and the houses are built close to the river; the rise of tide is great, and barges navigate under it: therefore, to render the access easy over the bridge, it became necessary to make it flat; and to admit of headroom under the arch, flatness again was necessary. This bridge was therefore made of cast iron. Its span is 86 feet, and its versed sine only $5\frac{1}{2}$ feet. The abutments have been well secured; and though many of the radii of the ribs broke, when the pavement was put on it, yet the rings are quite entire, and the bridge is as firm as can be wished.

In the course of the late improvements in Bristol harbour, two handsome cast iron bridges were erected over the New River there, in the years 1805 and 1806, under the direction of Messrs. Jessop. These two bridges are equal and alike in all respects. The arch in each is a circular segment, of 100 feet span, with a versed sine or rise of only 15 feet: the width of the bridge about 31 feet: the whole is of cast iron, of the strongest grey metal; amounting to 150 tons, viz. 100 tons in the ribs, pillars, bearers, balustrade, &c, and 50 tons in the plates for the roadway. The arch consists of two concentric circular rings or segments, firmly connected and bound together. Each of these is formed of 6 ribs, at 6 feet distance from each other, tied together by cross bars, at intervals of about $9\frac{1}{2}$ feet; as appears in the plan of the fabric here annexed on the following page. On the upper ring, of each rib, stand a number of pillars, in an upright position, or perpendicular to the horizon, their tops formed like a T, as bearers to support the plates for the roadway. All which, with the railing, or balustrade, as well as the disposition and coursing of the abutments, with piling underneath, appear in the represented elevation following; the courses of masonry very judiciously being laid inclining, as we have elsewhere recommended; and the whole seems otherwise very properly contrived. It would lead us too far here to enumerate all the ingenious particulars in the construction of this arch, with the dimensions of all the parts, and the practical methods of putting them together, and securing the whole in the firmest manner, as prescribed to the iron masters for their direction.



Plan of the Ribs and Piers.



Elevation.

Suffice it therefore to observe, that, from the mode of putting the bridge together, it is so contrived, that if any part be injured, it can be taken out, and replaced, without disturbing the main body of the bridge.

The cost of one bridge, independent of the digging and earth work, and making the roads to it, was nearly as below.

	£.
Piles - - - - - - - - -	250
Masonry, 3200 yards, at 18s. including stone - -	1600
Iron work, 100 tons, at 9l. 18s. and 50 tons, at 9l. -	1440
Covering with gravel, and paving, &c. - - -	292
Expences of erection and painting - - - -	418
	£4000

Thus has been given a short history of such iron bridges as have come to my knowledge: aware however that many others have been built, both for roads and for aqueducts in canals, &c: but none of these, that I have heard of, are remarkable either for their span or construction: so that it appears unnecessary to enter into any particular description of them. The projects also that have been made for bridges of this kind, but not executed, are numerous, and a short account may here be added of some of the more remarkable designs that have come to our notice; though our researches have not enabled us to trace any of them to a period prior to the execution of the bridge at Colebrook Dale.

A design was made in the year 1783, by whom, does not appear, for an arch, chiefly of iron, of 400 French feet in span, and 45 feet in the versed sine; answering to a circle of about 934 feet diameter. This design, with a memorial on the advantages of using iron, in the construction of bridges, was presented by the author to the unfortunate Louis of France, on the 5th of May 1783. It had two large ribs, partly of iron and partly of wood. These ribs were 30 feet deep at the springs, and 15 feet at the midde of the arch. Each rib was composed of 4 rings, drawn from different centres, the inner ring being the strongest; and they were connected together by pieces of iron in various fanciful forms, little adapted to give strength to the arch. Between the ribs were cills, or logs of timber, laid transversely, resting on the interior ring; and a floor of wood was proposed to cover them. So that the road was suspended by the ribs; and the upper part of the ribs was to answer the purpose of a parapet, similar to the wooden bridges in Switzerland.—It appears that this project possessed little merit beyond the boldness of its design; and we have never heard that any bridge has been constructed on this principle.

In the year 1791 a project was made by Mr. John Rennie, Civil Engineer, for an iron bridge, intended for the isle of Nevis. The span of the arch was to be 110 feet, and its versed sine $13\frac{3}{4}$; answering to a circle of 234 feet diameter. It was proposed that this arch was to have 6 ribs; each rib to consist of 3 rings, which were to be connected together by radii. The depth of the rib at the middle was $3\frac{1}{2}$ feet, and at the springs 6 feet. The ribs were to be connected together by transverse frames of iron, placed in the joints of the blocks of which the ribs were composed: the haunches to be filled with circular rings of iron; and the whole was to have been covered with plates of iron, to support the road.

In April 1794, he made another design for the same island of Nevis, in which the span was 80 feet, and the rise or versed sine $9\frac{1}{2}$ feet. This design was formed on the same principles as the former, except that the rib was $11\frac{1}{2}$ deep at the springs, though still only $3\frac{1}{2}$ in the middle. The radii were continued to the roadway; and the whole was to be covered with iron plates, as the former. Neither of these designs however was executed, as the French got possession of the island.

From the above period, no projects for iron bridges, except those above described, have come to my knowledge, till applications were made to parliament, for the purpose of improving the port of London, by means of wet docks. The House of Commons, after having heard a great deal of evidence, on the inadequacy of the Thames to accommodate the shipping, appointed a select committee, to take the whole into their consideration, and to report to the house the best means for giving relief to the extensive commerce of the metropolis. This committee, after having recommended the construction of the West India and London Docks, took up the consideration of the state of the Thames, and of London Bridge, which forms the great obstruction to the influx of tide, and greatly injures the navigation of this very important commercial river; and in the year 1799 they directed plans of London bridge to be made out, with correct descriptions of its construction and state of repair; from which it appeared to them, that a new bridge, of more waterway, was imperiously required: and in consequence encouragement was held out to artists, to bring forward designs, for the construction of a new bridge, instead of the old one. On this occasion many designs were made out, and presented to the committee. Some were for stone bridges, and some for iron. But as the object of this account relates to projects for iron bridges only, we shall here confine our attention to these last alone.

The encouragement held out, by the Select Committee, brought forward four designs of this kind: namely, one by Mr. Wilson, formerly mentioned, of 3 arches; the middle one of which was 240 feet span, having a versed sine of 37 feet; the two side arches of 220 feet span each, and their versed sine 30 feet. The height of the soffit of the middle arch 80 feet above the high-water of an ordinary neap tide. The principles of this design were so nearly the same as those of Sunderland bridge, that it is unnecessary to enter into any minute description of it.

Two other designs were brought forward by Messrs. Telford and Douglass: one to consist of 5 arches across the river, and the other of 3. The middle arch of the former was 180 feet span, with a versed sine of 38 feet; also two arches, each of 140 feet span, and two of 120 feet span each. The other had a middle arch of 240 feet span, with a versed sine of 48 feet; and two side arches, of 220 feet span each: the height of the soffit of the middle arch being 80 feet above the high water of neap tides, the same as that of Mr. Wilson's design.

The arches of both the designs of Messrs. Telford and Douglass were constructed in the same manner; therefore a description of one will serve for both. They were composed of ribs; each rib having an outer and inner ring: the inner ring much stronger than the outer, and they were connected together by radiated bars, which extended quite to the pieces that supported the roadway. In the large arches there were two portions of rings, to stay the radiated bars in the haunches; but in the small arches only one. Of how many pieces the ribs were composed, or in what manner to be joined, was not shown in the designs, nor mentioned in the descriptions. The great height given to these bridges, to admit of vessels passing under them, renders it necessary, particularly on the south side of the river, where the land is under the level of spring tides, that long approaches, or inclined planes, as the designers called them, should be made; and these they proposed to support on iron arches, constructed in a manner similar to those of the bridge. By the section it appears that there will be a rise of about 1 foot in 19, on the main approach from the Borough; so that, taking the height of the roadway on the bridge at 60 feet above the wharf of the Thames, this approach will extend 1140 feet into the Borough, High-street. Now a rise of 1 in 19 is almost double the rise in Ludgate-hill: so that, if it were to be made the same rise as Ludgate-hill, it would extend to a distance not much short of half a mile. The side approach upward, it appears also, would come within about 260 yards of Blackfriars bridge, and that downwards would extend to nearly opposite the Tower. So that a considerable part of the Borough would probably be subjected to great inconveniences and expences by these far extended approaches, which appear unavoidable. The additional labour too that would by this means be occasioned, would probably cost more, to the inhabitants of London and the Borough of Southwark, than all the advantage that might arise by bringing vessels up to Blackfriars bridge. These objections are not applicable to these designs alone, but in an equal degree to Mr. Wilson's also.

There can be no doubt but that both designs could be executed; whatever may have been the opinion of artists on the skill exercised in their mechanical construction. We have before shown, that the true principle on which an arch ought to be constructed, is to increase the depth of the voussoir, as it is called in masonry, towards the spring of the arch, so that the arch, with its load upon it, shall be in equilibrio in all its parts. This being accomplished, it does not appear that any good can result from extending the radii further; for as the roadway presses perpendicularly on the arch, it appears not the strongest mode to support this perpendicular load by inclined pieces; but rather the contrary. It seems proper, therefore, that the roadway should be sustained by upright pillars of iron, instead of inclined radii, though less elegant in appearance to the eye: nay we might even prefer the circular rings or eyes of Mr. Wilson, to this mode: though we are aware that a circle, pressed on four points, is by no means calculated to bear a very great pressure.

The Select Committee of the House of Commons, not being satisfied with any of the three designs, that have been described, directed Messrs. Dance and Jessop to report, whether any, and what advantages, would accrue to the navigation of the Thames, if it were to be considerably contracted. Accordingly these gentlemen reported, that if, instead of the channel of the Thames at London bridge being 740 feet wide, as it was proposed to be when the above designs were made, it were reduced to 600 feet, that great advantages would result to the navigation; since, by diminishing the width, the depth would be much increased.—It might be foreign to the purpose of the present work, to enter into any discussion on the propriety of this measure; for which reason we may leave that discussion to a future opportunity. In consequence of this opinion, Messrs. Telford and Douglass presented to the Committee a very elegant and magnificent design, for an arch of 600 feet span, having its versed sine about 65 feet; so that the circle of which this arch is a segment, must be about 1450 feet diameter.

The arch was composed of seven ribs; and each rib may be said to have 6 rings, the 3 lower concentric, and about 8 feet deep. The dimensions of the iron cannot be correctly taken by measurement from the plan, this being on a small scale. These rings were connected by radii about 18 inches asunder; the outer and inner are the strongest, and that in the middle appears light, and seems intended, it is presumed, chiefly to stiffen the radii, though doubtless it will also add to the strength of the bridge. The ribs are composed of frames of iron, each about 10 feet long, which extend quite to the entablature of the cornice. The other 3 rings are not concentric with those 3 lower, but each drawn from a larger radius than the other. The lowest of these three terminates in the upper ring of the three lower, at about 120 feet from the key, or the middle of the arch. The two above this unite at about the same distance from the middle of the arch, and are thence continued in one ring, till they reach within about 35 feet of the middle or key of the arch, where they join the said upper rib of the lower three. These three upper ribs are united to the third or upper ring, of those first described, by means of radii; but the spaces between these radii include the space of two of the lower radii; and, instead of being stiffened by a light ring, as the lower radii are, that object is effected by Gothic tracery. These seven ribs, above described, are set parallel to each other; and, to brace them horizontally, there are six others, or diagonal ribs, four of which cross the former diagonally, two terminating in the middle rib, and two in the adjoining ribs; and there are two outside ribs, that terminate each on the face of the exterior ones. So that, in fact, two of the seven have no diagonal rib terminating at their top. The whole of these last described ribs are therefore side or diagonal braces, to keep the seven principal ribs in their vertical position, and prevent the arch from racking sideways, as happened at Sunderland or Wearmouth bridge, before mentioned.—All these vertical and diagonal ribs are connected together by transverse frames, at the joints of each of the radiated frames or voussoirs. The top or platform, under the roadway, is covered, in the usual manner, with iron plates; and there is a light iron railing on each side, with Gothic ornaments.—The breadth of the roadway at the top, or middle of the arch, is 45 feet, and at the haunch or extremity of the arch 82 feet wide.—The arch springs from large frames of iron, set in abutments of masonry; and its approaches are similar to those before described for the designs of Messrs. Telford and Douglass.

The principles on which this arch is designed, may be found in a work published at Leyden, in the year 1721, entitled "Recueil de plusieurs machines de nouvelle invention, ouvrage posthume de M. Claude Perrault, &c. &c." and is described in pages 712, 13, 14 of that work, and represented in plates 10 and 11. It is described, "Pont de bois d'une seule arche de trente toises de diametre, pour traverser la Saine visavis le village de Sevre, ou l'on proposoit de la contruire." It may also be seen in the 1st vol. of the Machines approved by the Academy of Sciences, pa. 59, pl. 14. It may appear perhaps doubtful to some persons, whether this design is so proportioned as to be in perfect equilibrio, being remarkably heavy at the haunches; and that, were such an arch as there described to be erected over the Thames, whether it would permanently support itself.—The extension of the radii to the roadway has been before noticed as not well adapted to sustain the perpendicular pressure, with which it would be charged, and that unless its parts were in perfect equilibrio, the joints of the frames might open in such a manner, as to derange the whole fabric, and accelerate its destruction.—That an iron arch of 600 feet span might be constructed in such a manner, as to become a firm and stable fabric, it is not meant to be denied; but, according to the principles we have laid down, it should be rather differently constructed from that we have described. Indeed, if the weight of iron, mentioned in the estimate, be correct, the parts must be very slender indeed; and were the whole to be in equilibrio, this weight of the structure itself might bend the parts in such a manner, as in some measure to endanger its downfall.

We imagine that three distinct objects were proposed to be obtained by the improvements which the public have in view. These are, 1st. The maintaining of deeper water, from the lower part of the Thames to Blackfriars bridge, and upward.—2d. More clear space for the navigation of vessels under the bridge.—3d. Effecting this object with the least rise of road over it.

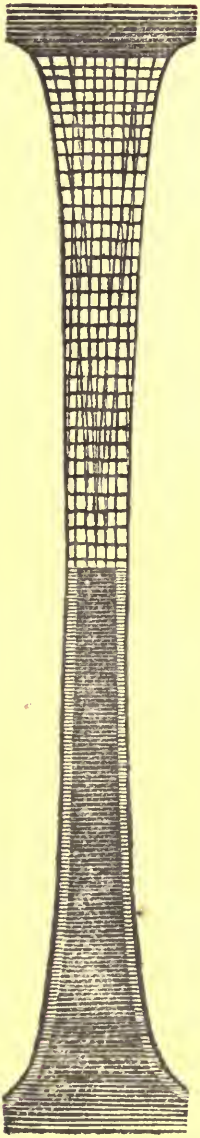
In respect to the first question, I have already declined entering into it; being of opinion it is a discussion rather foreign to the purpose of a book on bridges.—The second appears to come fully under the scope of the principles we have treated on.—The arch here proposed, as we have before seen, is of 600 feet span, with a versed sine or rise of 65 feet. Now, at the distance of 100 feet from the middle, the height is 58 feet; at 150 feet from the middle the height is 49 feet; and at 200 feet it is 37 feet in height. So that, only about 200 feet, or $\frac{1}{3}$ of the width of the river, can be accounted fit for the navigation of coasters: about another third may be fit for the ordinary barges; and the remaining third will be for little other purpose than the lug boats and wherries that ply on the river.

Vessels, therefore, in departing from the wharfs, must be drawn out nearly to the middle of the river, before they can take the advantage of the tide downwards: and those coming to a wharf, must fetch up in the river till they are hauled into it. This might do for vessels that frequent wharfs situated a considerable distance above the bridge: but those for wharfs that might be near it, must experience much trouble and inconvenience; and it is to be feared that they would frequently sustain damage in their masts and rigging, by striking against it, and might probably injure the bridge itself. Mr. Rennie has very properly noticed this, in his answer to one of the queries proposed by the Select Committee of the House of Commons: but he follows up his observations by saying, that, as the strength of the current will be chiefly in the middle of the river, the vessels will generally pass in that track. Now we may admit that, for a vessel sailing up or down the river, and going to some wharf near Blackfriars bridge, or departing from thence downward, that this will be the case: but when going to, or sailing from wharfs near the new bridge, it will be very much otherwise; as may be observed by any one who will attend to the vessels sailing to or from the wharfs below London bridge: and we should fear that, in order to prevent the accidents above noticed, dolphins, or some such contrivance, will be found absolutely necessary, to keep the vessels in the proper track, in passing through this arch.—Now, if we be right in our conjecture, it would probably be better to have two piers, and a bridge of three arches, than a bridge of one only; by which the height or space under the bridge, for vessels to pass, might be very much increased; and those wharfs which lie near the bridge not be subject to the inconveniences, nor the vessels to the risk before mentioned.

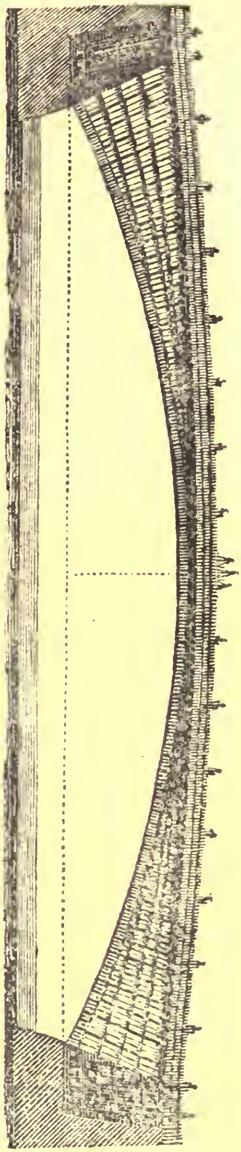
Thirdly, A bridge of three arches will not require the ribs to be so deep at the top, as a bridge of one arch, by at least 3 feet; and therefore so much will be gained in the height of the roadway over it. On the whole therefore it seems, that the design in question is not completely calculated to attain the objects the Select Committee of the House of Commons had in view: but, on the contrary, that it will appear to most thinking men, rather an injudicious idea, to effect by a great work, that which can at least as well, if not better, be accomplished by a work of less expence, and of more probable stability.

Our observations have been hitherto confined to the possibility and propriety of executing an iron arch, of 600 feet span, according to the design given with the report of the House of Commons. We may now add some observations on the practicability of building abutments, in this situation, sufficiently strong to resist the lateral pressure of this arch; which, according to our calculation, made on the supposition that the arch would be similar to one of stone, acting with a regular and uniform pressure upon it, would be of about 9000 tons. But when the effects of the vibration, which must necessarily take place in an arch of this magnitude, are taken into consideration, the lateral pressure, or rather vibrating push, will far exceed that quantity; and for this effort, as has been before noticed, provision must be made in the strength of the abutments: and though the thickness of these in the design, namely 85 feet, seems to be great, yet I am inclined to think it would be found too small, especially at the south end of the bridge, where I am informed the ground is very bad, being moorlog and soft mud to a considerable depth. Indeed I should fear that something of the kind of what happened at Staines would be likely to take place here, namely, the whole mass of masonry be forced back horizontally, by the great lateral push of the arch, in spite of every precaution that could be taken to prevent it. But we must observe, as we have before done in answer to the Queries in the Report of the Committee of the House of Commons, that the foundations of the abutments should be laid inclining towards the centre of the circle to which the arch is drawn, as a more likely mode of preventing them from sliding outwards, than if laid horizontally: but even with this precaution, if the substratum be moorlog or soft mud, it will be likely to give way; and if this ever take place, the abutment and arch must follow it.

The following is a rough sketch, on a very small scale, on the design, at least very elegant, which was given along with the above project.



Plan of the Ribs and the Roadway.



Elevation.

As in some degree and nature related to the foregoing account of iron arches, properly so called, we may here add a few words, just to notice two ingenious works lately executed, being a kind of straight or flat arch, for an iron aqueduct, supported on pillars, carried over rivers. These were, both of them, designed by Mr. Thomas Telord, engineer, and executed under his direction.

The former was a small aqueduct of cast iron, the first for a navigable canal, which was constructed in the year 1795, on the Shrewsbury canal, near Wellington in Shropshire. It is 180 feet in length ; and the surface of the water in the aqueduct is about 20 feet above that of low water in the river. The supporting pillars, in this case, are also of cast iron. There are no ribs under the bottom plates, these being connected with the side plates, shaped like the stones in a flat arch, which is also the case in the second instance, at Pontcysylte. The iron work of this aqueduct was cast at Ketley foundery, by Messrs. Reynolds.

The second instance was erected in the year 1805, at the Pontcysylte aqueduct. It having been found necessary to carry the Ellesmere canal across the river Dee, at the eastern termination of the vale of Llangollen, at the height of 126 feet 8 inches above the surface of low water in the river, Mr. Telford conceived the bold design of effecting this by means of an aqueduct constructed of cast iron, supported by stone pillars. These are 20 in number, including the abutments : the length of the aqueduct is 1020 feet, and the breadth across it 12 feet. It has been in constant use for the purposes of navigation ever since it was first opened, on the 26th of November 1805, and it answers every purpose perfectly well. The iron work was cast, and set up, by Mr. William Hazledine, of Shrewsbury. A small view of the elevation of this elegant structure is as here below.

TRACT VII.

A DISSERTATION ON THE NATURE AND VALUE OF INFINITE SERIES.

1. About the year 1780 I discovered a very general and easy method of valuing series, whose terms are alternately positive and negative, which equally applies to such series, whether they be converging, or diverging, or their terms all equal; together with several other properties relating to certain series: and as there may be occasion to deliver some of those matters in the course of these tracts, this opportunity is taken of premising a few ideas and remarks, on the nature and valuation of some of the classes of series, which form the object of those communications. This is done with a view to obviate any misconceptions that might perhaps be made, concerning the idea annexed to the term *value* of such series in those tracts, and the sense in which it is there always to be understood; which is the more necessary, as many controversies have been warmly agitated concerning these matters, not only of late, by some of our own countrymen, but also by others among the ablest mathematicians in Europe, at different periods in the course of the last century; and all this, it seems, through the want of specifying in what sense the term *value* or *sum* was to be understood in their dissertations. And in this discourse, I shall follow, in a great measure, the sentiments and manner of the late celebrated L. Euler, contained in a similar memoir of his in the fifth volume of the New Petersburgh Commentaries, adding and intermixing here and there other remarks and observations of my own.

2. By a converging series, is meant such a one whose terms continually decrease; and by a diverging series, that

whose terms continually increase. So that a series whose terms neither increase nor decrease, but are all equal, as they neither converge nor diverge, may be called a neutral series, as $a - a + a - a +$ &c. Now converging series, being supposed infinitely continued, may have their terms decreasing to 0 as a limit, as the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$ &c, or only decreasing to some finite magnitude as a limit, as the series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} +$ &c, which tends continually to 1 as a limit. So, in like manner, diverging series may have their terms tending to a limit, that is either finite or infinitely great: thus the terms $1 - 2 + 3 - 4 +$ &c, diverge to infinity; but the diverging terms $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$ &c, only to the finite magnitude 1. Hence then, as the ultimate terms of series which do not converge to 0, by supposing them continued *in infinitum*, may be either finite or infinite, there will be two kinds of such series, each of which will be further divided into two species, according as the terms shall either be all affected with the same sign, or have alternately the signs $+$ and $-$. We shall, therefore, have altogether four species of series which do not converge to 0, an example of each of which may be as here follows:

$$1. \quad - \quad - \quad \begin{cases} 1 + 1 + 1 + 1 + 1 + 1 + \text{\&c.} \\ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \text{\&c.} \end{cases}$$

$$2. \quad - \quad - \quad \begin{cases} 1 - 1 + 1 - 1 + 1 - 1 + \text{\&c.} \\ \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \text{\&c.} \end{cases}$$

$$3. \quad - \quad - \quad \begin{cases} 1 + 2 + 3 + 4 + 5 + 6 + \text{\&c.} \\ 1 + 2 + 4 + 8 + 16 + 32 + \text{\&c.} \end{cases}$$

$$4. \quad - \quad - \quad \begin{cases} 1 - 2 + 3 - 4 + 5 - 6 + \text{\&c.} \\ 1 - 2 + 4 - 8 + 16 - 32 + \text{\&c.} \end{cases}$$

3. Now concerning the sums of these species of series, there have been great dissensions among mathematicians; some affirming that they can be expressed by a certain sum, while others deny it. In the first place, however, it is evident that the sums of such series as come under the first of these species, will be really infinitely great, since by actually

collecting the terms, we can arrive at a sum greater than any proposed number whatever: and hence there can be no doubt but that the sums of this species of series may be exhibited by expressions of this kind $\frac{a}{0}$. It is concerning the other species, therefore, that mathematicians have chiefly differed; and the arguments which both sides allege in defence of their opinions, have been endued with such force, that neither party could be hitherto brought to yield to the other.

4. As to the second species, the celebrated Leibnitz was one of the first who treated of this series $1 - 1 + 1 - 1 + 1 - 1 +$ &c, and he concluded the sum of it to be $= \frac{1}{2}$, relying on the following cogent reasons. And first, that this series arises by resolving the fraction $\frac{1}{1 + a}$ into the series $1 - a + a^2 - a^3 + a^4 - a^5 +$ &c, by continual division in the usual way, and taking the value of a equal to unity. Secondly, for more confirmation, and for persuading such as are not accustomed to calculations, he reasons in the following manner: If the series terminate any where, and if the number of the terms be even, then its value will be $= 0$; but if the number of terms be odd, the value of the series will be $= 1$: but because the series proceeds *in infinitum*, and that the number of the terms cannot be reckoned either odd or even, we may conclude that the sum is neither $= 0$, nor $= 1$, but that it must obtain a certain middle value, equidifferent from both, and which is therefore $= \frac{1}{2}$. And thus, he adds, nature adheres to the universal law of justice, giving no partial preference to either side.

5. Against these arguments the adverse party make use of such objections as the following. First, that the fraction $\frac{1}{1 + a}$ is not equal to the infinite series $1 - a + a^2 - a^3 +$ &c, unless a be a fraction less than unity. For if the division be any where broken off, and the quotient of the remainder be added, the cause of the paralogism will be manifest;

for we shall then have $\frac{1}{1 + a} = 1 - a + a^2 - a^3 + \pm a^n \mp \frac{a^{n+1}}{1 + a}$; and that, although the number n should be made infinite, yet the supplemental fraction $\mp \frac{a^{n+1}}{1 + a}$ ought not to be omitted, unless it should become evanescent, which happens only in those cases in which a is less than 1, and the terms of the series converge to 0. But that in other cases there ought always to be included this kind of supplement $\mp \frac{a^{n+1}}{1 + a}$; and though it be affected with the dubious sign $\mp$, namely $-$ or $+$ according as n shall be an even or an odd number, yet if n be infinite, it may not therefore be omitted, under the pretence that an infinite number is neither odd nor even, and that there is no reason why the one sign should be used rather than the other; for it is absurd to suppose that there can be any integer number, even though it be infinite, which is neither odd nor even.

6. But this objection is rejected by those who attribute determinate sums to diverging series, because it considers an infinite number as a determinate number, and therefore either odd or even, when it is really indeterminate. For that it is contrary to the very idea of a series, said to proceed *in infinitum*, to conceive any term of it as the last, though infinite: and that therefore the objection above-mentioned, of the supplement to be added or subtracted, naturally falls of itself. Therefore, since an infinite series never terminates, we never can arrive at the place where that supplement must be joined; and therefore that the supplement not only may, but indeed ought to be neglected, because there is no place found for it.

And these arguments, adduced either for or against the sums of such series as above, hold also in the fourth species, which is not otherwise embarrassed with any further doubts peculiar to itself.

7. But those who dispute against the sums of such series,

think they have the firmest hold in the third species. For though the terms of these series continually increase, and that, by actually collecting the terms, we can arrive at a sum greater than any assignable number, which is the very definition of infinity; yet the patrons of the sums are forced to admit, in this species, series whose sums are not only finite, but even negative, or less than nothing. For since the fraction $\frac{1}{1-a}$, by evolving it by division, becomes $1 + a + a^2 + a^3 + a^4 +$ &c, we should have

$$\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + 16 + \&c,$$

$$\frac{1}{1-3} = -\tfrac{1}{2} = 1 + 3 + 9 + 27 + 81 + \&c,$$

which their adversaries, not undeservedly, hold to be absurd, since by the addition of affirmative numbers, we can never obtain a negative sum; and hence they urge that there is the greater necessity for including the before-mentioned supplement additive, since by taking it in, it is evident that

$$-1 \text{ is } = 1 + 2 + 4 + 8 \quad . \quad . \quad . \quad . \quad . \quad 2^n + \frac{2^{n+1}}{1-2},$$

though n should be an infinite number.

8. The defenders therefore of the sums of such series, in order to reconcile this striking paradox, more subtle perhaps than true, make a distinction between negative quantities; for they argue, that while some are less than nothing, there are others greater than infinite, or above infinity. Namely, that the one value of -1 ought to be understood, when it is conceived to arise from the subtraction of a greater number $a + 1$ from a less a; but the other value, when it is found equal to the series $1 + 2 + 4 + 8 +$ &c, and arising from the division of the number 1 by -1; for that in the former case it is less than nothing, but in the latter greater than infinite. For the more confirmation, they bring this example of fractions

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3}, \&c,$$

which, evidently increasing in the leading terms, it is inferred will continually increase; and hence they conclude that $\frac{1}{-1}$ is greater than $\frac{1}{0}$, and $\frac{1}{-2}$ greater than $\frac{1}{-1}$, and so on: and therefore as $\frac{1}{-1}$ is expressed by -1, and $\frac{1}{0}$ by ∞, or infinity, -1 will be greater than ∞, and much more will $= \frac{1}{-2}$ be greater than ∞. And thus they ingeniously enough repelled that apparent absurdity by itself.

9. But though this distinction seemed to be ingeniously devised, it gave but little satisfaction to the adversaries; and besides, it seemed to affect the truth of the rules of algebra. For if the two values of -1, namely $1-2$ and $\frac{1}{-1}$, be really different from each other, as we may not confound them, the certainty and the use of the rules, which we follow in making calculations, would be quite done away; which would be a greater absurdity than that for whose sake the distinction was devised: but if $1-2 = \frac{1}{-1}$, as the rules of algebra require, for by multiplication $-1 \times (1-2) = -1+2 = 1$, the matter in debate is not settled; since the quantity -1, to which the series $1+2+4+8+$ &c, is made equal, is less than nothing, and therefore the same difficulty still remains. In the mean time however, it seems but agreeable to truth, to say, that the same quantities which are below nothing, may be taken as above infinite. For we know, not only from algebra, but from geometry also, that there are two ways, by which quantities pass from positive to negative, the one through the cypher or nothing, and the other through infinity: and besides, that quantities, either by increasing or decreasing from the cypher, return again, and revert to the same term 0; so that quantities more than infinite are the same with quantities less than nothing, like as quantities less than infinite agree with quantities greater than nothing.

10. But, further, those who deny the truth of the sums

that have been assigned to diverging series, not only omit to assign other values for the sums, but even set themselves utterly to oppose all sums whatever belonging to such series, as things merely imaginary. For a converging series, as suppose this $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c, will admit of a sum $= 2$, because the more terms of this series we actually add, the nearer we come to the number 2: but in diverging series the case is quite different; for the more terms we add, the more do the sums which are produced differ from one another, neither do they ever tend to any certain determinate value. Hence they conclude, that no idea of a sum can be applied to diverging series, and that the labour of those persons who employ themselves in investigating the sums of such series, is manifestly useless, and indeed contrary to the very principles of analysis.

11. But notwithstanding this seemingly real difference, yet neither party could ever convict the other of any error, whenever the use of series of this kind has occurred in analysis; and for this good reason, that neither party is in an error, the whole difference consisting in words only. For if in any calculation we arrive at this series $1 - 1 + 1 - 1 +$ &c, and that we substitute $\frac{1}{2}$ instead of it, we shall surely not thereby commit any error; which however we should certainly incur if we substitute any other number instead of that series; and hence there remains no doubt but that the series $1 - 1 + 1 - 1 +$ &c, and the fraction $\frac{1}{2}$, are equivalent quantities, and that the one may always be substituted instead of the other without error. So that the whole matter in dispute seems to be reduced to this only, namely, whether the fraction $\frac{1}{2}$ can be properly called the *sum* of the series $1 - 1 + 1 - 1 +$ &c. Now if any persons should obstinately deny this, since they will not however venture to deny the fraction to be equivalent to the series, it is greatly to be feared they will fall into mere quarrelling about words.

12. But perhaps the whole dispute will easily be compromised, by carefully attending to what follows. Whenever, in analysis, we arrive at a complex function or expression,

either fractional or transcendental; it is usual to convert it into a convenient series, to which the remaining calculus may be more easily applied. And hence the occasion and rise of infinite series. So far only then do infinite series take place in analytics, as they arise from the evolution of some finite expression; and therefore, instead of an infinite series, in any calculus, we may substitute that formula, from whose evolution it arose. And hence, for performing calculations with more ease or more benefit, like as rules are usually given for converting into infinite series such finite expressions as are endued with less proper forms; so, on the other hand, those rules are to be esteemed not less useful, by the help of which we may investigate the finite expression from which a proposed infinite series would result, if that finite expression should be evolved by the proper rules: and since this expression may always, without error, be substituted instead of the infinite series, they must necessarily be of the same value: and hence no infinite series can be proposed, but a finite expression may, at the same time, be conceived as equivalent to it.

13. If, therefore, we only so far change the received notion of a sum as to say, that the sum of any series, is the finite expression by the evolution of which that series may be produced, all the difficulties, which have been agitated on both sides, vanish of themselves. For, first, that expression by whose evolution a converging series is produced, exhibits at the same time its sum, in the common acceptation of the term: neither, if the series should be divergent, could the investigation be deemed at all more absurd, or less proper, namely, the searching out a finite expression which, being evolved according to the rules of algebra, shall produce that series. And since that expression may be substituted in the calculation instead of this series, there can be no doubt but that it is equal to it. Which being the case, we need not necessarily deviate from the usual mode of speaking, but might be permitted to call that expression also the *sum*, which is *equal* to any series whatever, provided however,

that, in series whose terms do not converge to 0, we do not connect that notion with this idea of a sum, namely, that the more terms of the series are actually collected, the nearer we must approach to the value of the sum.

14. But if any person shall still think it improper to apply the term sum, to the finite expressions by whose evolution all series in general are produced; it will make no difference in the nature of the thing; and instead of the word sum, for such finite expression, he may use the term value, or function, or perhaps the term *radix* would be as proper as any other that could be employed for this purpose, as the series may justly be considered as issuing or growing out of it, like as a plant springs from its root, or from its seed. The choice of terms being in a great measure arbitrary, every person is at liberty to employ them in whatever sense he may think fit, or proper for the purpose in hand; provided always that he fix and determine the sense in which he understands or employs them. And as I consider any series, and the finite expression by whose evolution that series may be produced, as no more than two different ways of expressing one and the same thing, whether that finite expression be called the sum, or value, or function, or radix of the series; so in the following paper, and in some others which may perhaps hereafter be produced, it is in this sense I desire to be understood, when searching out the value of series, namely, that the object of the enquiry, is the radix by whose evolution the series may be produced, or else an approximation to the value of it in decimal numbers, &c.

TRACT VIII.

A NEW METHOD FOR THE VALUATION OF NUMERAL INFINITE SERIES, WHOSE TERMS ARE ALTERNATELY (+) PLUS AND (−) MINUS; BY TAKING CONTINUAL ARITHMETICAL MEANS BETWEEN THE SUCCESSIVE SUMS, AND THEIR MEANS.

ARTICLE 1.

THE remarkable difference between the facility which mathematicians have found, in their endeavours to determine the values of infinite series, whose terms are alternately affirmative and negative, and the difficulty of doing the same thing with respect to those series whose terms are all affirmative, is one of those striking circumstances in science which we can hardly persuade ourselves is true, even after we have seen many proofs of it; and which serve to put us ever after on our guard not to trust to our first notions, or conjectures, on these subjects, till we have brought them to the test of demonstration. For, at first sight it is very natural to imagine, that those infinite series whose terms are all affirmative, or added to the first term, must be much simpler in their nature, and much easier to be summed, than those whose terms are alternately affirmative and negative; which, however, we find, on examination, to be directly the reverse; the methods of finding the sums of the latter series being numerous and easy, and also very general, whereas those that have been hitherto discovered for the summation of the former series, are few and difficult, and confined to series whose terms are generated from each other according to some particular laws, instead of extending, as the other methods do, to all sorts of series, whose terms are connected together by addition, by whatever law their terms are formed. Of this remarkable difference between these two sorts of series, the new method of finding the sums of those whose terms are alternately positive and negative, which is the subject of the present tract, will afford us a striking instance, as it possesses the happy qualities of simplicity, ease, perspicuity, and universality; and yet, as the essence of it consists in the alternation of the signs $+$ and $-$, by which the terms are connected with the first term, it is of no use in the summation of those other series whose terms are all connected with each other by the sign $+$.

2. This method, so easy and general, is, in short, simply this: beginning at the first term a of the series $a - b + c - d + e - f +$ &c, which is to be summed, compute several successive values of it, by taking in successively more and more terms, one term being taken in at a time; so that the first value of the series shall be its first term a, or even 0 or nothing may begin the series of sums; the next value shall be its first two terms $a - b$, reduced to one number; its next value shall be the first three terms $a - b + c$, reduced to one number; its next value shall be the first four terms $a - b + c - d$, reduced also to one number; and so on. This, it is evident, may be done by means of the easy arithmetical operations of addition and subtraction. And then, having found a sufficient number of successive values of the series, more or less as the case may require, interpose between these values a set of arithmetical mean quantities or proportionals, and between these arithmetical means interpose a second set of arithmetical mean quantities; and between these arithmetical means of the second set, interpose a third set of arithmetical mean quantities; and so on as far as you please. By this process we soon find either the true value of the series proposed, when it has a determinate rational value, or otherwise we obtain several sets of values approximating nearer and nearer to the sum of the series, both in the columns and in the lines, either horizontal or obliquely de-

scending or ascending; namely, both of the several sets of means themselves, and the sets or series formed of any of their corresponding terms, as of all their first terms, of their second terms, of their third terms, &c, or of their last terms, of their penultimate terms, of their antepenultimate terms, &c: and if between any of these latter sets, consisting of the like or corresponding terms of the former sets of arithmetical means, we again interpose new sets of arithmetical means, as we did at first with the successive sums, we shall obtain other sets of approximating terms, having the same properties as the former. And thus we may repeat the process as often as we please, which will be found very useful in the more difficult diverging series, as we shall see hereafter. For this method, being derived only from the circumstance of the alternation of the signs of the terms, + and —, it is therefore not confined to converging series alone, but is equally applicable both to diverging series, and to *neutral* series, by which last name I shall take the liberty to distinguish those series, whose terms are all of the same constant magnitude; namely, the application is equally the same for all the three following sorts of series, viz.

Converging, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c.
Diverging, $1 - 2 + 3 - 4 + 5 - 6 +$ &c.
Neutral, $1 - 1 + 1 - 1 + 1 - 1 +$ &c.

As is demonstrated in what follows, and exemplified in a variety of instances.

It must be noted, however, that by the value of the series, I always mean such *radix*, or finite expression, as, by evolution, would produce the series in question; according to the sense we have stated in the former paper, on this subject; or an approximate value of such radix; and which radix, as it may be substituted instead of the series in any operation, I call the value of the series.

3. It is an obvious and well-known property of infinite series, with alternate signs, that when we seek their value by collecting their terms one after another, we obtain a series of successive sums, which approach continually nearer and

nearer to the true value of the proposed series, when it is a converging one, or one whose terms always decrease by some regular law; but in a diverging series, or one whose terms as continually increase, those successive sums diverge always more and more from the true value of the series. And from the circumstance of the alternate change of the signs, it is also a property of those successive sums, that when the last term which is included in the collection, is a positive one, then the sum obtained is too great, or exceeds the truth; but when the last collected term is negative, then the sum is too little, or below the truth. So that, in both the converging and diverging series, the first term alone, being positive, exceeds the truth; the second sum, or the sum of the first two terms, is below the truth; the third sum, or the sum of the three terms, is above the truth; the fourth sum, or the sum of four terms, is below the truth; and so on; the sum of any even number of terms being below the true value of the series, and the sum of any odd number, above it. All which is generally known, and evident from the nature and form of the series. So, of the series $a - b + c - d + e - f +$ &c, the first sum a is too great; the second sum $a - b$ too little; the third sum $a - b + c$ too great; and so on as in the following table, where s is the true value of the series, and 0 is placed before the collected sums, to complete the series, being the value when no terms are included:

	Successive sums.
s is greater than	0
s is less than	a
s is greater than	$a - b$
s is less than	$a - b + c$
s is greater than	$a - b + c - d$
s is less than	$a - b + c - d + e$
&c.	&c.

4. Hence the value of every alternate series s, is positive, and less than the first term a, the series being always supposed to begin with a positive term a; and consequently, if the signs of all the terms be changed, or if the series begin

with a negative term, the value s will still be the same, but negative, or the sign of the sum will be changed, and the value become $-s = -a + b - c + d -$ &c. Also, because the successive sums, in a converging series, always approach nearer and nearer to the true value, while they recede always farther and farther from it in a diverging one; it follows that, in a neutral series, $a - a + a - a +$ &c, which holds a middle place between the two former, the successive sums 0, a, 0, a, 0, a, &c, will neither converge nor diverge, but will be always at the same distance from the value of the proposed series $a - a + a - a +$ &c, and consequently that value will always be $= \frac{1}{2}a$, which holds every where the middle place between 0 and a.

I am not unaware that, though $a - a + a - a + -$ &c, may be produced by evolving $\frac{a^2}{a + a}$ by actual division, it will also arise by evolving several other functions in like manner; as

$$\frac{a^2}{a+a+a}, \text{ or } \frac{a^2}{a+a+a+a}, \text{ \&c, or } \frac{a^2+a^2+a^2+\text{\&c}}{a+a+a+a+\text{\&c}},$$

or any other similar function, in which the numerator has fewer terms than the denominator. Yet the preference among them all seems justly due to the first

$\frac{a^2}{a+a} = \frac{a^2}{2a} = \frac{a}{2} = \frac{1}{2}a$, for this reason, besides what is said above, viz, put s for the value of the series $a - a + a - a +$ &c: since

then $s = a - a + a - a + -$ &c,
and $a = a$, take the upper equ. from the under,
then $a - s = a - a + a - a +$ &c $= s$ by sup.
theref. $a - s = s$, and $2s = a$, or $s = \frac{1}{2}a$, as above.

5. Now, with respect to a converging series, $a - b + c - d +$ &c; because 0 is below, and a above s, the value of the series, but a nearer than 0 to the value s, it follows that s lies between a and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is less than s, and so nearer to s than 0 is. In like manner, because a is above, and $a - b$ below the value s, but $a - b$ nearer to that value than a is,

it follows that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is something above the value of s, but nearer to that value than a is. And thus, the same reasoning holding in every following pair of successive sums, the arithmetical means between them will form another series of terms, which are, like those sums, alternately less and greater than the value of the proposed series, but approximating nearer to that value than the several successive sums do, as every term of those means is nearer to the value s, than the corresponding preceding term in the sums is. And, like as the successive sums form a progression approaching always nearer and nearer to the value of the series; so, in like manner, their arithmetical means form another progression, coming nearer and nearer to the same value, and each term of the progression of means nearer than each term of the successive sums. Hence then we have the two following series, namely, of successive sums and their arithmetical means, in which each step approaches nearer to the value of s than the former, the latter progression being however nearer than the former, and the terms or steps of each alternately below and above the value s of the series $a - b + c - d +$ &c.

Successive sums.	Arithmetical means.
⊐ 0	⊐ $\frac{1}{2}a$
⊏ a	⊏ $a - \frac{1}{2}b$
⊐ $a - b$	⊐ $a - b + \frac{1}{2}c$
⊏ $a - b + c$	⊏ $a - b + c - \frac{1}{2}d$
⊐ $a - b + c - d$	⊐ $a - b + c - d + \frac{1}{2}e$
⊏ $a - b + c - d + e$	⊏ $a - b + c - d + e - \frac{1}{2}f$
&c.	&c.

where the mark ⊐, placed before any step, signifies that it is too little, or below the value s of the converging series $a - b + c - d +$ &c; and the mark ⊏ signifies the contrary, or too great. And hence $\frac{1}{2}a$, or half the first term of such a converging series, is less than s the value of the series.

6. And since these two progressions possess the same properties, but only the terms of the latter nearer to the truth than the former; for the very same reasons as before, the means between the terms of these first arithmetical means, will form a third progression, whose terms will approach still nearer to the value of s than the second progression, or the first means; and the means of these second means will approach nearer than the said second means do; and so on continually, every succeeding order of arithmetical means, approaching nearer to the value of s than the former. So that the following columns of sums and means will be each nearer to the value of s than the former, viz.

	Suc. sums.	1st means.	2d means.	3d means.
⊐	0	$\frac{a}{2}$	$\frac{3a-b}{4}$	$\frac{7a-4b+c}{8}$
⊏	a	$a-\frac{b}{2}$	$a-\frac{3b-c}{4}$	$a-\frac{7b-4c+d}{8}$
⊐	$a-b$	$a-b+\frac{c}{2}$	$a-b+\frac{3c-d}{4}$	$a-b+\frac{7c-4d+e}{8}$
⊏	$a-b+c$	$a-b+c-\frac{d}{2}$	$a-b+c-\frac{3d-e}{4}$	$a-b+c-$&c.
⊐	$a-b+c-d$	$a-b+$&c.	$a-b+$&c.	$a-b+$&c.
	&c.			

Where every column consists of a set of quantities, approaching still nearer and nearer to the value of s, the terms of each column being alternately below and above that value, and each succeeding column approaching nearer than the preceding one. Also every line, formed of all the first terms, all the second terms, all the third terms, &c, of the columns, forms also a progression whose terms continually approximate to the value of s, and each line nearer or quicker than the former; but differing from the columns, or vertical progressions, in this, namely, that whereas the terms in the columns are alternately below and above the value of s, those in each line are all on one side of the value s, namely, either all below or all above it; and the lines alternately too little and too great, namely, all the expressions in the first line too little, all those in the second line too great, those in the third line too little, and so on, every odd line being too little, and every even line too great.

7. Hence the expressions $\frac{a}{2}$, $\frac{3a-b}{4}$, $\frac{7a-4b+c}{8}$, $\frac{15a-11b+5c-a}{16}$, $\frac{31a-26b+16c-6d+e}{32}$, &c, are continual approximations to the value s, of the converging series $a-b+c-d+e-$ &c, and are all below the truth. But we can easily express all these several theorems by one general formula. For, since it is evident by the construction, that while the denominator in any one of them is some power (2^n) of 2 or $1+1$, the numeral co-efficients of a, b, c, &c, the terms in the numerator, are found by subtracting all the terms except the last term, one after another, from the said power 2^n or $(1+1)^n$, which is $=$

$1+n+n\cdot\frac{n-1}{2}+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}+$ &c, namely the coefficient of a equal to all the terms 2^n, minus the first term 1; that of b equal to all except the first two terms $1+n$; that of c equal to all except the first three; and so on, till the coefficient of the last term be $=1$, the last term of the power; it follows that the general expression for the several theorems, or the general approximate value of the converging series $b-a+c-d+$ &c, will be

$$\frac{2^n-1}{2^n}a-\frac{2^n-1-n}{2^n}b+\frac{2^n-1-n-n\cdot\frac{n-1}{2}}{2^n}c+$$

&c, continued till the terms vanish and the series break off, n being equal to 0 or any integer number. Or this general formula may be expressed by this series,

$$\frac{1}{2^n}\times[(2^n-1)a-(\text{A}-n)b+(\text{B}-n\cdot\frac{n-1}{2})c-(\text{C}-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3})d$$

&c]; where A, B, C, &c, denote the coefficients of the several preceding terms. And this expression, which is always too little, is the nearer to the true value of the series $a-b+c-d+$ &c, as the number n is taken greater; always excepting however those cases in which the theorem is accurately true, when n is some certain finite number. Also, with any value of n, the formula is nearer to the truth, as the terms a, b, c, &c, of the proposed series, are nearer to equality; so that the slower any proposed series converges, the more accurate is the formula, and the sooner does it afford a near value of that series: which is a very favourable circumstance, as it is in cases of very slow convergency that approximating formula are chiefly wanted. And, like as the formula approaches nearer to the truth as the terms of the series approach to an equality, so when the terms become quite equal, as in a neutral series, the formula becomes quite accurate, and always gives the same value $\frac{1}{2}a$ for s or the series, whatever integer number be taken for n. And further, when the proposed series, from being converging, passes through neutrality, when its terms are equal, and becomes diverging, the formula will still hold good, only it will then be alternately too great, and too little as long as the series diverges, as we shall presently see more fully. So that, in general, the value s of the series $a - b + c - d +$ &c, whether it be converging, diverging, or neutral, is less than the first term a; when the series converges, the value is above $\frac{1}{2}a$; when it diverges, it is below $\frac{1}{2}a$; and when neutral, it is equal to $\frac{1}{2}a$.

8. Take now the series of the first terms of the several orders of arithmetical means, which form the progression of continual approximating formulæ, being each nearer to the value of the series $a - b + c - d +$ &c, than the former, and place them in a column one under another; then take the differences between every two adjacent formulæ, and place them in another column by the side of the former, as here follows:

Approx. Formulæ.	Differences.
$\frac{a}{2}$	$\frac{a-b}{4}$
$\frac{3a-b}{4}$	$\frac{a-2b+c}{8}$
$\frac{7a-4b+c}{8}$	$\frac{a-3b+3c-d}{16}$
$\frac{15a-11b+5c-d}{16}$	$\frac{a-4b+6c-4d+e}{32}$
$\frac{31a-26b+16c-6d+e}{32}$	
&c.	&c.

From which it appears, that this series of differences consists of the very same quantities, which form the first terms of all the orders of differences of the terms of the proposed series $a-b+c-d+$ &c, when taken as usual in the differential method. And because the first of the above differences added to the first formula, gives the second formula; and the second difference added to the second formula, gives the third formula; and so on; therefore the first formula with all the differences added, will give the last formula; consequently our general formula, before mentioned,

$$\frac{1}{2^n} \times [(2^n-1)\,a-(\text{A}-n)\,b+(\text{B}-n\,.\,\frac{n-1}{2})\,c-\&\text{c}],$$

which approaches to the value of the series $a-b+c-d+$ &c, is also equivalent to, or reduces to this form,

$$\frac{a}{2}+\frac{a-b}{4}+\frac{a-2b+c}{8}+\frac{a-3b+3c-d}{16}+\&\text{c},$$

which, it is evident, agrees with the famous differential series. And this coincidence might be sufficient to establish the truth of our method, though we had not given other more direct proof of it. Hence it appears then, that our theorem is of the same degree of accuracy, or convergency, as the differential theorem; but admits of more direct and easy application, as the terms themselves are used, without the previous trouble of taking the several orders of differences. And our method will be rendered general for literal, as well as for numeral series, by supposing a, b, c, &c, to represent not barely the coefficients of the terms, but the whole terms, both the numeral and the literal part of them. However, as the chief use of this method is to obtain the numeral value of series, when a literal series is to be so summed, it is to be made numeral by substituting the numeral values of the letters instead of them. It is further evident, that we might easily derive our method of arithmetical means from the above differential series, by beginning with it, and receding back to our theorems, by a process counter to that above given.

9. Having, in Art. 5, 6, 7, 8, completed the investigations for the first or converging form of series, the first four articles being introductory to both forms in common; we may now proceed to the diverging form of series, for which we shall find the same method of arithmetical means, and the same general formula, as for the converging series; as well as the mode of investigation used in Art. 5 *et seq.* only changing sometimes greater for less, or less for greater. Thus then, reasoning from the table of successive sums in Art. 3, in which s is alternately above and below the expressions 0, a, $a - b$, $a - b + c$, &c, because 0 is below, and a above the value s of the series $a - b + c - d +$ &c, but 0 nearer than a to that value, it follows that s lies between 0 and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is greater than s, but nearer to s than a is. In like manner, because a is above, and $a - b$ below the value s, but a nearer that value than $a - b$ is, it follows, that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is below s, but that it is nearer to s than $a - b$ is. And thus, the same reasoning holding in every pair of successive sums, the arithmetical means between them will form another series of terms, which are alternately greater and less than s, the value of the proposed series; but here greater and less in the contrary way to what they were for the converging series, namely, those steps greater here which were less there, and less here which before were greater. And this first set of arithmetical means, will either converge to the value of s, or will at least diverge less from it than the progression of successive sums. Again, the same reasoning still holding good, by taking the arithmetical means of those first means, another set is found,

which will either converge, or else diverge less than the former. And so on as far as we please, every new operation gradually checking the first or greatest divergency, till a number of the first terms of a set converge sufficiently fast, to afford a near value of s the proposed series.

10. Or, by taking the first terms of all the orders of means, we find the same set of theorems, namely

$\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}$, &c, or in general,

$\frac{1}{2^n} \times [(2^n-1)\,a - (\mathrm{A}-n)\,b + (\mathrm{B}-n\,.\,\frac{n-1}{2})\,c - \text{\&c}]$,

which will be alternately above and below s, the value of the series, till the divergency is overcome. Then this series, which consists of the first terms of the several orders of means, may be treated as the snecessive sums, taking several orders of means of these again. After which, the first terms of these last orders may be treated again in the same manner; and so on as far as we please. Or the series of second terms, or third terms, &c, or sometimes, the terms ascending obliquely, may be treated in the same manner to advantage. And with a little practice and inspection of the several series, whether vertical, or horizontal, or oblique, for they all tend to the detection of the same value s, we shall soon learn to distinguish whereabouts the required quantity s is, and which of the series will soonest approximate to it.

11. To exemplify now this method, we shall take a few series of both sorts, and find their value, sometimes by actually going through the operations of taking the several orders of arithmetical means, and at other times by using some one of the theorems

$\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}$, &c, at once.

And to render the use of these theorems still easier, we shall here subjoin the following table, where the first line, consisting of the powers of 2, contains the denominators of the theorems in their order, and the figures in their perpendicular columns below them, are the coefficients of the several terms in the numerators of the theorems, namely, the upper figure, next below the power of 2, the coefficient of a; the next below, that of b; the third that of c, &c.

2	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}
1	3	7	15	31	63	127	255	511	1023	2047	1095	8191	16383	32767	65535	131071	262143	524287	1048575
	1	4	11	26	57	120	247	502	1013	2036	4083	8178	16369	32752	65519	131054	262125	524268	1048555
		1	5	16	42	99	219	466	968	1981	4017	8100	16278	32647	65399	130918	261972	524097	1048365
			1	6	22	64	163	382	848	1816	3797	7814	15914	32192	64839	130238	261156	523128	1047225
				1	7	29	93	256	638	1486	3302	7099	14913	30827	63019	127858	258096	519252	1042380
					1	8	37	130	386	1024	2510	5812	12911	27824	58651	121670	249528	507624	1026876
						1	9	46	176	562	1586	4096	9908	22819	50643	109294	230964	480492	938116
							1	10	56	232	794	2380	6476	16384	39203	89846	199140	430104	910596
								1	11	67	299	1093	3473	9949	26333	65536	155382	354522	784626
									1	12	79	378	1471	4944	14893	41226	106762	262144	616666
										1	13	92	470	1941	6885	21778	63004	169766	431910
											1	14	106	576	2517	9402	31180	94184	263950
												1	15	121	697	3214	12616	43796	137980
													1	16	137	834	4048	16664	60460
														1	17	154	988	5036	21700
															1	18	172	1160	6196
																1	19	191	1351
																	1	20	211
																		1	21
																			1

The construction and continuation of this table, is a business of little labour. For the numbers in the first horizontal line next below the line of the powers of 2, are those powers diminished each by unity. The numbers in the next horizontal line, are made from the numbers in the first, by subtracting from each the index of that power of 2 which stands above it. And for the rest of the table, the formation of it is obvious from this principle, which reigns through the whole, that every number in it is the sum of two others, namely, of the next to it on the left in the same horizontal line, and the next above that in the same vertical column. So that the whole table is formed from a few of its initial numbers, by easy operations of addition.

In converging series, it will be further useful, first to collect a few of the initial terms into one sum, and then apply our method to the following terms, which will be sooner valued, because they will converge slower.

12. For the first example, let us take the very slowly converging series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c, which is known to express the hyp. log. of 2, which is $= \cdot 69314718$.

Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, &c, and the value, as found by theorem the 1st, 2d, 3d, 4th, 10th, and 20th, will be thus:

1st, $\frac{a}{2} = \frac{1}{2} = \cdot 5$.

2d, $\frac{3a-b}{4} = \frac{3-\frac{1}{2}}{4} = \frac{2\frac{1}{2}}{4} = \cdot 625$.

3d, $\frac{7a-4b+c}{8} = \frac{7-2+\frac{1}{3}}{8} = \frac{5\frac{1}{3}}{8} = \frac{2}{3} = \cdot 666666$.

4th, $\frac{15a-11b+5c-d}{16} = \frac{15-5\frac{1}{2}+1\frac{2}{3}-\frac{1}{4}}{16} = \cdot 68229$.

10th, $\frac{1023a-1013b+\&c}{2^{10}} = \frac{709\cdot 698413}{4^5} = \cdot 693065$.

20th, $\frac{1048575a-1048555b+\&c}{2^{20}} = \frac{726817\cdot 45238043}{4^{10}} =$
$\cdot 69314714$.

Where it is evident that every theorem gives always a nearer value than the former: the 10th theorem gives the value true to the 3d figure, and the 20th theorem to the 7th figure. The operation for the 10th and 20th theorems, will be easily performed by dividing, mentally, the numbers in their respective columns in the table of coefficients in Art. 11, by the ordinate numbers 1, 2, 3, 4, 5, 6, &c, placing the quotients of the alternate terms below each other, then adding each up, and dividing the difference of the sums continually five or ten times successively by the number 4: after the manner as here placed below, where the operation is set down for both of them.

1. For the 10th Theorem.

	+	−
	1023	506·5
	322·666667	212
	127·6	64·333
	25·142857	7
	1·222222	0·1
	1499·631746	789·933
	789·933333	
4	709·698413	
4	177·424603	
4	44·356151	
4	11·089038	
4	2·772259	
	·693065	

2. For the 20th Theorem.

	+	−
	1048575	524277·5
	349455	261806·25
	208476	171146·
	141159·42857143	113824·5
	87180·66666667	61666·6
	39264·54545454	21995·83333333
	10613·84615385	4318·57142857
	1446·66666667	387·25
	79·47058824	11·72222222
	1·10526316	0·05
	1886251·72936456	1159434·27698413
	1159434·27698413	
4	726817·45238043	
4	181704·36309511	
4	45426·09077378	
4	11356·52269345	
4	2839·13067336	
4	709·78266834	
4	177·44566708	
4	44·36141677	
4	11·09035419	
4	2·77258855	
	69314714	

Again, to perform the operation by taking the successive sums, and the arithmetical means: let the terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c, be reduced to decimal numbers, by dividing the common numerator 1 by the denominators 2, 3, 4, &c, or rather by taking these out of the table printed at the end of this volume, which contains a table of the square roots and reciprocals of all the numbers, 1, 2, 3, 4, 5, 6, &c, to 1000, and which is of great use in such calculations as these. Then the operation will stand thus :

The terms.	Suc. sums	The several orders of means.					
+ 1	1						
− 0·5	0·5						
+ 333333	833333						
− 25	583333						
+ 2	783333						
− 166666	616666						
+ 142857	759524	688095	692560	693056			
− 125	634524	697024	693552	693205	693131	693144	
+ 111111	745635	690080	692858	693110	693158	693150	693147
− 1	645635	695635	693362	693173	693142		
+ 090909	736544	691090	692984				
− 083333	653211	694878					

Here, after collecting the first twelve terms, I begin at the bottom, and, ascending upwards, take a very few arithmetical means between the successive sums, placing them on the right of them : it being unnecessary to take the means of the whole, as any part of them will do the business, but the lower ones in a converging series best, because they are nearer the value sought, and approach sooner to it. I then take the means of the first means, and the means of these again, and so on, till the value is obtained as near as may be necessary. In this process we soon distinguish whereabouts the value lies, the limits or means, which are alternately above and below it, gradually contracting, and approaching towards each other. And when the means are reduced to a single one, and it is found necessary to get the value more exactly, I then go back to the columns of successive sums, and find another first mean, either next below or above those before found, and continue it through the 2d, 3d, &c, means, which makes now a duplicate in the last column of means, and the mean between them gives another single mean of the next order ; and so on as far as we see it necessary. By such a gradual progress we use no more terms nor labour than is quite requisite for the degree of accuracy required.

Or, after having collected the sum of any number of terms, we may apply any of the formulæ to the following terms. So, having as above found ·653211 for the sum of the first 12 terms, and calling the next or 13th term ·076923 $= a$, the

14th term ·0714285$=b$, the next, ·06666 &c $=c$, and so on: then the 2d theorem $\frac{3a-b}{4}$ gives ·039835, which added to ·653211 the sum of the first 12 terms, gives ·693046, the value of the series true in three places of figures; and the 3d theorem $\frac{7a-4b+c}{8}$ gives ·039927 for the following terms, and which added to ·653211 the sum of the first 12 terms, gives ·693138, the value of the series true in five places. And so on.

13. For a second example, let us take the slowly converging series $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{6}+$ &c, which is $=\frac{1}{2}+$ hyp. log. of $2=1·19314718$. Then the process will be thus.

Terms.	Suc. sums	Arithmetical means.					
+ 2	2						
− 1·5	0·5						
+ 1·333333	1·833333						
− 1·25	0·583333						
+ 1·2	1·783333						
− 1·166666	0·616666	1·188095					
+ 1·142857	1·759524	1·197024	1·192560				
− 1·125	0·634524	1·190080	1·193552	1·193056			
+ 1·111111	1·745635	1·195635	1·192858	1·193205	131		
− 1 1	0·645635	1·191090	1·193362	1·193110	157	144	
+ 1·090909	1·736544	1·194878	1·192934	1·193173	142	150	147
− 1·083333	0·653211						

Here, after the 3d column of means, the first four figures 1·193, which are common, are omitted, to save room and the trouble of writing them so often down; and in the last three columns, the process is repeated with the last three figures of each number; and the last of these 147, joined to the first four, give 1·193147 for the value of the series proposed. And the same value is also obtained by the theorems used as in the former example.

14. For the third example, let us take the converging series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$ &c, which is $=$·7853981 &c, or $\frac{1}{4}$ of the circumference of the circle whose diameter is 1. Here $a=1$, $b=\frac{1}{3}$, $c=\frac{1}{5}$, &c, then turning the terms into decimals, and proceeding with the successive sums and means as before, we obtain the 5th mean true within a unit in the 6th place as here below:

Terms.	S. sums	Arithmetical means.				
+ 1						
− 0·333333	666667					
+ 2	866667					
− 142857	723810					
+ 111111	834921					
− 090909	744012	782474				
+ 76923	820935	787601	785037			
− 66667	754268	783680	785641	785339		
+ 58823	813091	786775	785227	785434	785387	785397
− 52632	760459	784269	785522	785380	785407	
+ 47619	808078					

15. To find the value of the converging series

$$1-\frac{1^2}{2^2}+\frac{1^2\,.\,3^2}{2^2\,.\,4^2}+\frac{1^2\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2}+\frac{1^2\,.\,3^2\,.\,5^2\,.\,7^2}{2^2\,.\,4^2\,.\,6^2\,.\,8^2}-\&c,$$

which occurs in the expression for determining the time of a body's descent down the arc of a circle.

The first terms of this series I find ready computed by Mr. Baron Maseres, pa. 219 Philos. Trans. 1777; these being arranged under one another, and the sums collected, &c, as before, give ·834625 for the value of that series, being only 1 too little in the last figure.

Terms.	S. sums	Arithmetical means.				
+ 1						
− 0·25	75					
+ 140625	890625					
− 97656	792969					
+ 74768	867737					
− 60562	807175	832620				
+ 50889	858064	836124	834372			
− 43879	814185	833468	834796	834584		
+ 38565	852750	835550	834509	834652	834618	834625
− 34399	818351	833873	834711	834610	834631	
+ 31045	849396					

16. To find the value of $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-$ &c, consisting of the reciprocals of the natural series of square numbers.

Terms.	S. sums	Arithmetical means.					
+ 1							
− 0 25	75						
+ 111111	861111						
− 625	798611						
+ 4	838611						
− 27778	810833						
+ 20408	831241	823429					
− 15625	815616	821789	822609				
+ 12346	827962	822962	822376	822492	822472		
− 1	817962	822094	822528	822452	822464	822468	822467
+ 08264	826226	822754	822424	822476	822468	822466	
− 6944	819282	822240	822497	822460			
+ 5917	825199						

The last mean ·822467 is true in the last figure, the more accurate value of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} +$ &c, being ·8224670 &c.

17. Let the diverging series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$ &c, be proposed; where the terms are the reciprocals of those in Art. 13.

Terms.	Suc. sums.	Arithmetical means.					
+ 5	+ 5						
— 666666	— 166666						
+ 75	+ 583333						
— 8	— 216666						
+ 833333	+ 616666						
— 857143	— 240476	188095					
+ 875	+ 634524	197024	192560				
— 888889	— 254365	190080	193552	193056			
+ 9	+ 645635	195635	192858	193205	151		
— 909091	— 263456	191090	193362	193110	157	144	
+ 916667	+ 653211	194878	192984	193173	142	150	147

Here the successive sums are alternately + and —, as well as the terms themselves of the proposed series, but all the arithmetical means are positive. The numbers in each column of means are alternately too great and too little, but so as visibly to approach towards each other. The same mutual approximation is visible in all the oblique lines from left to right, so that there is a general and mutual tendency, in all the columns, and in all the lines, to the limit of the value of the series. But with this difference, that all the numbers in any line descending obliquely from left to right, are on one side of the limit, and those in the next line in the same direction, all on the other side, the one line having its numbers all too great, while those in the next line are all too little; but, on the contrary, the lines which ascend from below obliquely towards the right, have their numbers alternately too great and too little, after the manner of those in the columns, but approximating quicker than those in the columns. So that, after having continued the columns of arithmetical means to any convenient extent, we may then select the terms in the last, or any other line obliquely ascending from left to right, or rather beginning with the last found mean on the right, and descending towards the left; then arrange those terms below one another in a column, and take their continual arithmetical means, like as was done with the first succesive sums, to such extent as the case may require. And if neither these new columns, nor the oblique lines approach near enough to each other, a new set may be formed from one of their oblique lines which has its terms alternately too great and too little. And thus we may proceed as far as we please. These repetitions will be more necessary in treating series which diverge more; and having here once for all described the properties attending the series, with the method of repetition, we shall only have to refer to them as occasion shall offer. In the present instance, the last two or three means vary or differ so little, that the limit may be concluded to lie nearly in the middle between them, and therefore the mean between the two last 144 and 150, namely 147, may be concluded to be very near the truth, in the last three figures; for as to the first three figures 193, repetition of them is omitted after the first three columns of means, both to save space, and the trouble of writing them so often over again. So that the value of the series in question may be concluded to be ·193147 very nearly, which is $= -\frac{1}{2} +$ the hyp. log. of 2; or 1 less than its reciprocal series in Art. 13.

18. Take the diverging series $\frac{5}{4} - \frac{5.7}{4.6} + \frac{5.7.9}{4.6.8} - \frac{5.7.9.11}{4.6.8.10} +$ &c. Here, first using some of the formulæ, we have by the

1st, $\frac{a}{2} = \cdot 625.$

2d, $\frac{3a-b}{4} = \cdot 57292.$

3d, $\frac{7a-4b+c}{8} = \cdot 56966.$

4th, $\frac{15a-11b+5c-d}{16} = \cdot 56917.$

5th, $\frac{31a-26b+16c-6d+e}{32} = \cdot 56907.$ &c.

Or, thus, taking the several orders of means, &c.

Terms.	Suc. sums.	Arithmetical means.						
+ 1·25	+ 1·25							
— 1·458333	— 0·208333	520833	566406	8685				
+ 1·640625	+ 1·432292	611980	570964	9255	8970	021		
— 1·804688	— 0·372396	529948	567546	8889	9072	043	032	035
+ 1·955079	+ 1·582683	605144	570232	9141	9015	033	038	
— 2·094727	— 0·512044	535320	568050	8959	9050			
+ 2·225647	+ 1·713603	600780	569868					
— 2·349294	— 0·635691	538956						

Here the successive sums are alternately + and —, but the arithmetical means are all +. After the second column of means, the first two figures 56 are omitted, being common; and in the last three columns the first three figures 569, which are common, are omitted. Towards the end, all the numbers, both oblique and vertical, approach so near together, that we may conclude that the last three figures 035 are all true; and these being joined to the first three 569, we have ·569035 for the value of the series, which is otherwise found $=$

$$\frac{2+\sqrt{2}}{6} = \cdot 56903559 \text{ \&c.}$$

19. Let us take the diverging series

$\frac{2^2}{1} - \frac{3^2}{2} + \frac{4^2}{3} - \frac{5^2}{4} + \&c$, or $\frac{4}{1} - \frac{9}{2} + \frac{16}{3} - \frac{25}{4} + \&c$, or $4 - 4\frac{1}{2} + 5\frac{1}{3} - 6\frac{1}{4} + 7\frac{1}{5} - 8\frac{1}{6} + \&c$.

Terms.	Sums.	Arithmetical means.					
+ 4	+ 4·						
— 4·5	— 0·5						
+ 5·333333	+ 4·833333						
— 6·25	— 1·416666						
+ 7·2	+ 5·783333						
— 8·166666	— 2·383333	2·188096					
+ 9·142857	+ 6·759524	1·697024	1·942560	059			
— 10·125	— 3·365476	2·190080	1·943557	207	128	143	
+ 11·111111	+ 7·745635	1·695635	1·942857	110	158	150	147
— 12·1	— 4·354365	2·191089	1·943362	173	142		
+ 13·090909	+ 8·736544	1·694877	1·942983				
— 14·083333	— 5·346789						

After the second column of means, the first four figures 1·943 are omitted, being common to all the following columns; to these annexing the last three figures 147 of the last mean, we have 1·943147 for the sum of the series, which we otherwise know is equal to $\frac{5}{4}$ + hyp. log. of 2. See Simp. Dissert. Ex. 2. p. 75 and 76.

And the same value might be obtained by means of the formulæ, using them as before.

20. Taking the diverging series $1 - 2 + 3 - 4 + 5 -$ &c, formed from the radix $(\frac{1}{1+1})^2 = \frac{1}{1+2+1} = \frac{1}{4}$, by dividing 1 by $1 + 2 + 1$; the method of means gives us the following.

Terms.	Sums.	Means.	
+ 1	+ 1	0	
− 2	− 1	$\frac{1}{2}$	$\frac{1}{4}$
+ 3	+ 2	0	$\frac{1}{4}$
− 4	− 2	$\frac{1}{2}$	$\frac{1}{4}$
+ 5	+ 3	0	$\frac{1}{4}$
− 6	− 3		

Where the second, and every succeeding column of means, gives $\frac{1}{4}$ for the value of the series proposed.

In like manner, using the theorems, the first gives $\frac{1}{2}$, but the second, third, fourth, &c, give each of them the same value $\frac{1}{4}$; thus:

$$\frac{a}{2} = \frac{1}{2}$$

$$\frac{3a-b}{4} = \frac{3-2}{4} = \frac{1}{4}$$

$$\frac{7a-4b+c}{8} = \frac{7-8+3}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\frac{15a-11b+5c-d}{16} = \frac{15-22+15-4}{16} = \frac{4}{16} = \frac{1}{4}$$

21. Taking the series $1 - 4 + 9 - 16 + 25 - 36 +$ &c, whose terms consist of the squares of the natural series of numbers, we have, by the arithmetical means,

Terms.	Sums.	Arithmetical means.		
+ 1	+ 1	− 1		
− 4	− 3	+ $1\frac{1}{2}$	+ $\frac{1}{4}$	
+ 9	+ 6	− 2	− $\frac{1}{4}$	0
− 16	− 10	+ $2\frac{1}{2}$	+ $\frac{1}{4}$	0
+ 25	+ 15	− 3	− $\frac{1}{4}$	0
− 36	− 21			

Where it is only in the second column of means that the divergency is counteracted; after that the third and all the other orders of means give 0 for the value of the series $1 - 4 + 9 - 16 +$ &c.

The same thing takes place on using the formulæ, for

$$\frac{a}{2} = \tfrac{1}{2}$$

$$\frac{3a-b}{4} = \frac{3-4}{4} = -\frac{1}{4}$$

$$\frac{7a-4b+c}{8} = \frac{7-16+9}{8} = \frac{0}{8} = 0$$

$$\frac{15a-11b+5c-d}{16} = \frac{15-44+45-16}{16} = \frac{0}{16} = 0$$

where the third and all after it give the same value 0.

22. Taking the geometrical series of terms $1-2+4-8+$ &c, derived from the radix $\frac{1}{1+2} = \frac{1}{3}$, by actually dividing 1 by $1+2$.

Terms.	Sums.	Arithmetical means.				
+ 1	+ 1	+ $\frac{1}{2}$	+ $\frac{1}{4}$	$\frac{3}{8}$		
− 2	− 1	0	+ $\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{16}$	
+ 4	+ 3	+ 1	0	$\frac{1}{2}$	$\frac{3}{8}$	
− 8	− 5	− 1	+ 1	0	$\frac{1}{4}$	
+ 16	+ 11	+ 3	− 1	+ 1	$\frac{1}{2}$	&c.
− 32	− 21	− 5	+ 3	− 1	0	
+ 64	+ 43	+ 11	− 5	+ 3	+ 1	
− 128	− 85	− 21	+ 11	− 5	− 1	
+ 256	+ 171	+ 43			&c.	

Here the lower parts of all the columns of means, from the cipher 0 downwards, consist of the same series of terms $+1-1+3-5+11-21+43-85+$ &c, and the other part of the columns, from the cipher upwards, as well as each line of oblique means, parallel to, and above the line of ciphers, forms a series of terms $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{5}{16}$ . . . . $\frac{1}{3}\cdot\frac{2^n \pm 1}{2^n}$, alternately above and below the value of the series, $\frac{1}{3}$, and approaching continually nearer and nearer to it, and which, when infinitely continued, or when n is infinite, the term becomes $\frac{1}{3}$ for the value of the geometrical series, $1-2+4-8+16-$ &c.

And the same set of terms would be given by each of the formulæ.

23. Taking the geometrical series $1-3+9-27+81-$ &c, obtained from the radix $\frac{1}{1+3} = \frac{1}{4}$, by dividing 1 by $1+3$.

Terms.	Sums.	Arithmetical means.							
$+1$	$+1$	$+\frac{1}{2}$	0						
-3	-2	$-\frac{1}{2}$	$+1$	$+\frac{1}{2}$	0				
$+9$	$+7$	$+2\frac{1}{2}$	-2	$-\frac{1}{2}$	$+1$	$+\frac{1}{2}$	0		
-27	-20	$-6\frac{1}{2}$	$+7$	$+2\frac{1}{2}$	-2	$-\frac{1}{2}$	$+1$	$+\frac{1}{2}$	0
$+81$	$+61$	$+20\frac{1}{2}$	-20	$-6\frac{1}{2}$	$+7$	$+2\frac{1}{2}$	-2	$-\frac{1}{2}$	

Here the column of successive sums, and every second column of the arithmetical means, below the 0, consists of the same series of terms 1, -2, $+7$, -20, $+$ &c, while all the other columns of means consist of this other set of terms $\frac{1}{2}$, $-\frac{1}{2}$, $+2\frac{1}{2}$, $-6\frac{1}{2}$, $+$ &c; also the first oblique line of means, $\frac{1}{2}$, 0, $\frac{1}{2}$, 0, $\frac{1}{2}$, 0, &c, consists of the terms $\frac{1}{2}$ and 0 alternately, which are all at equal distance from the value of the series proposed $1-3+9-27+81-$ &c, as indeed are the terms of all the other oblique descending lines. And the mean between every two terms gives $\frac{1}{4}$ for that value. And the same terms would be given by the formulæ, namely alternately $\frac{1}{2}$ and 0.

And thus the value of any geometrical series, whose ratio or second term is r, will be found to be $=\frac{1}{1+r}$.

24. Finally, let there be taken the hypergeometrical series $1-1+2-6+24-120+$ &c $=1-1\text{A}+2\text{B}-3\text{C}+4\text{D}-5\text{E}+$ &c; which difficult series has been honoured by a very considerable memoir written on the valuation of it by the celebrated L. Euler, in the New Petersburg Commentaries, vol. v, where the value of it is at length determined to be ·5963473 &c.

To simplify this series, let us omit the first two terms $1-1=0$, which will not alter the value, and divide the remaining terms by 2, and the quotients will give $1-3+12-60+360-2520+$ &c; which, being half the proposed series, ought to have for its value the half of ·596347 &c, namely ·298174 nearly.

Now, ranging the terms in a column, and taking the sums and means as usual, we have

Terms.	Sums.	Arithmetical means.						
+ 1	+ 1	+ $\frac{1}{2}$	0	$\frac{7}{8}$	1·1250	4·58125	14·4375	59·023438
− 3	− 2	− $\frac{1}{2}$	$1\frac{3}{4}$	$3\frac{1}{8}$	10·1875	33·40625	132·484375	
+ 12	+ 10	+ 4	8	$23\frac{1}{2}$	77·	298·375		
− 60	− 50	− 20	55	$177\frac{1}{2}$	673·75			
+ 360	+ 310	+ 130	410	1525				
− 2520	− 2210	− 950	3460					
+ 20160	+ 17950	+ 7870						

Where it is evident, that the diverging is somewhat diminished, but not quite counteracted, in the columns and oblique descending lines, from beginning to end, as the terms in those directions still increase, though not quite so fast as the original series ; and that the signs of the same terms are alternately + and −, while those of the terms in the other lines obliquely ascending from left to right, are alternately one line all +, and another line all −, and these terms continually decreasing. The terms in the oblique descending lines, being alternately too great and too little, are the fittest to proceed with again. Taking therefore any one of those lines, as suppose the first, and ranging it vertically, take the means as before, and they will approach nearer to the value of the series, thus :

+ ·5	+ ·25	+ ·34375	+ ·25	+ ·361328	·194336	·492066
− ·0	+ ·4375	+ ·15625	+ ·472656	+ ·027344	·759795	
+ ·875	− ·1250	+ ·789062	− ·417969	+ 1·552246		
− 1·125	+ 1·703125	− 1·625	+ 3·522461			
+ 4·53125	− 4·953125	+ 8·669922				
− 14·4375	+ 22·292969					
+ 59·023438						

Here the same approximation in the lines and columns, towards the value of the series, is observable again, only in a higher degree : also the terms in the columns and oblique descending lines, are again alternately too great and too little, but now within narrower limits, and the signs of the terms are more of them positive ; also the terms in each oblique ascending line, are still either all above or all below the value of the series, and that alternately one line after another, as before. But the descending lines will again be the fittest to use, because the terms in each are alternately above and below the value sought. Taking therefore again the first of these oblique descending lines, and treating it as before, we obtain sets of terms approaching still nearer to the value, thus :

25	296875	296875	299073	297791	298306
34375	296875	301271	296509	298821	
25	305654	291748	301132		
361328	277832	310516			
194336	343201				
492066					

Here the approach to an equality, among all the lines and columns, is still more visible, and the deviations restricted within narrower limits, the terms in the oblique ascending lines still on one side of the value, and gradually increasing, while the columns and the oblique descending lines, for the most part, have their terms alternately too great and too little, as is evident from their alternately becoming greater and less than each other : and from an inspection of the whole, it is easy to pronounce that the first three figures of the number sought, will be 298. Taking therefore the last four terms of the first descending line, and proceeding as before, we have

296875	297974	298203	298222
299073	298432	298240	
297791	298048		
298306			

And, finally, taking the lowest ascending line, because it has most the appearance of being alternately too great and too little, proceed with it as before, thus :

298306	298177	298161	298174
298048	298144	298187	
298240	298231		
298222			

where the numbers in the lines and columns gradually approach nearer together, till the last mean is true to the nearest unit in the last figure, giving us ·298174 for the value of the proposed hypergeometrical series $1 - 3 + 12 - 60 + 360 - 2520 + 20160 - \&c.$

And in like manner are we to proceed with any other series whose terms have alternate signs.

Royal Military Academy,
Woolwich, May, 1780.

POSTSCRIPT.

Since the foregoing method was discovered, and made known to several friends, two passages have been offered to my consideration, which I shall here mention, in justice to their authors, Sir I. Newton, and the late learned Mr. Euler.

The first of these is in Sir Isaac's letter to Mr. Oldenburg, dated October 24, 1676, and may be seen in Collins's *Commercium Epistolicum*, p. 177, the last paragraph near the bottom of the page, namely, *Per seriem Leibnitii etiam*, si ultimo loco dimidium termini adjiciatur, *et alia quædam similia artificia adhibeantur, potest computum produci ad multas figuras.* The series here alluded to, is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ &c, denoting the area of the circle whose diameter is 1; and Sir Isaac here directs to add in half the last term, after having collected all the foregoing, as the means of obtaining the sum a little exacter. And this, indeed, is equivalent to taking one arithmetical mean between two successive sums, but it does not reach the idea contained in my method. It appears also, both by the other words, *et alia quædam similia artificia adhibeantur*, contained in the above extract, and by these, *alias artes adhibuissem*, a little higher up in the same pa. 177, that Sir Isaac Newton had several other contrivances for obtaining the sums of slowly converging series; but what they were, it may perhaps be now impossible to determine.

The other is a passage in the *Novi Comment. Petropol.* tom. v. p. 226, where Mr. Euler gives an instance of taking one set of arithmetical means between a series of quantities which are gradually too little and too great, to obtain a nearer value of the sum of a series in question. But neither does this reach the idea contained in our method. However, I have thought it but justice to the characters of these two eminent men, to make this mention of their ideas, which have some relation to my own, though unknown to me at the time of my discovery.

TRACT IX.

A METHOD OF SUMMING THE SERIES $a + bx + cx^2 + dx^3 + ex^4 +$ &c, WHEN IT CONVERGES VERY SLOWLY, NAMELY, WHEN x IS NEARLY EQUAL TO 1, AND THE COEFFICIENTS a, b, c, d, &c, DECREASE VERY SLOWLY: THE SIGNS OF ALL THE TERMS BEING POSITIVE.

ARTICLE 1.

WHEN there is occasion to find the sum of such series as that above-mentioned, having the coefficients a, b, c, d, &c, of the terms, decreasing very slowly, and the converging quantity x pretty large; the sum cannot be found by collecting the terms together, on account of the immense number of them which it would be necessary to collect; neither can it be summed by means of the differential series, because the powers of the quantity $\frac{x}{1-x}$ will then diverge faster than the differential coefficients converge. In such case then we must have recourse to some other method of transforming it into another series which shall converge faster. The following is a method by which the proposed series is changed into another, which converges so much the quicker as the original series is slower.

2. The method is thus. Assume $\frac{a^2}{\mathrm{D}} =$ the given series

$$a + bx + cx^2 + dx^3 + \&c. \text{ Then shall}$$

D be $= \frac{a^2}{a+bx+cx^2+\&c}$; which, by actual division, is $= a - bx - (c - \frac{b^2}{a})x^2 - (d - \frac{2bc}{a} + \frac{b^3}{a^2})x^3 - (e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3})\, x^4 -$ &c. Consequently a^2 divided by this series will be equal to the series proposed; and this new series will be very easily summed, in comparison with the original one, because all the coefficients after the second term are evidently very small; and indeed they are so much the smaller, and fitter for summation, by how much the coefficients of the original series are nearer to equality; so that, when these a, b, c, d, &c, are quite equal, then the third, fourth, &c, coefficients of the new series become equal to nothing, and the sum accurately equal to $\frac{a^2}{a-bx} = \frac{a^2}{a-ax} = \frac{a}{1-x}$; which is also known to be true from other principles.

3. Though the first two terms, $a - bx$, of the new series, be very great in comparison with each of the following terms, yet these latter may not always be small enough to be entirely rejected when much accuracy is required in the summation. And in such case it will be necessary to collect a great number of them, to obtain their sum pretty near the truth; because their rate of converging is but small, it being indeed pretty much like to the rate of the original series, but only the terms, each to each, are much smaller, and that commonly in a degree to the hundredth or thousandth part.

4. The resemblance of this new series however, beginning with the third term, to the original one, in the law of progression, intimates to us that it will be best to sum it in the very same manner as the former. Hence then putting

$$a' = c - \frac{b^2}{a},$$

$$b' = d - \frac{2bc}{a} + \frac{b^3}{a^2},$$

$$c' = e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3},$$

&c,

and consequently the proposed series $a + bx + cx^2 + \&c$,

$$= \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 \&c} = \frac{a^2}{a - bx - x \times (a' + b'x + c'x^2 \&c)},$$

by taking the sum of the series $a' + b'x + c'x^2 + \&c$, by the very same theorem as before, the sum s of the original series will then be expressed thus, s =

$$\cfrac{a^2}{a - bx - \cfrac{a'^2 x^2}{a' - b'x - (c' - \frac{b'^2}{a'})x^2 - (d' - \frac{2b'c'}{a'} + \frac{b'^3}{a'^2})x^3 - \&c}};$$

where the series in the last denominator, having again the same properties with the former one, will have its first two terms very large in respect of the following terms. But these first two terms, $a' - b'x$, are themselves very small in comparison with the first two, $a - bx$, of the former series; and therefore much more are the third, fourth, &c, terms of this last denominator, very small in comparison with the same $a - bx$: for which reason they may now perhaps be small enough to be neglected.

5. But if these last terms be still thought too large to be omitted, then find the sum of them by the very same theorem: and thus proceed, by repeating the operation in the same manner, till the required degree of accuracy is obtained. Which it is evident, will happen after a small number of repetitions, because that, in each new denominator, the third, fourth, &c, terms, are commonly depressed, in the scale of numbers, two or three places lower than the first and second terms are. And the general theorem, denoting the sum s when the process is continually repeated, will be this,

$$\cfrac{aa}{a - bx - \cfrac{a'a'xx}{a' - b'x - \cfrac{a''a''xx}{a'' - b''x - \cfrac{a'''a'''xx}{a''' - b'''x - \cfrac{a^{\text{iv}}a^{\text{iv}}xx}{a'''' - b''''x \ \&c.}}}}}$$

6. But the general denominator D in the fraction $\frac{a^2}{D}$, which is assumed for the value of s or $a + bx + cx^2 + \&c$, may be otherwise found as below; from which the general law of

the values of the coefficients will be rendered visible. Assume s or $a + bx + cx^2 + \&c$,

$$\text{or } \frac{a^2}{\text{D}} = \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 - \&c}; \text{ then shall}$$

$$\begin{aligned}
a^2 = {} & a + bx + cx^2 + \&c \times a - bx - a'x^2 - b'x^3 - \&c \\
= {} & a^2 + abx + acx^2 + adx^3 + aex^4 + afx^5 + \&c \\
& \quad - ab \quad - bb \quad - bc \quad - bd \quad - be \\
& \qquad\qquad - a'a \quad - a'b \quad - a'c \quad - a'd \\
& \qquad\qquad\qquad\qquad - b'a \quad - b'b \quad - b'c \\
& \qquad\qquad\qquad\qquad\qquad\quad - c'a \quad - c'b \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad - d'a.
\end{aligned}$$

Hence, by equating the coefficients of the like terms to nothing, we obtain the following general values:

$$\begin{aligned}
a' &= c - \frac{bb}{a}, \\
b' &= d - \frac{ba' + cb}{a}, \\
c' &= e - \frac{bb' + ca' + db}{a}, \\
d' &= f - \frac{bc' + cb' + da' + eb}{a}, \\
e' &= g - \frac{bd' + cc' + db' + ea' + fb}{a}, \\
&\&c.
\end{aligned}$$

Where the values of the coefficients are the same in effect as before found, but here the law of their continuation is manifest.

7. To exemplify now the use of this method, let it be proposed to sum the very slow series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \&c.$ when $x = \frac{9}{10} = \cdot 9$, denoting the hyp. log. of $\frac{1}{1-x}$, or, in this case, of 10.

Now it will be proper, in the first place, to collect a few of the first terms together, and then apply the theorem to the remaining terms, which, by being nearer to an equality than the terms are near the beginning of the series, will be fitter to receive the application of the theorem. Thus to collect the first 12 terms :

No.	Powers of x.	The first 12 terms, found by dividing x, x^2, x^3, &c, by the numbers 1, 2, 3, &c,
1	·9 - - - - -	·9
2	·81 - - - -	·405
3	·729 - - - -	·243
4	·6561 - - -	·164025
5	·59049 - - -	·118098
6	·531441 - -	·0885735
7	·4782969 - -	·06832812857
8	·43046721 - -	·05380840125
9	·387420489 -	·043046721
10	·3486784401 -	·03486784401
11	·31381059609 -	·02852823601
12	·282429536481	·02353579471
13	·2541865828329	2·17081162555 the sum of 12 terms.

Then we have to find the sum of the rest of the terms after these first 12, namely of $x^{13} \times (\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \frac{1}{16}x^3 + \&c)$, in which $x = \cdot 9$, and $x^{13} = \cdot 2541865828329$; also $a = \frac{1}{13}$, $b = \frac{1}{14}$, $c = \frac{1}{15}$, &c, and the first application of our rule, gives, for the value of $\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \&c$, or s,

$$\frac{(\frac{1}{13})^2 = \cdot 005917159763 \;\&c}{\cdot 012637363 - x^2 \times : \cdot 000340136 + \cdot 000279397x + \cdot 000233592x^2 + \&c};$$

the second gives

$$\frac{\cdot 00591715976}{\cdot 012637363 - \dfrac{\cdot 000340136^2 x^2}{\cdot 000088678 - x^2 \times : \cdot 000004086 + \cdot 000003060x + \&c}};$$

the third gives

$$\frac{\cdot 00591715976}{\cdot 012637363 - \dfrac{\cdot 000340136^2 x^2}{\cdot 000088678 - \dfrac{\cdot 000004087^2 x^2}{\cdot 000001333 - x^2 \times : \cdot 000000089 + \&c}}};$$

the fourth gives

$$\frac{\cdot 00591715976}{\cdot 012637363 - \dfrac{\cdot 000340136^2 x^2}{\cdot 000088678 - \dfrac{000004087^2 x^2}{\cdot 000001333 - \dfrac{\cdot 000000089^2 x^2}{\cdot 000000344}}}}.$$

Or, when the terms in the numerators are squared, it is

$$\cfrac{\cdot 00591715976}{\cdot 012637363 - \cfrac{\cdot 000000093710985}{\cdot 000088678 - \cfrac{\cdot 0000000000013526212}{\cdot 000001333 - \cfrac{\cdot 000000000000006416}{\cdot 0000000344}}}}.$$

Or, by omitting a proper number of ciphers, it is

$$\cfrac{\cdot 0591715976}{\cdot 12637363 - \cfrac{\cdot 0093710985}{\cdot 88678 - \cfrac{\cdot 013526212}{\cdot 1333 - \cfrac{\cdot 006416}{\cdot 344 - z}}}}.$$

An unknown quantity z is here placed after the last denominator, to represent the small quantity to be subtracted from the said denominator 344. Now, rejecting the small quantity z, and beginning at the last fraction to calculate, their values will be as here ranged in the first annexed column.

Fractions.	1. Ra.	2. Ra.	3. Ratio.	4. Ratio.
·518200000	425			
1218931	106	4·01		
11799	63	1·68	2·39	
187	$\frac{187}{z}$	$\frac{63z}{187}$	$\frac{1\cdot 68 \times 187}{63z}$	$\frac{2\cdot 39 \times 63z}{1\cdot 68 \times 187}$
z				
$4\frac{3}{10}$	44	1·43	1·18	2·03

placing z below them for the next unknown fraction. Divide then every fraction by the next below it, placing the quotients or ratios in the next column. Then take the quotients or ratios of these; and so on till the last ratio $\frac{2\cdot 39 \times 63z}{1\cdot 68 \times 187}$; which, from the nature of the series of the first terms of every column, must be less than the next preceding one 2·39: consequently z must be less than $\frac{1\cdot 68 \times 187}{63}$, or less than 5. But, from the nature of the series in the vertical row, or column of first ratios, $\frac{187}{z}$ must be less than 63; and consequently z must be greater than $\frac{187}{63}$, or greater than 3. Since then

z is less than 5 and greater than 3, it is probable that the mean value 4 is near the truth: and accordingly taking 4 for z, or rather 4·3, as z appears to be nearer 5 than 3, and taking the continual ratios, as placed along the last line of the table, their values are found to accord very well with the next preceding numbers, both in the columns and oblique rows.

Hence, using ·043 for z in the denominator $·344-z$ of the last fraction of the general expression, and computing from the bottom, upwards through the whole, the quotients, or values of the fractions, in the inverted order, will be

213
12079
1223397
·518414000

of which the last must be nearly the value of the series $\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 +$ &c, when $x = ·9$.

Then this value ·518414 of the series, being multiplied by x^{13} or ·2541865828329, gives ·1317738 for the sum of all the terms of the original series after the first 12 terms; to which therefore the sum of the first 12 terms, or 2·17081162, being added, we have 2·30258542 for the sum of the original series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. Which value is true within about 3 in the 8th place of figures, the more accurate value being 2·30258509 &c, or the hyp. log. of 10.

N. B. By prop. 8 Stirling's Summat.; and by cor. 4, p. 65 Simpson's Dissert. the series $a + bx + cx^2 + dx^3 +$ &c, transforms to

$$\frac{1}{1-x} \times [a - \text{D}(\frac{x}{1-x}) + \text{D}'(\frac{x}{1-x})^2 - \text{D}''(\frac{x}{1-x})^3 + \text{D}'''(\frac{x}{1-x})^4 -].$$

And thus the series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$ &c, becomes

$$\frac{x}{1-x} \times [1 - \tfrac{1}{2}(\frac{x}{1-x}) + \tfrac{1}{3}(\frac{x}{1-x})^2 - \tfrac{1}{4}(\frac{x}{1-x})^3 + \text{ \&c}],$$ which

may be summed by our method.

TRACT X.

THE INVESTIGATION OF CERTAIN EASY AND GENERAL RULES, FOR EXTRACTING ANY ROOT OF A GIVEN NUMBER.

1. The roots of given numbers are commonly to be found, with much ease and expedition, by means of logarithms, when the indices of such roots are simple numbers, and the roots are not required to a great number of figures. And the square or cubic roots of numbers, to a good practical degree of accuracy, may be obtained, by inspection, by means of my tables of squares and cubes, published by order of the Commissioners of Longitude, in the year 1781. But when the indices of such roots are complex or irrational numbers; or when the roots are required to be found to a great many places of figures; it is necessary to make use of certain approximating rules, by means of the ordinary arithmetical computations. Such rules as are here alluded to, have only been discovered since the great improvements in the modern algebra: and the persons who have best succeeded in their enquiries after such rules, have been successively Sir Isaac Newton, Mr. Raphson, M. de Lagney, and Dr. Halley; who have shown, that the investigation of such theorems is also useful in discovering rules for approximating to the roots of all sorts of compound algebraical equations, to which the former rules, for the roots of all simple equations, bear a considerable affinity. It is presumed that the following short tract contains some advantages over any other method that has hitherto been given, both as to the ease and universality of the conclusions, and the general way in which the investigations are made: for here, a theorem is discovered, which, though it be general for all roots whatever, is at the same time

very accurate, and so simple and easy to use and to keep in mind, that nothing more so can be desired or hoped for; and further, that instead of searching out rules severally for each root, one after another, our investigation is at once for any indefinite possible root, by whatever quantity the index is expressed, whether fractional, or irrational, or simple, or compound.

2. In every theorem, or rule, here investigated,

N denotes the given number, whose root is sought,

n the index of that root,

a its nearest rational root, or a^n the nearest rational power to N, whether greater or less,

x the remaining part of the root sought, which may be either positive or negative, namely, positive when N is greater than a^n, otherwise negative. Hence then, the given number

N is $= (a + x)^n$, and the required root $\mathrm{N}^{\frac{1}{n}} = a + x$.

3. Now, for the first rule, expand the quantity $(a+x)^n$ by the binomial theorem, so shall we have

$$\mathrm{N} = (a + x)^n = a^n + na^{n-1}x + n \cdot \frac{n-1}{2}a^{n-2}x^2 + \&c.$$

Subtract a^n from both sides, so shall

$$\mathrm{N} - a^n = n\,a^{n-1}x + n \cdot \frac{n-1}{2}a^{n-2}x^2 + \&c.$$

Divide by na^{n-1}, so shall

$$\frac{\mathrm{N}-a^n}{n\,a^{n-1}} \text{ or } \frac{\mathrm{N}-a^n}{n\,a^n} \times a = x + \frac{n-1}{2} \cdot \frac{x^2}{a} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a^2} + \&c.$$

Here, on account of the smallness of the quantity x in respect of a, all the terms of this series, after the first term, will be very small, and may therefore be neglected without much error, which gives $\frac{\mathrm{N}-a^n}{n\,a^n}a$ for a near value of x, being only a small matter too great. And consequently

$a + x = \frac{\mathrm{N}+(n-1)a^n}{n\,a^n}a$ is nearly $= \mathrm{N}^{\frac{1}{n}}$ the root sought. And this may be accounted the first theorem.

4. Again, let the equation $N = a^n + n a^{n-1} x + \&c$, be multiplied by $n - 1$, and a^n added to each side, so shall we have

$(n-1) N + a^n = n a^n + (n-1) . n a^{n-1} x + \&c$, for a divisor: Also multiply the sides of the same equation by a and subtract a^{n+1} from each, so shall we have

$(N - a^n) a = n a^n . x + n . \frac{n-1}{2} a^{n-1} x^2 + \&c$, for a dividend:

Divide now this dividend by the divisor, so shall

$$\frac{N - a^n}{(n-1) N + a^n} a = x - \frac{n-1}{2} . \frac{x^2}{a} + \frac{n-1}{2} . \frac{n-2}{3} . \frac{x^3}{a} + \&c.$$

Which will be nearly equal to x, for the same reason as before; and this expression is about as much too little as the former expression was too great. Consequently, by adding a, we have $a + x$ or $N^{\frac{1}{n}}$ nearly $= \frac{nNa}{(n-1) N + a^n}$, for a second theorem, and which is nearly as much in defect as the former was in excess.

5. Now because the two foregoing theorems differ from the truth by nearly equal small quantities, if we add together the two numerators and the two denominators of the foregoing two fractional expressions, namely

$\frac{N + (n-1) a^n}{n a^n} a$ and $\frac{nN}{(n-1) . N + a^n} a$, the sums will be the numerator and denominator of a new fraction, which will be much nearer than either of the former. The fraction so found is $\frac{\overline{n+1} . N + \overline{n-1} . a^n}{\overline{n-1} . N + \overline{n+1} . a^n} a$; which will be very nearly equal to $N^{\frac{1}{n}}$, or $a + x$, the root sought; for, by division, it is found to be equal to $a + x * - \frac{n-1}{2} . \frac{n+1}{6} . \frac{x^3}{a^2} + \&c$, where the term is wanting which contains the square of x, and the following terms are very small. And this is the third theorem.

6. A fourth theorem might be found by taking the arithmetical mean between the first and second, which would be

$(\frac{\text{N}+\overline{n-1}\,.\,a^n}{n a^n} + \frac{n\text{N}}{\overline{n-1}\,.\,\text{N}+a^n}) \times \frac{a}{2}$; which will be nearly of the same value, though not so simple, as the third theorem; for this arithmetical mean is found equal to

$$a + x * + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a^2} + \&c.$$

7. But the third theorem may be investigated in a more general way, thus: Assume a quantity of this form $\frac{p\text{N} + q\,a^n}{q\text{N} + p\,a^n}a$, with coefficients p and q to be determined from the process; the other letters N, a, n, representing the same things as before; then divide the numerator by the denominator, and make the quotient equal to $a + x$, so shall the comparison of the coefficients determine the relation between p and q required. Thus,

$$p\text{N} + q\,a^n = (p+q)a^n + pn\,a^{n-1}x + pn\,.\,\frac{n-1}{2}a^{n-2}x^2 + \&c.$$

$$q\text{N} + pa^n = (p+q)a^n + qn\,a^{n-1}x + qn\,.\,\frac{n-1}{2}a^{n-2}x^2 + \&c.$$

then dividing the former of these by the latter, we have

$$\frac{p\text{N}+qa^n}{q\text{N}+pa^n}a \text{ or } a+x = a + \frac{p-q}{p+q}nx + \frac{p-q}{p+q}n\left(\frac{n-1}{2} - \frac{qn}{p+q}\right)\frac{x^2}{a} + \&c.$$

Then, by equating the corresponding terms, we obtain these three equations

$$a = a,$$

$$\frac{p-q}{p+q}n = 1,$$

$$\frac{n-1}{2} - \frac{qn}{p+q} = 0.$$

From which we find $\frac{p-q}{p+q} = \frac{1}{n}$ and $p : q :: n+1 : n-1$.

So that, by substituting $n+1$ and $n-1$, or any quantitities proportional to them, for p and q, we shall have

$\frac{\overline{n+1}\,.\,\text{N}+\overline{n-1}\,.\,a^n}{\overline{n-1}\,.\,\text{N}+\overline{n+1}\,.\,a^n}a$ for the value of the assumed quantity

$\frac{p\text{N}+qa^n}{q\text{N}+pa^n}a$, which is supposed nearly equal to $a+x$, the required root of the quantity N.

8. Now this third theorem $\frac{n+1\,.\,\text{N}+n-1\,.\,a^n}{n-1\,.\,\text{N}+n+1\,.\,a^n}a = \text{N}^{\frac{1}{n}}$, which is general for roots, whatever be the value of n, and whether a^n be greater or less than N, includes all the rational formulas of De Lagney and Halley, which were separately investigated by them; and yet this general formula is perfectly simple and easy to apply, and easier kept in mind than any one of the said particular formulas. For, in words at length, it is simply this: to $n+1$ times N add $n-1$ times a^n, and to $n-1$ times N add $n+1$ times a^n, then the former sum multiplied by a and divided by the latter sum, will give the root $\text{N}^{\frac{1}{n}}$ nearly; or, as the latter sum is to the former sum, so is a, the assumed root, to the required root, nearly. Where it is to be observed that a^n may be taken either greater or less than N, but that the nearer it is to it, the better.

9. By substituting for n, in the general theorem, severally the numbers 2, 3, 4, 5, &c, we shall obtain the following particular theorems, as adapted to the 2d, 3d, 4th, 5th, &c, roots, namely, for the

2d or square root,	$\frac{3\text{N}+a^2}{\text{N}+3a^2}a$	- - - -	$=\text{N}^{\frac{1}{2}}$
3d or cube root,	$\frac{4\text{N}+2a^3}{2\text{N}+4a^3}a$, or	$\frac{2\text{N}+a^3}{\text{N}+2a^3}a$	$=\text{N}^{\frac{1}{3}}$
4th root - - -	$\frac{5\text{N}+3a^4}{3\text{N}+5a^4}a$	- - - -	$=\text{N}^{\frac{1}{4}}$
5th root - - -	$\frac{6\text{N}+4a^5}{4\text{N}+6a^5}a$, or	$\frac{3\text{N}+2a^5}{2\text{N}+3a^5}a$	$=\text{N}^{\frac{1}{5}}$
6th root - - -	$\frac{7\text{N}+5a^6}{5\text{N}+7a^6}a$	- - - -	$=\text{N}^{\frac{1}{6}}$
7th root - - -	$\frac{8\text{N}+6a^7}{6\text{N}+8a^7}a$, or	$\frac{4\text{N}+3a^7}{3\text{N}+4a^7}a$	$=\text{N}^{\frac{1}{7}}$
		&c.	

10. To exemplify now our formula, let it be first required to extract the square root of 365. Here $N = 365$, $n = 2$, the nearest square is 361, whose root is 19.

$$\text{Hence } 3N + a^2 = 3 \times 365 + 361 = 1456,$$
$$\text{and } N + 3a^2 = 365 + 3 \times 361 = 1448;$$

then as $1448 : 1456 :: 19 : \frac{19 \times 182}{181} = 19\frac{19}{181} = 19 \cdot 10497$ &c.

Again, to approach still nearer, substitute this last found root $\frac{19 \times 182}{181}$ for a, the values of the other letters, remaining as before, we have $a^2 = \frac{19^2 \times 182^2}{181^2} = \frac{3458^2}{181^2}$; then

$$3N + a^2 = 3 \times 365 + \frac{3458^2}{181^2} = \frac{47831059}{32761},$$
$$N + 3a^2 = 365 + \frac{3 \times 3458^2}{181^2} = \frac{47831057}{32761}; \text{ hence}$$

$47831057 : 47831059 :: \frac{19 \times 182}{181}$ or $\frac{3458}{181} : \frac{3458 \times 47831059}{181 \times 47831057}$ = the root of 365 very exact, which being brought into decimals, would be true to about 20 places of figures.

11. For a second example, let it be proposed to double the cube, or to find the cube root of the number 2.

Here $N = 2$, $n = 3$, the nearest root $a = 1$, also $a^3 = 1$.

$$\text{Hence } 2N + a^3 = 4 + 1 = 5,$$
$$\text{and } N + 2a^3 = 2 + 2 = 4;$$

then as $4 : 5 :: 1 : \frac{5}{4} = 1 \cdot 25$ = the first approximation.

Again, take $a = \frac{5}{4}$, and consequently $a^3 = \frac{125}{64}$;

$$\text{Hence } 2N + a^3 = 4 + \frac{125}{64} = \frac{381}{64},$$
$$\text{and } N + 2a^3 = 2 + \frac{250}{64} = \frac{378}{64};$$

then $378 : 381$, or as $126 : 127 :: \frac{5}{4} : \frac{5}{4} \times \frac{127}{126} = \frac{635}{504} = 1 \cdot 259921$, for the cube root of 2, which is true in the last figure.

And by taking $\frac{635}{504}$ for the value of a, and repeating the process, a great many more figures may be found.

12. For a third example let it be required to find the 5th root of 2.

Here $N = 2$, $n = 5$, the nearest root $a = 1$.

$$\text{Hence } 3N + 2a^5 = 6 + 2 = 8,$$
$$\text{and } 2N + 3a^5 = 4 + 3 = 7;$$

then as $7 : 8 :: 1 : \frac{8}{7} = 1\frac{1}{7}$ for the first approximation.

Again, taking $a = \frac{8}{7}$, we have

$$3N + 2a^5 = 6 + \frac{65536}{16807} = \frac{166378}{16807},$$
$$2N + 3a^5 = 4 + \frac{98304}{16807} = \frac{165532}{16807};$$

then $165532 : 166378 :: \frac{8}{7} : \frac{8}{7} \times \frac{83189}{82766} = \frac{4}{7} \times \frac{83189}{41383} = \frac{332756}{289681}$ $= 1{\cdot}148698$ &c, for the 5th root of 2, true in the last figure.

13. To find the 7th root of $126\frac{1}{5}$.

Here $N = 126\frac{1}{5}$, $n = 7$, the nearest root $a = 2$, also $a^7 = 128$.

$$\text{Hence } 4N + 3a^7 = 504\tfrac{4}{5} + 384 = 888\tfrac{4}{5} = \frac{4444}{5},$$
$$\text{and } 3N + 4a^7 = 378\tfrac{3}{5} + 512 = 890\tfrac{3}{5} = \frac{4453}{5};$$

then $4453 : 4444 :: 2 : \frac{8888}{4453} = 1{\cdot}995957$, root very exact by one operation, being true to the nearest unit in the last figure.

14. To find the 365th root of $1{\cdot}05$, or the amount of 1 pound for 1 day, at 5 per cent. per annum, compound interest.

Here $N = 1{\cdot}05$, $n = 365$, $a = 1$ the nearest root.

$$\text{Hence } 366N + 364a = 748{\cdot}3,$$
$$\text{and } 364N + 366a = 748{\cdot}2;$$

then as $748\cdot2 : 748\cdot3 :: 1 : \frac{7483}{7482} = 1\frac{1}{7482} = 1\cdot00013366$, the root sought, very exact at one operation.

15. Required to find the value of the quantity $(5\frac{1}{4})^{\frac{2}{3}}$ or $(\frac{21}{4})^{\frac{2}{3}}$. Now this may be done two ways; either by finding the $\frac{2}{3}$ power or $\frac{3}{2}$ root of $\frac{21}{4}$ at once; or else by finding the 3d or cubic root of $\frac{21}{4}$, and then squaring the result.

By the first way:—Here it is easy to see that a is nearly $= 3$, because $3^{\frac{3}{2}} = \sqrt{27} = 5 +$ some small fraction. Hence, to find nearly the square root of 27, or $\sqrt{27}$, the nearest power to which is $25 = a^2$ in this case:

$$\text{Hence } 3\text{N} + a^2 = 3 \times 27 + 25 = 106,$$
$$\text{and } \text{N} + 3a^2 = 27 + 3 \times 25 = 102;$$

then $102 : 106$, or $51 : 53 :: 5 : \frac{5 \times 53}{51} = \frac{265}{51} = \sqrt{27}$ nearly.

Then having $\text{N} = \frac{21}{4}$, $n = \frac{3}{2}$, $a = 3$, and $a^{\frac{3}{2}} = \frac{265}{51}$ nearly;

$$\text{it will be } \tfrac{5}{2}\text{N} + \tfrac{1}{2}a^{\frac{3}{2}} = \frac{5}{2} \times \frac{21}{4} + \frac{1}{2} \times \frac{265}{51} = \frac{6415}{408},$$

$$\text{and } \tfrac{1}{2}\text{N} + \tfrac{5}{2}a^{\frac{3}{2}} = \frac{1}{2} \times \frac{21}{4} + \frac{5}{2} \times \frac{265}{51} = \frac{6371}{408},$$

hence $6371 : 6415 :: 3 : \frac{19245}{6371} = 3\frac{134}{6371} = 3\cdot020719$, for the value of the quantity sought nearly, by this way.

Again, by the other method, in finding first the value of $(\frac{21}{4})^{\frac{1}{3}}$, or the cube root of $\frac{21}{4}$. It is evident that 2 is the nearest integer root, being the cube root of $8 = a^3$.

$$\text{Hence } 2\text{N} + a^3 = \tfrac{21}{2} + 8 = \tfrac{74}{4},$$
$$\text{and } \text{N} + 2a^3 = \tfrac{21}{4} + 16 = \tfrac{85}{4};$$

then $85 : 74 :: 2 : \frac{148}{85}$, or $= \frac{7}{4}$ nearly. Then taking $\frac{7}{4}$ for a,

$$\text{we have } 2\text{N} + a^3 = \frac{21}{2} + \frac{343}{64} = \frac{1015}{64},$$

$$\text{and } \text{N} + 2a^3 = \frac{21}{4} + \frac{2.343}{64} = \frac{1022}{64};$$

hence 1022 : 1015, or 146 : 145 $:: \frac{7}{4} : \frac{7}{4} \times \frac{145}{146} = (\frac{21}{4})^{\frac{1}{3}}$ nearly.

Consequently the square of this, or $(\frac{21}{4})^{\frac{2}{3}}$ will be $=$ $\frac{7^2}{4^2} \times \frac{145^2}{146^2} = \frac{1030225}{341056} = 3\frac{7057}{341056} = 3{\cdot}020690$, the quantity sought more nearly, being true in the last figure.

TRACT XI.

A NEW METHOD OF FINDING, IN FINITE AND GENERAL TERMS, NEAR VALUES OF THE ROOTS OF EQUATIONS OF THIS FORM, $x^n - px^{n-1} + qx^{n-2} - \&c = 0$; NAMELY, HAVING THE TERMS ALTERNATELY PLUS AND MINUS.

1. THE following is one method more, to be added to the many we are already possessed of, for determining the roots of the higher equations. By means of it we readily find a root, which is sometimes accurate; and when not so, it is at least near the truth, and that by an easy finite formula, which is general for all equations of the above form, and of the same dimension, provided that root be a real one. This is of use for depressing the equation down to lower dimensions, and thence for finding all the roots, one after another, when the formula gives the root sufficiently exact; and when not, it serves as a ready means of obtaining a near value of a root, by which to commence an approximation still nearer, by the previously known methods of Newton, or Halley, or others. This method is further useful in elucidating the nature of equations, and certain properties of numbers; as will appear in some of the following articles. We have already easy methods for finding the roots of simple and quadratic equations.

I shall therefore begin with the cubic equation, and treat of each order of equations separately, in ascending gradually to the higher dimensions.

2. Let then the cubic equation $x^3 - px^2 + qx - r = 0$ be proposed. Assume the root $x = a$, either accurately or approximately, as it may happen, so that $x - a = 0$, accurately or nearly. Raise this $x - a = 0$ to the third power, the same dimension with the proposed equation,

so shall $x^3 - 3ax^2 + 3a^2x - a^3 = 0$;

but the proposed equation is $x^3 - px^2 + qx - r = 0$;

therefore the one of these is equal to the other. But the first term (x^3) of each is the same; and hence, if we assume the second terms equal between themselves, it will follow that the sum of the two remaining terms will also be equal, and give a simple equation by which the value of x is determined. Thus, $3ax^2$ being $= px^2$, or $a = \frac{1}{3}p$, and $3a^2x - a^3 = qx - r$, we hence have

$x = \frac{a^3 - r}{3a^2 - q} = \frac{(\frac{1}{3}p)^3 - r}{3 \times (\frac{1}{3}p)^2 - q} = \frac{p^3 - 27r}{p^2 - 3q} \times \frac{1}{9}$ by substituting $\frac{1}{3}p$, the value of a, instead of it.

3. Now this value of x here found, will be the middle root of the proposed cubic equation. For because a is assumed nearly or accurately equal to x, and also equal to $\frac{1}{3}p$, therefore x is $= \frac{1}{3}p$ nearly or accurately, that is, $\frac{1}{3}$ of the sum of the three roots, to which the coefficient p, of the second term of the equation, is always equal; and thus, being a medium among the three roots, it will be either nearly or accurately equal to the middle root of the proposed equation, when that root is a real one.

4. Now this value of x will always be the middle root *accurately*, whenever the three roots are in arithmetical progression; otherwise, only *approximately*. For when the three roots are in arithmetical progression, $\frac{1}{3}p$ or $\frac{1}{3}$ of their sum, it is well known, is equal to the middle term or root. In the other cases, therefore, the above-found value of x is only *near* the middle root.

5. When the roots are in arithmetical progression, because the middle term or root is then $=\frac{1}{3}p$, and also $=\frac{1}{9}\times\frac{p^3-27r}{p^2-\ 3q}$, therefore $\frac{1}{3}p=\frac{1}{9}\times\frac{p^3-27r}{p^2-\ 3q}$, or $2p^3=9pq-27r=9\times(pq-3r)$, an equation expressing the general relation of p, q, and r; where p is the sum of any three terms in arithmetical progression, q the sum of their three rectangles, and r the product of all the three. For, in any equation, the coefficient p of the second term, is the sum of the roots; the coefficient q of the third term, is the sum of the rectangles of the roots; and the coefficient r of the fourth term, is the sum of the solids of the roots, which in the case of the cubic equation is only one:—Thus, if the roots, or arithmetical terms, be 1, 2, 3. Here $p = 1 + 2 + 3 = 6$, $q = 1 \times 2 + 1 \times 3 + 2 \times 3 = 2 + 3 + 6 = 11$, $r = 1 \times 2 \times 3 = 6$; then $2p^3 = 2 \times 6^3 = 432$, and $9 \times (pq - 3r) = 9 \times 48 = 432$ also.

6. To illustrate now the rule $x = \frac{1}{9} \times \frac{p^3-27r}{p^2-\ 3q}$ by some examples; let us in the first place take the equation $x^3-6x^2+11x-6=0$. Here $p=6$, $q=11$, and $r=6$; consequently $x=\frac{1}{9}\times\frac{p^3-27r}{p^2-\ 3q}=\frac{1}{9}\times\frac{6^3-27\times 6}{6^2-\ 3\times 11}=\frac{8-\ 6}{12-11}=\frac{2}{1}=2.$ This being substituted for x in the given equation, makes all the terms to vanish, and therefore it is an exact root, and the roots will be in arithmetical progression. Dividing therefore the given equation by $x - 2 = 0$, the quotient is $x^2 - 4x + 3 = 0$, the roots of which quadratic equation are 3 and 1, which are the other two roots of the proposed equation $x^3 - 6x^2 + 11x - 6 = 0$.

7. If the equation be $x^3 - 39x^2 + 479x - 1881 = 0$; we shall have $p = 39$, $q = 479$, and $r=1881$; then $x = \frac{1}{9} \times \frac{p^3-27r}{p^2-\ 3q}=\frac{1}{9}\times\frac{39^3-27\times 1881}{39^2-\ 3\times\ 479}=\frac{13^3-1881}{13^2-3\times 479}=\frac{316}{28}=\frac{79}{7}=11\frac{2}{7}$ Then, substituting $11\frac{2}{7}$ for x in the proposed equation, the

negative terms are found to exceed the positive terms by 5, thus showing that $11\frac{2}{7}$ is very near, but something above, the middle root, and that therefore the roots are not in arithmetical progression. It is therefore probable that 11 may be the true value of the root, and on trial it is found to succeed. Then dividing $x^3 - 39x^2 + 479x - 1881$ by $x - 11$, the quotient is $x^2 - 28x + 171 = 0$, the roots of which quadratic equation are 9 and 19, the two other roots of the proposed equation.

8. If the equation be $x^3 - 6x^2 + 9x - 2 = 0$; we shall have $p = 6$, $q = 9$, and $r = 2$; then $x =$

$$\frac{1}{9} \times \frac{p^3-27r}{p^2-3q} = \frac{1}{9} \times \frac{6^3-27\times 2}{6^2-3\times 9} = \frac{2^3-2}{12-9} = \frac{6}{3} = 2.$$

This value of x being substituted for it in the proposed equation, causes all the terms to vanish, as it ought, thus showing that 2 is the middle root, and that the roots are in arithmetical progression. Accordingly, dividing the given quantity $x^3 - 6x^2 + 9x - 2$ by $x - 2$, the quotient is $x^2 - 4x + 1 = 0$, a quadratic equation, whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, the two other roots of the equation proposed.

9. If the equation be $x^3 - 5x^2 + 5x - 1 = 0$; we shall have $p = 5$, $q = 5$, and $r = 1$; then $x =$

$$\frac{1}{9} \times \frac{5^3-27\times 1}{5^2-3\times 5} = \frac{1}{9} \times \frac{125-27}{25-15} = \frac{1}{9} \times \frac{98}{10} = \frac{49}{45} = 1\tfrac{4}{45}.$$

From which one might guess the root ought to be 1, and which being tried, is found to succeed. But without such trial, we might make use of this value $1\frac{4}{45}$, or $1\frac{1}{11}$ nearly, and approximate with it in the common way.

Having found the middle root to be 1, divide the given quantity $x^3 - 5x^2 + 5x - 1$ by $x - 1$, and the quotient is $x^2 - 4x + 1 = 0$, the roots of which are $2 + \sqrt{2}$, and $2 - \sqrt{2}$, the two other roots, as in the last article.

10. If the equation be $x^3 - 7x^2 + 18x - 18 = 0$; we shall have $p = 7$, $q = 18$, and $r = 18$; then $x =$

$$\frac{1}{9} \times \frac{7^3-27\times 18}{7^2-3\times 18} = \frac{1}{9} \times \frac{343-486}{49-54} = \frac{143}{45} = 3\tfrac{8}{45} \text{ or } 3 \text{ nearly.}$$

Then trying 3 for x, it is found to succeed. And dividing $x^3-7x^2+18x-18$ by $x-3$, the quotient is $x^2-4x+6=0$, a quadratic equation whose roots are $2+\sqrt{-2}$ and $2-\sqrt{-2}$, the two other roots of the proposed equation, which are both impossible or imaginary.

11. If the equation be $x^3-6x^2+14x-12=0$; we shall have $p=6$, $q=14$, and $r=12$; then $x=\frac{1}{9}\times\frac{6^3-27\times12}{6^2-3\times14}=\frac{1}{9}\times\frac{216-324}{36-42}=\frac{108}{54}=2$. Which being substituted for x, it is found to answer, the sum of the terms coming out $=0$. Therefore the roots are in arithmetical progression. And, accordingly, by dividing $x^3-6x^2+14x-12$ by $x-2$, the quotient is $x^2-4x+6=0$, the roots of which quadratic equation are $2+\sqrt{-2}$ and $2-\sqrt{-2}$, the two other roots of the proposed equation, and the common difference of the three roots is $\sqrt{-2}$.

12. But if the equation be $x^3-8x^2+22x-24=0$; we shall have $p=8$, $q=22$, and $r=24$; then $x=\frac{1}{9}\times\frac{8^3-27\times24}{8^2-3\times22}=\frac{1}{9}\times\frac{512-648}{64-66}=\frac{136}{18}=\frac{68}{9}=7\frac{5}{9}$. Which being substituted for x in the proposed equation, the sum of the terms differs very widely from the truth, thereby showing that the middle root of the equation is an imaginary one, as it is indeed, the three roots being 4, and $2+\sqrt{-2}$, and $2-\sqrt{-2}$.

13. In Art. 2 the value of x was determined by assuming the second terms of the two equations, equal to each other. But a like near value might be determined by assuming either the two third terms, or the two fourth terms equal.

Thus the equations being $\left\{\begin{array}{l} x^3-3ax^2+3a^2x-a^3=0, \\ x^3-px^2+qx-r=0, \end{array}\right.$

if we assume the third terms $3a^2x$ and qx equal, or $a=\sqrt{\frac{1}{3}q}$, the sums of the second and fourth terms will be equal, namely, $3ax^2+a^3=px^2+r$; and hence we find

$$x=\sqrt{\frac{a^3-r}{p-3a}}=\sqrt{\frac{(\sqrt{\frac{1}{3}q})^3-r}{p-3\sqrt{\frac{1}{3}q}}},$$

by substituting $\sqrt{\frac{1}{3}q}$ the value of a instead of it.

And if we assume the fourth terms equal, namely $a^3 = r$, or $a = \sqrt[3]{r}$, then the sums of the second and third terms will be equal, namely, $3ax - 3a^2 = px - q$; and hence $x = \frac{q - 3a^2}{p - 3a} = \frac{q - 3r^{\frac{2}{3}}}{p - 3r^{\frac{1}{3}}}$, by substituting $r^{\frac{1}{3}}$ instead of a. And either of these two formulas will give nearly the same value of the root as the first formula, at least when the roots do not differ very greatly from one another.

But if they differ very much among themselves, the first formula will not be so accurate as these two others, because that in them the roots were more complexly mixed together; for the second formula is drawn from the coefficient of the third term, which is the sum of all the rectangles of the roots; and the third formula is drawn from the coefficient of the last term, which is equal to the continual product of all the roots; while the first formula is drawn from the coefficient of the second term, which is simply the sum of the roots. And indeed the last theorem is commonly the nearest of all. So that when we suspect the roots to be very wide of each other, it may be best to employ either the second or third formula.

Thus, in the equation $x^3 - 23x^2 + 62x - 40 = 0$, whose three roots are 1, 2, and 20. Here $p = 23$, $q = 62$, $r = 40$; and by the

$$\text{1st th. } x = \frac{1}{9} \times \frac{23^3 - 27 \times 40}{23^2 - 3 \times 62} = \frac{1}{9} \times \frac{12167 - 1080}{529 - 186} = 3\tfrac{3}{5} \text{ nearly,}$$

$$\text{2d th. } x = \sqrt{\frac{(\frac{62}{3})^{\frac{3}{2}} - 40}{23 - 3\sqrt{\frac{62}{3}}}} = \sqrt{\frac{94 - 40}{23 - 12{\cdot}87}} = \sqrt{5{\cdot}34} = 2\tfrac{1}{3} \text{ nearly.}$$

$$\text{3d th. } x = \frac{62 - 3 \times 40^{\frac{2}{3}}}{23 - 3 \times 40^{\frac{1}{3}}} = \frac{62 - 35{\cdot}1}{26 - 10\frac{1}{4}} = \frac{12}{7} = 1\tfrac{5}{7} \text{ nearly.}$$

Where the two latter are much nearer the middle root (2) than the first. And the mean between these two is $2\frac{1}{42}$, which is very near to that root. And this is commonly the case; the one being nearly as much too great as the other is too little.

14. To proceed now, in like manner, to the biquadratic equation, which is of this general form

$$x^4 - px^3 + qx^2 - rx + s = 0.$$

Assume the root $x = a$, or $x - a = 0$, and raise this equation $x - a = 0$ to the fourth power, or the same height with the proposed equation, which will give
$x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = 0$; but the proposed equation is $x^4 - px^3 + qx^2 - rx + s = 0$; therefore these two are equal to each other. Now if we assume the second terms equal, namely $4a = p$, or $a = \frac{1}{4}p$, then the sums of the three remaining terms will also be equal, namely,

$6a^2x^2 - 4a^3x + a^4 = qx^2 - rx + s$; and hence
$(6a^2 - q)\,x^2 - (4a^3 - r)\,x = s - a^4$, or
$(\frac{3}{8}p^2 - q)\,x^2 - (\frac{1}{16}p^3 - r)\,x = s - \frac{1}{256}p^4$ by substituting $\frac{1}{4}p$ instead of a: then, resolving this quadratic equation, we find its roots to be thus

$$x = \frac{p^3 - 16r \pm \sqrt{[(p^3 - 16r)^2 - (\frac{3}{2}p^2 - 4q) \times (p^4 - 256s)]}}{8 \times (\frac{3}{2}p^2 - 4q)};$$

or if we put $\text{A} = \frac{3}{2}p^2 - 4q$,
$\text{B} = p^3 - 16r$,
$\text{C} = p^4 - 256s$,

the two roots will be $x = \dfrac{\text{B} \pm \sqrt{(\text{B}^2 - \text{AC})}}{8\text{A}}$.

15. It is evident that the same property is to be understood here, as for the cubic equation in Art. 3, namely, that the two roots above found, are the middle roots of the four which belong to the biquadratic equation, when those roots are real ones; for otherwise the formulæ are of no use. But however those roots will not be accurate, when the sum of the two middle roots, of the proposed equation, is equal to the sum of the greatest and least roots, or when the four roots are in arithmetical progression; because that, in this case, $\frac{1}{4}p$, the assumed value of a, is neither of the middle roots exactly, but only a mean between them.

16. To exemplify this formula $x = \dfrac{\text{B} \pm \sqrt{(\text{B}^2 + \text{AC})}}{8\text{A}}$, let the proposed equation be $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$. Then

$$\begin{aligned}
A &= \tfrac{3}{2}p^2 - 4q = 12^2 \times \tfrac{3}{2} - 4 \times 49 = 216 - 196 = 20,\\
B &= p^3 - 16r = 12^3 - 16 \times 78 = 1728 - 1248 = 480,\\
C &= p^4 - 256s = 12^4 - 256 \times 40 = 20736 - 10240 = 10496.
\end{aligned}$$

Hence $x = \frac{B \pm \sqrt{(B^2 - AC)}}{8A} = \frac{480 \pm \sqrt{(480^2 - 20 \times 10496)}}{8 \times 20} = \frac{15 \pm \sqrt{40}}{5} = 3 \pm 1\frac{1}{4}$ nearly, or $4\frac{1}{4}$ and $1\frac{3}{4}$ nearly, or nearly 4 and 2, whose sum is 6. And trying 4 and 2, they are both found to answer, and therefore they are the two middle roots.

Then $(x - 4) \times (x - 2) = x^2 - 6x + 8$, by which dividing the given equation $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, the quotient is $x^2 - 6x + 5 = 0$, the roots of which quadratic equation are 5 and 1, and which therefore are the greatest and least roots of the equation proposed.

17. If the equation be $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$; then

$$\begin{aligned}
A &= \tfrac{3}{2}p^2 - 4q = 12^2 \times \tfrac{3}{2} - 4 \times 47 = 216 - 188 = 28,\\
B &= p^3 - 16r = 12^3 - 16 \times 72 = 1728 - 1152 = 576,\\
C &= p^4 - 256s = 12^4 - 256 \times 36 = 20736 - 9216 = 11520.
\end{aligned}$$

Hence $x = \frac{B \pm \sqrt{(B^2 - AC)}}{8A} = \frac{576 \pm \sqrt{(576^2 - 28 \times 11520)}}{8 \times 28} = \frac{18 \pm 3}{7} = 3$ and $2\frac{1}{7}$, or 3 and 2 nearly; both of which answer on trial; and therefore 3 and 2 are the two middle roots.

Then $(x - 3) \times (x - 2) = x^2 - 5x + 6 = 0$, by which dividing the given quantity $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the quotient is $x^2 - 7x + 6 = 0$, the roots of which quadratic equation are 6 and 1, which therefore are the greatest and least roots of the equation proposed.

18. If the equation be $x^4 - 7x^3 + 15x^2 - 11x + 3 = 0$; then

$$\begin{aligned}
A &= \tfrac{3}{2}p^2 - 4q = 7^2 \times \tfrac{3}{2} - 4 \times 15 = 73\tfrac{1}{2} - 60 = 13\tfrac{1}{2},\\
B &= p^3 - 16r = 7^3 - 16 \times 11 = 343 - 176 = 167,\\
C &= p^4 - 256s = 7^4 - 256 \times 3 = 2401 - 768 = 1633.
\end{aligned}$$

Hence $x = \frac{B \pm \sqrt{(B^2 - AC)}}{8A} = \frac{167 \pm \sqrt{(167^2 - 13\frac{1}{2} \times 1633)}}{8 \times 13\frac{1}{2}} = \frac{167 + 76}{108} = 2\frac{1}{3}$ and $\frac{9}{10}$ nearly, or nearly 2 and 1; both which are found, on trial, to answer; and therefore 2 and 1 are the two middle roots sought.

Then $(x-2)\times(x-1)=x^2-3x+2$, by which dividing the given equation $x^4-7x^3+15x^2-11x+3=0$, the quotient is $x^2-4x+1=0$, the roots of which quadratic equation are $2+\sqrt{2}$ and $2-\sqrt{2}$, and which therefore are the greatest and least roots of the proposed equation.

19. But if the equa. be $x^4-9x^3+30x^2-46x+24=0$; then

$$\text{A}=\tfrac{3}{2}p^2-\ \ 4q=9^2\times\tfrac{3}{2}-\ \ 4\times30=\ 121\tfrac{1}{2}-\ 120=\ 1\tfrac{1}{2},$$
$$\text{B}=\ p^3-\ 16r=9^3\ \ -\ 16\times46=\ 729\ -\ 736=-7,$$
$$\text{C}=\ p^4-256s=9^4\ \ -256\times24=6561\ -6144=417.$$

Hence $x=\dfrac{\text{B}\pm\sqrt{(\text{B}^2-\text{AC})}}{8\text{A}}=\dfrac{-7\pm\sqrt{(49-625\frac{1}{2})}}{8\times1\frac{1}{2}}=$

$\dfrac{-7\pm\sqrt{-576\frac{1}{2}}}{12}$, an imaginary quantity, showing that the two middle roots are imaginary, and therefore the formula is of no use in this case, the four roots being 1, $2+\sqrt{-2}$, $2-\sqrt{-2}$, and 4.

20. And thus in other examples the two middle roots will be found when they are rational, or a near value when irrational, which in this case will serve for the foundation of a nearer approximation, to be made in the usual way.

We might also find another formula for the biquadratic equation, by assuming the last terms as equal to each other; for then the sum of the 2d, 3d, and 4th terms of each would be equal, and would form another quadratic equation, whose roots would be nearly the two middle roots of the biquadratic proposed.

21. Or a root of the biquadratic equation may easily be found, by assuming it equal to the product of two squares, as $(x-a)^2\times(x-b)^2=x^4-2(a+b)x^3+[2ab+(a+b)^2]x^2-2ab(a+b)x+a^2b^2=0$. For, comparing the terms of this with the terms of the equation proposed, in this manner, namely, making the second terms equal, then the third terms equal, and lastly the sums of the fourth and fifth terms equal, these equations will determine a near value of x by a simple equation. For those equations are

$p = 2(a + b)$, or $\frac{1}{2}p = a + b$,
$q = 2ab + (a + b)^2 = 2ab + \frac{1}{4}p^2$, or $2ab = q - \frac{1}{4}p^2$,
$rx - s = 2ab(a + b)x - a^2b^2 = \frac{1}{2}p(q - \frac{1}{4}p^2)x - \frac{1}{4}(q - \frac{1}{4}p^2)^2$.
Then the values of ab and $a + b$, found from the first and second of these equations, and substituted in the third, give $x = \frac{s - (\frac{1}{2}q - \frac{1}{8}p^2)^2}{r - p(\frac{1}{2}q - \frac{1}{8}p^2)} = \frac{64s - (4q - p^2)^2}{64r - 8p(4q - p^2)}$, a general formula for one of the roots of the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$.

22. To exemplify now this formula, let us take the same equation as in Art. 17, namely, $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the roots of which were there found to be 1, 2, 3, and 6. Then, by the last formula we shall have $x = \frac{64s - (4q - p^2)^2}{64r - 8p(4q - p^2)} = \frac{64 \times 36 - (4 \times 47 - 12^2)^2}{64 \times 72 - 96(4 \times 47 - 12^2)} = \frac{64 \times 36 - 44 \times 44}{64 \times 72 - 96 \times 44} = \frac{23}{24}$, or nearly 1, which is the least root.

23. Again, in the equation $x^4 - 7x^3 + 15x - 11x^2 + 3 = 0$, whose roots are 1, 2, $2 + \sqrt{2}$, and $2 - \sqrt{2}$, we have $x = \frac{64 \times 3 - (60 - 49)^2}{64 \times 11 - 56(60 - 49)} = \frac{64 \times 3 - 11 \times 11}{64 \times 11 - 56 \times 11} = \frac{192 - 121}{704 - 616} = \frac{71}{88} = \frac{4}{5}$ nearly, which is nearly a mean between the two least roots 1 and $2 - \sqrt{2}$ or $\frac{3}{5}$ nearly.

24. But if the equation be $x^4 - 9x^3 + 30x^2 - 46x + 24 = 0$, which has impossible roots, the four roots being 1, $2 + \sqrt{-2}$, $2 - \sqrt{-2}$, and 4; we shall have $x = \frac{64 \times 24 - (120 - 81)^2}{64 \times 46 - 72(120 - 81)} = \frac{64 \times 24 - 39 \times 39}{64 \times 46 - 72 \times 39} = \frac{1536 - 1521}{2944 - 2808} = \frac{15}{136} = \frac{1}{9}$ nearly, which is of no use in this case of imaginary roots.

25. This formula will also sometimes fail when the roots are all real. As if the equation be $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, the roots of which are 1, 2, 4, and 5. For here $x = \frac{64 \times 40 - (196 - 144)^2}{64 \times 78 - 96(196 - 144)} = \frac{64 \times 40 - 52 \times 52}{64 \times 78 - 96 \times 52} = \frac{16 \times 10 - 13 \times 13}{16 \times 19\frac{1}{2} - 24 \times 13} = \frac{160 - 169}{312 - 312} = \frac{-9}{0}$, which is of no use, being infinite.

26. For equations of higher dimensions, as the 5th, the 6th, the 7th, &c, we might, in imitation of this last method, combine other forms of quantities together. Thus, for the 5th power, we might compare it either with $(x-a)^4 \times (x-b)$, or with $(x-a)^3 \times (x-b)^2$, or with $(x-a)^3 \times (x-b) \times (x-c)$, or with $(x-a)^2 \times (x-b)^2 \times (x-c)$. And so for the other powers.

TRACT XII.

OF THE BINOMIAL THEOREM. WITH A DEMONSTRATION OF THE TRUTH OF IT IN THE GENERAL CASE OF FRACTIONAL EXPONENTS.

1. It is well known that this celebrated theorem is called *binomial*, because it contains a proposition of a quantity consisting of *two* terms, as a radix, to be expanded in a series of equal value. It is also called emphatically the Newtonian theorem, or Newton's binomial theorem, because he has commonly been reputed the author of it, as he was indeed for the case of fractional exponents, which is the most general of all, and includes all the other particular cases, of powers, or divisions, &c.

2. The binomial, as proposed in its general form, was, by Newton, thus expressed $\overline{\mathrm{P}+\mathrm{PQ}}^{\frac{m}{n}}$; where P is the first term of the binomial, Q the quotient of the second term divided by the first, and consequently PQ is the second term itself; or PQ may represent all the terms of a multinomial, after the first term, and consequently Q the quotient of all those terms, except the first term, divided by that first term, and may be either positive or negative; also $\frac{m}{n}$ represents the exponent of the binomial, and may denote any quantity, integral or

fractional, positive or negative, rational or surd. When the exponent is integral, the denominator n is equal to 1, and the quantity then in this form $(P + PQ)^m$, denotes a binomial to be raised to some power; the series for which was fully determined before Newton's time, as will be shown in the course of the 19th Tract of this volume. When the exponent is fractional, m and n may be any quantities whatever, m denoting the index of some power to which the binomial is to be raised, and n the index of the root to be extracted of that power: and to this case it was first extended and applied by Newton. When the exponent is negative, the reciprocal of the same quantity is meant; as

$(P+PQ)^{-\frac{m}{n}}$ is equal to $\dfrac{1}{(P + PQ)^{\frac{m}{n}}}$.

3. Now when the radical binomial is expanded in an equivalent series, it is asserted that it will be in this general form, namely $(P + PQ)^{\frac{m}{n}}$ or $P^{\frac{m}{n}} \times (1 + Q)^{\frac{m}{n}} =$

$$P^{\frac{m}{n}} \times 1 + \frac{m}{n}Q + \frac{m}{n} \cdot \frac{m-n}{2n}Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n}Q^3 + \&c),$$

$$\text{or } P^{\frac{m}{n}} \times 1 + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

where the law of the progression is visible, and the quantities P, m, n, Q, include their signs + or —, the terms of the series being all positive when Q is positive, and alternately positive and negative when Q is negative, independent however of the effect of the coefficients made up of m and n: also A, B, C, D, &c, in the latter form, denote each preceding term. This latter form is the easier in practice, when we want to collect the sum of the terms of a series; but the former is the fitter for showing the law of the progression of the terms.

4. The truth of this series was not demonstrated by Newton, but only inferred by way of induction. Since his time however, several attempts have been made to demonstrate it, with various success, and in various ways; of which however

those are justly preferred, which proceed by pure algebra, and without the help of fluxions. And such has been esteemed the difficulty of proving the general case, independent of the doctrine of fluxions, that many eminent mathematicians to this day account the demonstration not fully accomplished, and still a thing greatly to be desired. Such a demonstration I think is here effected. But before delivering it, it may not be improper to premise somewhat of the history of this theorem, its rise, progress, extension, and demonstrations.

5. Till very lately the prevailing opinion has been, that the theorem was not only invented by Newton, but first of all by him; that is, in that state of perfection in which the terms of the series, for any assigned power whatever, can be found independently of the terms of the preceding powers; namely, the second term from the first, the third term from the second, the fourth term from the third, and so on, by a general rule. Upon this point I have already given an opinion in the history to my logarithms, above cited, and I shall here enlarge somewhat further on the same head.

That Newton invented it himself, I make no doubt. But that he was not the first inventor, is at least as certain. It was described by Briggs, in his Trigonometria Britannica, long before Newton was born; not indeed for fractional exponents, for that was the application of Newton, but for any integral power whatever, and that by the general law of the terms as laid down by Newton, independent of the terms of the powers preceding that which is required. For as to the generation of the coefficients of the terms of one power from those of the preceding powers, successively one after another, it was remarked by Vieta, Oughtred, and many others, and was not unknown to much more early writers on arithmetic and algebra, as will be manifest by a slight inspection of their works, as well as the gradual advance the property made, both in extent and perspicuity, under the hands of the successive masters in arithmetic, every one adding somewhat more towards the perfection of it.

6. Now the knowledge of this property of the coefficients of the terms in the powers of a binomial, is at least as old as the practice of the extraction of roots; for this property was both the foundation and the principle, as well as the means of those extractions. And as the writers on arithmetic became acquainted with the nature of the coefficients in powers still higher, just so much higher did they extend the extraction of roots, still making use of this property. At first it seems they were only acquainted with the nature of the square, which consists of these three terms, 1, 2, 1; and accordingly they extracted the square roots of numbers by means of them; but went no further. The nature of the cube next presented itself, which consists of these four terms, 1, 3, 3, 1; and by means of these they extracted the cubic roots of numbers, in the same manner as we do at present. And this was the extent of their extractions in the time of Lucas de Burgo, an Italian, who, from 1470 to 1500, wrote several tracts on arithmetic, containing the sum of what was then known of this science, which chiefly consisted in the doctrine of the proportions of numbers, the nature of figurate numbers, and the extraction of roots, as far as the cubic root inclusively.

7. It was not long however before the nature of the coefficients of all the higher powers became known, and tables formed for constructing them indefinitely. For in the year 1544 came out, at Norimberg, an excellent treatise of arithmetic and algebra, by Michael Stifelius, a German divine, and an honest, but a weak, disciple of Luther. In this work, *Arithmetica Integra*, of Stifelius, are contained several curious things, some of which have been ascribed to a much later date. He here treats, pretty fully and ably, of progressional and figurate numbers, and in particular of the following table for constructing both them and the coefficients of the terms of all powers of a binomial, which has been so often used since his time for these and other purposes, and which more than a century after was, by Pascal, otherwise called the

arithmetical triangle, but who only mentioned some additional properties of the table.

1							
2							
3	3						
4	6						
5	10	10					
6	15	20					
7	21	35	35				
8	28	56	70				
9	36	84	126	126			
10	45	120	210	252			
11	55	165	330	462	462		
12	66	220	495	792	924		
13	78	286	715	1287	1716	1716	
14	91	364	1001	2002	3003	3432	
15	105	455	1365	3003	5005	6435	6435
16	120	560	1820	4368	8008	11440	12870
17	136	680	2380	6188	12376	19448	24310

Stifelius here observes that the horizontal lines of this table furnish the coefficients of the terms of the correspondent powers of a binomial; and teaches how to use them in extracting the roots of all powers whatever. And after him the same table was used for the same purpose, by Cardan, and Stevin, and the other writers on arithmetic. I suspect however, that the nature of this table was known much earlier than the time of Stifelius, at least so far as regards the progressions of figurate numbers, a doctrine amply treated of by Nichomachus, who lived, according to some, before Euclid, but not till long after him according to others. His work on arithmetic was published at Paris in 1538; and it is supposed was chiefly copied into the treatise on the same subject by Boethius: but I have never seen either of these two works. Though indeed Cardan seems to ascribe the invention of the table to Stifelius; but I suppose that is only to be understood of its application to the extraction of roots. See Cardan's *Opus Novum de Proportionibus*, where he quotes it, and extracts the table and its use from Stifelius's book. Cardan also, at p. 185, *et seq.* of the same work, makes use of a like table to find the number of variations of things, or conjugations as he calls them.

8. The contemplation of this table has probably been attended with the invention and extension of some of our most curious discoveries in mathematics, both in regard to the powers of a binomial, with the consequent extraction of roots, the doctrine of angular sections by Vieta, and the differential method by Briggs and others. For, one or two of the powers or sections being once known, the table would be of excellent use in discovering and constructing the rest. And accordingly we find this table used on many occasions by Stifelius, Cardan, Stevin, Vieta, Briggs, Oughtred, Mercator, Pascal, &c, &c.

9. On this occasion I cannot help mentioning the ample manner in which I see Stifelius, at fol. 35, *et seq.* of the same book, treats of the nature and use of logarithms, though not under the same name, but under the idea of a series of arithmeticals, adapted to a series of geometricals. He there explains all their uses; such as, that the addition of them, answers to the multiplication of their geometricals; subtraction to division; multiplication of exponents, to involution; and dividing of exponents, to evolution. And he exemplifies the use of them in cases of the Rule-of-Three, and in finding mean proportionals between given terms, and such like, exactly as is done in logarithms. So that he seems to have been in the full possession of the idea of logarithms, and wanted only the necessity of troublesome calculations to induce him to make a table of such numbers.

10. But though the nature and construction of this table, namely of figurate numbers, was thus early known, and employed in the raising of powers, and extracting of roots; yet it was only by raising the numbers one from another by continual additions, and then taking them from the table for use when wanted; till Briggs first pointed out the way of raising any horizontal line in the foregoing table by itself, without any of the preceding lines; and thus teaching to raise the terms of any power of a binomial, independent of any other

powers; and so gave the substance of the binomial series in words, wanting only the notation in symbols; as it is shown at large in the 19th Tract, in this volume.

11. Whatever was known however of this matter, related only to pure or integral powers, no one before Newton having thought of extracting roots by infinite series. He happily discovered, that, by considering powers and roots in a continued series, roots being as powers having fractional exponents, the same binomial series would equally serve for them all, whether the index should be fractional or integral, or the series be finite or infinite.

12. The truth of this method however was long known only by trial in particular cases, and by induction from analogy. Nor does it appear that even Newton himself ever attempted any direct proof of it. But various demonstrations of this theorem have been since given by the more modern mathematicians, of which some are by means of the doctrine of fluxions, and others, more legally, from the pure principles of algebra only. Some of which I shall here give a short account of.

13. One of the first demonstraters of this theorem, was Mr. James Bernoulli. His demonstration is, among several other curious things, contained in this little work called *Ars Conjectandi*, which has been improperly omitted in the collection of his works published by his nephew Nicholas Bernoulli. This is a strict demonstration of the binomial theorem in the case of integral and affirmative powers, and is to this effect. Supposing the theorem to be true in any one power, as for instance, in the cube, it must be true in the next higher power; which he demonstrates. But it is true in the cube, in the fourth, fifth, sixth, and seventh powers, as will easily appear by trial, that is by actually raising those powers by continual multiplications. Therefore it is true in all higher powers. All this he shows in a regular and legitimate manner, from the principles of multiplication, and without the

help of fluxions. But he could not extend his proof to the other cases of the binomial theorem, in which the powers are fractional. And this demonstration has been copied by Mr. John Stewart, in his commentary on Newton's quadrature of curves. To which he has added, from the principles of fluxions, a demonstration of the other case, for roots or fractional exponents.

14. In No. 230 of the Philosophical Transactions for the year 1697, is given a theorem, by Mr. De Moivre, in imitation of the binomial theorem, which is extended to any number of terms, and thence called the multinomial theorem; which is a general expression in a series, for raising any multinomial quantity to any power. His demonstration of the truth of this theorem, is independent of the truth of the binomial theorem, and contains in it a demonstration of the binomial theorem as a subordinate proposition, or particular case of the other more general theorem. And this demonstration may be considered as a legitimate one, for pure powers, founded on the principles of multiplication, that is, on the doctrine of combinations and permutations. And it proves that the law of the continuation of the terms, must be the same in the terms not computed, or not set down, as in those that are written down.

15. The ingenious Mr. Landen has given an investigation of the binomial theorem, in his *Discourse concerning the Residual Analysis*, printed in 1758, and in the *Residual Analysis* itself, printed in 1764. The investigation is deduced from this lemma, namely, if m and n be any integers, and $q = \frac{v}{x}$, then is

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + q + q^2 + q^3 \;\; - \;\; - \;\; (m)}{1 + q^{\frac{m}{n}} + q^{\frac{2m}{n}} + q^{\frac{3m}{n}} \;\; - \;\; - \;\; (n)}$$

which theorem is made the principal basis of his Residual Analysis.

The investigation is thus: the binomial proposed being $(1+x)^{\frac{m}{n}}$, assume it equal to the following series $1+ax+bx^2+cx^3$ &c, with indeterminate coefficients. Then for the same reason

$$\text{as } (1+x)^{\frac{m}{n}} \text{ is } = 1+ax+bx^2+cx^3 \text{ \&c,}$$

$$\text{will } (1+y)^{\frac{m}{n}} \text{ be } = 1+ay+by^2+cy^3 \text{ \&c.}$$

Then, by subtraction,

$$(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}} = a(x-y)+b(x^2-y^2)+c(x^3-y^3) \text{ \&c.}$$

And, dividing both sides by $x-y$, and by the lemma, we have

$$\frac{(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}}}{x-y} = (1+x)^{\frac{m}{n}-1} \times$$

$$\frac{1+\frac{1+y}{1+x}+\left(\frac{1+y}{1+x}\right)^2+\left(\frac{1+y}{1+x}\right)^3 \quad - \; - \; - \; (m)}{1+\left(\frac{1+y}{1+x}\right)^{\frac{m}{n}}+\left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}}+\left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} \quad - \; - \; - \; (n)}$$

$$= a+b(x+y)+c(x^2+xy+y^2)+d(x^3+x^2y+xy^2+y^3) \text{ \&c.}$$

Then, as this equation must hold true whatever be the value of y, take $y=x$, and it will become

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a+2bx+3cx^2+4cx^3 \text{ \&c.}$$

Consequently, multiplying by $1+x$, we have

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}}, \text{ or its equal by the assumption,}$$

$$\text{viz. } \frac{m}{n}+\frac{m}{n}ax+\frac{m}{n}bx^2+\frac{m}{n}cx^3 \text{ \&c.}$$

$$= a+\left.\begin{matrix}2b\\a\end{matrix}\right\}x+\left.\begin{matrix}3c\\2b\end{matrix}\right\}x^2+\left.\begin{matrix}4d\\3c\end{matrix}\right\}x^3 \text{ \&c.}$$

Then, by comparing the homologous terms, the value of the coefficients a, b, c, &c, are deduced for as many terms as are compared.

A large account is also given of this investigation by the learned Dr. Hales, in his *Analysis Equationum*, lately published at Dublin.

Mr. Landen then contrasts this investigation with that by the method of fluxions, which is as follows. Assume as before;

$$(1+x)^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 + dx^4 \text{ \&c.}$$

Take the fluxion of each side, and we have

$$\frac{m}{n} \times (1 + x)^{\frac{m}{n}-1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} \text{ \&c.}$$

Divide by $\dot{x}$, or take it $= 1$, so shall

$$\frac{m}{n} \times (1 + x)^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3 \text{ \&c.}$$

Then multiply by $1 + x$, and so on as above in the other way.

16. Besides the above, and an investigation by the celebrated M. Euler, which are the principal demonstrations and investigations that have been given of this important theorem, I have been shown an ingenious attempt of Mr. Baron Maseres, to demonstrate this theorem in the case of roots or fractional exponents, by the help of De Moivre's multinomial theorem. But, not being quite satisfied with his own demonstration, as not expressing the law of continuation of the terms which are not actually set down, he was pleased to urge me to attempt a more complete and satisfactory demonstration of the general case of roots, or fractional exponents. And he further proposed it in this form, namely, that if Q be the coefficient of one of the terms of the series which is equal to $(1 + x)^{\frac{1}{r}}$, and P the coefficient of the next preceding term, and R the coefficient of the next following term; then, if Q be $= \frac{a}{b} \times$ P, it is required to prove that R will be $= \frac{a-n}{b+n}$ $\times$ Q. This he observed would be quite perfect and satisfactory, as it would include all the terms of the series, as well those that are omitted, as those that are actually set down. And I was, in my demonstration, to suppose, if I pleased, the truth of the binomial and multinomial theorems for integral powers, as truths that had been previously and perfectly proved.

In consequence I sent him soon after the substance of the following demonstration; with which he was quite satisfied; and which I now proceed to explain at large.

17. Now the binomial integral is $(1+x)^n =$

$$1+\overset{a}{\frac{n}{1}}x+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}}^{b}x^2+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}}^{c}x^3+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}}^{d}x^4\ \&c,$$

$$\text{or } 1+\frac{n}{1}x+\frac{n-1}{2}ax^2+\frac{n-2}{3}bx^3+\frac{n-3}{4}cx^4+\&c,$$

where a, b, c, &c, denote the whole coefficients of the 2d, 3d, 4th, &c, terms, over which they are placed; and in which the law is this, namely, if P, Q, R, be the coefficients of any three terms in succession, and if

$\frac{g}{h}$P bc = Q, then is $\frac{g-1}{h+1}$Q = R; as is evident; and which, it is granted, has been proved.

18. And the binomial fractional is $(1+x)^{\frac{1}{n}} =$

$$1+\overset{a}{\frac{1}{n}}x+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}}^{b}x^2+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}}^{c}x^3+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}\cdot\frac{1-3n}{4n}}^{d}x^4$$

$$\&c,\ \text{ or } 1+\frac{1}{n}x+\frac{1-n}{2n}ax^2+\frac{1-2n}{3n}bx^3+\frac{1-3n}{4n}cx^4+\&c;$$

in which the law is this, namely, if P, Q, R be the coefficients of three terms in succession; and if

$\frac{g}{h}$P bc = Q, then is $\frac{g-n}{h+n}$Q = R. Which is the property to be proved.

19. Again, the multinomial integral $(1+\text{A}x+\text{B}x^2+\text{C}x^3\&\text{c})^n$, is $=$ - - - 1

$$(a)\quad +\frac{n}{1}\text{A}x$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\text{A}^2x^2$$

$$(b)\quad +\frac{n}{1}\text{B}$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\text{A}^3x^3$$

$$(c)\quad +\frac{n}{1}\cdot\frac{n-1}{1}\text{AB}$$

$$+\frac{n}{1}\text{C}$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\text{A}^4x^4$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{1}\text{A}^2\text{B}$$

$$(d)\quad +\frac{n}{1}\cdot\frac{n-1}{1}\text{AC}$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\text{B}^2$$

$$+\frac{n}{1}\text{D}$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{n-4}{5}\text{A}^5x^5$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{1}\text{A}^3\text{B}$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{1}\text{A}^2\text{C}$$

$$(e)\quad +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{2}\text{AB}^2$$

$$+\frac{n}{1}\cdot\frac{n-1}{2}\text{AD}$$

$$+\frac{n}{1}\cdot\frac{n-1}{1}\text{BC}$$

$$+\frac{n}{1}\text{E}$$

&c.

Or, if we put a, b, c, d, &c, for the coefficients of the 2d, 3d, 4th, 5th, &c, terms, or powers of x, the last series, by substitution, will be changed into this form,

$$(1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&c)^n = \;\text{- - - - - - -}\; 1$$

$$(a) \qquad + \frac{n\text{A}}{1}x$$

$$(b) \qquad + \frac{2n\text{B} + (n-1)\text{A}a}{2}x^2$$

$$(c) \qquad + \frac{3n\text{C} + (2n-1)\text{B}a + (n-2)\text{A}b}{3}x^3$$

$$(d) \qquad + \frac{4n\text{D} + (3n-1)\text{C}a + (2n-2)\text{B}b + (n-3)\text{A}c}{4}x^4$$

$$(e) \qquad + \frac{5n\text{E} + (4n-1)\text{D}a + (3n-2)\text{C}b + (2n-3)\text{B}c + (n-4)\text{A}d}{5}x^5$$

&c.

20. Now, to find the series in Art. 18, assume the proposed binomial equal to a series with indeterminate coefficients, as

$$(1 + x)^{\frac{1}{n}} = 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \text{D}x^4 + \&c.$$

Then raise each side to the n power, so shall

$$1 + x = (1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&c)^n.$$

But it is granted that the multinomial raised to any integral power is proved, and known to be, as in the last Art. viz,

$$1 + x = (1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&c)^n =$$

$$1 + \overbrace{\frac{n\text{A}}{1}}^{a}x + \overbrace{\frac{2n\text{B} + (n-1)\text{A}a}{2}}^{b}x^2 + \overbrace{\frac{3n\text{C} + (2n-1)\text{B}a + (n-2)\text{A}b}{3}}^{c}x^3$$

&c.

It follows then, that if this last series be equal to $1 + x$, by equating the homologous coefficients, all the terms after the second must vanish, or all the coefficients b, c, d, &c, after the second term, must be each $= 0$. Writing therefore, in this series, 0 for each of the letters b, c, d, &c, it will become of this more simple form, viz, $1 + x =$

$$1 + \overbrace{\frac{n\text{A}}{1}}^{a}x + \overbrace{\frac{2n\text{B} + (n-1)\text{A}a}{2}}^{b\,=\,0}x^2 + \overbrace{\frac{3n\text{C} + (2n-1)\text{B}a}{3}}^{c\,=\,0}x^3 + \&c.$$

Put now each of the coefficients, after the second term, $= 0$, and we shall have these equations

$$2n\text{B} + (1n - 1)\,\text{A}a = 0$$
$$3n\text{C} + (2n - 1)\,\text{B}a = 0$$
$$4n\text{D} + (3n - 1)\,\text{C}a = 0$$
$$5n\text{E} + (4n - 1)\,\text{D}a = 0$$
&c.

The resolution of which equations gives the following values of the assumed indeterminate coefficients, namely,

$$\text{B} = \frac{1-n}{2n}\text{A}a,\ \text{C} = \frac{1-2n}{3n}\text{B}a,\ \text{D} = \frac{1-3n}{4n}\text{C}a,\ \text{E} = \frac{1-4n}{5n}\text{D}a,\ \&\text{c};$$

which coefficients are according to the law proposed, namely, when $\frac{g}{h}\text{P}$ is $= \text{Q}$, then is $\frac{g-n}{h+n}\text{Q} = \text{R}$. Q. E. D.

21. Also, by equating the second coefficients, namely, $1 = a = n\text{A}$, we find $\text{A} = \frac{1}{n}$. This being written for A in the above values of B, C, D, &c, will give the proper series for the binomial in question, namely, $(1 + x)^{\frac{1}{n}}$

$$= 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c},$$
$$= 1 + \frac{1}{n}x + \frac{1-n}{2n}ax^2 + \frac{1-2n}{3n}bx^3 + \&\text{c},$$
$$= 1 + \frac{1}{n}x + \frac{1}{n}\cdot\frac{1-n}{2n}x^2 + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}x^3 + \&\text{c}.$$

Of the Form of the Assumed Series.

22. In the demonstrations or investigations of the truth of the binomial theorem, the butt or object has always been the law of the coefficients of the terms: the form of the series, as to the powers of x, having never been disputed, but taken for granted, either as incapable of receiving a demonstration, or as too evident to need one. But since the demonstration of the law of the coefficients has been accomplished, in which the main, if not the only, difficulty was supposed to consist, we have extended our researches still further, and have even doubted or queried the very *form* of the terms themselves,

namely, $1 + Ax + Bx^2 + Cx^3 + Dx^4 +$ &c, increasing by the regular integral series of the powers of x, as assumed to denote the quantity $(1 + x)^{\frac{1}{n}}$, or the n root of $1 + x$. And in consequence of these scruples, I have been required, by a learned friend, to vindicate the propriety of that assumption. Which I think is effectually done as follows.

23. To prove then, that any root of the binomial $1 + x$ *can* be represented by a series of this form $1 + x + x^2 + x^3 + x^4$ &c, where the coefficients are omitted, our attention being now employed only on the powers of x; let the series representing the value of $(1 + x)^{\frac{1}{n}}$ be $1 + A + B + C + D +$ &c; where A, B, C, &c, now represent the whole of the 2d, 3d, 4th, &c, terms, both their coefficients and the powers of x, whatever they may be, only increasing from the less to the greater, because they increase in the terms $1 + x$ of the given binomial itself; and in which the first term must evidently be 1, the same as in the given binomial.

Raise now $(1 + x)^{\frac{1}{n}}$, and its equivalent series $1 + A + B + C +$ &c, both to the n power, by the multinomial theorem, and we shall have, as before,

$$\begin{array}{llll} 1 + x = 1 + \frac{n}{1}A & + \frac{n}{1}\cdot\frac{n-1}{2}A^2 & + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3 & + \text{\&c.} \\ & \quad \frac{n}{1}B & \quad \frac{n}{1}\cdot\frac{n-1}{1}AB & \\ & & \quad \frac{n}{1}C & \end{array}$$

Then equate the corresponding terms, and we have the first term $1 = 1$.

Again, the second term of the series $\frac{n}{1}A$, must be equal to the second term x of the binomial. For none of the other terms of the series are equipollent, or contain the same power of x, with the term $\frac{n}{1}A$. Not any of the terms A^2, A^3, A^4, &c; for they are double, triple, quadruple, &c, in power to A. Nor yet any of the terms containing B, C, D, &c; because, by the supposition, they contain all different and increasing powers. It follows therefore, that $\frac{n}{1}$A makes up the whole value of the second term x of the given binomial. Consequently the second term A of the assumed series, contains only the first power of x; and the whole value of that term A is $= \frac{1}{n}x$.

But all the other equipollent terms of the expanded series must be equal to nothing, which is the general value of the terms, after the second, of the given quantity $1 + x$ or $1 + x + 0 + 0 + 0 +$ &c. Our business is therefore to find the several orders of equipollent terms of the expanded series. And these it is asserted will be as they are arranged above, in which B is equipollent with A^2, C with A^3, D with A^4, and so on.

Now that B is equipollent with A^2, is thus proved. The value of the third term is 0. But $\frac{n}{1}\cdot\frac{n-1}{2}A^2$ is a part of the third term. And it is only a part of that term: otherwise $\frac{n}{1}\cdot\frac{n-1}{2}$ would be $= 0$, which it is evident cannot happen in every value of n, as it ought; for indeed it happens only when n is $= 1$. Some other quantity then must be equipollent with $\frac{n}{1}\cdot\frac{n-1}{2}A^2$, and must be joined with it, to make up the whole third term equal to 0. Now that supplemental quantity can be no other than $\frac{n}{1}$B: for all the other following terms are evidently plupollent than B. It follows therefore, that B is equipollent with A^2, and contains the second power of x; or that $\frac{n}{1}\cdot\frac{n-1}{2}A^2 + \frac{n}{1}B = 0$, and consequently $\frac{n-1}{2}A^2 + B = 0$, or $B = \frac{1-n}{2}A^2 = \frac{1-n}{2n}Ax = \frac{1}{n}\cdot\frac{1-n}{2n}x^2$.

Again, the fourth term must be $= 0$. But the quantities $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3 + \frac{n}{1}\cdot\frac{n-1}{2}AB$ are equipollent, and make up part of that fourth term. They are equipollent, or A^3 equipollent with AB, because A^2 and B are equipollent. And they do not constitute the whole of that term; for if they did, then would $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3 + \frac{n}{1}\cdot\frac{n-1}{2}AB$ be $=0$ in all values of n, or $\frac{n-2}{3n}A^2 + B = 0$: but it has been just shown above, that $\frac{n-1}{2}A^2 + B = 0$; it would therefore follow that $\frac{n-2}{3}$ would be $= \frac{n-1}{2}$, a circumstance which can only happen when $n = -1$, instead of taking place for every value of n. Some other quantity must therefore be joined with these to make up the whole of the fourth term. And this supplemental quantity can be no other than $\frac{n}{1}C$, because all the other following quantities are evidently plupollent than A^3 or AB. It follows therefore, that C is equipollent with A^3, and therefore contains the 3d power of x. And the whole value of C is

$$\frac{1-n}{2}\cdot\frac{n-2}{3}A^3 + \frac{1-n}{1}AB = \frac{1-2n}{3}AB = \frac{1-2n}{3n}Bx = \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}x^3.$$

And the process is the same for all the other following terms. Thus then we have proved the law of the whole series, both with respect to the coefficients of its terms, and to the powers of the letter x.

Since the above account was first written, almost 30 years ago, other demonstrations have been given by several ingenious and learned writers; which may be seen in some of the later volumes of the Philos. Trans. and elsewhere.

TRACT XIII.

ON THE COMMON SECTIONS OF THE SPHERE AND CONE. WITH THE DEMONSTRATION OF SOME OTHER NEW PROPERTIES OF THE SPHERE, WHICH ARE SIMILAR TO CERTAIN KNOWN PROPERTIES OF THE CIRCLE.

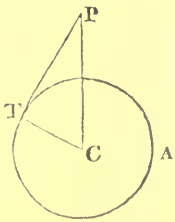
THE study of the mathematical sciences is useful and profitable, not only on account of the benefit derivable from them to the affairs of mankind in general; but are most eminently so, for the pleasure and delight which the human mind feels in the discovery and contemplation of the endless number of truths, that are continually presenting themselves to our view. These meditations are of a sublimity far above all others, whether they be purely intellectual, or whether they respect the nature and properties of material objects; they methodize, strengthen, and extend the reasoning faculties in the most eminent degree, and so fit the mind the better for understanding and improving every other science; but, above all, they furnish us with the purest and most permanent delight, from the contemplation of truths peculiarly certain and immutable, and from the beautiful analogy which reigns through all the objects of similar inquiry. In the mathematical sciences, the discovery, often accidental, of a plain and simple property, is but the harbinger of a thousand others of the most sublime and beautiful nature, to which we are gradually led, delighted, from the more simple, to the more compound and general, till the mind becomes quite enraptured at the full blaze of light bursting upon it from all directions.

Of these very pleasing subjects, the striking analogy that prevails among the properties of geometrical figures, or figured extension, is not one of the least. Here we often find that a plain and obvious property of one of the simplest figures, leads us to, and forms only a particular case of, a property in some other figure, less simple; afterwards this again turns out to be no more than a particular case of another still more general; and so on, till at last we often trace the tendency to end in a general property of all figures whatever.

The few properties which make a part of this paper, constitute a small specimen of the analogy, and even identity, of some of the more remarkable properties of the circle, with those of the sphere. To which are added some properties of the lines of section, and of contact, between the sphere and cone. Both which may be further extended as occasions may offer: like as all of these properties have occurred from the circumstance, mentioned near the end of the paper, of considering the inner surface of a hollow spherical vessel, as viewed by an eye, or as illuminated by rays, from a given point.

PROPOSITION 1.

All the tangents are equal, which are drawn, from a given point without a sphere, to the surface of the sphere, quite around.



Demons.—For, let PT be any tangent from the given point P; and draw PC to the centre C, and join TC. Also let CTA be a great circle of the sphere in the plane of the triangle TPC. Then, CP and CT, as well as the angle T, which is right (Eucl. iii. 18), being constant, in every position of the tangent, or of the point of contact T; the square of PT will be every where equal to the difference of the squares of the constant lines CP, CT, and therefore constant; and consequently the line or tangent PT itself of a constant length, in every position, quite round the surface of the sphere.

PROP. 2.

If a tangent be drawn to a sphere, and a radius be drawn from the centre to the point of contact, it will be perpendicular to the tangent; and a perpendicular to the tangent will pass through the centre.

Demons.—For, let PT be the tangent, TC the radius, and CTA a great circle of the sphere, in the plane of the triangle TPC, as in the foregoing proposition. Then, PT touching the circle in the point T, the radius TC is perpendicular to the tangent PT, by Eucl. iii. 18, 19.

PROP. 3.

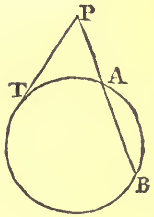
If any line or chord be drawn in a sphere, its extremes terminating in the circumference; then a perpendicular drawn to it, from the centre, will bisect it: and if the line drawn from the centre, bisect it, it is perpendicular to it.

Demons.—For, a plane may pass through the given line and the centre of the sphere; and the section of that plane with the sphere, will be a great circle (Theodos. i. 1), of which the given line will be a chord. Therefore (Eucl. iii. 3) the perpendicular bisects the chord, and the bisecting line is perpendicular.

Corol.—A line drawn from the centre of the sphere, to the centre of any lesser circle, or circular section, is perpendicular to the plane of that circle. For, by the proposition, it is perpendicular to all the diameters of that circle.

PROP. 4.

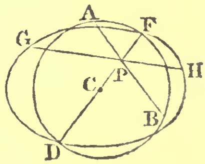
If from a given point, a right line be drawn in any position through a sphere, cutting its surface always in two points; the rectangle contained under the whole line and the external part, that is the rectangle contained by the two distances between the given point, and the two points where the line meets the surface of the sphere, will always be of the same constant magnitude, namely, equal to the square of the tangent drawn from the same given point.



Demons.—Let P be the given point, and AB the two points in which the line PAB meets the surface of the sphere; through PAB and the centre let a plane cut the sphere in the great circle TAB, to which draw the tangent PT. Then the rectangle PA . PB is equal to the square of PT (Eucl. iii. 36); but PT, and consequently its square, is constant by Prop. 1; therefore the rectangle PA . PB, which is always equal to this square, is every where of the same constant magnitude.

PROP. 5.

If any two lines intersect each other within a sphere, and be terminated at the surface on both sides; the rectangle of the parts of the one line, will be equal to the rectangle of the parts of the other. And, universally, the rectangles of the two parts of all lines passing through the point of intersection, are all of the same magnitude.



Demons.—Through any one of the lines, as AB, conceive a plane to be drawn through the centre C of the sphere, cutting the sphere in the great circle ABD; and draw its diameter DCPF through the points of intersection P of all the lines. Then the rectangle AP . PB is equal to the rectangle DP . PL (Eucl. iii. 35).

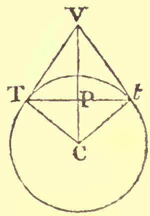
Again, through any other of the intersecting lines GH, and the centre, conceive another plane to pass, cutting the sphere in another great circle DGFH. Then, because the points C and P are in this latter plane, the line CP, and consequently the whole diameter DCPF, is in the same plane; and therefore it is a diameter of the circle DGFH, of which GPH is a chord. Therefore, again, the rectangle GP . PH is equal to the rectangle DP . PF (Eucl. iii. 35).

Consequently all the rectangles AP . PB, GP . PH, &c, are equal, being each equal to the constant rectangle DP . PF.

Corol.—The great circles passing through all the lines or chords which intersect in the point P, will all intersect in the common diameter DPF.

PROP. 6.

If a sphere be placed within a cone, so as to touch it in two points; then shall the outside of the sphere, and the inside of the cone, mutually touch quite around, and the line of contact will be a circle.



Demons.—Let V be the vertex of the cone, C the centre of the sphere, T one of the two points of contact, and TV a side of the cone. Draw CT, CV. Then TVC is a triangle right-angled at T (Prop. 2). In like manner, *t* being another point of contact, and C*t* being drawn, the triangle *t*VC will be right-angled at *t*. These two triangles then, TVC, *t*VC, having the two sides CT, TV, equal to the two C*t*, *t*V (Prop. 1), and the included angle T equal to the included angle *t*, will be equal in all respects (Eucl. i. 4), and consequently have the angle TVC equal to the angle *t*VC.

Again, let fall the perpendiculars TP, *t*P. Then the two triangles TVP, *t*VP, having the two angles TVP and TPV equal to the two *t*VP and *t*PV, and the side TV equal to the side *t*V (Prop. 1), will be equal in all respects (Eucl. i. 26); consequently TP is equal to *t*P, and VP equal to VP. Hence PT, P*t* are radii of a little circle of the sphere, whose plane is perpendicular to the line CV, and its circumference every where equidistant from the point C or V. This circle is therefore a circular section both of the sphere and of the cone, and is therefore the line of their mutual contact. Also CV is the axis of the cone.

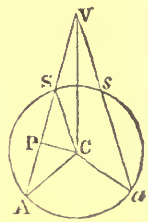
Corol. 1.—The axis of a cone, when produced, passes through the centre of the inscribed sphere.

Corol. 2.—Hence also, every cone circumscribing a sphere, so that their surfaces touch quite around, is a right cone;

nor can any scalene or oblique cone touch a sphere in that manner.

PROP. 7.

The two common sections of the surfaces of a sphere and a right cone, are the circumferences of circles, if the axis of the cone pass through the centre of the sphere.



Demons.—Let V be the vertex of the cone, C the centre of the sphere, and S one point of the less or nearer section; draw the lines CS, CV. Then, in the triangle CSV, the two sides CS, CV, and the included angles CV, are constant, for all positions of the side VS; and therefore the side VS is of a constant length for all positions, and is consequently the side of a right cone having a circular base; therefore the locus of all the points S, is the circumference of a circle perpendicular to the axis CV, that is, the common section of the surfaces of the sphere and cone, is that circumference.

In the same manner it is proved that, if A be any point in the farther or greater section, and CA be drawn; then VA is constant for all positions, and therefore, as before, is the side of a cone cut off by a circular section whose plane is perpendicular to the axis.

And these circles, being both perpendicular to the axis, are parallel to each other. Or, they are parallel because they are both circular sections of the cone.

Corol. 1.—Hence SA = *sa*, because VA = V*a*, and VS = V*s*.

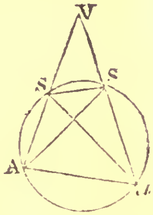
Corol. 2.—All the intercepted equal parts SA, *sa*, &c, are equally distant from the centre. For, all the sides of the triangle SCA are constant, and therefore the perpendicular CP is constant also. And thus all the equal right lines or chords in a sphere, are equally distant from the centre.

Corol. 3.—The sections are not circles, and therefore not in planes, if the axis pass not through the centre. For then some of the points of section are farther from the vertex than others.

PROP. 8.

Of the two common sections of a sphere and an oblique cone, if the one be a circle, the other will be a circle also.

Demons.—Let S A *as* and A S V *a* be sections of the sphere and cone, made by a common plane passing through the axes of the cone and the sphere; also S*s*, A*a* the diameters of the two sections. Now, by the supposition, one of these, as A*a*, is the diameter of a circle. But the angle V S *s* = the angle V*a*A (Eucl. i. 13, and iii. 22), therefore S*s* cuts the cone in sub-contrary position to A*a*; and consequently, if a plane pass through S*s*, and perpendicular to the plane A V *a*, its section with the oblique cone will be a circle, whose diameter is the line S*s* (Apol. i. 5). But the section of the same plane and the sphere, is also a circle whose diameter is the same line S*s* (Theod. i. 1). Consequently the circumference of the same circle, whose diameter is S*s*, is in the surface both of the cone and sphere; and therefore that circle is the common section of the cone and sphere.



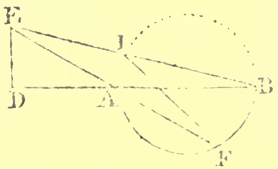
In like manner, if the one section be a circle whose diameter is S*a*, the other section will be a circle whose diameter is *s*A.

Corol. 1.—Hence, if the one section be not a circle, neither of them is a circle; and consequently they are not in planes; for the section of a sphere by a plane, is a circle.

Corol. 2.—When the sections of a sphere and oblique cone are circles, the axis of the cone does not pass through the centre of the sphere, (except when one of the sections is a great circle, or passes through the centre). For, the axis passes through the centre of the base, but not perpendicularly; whereas a line drawn from the centre of the sphere to the centre of the base, is perpendicular to the base, by cor. to prop. 3.

Corol. 3.—Hence, if the inside of a bowl, which is a hemisphere, or any segment of the sphere, be viewed by an eye not situated in the axis produced, which is perpendicular to the section or brim; the lower, or extreme part of the internal surface which is visible, will be bounded by a circle of the sphere; and the part of the surface seen by the eye, will be included between the said circle, and border or brim, which it intersects in two points. For the eye is in the place of the vertex of the cone; and the rays from the eye to the brim of the bowl, and thence continued from the nearer part of the brim, to the opposite internal surface, form the sides of the cone; which, by the proposition, will form a circular arc on the said internal surface; because the brim, which is the one section, is a circle.

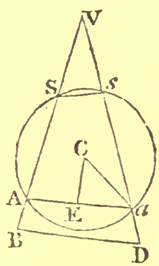
And hence, the place of the eye being given, the quantity of internal surface that can be seen, may be easily determined. For the distance and height of the eye, with respect to the brim, will give the greatest distance of the section below the brim, together with its magnitude and inclination to the plane of the brim; which being known, common mensuration furnishes us with the measure of the surface included between them. Thus, if AB be the diameter in the vertical plane passing through the eye at E, also AFB the section of the bowl by the same plane, and AIB the supplement of that arc. Draw EAF, EIB, cutting this vertical circle in F and I; and join IF. Then shall IF be the diameter of the section or extremity of the visible surface, and BF its greatest distance below the brim, an arc which measures an angle double the angle at A.



Corol. 4.—Hence also, and from Proposition 4, it follows, that if through every point in the circumference of a circle, lines be drawn to a given point E out of the plane of the circle, so that the rectangle contained under the parts between the point E and the circle, and between the same point E and some other point F, may always be of a certain given magnitude; then the locus of all the points F will also be a circle, cutting the former circle in the two points where the lines drawn from the given point E, to the several points in the circumference of the first circle, change from the convex to the concave side of the circumference. And the constant quantity, to which the rectangle of the parts is always equal, is equal to the square of the line drawn from the given point E to either of the said two points of intersection.—And thus the loci of the extremes of all such lines, are circles.

PROP. 9.

Prob.—To place a given sphere, and a given oblique cone, in such positions, that their mutual sections shall be circles.



Let V be the vertex, VB the least side, and VD the greatest side of the cone. In the plane of the triangle VBD it is evident will be found the centre of the sphere. Parallel to BD draw A*a* the diameter of a circular section of the cone, so that it be not greater than the diameter of the sphere. Bisect A*a* with the perpendicular EC; with the centre A and radius of the sphere, cut EC in C, which will be the centre of the sphere; from which therefore describe a great circle of it, cutting the sides of the cone in the points S, *s*, A, *a*: so shall S*s* and A*a* be the diameters of circular sections which are common to both the sphere and cone.

July 29, 1785.

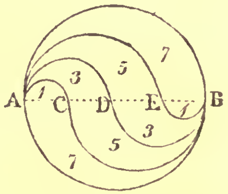
TRACT XIV.

ON THE GEOMETRICAL DIVISION OF CIRCLES AND ELLIPSES INTO ANY NUMBER OF PARTS, AND IN ANY PROPOSED RATIOS.

ARTICLE 1.

In the year 1774 was published a pamphlet in 8vo, with this title, *A Dissertation on the Geometrical Analysis of the Antients. With a Collection of Theorems and Problems, without Solutions, for the Exercise of Young Students.* This pamphlet was anonymous; it was however well known to myself, and to several other persons, that the author of it was the late Mr. John Lawson, B. D. rector of Swanscombe in Kent, an ingenious and learned geometrician, and, what is still more estimable, a most worthy and good man; one in whose heart was found no guile, and whose pure integrity, joined to the most amiable simplicity of manners, and sweetness of temper, gained him the affection and respect of all who had the happiness to be acquainted with him. His collection of problems in that pamphlet concluded with this singular one, "To divide a circle into any number of parts, which shall be as well equal in area as in circumference.—N. B. *This may seem a paradox, however it may be effected in a manner strictly geometrical.*" The solution of this seeming paradox he reserved to himself, as far as I know; but I fell upon the discovery of it soon after; and my solution was published in an account which I gave of the pamphlet in the Critical Review for 1775, vol. xl, and which the author afterwards informed me was on the same principle as his own. This account is in page 21 of that volume, and in the following words:

2. "We have no doubt but that our mathematical readers will agree with us in allowing the truth of the author's remark concerning the seeming paradox of this problem; because there is no geometrical method of dividing the circumference of a circle into any proposed number of parts taken at pleasure, and it does not readily appear that there can be any other way of resolving the problem, than by drawing radii to the points of equal division in the circumference. However another method there is, and that strictly geometrical, which is as follows.

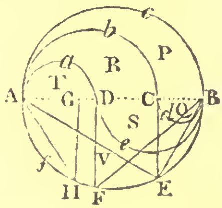


" Divide the diameter AB of the given circle into as many equal parts as the circle itself is to be divided into, at the points C, D, E, &c. Then on the lines AC, AD, AE, &c, as diameters, as also on BE, BD, BC, &c, describe semicircles, as in the annexed figure: and they will divide the whole circle in the manner as required.

" For, the several diameters being in arithmetical progression, of which the common difference is equal to the least of them, and the diameters of circles being as their circumferences, these will also be in arithmetical progression. But, in such a progression, the sum of the extremes is equal to the sum of each pair of terms equally distant from them; therefore the sum of the circumferences on AC and CB, is equal to the sum of those on AD and DB, and of those on AE and EB, &c, and each sum equal to the semi-circumference of the given circle on the diameter AB. Therefore all the parts have equal perimeters; and each is equal to the whole circumference of the proposed circle. Which satisfies one of the conditions in the problem.

" Again, the same diameters being as the numbers 1, 2, 3, 4, &c, and the areas of circles being as the squares of their diameters, the semicircles will be as the square numbers 1, 4, 9, 16, &c, and consequently the differences between all the adjacent semicircles are as the terms of the arithmetical progression 1, 3, 5, 7, &c; and here again the sums of the extremes, and of every two equidistant means, make up the several equal parts of the circle. Which is the other condition."

3. But this subject admits of a more geometrical form, and is capable of being rendered very general and extensive, and is moreover very fruitful in curious consequences. For first, in whatever ratio the whole diameter is divided, whether into equal or unequal parts, and whatever be the number of parts, the perimeters of the spaces will always be equal. For since the circumferences of circles are in the same ratio as their diameters, and because AB and AD + DB and AC + CB are all equal, therefore the semi-circumferences *c* and *b* + *d* and *a* + *e* are all equal, and constant, by the same, whatever be the ratio of the parts AD, DC, CB, of the diameter. We shall presently find too that the spaces TV, RS, and PQ, will be universally as the same parts AD, DC, CB, of the diameter.



4. The semicircles having been described as before mentioned, erect CE perpendicular to AB, and join BE. Then will the circle on the diameter BE, be equal to the space PQ. For, join AE.

Now the space P = semicircle on AB − semicircle on AC;
but the semicir. on AB = semicir. on AE + semicir. on BE,
and the semicir. on AC = semicir. on AE − semicir. on CE,
theref. semic. AB − semic. AC = semic. BE + semicir. CE,
that is, the space P is = semic. BE + semicir. CE;
to each of these add the space Q, or the semicircle on EC,
then P + Q = semic. BE + semic. CE + semic. BC,
that is P + Q = double the semic. BE, or = the whole circle on BE.

5. In like manner, the two spaces PQ and RS together, or the whole space PQRS, is equal to the circle on the diameter BF. And therefore the space RS alone, is equal to the difference, or the circle on BF *minus* the circle on BE.

6. But, circles being as the squares of their diameters, BE^2, BF^2, and these again being as the parts or lines BC, BD,

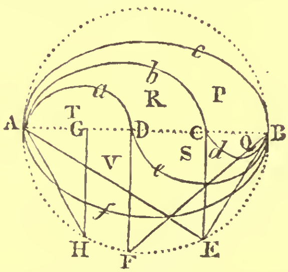
therefore the spaces PQ, PQRS, RS, TV,
are respectively as the lines BC, BD, CD, AD.

And if BC be equal to CD, then will PQ be equal to RS, as in the first or simplest case.

7. Hence, to find a circle equal to the space RS, where the points D and C are taken at random: From either end of the diameter, as A, take AG equal to DC, erect GH perpendicular to AB, and join AH; then the circle on AH will be equal to the space RS. For, the space PQ : the space RS :: BC : CD or AG, that is as $BE^2 : AH^2$ the squares of the diameters, or as the circle on BE to the circle on AH; but the circle on BE is equal to the space PQ, and therefore the circle on AH is equal to the space RS.

8. Hence, to divide a circle in this manner, into any proposed number of parts, that shall be in any ratios to one another: Divide the diameter into as many parts, at the points D, C, &c, and in the same ratios as those proposed; then on the several distances of these points, from the two ends A and B, as diameters, describe the alternate semicircles on the different sides of the whole diameter AB: and they will divide the whole circle in the manner proposed. That is, the spaces TV, RS, PQ, will be as the lines AD, DC, CB.

9. But these properties are not confined to the circle alone. They are found also in the ellipse, as the genus of which the circle is only a species. For if the annexed figure be an ellipse described on the axis AB, the area of which is, in like manner, divided by similar semi-ellipses, described on AD, AC, BC, BD, as axes, all the semi-perimeters *f*, *ae*, *bd*, *c*, will be equal to one another, for the same reason as before in Art. 3, namely, because the peripheries of ellipses are as their diameters. And the same property



would still hold good, if AB were any other diameter of the ellipse, instead of the axis; describing on the parts of it semi-ellipses which shall be similar to those into which the diameter AB divides the given ellipse.

10. And further, if a circle be described about the ellipse, on the diameter AB, and lines be drawn similar to those in the second figure; then, by a process the very same as in Art. 4, *et seq.* substituting only semiellipse for semicircle, it is found that the space

PQ is equal to the similar ellipse on the diameter BE,
PQRS is equal to the similar ellipse on the diameter BF,
RS is equal to the similar ellipse on the diameter AH,
or to the difference of the ellipses on BF and BE;
also the elliptic spaces - - - PQ, PQRS, RS, TV,
are respectively as the lines - BC, BD, DC, AD,

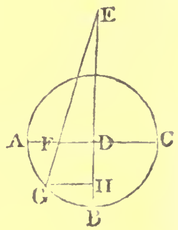
the same ratio as the circular spaces. And hence an ellipse is divided into any number of parts, in any assigned ratios, in the same manner as the circle is divided, namely, dividing the axis, or any diameter in the same manner, and on the parts of it describing similar semiellipses.

TRACT XV.

AN APPROXIMATE GEOMETRICAL DIVISION OF THE CIRCLE.

The solution, here improved, of the following problem, I first gave in my Miscellanea Mathematica, published in the year 1775, pa. 311. The problem is as follows.

To find whether there is any such fixed point E, in the radius BD produced, bisecting the semicircle ABC, so that any line EFG being drawn from it, this line shall always cut the perpendicular radius AD and the quadrantal arc AB, proportionally in the two points F and G; viz. so that DF shall be to BG in a constant ratio.



Solution.—Put the radius AD or DB $= r$, DE $= ar$, the arc BG $= z$, GH $= y$, and DF $= v$. Now, if z to v be a constant ratio, then $\dot{z}$ to $\dot{v}$ will also be constant; and the contrary. But, by similar triangles, EH $= ar + \sqrt{(r^2-y^2)}$: GH $= y$: : ED $= ar$: DF $= \frac{ary}{ar+\sqrt{(r^2-y^2)}} = v$; the fluxion of which is $ar\dot{y} \times \frac{r^2+arw}{w(ar+w)^2} = \dot{v}$; putting $w = \sqrt{(r^2-y^2)} =$ DH; also $\dot{z} = \dot{y} \times \frac{r}{\sqrt{(r^2-y^2)}} = \dot{y} \times \frac{r}{w}$. Hence then $\dot{z} : \dot{v} :: \frac{r}{w} : ar \times \frac{r^2+arw}{w(ar+w)^2} :: 1 : ar \times \frac{r+aw}{(ar+w)^2}$; which is evidently a variable ratio. Therefore there is no such fixed point E, as that mentioned in the problem.

Corollary 1.—Hence then it appears, that the common method of finding the side of a polygon inscribed in a circle, by drawing a line from a certain fixed point E, through F and G, making AF to AC as 2 is to the number of sides of the polygon, is not generally true.

Corol. 2.—But such a point E may be found, as shall render that construction at least *nearly* true, in the following manner. Suppose the line EFG to revolve about E, from B to A: at B, the arc BG and the line DF arise in the ratio of BE to DE; and at A they are in the ratio of BA to AD or DB; therefore make these two ratios equal to each other, and it will determine the point E, so that the ratios in all the intermediate points, or situations, will be nearly equal: thus then, BE : DE : : BA : AD : : $p : 2$, making $p = 3\cdot1416$; or BD : DE : : $p - 2 : 2$; hence DE $= \frac{2}{p-2} \times$ BD $= 1\cdot752$ BD $= \frac{7}{4}$BD very nearly. If, therefore, DE be taken to DA as 7 to 4; then any line drawn from E, to cut the diameter AC, and the semicircumference ABC, it will very nearly cut them proportionally. Therefore, if a polygon is to be inscribed, or if the whole circumference is to be divided into any number of equal parts; first divide the diameter into the same number of parts, and through the 2d point of division draw EFG, so will AG be one of the equal parts very nearly.

Corol. 3.—The number 1·752 being equal to $\sqrt{3}$ nearly, for $\sqrt{3} = 1{\cdot}732$; therefore, if DE be taken to DA as $\sqrt{3}$ to 1, the point E will be found answering the same purpose as before, but not quite so near as the former. And here, because DA : DE : : 1 : $\sqrt{3}$, therefore DE is the perpendicular of an equilateral triangle described on AC. Hence then, if with the centres A, C, and radius AC, two arcs be described, they will intersect in the point E, nearly the same as before. And this is the method in common practice; but it is not so near the truth as the construction in the 2d Corollary.

Corol. 4.—Hence also a right line is found equal to the arc of a circle nearly: for BG is $= \frac{11}{7}$ DF nearly. And this is the same as the ratio of 11 to 7, which Archimedes gave for the ratio of the semicircumference to the diameter, or 22 to 7 the ratio of the whole circumference to the diameter. But the proportion is here rendered general for any arc of the circle, as well as for the whole circumference.

TRACT XVI.

ON PLANE TRIGONOMETRY WITHOUT TABLES.

THE cases of trigonometry are usually calculated by means of tables of sines, tangents or secants, either of their natural numbers, or their logarithms. But the calculations may also be made without any such tables, to a tolerable degree of accuracy, by means of the theorems and rules contained in the following propositions and corollaries.

PROPOSITION.

If $2a$ denote a side of any triangle, A the number of degrees contained in its opposite angle, and r the radius of the circle circumscribing the triangle: Then the value of A is equal to $57{\cdot}2957795 \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.

For, since $2a$ is the chord of the arc on which the angle, whose measure is A, insists; a will be the sine of half that arc, or the sine of the angle to the radius r, since an angle in the circumference of a circle is measured by half the arc on which it stands; now it is well known that the said half arc z is equal to

$a + \frac{a^3}{2.3r^2} + \frac{3a^5}{2.4.5r^4} + \frac{3.5a^7}{2.4.6.7r^6}$ &c; and, $3{\cdot}14159r$ denoting half the circumference of the same circle, or the arc of 180 degrees, it will be

as $3{\cdot}14159r : 180^\circ :: z : \frac{180z}{3{\cdot}14159r} = \frac{57{\cdot}2957795z}{r}$

$= 57{\cdot}2957795 \times (\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7}$ &c,) the degrees in the angle or half arc.

Corollary 1.—By reverting the above series, we obtain

$$\frac{a}{r} = \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \text{ \&c};$$

$$\text{putting } n = 57{\cdot}2957795 = \frac{180}{3{\cdot}14159 \text{ \&c.}}.$$

Corollary 2.—If $2a$ be the hypothenuse of a right-angled triangle, a will be $= r$, and then the general series will become $n \times (a +$

$\frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7}$ &c) $= 90$, or $\frac{90}{n} = \frac{90 \times 3{\cdot}14159 \text{ \&c}}{180} =$

$$\frac{3.14159 \text{ \&c}}{2} = 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{2.5.7}{2.4.6.8.9} \text{ \&c.}$$

Corol. 3.—Since the chord of 60 degrees is = the radius, or the sine of 30 degrees = half the radius, putting a for $\frac{1}{2}r$ in the general series, will give $n \times (\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7}$ &c $= 30$; and hence the sum of the infinite series

$$\frac{1}{2}+\frac{1}{2.3.2^3}+\frac{3}{2.4.5.2^5}+\frac{3.5}{2.4.6.7.2^7}\ \&c,$$

$$\text{is} = \frac{30}{n} = \frac{30 \times 3{\cdot}14159\ \&c}{180} = \frac{3{\cdot}14159\ \&c}{6} =$$

$\frac{1}{6}$th of the circumference of the circle whose diameter is 1.

Corol. 4.—It might easily be shown, from the principles of common geometry, that the sine of 60 degrees is to the radius, as $\frac{1}{2}\sqrt{3}$ is to 1; substituting then $\frac{1}{2}r\sqrt{3}$ for a in the general series, we shall have $n\sqrt{3} \times (\frac{1}{2}+\frac{3}{2.3.2^3}+\frac{3.3^2}{2.4.5.2^5}+\frac{3.5.3^3}{2.4.6.7.2^7}$ &c) $=$ 60; and hence the sum of the infinite series

$$\frac{1}{2}+\frac{3}{2.3.2^3}+\frac{3.3^2}{2.4.5.2^5}+\frac{3.5.3^3}{2.4.6.7.2^7}\ \&c, \text{ will be}$$

$= \frac{60}{n\sqrt{3}} = \frac{60 \times 3{\cdot}14159\ \&c}{180\sqrt{3}} = \frac{3{\cdot}14159\ \&c}{3\sqrt{3}}$, and is therefore to the infinite series in the 3d corollary, as 2 is to $\sqrt{3}$.

Corol. 5.—If b, c be the halves of the other two sides of the triangle, and B, C the degrees contained in their opposite angles; since $\text{B} = n \times (\frac{b}{r}+\frac{b^3}{2.3r^3}+\frac{3b^5}{2.4.5r^5}$ &c), and $\text{C} = n \times (\frac{c}{r}+\frac{c^3}{2.3r^3}$ &c, and the 3 angles of any triangle are equal to 180 degrees; we shall have $180 = \text{A}+\text{B}+\text{C} = n \times (\frac{a+b+c}{r}+\frac{a^3+b^3+c^3}{2.3r^3}$ &c), or the sum of the infinite series

$$\frac{a+b\ c}{r}+\frac{1}{2.3}\cdot\frac{a^3+b^3+c^3}{r^3}+\frac{3}{2.4.5}\cdot\frac{a^5+b^5+c^5}{r^5}+\frac{3.5}{2.4.6.7}\cdot\frac{a^7+b^7+c^7}{r^7}$$

&c, will be $= \frac{180}{n} = \frac{180 \times 3{\cdot}14159\ \&c}{180} = 3.14159$ &c $=$ the circumference of a circle whose diameter is 1; a, b, c, being the halves of the three sides of any triangle, and r the radius of its circumscribing circle.

Corol. 6.—Since, by theor. 3, $b : a+c :: a-c : \frac{aa-cc}{b} =$ half the difference of the segments of the base (b) made by a perpendicular demitted from its opposite angle, and $b + \frac{aa-cc}{b} = \frac{aa+bb-cc}{b}$ = the segment adjoining to the side $2a$, we shall have $\sqrt{(4a^2 - \frac{(aa+bb-cc)^2}{bb})} = \frac{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}{b}$ for the value of the said perpendicular to the base; and hence $\frac{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}{b} : 2a :: c : \frac{2abc}{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}} = r$ the radius of the circumscribing circle.

Having now found the value of r, we can calculate all the cases of trigonometry without any tables, and without reducing oblique triangles to right-angled ones; for, having any three parts (except the three angles) given, we can find the rest from these five equations following:

$$1.\ r = \frac{2abc}{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}.$$

$$2.\ \text{A} = n \times (\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}\ \&c.)$$

$$3.\ \text{B} = n \times (\frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5} + \frac{3.5b^7}{2.4.6.7r^7} + \frac{3.5.7b^9}{2.4.6.8.9r^9}\ \&c.)$$

$$4.\ \text{C} = n \times (\frac{c}{r} + \frac{c^3}{2.3r^3} + \frac{3c^5}{2.4.5r^5} + \frac{3.5c^7}{2.4.6.7r^7} + \frac{3.5.7c^9}{2.4.6.8.9r^9}\ \&c.)$$

$$5.\ \text{A} + \text{B} + \text{C} = 180.$$

And, for the more convenience, we may add the three following, which are derived from the 2d, 3d, and 4th, by reversion of series.

$$6.\ a = r \times (\frac{\text{A}}{n} - \frac{\text{A}^3}{2.3n^3} + \frac{\text{A}^5}{2.3.4.5n^5} - \frac{\text{A}^7}{2.3.4.5.6.7n^7}\ \&c.)$$

$$7.\ b = r \times (\frac{\text{B}}{n} - \frac{\text{B}^3}{2.3n^3} + \frac{\text{B}^5}{2.3.4.5n^5} - \frac{\text{B}^7}{2.3.4.5.6.7n^7}\ \&c.)$$

$$8.\ c = r \times (\frac{\text{C}}{n} - \frac{\text{C}^3}{2.3n^3} + \frac{\text{C}^5}{2.3.4.5n^5} - \frac{\text{C}^7}{2.3.4.5.6.7n^7}\ \&c.)$$

Where $n = 57{\cdot}2957795$ &c.

EXAMPLE.

Suppose we take here the following example, in which are given the two sides $2b = 345$, $2c = 232$, and the angle opposite to $2c = 37° 20' = 37\frac{1}{3}$ degrees $= \text{C}$. Then since $\frac{\text{C}}{n} = \frac{37\frac{1}{3} \times 3{\cdot}14159 \ \&c}{180} = {\cdot}651589587$, we have $c = \frac{232}{2}$ $= 116 = r \times ({\cdot}651589587 - {\cdot}04610744 + {\cdot}00097879 - {\cdot}000009894 + {\cdot}000000058 \ \&c) = r \times ({\cdot}652568435 - {\cdot}046117334) = {\cdot}6064511r$. Hence $r = \frac{116}{{\cdot}6064511} = 191{\cdot}27677$;

and $\frac{b}{r} = \frac{345 \times {\cdot}6064511}{2 \times 116} = {\cdot}9018346$.

Again, $\text{B} = 57{\cdot}2957795 \times 1{\cdot}12402$ (the sum of the series in the 3d equation) $= 64{\cdot}4016$ degrees $= 64° 24'$.

And $\text{A} = 180 - 37\frac{1}{3} - 64{\cdot}4016 = 180 - 101{\cdot}735 = 78°{\cdot}265$ $= 78° 16'$ nearly.

Lastly, $\frac{\text{A}}{n}$ being $= \frac{78{\cdot}265}{57{\cdot}2957795} = 1{\cdot}365982$, and $r =$ $191{\cdot}27677$, from the 5th equation we have $a = 191{\cdot}27677 \times$ $(1{\cdot}365982 - {\cdot}4247992 + {\cdot}0396379 - {\cdot}0017607 + {\cdot}0000288$ $- {\cdot}0000005) = 191{\cdot}27677 \times {\cdot}9790883 = 187{\cdot}27684$.
And hence $2a = 374{\cdot}55368 =$ the third side of the triangle.

Corol. 7.—As the series by which an angle is found, often converges very slowly, I have inserted the following approximation of it; viz.

$\text{A} = n \times (\frac{4}{3}\sqrt{(2 - 2\sqrt{(1 - \frac{aa}{rr})})} - \frac{a}{3r})$ nearly; where the letters denote the same quantities as in the above series. For since $\text{P} = \sqrt{(2 - 2\sqrt{(1 - \frac{aa}{rr})})}$ is $= \frac{a}{r} + \frac{a^3}{2.4r^3} + \frac{7a^5}{2.4.16r^5}$ &c,

and $\frac{\text{A}}{n}$ is $= \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5^5}$ &c,

we shall have, by taking the former of these from the latter, $\frac{\text{A}}{n} - \text{P} = \frac{a^3}{24r^3} + \frac{13a^5}{640r^5}$ &c. But, from the first series, $\frac{1}{3}\text{P} - \frac{a}{3r} = \frac{a^3}{24r^3} + \frac{7a^5}{384r^5}$ &c; hence, by subtracting the latter from the former, it gives

$\frac{A}{n} - P - \frac{1}{3}P + \frac{a}{3r} = \frac{a}{n} - \frac{4}{3}P + \frac{a}{3r} = \frac{a^5}{480r^5}$ &c ; and

$A = n \times (\frac{4}{3}P - \frac{a}{3r} = n \times (\frac{4}{3}\sqrt{(2-2\sqrt{(1-\frac{aa}{rr})})} - \frac{a}{3r})$ nearly.

Corol. 7.—And again, since $\frac{4}{105} \times (P - q - \frac{1}{8}q^3 = \frac{1}{480}q^5$ &c ; where q is $= \frac{a}{r}$; by subtracting this from $\frac{A}{n} - \frac{4P - q}{3} = \frac{1}{480}q^6$ &c, and reducing, there will be obtained $A = \frac{n}{105} \times (144P - 39q - \frac{1}{2}q^3) = \frac{n}{105} \times (144\sqrt{2-2\sqrt{(1-q^2)}}) - 39q - \frac{1}{2}q^5$, which will commonly give the angle exact to within a minute of the truth. Where note, that the constant quantity $\frac{n}{105}$ is $= \cdot 54567409$. And from the whole may be drawn the following general problem.

PROBLEM.

To perform all the Cases of Trigonometry without any Tables.

Having any three parts of a triangle given, except the three angles, the other three parts may be found, by some of the following six general theorems.

1. $A = \frac{1}{3}n \times (4\sqrt{(2-2\sqrt{(-\frac{a^2}{r^2})})} - \frac{a}{r})$ nearly. Or

$A = \frac{n}{105} \times (144\sqrt{(2-2\sqrt{(1-\frac{a^2}{r^2})})} - 39\frac{a}{r} - \frac{a^3}{2r^3})$ more nearly.

2. $A = n \times (\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.)

3. $a = r \times (\frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7}$ &c.)

4. $r = \dfrac{a}{\frac{A}{n} - \frac{A^3}{2.3.n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \text{ &c.}}$.

5. $r = \dfrac{2abc}{\sqrt{(a^2b^2 - (a^2+b^2-c^2)^2)}}$

$= \dfrac{2abc}{\sqrt{[(a+b+c)\times(a+b-c)\times(a-b+c)\times(-a+b+c)]}}.$

6. $c = \sqrt{[a^2+b^2-2ab\sqrt{(1-(\frac{\text{C}}{n}-\frac{\text{C}^3}{2.3n^3}-\frac{\text{C}^5}{2.3.4.5n^5}\ \&c)^2}]}.$

Where a, b, c, are the halves of the three sides of the triangle, and A the number of degrees in the angle opposite the side $2a$, and C the degrees in the angle opposite the side $2c$; also r is the radius of the circumscribed circle;

and $n = \frac{180}{3\cdot14159} = 57\cdot2957795$, or $\frac{n}{105} = \cdot54567409$.

EXAMPLE.

Thus, if the three sides be given, as for example $a = 13$, $b = 14$, $c = 15$. Then is $r = 16\frac{1}{4}$, and the angles by these theorems come out as follow; viz.

Angles by the Theor.		The true Angles.
53° 7′	- - angle A - -	53° 7′$\frac{4}{5}$
59 28	- - angle B - -	59 29$\frac{2}{5}$
67 19	- - angle C - -	67 22$\frac{4}{5}$
179 54	sum of all	180 00

TRACT XVII.

ON MACHIN'S QUADRATURE OF THE CIRCLE.

SINCE the chief advantage of this method consists in taking small arcs, whose tangents shall be numbers easy to manage, Mr. Machin very properly considered, that as the tangent of 45° is 1; and that the tangent of any arc being given, the tangent of double that arc can easily be found; if there be assumed some small simple number for the tangent of an arc, and then the tangent of the double arc be continually taken, till a tangent be found nearly equal to 1, the tangent of 45°, by taking the tangent answering to the small difference between 45° and this multiple, there would be obtained two very small tangents, viz. the tangent first assumed, and the tangent of the difference between 45° and the multiple arc; and that therefore the lengths of the arcs corresponding to these two tangents being calculated, and the arc belonging to the tangent first assumed being as often doubled as the multiple denotes, the result increased or diminished by the other arc, would be the arc of 45°, according as the multiple arc should be below or above it.

Having thus thought of his method, by a few trials he was lucky enough to find a number, and perhaps the only one, proper for this purpose, viz, knowing that the tangent of $\frac{1}{4}$ of 45° is nearly $= \frac{1}{5}$, he assumed $\frac{1}{5}$ as the tangent of an arc: then since, if t be the tangent of an arc, the tangent of the double arc will be $\frac{2t}{1-tt}$ the radius being 1; the tangent of an arc double to that of which $\frac{1}{5}$ is the tangent, will be $\frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{10}{24} = \frac{5}{12}$, and the tangent of the double of this last is $\frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{120}{119}$; which, being very near equal to 1, shows that the arc which is equal to 4 times the first, is very near 45°. Then, since the tangent of the difference between 45° and an arc whose tangent is T, is $\frac{\text{T}-1}{\text{T}+1}$, we shall have the tangent of the difference between 45° and the arc whose tangent is $\frac{120}{119}$ equal to $\frac{\frac{120}{119}-1}{\frac{120}{119}+1} = \frac{120-119}{120+119} = \frac{1}{239}$.

Now by calculating, from the general series, the arcs whose tangents are $\frac{1}{5}$ and $\frac{1}{239}$, which may be quickly done, by reason of the smallness and the simplicity of the numbers, and taking the latter arc from 4 times the former, the remainder will be the arc of 45°. And this is Mr. Machin's ingenious quadrature of the circle.

But it was by means of Dr. Halley's method that Mr.

Machin found the circumference of a circle, whose diameter is 1, to be

3·1415926535,8979323846,2643383279,5028841971,6939937510,
5820974944,5923078164,0628620899,8628034825,3421170679 +,

true to above 100 places of figures.

Or, by substituting the above numbers in Machin's series, we get the series $(\frac{16}{5}-\frac{4}{239})-\frac{1}{3}(\frac{16}{5^3}-\frac{4}{239^3})+\frac{1}{5}(\frac{16}{5^5}-\frac{4}{239^5}$ &c, equal to the semicircumference whose radius is 1, or the whole circumference whose diameter is 1. Being the series published by Mr. Jones, and which he acknowledges he received from Mr. Machin.

But because the arc whose tangent is $\frac{1}{5}$, is $=$ 2 times the arc whose tangent is $\frac{1}{10}$, minus the arc to tangent $\frac{1}{515}$; (for $\frac{\frac{2}{10}}{1-\frac{1}{100}}=\frac{20}{99}=$ tangent of twice the arc to tangent $\frac{1}{10}$, and $\frac{\frac{20}{99}-\frac{1}{5}}{1+\frac{4}{99}}=\frac{1}{515}=$ tang. of diff. between the arcs whose tangents are $\frac{20}{99}$ and $\frac{1}{5}$); therefore 8 times arc to tangent $\frac{1}{10}-4$ times arc to tang. $\frac{1}{515}$ − arc to tang. $\frac{1}{239}=$ arc of 45°, or whose tang. is 1. Which is much easier than Machin's way. And various other methods may easily be discovered from the same principles.

TRACT XVIII.

A NEW AND GENERAL METHOD OF FINDING SIMPLE AND QUICKLY-CONVERGING SERIES; BY WHICH THE PROPORTION OF THE DIAMETER OF A CIRCLE TO ITS CIRCUMFERENCE MAY EASILY BE COMPUTED TO A GREAT MANY PLACES OF FIGURES.

In examining the methods of Mr. Machin and others, for computing the proportion of the diameter of a circle to its circumference, I discovered the method explained in this paper. This method is very general, and discovers many series that are fit for the abovementioned purpose. The advantage of this method is chiefly owing to the simplicity of the series by which an arc is found from its tangent. For, if t denote the tangent of an arc a, the radius being 1, then it is well known, that the arc a is denoted by the infinite series, $t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 -$ &c; where the form is as simple as can be desired. And it is evident that nothing further is required, than to contrive matters so, as that the value of the quantity t, in this series, may be both a small and a very simple number. Small, that the series may be made to converge sufficiently fast; and simple, that the several powers of t may be raised by easy multiplications, or easy divisions.

Since the first discovery of the above series, many authors have used it, and that after different methods, for determining the length of the circumference to a great number of figures. Among these were, Dr. Halley, Mr. Abra. Sharp, Mr. Machin, and others, of our own country; and M. de Lagney, M. Euler, &c, abroad. Dr. Halley used the arc of 30°, or $\frac{1}{12}$th of the circumference, the tangent of which being $=\sqrt{\frac{1}{3}}$, by substituting $\sqrt{\frac{1}{3}}$ for t in the above series, and multiplying by 6, the semicircumference is =

$6\sqrt{\frac{1}{3}} \times (1 - \frac{1}{3.3} + \frac{1}{5.3^2} + \frac{1}{7.3^3} + \frac{1}{9.3^4} -$ &c); which series is, to be sure, very simple; but its rate of converging is not very great, on which account a great many terms must be used to compute the circumference to many places of figures. By this very series however, the industrious Mr. Sharp computed the circumference to 72 places of figures; Mr. Machin extended it to 100; and M. de Lagney, still by the same series, continued it to 128 places of figures. But though this series, from the 12th part of the circumference, does not converge very quickly, it is perhaps the best aliquot part of the circumference which can be employed for this purpose; for when smaller arcs, which are exact aliquot parts, are used, their tangents, though smaller, are so much more complex, as to render them, on the whole, more operose in the application: this will easily appear, by inspecting some instances that have been given in the introductions to logarithmic tables. One of these methods is from the arc of 18°, the tangent of which is $\sqrt{(1-2\sqrt{\frac{1}{5}})}$; another is from the arc of $22°\frac{1}{2}$, the tangent of which is $\sqrt{2}-1$; and a third is from the arc of 15°, the tangent of which is $2-\sqrt{3}$. All of which are evidently too complex to afford an easy application to the general series.

In order to a still further improvement of the method by the above general series, Mr. Machin, by a very singular and excellent contrivance, has greatly reduced the labour naturally attending it. I have given an analysis of his method, or a conjecture concerning the manner in which it is probable Mr. Machin discovered it, in my Treatise on Mensuration; which, I believe, is the only book in which that method has been investigated, as it is repeated in the foregoing Tract. For though the series discovered by that method were published by Mr. Jones, in his "Synopsis Palmariorum Matheseos," which was printed in the year 1706, he has given them merely by themselves, without the least hint of the manner in which they were obtained. The result shows, that the proportion of the diameter to the circumference, is equal to that of 1 to quadruple the sum of the two series,

$$\frac{4}{5}\times(1-\frac{1}{3.5^2}+\frac{1}{5.5^4}-\frac{1}{7.5^6}+\frac{1}{9.5^8}\ \&c)\text{ and}$$

$$\frac{1}{239}\times(1-\frac{1}{3.239^2}+\frac{1}{5.239^4}-\frac{1}{7.239^6}+\frac{1}{9.239^8}\ \&c).$$

The slower of which series converges almost thrice as fast as Dr. Halley's, raised from the tangent of 30°. The latter of these two series converges still a great deal quicker; but then the large prime number 239, by the reciprocals of the powers of which the series converges, occasions such long and tedious divisions, as to counter-balance its quickness of convergency; so that the former series is summed with rather more ease than the latter, to the same number of places of figures. Mr. Jones, in his "Synopsis," mentions other series besides this, which he had received from Mr. Machin for the same purpose, and drawn from the same principle.

But we may conclude this to be the best of them all, as he did not publish any other besides it.

M. Euler too, in his "Introductio in Analysin Infinitorum," by a contrivance something like Mr. Machin's, discovers, that $\frac{1}{2}$ and $\frac{1}{3}$ are the tangents of two arcs, the sum of which is just 45°; and that therefore the diameter is to the circumference, as 1 to quadruple the sum of the two following series,

$$\frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} \text{ \&c) and}$$

$$\frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} \text{ \&c).}$$

Both which series converge much faster than Dr. Halley's, and are yet at the same time made to converge by the powers of numbers producing only short divisions; that is, divisions performed in one line, or without writing down any thing besides the quotients.

I come now to explain my own method, which indeed bears some little resemblance to the methods of Machin and Euler; but then it is more general, and discovers, as particular cases of it, both the series of those gentlemen, and many others, some of which are fitter for this purpose than theirs are.

This method then consists in finding out such small arcs, as have for tangents some small and simple vulgar fractions, the radius being denoted by 1, and such also that some multiple of those arcs shall differ from an arc of 45°, the tangent of which is equal to the radius, by other small arcs, which also shall have tangents denoted by other such small and simple vulgar fractions. For it is evident, that if such a small arc can be found, some multiple of which has such a proposed difference, from an arc of 45°, then the lengths of these two small arcs will be easily computed from the general series, because of the smallness and simplicity of their tangents; after which, if the proper multiple of the first arc be increased or diminished by the other arc, the result will be the length of an arc of 45°, or $\frac{1}{8}$th of the circumference. And the manner in which I discover such arcs is thus:

Let T, t, denote any two tangents, of which T is the greater, and t the less: then it is known, that the tangent of the difference of the corresponding arcs is equal to $\frac{T-t}{1+Tt}$. Hence, if t, the tangent of the smaller arc, be successively denoted by each of the simple fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c, the general expression for the tangent of the difference between the arcs will become respectively

$\frac{2T-1}{2+T}$, $\frac{3T-1}{3+T}$, $\frac{4T-1}{4+T}$, $\frac{5T-1}{5+T}$, &c; so that if T be expounded by any given number, then these expressions will give the tangent of the difference of the arcs in known numbers, according to the values of t, severally assumed respectively. And if, in the first place, T be equal to 1, the tangent of 45°, the foregoing expressions will give the tangent of an arc, which is equal to the difference between that of 45° and the first arc; or that of which the tangent is one of the numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. Then, if the tangent of this difference, just now found, be taken for T, the same expressions will give the tangent of an arc, which is equal to the difference between the arc of 45° and the double of the first arc. Again, if for T we take the tangent of this last found difference, then the foregoing expressions will give the tangent of an arc, equal to the difference between that of 45° and the triple of the first arc. And again taking this last found tangent for T, the same theorem will produce the tangent of an arc equal to the difference between that of 45° and the quadruple of the first arc; and so on, always taking for T the tangent last found, the same expressions will give the tangent of the difference between the arc of 45° and the next greater multiple of the first arc; or that of which the tangent was at first assumed equal to one of the small numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. This operation, being continued till some of the expressions give such a fit, small, and simple fraction as is required, is then at an end, for we have then found two such small tangents as were required, viz, the tangent last found, and the tangent first assumed.

Here follow the several operations adapted to the several values of t. The letters a, b, c, d, &c, denote the several successive tangents.

1. Take $t = \frac{1}{2}$, then the theorem $\frac{2T-1}{2+T}$ gives $a=\frac{1}{3}$, $b=-\frac{1}{7}$. Therefore the arc of 45°, or $\frac{1}{8}$th of the circumference, is either equal to the sum of the two arcs of which $\frac{1}{2}$ and $\frac{1}{3}$ are the tangents, or to the difference between the arc of which the tangent is $\frac{1}{7}$, and the double of the arc of which the tangent is $\frac{1}{2}$; that is, putting A = the arc of 45°, then

$$A = \begin{cases} \frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} - \text{\&c.}) \\ + \frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} - \text{\&c.}) \end{cases}$$

$$\text{Or, } A = \begin{cases} 1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} - \text{\&c,} \\ - \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \frac{1}{9.49^4} - \text{\&c).} \end{cases}$$

The former of these values of A is the same with that before mentioned, as given by M. Euler; but the latter is much better, as the powers of $\frac{1}{49}$ converge much faster than those of $\frac{1}{9}$.

Corol.—From double the former of these values of A, subtracting the latter, the remainder is,

$$A = \begin{cases} \frac{2}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \text{\&c.}) \\ + \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \text{\&c.}) \end{cases}$$

which is a much better theorem than either of the former.

2. If t be taken $= \frac{1}{3}$, then the expression $\frac{3T-1}{3+T}$ gives $a = \frac{1}{2}$, $b = \frac{1}{7}$. Here the value of $a = \frac{1}{2}$ gives the same expression for the value of A as the first in the foregoing case, and the value of $b = \frac{1}{7}$ gives the value of A the very same as in the corollary to the case above.

3. Taking $t = \frac{1}{4}$, the expression $\frac{4T-1}{4+T}$ gives $a = \frac{3}{5}$, $b = \frac{7}{23}$, $c = \frac{5}{99}$, $d = -\frac{79}{401}$. Where it is evident that the value

of $c = \frac{5}{99}$ is the fittest number afforded by this case; and hence it appears, that the arc of 45° is equal to the sum of the arc of which the tangent is $\frac{5}{99}$, and the triple of the arc of which the tangent is $\frac{1}{4}$.

$$\text{Or that A} = \begin{cases} \frac{3}{4} \times (1 - \frac{1}{3.16} + \frac{1}{5.16^2} - \frac{1}{7.16^3} + \&c) \\ + \frac{5}{99} \times (1 - \frac{5^2}{3.99^2} + \frac{5^4}{5.99^4} - \frac{5^6}{7.99^6} + \&c). \end{cases}$$

Which is the best theorem that we have yet found, because the number 99 resolves into the two easy factors 9 and 11.

4. Let now t be taken $= \frac{1}{5}$; then the expression $\frac{5\text{T}-1}{5+\text{T}}$ gives $a = \frac{2}{3}$, $b = \frac{7}{17}$, $c = \frac{9}{46}$, $d = -\frac{1}{239}$. Where it is evident that the last number, or the value of d, is the fittest of those produced in this case; and from which it appears, that the arc of 45° is equal to the difference between the arc of which the tangent is $\frac{1}{239}$, and quadruple the arc of which the tangent is $\frac{1}{5}$. Or that

$$\text{A} = \begin{cases} \frac{4}{5} \times (1 - \frac{1}{3.5^2} + \frac{1}{5.5^4} - \frac{1}{7.5^6} + \&c.) \\ -\frac{1}{239} \times (1 - \frac{1}{3.239^2} + \frac{1}{5.239^4} - \frac{1}{7.239^6} + \&c). \end{cases}$$

Which is the very theorem that was invented by Mr. Machin, as we have before mentioned.

5. Take now $t = \frac{1}{6}$; then the expression $\frac{6\text{T}-1}{6+\text{T}}$ gives $a = \frac{5}{7}$, $b = \frac{23}{47}$, $c = \frac{91}{305}$, $d = \frac{241}{1921}$, $e = \frac{-475}{11767}$. Of which numbers it is evident that none are fit for our purpose.

6. Again, take $t = \frac{1}{7}$, and the expression $\frac{7\text{T}-1}{7+\text{T}}$ will give $a = \frac{3}{4}$, $b = \frac{17}{31}$, $c = \frac{11}{28}$, $d = \frac{49}{205}$, $e = \frac{69}{742}$, $f = -\frac{259}{5263}$. Neither are any of these numbers fit for our purpose.

7. In like manner take $t = \frac{1}{8}$, so shall $\frac{8\text{T}-1}{8+\text{T}}$ give $a = \frac{7}{9}$, $b = \frac{47}{79}$, $c = \frac{297}{679}$, $d = \frac{1697}{5729}$, $e = \frac{7847}{47529}$, $f = \frac{14047}{388079}$.

8. And if t be taken $= \frac{1}{9}$, the expression $\frac{9\text{T}-1}{9+\text{T}}$ will give $a = \frac{4}{5}$, $b = \frac{31}{49}$, $c = \frac{115}{236}$, $d = \frac{799}{2239}$, $e = \frac{2467}{10475}$, &c.

9. Also, if we take $t = \frac{1}{10}$, the expression $\frac{10\text{T}-1}{10+\text{T}}$ will give $a = \frac{9}{11}$, $b = \frac{79}{119}$, $c = \frac{671}{1269}$, $d = \frac{5441}{13361}$, $e = \frac{41049}{139051}$, &c.

10. Further, if we take $t = \frac{1}{11}$, the expression $\frac{11\text{T}-1}{11+\text{T}}$ gives $a = \frac{5}{6}$, $b = \frac{49}{71}$, $c = \frac{234}{415}$, $d = \frac{2159}{4799}$, $e = \frac{9475}{27474}$, &c.

11. Lastly, if we take $t = \frac{1}{12}$, the expression $\frac{12\text{T}-1}{12+\text{T}}$ gives $a = \frac{11}{13}$, $b = \frac{113}{167}$, $c = \frac{41}{73}$, $d = \frac{419}{917}$, $e = \frac{4111}{11423}$, &c.

Here it is evident, that none of these latter cases afford any numbers that are fit for this purpose. And to try any other fractions less than $\frac{1}{12}$ for the value of t, does not seem likely to answer any good purpose, especially as the divisors after 12 become too large to be managed in the easy way of short division in one line.

By the foregoing means it appears then, that we have discovered five different forms of the value of A, or $\frac{1}{4}$th of the semicircumference, all of which are very proper for readily computing its length; viz, three forms in the first case and its corollary, one in the 3d case, and one in the 4th case. Of these, the first and last are the same as those invented by Euler and Machin respectively, and the other three are quite new, as far as I know.

But another remarkable excellence attending the first three of the before mentioned series, is, that they are capable of being changed into others which not only converge still faster, but in which the converging quantity shall be $\frac{1}{10}$, or some multiple or sub-multiple of it, and so the powers of it raised with the utmost ease. The series, or theorems, here meant are these three:

$$1\text{st, A} = \begin{cases} \frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \&c) \\ + \frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \&c). \end{cases}$$

$$2\text{dly, A} = \begin{cases} 1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \&c \\ - \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \&c). \end{cases}$$

$$3\text{dly, A} = \begin{cases} \frac{2}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \&c) \\ + \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \&c). \end{cases}$$

Now if each of these be transformed, by means of the differential series, in cor. 3 p. 64 of the late Mr. Simpson's Mathematical Dissertations, they will become of these very commodious forms, viz,

$$1\text{st, A} = \begin{cases} \frac{4}{10} \times (1 + \frac{4}{3.10} + \frac{8\alpha}{5.10} + \frac{12\beta}{7.10} + \&c) \\ + \frac{3}{10} \times (1 + \frac{2}{3.10} + \frac{4\alpha}{5.10} + \frac{6\beta}{7.10} + \&c). \end{cases}$$

$$2\text{dly, A} = \begin{cases} \frac{4}{5} \times (1 + \frac{4}{3.10} + \frac{8\alpha}{5.10} + \frac{12\beta}{7.10} + \&c) \\ - \frac{7}{50} \times (1 + \frac{4}{3.100} + \frac{8\alpha}{5.100} + \frac{12\beta}{7.100} + \&c). \end{cases}$$

$$3\text{dly, A} = \begin{cases} \frac{6}{10} \times (1 + \frac{2}{3.10} + \frac{4\alpha}{5.10} + \frac{6\beta}{7.10} + \&c) \\ + \frac{7}{50} \times (1 + \frac{2}{3.50} + \frac{4\alpha}{5.50} + \frac{6\beta}{7.50} + \&c). \end{cases}$$

Where α, β, γ, &c, denote always the preceding terms in each series.

Now it is evident that all these latter series are much easier than the former ones, to which they respectively correspond; for, because of the powers of 10 here concerned, we have little more to do than to divide by the series of odd numbers 1, 3, 5, 7, 9, &c.

Of all these three forms, the 2d is the fittest for computing the required proportion ; because, of the two series of which it consists, the several terms of the one are found from the like terms of the other, by dividing these latter by 10, and its several successive powers, 100, 1000, &c ; that is, the terms of the one consist of the same figures as the terms of the other, only removed a certain number of places farther towards the right hand, in the decuple scale of numbers; and the number of places by which they must be removed, is the same as the distance of each term from the first term of the series, viz, in the 2d term the figures must be moved one place lower, in the 3d term two, in the 4th term three, &c; so that the latter series will consist of but about half the number of the terms of the former. Thus then this method may be said to effect the business by one series only, in which there is little more to do, than to divide by the several numbers 1, 3, 5, 7, &c; for as to the multiplications by the numbers in the numerators of the terms, after they become large, they are easily performed by barely multiplying by the number 2, and subtracting one number from another: for since every numerator is less by 2 than the double of its denominator, if d denote any denominator, exclusive always of the powers of 10, then the coefficient of that term is $\frac{2d-2}{d}$ or $2 - \frac{2}{d}$, by which the preceding term is to be multiplied; to do which therefore, multiply it by 2, that is double it, and divide that double by the divisor d, and subtract the quotient from the said double.

TRACT XIX.

HISTORY OF TRIGONOMETRICAL TABLES, &c.

NECESSITY, the fruitful mother of most useful inventions; gave birth to the various numeral tables employed in trigonometry, astronomy, navigation, &c. Astronomy has been cultivated from the earliest ages. The progress of that science, requiring numerous arithmetical computations of the sides and angles of triangles, both plain and spherical, gave rise to trigonometry; for those frequent calculations suggested the necessity of performing them by the property of similar triangles; and for the ready application of this property, it was necessary that certain lines described in and about circles, to a determinate radius, should be computed, and disposed in tables. Navigation, and the continually improving accuracy of astronomy, have also occasioned as continual an increase in the accuracy and extent of those tables. And this, it is evident, must ever be the case, the improvement of trigonometry uniformly following the improvement of those other useful sciences, for the sake of which it is more especially cultivated.

The ancients performed their trigonometry by means of the chords of arcs, which, with the chords of their supplemental arcs, and the constant diameter, formed all species of right-angled triangles. Beginning with the radius, and the arc whose chord is equal to the radius, they divided them both into 60 equal parts, and estimated all other arcs and chords by those parts, namely all arcs by 60ths of that arc, and all chords by 60ths of its chord or of the radius. At least this method is as old as the writings of Ptolemy, who used the sexagenary arithmetic for this division of chords and arcs, and for astronomical purposes.—And this, by-the-bye, may be the reason why the whole circumference is divided into 360, or 6 times 60, equal parts or degrees, the whole circumference being equal to 6 times the first arc, whose chord is equal to the radius: unless perhaps we are rather to seek for the division of the circle in the number of days in the year; for thus, the ancient year consisting of 360 days, the sun or earth in each day described the 360th part of the orbit; and thence might arise the method of dividing every circle into 360 parts; and, radius being equal to the chord of 60 of those parts, the sexagesimal division, both of the radius and of the parts, might thence follow. Trigonometry however must have been cultivated long before the time of Ptolemy; and indeed Theon, in his commentary on Ptolemy's Almagest, l. i. ch 9, mentions a work of the philosopher Hipparchus, written about a century and a half before Christ, consisting of 12 books on the chords of circular arcs; which must have been a treatise on trigonometry. And Menelaus also, in the first century of Christ, wrote 6 books concerning subtenses or chords of arcs. He used the word *nadir*, of an arc, which he defined to be the right line subtending the double of the arc; so that his nadir of an arc was the double of our sine of the same arc, or the chord of the double arc; and therefore whatever he proves of the former, may be applied to the latter, substituting the double sine for the nadir.

The radius has since been decimally divided; but the sexagesimal divisions of the arc have continued in use to this day. Indeed our countrymen Briggs and Gellibrand, having a general dislike to all sexagesimal divisions, made an attempt at some reformation of this custom, by dividing the degrees of the arcs, in their tables, into centesms or hundredth parts, instead of minutes or 60th parts. The same was also recommended by Vieta and others; and a decimal division of the whole quadrant might perhaps soon have followed, had it not been for the tables of Vlacq, which came out a little after, to every 10 seconds, or 6th parts of minutes.—But the complete reformation would be, to express all arcs by their real lengths, namely in equal parts of the radius decimally divided: according to which method I have nearly completed a table of sines and tangents.

It is not to be doubted that many of the ancients wrote on the subject of trigonometry, being a necessary part of astronomy; though few of their labours on that branch have come to our knowledge, and still fewer of the writings themselves have been handed down to us.

We are in possession of the three books of Menelaus, on spherical trigonometry; but the six books are lost which he wrote upon chords, being probably a treatise on the construction of trigonometrical tables.

The trigonometry of Menelaus was much improved by Ptolemy (Claudius Ptolemæus) the celebrated philosopher and mathematician. He was born at Pelusium; taught astronomy at Alexandria in Egypt; and died in the year of Christ 147, being the 78th year of his age. In the first book of his Almagest, Ptolemy delivers a table of arcs and chords, with the method of construction. This table contains 3 columns; in the 1st are the arcs to every half degree or 30 minutes; in the 2d are their chords, expressed in degrees, minutes and seconds; of which degrees the radius contains 60; and in the 3d column are the differences of the chords answering to 1 minute of the arcs, or the 30th part of the differences between the chords in the 2d column. In the construction of this table, among other theorems, Ptolemy shows, for the first time that we know of, this property of any quadrilateral inscribed in a circle, namely, that the rectangle under the two diagonals, is equal to the sum of the two rectangles under the opposite sides.

This method of computation, by the chords, continued in use till about the middle centuries after Christ; when it was changed for that of the sines, which were about that time introduced into trigonometry by the Arabians, who in other respects much improved this science, which they had received from the Greeks, introducing, among other things, the three or four theorems, or axioms, which we make use of at present, as the foundation of our modern trigonometry.

The other great improvements, that have been made in this branch, are due to the Europeans. These improvements they have gradually introduced since they received this science from the Arabians. And though these latter people had long used the Indian or decimal scale of arithmetic, it does not appear that they varied from the Greek or sexagesimal division of the radius, by which the chords and sines had been expressed.

This alteration, it is said, was first made by George Purbach, who was so called from his being a native of a place of that name, between Austria and Bavaria. He was born in 1423, and studied mathematics and astronomy at the university of Vienna, where he was afterwards professor of those sciences, though but for a short time, the learned world quickly suffering a great loss by his immature death, which happened in 1462, at the age of 39 years only. Purbach, besides enriching trigonometry and astronomy with several new tables, theorems, and observations, conceived the radius to be divided into 600,000 equal parts, and computed the sines of the arcs, for every 10 minutes, in such equal parts of the radius, by the decimal notation.

This project of Purbach was completed by his disciple, companion, and successor, John Muller, or Regiomontanus, being so called from the place of his nativity, the little town of Mons Regius, or Koningsberg, in Franconia, where he was born in the year 1436. Regiomontanus not only extended the sines to every minute, the radius being 600,000, as designed by Purbach, but afterwards disliking that scheme as evidently imperfect, he computed them also to the radius 1,000,000, for every minute of the quadrant. He also introduced the tangents into trigonometry, the canon of which he called *fœcundus*, because of the many and great advantages arising from them. Besides these, he enriched trigonometry with many theorems and precepts. Through the benefit of all these improvements, except for the use of logarithms, the trigonometry of Regiomontanus is but little inferior to that of our own time. His treatise, on both plane and spherical trigonometry, is in 5 books; it was written about the year 1464, and was printed in folio at Nuremberg, in 1533. And in the 5th book are also various problems concerning rectilinear triangles, some of which are resolved by means of algebra—a proof that this science was not wholly unknown in Europe before the treatise of Lucas de Burgo. Regiomontanus died in 1476, at the age of 40 years only; being then at Rome, whither he had been invited by the Pope, to assist in the reformation of the calendar, and where it was suspected he was poisoned by the sons of George Trebizonde, in revenge for the death of their father, which was said to have been caused by the grief he felt on account of the criticisms made by Regiomontanus on his translation of Ptolemy's Almagest.

Soon after this, several other mathematicians contributed to the improvement of trigonometry, by extending and enlarging the tables, though few of their works have been printed; and particularly John Werner of Nuremberg, who was born in 1468, and died in 1528, and who it seems wrote five books on triangles.

About the year 1500, Nicholas Copernicus, the celebrated modern restorer of the true solar system, wrote a brief treatise on trigonometry, both plane and spherical, with the description and construction of the canon of chords, or their halves, nearly in the manner of Ptolemy; to which is subjoined a canon of sines, with their differences, for every 10 minutes of the quadrant, to the radius 100,000. This tract is inserted in the first book of his "Revolutiones Orbium Cœlestium," first printed in folio at Nuremberg, 1543. It is remarkable that he does not call these lines *sines*, but *semisses subtensarum*, namely of the double arcs.—Copernicus was born at Thorn in 1473, and died in 1543.

In 1553 was published the "Canon Fœcundus," or table of tangents, of Erasmus Reinhold, professor of mathematics in the academy of Wurtemburg. He was born at Salfieldt in Upper Saxony, in the year 1511, and died in 1553.

To Francis Maurolyc, abbot of Messina in Sicily, we owe the introduction of the "Tabula Benefica," or canon of secants, which came out about the same time, or a little before. But Lansberg erroneously ascribes this to Rheticus. And the tangents and secants are both ascribed to Reinhold, by Briggs, in his "Mathematica ab antiquis minus cognita," (p. 30, Appendix to Ward's Lives of the Professors of Gresham College.)

Francis Vieta was born in 1540 at Fontenai, or Fontenai-le-Comte, in Lower Poitou, a province of France. He was master of requests at Paris, where he died in 1603, being the 63d year of his age. Among other branches of learning in which he excelled, he was one of the most respectable mathematicians of the 16th century, or indeed of any age. His writings abound with marks of great originality, and the finest genius, as well as intense application. Among them are several pieces relating to trigonometry, which may be found in the collection of his works published at Leyden in 1646, by Francis Schooten, besides another large and separate volume in folio, published in the author's lifetime at Paris in 1579, containing trigonometrical tables, with their construction and use; very elegantly printed, by the king's mathematical printer, with beautiful types and rules; the differences of the sines, tangents and secants, and some other parts, being printed with red ink, for the better distinction; but it is inaccurately executed, as he himself testifies in page 323 of his other works above mentioned. The first part of this curious volume is entitled "Canon Mathematicus, seu ad Triangula, cum Appendicibus," and it contains a great variety of tables useful in trigonometry. The first of these is what he more peculiarly calls "Canon Mathematicus, seu ad Triangula," which contains all the sines, tangents, and secants for every minute of the quadrant, to the radius 100,000, with all their differences; and towards the end of the quadrant the tangents and secants are extended to 8 or 9 places of figures. They are arranged like our tables at present, increasing on the left-hand side to 45 degrees, and then returning upwards by the right hand side to 90 degrees; so that each number and its complement stand on the same line. But here the canon of what we now call tangents is denominated *fœcundus*, and that of the secants *fœcundissimus*. For the general idea prevailing in the form of these tables, is, not that the lines represented by the numbers are those which are drawn in and about a circle, as sines, tangents, and secants, but the three sides of right-angled triangles; this being the way in which those lines had always been considered, and which still continued for some time longer. Hence it is that he considers the canon as a series of plane right-angled triangles, one side being constantly 100,000; or rather as three series of such triangles, for he makes a distinct series for each of the three varieties, namely, according as the hypotenuse, or the base, or the perpendicular, is represented by the constant number 100,000, which is similar to the radius. Making each side constantly 100,000, the other two sides are computed to every magnitude of the acute angle at the base, from 1 minute up to 90 degrees, or the whole quadrant. Each of the three series therefore consists of two parts, representing the two variable sides of the triangle. When the hypotenuse is made the constant number 100,000, the two variable sides of the triangle are the perpendicular and base, or our sine and cosine; when the base is 100,000, the perpendicular and hypotenuse are the variable parts, forming the *canon fœcundus et fœcundissimus*, or our tangent and secant; and when the perpendicular is made the constant 100,000, the series contains the variable base and hypotenuse, or also *canon fœcundus et fœcundissimus*, or our cotangent and cosecant. Of course, therefore, the table consists of 6 columns, 2 for each of the three series, besides the two columns on the right and left for minutes, from 0 to 60 in each degree.

The second of these tables is similar to the first, but all in rational numbers, consisting, like *it*, of three series of two columns each; the radius, or constant side of the triangle, in each series, being 100,000, as before; and the other two sides *accurately* expressed in integers and rational vulgar fractions. So that we have here the canon of *accurate* sines, tangents, and secants, or a series of about 4300 rational right-angled triangles. But then the several corresponding arcs of the quadrant, or angles of those triangles, are not expressed. Instead of them, are inserted, in the first column next the margin, a series of numbers decreasing from the beginning to the end of the quadrant, which are called *numeri primi baseos*. It is from these numbers that Vieta constructs the sides of the three series of right-angled triangles, one side in each series being the constant number 100,000, as before. The theorems by which these series of rational triangles are computed from the *numeri primi baseos*, or marginal numbers, are inserted all in one page at the end of this second table, and in the modern notation they may be briefly expressed thus: Let p denote the primary or marginal number on any line, and r the constant radius or number 100,000: then if r denote the hypotenuse of the right-angled triangle, the perpendicular and base, or the sine and cosine will be respectively,

$\frac{pr}{\frac{1}{4}p^2+1}$ and $r - \frac{2r}{\frac{1}{4}p^2+1}$, (which last we may reduce to $\frac{\frac{1}{4}p^2-1}{\frac{1}{4}p^2+1}r$).

When r denotes the base of the right-angled triangle, the perpendicular and hypotenuse, or the tangent and secant, are expressed by

$\frac{pr}{\frac{1}{4}p^2-1}$ and $r + \frac{2r}{\frac{1}{4}p^2-1}$, (which last we may reduce to $\frac{\frac{1}{4}p^2+1}{\frac{1}{4}p^2-1}r$);

and when r denotes the perpendicular of the right-angled triangle, the base and hypotenuse, or the cotangent and cosecant, are then expressed by

$$\tfrac{1}{4}pr - \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2-1}{p}r\text{), and } \tfrac{1}{4}pr + \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2+1}{p}r\text{).}$$

So that Vieta's general values will be as we have here collected them together in the following expressions, immediately under the words sine, cosine, &c; and just below Vieta's forms I have here placed the others, to which they reduce and are equivalent, which are more contracted, though not so well adapted to the expeditious computation as Vieta's forms.

Sine	Cosine	Tangent	Secant	Cotangent	Cosecant
$\frac{pr}{\frac{1}{4}p^2+1}$	$r - \frac{2r}{\frac{1}{4}p^2+1}$	$\frac{pr}{\frac{1}{4}p^2-1}$	$r + \frac{2r}{\frac{1}{4}p^2-1}$	$\frac{1}{4}pr - \frac{r}{p}$	$\frac{1}{4}pr + \frac{r}{p}$
$\frac{p}{\frac{1}{4}p^2+1}r$	$\frac{\frac{1}{4}p^2-1}{\frac{1}{4}p^2+1}r$	$\frac{p}{\frac{1}{4}p^2-1}r$	$\frac{\frac{1}{4}p^2+1}{\frac{1}{4}p^2-1}r$	$\frac{\frac{1}{4}p^2-1}{p}r$	$\frac{\frac{1}{4}p^2+1}{p}r$

All these expressions, it is evident, are rational; and by assuming p of different values, from the first theorems Vieta computed the corresponding sides of the triangles, and so expressed them all in integers and rational fractions.

To the foregoing principal tables are subjoined several other smaller tables, or short specimens of large ones: as, a table of the sines, tangents and secants, for every single degree of the quadrant, with the corresponding lengths of the arcs, the radius being 100,000,000; another table of the sines, tangents, and secants, for each degree also, expressed in sexagesimal parts of the radius, as far as the third order of part ; also two other tables for the multiplication and reduction of sexagesimal quantities.

The second part of this volume is entitled "Universalium Inspectionum ad Canonem Mathematicum Liber singularis." It contains the construction of the tables, a compendious treatise on plane and spherical trigonometry, with the application of them to a great variety of curious subjects in geometry and mensuration, treated in a very learned manner; as also many curious observations concerning the quadrature of the circle, the duplication of the cube, &c. Computations are here given of the ratio of the diameter of a circle to the circumference, and of the length of the sine of 1 minute, both to many places of figures; by which he found that the sine of 1 minute is between 2,908,881,959 and 2,908,882,056; also, the diameter of a circle being 1000 &c, that the perimeter of the inscribed and circumscribed polygon of 393,216 sides, will be as follows:

perimeter of the inscrib. polygon 314,159,265,35,

perimeter of the circum. polygon 314,159,265,37,

and that therefore the circumference of the circle lies between those two numbers.

Though no author's name appears to the volume we have been describing, there can be no doubt of its being the performance of Vieta; for, besides bearing evident marks of his masterly hand, it is mentioned by himself in several parts of his other works collected by Schooten, and in the preface to those works by Elzevir, the printer of them; as also in Montucla's "Histoire des Mathematiques;" which are the only notices I have ever seen or heard of concerning this book, the copies of which are so rare, that I never saw one besides that which is in my own possession.

In the other works of Vieta, published at Leyden in 1646, by Schooten, as mentioned above, there are several other pieces of trigonometry; some of which, on account of their originality and importance, are very deserving of particular notice in this place. And first, the very excellent theorems, here first of all given by our author, relating to angular sections, the geometrical demonstrations of which are supplied by that ingenious geometrician, Alexander Anderson, then professor of mathematics at Paris, but a native of Aberdeen, and cousin-german to Mr. David Anderson, of Finzaugh, whose daughter was the mother of the celebrated Mr. James Gregory, inventor of the Gregorian telescope. We find here, theorems for the chords, and consequently sines, of the sums and differences of arcs; and for the chords of arcs that are in arithmetical progression, namely, that the 1st or least chord is to the 2d, as any one after the 1st is to the sum of the two next less and greater: for example, as the 2d to the sum of the 1st and 3d, and as the 3d to the sum of the 2d and 4th, and as the 4th to the sum of the 3d and 5th, &c; so that the 1st and 2d being given, all the rest are found from them by one subtraction, and one proportion for each, in which the 1st and 2d terms are constantly the same. Next are given theorems for the chords of any multiples of a given arc or angle, as also the chords of their supplements to a semicircle, which are similar to the sines and cosines of the multiples of given angles; and the conclusions from them are expressed in this manner: 1st, that if c be the chord of the supplement of a given arc a, to the radius 1; then the chords of the supplements of the multiple arcs will be as in the annexed table: where the author observes, that the signs are alternately + and −; that the vertical columns of numeral coefficients to the terms of the chords, are the several orders of figurate numbers, which he calls triangular, pyramidal, triangulo-triangular, triangulo-pyramidal, &c, *generated in the ordinary way by continual additions; not indeed from unity*, AS IN THE GENERATION OF POWERS, but beginning with the number 2; and that the powers observe always the same progression: secondly, that if the chord of an arc a be called 1, and d the chord of the double arc $2a$, then the chords of the series of multiple arcs will be as in this table; where the author remarks as before on the law of the powers, signs, and coefficients, these being the orders of figurate numbers, raised from unity by continual additions, *after the manner of the genesis of powers*, which generation in that way he speaks of as a thing generally known, but without giving any hint how the coefficients of the terms of any power may be found from one another only, and independent of those of any other power, as it was afterwards, and first of all, I believe, done by Henry Briggs, about the year 1600: and 3dly, that if c be the chord of any arc a, to the radius 1,

Arcs	Chords of the Sup.
$1a$	c
$2a$	c^2-2
$3a$	c^3-3c
$4a$	c^4-4c^2+2
$5a$	c^5-5c^3+5c
$6a$	$c^6-6c^4+9c^2-2$
$7a$	$c^7-7c^5+14c^3-7c$
&c.	&c.

Arcs	Chords.
$1a$	1
$2a$	d
$3a$	d^2-1
$4a$	d^3-2d
$5a$	d^4-3d^2+1
$6a$	d^5-4d^3+3d
$7a$	$d^6-5d^4+6d^2-1$
$8a$	$d^7-6d^5+10d^3-4d$
&c.	&c.

then the series of the chords and supplemental chords of the multiple arcs will be thus; where the values are alternately

Arcs	Chords and Chords of Sup.
$1a$	Chord $= + c$
$2a$	Sup. ch. $= - c^2 + 2$
$3a$	Chord $= - c^3 + 3c$
$4a$	Sup. ch. $= + c^4 - 4c^2 + 2$
$5a$	Chord $= + c^5 - 5c^3 + 5c$
$6a$	Sup. ch. $= - c^6 + 6c^4 - 9c^2 + 2$
$7a$	Chord $= - c^7 + 7c^5 - 14c^3 + 7c$
&c.	&c.

chords, and chords of the supplements of the arcs on the same line, and the law of the powers and coefficients as before, but every alternate couplet of lines having their signs changed.

Another curious theorem is added to the above, for finding the sum of all these chords drawn in a semicircle, from one end of the diameter to every point in the circumference, those points dividing the circumference into any number of equal parts; namely, as the least chord is to the diameter, so is the sum of the said least chord and diameter and greatest chord, to double the sum of all the chords, including the diameter as one of them.

As the above theorems are chiefly adapted for the chords of multiple angles, a few problems and remarks are then added (whether by Vieta or Anderson does not clearly appear, but I think by the latter) concerning the application of them, to the section of angles into submultiples, and thence to the computation of the chords or sines, or a canon of triangles. The general precept for the angular sections is this: select one of the above equations adapted to the proper number of the section, in which will be concerned the powers of the unknown or required quantity, as high as the index of the section; and from this equation find that quantity by the known methods for the resolution of equations. Examples are given of three different sections, namely, for 3, 5, and 7 equal parts, the forms of which are respectively these,

$$3c - c^3 \quad . \quad . \quad . \quad . \quad = g$$
$$5c - 5c^3 + c^5 \quad . \quad . \quad = g$$
$$7c - 14c^3 + c^5 - c^7 = g$$

where g is the chord of the given arc or angle, and c the required chord of the 3d, 5th, or 7th part of it. And it is shown, geometrically, that the first of these equations has 2 real positive roots, the second 3, and the last 4; also, from the same principles, the relations of these roots are pointed out.

The method then annexed for constructing the canon of sines, from the foregoing theorems is thus: By dividing the radius in extreme-and-mean ratio, is obtained the sine of 18 degrees; this quinquisected, gives the sine of 3° 36′. Again, by trisecting the arc of 60°, there is obtained the sine of 20°; this again trisected gives that of 6° 40′; and this bisected gives that of 3° 20′: Then, by the theorem for the difference of two arcs, there will be found the sine of 16′, the difference between 3° 36′ and 3° 20′: Lastly, by four successive bisections, will at length be found the sines of 8′, 4′, 2′, and 1′. This last being found, the sines of its multiples, and again of the multiples of these multiples, &c, throughout the quadrant, are to be taken by the proper theorems before laid down.—And the same subject is still further pursued and explained, in the tract containing the answer given by Vieta, to the problem proposed to the whole world by Adrianus Romanus. In the same collection of Vieta's works, from page 400 to 432, is given a complete treatise on practical trigonometry, containing rules for resolving all the cases of plane and spherical triangles, by the *Canon Mathematicus*, or table of sines, tangents and secants.

The next authors whose labours in this way have been printed, are Rheticus, Otho, and Pitiscus: to all of whom we owe very great improvements in trigonometry.—George Joachim Rheticus, professor of mathematics in the university of Wittemberg, and sometime pupil to Copernicus, died in 1576, in the 60th year of his age. He conceived, and executed, the great design of computing the triangular canon for every 10 seconds of the quadrant, to the radius 1000000000000000, consisting of 1, followed by 15 ciphers. The series of sines which Rheticus computed to this radius, for every 10 seconds, and for every single second in the first and last degree of the quadrant, was published in folio at Francfort, 1613, by Pitiscus, who himself added a few of the first sines computed to the radius 1000000000000000000000.

But the large work, or whole trigonometrical canon computed by Rheticus, was published in 1596 by Valentine Otho, mathematician to the Electoral Prince Palatine. This vast work contains all the three series for the whole canon of right-angled triangles (being similar to the sines, tangents and secants, by which names I shall call them), with all the differences of the numbers, to the radius 10000000000.

Prefixed to these tables, are several books on their construction and use, in plane and spherical trigonometry, &c. Of these, the first three are by Rheticus himself; namely, book the 1st, containing the demonstrations of 9 lemmas, concerning the properties of certain lines drawn in and about circles: the 2d book contains 10 propositions, relating to the sines and cosines of arcs, together with those of their sums and differences, their halves and doubles, &c. The 3d book teaches, in 13 propositions, the construction of the canon to the radius 1000000000000000. By some of the common properties of geometry, having determined the sines of a few principal arcs, as 30°, 36°, &c, in the first proposition, by continual bisections, he finds the sines of various other arcs, down to 45 minutes. Then, in the 2d proposition, by the theorems for the sums and differences of arcs, he finds all the sines and cosines, up to 90 degrees, in a series of arcs differing by 1° 30′. And, in the 3d proposition, by the continual addition of 45′, he obtains all the sines and cosines in the series whose common difference is 45′. In the 4th proposition, beginning with 45′, and continually bisecting, he finds the sines and cosines of the series of half arcs, till he arrives at the arc of $14^{viii}\ 19^{ix}$, the sine of which is found to be 1, and its cosine 999999999999999. In the 5th proposition are computed the sine and cosine of 30″, or half a minute. In the 6th and 7th propositions are computed the sines and cosines for every minute, from 1′ to 45′, as well as of many larger arcs. The 8th proposition extends the computation for single minutes much farther. In propositions 9 and 10 are computed the tangents and secants for all arcs in the series whose common difference is 45′; and these are deduced from the sines of the same arcs by one proportion for each. In the remaining three propositions, 11, 12, 13, are computed the tangents and secants for several small angles. And from all these primary sines, tangents, and secants, the whole canon is deduced and completed.

The remaining books in this work are by the editor Otho; namely, a treatise, in one book, on right-angled plane triangles, the cases of which are resolved by the tables: then right-angled spherical trigonometry, in four books; next oblique spherical trigonometry, in five books; and lastly several other books, containing various spherical problems.

Next after the above are placed the tables themselves, containing the sines, tangents, and secants, for every 10 seconds in the quadrant, with all the differences annexed to each, in a smaller character. The numbers however are not called sines, tangents, and secants, but, like Vieta's, before described, they are considered as representing the sides of right-angled triangles, and are titled accordingly. They are also, in like manner, divided into three series, namely, according as the radius, or constant side of the triangle, is made the hypotenuse, or the greater leg, or the less leg of the triangle. When the hypotenuse is made the constant radius 10000000000, the two columns of this case, or series, are called the perpendicular and base, which are our sine and cosine; when the greater leg is the constant radius, the two columns on this series are titled hypotenuse and perpendicular, which are our secant and tangent; and when the less leg is constant, the two columns in this case are called hypotenuse and base; which are our cosecant and cotangent. After this large canon, is printed another smaller table, which is said to be the two columns of the third series, or cosecants and cotangents, with their differences, but to 3 places of figures less, or to the radius 10000000. But I cannot discover the reason for adding this less table, even if it were correct, which is very far from being the case, the numbers being uniformly erroneous, and different from the former through the greatest part of the table.

Towards the close of the 16th century, many persons wrote on the subject of trigonometry, and the construction of the triangular canon. But, their writings being seldom printed till many years afterwards, it is not easy to assign their order in respect of time. I shall therefore mention but a few of the principal authors, and that without pretending to any great precision on the score of chronological precedence.

In 1591 Philip Lansberg first published his "Geometria Triangulorum," in four books, with the canon of sines, tangents, and secants; a brief, but very elegant work; the whole being clearly explained: and it is perhaps the first set of tables titled with those words. The sines, tangents, and secants of the arcs to 45 degrees, with those of their complements, are each placed in adjacent columns, in a very commodious manner, continued forwards and downwards to 45 degrees, and then returning backwards and upwards to 90 degrees: the radius is 10000000, and a specimen of the first page of the table is as follows:

0	Sinus		Tangens		Secans		
0	0	10000000	0	infinitum.	10000000	infinitum.	60
1	2909	9999999	2909	34377466738	10000000	34377468193	59
2	5818	9999998	5818	17188731915	10000002	17188734824	58
3	8727	9999996	8727	11459152994	10000004	11459157357	57
4	11636	9999993	11636	8594363048	10000007	8594368866	56
5	14544	9999989	14544	6875488693	10000011	6875495966	55
&c.							&c.
							89

Of this work, the first book treats of the magnitude and relations of such lines as are considered in and about the circle, as the chords, sines, tangents, and secants. In the second book is delivered the construction of the trigonometrical canon, by means of the properties laid down in the first book : After which follows the canon itself. And in the third and fourth books is shown the application of the table, in the resolution of plane and spherical triangles.—Lansberg, who was born in Zealand 1561, was many years a minister of the gospel, and died at Middleburg in 1632.

The trigonometry of Bartholomew Pitiscus was first published at Francfort in the year 1599. This is a very complete work; containing, besides the triangular canon, with its construction and use in resolving triangles, the application of trigonometry to problems of surveying, altimetry, architecture, geography, dialling, and astronomy. The construction of the canon is very clearly described : And, in the third edition of the book, in the year 1612, he boasts to have added, in this part, arithmetical rules for finding the chords of the 3d, 5th, and other uneven parts of an arc, from the chord of that arc being given; saying, that it had been heretofore thought impossible to give such rules : But, after all, those boasted methods are only the application of the double rule of False-Position to the then known rules for finding the chords of multiple arcs; namely, making the supposition of some number for the required chord of a submultiple of any given arc, then from this assumed number computing what will be the chord of its multiple arc, which is to be compared with that of the given arc; then the same operation is performed with another supposition; and so on, as in the double rule of position. The canon contains the sine, tangent, and secant, for every minute of the quadrant, in some parts to 7 places of figures, in others to 8; as also the differences for every 10 seconds. The sines, tangents, and secants, are also given for every 10 seconds in the first and last degree of the quadrant, for every 2 seconds in the first and last 10 minutes, and for every single second in the first and last minute. In this table, the sines, tangents, and secants, are continued downwards on the left-hand pages, as far as to 45 degrees, and then returned upwards on the right-hand pages, so that the complements are always on the same line in the opposite or facing pages.

The mathematical works of Christopher Clavius (a German jesuit, who was born at Bamberg in 1537) in five large folio volumes, were printed at Moguntia, or Mentz, in 1612, the year in which the author died, at the age of 75. In the first volume we find a very ample and circumstantial treatise on trigonometry, with Regiomontanus's canon of sines, for every minute, as also canons of tangents and secants, each in a separate table, to the radius 10000000, and in a form continued forwards all the way up to 90 degrees. The explanation of the construction of the tables is very complete, and is chiefly extracted from Ptolemy, Purbach, and Regiomontanus. The sines have the differences set down for each second, that is, the quotients arising from the differences of the sines divided by 60.

About the year 1600, Ludolph van Collen, or à Ceulen, a respectable Dutch mathematician, wrote his book " de circulo et adscriptis," in which he treats fully and ably of the properties of lines drawn in and about the circle, and especially of chords or subtenses, with the construction of the canon of sines. The geometrical properties from which these lines are computed, are the same as those used by former writers; but his mode of computing and expressing them is different from theirs; for they actually extracted all the roots, &c, at every step, or single operation, in decimal numbers; but he retained the radical expressions to the last, making them however always as simple as possible: thus, for instance, he determines the sides of the polygons of 4, 8, 16, 32, &c, sides, inscribed in the circle whose radius is 1, to be as in the table here annexed: where the point before any figure, as $\surd .2$ signifies the

No. of Sides.	Length of each side.
4	$\surd 2$
8	$\surd .2 - \surd 2$
16	$\surd .2 - \surd .2 + \surd 2$
32	$\surd .2 - \surd .2 + \surd .2 - \surd 2$
&c.	&c.

root of all that follows it; so the last line is in our notation the same as $\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$. And as the perfect management of such surds was then not generally known, he added a very neat tract on that subject, to facilitate the computations. These, together with other dissertations on similar geometrical matters, were translated from the Dutch language, into Latin, by Willebrord Snell, and published at (Lugd. Batav.) Leyden in 1619. It was in this work that Ludolph determined the ratio of the diameter to the circumference of the circle, to 36 figures, showing that, if the diameter be 1, the circumference will be
greater than 3·14159 26535 89793 23846 26433 83279 50288,
but less than 3·14159 26535 89793 23846 26433 83279 50289,
which ratio was, by his order, in imitation of Archimedes, engraven on his tomb-stone, as is witnessed by the said Snell, pa. 54, 55, "Cyclometricus," published at Leyden two years after, in which he treats the same subject in a similar manner, recomputing and verifying Ludolph's numbers. And, in the same book, he also gives a variety of geometrical approximations, or mechanical solutions, to determine very nearly the lengths of arcs, and the areas of sectors and segments of circles.

Besides the "Cyclometricus," and another geometrical work (Apollonius Battavus) published in 1608, the same Snell wrote also four others "doctrinæ triangulorum canonicæ," in which is contained the canon of secants, and in which the construction of sines, tangents, and secants, together with the dimension or calculation of triangles, both plane and spherical, are briefly and clearly treated. After the author's death, this work was published in 8vo, at Leyden, 1627, by Martinus Hortensius, who added to it a tract on surveying and spherical problems. Willebrord Snell was born in 1591 at Royen, and died in 1626, being only 35 years of age. He was professor of mathematics in the university of Leyden, as was also his father Rodolph Snell.

Also in 1627, Francis van Schooten published, at Amsterdam, in a small neat form, tables of sines, tangents, and secants, for every minute of the quadrant, to 7 places of figures, the radius being 10000000; together with their use in plane trigonometry. These tables have a great character for their accuracy, being declared by the author to be without one single error. This however must not be understood of the last figure of the numbers, which I find are very often erroneous, sometimes in excess and sometimes in defect, by not being always set down to the nearest unit. Schooten died in 1659, while the second volume of his second edition of Descartes' geometry was in the press. He was also author of several other valuable works in geometry, and other branches of the mathematics.

The foregoing are the principal writers on the tables of sines, tangents, and secants, before the invention of logarithms, which happened about this time, namely, soon after the year 1600. Tables of the natural numbers were now all completed, and the methods of computing them nearly perfected: And therefore, before entering on the discovery and construction of logarithms, I shall stop here awhile to give a summary of the manner in which the said natural sines, tangents, and secants, were actually computed, after having been gradually improved from Hipparchus, Menelaus, and Ptolemy, who used only the chords, down to the beginning of the 17th century, when sines, tangents, secants, and versed sines were in use, and when the method hitherto employed had received its utmost improvement. In this explanation, we may here first enumerate the theorems by which the calculations were made, and then describe the application of them to the computation itself.

Theorem 1.—The square of the diameter of a circle, is equal to the sum of the squares of the chord of an arc, and of the chord of its supplement to a semicircle.—2. The rectangle under the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the two rectangles under the opposite sides.—3. The sum of the squares of the sine and cosine, hitherto called the sine of the complement, is equal to the square of the radius.—4. The difference between the sines of two arcs that are equally distant from 60 degrees, or $\frac{1}{6}$ of the whole circumference, the one as much greater as the other is less, is equal to the sine of half the difference of those arcs, or of the difference between either arc and the said arc of 60 degrees.—5. The sum of the cosine and versed sine, is equal to the radius.—6. The sum of the squares of the sine and versed sine, is equal to the square of the chord, or to the square of double the sine of half the arc.—7. The sine is a mean proportional between half the radius and the versed sine of double the arc.—8. A mean proportional between the versed sine and half the radius, is equal to the sine of half the arc.—9. As radius is to the sine, so is twice the cosine to the sine of twice the arc.—10. As the chord of an arc, is to the sum of the chords of the single and double arc, so is the difference of those chords, to the chord of thrice the arc.—11. As the chord of an arc, is to the sum of the chords of twice and thrice the arc, so is the difference of those chords, to the chord of five times the arc.—12. And in general, as the chord of an arc, is to the sum of the chords of n times and $n + 1$ times the arc, so is the difference of those chords, to the chord of $2n + 1$ times the arc.—13. The sine of the sum of two arcs, is equal to the sum of the products of the sine of each multiplied by the cosine of the other, and divided by the radius.—14. The sine of the difference of two arcs, is equal to the difference of the said two products divided by radius.—15. The cosine of the sum of two arcs, is equal to the difference between the products of their sines and of their cosines, divided by radius.—16. The cosine of the difference of two arcs, is equal to the sum of the said products divided by radius.—17. A small arc is equal to its chord or sine, nearly.—18. As cosine is to sine, so is radius to tangent.—19. Radius is a mean proportional between the tangent and cotangent.—20. Radius is a mean proportional between the secant and cosine.—21. Radius is a mean proportional between the sine and cosecant.—22. Half the difference between the tangent and cotangent of an arc, is equal to the tangent of the difference between the arc and its complement. Or, the sum arising from the addition of double the tangent of an arc with the tangent of half its complement, is equal to the tangent of the sum of that arc and the said half complement.—23. The square of the secant of an arc, is equal to the sum of the squares of the radius and tangent.—24. The secant of an arc, is equal to the sum of its tangent and the tangent of half its complement. Or, the secant of the difference between an arc and its complement, is equal to the tangent of the said difference added to the tangent of the less arc.—25. The secant of an arc, is equal to the difference between the tangent of that arc and the tangent of the arc added to half its complement. Or, the secant of the difference between an arc and its complement, is equal to the difference between the tangent of the said difference and the tangent of the greater arc.

From some of these 25 theorems, extracted from the writers before mentioned, and a few propositions of Euclid's elements, they compiled the whole table of sines, tangents, and secants, nearly in the following manner. By the elements were computed the sides of a few of the regular figures inscribed in a circle, which were the chords of such parts of the whole circumference as are expressed by the number of sides, and therefore the halves of those chords the sines of the halves of the arcs. So, if the radius be 10000000, the sides of the following figures will give the annexed chords and sines.

The figure.	Arcs subtended	Its chord or side.	Half arc.	Its sine or ½ chord.
Triangle	120°	17320508	60°	8660254
Square	90	14142136	45	7071068
Pentagon	72	11755705	36	5877853
Hexagon	60	10000000	30	5000000
Decagon	36	6180340	18	3090170
Quindecagon	24	4158234	12	2079117

Of some, or all of these, the sines of the halves were continually taken, by theorem the 6th, 7th, or 8th, and of their complements by the 3d ; then the sines of the halves of these, and of their complements, by the same theorems ; and so on, alternately, of the halves and complements, till they arrived at an arc which is nearly equal to its sine. Thus, beginning with the above arc of 12 degrees, and its sine, the halves were obtained as follows :

The halves.		Sines.
6°	′	1045285
3		523360
1	30	261769
	45	130896
The Comp. of these.		
84		9945218
87		9986295
88	30	9996573
89	15	9999143
The halves of these.		
42		6691306
21		3583679
10	30	1822355
5	15	915016
43	30	6883545
21	45	3705574
44	15	6977905

The comp. of these.		Sines.
48°	′	7431448
69		9335804
79	30	9832549
84	45	9958049
46	30	7253744
68	15	9288095
45	45	7163019
The halves of these.		
24		4067366
34	30	5664062
17	15	2965416
39	45	6394390
23	15	3947439
The comp.		
66		9135455
55	30	8241262
72	45	9550199
50	15	7688418
66	45	9187912

The halves.		Sines.
33°	′	5446390
16	30	2840153
8	15	1434926
27	45	4656145
Comps.		
57		8386706
73	30	9588197
81	45	9896514
62	15	8849876
Halves.		
28	30	4771588
14	15	2461533
36	45	5983246
Comps.		
61	30	8788171
75	45	9692309
53	15	8012538
Half.		
30	45	5112931
Comp.		
59	15	8594064

The sines of small arcs are then deduced in this manner. From the sine of 45′, above determined, are found the halves, which will be thus :

45′	0″	- - - -	130896
22	30	- - - -	65449,4
11	15	- - - -	32724,8

Now these last two sines being evidently in the same ratio as their arcs, the sines of all the less single minutes will be found by single proportion. So the 45th part of the sine of 45′, gives 2909 for the sine of 1′; which may be doubled, tripled, &c, for the sines of 2′, 3′, &c, up to 45′.

Then, from all the foregoing primary sines, by the theorems for halving, doubling, or tripling, and by those for the sums and differences, the rest of the sines are deduced, to complete the quadrant.

But having thus determined the sines and cosines of the first 30° of the quadrant, that is, the sines of the first and last 30°, those of the intermediate 30° are, by theor. 4, found by one single subtraction for each sine.

The sines of the whole quadrant being thus completed, the tangents are found by theor. 18, 19, 22, namely, for one half of the quadrant by the 18th and 19th, and the other half, by one single addition or subtraction for each, by the 22d theorem. And lastly, by theor. 24 and 25, the secants are deduced from the tangents, by addition and subtraction only.

Among the various means used for constructing the canon of sines, tangents, and secants, the writers above enumerated seem not to have been possessed of the method of differences, so profitably used since, and first of all I believe by Briggs, in computing his trigonometrical canon and his logarithms, as we shall see hereafter, when we come to describe those works. They took however the successive differences of the numbers, after they were computed, to verify or prove the truth of them; and if found erroneous, by any irregularity in the last differences, from thence they had a method of correcting the original numbers themselves. At least, this method is used by Pitiscus, Trig. lib. 2, where the differences are extended to the third order.—In page 44 of the same book also is described, for the first time that I know of, the common notation of decimal fractions, as now used. And this same notation was afterwards described and used by baron Napier, in positio 4 and 5 of his posthumous works, on the construction of logarithms, published by his son in the year 1619. But the decimal fractions themselves may be considered as having been introduced by Regiomontanus, by his decimal division of the radius, &c, of the circle; and from that time gradually brought into use; but continued long to be denoted after the manner of vulgar fractions, by a line drawn between the numerator and denominator, which last however was soon omitted, and only the numerator set down, with the line below it: thus, it was first $31\frac{35}{100}$, then $31\underline{35}$; afterwards, omitting the line, it became 31^{35}, and lastly 31_{35}, or 31.35, or 31·35: as may be traced in the works of Vieta, and others since his time, gradually into the present century.

Having often heard it remarked, that the word *sine*, or in Latin and French *sinus*, is of doubtful origin; and as the various accounts which I have seen of its derivation are very different from one another, it may not be amiss here to employ a few lines on this matter. Some authors say, this is an Arabic word, others that it is the single Latin word *sinus;* and in Montucla's "Histoire des Mathematiques" it is conjectured to be an abbreviation of two Latin words. The conjecture is thus expressed by the ingenious and learned author of that excellent history, at p. xxxiii, among the additions and corrections of the first volume: "A l'occasion des sinus dont on parle dans cette page, comme d'une invention des Arabes, voici une étymologie de ce nom, tout-a-fait heureuse et vraisemblable. Je la dois à M. Godin, de l'Académie Royale des Sciences, Directeur de l'Ecole de Marine de Cadix. Les sinus sont, comme l'on scait, des moitiés de cords; et les cordes en Latin se nomment *inscriptæ.* Les sinus sont donc *semisses inscriptarum,* ce que probablement on écrivit ainsi pour abréger, S. Ins. Delà ensuite s'est fait par abus le mot de sinus." Now, ingenious as this conjecture is, there appears to be little or no probability for the truth of it. For, in the first place, it is not in the least supported by quotations from any of the more early books, to show that it ever was the practice to write or print the words thus, *S. Ins.* upon which the conjecture is founded. Again, it is said the chords are called in Latin *inscriptæ;* and it is true that they sometimes are so: but I think they are more frequently called *subtensæ*, and the sines *semisses subtensarum* of the double arcs, which will not abbreviate into the word *sinus.* This conjecture the learned author has relinquished in the new edition of his history. But it may be said, what reason have we to suppose that this word is either a Latin word, or the abbreviation of any Latin words whatever? and that it seems but proper to seek for the etymology of *words* in the language of the inventors of the *things*. For which reason it is, that we find the two other words, *tangens* and *secans*, are Latin, as they were invented and used by authors who wrote in that language. But the sines are acknowledged to have been invented and introduced by the Arabians, and thence by analogy it would seem probable that this is a word of *their* language, and from them adopted, together with the use of it, by the Europeans. And indeed Lansberg, in the second page of his trigonometry abovementioned, expressly says, that it *is* Arabic: His words are, *Vox sinus Arabica est, et proinde barbara; sed cum longo usu approbata sit, et commodior non suppetat, nequaquam repudianda est: faciles enim in verbis nos esse oportet, cùm de rebus convenit.* And Vieta says something to the same purport, in page 9 of his "Universalium Inspectionum ad Canonem Mathematicum Liber:" His words are, *Breve sinus vocabulum, cùm sit artis, Saracenis præsertim quàm familiare, non est ab artificibus explodendum, ad laterum semissium inscriptorum denotationem, &c.*

Guarinus also is of the same opinion: in his "Euclides Adauctus," &c. tract xx. pa. 307, he says, SINUS *vero est nomen Arabicum usurpatum in hanc significationem à mathematicis;* though he was aware that a Latin origin was ascribed to it by Vitalis, for he immediately adds, *Licet Vitalis in suo Lexico Mathematico ex eo velit sinum appellatum, quòd claudat curvitatem arcus.*

Long before I either saw or heard of any conjecture, or observation, concerning the etymology of the word *sinus*, I remember that I *imagined* it to be taken from the same Latin word, signifying breast or bosom, and that our sine was so called allegorically. I had observed, that several of the terms in trigonometry were derived from a bow to shoot with, and its appendages; as *arcus* the bow, *chorda* the string, and *sagitta* the arrow, by which name the versed sine, which represents it, was sometimes called; also, that the *tangens* was so called from its office, being a line *touching* the circle, and *secans* from its *cutting* the same: I therefore imagined that the *sinus* was so called, either from its resemblance to the breast or bosom, or from its being a line drawn within the bosom *(sinus)* of the arc, or from its being that part of the string *(chorda)* of a bow *(arcus)* which is drawn near the breast *(sinus)* in the act of shooting. And perhaps Vitalis's definition, above-quoted, has some allusion to the same similitude.

Also Vieta seems to allude to the same thing, in calling *sinus* an allegorical word, in page 417 of his works, as published by Schooten, where, with his usual judgment and precision, he treats of the propriety of the terms used in trigonometry for certain lines drawn in and about the circle; of which, as it very well deserves, I shall here extract the principal part, to show the opinion and arguments of so great a man on those names. "Arabes autem semisses inscriptas duplo, numeris præsertim æstimatas, vocaverunt allegoricè Sinus, atque ideo ipsam semi-diametrum, quæ maxima est semissium inscriptarum, Sinum Totum. Et de iis suâ methodo canones exiverunt qui circumferuntur, supputante præsertim Regiomontano benè justè et accuratè, in iis etiam particulis qualium semidiameter adsumitur 10,000,000.

"Ex canonibus deinde sinuum derivaverunt recentiores canonem semissium circumscriptarum, quem dixêre *Fœcundum;* et canonem eductarum è centro, quem dixêre *Fœcundissimum* et *Beneficum*, hypotenusis addictum. Atque adeò semisses circumscriptas, numeris præsertim æstimatas, vocaverunt *Fœcundos*, Sinus numerósve videlicet; quanquam nihil vetat *Fœcundi* nomen substantivè accipi. Hypotenusas autem Beneficas, vel etiam simpliciter Hypotenusas: quoniam hypotenusa in primâ serie sinûs totius nomen retinet. Itaque ne novitate verborum res adumbretur, et alioqui sua artificibus, eo nomine dibita, præripiatur gloria, præposita in Canone Mathematico canonicis numeris inscriptio, candidè admonet primam seriem esse Canonem Sinuum. In secundâ vero, partem canonis fœcundi, partem canonis fœcundissimi, cotineri. In tertiâ, reliquam.

" Sanè præter inscriptas et circumscriptas, circulum etiàm adficiunt aliæ lineæ rectæ, velut Incidentes, Tangentes, et Secantes. Verùm illæ voces substantivæ sunt, non peripheriarum relativæ. Ac secare quidem circulum linea recta tunc intelligitur, cum in duobus punctis secat. Itaque non loquuntur benè geometricè, qui eductas è centro ad metas circumscriptarum vocant secantes improprìè, cum secantes et tangentes ad certos angulos vel peripherias referunt. Immò verò artem confundunt, cum his vocibus necesse habeat uti geometra abs relatione.

" Quare si quibus arrideat Arabum metaphora; quæ quidem aut omninò retinenda videtur, aut omninò explodenda; ut semisses inscriptas, Arabes vocant sinus; sic semisses circumscriptæ, vocentur Prosinus Amsinusve; et eductæ è centro Transinuosæ. Sin allegoria displiceat, geometrica sane inscriptarum et circumscriptarum nomina retineantur. Et cum eductæ è centro ad metas circumscriptarum, non habeant hactenus nomen certum neque elegans, voceantur sanè prosemidiametri, quasi protensæ semidiametri, se habentes ad suas circumscriptas, sicut semidiametri ad inscriptas."

Against the Arabic origin however of this word *(sinus)* may be urged its being varied according to the fourth declension of Latin nouns, like *manus;* and that if it were an Arabic word latinized, it would have been ranked under either the first, second, or third declension, as is usual in such adopted words.

So that, upon the whole, it will perhaps rather seem probable, that the term *sinus* is the Latin word answering to the name by which the Saracens called that line, and not their word itself. And this conjecture seems to be rendered still more probable by some expressions in pa. 4 and 5 of Otho's " Preface to Rheticus's Canon," where it is not only said, that the Saracens called the half-chord of double the arc *sinus*, but also that they called the part of the radius lying between the sine and the arc *sinus versus, vel sagitta*, which are evidently Latin words, and seem to be intended for the Latin translations of the names by which the Arabians called these lines, or the numbers expressing the lengths of them.

And this conjecture has been confirmed and realised, by a reference to Golius's Lexicon of the Arabic and Latin languages. In consequence I find that the Arabic and Latin writers on trigonometry do both of them use those words in the same allegorical sense, the latter being the Latin translations of the former, and not the Arabic words corrupted. Thus, the true Arabic word to denote the trigonometrical sine is جيب, pronounced *Jeib*, (reading the vowels in the French manner), meaning *sinus indusii, vestisque*, the bosom part of the garment; the versed sine is سهم, *Sehim*, which is *sagitta*, the arrow; the arc is قوس, which is *arcus*, the arc; and the chord is وتر, *Vitr*, that is *chorda*, the chord.

TRACT XX.

HISTORY OF LOGARITHMS.

The trigonometrical canon, of natural sines, tangents, and secants, being now brought to a considerable degree of perfection; the great length and accuracy of the numbers, together with the increasing delicacy and number of astronomical problems, and spherical triangles, to the solution of which the canon was applied, urged many persons, conversant in those matters, to endeavour to discover some means of diminishing the great labour and time, requisite for so many multiplications and divisions, in such large numbers as the tables then consisted of. And their chief aim was, to reduce the multiplications and divisions to additions and subtractions, as much as possible.

For this purpose, Nicholas Raymer Ursus Dithmarsus invented an ingenious method, which serves for one case in the sines, namely, when radius is the first term in the proportion, and the sines of two arcs are the second and third terms; for he showed, that the fourth term, or sine, would be found by only taking half the sum or difference of the sines of two other arcs, which should be the sum and difference of the less of the two former given arcs, and the complement of the greater. This is no more, in effect, than the following well-known theorem in trigonometry: as half radius is to the sine of one arc, so is the sine of another arc, to the cosine of the difference minus the cosine of the sum of the said arcs. The author published this ingenious device, in 1588, in his "Fundamentum Astronomiæ." And three or four years afterwards it was greatly improved by Clavius, who adapted it to all proportions in the solution of spherical triangles, for sines, tangents, secants, versed sines, &c; and that whether radius be in the proportion or not. All which he explains very fully in lem. 53, lib. 1, of his treatise on the Astrolabe. See more on this subject in Longomont. Astron. Danica. pa. 7, et seq. This method, though ingenious enough, depends not on any abstract property of numbers, but only on the relations of certain lines, drawn in and about the circle; for which reason it was rather limited, and sometimes attended with trouble in the application.

After perhaps various other contrivances, incessant endeavours at length produced the happy invention of logarithms, which are of direct and universal application to all numbers abstractedly considered, being derived from a property inherent in numbers themselves. This property may be considered, either as the relation between a geometrical series of terms and a corresponding arithmetical one, or as the relation between ratios and the measures of ratios, which comes to much the same thing, having been conceived in one of these ways by some of the writers on this subject, and in the other by the rest of them, as well as in both ways at different times by the same writer. A succinct idea of this property, and of the probable reflections made on it by the first writers on logarithms, may be to the following effect:

The learned calculators, about the close of the 16th, and beginning of the 17th century, finding the operations of multiplication and division by very long numbers, of 7 or 8 places of figures, which they had frequently occasion to perform, in resolving problems relating to geography and astronomy, to be exceedingly troublesome, set themselves to consider, whether it was not possible to find some method of lessening this labour, by substituting other easier operations in their stead. In pursuit of this object, they reflected, that since, in every multiplication by a whole number, the ratio, or proportion, of the product to the multiplicand, is the same as the ratio of the multiplier to unity, it will follow that the ratio of the product to unity (which, according to Euclid's definition of compound ratios, is compounded of the ratios of the said product to the multiplicand and of the multiplicand to unity), must be equal to the sum of the two ratios of the multiplier to unity and of the multiplicand to unity. Consequently, if they could find a set of artificial numbers that should be the representatives of, or should be proportional to, the ratios of all sorts of numbers to unity, the addition of the two artificial numbers that should represent the ratios of any multiplier and multiplicand to unity, would answer to the multiplication of the said multiplicand by the said multiplier, or the sum arising from the addition of the said representative numbers, would be the representative number of the ratio of the product to unity; and consequently, the natural number to which it should be found, in the table of the said artificial or representative numbers, that the said sum belonged, would be the product of the said multiplicand and multiplier. Having settled this principle, as the foundation of their wished-for method of abridging the labour of calculations, they resolved to compose a table of such artificial numbers, or numbers that should be representatives of, or proportional to, the ratios of all the common or natural numbers to unity.

The first observation that naturally occurred to them in the pursuit of this scheme was, that whatever artificial numbers should be chosen to represent the ratios of other whole numbers to unity, the ratio of equality, or of unity to unity, must be represented by 0; because *that* ratio has properly no magnitude, since, when it is added to, or subtracted from, any other ratio, it neither increases nor diminishes it.

The second observation that occurred to them was, that any number whatever might be chosen at pleasure for the representative of the ratio of any given natural number to unity; but that, when once such choice was made, all the other representative numbers would be thereby determined, because they must be greater or less than that first representative number, in the same proportions in which the ratios represented by them, or the ratios of the corresponding natural numbers to unity, were greater or less than the ratio of the said given natural number to unity. Thus, either 1, or 2, or 3, &c, might be chosen for the representative of the ratio of 10 to 1. But, if 1 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1, which are double and triple of the ratio of 10 to 1, must be 2 and 3, and cannot be any other numbers; and, if 2 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1, will be 4 and 6, and cannot be any other numbers; and, if 3 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1, will be 6 and 9, and cannot be any other numbers; and so on.

The third observation that occurred to them was, that, as these artificial numbers were representatives of, or proportional to, ratios of the natural numbers to unity, they must be expressions of the numbers of some smaller equal ratios that are contained in the said ratios. Thus, if 1 be taken for the representative of the ratio of 10 to 1, then 3, which is the representative of the ratio of 1000 to 1, will express the number of ratios of 10 to 1 that are contained in the ratio of 1000 to 1. And if, instead of 1, we make 10,000,000, or ten millions, the representative of the ratio of 10 to 1, (in which case 1 will be the representative of a very small ratio, or *ratiuncula*, which is only the ten-millionth part of the ratio of 10 to 1, or will be the representative of the 10,000,000th root of 10, or of the first or smallest of 9,999,999 mean proportionals interposed between 1 and 10), the representative of the ratio of 1000 to 1, which will in this case be 30,000,000, will express the number of those *ratiunculæ*, or small ratios of the 10,000,000th root of 10 to 1, which are contained in the said ratio of 1000 to 1. And the like may be shown of the representative of the ratio of any other number to unity. And therefore they thought these artificial numbers, which thus represent, or are proportional to, the magnitudes of the ratios of the natural numbers to unity, might not improperly be called the LOGARITHMS of those ratios, since they express the numbers of smaller ratios of which they are composed. And then, for the sake of brevity, they called them the *Logarithms of the said natural numbers themselves*, which are the antecedents of the said ratios to unity, of which they are in truth the representatives.

The foregoing method of considering this property leads to much the same conclusions as the other way, in which the relations between a geometrical series of terms, and their exponents, or the terms of an arithmetical series, are contemplated. In this latter way, it readily occurred that the addition of the terms of the arithmetical series corresponded to the multiplication of the terms of the geometrical series; and that the arithmeticals would therefore form a set of artificial numbers, which, when arranged in tables, with their geometricals, would answer the purposes desired, as has been explained above.

From this property, by assuming four quantities, two of them as two terms in a geometrical series, and the others as the two corresponding terms of the arithmeticals, or artificials, or logarithms, it is evident that all the other terms of both the two series may thence be generated. And therefore there may be as many sets or scales of logarithms as we please, since they depend entirely on the arbitrary assumption of the first two arithmeticals. And all possible natural numbers may be supposed to coincide with some of the terms of any geometrical progression whatever, the logarithms or arithmeticals determining which of the terms in that progression they are.

It was proper however that the arithmetical series should be so assumed, as that the term 0 in it might answer to the term 1 in the geometricals; otherwise the sum of the logarithms of any two numbers would be always to be diminished by the logarithm of 1, to give the logarithm of the product of those numbers: for which reason, making 0 the logarithm of 1, and assuming any quantity whatever for the value of the logarithm of any one number, the logarithms of all other numbers were thence to be derived. And hence, like as the multiplication of two numbers is effected by barely adding their logarithms, so division is performed by subtracting the logarithm of the one from that of the other, raising of powers by multiplying the logarithm of the given number by the index of the power, and extraction of roots by dividing the logarithm by the index of the root. It is also evident that, in all scales or systems of logarithms, the logarithm of 0 will be infinite; namely, infinitely negative if the logarithms increase with the natural numbers, but infinitely positive if the contrary; because that, while the geometrical series must decrease through infinite divisions by the ratio of the progression, before the quotient come to 0 or nothing; the logarithms, or arithmeticals, will in like manner undergo the corresponding infinite subtractions or additions of the common equal difference; which equal increase or decrease, thus indefinitely continued, must needs tend to an infinite result.

This however was no newly-discovered property of numbers, but what was always well known to all mathematicians, being treated of in the writings of Euclid, as also by Archimedes, who made great use of it in his Arenarius, or treatise on the number of the sands, namely, in assigning the rank or place of those terms, of a geometrical series, produced from the multiplication together of any of the foregoing terms, by the addition of the corresponding terms of the arithmetical series, which served as the indices or exponents of the former. Stifelius also treats very fully of this property at folio 35 et seq. and there explains all its principal uses as relating to the logarithms of numbers, only without the name; such as, that addition answers to multiplication, subtraction to division, multiplication of exponents to involution, and dividing of exponents to evolution; all which he exemplifies in the rule-of-three, and in finding several mean proportionals, &c, exactly as is done in logarithms. So that he seems to have been in the full possession of the idea of logarithms, but without the necessity of making a table of such numbers. For the reason why tables of these numbers were not sooner composed, was, that the accuracy and trouble of trigonometrical computations had not sooner rendered them necessary. It is therefore not to be doubted that, about the close of the sixteenth and beginning of the seventeenth century, many persons had thoughts of such a table of numbers, besides the few who are said to have attempted it.

It has been said by some, that Longomontanus invented logarithms: but this cannot well be supposed to have been any more than in idea, since he never published any thing of the kind, nor ever laid claim to the invention, though he lived thirty-three years after they were first published by baron Napier, as he died only in 1647, when they had been long known and received all over Europe. Nay more, Longomontanus himself ascribes the invention to Napier: vid. Astron. Danica, p. 7, &c. Some circumstances of this matter are indeed related by Wood in his "Athenæ Oxonienses," under the article Briggs, on the authority of Oughtred and Wingate, viz. "That one Dr. Craig, a Scotchman, coming out of Denmark into his own country, called upon Joh. Neper baron of Marcheston near Edenburgh, and told him among other discourses, of a new invention in Denmark (by Longomontanus as 'tis said) to save the tedious multiplication and division in astronomical calculations. Neper being solicitous to know farther of him concerning this matter, he could give no other account of it, than that it was by proportionable numbers. Which hint Neper taking, he desired him at his return to call upon him again. Craig, after some weeks had passed, did so, and Neper then showed him a rude draught of that he called *Canon mirabilis Logarithmorum.* Which draught, with some alterations, he printing in 1614, it came forthwith into the hands of our author Briggs, and into those of Will. Oughtred, from whom the relation of this matter came."

Kepler also says, that one Juste Byrge, assistant astronomer to the landgrave of Hesse, invented or projected logarithms long before Neper did; but that they had never come abroad, on account of the great reservedness of their author with regard to his own compositions. It is also said, that Byrge computed a table of natural sines for every two seconds of the quadrant.

But whatever may have been said, or conjectured, concerning any thing that may have been done by others, it is certain that the world is indebted, for the first publication of logarithms, to John Napier, or Nepair*, or in Latin, Neper, baron of Merchiston, or Markinston, in Scotland, who died the 3d of April 1618, at 67 years of age. Baron Napier added considerable improvements to trigonometry, and the frequent numeral computations he performed in this branch, gave occasion to his invention of logarithms, in order to save part of the trouble attending those calculations; and for this reason he adapted his tables peculiarly to trigonometrical uses.

* The origin of which name, Crawfurd informs us, was from a (less) peer*less* action of one of his ancestors, viz. Donald, second son of the earl of Lenox, in the time of David the Second. " Some English writers, mistaking the import of the term *baron*, having called this celebrated person lord Napier, a Scotch nobleman. He was not indeed a peer of Scotland: but the peerage of Scotland informs us, that he was of a very ancient, honourable, and illustrious family; that his ancestors, for many generations, had been possessed of sundry baronies, and, amongst others, of the barony of Merchistoun, which descended to him by the death of his father in 1608. Mr. Briggs, therefore, very properly styles him *Baro Merchestonii.* Now, according to Skene, *de verborum significatione,* ' In this realm (of Scotland) he is called a Baronne, quha haldis his landes immediatelie in chiefe of the king, and hes power of Pit and Gallows; *Fossa et Furca;* quhilk was first institute and granted be king Malcome, quha gave power to the Barrones to have ane Pit, quhairin wemen condemned for thieft suld be drowned, and ane Gallows, whereupon men thieves and trespassowres suld be hanged, conforme to

This discovery he published in 1614, in his book intitled "Mirifici Logarithmorum Canonis Descriptio," reserving the construction of the numbers till the sense of the learned concerning his invention should be known. And, excepting the construction, this is a perfect work on this kind of logarithms, containing in effect the logarithms of all numbers, and the logarithmic sines, tangents, and secants, for every minute of the quadrant, together with the description and uses of the tables, as also his definition and idea of logarithms.

Napier explains his notion of logarithms by lines described or generated by the motion of points, in this manner: He first conceives a line to be generated by the equable motion of a point, which passes over equal portions of it in equal small moments or portions of time: he then considers another line as generated by the unequal motion of a point, in such manner, that, in the aforesaid equal moments or portions of time, there may be described or cut off, from a given line, parts which shall be continually in the same proportion with the respective remainders, of that line, which had before been left: then are the several lengths of the first line, the logarithms of the corresponding parts of the latter. Which description of them is similar to this, that the logarithms are a series of quantities or numbers in arithmetical progression, adapted to another series in geometrical progression. The

the doome given in the Baron Court thereanent.' So that a Scotch baron, though no peer, was nevertheless a very considerable personage, both in dignity and power." *Reid's Essay on Logarithms.*—The name of the illustrious inventor of logarithms, has been variously written at different times, and on different occasions. In his own Latin works, and in (perhaps) all other books in Latin, it is *Neper*, or *Neperus Baro Merchestonii:* By Briggs, in a letter to Archbishop Usher, he is called *Naper, lord of Markinston:* In Wright's translation of the logarithms, which was revised by the author himself, and published in 1616, he is called *Nepair, baron of Merchiston;* and the same by Crawfurd and some others: But M'Kenzie and others write it *Napier, baron of Merchiston;* which, being also the orthography now used by the family, I shall adopt in this work. I observe also, that the Scotch Compendium of Honour says he was only Sir John Napier, and that his son and heir Archibald, was the first lord, being raised to that dignity in 1626. Be this however as it may, I shall conform to the common modes of expression, and call him indifferently, *Baron Napier*, or *Lord Napier.*

first or whole length of the line, which is diminished in geometrical progression, he makes the radius of a circle, and its logarithm 0 or nothing, representing the beginning of the first or arithmetical line; and the several proportional remainders of the geometrical line, are the natural sines of all the other parts of the quadrant, decreasing down to nothing, while the successive increasing values of the arithmetical line, are the corresponding logarithms of those decreasing sines: so that, while the natural lines decrease from radius to nothing, their logarithms increase from nothing to infinite. Napier made the logarithm of radius to be 0, that he might save the trouble of adding or subtracting it, in trigonometrical proportions, in which it so frequently occurred; and he made the logarithms of the sines, from the entire quadrant down to 0, to increase, that they might be positive, and so in his opinion the easier to manage, the sines being of more frequent use than the tangents and secants, of which the whole of the latter and half the former would, in his way, be of a different affection from the sines; for it is evident that the logarithms of all the secants in the quadrant, and of all the tangents above 45°, or the half quadrant, would be negative, being the logarithms of numbers greater than the radius, whose logarithm is made equal to 0 or nothing.

As to the contents of Napier's table; it consists of the natural sines and their logarithms, for every minute of the quadrant. Like most other tables, the arcs are continued to 45 degrees from top to bottom on the left-hand side of the pages, and then returned backwards from bottom to top on the right-hand side of the pages: so that the arcs and their complements, with the sines, natural and logarithmic, stand on the same line of the page, in six columns; and in another column, in the middle of the page, are placed the differences between the logarithmic sines and cosines, on the same lines, and in the adjacent columns on the right and left; thus making in all seven columns in each page. Of these columns, the first and seventh contain the arc and its complement, in degrees and minutes; the second and sixth, the natural sine and cosine of each arc; the third and fifth, the logarithmic sine and cosine; and the fourth, or middle column, the difference between the logarithmic sine and cosine which are in the third and fifth columns. To elucidate the description, the first page of the table is here inserted, as follows.

Gr. min.	0 Sinus.	Logarithmi.	+ | − Differentiæ.	Logarithmi.	Sinus.	
0	0	Infinitum.	Infinitum.	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74404211	2	9999998	58
3	8727	70439560	70439560	4	9999996	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55
6	17453	63508099	63508083	16	9999984	54
7	20362	61966595	61966573	22	9999980	53
8	23271	60631284	60631256	28	9999974	52
9	26180	59453453	59453418	35	9999967	51
10	29088	58399857	58399814	43	9999959	50
11	31997	57446759	57446707	52	9999950	49
12	34906	56576646	56576584	62	9999940	48
13	37815	55776222	55776149	73	9999928	47
14	40724	55035148	55035064	84	9999917	46
15	43632	54345225	54345129	96	9999905	45
16	46541	53699843	53699734	109	9999892	44
17	49450	53093600	53093577	123	9999878	43
18	52359	52522019	52521881	138	9999863	42
19	55268	51981356	51981202	154	9999847	41
20	58177	51468431	51468361	170	9999831	40
21	61086	50980537	50980450	187	9999813	39
22	63995	50515342	50515137	205	9999795	38
23	66904	50070827	50070603	224	9999776	37
24	69813	49645239	49644995	244	9999756	36
25	72721	49237030	49236765	265	9999736	35
26	75630	48844826	48844539	287	9999714	34
27	78539	48467431	48467122	309	9999692	33
28	81448	48103763	48103431	332	9999668	32
29	84357	47752859	47752503	356	9999644	31
30	87265	47413852	47413471	381	9999619	30

Besides the columns which are actually contained in this table, as above exhibited and described, namely, the natural and logarithmic sines, and their differences, the same table is made to serve also for the logarithmic tangents and secants of the whole quadrant, and for the logarithms of common numbers. For, the fourth or middle column contains the logarithmic tangents, being equal to the differences between the logarithmic sines and cosines, when the logarithm of radius is 0, because cosine : sine : : radius : tangent, that is, in logarithms, tangent = sine − cosine. Also the logarithmic sines, made negative, become the logarithmic cosecants, and the logarithmic cosines made negative, are the logarithmic secants; because sine : radius : : radius : cosecant, and cosine : radius : : radius : secant; that is, in logarithms, cosecant = 0 − sine = − sine, and secant = 0 − cosine = − cosine. And to make it answer the purpose of a table of logarithms of common numbers, the author directs to proceed thus: A number being given, find that number in any table of natural sines, or tangents, or secants, and note the degrees and minutes in its arc; then in his table find the corresponding logarithmic sine, or tangent, or secant, to the same number of degrees and minutes; and it will be the required logarithm of the given number.

After his definitions and descriptions of logarithms, Napier explains his table, and illustrates the precepts with examples, showing how to take out the logarithms of sines, tangents, secants, and of common numbers; as also how to add and subtract logarithms. He then proceeds to teach the uses of those numbers; and first, in finding any of the terms of three or four proportionals, showing how to multiply and divide, and to find powers and roots, by logarithms: 2dly, in trigonometry, both plane and spherical, but especially the latter, in which he is very explicit, turning all the theorems for every case into logarithms, computing examples to each in numbers, and then enumerating a set of astronomical problems of the sphere which properly belong to each case. Napier here teaches also some new theorems in spherical trigonometry, particularly, that the tangent of half the base : tang. $\frac{1}{2}$ sum legs :: tang. $\frac{1}{2}$ dif. legs : tang. $\frac{1}{2}$ the alternate base; and the general theorem for what are called his five circular parts, by which he condenses into one rule, in two parts, the theorems for all the cases of right-angled spherical triangles, which had been separately demonstrated by Pitiscus, Lansbergius, Copernicus, Regiomontanus, and others.

The description and use of Napier's canon being in the Latin language, they were translated into English by Mr. Edward Wright, an ingenious mathematician, and inventor of the principles of what has commonly, though erroneously, been called Mercator's Sailing. He sent the translation to the author, at Edinburgh, to be revised by him before publication; who having carefully perused it, returned it with his approbation, and a few lines introduced besides into the translation. But, Mr. Wright dying soon after he received it back, it was after his death published, together with the tables, but each number to one figure less, in the year 1616, by his son Samuel Wright, accompanied with a dedication to the East-India Company, as also a preface by Henry Briggs, of whom we shall presently have occasion to speak more at large, on account of the great share he bore in perfecting the logarithms. In this translation, Mr. Briggs gave also the description and draught of a scale that had been invented by Mr. Wright, and several other methods of his own, for finding the proportional parts to intermediate numbers, the logarithms having been only printed for such numbers as were the natural sines of each minute. And the note which Baron Napier inserted in this English edition, and which was not in the original, was as follows: "But because the addition and subtraction of these former numbers may seem somewhat painful, I intend (if it shall please God) in a second edition, to set out such logarithms as shall make those numbers above written to fall upon decimal numbers, such as 100,000,000, 200,000,000, 300,000,000, &c, which are easie to be added or abated to or from any other number." This note had reference to the alteration of the scale of logarithms, in such manner, that

1 should become the logarithm of the ratio of 10 to 1, instead of the number 2.3025851, which Napier had made that logarithm in his table, and which alteration had before been recommended to him by Briggs, as we shall see presently. Napier also inserted a similar remark in his "Rabdologia," which he printed at Edinburgh in 1617.

The following is the preface to * Wright's book, which, as far as where it mentions the change from the Latin into English, is a literal translation of the preface to Napier's original; but what follows that, is added by Napier himself. And I willingly insert it here, as it contains a declaration of the motives which led to this discovery, and as the book itself is very scarce. "Seeing there is nothing (right well beloved students in the mathematics) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, than the Multiplications, Divisions, square and

* Of this ingenious man I shall here insert in a note the following memoirs, as they have been translated from a Latin piece taken out of the annals of Gonvile and Caius College at Cambridge, viz. "This year (1615) died at London, Edward Wright of Garveston in Norfolk, formerly a fellow of this college; a man respected by all for the integrity and simplicity of his manners, and also famous for his skill in the mathematical sciences: insomuch that he was deservedly styled a most excellent mathematician by Richard Hackluyt, the author of an original treatise of our English navigations. What knowledge he had acquired in the science of mechanics, and how usefully he employed that knowledge to the public as well as private advantage, abundantly appear both from the writings he published, and from the many mechanical operations still extant, which are standing monuments of his great industry and ingenuity. He was the first undertaker of that difficult but useful work, by which a little river is brought from the town of Ware in a new canal, to supply the city of London with water; but by the tricks of others he was hindered from completing the work he had begun. He was excellent both in contrivance and execution; nor was he inferior to the most ingenious mechanic in the making of instruments, either of brass, or any other matter. To his invention is owing whatever advantage Hondius's geographical charts have above others; for it was our Wright that taught Jodocus Hondius the method of constructing them, which was till then unknown: but the ungrateful Hondius concealed the name of the true author, and arrogated the glory of the invention to himself. Of this fraudulent practice the good man could not help complaining, and justly enough, in the preface to his Treatise of the Correction of Errors in the Art of Navigation; which he composed with excellent judgment, and after long experience, to the great advancement of naval

cubical Extractions of great numbers, which, besides the tedious expence of time, are for the most part subject to many slippery errors: I began therefore to consider in my minde, by what certaine and ready Art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent briefe rules to be treated of (perhaps) hereafter. But amongst all, none more profitable then this, which together with the hard and tedious Multiplications, Divisions, and Extractions of rootes, doth also cast away from the worke it selfe, even the very numbers themselves that are to be multiplied, divided, and resolved into rootes, and putteth other numbers in their place, which performe as much as they can do, onely by Addition and Subtraction, Division by two, or Division by three; which secret invention, being (as all other good things are) so much the better as it shall be the more common; I thought good

affairs. For the improvement of this art he was appointed mathematical lecturer by the East India Company, and read lectures in the house of that worthy knight Sir Thomas Smith, for which he had a yearly salary of 50 pounds. This office he discharged with great reputation, and much to the satisfaction of his hearers. He published in English, a book on the doctrine of the sphere, and another concerning the construction of sun-dials. He also prefixed an ingenious preface to the learned Gilbert's book on the loadstone. By these and other his writings, he has transmitted his fame to latest posterity. While he was yet a fellow of this college, he could not be concealed in his private study, but was called forth to the public business of the kingdom, by the queen's majesty, about the year 1593. He was ordered to attend the earl of Cumberland in some maritime expeditions. One of these he has given a faithful account of, in the way of a journal or ephemeris, to which he has prefixed an elegant hydrographical chart of his own contrivance. A little before his death, he employed himself about an English translation of the book of logarithms, then lately found out by the honourable Baron Napier, a Scotchman, who had a great affection for him. This posthumous work of his was published soon after, by his only son Samuel Wright, who was also a scholar of this college. He had formed many other useful designs, but was hindered by death from bringing them to perfection. Of him it may be truly said, that he studied more to serve the public than himself; and though he was rich in fame, and in the promises of the great, yet he died poor, to the scandal of an ungrateful age."

Other anecdotes of him, as well as many other mathematical authors, may be found in the curious history of navigation by Dr. James Wilson, prefixed to Mr. Robertson's excellent treatise on that subject.

heretofore to set forth in Latine for the publique use of Mathematicians. But now some of our Countrymen in this Island well affected to these studies, and the more publique good, procured a most learned Mathematician to translate the same into our vulgar English tongue, who after he had finished it, sent the Coppy of it to me, to be seene and considered on by myself. I having most willingly and gladly done the same, finde it to bee most exact and precisely conformable to my minde and the originall. Therefore it may please you who are inclined to these studies, to receive it from me and the Translator, with as much good will as we recommend it unto you. Fare yee well."

There are also extant copies of Wright's translation with the date 1618 in the title: but this is not properly a new edition, being only the old work with a new title-page adapted to it (the old one being cancelled), together with the addition of sixteen pages of new matter, called "An Appendix to the Logarithms, shewing the practice of the calculation of triangles, and also a new and ready way for the exact finding out of such lines and logarithmes as are not precisely to be found in the canons." But we are not told by what author: probably it was by Briggs.

Besides the trouble attending Napier's canon, in finding the proportional parts, when used as a table of the logarithms of common numbers, and which was in part remedied by the fore-mentioned contrivances of Wright and Briggs, it was also accompanied with another inconvenience, which arose from the logarithms being sometimes + or additive, and sometimes − or negative, and which required therefore the knowledge of algebraic addition and subtraction. And this inconvenience was occasioned, partly by making the logarithm of radius to be 0, and the sines to decrease, and partly by the compendious manner in which the author had formed the table; making the three columns of sines, cosines and tangents, to serve also for the other three of cosecants, secants, and cotangents.

But this latter inconvenience was well remedied by John Speidell, in his New Logarithms, first published in 1619, which contained all the six columns, and in this order; sines, cosines, tangents, cotangents, secants, cosecants: and they were besides made all positive, by being taken the arithmetical complements of Napier's, that is, they were the remainders left by subtracting each of these latter from 10000000. And the former inconvenience was more effectually removed by the said Speidell, in an additional table, given in the sixth impression of the former work, in the year 1624. This was a table of Napier's logarithms for the round or integer numbers 1, 2, 3, 4, 5, &c, to 1000, together with their differences and arithmetical complements; as also the halves of the said logarithms, with their differences and arithmetical complements; which halves consequently were the logarithms of the square roots of the said numbers. These logarithms are however a little varied in their form from Napier's, namely, so as to increase *from* 1, whose logarithm is 0, instead of decreasing *to* 1, or radius, whose logarithm was made 0 likewise; that is, Speidell's logarithm of any number n, is equal to Napier's logarithm of its reciprocal $\frac{1}{n}$: so that in this last table of Speidell's, the logarithm of 1 being 0, the logarithm of 10 is 2302584, the logarithm of 100 is twice as much, or 4605168, and that of 1000 thrice as much, or 6907753.

This table is now commonly called *hyperbolic* logarithms, because the numbers express the areas between the asymptote and curve of the hyperbola, those areas being limited by ordinates parallel to the other asymptote, and the ordinates decreasing in geometrical progression. But this is not a very proper method of denominating them, as such areas may be made to denote any system of logarithms whatever, as will be shown more at large in the proper place.

In the year 1619, Robert Napier, son of the inventor of logarithms, published a new edition of his late father's "Logarithmorum Canonis Descriptio," together with the promised "Logarithmorum Canonis Constructio," and other miscellaneous pieces, written by his father and Mr. Briggs.—Also one Bartholomew Vincent, a bookseller at Lugdunum, or Lyons, in France, printed there an exact copy of the same two works in one volume, in the year 1620; which was four years before the logarithms were carried to France by Wingate, who was therefore erroneously said to have first introduced them into that country. But we shall treat more particularly of the contents of this work, after having enumerated the other writers on this sort of logarithms.

In 1618 or 1619, Benjamin Ursinus, mathematician to the Elector of Brandenburg, published, at Cologn, his "Cursus Mathematicus," in which is contained a copy of Napier's logarithms, with the addition of some tables of proportional parts. And in 1624, he printed at the same place, his "Trigonometria," with a table of natural sines and their logarithms, of the Napierian kind and form, to every ten seconds in the quadrant; which he had been at much pains in computing.

In the same year 1624, logarithms, of nearly the same kind, were also published, at Marpurg, by the celebrated John Kepler, mathematician to the Emperor Ferdinand the Second, under the title of "Chilias Logarithmorum ad Totidem Numeros Rotundos, præmissa Demonstratione legitima Ortus Logarithmorum eorumque Usus," &c; and the year following, a supplement to the same; being applied to round or integer numbers, and to such natural sines as nearly coincide with them. These are exactly the same kind of logarithms as Napier's, being the same logarithms of the natural sines of arcs, beginning from the quadrant, whose sine or radius is 10,000,000, the logarithm of which is made 0, and from thence the sines decreasing by equal differences, down to 0, or the beginning of the quadrant, while their logarithms increase to infinity. So that the difference between this table and Napier's, consists only in this, namely, that in Napier's table the *arc* of the quadrant is divided into equal parts, differing by one minute each, and consequently their sines, to which the logarithms are adapted, are irrational or interminate numbers, and only expressed by approximate decimals; whereas in Kepler's table, the *radius* is divided into equal parts, which are considered as perfect and terminate sines, having equal differences, and to which terminate sines the logarithms are here adapted. By this means indeed the proportions for intermediate numbers and logarithms are easier made; but then the corresponding arcs are not terminate, being irrational, and only set down to an approximate degree. So that Kepler's table is more convenient as a table of the logarithms of common numbers, and Napier's as the logarithmic sines of the arcs of the quadrant. In both tables, the logarithm of the ratio of 10 to 1, is the same quantity, namely 23025852; and as the radius, or greatest sine, is 10,000,000, whose logarithm is made 0, the logarithms of the decuple parts of it will be found by adding 23025852 continually, or multiplying this logarithm by 2, 3, 4, &c; and hence the logarithm of 1, the first number, or smallest sine, in the table, is 161180959, or 7 times 2302 &c.

Besides the two columns, of the natural sines and their logarithms, with the differences of the logarithms, this table of Kepler's consists also of three other columns; the first of which contains the nearest arcs, belonging to those sines, expressed in degrees, minutes and seconds; and the other two express what parts of the radius each sine is equal to, namely, the one of them in 24th parts of the radius, and minutes and seconds of them; and the other in 60th parts of the radius, and minutes of them. The following specimen is extracted from the last page of the table, printed exactly as in the work itself.

Arcus Circuli cum differentiis.	Sinus seu numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
19. 34			101.58	
80. 3. 46	98500.00	23. 38. 24	1511.36+	59. 6
20. 12			101.47	
80. 23. 58	98600.00	23. 39. 50	1409.89+	59. 10
20. 53			101.37	
80. 44. 51	98700.00	23. 41. 17	1308.52+	59. 13
21. 42			101.26	
81. 6. 33	98800.00	23. 42. 43	1207.26	59. 17
22. 53			101.17	
81. 29. 26	98900.00	23. 44. 10	1106.09+	59. 20
24. 6			101.06	
81. 53. 32	99000.00	23. 45. 36	1005.03+	59. 24
25. 6			100.96	
82. 18. 38	99100.00	23. 47. 2	904.07+	59. 28
26. 28			100.85	
82. 45. 6	99200.00	23. 48. 29	803.22+	59. 31
27. 54			100.76	
83. 13. 0	99300.00	23. 49. 55	702.46	59. 35
30. 20			100.65	
83. 43. 20	99400.00	23. 51. 22	601.81	59. 38
32. 40			100.56	
84. 16. 0	99500.00	23. 52. 48	501.25+	59. 42
36. 30			100.45	
84. 52. 30	99600.00	23. 54. 14	400.80	59. 46
41. 9			100.35	
85. 33. 39	99700.00	23. 55. 41	300.45	59. 49
48. 54			100.25	
86. 22. 33	99800.00	23. 57. 7	200.20	59. 53
1. 3. 42			100.15	
87. 26. 15	99900.00	23. 58. 34	100.05	59. 56
2. 33. 45			100.05	
90. 0. 0	100000.00	24. 0. 0	000000.00	60. 0

To the table, Kepler prefixes a pretty considerable tract, containing the construction of the logarithms, and a demonstration of their properties and structure, in which he considers logarithms, in the true and legitimate way, as the measures of ratios, as shall be shown more particularly hereafter in the next tract, where the construction of logarithms is fully treated on.

Kepler also introduced the logarithmic calculus into his Rudolphine tables, published in 1627; and inserted in that work several logarithmic tables; as, first a table similar to that above described, except that the second, or column of sines, or of absolute numbers, is omitted, and, instead of it, another column is added, showing what part of the quadrant each arc is equal to, namely the quotient, expressed in integers and sexagesimal parts, arising from dividing the whole quadrant by each given arc; 2dly, Napier's table of logarithmic sines, to every minute of the quadrant; also two other smaller tables, adapted to the purposes of eclipses and the latitudes of the planets. In this work also, Kepler gives a succinct account of logarithms, with the description and use of those that are contained in these tables. And here it is that he mentions Justus Byrgius, as having had logarithms before Napier published them.

Besides the above, some few others published logarithms of the same kind, about this time. But let us now return to treat of the history of the common or Briggs's logarithms, so called because he first computed them, and first mentioned them, and recommended them to Napier, instead of the first kind by him invented.

Mr. Henry Briggs, not less esteemed for his great probity, and other eminent virtues, than for his excellent skill in mathematics, was, at the time of the publication of Napier's logarithms, in 1614, professor of geometry in Gresham college in London, having been appointed the first professor after its institution: which appointment he held till January 1620, when he was chosen, also the first, Savilian professor of geometry at Oxford, where he died January the 26th, 163$\frac{0}{1}$, aged about 74 years.

On the publication of Napier's logarithms, Briggs immediately applied himself to the study and improvement of them. In a letter to Mr. (afterwards Archbishop) Usher, dated the

10th of March 1615, he writes, "that he was wholly taken up and employed about the noble invention of logarithms, lately discovered." And again, "Napier lord of Markinston hath set my head and hands at work with his new and admirable logarithms: I hope to see him this summer, if it please God; for I never saw a book which pleased me better, and made me more wonder." Thus we find that Briggs began very early to compute logarithms: but these were not of the same kind with Napier's, in which the logarithm of the ratio of 10 to 1 was 2.3025851 &c; for, in Briggs's first attempt he made 1 the logarithm of that ratio; and, from the evidence we have, it appears that he was the first person who formed the idea of this change in the scale, which he presently and liberally communicated, both to the public in his lectures, and to lord Napier himself, who afterwards said that he also had thought of the same thing; as appears by the following extract, translated from the preface to Briggs's "Arithmetica Logarithmica:" "Wonder not (says he) that these logarithms are different from those which the excellent baron of *Marchiston* published in his Admirable Canon. For when I explained the doctrine of them to my auditors at Gresham college in London, I remarked that it would be much more convenient, the logarithm of the sine total or radius being 0 (as in the *Canon Mirificus*), if the logarithm of the 10th part of the said radius, namely, of 5° 44′ 21″, were 100000 &c; and concerning this I presently wrote to the author; also, as soon as the season of the year and my public teaching would permit, I went to Edinburgh, where being kindly received by him, I staid a whole month. But when we began to converse about the alteration of them, he said that he had formerly thought of it, and wished it; but that he chose to publish those that were already done, till such time as his leisure and health would permit him to make others more convenient. And as to the manner of the change, he thought it more expedient that 0 should be made the logarithm of 1, and 100000 &c the logarithm of radius; which I could not but acknowledge was much better. Therefore, rejecting those which I had before prepared, I proceeded, at his exhortation, to calculate these: and the next summer I went again to Edinburgh, to shew him the principle of them; and should have been glad to do the same the third summer, if it had pleased God to spare him so long."

So that it is plain that Briggs was the inventor of the present scale of logarithms, in which 1 is the logarithm of the ratio of 10 to 1, and 2 that of 100 to 1, &c; and that the share which Napier had in them, was only advising Briggs to begin at the lowest number 1, and make the logarithms, or artificial numbers, as Napier had also called them, to *increase* with the natural numbers, instead of *decreasing*; which made no alteration in the figures that expressed Briggs's logarithms, but only in their affection or signs, changing them from negative to positive; so that Briggs's first logarithms to the numbers in the second column of the annexed tablet, would have been as in the first column; but after they were changed, as they are here in the third column; which is a change of no essential difference, as the logarithm of the ratio of 10 to 1, the radix of the natural system of numbers, continues the same; and a change in the logarithm of that ratio being the only circumstance that can essentially alter the system of logarithms, the logarithm of 1 being 0. And the reason why Briggs, after that interview, rejected what he had before done, and began anew, was probably because he had adapted his new logarithms to the approximate sines of arcs, instead of to the round or integer numbers; and not from their being logarithms of another system, as were those of Napier.

B	Num.	N
n	$\cdot 01^n$	$-n$
3	·001	−3
2	·01	−2
1	·1	−1
0	1	0
−1	10	1
−2	100	2
−3	1000	3
$-n$	10^n	n

On Briggs's return from Edinburgh to London the second time, namely, in 1617, he printed the first thousand logarithms, to eight places of figures, besides the index, under the title of "Logarithmorum Chilias Prima." Though these seem not to have been published till after death of Napier, which happened on the 3d of April 1618, as before said; for, in the preface to them, Briggs says, "Why these logarithms differ from those set forth by their most illustrious inventor, of ever respectful memory, in his 'Canon Mirificus,' IT IS TO BE HOPED his posthumous work will shortly make appear." And as Napier, after communication had with Briggs on the subject of altering the scale of logarithms, had given notice, both in Wright's translation, and in his own "Rabdologia," printed in 1617, of his intention to alter the scale, (though it appears very plainly that he never intended to compute any more), without making any mention of the share which Briggs had in the alteration, this gentleman modestly gave the above hint. But not finding any regard paid to it in the said posthumous work, published by lord Napier's son in 1619, where the alteration is again adverted to, but still without any mention of Briggs; this gentleman thought he could not do less than state the grounds of that alteration himself, as they are above extracted from his work published in 1624.

Thus, upon the whole matter, it seems evident that Briggs, whether he had thought of this improvement in the construction of logarithms, of making 1 the logarithm of the ratio of 10 to 1, before lord Napier, or not (which is a secret that could be known only to Napier himself), was the first person who communicated the idea of such an improvement to the world; and that he did this in his lectures to his auditors at Gresham college in the year 1615, very soon after his perusal of Napier's "Canon Mirificus Logarithmorum," published in the year 1614. He also mentioned it to Napier, both by letter in the same year, and on his first visit to him in Scotland in the summer of the year 1616, when Napier approved the idea, and said it had already occurred to himself, and that he had determined to adopt it. It appears therefore, that it would have been more candid in lord Napier to have told the world, in the second edition of this book, that Mr. Briggs had mentioned this improvement to him, and that he had thereby been confirmed in the resolution he had already taken, before

Mr. Briggs's communication with him (if indeed that was the fact), to adopt it in that his second edition, as being better fitted to the decimal notation of arithmetic which was in general use. Such a declaration would have been but an act of justice to Mr. Briggs; and the not having made it, cannot but incline us to suspect that lord Napier was desirous that the world should ascribe to him alone the merit of this very useful improvement of the logarithms, as well as that of having originally invented them; though, if the having first communicated an invention to the world be sufficient to entitle a man to the honour of having first invented it, Mr. Briggs had the better title to be called the first inventor of this happy improvement of logarithms.

In 1620, two years after the "Chilias Prima" of Briggs came out, Mr. Edmund Gunter published his "Canon of Triangles," which contains the artificial or logarithmic sines and tangents, for every minute, to seven places of figures, besides the index, the logarithm of radius being 10·0 &c. These logarithms are of the kind last agreed upon by Napier and Briggs, and they were the first tables of logarithmic sines and tangents that were published of this sort. Gunter also, in 1623, reprinted the same in his book "De Sectore et Radiâ," together with the "Chilias Prima" of his old colleague Mr. Briggs, he being professor of astronomy at Gresham college when Briggs was professor of geometry there, Gunter having been elected to that office the 6th of March 1619, and enjoyed it till his death, which happened on the 10th of December 1626, about the forty-fifth year of his age. In 1623, also, Gunter applied these logarithms of numbers, sines, and tangents, to straight lines drawn on a ruler; with which, proportions in common numbers and trigonometry were resolved by the mere application of a pair of compasses; a method founded on this property, that the logarithms of the terms of equal ratios are equidifferent. This instrument, in the form of a two-foot scale, is now in common use for navigation and other purposes, and is commonly called the Gunter. He also greatly improved the sector for the same uses. Gunter was the first who used the word *cosine* for the sine of the complement of an arc. He also introduced the use of arithmetical complements into the logarithmical arithmetic, as is witnessed by Briggs, chap. 15, Arith. Log. And it has been said, that he started the idea of the logarithmic curve, which was so called because the segments of its axis are the logarithms of the corresponding ordinates.

The logarithmic lines were afterwards drawn in various other ways. In 1627, they were drawn by Wingate on two separate rulers sliding against each other, to save the use of compasses in resolving proportions. They were also, in 1627, applied to concentric circles, by Oughtred. Then in a spiral form, by a Mr. Milburne of Yorkshire, about the year 1650. And, lastly, in 1657, on the present sliding rule, by Seth Partridge.

The discoveries relating to logarithms were carried to France by Mr. Edmund Wingate, but not first of all, as he erroneously says in the preface to his book. He published at Paris, in 1624, two small tracts in the French language; and afterwards at London, in 1626, an English edition of the same, with improvements. In the first of these, he teaches the use of Gunter's rules; and in the other, that of Briggs's logarithms, and the artificial sines and tangents. Here are contained, also, tables of those logarithms, sines, and tangents, copied from Gunter. The edition of these logarithms printed at London in 1635, and the former editions also I suppose, has the units figures disposed along the tops of the columns, and the tens down the margins, like our tables at present; with the whole logarithm, which was only to fix places of figures, in the angle of meeting: which is the first instance that I have seen of this mode of arrangement.

But proceed we now to the larger structure of logarithms. Briggs had continued from the beginning to labour with great industry at the computation of those logarithms of which he before published a short specimen in small numbers. And, in 1624, he produced his "Arithmetica Logarithmica"—a stupendous work for so short a time!—containing the logarithms of 30000 natural numbers, to fourteen places of figures besides the index, namely, from 1 to 20000, and from 90000 to 100000; together with the differences of the logarithms. Some writers say that there was another *chiliad*, namely, from 100000 to 101000; but none of the copies that I have seen have more than the 30000 above mentioned, and they were all regularly terminated in the usual way with the word FINIS. The preface to these logarithms contains, among other things, an account of the alteration made in the scale by Napier and himself, from which we have given an extract; and an earnest solicitation to others to undertake the computation for the intermediate numbers, offering to give instructions, and paper ready ruled for that purpose, to any persons so inclined to contribute to the completion of so valuable a work. In the introduction, he gives also an ample treatise on the construction and uses of these logarithms, which will be particularly described hereafter.—By this invitation, and other means, he had hopes of collecting materials for the logarithms of the intermediate 70000 numbers, while he should employ his own labour more immediately on the canon of logarithmic sines and tangents, and so carry on both works at once; as indeed they were both equally necessary, and he himself was now pretty far advanced in years.

Soon after this however, Adrian Vlacq, or Flack, of Gouda in Holland, completed the intermediate seventy chiliads, and republished the "Arithmetica Logarithmica" at that place, in 1627 and 1628, with those intermediate numbers, making in the whole the logarithms of all numbers to 100000, but only to ten places of figures. To these was added a table of artificial sines, tangents, and secants, to every minute of the quadrant.

Briggs himself lived also to complete a table of logarithmic sines and tangents for the hundredth part of every degree, to fourteen places of figures besides the index; together with a table of natural sines for the same parts to fifteen places, and the tangents and secants for the same to ten places; with the construction of the whole. These tables were printed at

Gouda, under the care of Adrian Vlacq, and mostly finished off before 1631, though not published till 1633. But his death, which then happened, prevented him from completing the application and uses of them. However, the performing of this office, when dying, he recommended to his friend Henry Gellibrand, who was then professor of astronomy in Gresham college, having succeeded Mr. Gunter in that appointment. Gellibrand accordingly added a preface, and the application of the logarithms to plain and spherical trigonometry, &c; and the whole was printed at Gouda by the same printer, and brought out in the same year, 1633, as the "Trigonometria Artificialis" of Vlacq, who had the care of the press as above said. This work was called "Trigonometria Britannica;" and besides the arcs in degrees and centesms of degrees, it has another column, containing the minutes and seconds answering to the several centesms in the first column.

In 1633, as mentioned above, Vlacq printed at Gouda, in Holland, his "Trigonometria Artificialis; sive Magnus Canon Triangulorum Logaritimicus ad Decadas Secundorum Scrupulorum constructus." This work contains the logarithmic sines and tangents to ten places of figures, with their differences, for every ten seconds in the quadrant. To them is also added Briggs's table of the first 20000 logarithms, but carried only to ten places of figures besides the index, with their differences. The whole is preceded by a description of the tables, and the application of them to plane and spherical trigonometry, chiefly extracted from Briggs's "Trigonometria Britannica," mentioned above.

Gellibrand published also, in 1635, "An Institution Trigonometricall," containing the logarithms of the first 10000 numbers, with the natural sines, tangents, and secants, and the logarithmic sines and tangents, for degrees and minutes, all to seven places of figures, besides the index; as also other tables proper for navigation; with the uses of the whole. Gellibrand died the 9th of February 1636, in the 40th year of his age, to the great loss of the mathematical world.

Besides the persons hitherto mentioned, who were mostly computers of logarithms, many others have also published tables of those artificial numbers, more or less complete, and sometimes improved and varied in the manner and form of them. We may here just advert to a few of the principal of these.

In 1626, D. Henrion published, at Paris, a treatise concerning Brigg's logarithms of common numbers, from 1 to 20000, to eleven places of figures; with the sines and tangents to eight places only.

In 1631, was printed, at London, by one George Miller, a book containing Briggs's logarithms, with their differences, to ten places of figures besides the index, for all numbers to 100000; as also the logarithmic sines, tangents, and secants, for every minute of the quadrant; with the explanation and uses in English.

The same year, 1631, Richard Norwood published his "Trigonometria;" in which we find Briggs's logarithms for all numbers to 10000, and for the sines, tangents, and secants, to every minute, both to seven places besides the index.—In the conclusion of the trigonometry, he complains of the unfair practices of printing Vlacq's book in 1627 or 1628, and the book mentioned in the last article. His words are, "Now, whereas I have here, and in sundry places in this book, cited Mr. Briggs his 'Arithmetica Logarithmica,' (lest I may seem to abuse the reader) you are to understand not the book put forth about a month since in English, as a translation of his, and with the same title; being nothing like his, nor worthy his name; but the book which himself put forth with this title in Latin, being printed at London anno 1624. And here I have just occasion to blame the ill dealing of these men, both in the matter before mentioned, and in printing a second edition of his 'Arithmetica Logarithmica' in Latin, whilst he lived, against his mind and liking; and brought them over to sell, when the first were unsold; so frustrating those additions which Mr. Briggs intended in his second edition, and moreover leaving out some things that were in the first edition, of special moment: a practice of very ill consequence, and tending to the great disparagement of such as take pains in this kind."

Francis Bonaventure Cavalerius published at Bologna, in 1632, his "Directorium Generale Uranometricum," in which are tables of Briggs's logarithms of sines, tangents, secants, and versed sines, each to eight places, for every second of the first five minutes, for every five seconds from five to ten minutes, for every ten seconds from ten to twenty minutes, for every twenty seconds from twenty to thirty minutes, for every thirty seconds from 30′ to 1° 30′, and for every minute in the rest of the quadrant; which is the first table of logarithmic versed sines that I know of. In this book are contained also the logarithms of the first ten chiliads of natural numbers, namely, from 1 to 10000, disposed in this manner: all the twenties at top, and from 1 to 19 on the side, the logarithm of the sum being in the square of meeting. In this work also, I think Cavalerius gave the method of finding the area or spherical surface contained by various arcs described on the surface of a sphere; which had before been given by Albert Girard, in his Algebra, printed in the year 1629.

Also, in the "Trigonometria" of the same author, Cavalerius, printed in 1643, besides the logarithms of numbers from 1 to 1000, to eight places, with their differences, we find both natural and logarithmic sines, tangents, and secants, the former to seven, and the latter to eight places; namely, to every 10″ of the first 30 minutes, to every 30″ from 30′ to 1°; and the same for their complements, or backwards through the last degree of the quadrant; the intermediate 88° being to every minute only.

Mr. Nathaniel Roe, "Pastor of Benacre in Suffolke," also reduced the logarithmic tables to a contracted form, in his "Tabulæ Logarithmicæ," printed at London in 1633. Here we have Briggs's logarithms of numbers from 1 to 100000, to eight places; the fifties placed at top, and from 1 to 50 on the side; also the first four figures of the logarithms at top, and the other four down the columns. They contain also the logarithmic sines and tangents to every 100th part of degrees, to ten places.

Ludovicus Frobenius published at Hamburgh, in 1634, his "Clavis Universa Trigonometriæ," containing tables of Briggs's logarithms of numbers, from 1 to 2000; and of sines, tangents, and secants, for every minute; both to seven places.

But the table of logarithms of common numbers was reduced to its most convenient form by John Newton, in his "Trigonometria Britannica," printed at London in 1658, having availed himself of both the improvements of Wingate and Roe, namely, uniting Wingate's disposition of the natural numbers with Roe's contracted arrangement of the logarithms, the numbers being all disposed as in our best tables at present, namely, the units along the top of the page, and the tens down the left-hand side, also the first three figures of each logarithm in the first column, and the remaining five figures in the other columns, the logarithms being to eight places. This work contains also the logarithmic sines and tangents, to eight figures besides the index, for every 100th part of a degree, with their differences, and for 1000th parts in the first three degrees.—In the preface to this work, Newton takes occasion, as Wingate and Norwood had done before, as well as Briggs himself, to censure the unfair practices of some other publishers of logarithms. He says, "In the second part of this institution, thou art presented with Mr. Gellibrand's Trigonometrie, faithfully translated from the Latin copy, that which the author himself published under the title of 'Trigonometria Britannica,' and not that which Vlacq the Dutchman styles 'Trigonometria Artificialis,' from whose corrupt and imperfect copy that seems to be translated which is amongst us generally known by the name of 'Gellibrand's Trigonometry;' but those who either knew him, or have perused his writings, can testify that he was no admirer of the old sexagenary way of working; nay, that he did preferre the decimal way before it, as he hath abundantly testified in all the examples of this his Trigonometry, which differs from that other which Vlacq hath published, and that which hath hitherto borne his name in English, as in the form, so likewise in the matter of it; for in the two last-mentioned editions, there is something left out in the second chapter of plain triangles, the third chapter wholly omitted, and a part of the third in the spherical; but in this edition nothing: something we have added to both, by way of explanation and demonstration."

In 1670, John Caramuel published his "Mathesis Nova," in which are contained 1000 logarithms both of Napier's and Briggs's form, as also 1000 of what he calls the Perfect Logarithms, namely, the same as those which Briggs first thought of, which differ from the last only in this, that the one increases while the other decreases, the radix or logarithm of the ratio of 10 to 1 being the same in both.

The books of logarithms have since become very numerous, but the logarithms are mostly of that sort invented by Briggs, and which are now in common use. Of these, the most noted for their accuracy or usefulness, besides the works above mentioned, are Vlacq's small volume of tables, particularly that edition printed at Lyons, in 1670; also tables printed at the same place in 1760; but most especially the tables of Sherwin and Gardiner, particularly my own improved editions of them. Of these, Sherwin's "Mathematical Tables," in 8vo, formed, till lately, the most complete collection of any, containing, besides the logarithms of all numbers to 101000, the sines, tangents, secants, and versed sines, both natural and logarithmic, to every minute of the quadrant, though not conveniently arranged. The first edition was in 1706; but the third edition, in 1742, which was revised by Gardiner, is esteemed the most correct of any, though containing many thousands of errors in the final figures, as well as all the former editions: as to the last or fifth edition, in 1771, it is so erroneously printed that no dependance can be placed in it, being the most inaccurate book of tables I ever knew; I have a list of several thousand errors which I have corrected in it, as well as in Gardiner's octavo edition, and in Sherwin's edition.

Gardiner also printed at London, in 1742, a quarto volume of "Tables of Logarithms, for all numbers from 1 to 102100, and for the sines and tangents to every ten seconds of each degree in the quadrant; as also, for the sines of the first 72 minutes to every single second: with other useful and necessary tables;" namely a table of Logistical Logarithms, and three smaller tables to be used for finding the logarithms of numbers to twenty places of figures. Of these tables of Gardiner, only a small number was printed, and that by subscription; and they have always been held in great estimation for their accuracy and usefulness.

An edition of Gardiner's collection was also elegantly printed at Avignon in France, in 1770, with some additions, namely, the sines and tangents for every single second in the first four degrees, and a small table of hyperbolic logarithms, copied from a treatise on Fluxions by the late ingenious Mr. Thomas Simpson: but this is not quite so correct as Gardiner's own edition. The tables in all these books are to seven places of figures.

Lastly, my own Mathematical Tables, being the most accurate and best arranged set of logarithmic tables ever before given; preceded also by a large and critical history of Trigonometry and Logarithms, and terminating with a copious list of the errors discovered in the principal other tables of this kind.

There have also lately appeared the following accurate and elegant books of logarithms; viz. 1. "Logarithmic Tables," by the late Mr. Michael Taylor, a pupil of mine, and author of "The Sexagesimal Table." His work consists of three tables; 1st, The Logarithms of Common Numbers from 1 to 1260, each to 8 places of figures; 2dly, The Logarithms of all Numbers from 1 to 101000, each to 7 places; 3dly, The Logarithmic Sines and Tangents to every Second of the Quadrant, also to 7 places of figures: a work that must prove highly useful to such persons as may be employed in very nice and accurate calculations, such as astronomical tables, &c. The author dying when the tables were nearly all printed off, the Rev. Dr. Maskelyne, astronomer royal, supplied a preface, containing an account of the work, with excellent precepts for the explanation and use of the tables: the whole very accurately and elegantly printed on large 4to, 1792.

2. "Tables Portatives de Logarithmes, publiées à Londres par Gardiner," &c. This work is most beautifully printed in a neat portable 8vo volume, and contains all the tables in Gardiner's 4to volume, with some additions and improvements, and with a considerable degree of accuracy. Printed at Paris, by Didot, 1793. On this, as well as several other occasions, it is but justice to remark the extraordinary spirit and elegance with which the learned men, and the artisans of the French nation, undertake and execute works of merit.

3. A second edition of the "Tables Portatives de Logarithmes," &c. printed at Paris with the stereotypes, of solid pages, in 8vo, 1795, by Didot. This edition is greatly enlarged, by an extension of the old tables, and many new ones; among which are the logarithm sines and tangents to every ten thousandth part of the quadrant, viz. in which the quadrant is first divided into 100 equal parts, and each of these into 100 parts again.

4. Other more extensive tables, by Borda and Delambre, were published at Paris in 1801. Besides the usual table of the logarithms of common numbers, and a large introduction, on the nature and construction of them, this work contains very extensive tables of decimal trigonometry, arranged in a new and curious way, and containing the log. sines, tangents, and secants, of the quadrant, divided first into 100 degrees, each degree into 100 minutes, and each minute into 100 seconds.

The logarithmic canon serves to find readily the logarithm of any assigned number; and we are told by Dr. Wallis, in the second volume of his Mathematical Works, that an antilogarithmic canon, or one to find as readily the number corresponding to every logarithm, was begun, he thinks, by Harriot the algebraist, who died in 1621, and completed by Walter Warner, the editor of Harriot's works, before 1640;

which ingenious performance, it seems, was lost, for want of encouragement to publish it.

A small specimen of such numbers was published in the Philosophical Transactions for the year 1714, by Mr. Long of Oxford; but it was not till 1742 that a complete antilogarithmic canon was published by Mr. James Dodson, wherein he has computed the numbers corresponding to every logarithm from 1 to 100000, for 11 places of figures.

TRACT XXI.

THE CONSTRUCTION OF LOGARITHMS, &c.

Having, in the last Tract, described the several kinds of logarithms, their rise and invention, their nature and properties, and given some account of the principal early cultivators of them, with the chief collections that have been published of such tables; proceed we now to deliver a more particular account of the ideas and methods employed by each author, and the peculiar modes of construction made use of by them. And first, of the great inventor himself, Lord Napier.

Napier's Construction of Logarithms.

The inventor of logarithms did not adapt them to the series of natural numbers 1, 2, 3, 4, 5, &c, as it was not his principal idea to extend them to all arithmetical operations in general: but he confined his labours to that circumstance which first suggested the necessity of the invention, and adapted his logarithms to the approximate numbers which express the natural sines of every minute in the quadrant, as they had been set down by former writers on trigonometry.

The same restricted idea was pursued through his method of constructing the logarithms. As the lines of the sines of all arcs are parts of the radius, or sine of the quadrant, which was therefore called the *sinus totus*, or whole sine, he conceived the line of the radius to be described, or run over, by a point moving along it in such a manner, that in equal portions of time it generated, or cut off, parts in a decreasing geometrical progression, leaving the several remainders, or sines, in geometrical progression also; while another point, in an indefinite line, described equal parts of *it* in the same equal portions of time; so that the respective sums of these, or the whole line generated, were always the arithmeticals or logarithms of these sines. Thus, az is the given radius on which all the sines are to be taken, and A&c the indefinite line containing the logarithms; these lines being each generated by the motion of points, beginning at A, a. Now, at the end of the 1st, 2d, 3d, &c, moments, or equal small portions of time, the moving points being found at the places marked 1, 2, 3, &c; then za, $z1$, $z2$, $z3$, &c, will be the series of natural sines, and A0, or 0, A1, A2, A3, &c, will be their logarithms; supposing the point which generates az to move every where with a velocity decreasing in proportion to its distance from z, namely, its velocity in the points 0, 1, 2, 3, &c, to be respectively as the distances $z0$, $z1$, $z2$, $z3$, &c, while the velocity of the point generating the logarithmic line A&c remains constantly the same as at first in the point A or 0.

Hitherto the author had not fully limited his system or scale of logarithms, having only supposed one condition or limitation, namely, that the logarithm of the radius az should be 0: whereas two independent conditions, no matter what, are necessary to limit the scale or system of logarithms. It did not occur to him that it was proper to form the other limit, by affixing some particular value to an assigned number, or part of the radius: but, as another condition was necessary, he assumed *this* for it, namely, that the two generating points should begin to move at a and A with equal velocities; or that the increments $a1$ and A1, described in the first moments, should be equal; as he thought this circumstance would be attended with some little ease in the computation. And this is the reason that, in his table, the natural sines and their logarithms, at the complete quadrant, have equal differences; and this is also the reason why his scale of logarithms happens accidentally to agree with what have since been called the hyperbolic logarithms, which have numeral differences equal to those of their natural numbers, at the beginning; except only that these latter increase with the natural numbers, and his on the contrary decrease; the logarithm of the ratio of 10 to 1 being the same in both, namely, 2·30258509.

And here, by the way, it may be observed, that Napier's manner of conceiving the generation of the lines of the natural numbers, and their logarithms, by the motion of points, is very similar to the manner in which Newton afterwards considered the generation of magnitudes in his doctrine of fluxions; and it is also remarkable, that, in art. 2, of the "Habitudines Logarithmorum et suorum naturalium numerorum invicem," in the appendix to the "Constructio Logarithmorum," Napier speaks of the velocities of the increments or decrements of the logarithms, in the same way as Newton does of his fluxions, namely, where he shows that those velocities, or fluxions, are inversely as the sines or natural numbers of the logarithms; which is a necessary consequence of the nature of the generation of those lines as described above; with this alteration, however, that now the radius az must be considered as generated by an equable motion of the point, and the indefinite line A&c by a motion increasing in the same ratio as the other before decreased; which is a supposition that Napier must have had in view when he stated that relation of the fluxions.

Having thus limited his system, Napier proceeds, in the posthumous work of 1619, to explain his construction of the logarithmic canon; and this he effects in various ways, but chiefly by generating, in a very easy manner, a series of proportional numbers, and their arithmeticals or logarithms; and then finding, by proportion, the logarithms to the natural sines, from those of the nearest numbers among the original proportionals.

After describing the necessary cautions he made use of, to preserve a sufficient degree of accuracy, in so long and complex a process of calculation; such as annexing several ciphers, as decimals separated by a point, to his primitive numbers, and rejecting the decimals thence resulting after the operations were completed; setting the numbers down to the nearest unit in the last figure; and teaching the arithmetical processes of adding, subtracting, multiplying, and dividing the limits, between which certain unknown numbers must lie, so as to obtain the limits between which the results must also fall; I say, after describing such particulars, in order to clear and smooth the way, he enters on the great field of calculation itself. Beginning at radius 10000000, he first constructs several descending geometrical series, but of such a nature, that they are all quickly formed by an easy continual subtraction, and a division by 2, or by 10, or 100, &c, which is done by only removing the decimal point so many places towards the left-hand, as there are ciphers in the divisor. He constructs three tables of such series: The first of these consists of 100 numbers, in the proportion of radius to radius minus 1, or of 10000000 to 9999999; all which are found by only subtracting from each its 10000000th part, which part is also found by only removing each figure seven places lower: the last of these 100 proportionals is found to be 9999900·0004950.

The 2d table contains 50 numbers, which are in the continual proportion of the first to the last in the first table, namely, of 10000000·0000000 to 9999900 0004950, or nearly the proportion of 100000 to 99999; these therefore are found by only removing the figures of each number 5 places lower, and subtracting them from the same number: the last of these he finds to be 9995001·222927. And a specimen of these two tables is here annexed.

No.	First Table.	Second Table.
1	10000000.0000000	10000000.000000
2	9999999.0000000	9999900.000000
3	9999998.0000001	9999800.001000
4	9999997.0000003	9999700.003000
&c.	&c till the 100th	&c to the 50th
50	term, which will be	term.
100	9999900.0004950	9995001.222927

The 3d table consists of 69 columns, and each column of 21 numbers or terms, which terms, in every column, are in the continual proportion of 10000 to 9995, that is, nearly as the first is to the last in the 2d table; and as 10000 exceeds 9995 by the 2000th part, the terms in every column will be constructed by dividing each upper number by 2, removing the figures of the quotient 3 places lower, and then subtracting them; and in this way it is proper to construct only the first column of 21 numbers, the last of which will be 9900473·5780: but the 1st, 2d, 3d, &c, numbers, in all the columns, are in the continual proportion of 100 to 99, or nearly the proportion of the first to the last in the first column; and therefore these will be found by removing the figures of each preceding number two places lower, and subtracting them, for the like number in the next column. A specimen of this 3d table is as here below.

The Third Table.					
Terms	1st Column.	2d Column.	3d Column.	&c till the	69th Col.
1	10000000.0000	9900000.0000	9801000.0000	&c for	5048858.8900
2	9995000.0000	9895050.0000	9796099.5000	the 4th	5046334.4605
3	9990002.5000	9890102.4750	9791201.4503	5th, 6th,	5043811.2932
4	9985007.4987	9885157.4237	9786305.8495	7th, &c	5041289.3879
5	9980014.9950	9880214.8451	9781412.6967	col. till	5038768.7435
&c	&c till	&c	&c	the last	&c
21	9900473.5780	9801468.8423	9703454.1539	or	4998609.4034

Thus he had, in this 3d table, interposed between the radius and its half, 68 numbers in the continual proportion of 100 to 99; and interposed between every two of these, 20 numbers in the proportion of 10000 to 9995: and again, in the 2d table, between 10000000 and 9995000, the two first of the 3d table, he had 50 numbers in the proportion of 100000 to 99999; and lastly, in the 1st table, between 10000000 and 9999900, or the two first in the 2d table, 100 numbers in the proportion of 10000000 to 9999999; that is in all, about 1600 proportionals; all found in the most simple manner, by little more than easy subtractions; which proportionals nearly coincide with all the natural sines from 90° down to 30°.

To obtain the logarithms of all those proportionals, he demonstrates several properties and relations of the numbers and logarithms, and illustrates the manner of applying them. The principal of these properties are as follow: 1st, that the logarithm of any sine is greater than the difference between that sine and the radius, but less than the said difference when increased in the proportion of the sine to radius*; and 2dly, that the difference between the logarithms of two sines, is less than the difference of the sines increased in the proportion of the less sine to radius, but greater than the said difference of the sines increased in the proportion of the greater sine to radius †.

Hence, by the 1st theorem, the logarithm of 10000000, the radius or first term in the first table, being 0, the logarithm of 9999999, the 2d term, will be between 1 and 1·0000001, and will therefore be equal to 1·00000005 very nearly: and this will be also the common difference of all the terms or proportionals in the first table; therefore, by the continual addition of this logarithm, there will be obtained the logarithms of all these 100 proportionals; consequently 100 times the said first logarithm, or the last of the above sums, will

* By this first theorem, r being radius, the logarithm of the sine s is between $r-s$ and $\frac{r-s}{s}r$; and therefore, when s differs but little from r, the logarithm of s will be nearly equal to $\frac{(r+s)\times(r-s)}{2s}$, the arithmetical mean between the limits $r-s$ and $\frac{r-s}{s}r$; but still nearer to $(r-s)\sqrt{\frac{r}{s}}$ or $\frac{r-s}{s}\sqrt{rs}$, the geometrical mean between the said limits.

† By this second theorem, the difference between the logarithms of the two sines S and s, lying between the limits $\frac{S-s}{s}r$ and $\frac{S-s}{S}r$, will, when those sines differ but little, be nearly equal to $\frac{S^2-s^2}{2Ss}r$ or $\frac{(S+s)\times(S-s)}{2Ss}r$, their arithmetical mean; or nearly $\frac{S-s}{\sqrt{Ss}}r$, the geometrical mean; or nearly $=\frac{S-s}{S+s}2r$, by substituting in the last denominator, $\frac{1}{2}(S+s)$ for $\sqrt{Ss}$, to which it is nearly equal.

give 100·000005, for the logarithm of 9999900·0004950, the last of the said 100 proportions.

Then, by the 2d theorem, it easily appears, that ·0004950 is the difference between the logarithms of 9999900·0004950 and 9999900, the last term of the first table, and the 2d term of the second table; this then being added to the last logarithm, gives 100·0005000 for the logarithm of the said 2d term, as also the common difference of the logarithms of all the proportions in the second table; and therefore, by continually adding it, there will be generated the logarithms of all these proportionals in the second table; the last of which is 5000·025, answering to 9995001·222927, the last term of that table.

Again, by the 2d theorem, the difference between the logarithms of this last proportional of the second table, and the 2d term in the first column of the third table, is found to be 1·2235387; which being added to the last logarithm, gives 5001·2485387 for the logarithm of 9995000, the said 2d term of the third table, as also the common difference of the logarithms of all the proportionals in the first column of that table; and that this, therefore, being continually added, gives all the logarithms of that first column, the last of which is 100024·97077, the logarithm of 9900473·5780, the last term of the said column.

Finally, by the 2d theorem again, the difference between the logarithms of this last number and 9900000, the 1st term in the second column, is 478·3502; which being added to the last logarithm, gives 100503·3210 for the logarithm of the said 1st term in the second column, as well as the common difference of the logarithms of all the numbers on the same line in every line of the table, namely, of all the 1st terms, of all the 2d, of all the 3d, of all the 4th, &c, terms, in all the columns; and which, therefore, being continually added to the logarithms in the first column, will give the corresponding logarithms in all the other columns.

And thus is completed what the author calls the radical table, in which he retains only one decimal place in the logarithms (or *artificials*, as he always calls them in his tract on the construction), and four in the naturals. A specimen of the table is as here follows:

Radical Table.

Terms	1st Column.		2d Column.		69th Column.	
	Naturals.	Artificals	Naturals.	Artific.	Naturals.	Artificials
1	10000000.0000	0	9900000.0000	100503.3	5048858.8900	6834225.8
2	9995000.0000	5001.2	9895050.0000	105504.6	5046333.4605	6839227.1
3	9990002.5000	10002.5	9890102.4750	110505.8	5043811.2932	6844228.3
4	9985007.4987	15003.7	9885157.4237	115507.1	5041289.3879	6849229.6
5	9980014.9950	20005.0	9880214.8451	120508.3	5038768.7435	6854230.8
&c	&c till	&c	&c	&c	&c	&c
21	9900473.5780	100025.0	9801468.8423	200528.2	4998609.4034	6934250.8

Having thus, in the most easy manner, completed the radical table, by little more than mere addition and subtraction, both for the natural numbers and logarithms; the logarithmic sines were easily deduced from it by means of the 2d theorem, namely, taking the sum and difference of each tabular sine and the nearest number in the radical table, annexing 7 ciphers to the difference, dividing the result by the sum, then half the quotient gives the difference between the logarithms of the said numbers, namely, between the tabular sine and radical number; consequently, adding or subtracting this difference, to or from the given logarithm of the radical number, there is obtained the logarithmic sine required. And thus the logarithms of all the sines, from radius to the half of it, or from 90° to 30°, were perfected.

Next, for determining the sines of the remaining 30 degrees, he delivers two methods. In the first of these he proceeds in this manner: Observing that the logarithm of the ratio of 2 to 1, or of half the radius, is 6931469·22, of 4 to 1 is the double of this, of 8 to 1 is triple of it, &c; that of 10 to 1 is 23025842.34, of 20 to 1 is the sum of the logarithms of 2 and 10; and so on, by composition for the logarithms of the ratios between 1 and 40, 80, 100, 200, &c, to 10000000; he multiplies any given sine, for an arc less than 30 degrees, by some of these numbers, till he finds the product nearly equal to one of the tabular numbers; then by means of this and the second theorem, the logarithm of this product is found; to which adding the logarithm that answers to the multiple above mentioned, the sum is the logarithm sought. But the other method is still much easier, and is derived from this property, which he demonstrates, namely, as half radius is to the sine of half an arc, so is the cosine of the said half arc, to the sine of the whole arc; or as $\frac{1}{2}$ radius : sine of an arc :: cosine of the arc : sine of double arc; hence the logarithmic sine of an arc is found, by adding together the logarithms of half radius and of the sine of the double arc, and then subtracting the logarithmic cosine from the sum.

And thus the remainder of the sines, from 30° down to 0, are easily obtained. But in this latter way, the logarithmic sines for full one half of the quadrant, or from 0 to 45 degrees, he observes, may be derived; the other half having already been made by the general method of the radical table, by one easy division and addition or subtraction for each.

We have dwelt the longer on this work of the inventor of logarithms, because I have not seen, in any author, an account of his method of constructing his table, though it is perfectly different from every other method used by the later computers, and indeed almost peculiar to his species of logarithms. The whole of this work manifests great ingenuity in the designer, as well as much accuracy. But notwithstanding the caution he took to obtain his logarithms true to the nearest unit in the last figure set down in the tables, by extending the numbers in the computations to several decimals, and other means; he had been disappointed of that end, either by the inaccuracy of his assistant computers or transcribers, or through some other cause; as the logarithms in the table are commonly very inaccurate. It is remarkable too, that in this tract on the construction of the logarithms, Lord Napier never calls them logarithms, but every where *artificials*, as opposed in idea to the natural numbers: and this notion, of natural and artificial numbers, I take to have been his first idea of this matter, and that he altered the word *artificials* to *logarithms* in his first book, on the description of them, when he printed it, in the year 1614, and that he would also have altered the word every where in this posthumous work, if he had lived to print it: for in the two or three pages of appendix, annexed to the work by his son, from Napier's papers, he again always calls them logarithms. This appendix relates to the change of the logarithms to that scale in which 1 is the logarithm of the ratio of 10 to 1, the logarithm of 1, with or without ciphers, being 0; and it appears to have been written after Briggs communicated to him his idea of that change.

Napier here in this appendix also briefly describes some methods, by which this new species of logarithms may be constructed. Having supposed 0 to be the logarithm of 1, and 1, with any number of ciphers, as 10000000000, the logarithm of 10; he directs to divide this logarithm of 10, and the successive quotients, ten times by 5; by which divisions there will be obtained these other ten logarithms, viz. 2000000000, 400000000, 80000000, 16000000, 3200000, 640000, 128000, 25600, 5120, 1024: then this last logarithm, and its quotients, being divided ten times by 2, will give these other ten logarithms, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. And the numbers answering to these twenty logarithms, we are directed to find in this manner; namely, extract the 5th root of 10, with ciphers, then the 5th root of that root, and so on, for ten continual extractions of the 5th root; so shall these ten roots be the natural numbers belonging to the first ten logarithms, above found in continually dividing by 5: next, out of the last 5th root we are to extract the square root, then the square root of this last root, and so on, for ten successive extractions of the square root; so shall these last ten roots be the natural numbers corresponding to the logarithms or quotients arising from the last ten divisions by the number 2. And from these twenty logarithms, 1, 2, 4, 8, 16, &c, and their natural numbers, the author observes that other logarithms and their numbers may be formed, namely, by adding the logarithms, and multiplying their corresponding numbers. It is evident that this process would generate rather an antilogarithmic canon, such as Dodson's, than the table of Briggs; and that the method would also be very laborious, since, besides the very troublesome original extractions of the 5th roots, all the numbers would be very large, by the multiplication of which the successive secondary natural numbers are to be found.

Our author next mentions another method of deriving a few of the primitive numbers and their logarithms, namely, by taking continually geometrical means, first between 10 and 1, then between 10 and this mean, and again between 10 and the last mean, and so on; and taking the arithmetical means between their corresponding logarithms. He then lays down various relations between numbers and their logarithms; such as, that the products and quotients of numbers answer to the sums and differences of their logarithms, and that the powers and roots of numbers answer to the products and quotients of the logarithms by the index of the power or root, &c; as also that, of any two numbers whose logarithms are given, if each number be raised to the power denoted by the logarithm of the other, the two results will be equal. He then delivers another method of making the logarithms to a few of the prime integer numbers, which is well adapted for constructing the common table of logarithms. This method easily follows from what has been said above; and it depends on this property, that the logarithm of any number in this scale, is 1 less than the number of places or figures contained in that power of the given number whose exponent is 10000000000, or the logarithm of 10, at least as to integer numbers, for they really differ by a fraction, as is shown by Mr. Briggs in his illustrations of these properties, printed at the end of this appendix to the construction of logarithms. We shall here just notice one more of these relations, as the manner in which it is expressed is exactly similar to that of fluxions and fluents, and it is this: Of any two numbers, as the greater is to the less, so is the velocity of the increment or decrement of the logarithms at the less, to the velocity of the increment or decrement of the logarithms at the greater: that is, in our modern notation, as $X : Y :: \dot{y}$ to $\dot{x}$, where $\dot{x}$ and $\dot{y}$ are the fluxions of the logarithms of X and Y.

Kepler's Construction of Logarithms.

The logarithms of Briggs and Kepler were both printed the same year, 1624; but as the latter are of the same sort as Napier's, we may first consider this author's construction of them, before proceeding to that of Briggs's.

We have already, in the last Tract, described the nature and form of Kepler's logarithms; showing that they are of the same kind as Napier's, but only a little varied in the form of the table. It may also be added, that, in general, the ideas which these two masters had on this subject, were of the same nature; only they were more fully and methodically laid down by Kepler, who expanded, and delivered in a regular science, the hints that were given by the illustrious inventor. The foundation and nature of their methods of construction are also the same, but only a little varied in their modes of applying them. Kepler here, first of any, treats of logarithms in the true and genuine way of the measures of ratios, or proportions*, as he calls them, and that in a very full and scientific manner: and this method of his was afterwards followed and abridged by Mercator, Halley, Cotes, and others, as we shall see in the proper places. Kepler first erects a regular and purely mathematical system of proportions, and the measures of proportions, treated at considerable length in a number of propositions, which are fully and chastely demonstrated by genuine mathematical reasoning, and illustrated by examples in numbers. This part contains and demonstrates both the nature and the principles of the struc-

* Kepler almost always uses the term *proportion* instead of *ratio,* which we shall also do in the account of his work, as well as conform in expressions and notations to his other peculiarities. It may also be here remarked, that I observe the same practice in describing the works of other authors, the better to convey the idea of their several methods and style. And this may serve to account for some seeming inequalities in the language of this history.

ture of logarithms. And in the second part the author applies those principles in the actual construction of his table, which contains only 1000 numbers, and their logarithms, in the form as we before described: and in this part he indicates the various contrivances made use of in deducing the logarithms of proportions one from another, after a few of the leading ones had been first formed, by the general and more remote principles. He uses the name *logarithms*, given them by the inventor, being the most proper, as expressing the very nature and essence of those artificial numbers, and containing as it were a definition in the very name of them; but without taking any notice of the inventor, or of the origin of those useful numbers.

As this tract is very curious and important in itself, and is besides very rare and little known, instead of a particular description only, we shall here give a brief translation of both the parts, omitting only the demonstrations of the propositions, and some rather long illustrations of them. The book is dedicated to Philip, landgrave of Hesse, but is without either preface or introduction, and commences immediately with the subject of the first part, which is intitled "The Demonstration of the Structure of Logarithms;" and the contents of it are as follow.

Postulate 1. That all proportions that are equal among themselves, by whatever variety of couplets of terms they may be denoted, are measured or expressed by the same quantity.

Axiom 1. If there be any number of quantities of the same kind, the proportion of the extremes is understood to be composed of all the proportions of every adjacent couplet of terms, from the first to the last.

1 *Proposition.* The mean proportional between two terms, divides the proportion of those terms into two equal proportions.

Axiom 2. Of any number of quantities regularly increasing, the means divide the proportion of the extremes into one proportion more than the number of the means.

Postulate 2. That the proportion between any two terms is divisible into any number of parts, until those parts become less than any proposed quantity.

An example of this section is then inserted in a small table, in dividing the proportion which is between 10 and 7 into 1073741824 equal parts, by as many mean proportionals wanting one, namely, by taking the mean proportional between 10 and 7, then the mean between 10 and this mean, and the mean between 10 and the last, and so on for 30 means, or 30 extractions of the square root, the last or 30th of which roots is 99999999966782056900; and the 30 power of 2, which is 1073741824, shows into how many parts the proportion between 10 and 7, or between 1000 &c, and 700 &c, is divided by 1073741824 means, each of which parts is equal to the proportion between 1000 &c, and the 30th mean 999&c, that is, the proportion between 1000&c, and 999&c, is the 1073741824th part of the proportion between 10 and 7. Then by assuming the small difference 00000000033217943100, for the measure of the very small element of the proportion of 10 to 7, or for the measure of the proportion of 1000&c, to 999&c, or for the logarithm of this last term, and multiplying it by 1073741824, the number of parts, the product gives 35667.49481.37222.14400, for the logarithm of the less term 7 or 700 &c.

Postulate 3. That the extremely small quantity or element of a proportion, may be measured or denoted by any quantity whatever; as for instance, by the difference of the terms of that element.

2 *Proposition.* Of three continued proportionals, the difference of the two first has to the difference of the two latter, the same proportion which the first term has to the 2d, or the 2d to the 3d.

3 *Prop.* Of any continued proportionals, the greatest terms have the greatest difference, and the least terms the least.

4 *Prop.* In any continued proportionals, if the difference of the greatest terms be made the measure of the proportion between *them*, the difference of any other couplet will be less than the true measure of *their* proportion.

5 *Prop.* In continued proportionals, if the difference of the greatest terms be made the measure of their proportion, then the measure of the proportion of the greatest to any other term will be greater than *their* difference.

6 *Prop.* In continued proportionals, if the difference of the greatest term and any one of the less, taken not immediately

next to it, be made the measure of their proportion, then the proportion which is between the greatest and any other term greater than the one before taken, will be less than the difference of those terms; but the proportion which is between the greatest term, and any one less than that first taken, will be greater than their difference.

7 *Prop.* Of any quantities placed according to the order of their magnitudes, if any two successive proportions be equal, the three successive terms which constitute them, will be continued proportionals.

8 *Prop.* Of any quantities placed in the order of their magnitudes, if the intermediates lying between any two terms be not among the mean proportionals which can be interposed between the said two terms, then such intermediates do not divide the proportion of those two terms into commensurable proportions.

Besides the demonstrations, as usual, several definitions are here given; as of commensurable proportions, &c.

9 *Prop.* When two expressible lengths are not to one another as two figurate numbers of the same species, such as two squares, or two cubes, there cannot fall between them other expressible lengths, which shall be mean proportionals, and as many in number as that species requires, namely, one in the squares, two in the cubes, three in the biquadrats, &c.

10 *Prop.* Of any expressible quantities, following in the order of their magnitudes, if the two extremes be not in the proportion of two square numbers, or two cubes, or two other powers of the same kind, none of the intermediates divide the proportion into commensurables.

11 *Prop.* All the proportions, taken in order, which are between expressible terms that are in arithmetical proportion, are incommensurable to one another. As between 8, 13, 18.

12 *Prop.* Of any quantities placed in the order of their magnitude, if the difference of the greatest terms be made the measure of their proportion, then the difference between any two others will be less than the measure of *their* propor-

tion; and if the difference of the two least terms be made the measure of their proportion, then the differences of the rest will be greater than the measure of the proportion between *their* terms.

Corol. If the measure of the proportion between the greatest exceed their difference, then the proportion of this measure to the said difference, will be less than that of a following measure to the difference of its terms. Because proportionals have the same ratio.

13 *Prop.* If three quantities follow one another in the order of magnitude, the proportion of the two least will be contained in the proportion of the extremes, a less number of times than the difference of the two least is contained in the difference of the extremes: And, on the contrary, the proportion of the two greatest will be contained in the proportion of the extremes, oftener than the difference of the former is contained in that of the latter.

Corol. Hence, if the difference of the two greater be equal to the difference of the two less terms, the proportion between the two greater will be less than the proportion between the two less.

14 *Prop.* Of three equidifferent quantities, taken in order, the proportion between the extremes is more than double the proportion between the two greater terms.

Corol. Hence it follows, that half the proportion of the extremes is greater than the proportion of the two greatest terms, but less than the proportion of the two least.

15 *Prop.* If two quantities constitute a proportion, and each quantity be lessened by half the greater, the remainders will constitute a proportion greater than double the former.

16 *Prop.* The aliquot parts of incommensurable proportions are incommensurable to each other.

17 *Prop.* If one thousand numbers follow one another in the natural order, beginning at 1000, and differing all by unity, viz. 1000, 999, 998, 997, &c: and the proportion between the two greatest 1000, 999, by continual bisection, be cut into parts that are smaller than the excess of the propor-

tion between the next two 999, 998, over the said proportion between the two greatest 1000, 999; and then for the measure of that small element of the proportion between 1000 and 999, there be taken the difference of 1000 and that mean proportional which is the other term of the element. Again, if the proportion between 1000 and 998 be likewise cut into double the number of parts which the former proportion, between 1000 and 999, was cut into; and then for the measure of the small element in this division, be taken the difference of its terms, of which the greater is 1000. And, in the same manner, if the proportion of 1000 to the following numbers, as 997, &c, by continual bisection, be cut into particles of such magnitude, as may be between $\frac{1}{2}$ and $\frac{3}{4}$ of the element arising from the section of the first proportion between 1000 and 999, the measure of each element will be given from the difference of its terms. Then, this being done, the measure of any one of the 1000 proportions will be composed of as many measures of its element, as there are of those elements in the said divided proportion. And all these measures, for all the proportions, will be sufficiently exact for the nicest calculations.

All these sections and measures of proportions are performed in the manner of that described at postulate 2, and the operation is abundantly explained by numerical calculations.

18 *Prop.* The proportion of any number, to the first term 1000, being known; there will also be known the proportion of the rest of the numbers in the same continued proportion, to the said first term.

So, from the known proportion between 1000 and 900, there is also known the prop. of 1000 to 810, and to 729;

And from 1000 to 800, also 1000 to 640, and to 512;
And from 1000 to 700, also 1000 to 490, and to 343;
And from 1000 to 600, also 1000 to 360, and to 216;
And from 1000 to 500, also 1000 to 250, and to 125.

Corol. Hence arises the precept for squaring, cubing, &c; as also for extracting the square root, cube root, &c. For it will be, as the greatest number of the chiliad, as a denominator, is to the number proposed as a numerator, so is this fraction to the square of it, and so is this square to the cube of it.

19 *Prop.* The proportion of a number to the first, or 1000, being known; if there be two other numbers in the same proportion to each other, then the proportion of one of these to 1000 being known, there will also be known the proportion of the other to the same 1000.

Corol. 1. Hence, from the 15 proportions mentioned in prop. 18, will be known 120 others below 1000, to the same 1000.

For so many are the proportions, equal to some one or other of the said 15, that are among the other integer numbers which are less than 1000.

Corol. 2. Hence arises the method of treating the Rule-of-Three, when 1000 is one of the given terms.

For this is effected by adding to, or subtracting from, each other, the measures of the two proportions of 1000 to each of the other two given numbers, according as 1000 is, or is not, the first term in the Rule-of-Three.

20 *Prop.* When four numbers are proportional, the first to the second as the third to the fourth, and the proportions of 1000 to each of the three former are known, there will also be known the proportion of 1000 to the fourth number.

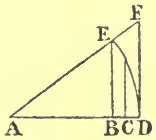
Corol. 1. By this means other chiliads are added to the former.

Corol. 2. Hence arises the method of performing the Rule-of-Three, when 1000 is not one of the terms. Namely from the sum of the measures of the proportions of 1000 to the second and third, take that of 1000 to the first, and the remainder is the measure of the proportion of 1000 to the fourth term.

Definition. The measure of the proportion between 1000 and any less number, as before described, and expressed by a number, is set opposite to that less number in the chiliad, and is called its LOGARITHM, that is, the number (*αριθμος*) indicating the proportion (λογον) which 1000 bears to that number, to which the logarithm is annexed.

21 *Prop.* If the first or greatest number be made the radius

of a circle, or *sinus totus;* every less number, considered as the cosine of some arc, has a logarithm greater than the versed sine of that arc, but less than the difference between the radius and secant of the arc; except only in the term next after the radius, or greatest term, the logarithm of which, by the hypothesis, is made equal to the versed sine.



That is, if CD be made the logarithm of AC, or the measure of the proportion of AC to AD; then the measure of the proportion of AB to AD, that is the logarithm of AB, will be greater than BD, but less than EF. And this is the same as Napier's first rule in page 345.

22 *Prop.* The same things being supposed; the sum of the versed sine and excess of the secant over the radius, is greater than double the logarithm of the cosine of an arc.

Corol. The log. cosine is less than the arithmetical mean between the versed sine and the excess of the secant.

Precept 1. Any sine being found in the canon of sines, and its defect below radius to the excess of the secant above radius, then shall the logarithm of the sine be less than half that sum, but greater than the said defect or coversed sine.

Let there be the sine	99970.1490	of an arc:
Its defect below radius is	29.8510	the covers. and less than the log. sine:
Add the excess of the secant	29.8599	
Sum	59.7109	
its half or	29.8555	greater than the logarithm.
Therefore the log. is between	29.8510	
and	29.8555	

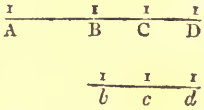
Precept 2. The logarithm of the sine being found, there will also be found nearly the logarithm of the round or integer number, which is next less than the sine with a fraction, by adding that fractional excess to the logarithm of the said sine.

Thus, the logarithm of the sine 99970.149 is found to be about 29.854; if now the logarithm of the round number 99970.000 be required, add 149, the fractional part of the sine, to its logarithm, observing the point, thus,

29.854
149

the sum 30.003 is the log. of the round number 99970.000 nearly.

23 *Prop.* Of three equidifferent quantities, the measure of the proportion between the two greater terms, with the measure of the proportion between the two less terms, will constitute a proportion, which will be greater than the proportion of the two greater terms, but less than the proportion of the two least.



Thus if AB, AC, AD be three quantities having the equal differences BC, CD; and if the measure of the proportion of AD, AC, *bc*, *cd*, and that of AC, AB be *bc*; then the proportion of *cd* to *cb* will be greater than the proportion of AC to AD, but less than the proportion of AB to AC.

24 *Prop.* The said proportion between the two measures is less than half the proportion between the extreme terms. That is, the proportion between *bc*, *cd*, is less than half the proportion between AB, AD.

Corol. Since therefore the arithmetical mean divides the proportion into unequal parts, of which the one is greater, and the other less, than half the whole; if it be inquired what proportion is between these proportions, the answer is, that it is a little less than the said half.

An Example of finding nearly the limits, greater and less, to the measure of any proposed proportion.

It being known that the measure of the proportion between 1000 and 900 is 10536.05, required the measure of the proportion 900 to 800, where the terms 1000, 900, 800, have equal differences. Therefore as 9 to 10, so 10536.05 to 11706.72, which is less than 11773.30 the measure of the proportion 9 to 8. Again, as the mean proportional between 8 and 10 (which is 8.9442719) is to 10, so 10536.05 to 11779.65, which is greater than the measure of the proportion between 9 and 8.

Axiom. Every number denotes an expressible quantity.

25 *Prop.* If the 1000 numbers, differing by 1, follow one another in the natural order; and there be taken any two adjacent numbers, as the terms of some proportion; the measure of this proportion will be to the measure of the proportion between the two greatest terms of the chiliad, in a proportion greater than that which the greatest term 1000 bears to the

greater of the two terms first taken, but less than the proportion of 1000 to the less of the said two selected terms.

So, of the 1000 numbers, taking any two successive terms, as 501 and 500, the logarithm of the former being 69114.92, and of the latter 69314.72, the difference of which is 199.80. Therefore, by the definition, the measure of the proportion between 501 and 500 is 199.80. In like manner, because the logarithm of the greatest term 1000 is 0, and of the next 999 is 100.05, the difference of these logarithms, and the measure of the proportion between 1000 and 999, is 100.05. Couple now the greatest term 1000 with each of the selected terms 501 and 500; couple also the measure 199.80 with the measure 100.05; so shall the proportion between 199.80 and 100.05, be greater than the proportion between 1000 and 501, but less than the proportion between 1000 and 500.

Corol. 1. Any number below the first 1000 being proposed, as also its logarithm, the differences of any logarithms antecedent to that proposed, towards the beginning of the chiliad, are to the first logarithm (viz. that which is assigned to 999) in a greater proportion than 1000 to the number proposed; but of those which follow towards the last logarithm, they are to the same in a less proportion.

Corol. 2. By this means, the places of the chiliad may easily be filled up, which have not yet had logarithms adapted to them by the former propositions.

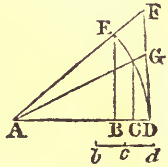
26 *Prop.* The difference of two logarithms, adapted to two adjacent numbers, is to the difference of these numbers, in a proportion greater than 1000 bears to the greater of those numbers, but less than that of 1000 to the less of the two numbers.

This 26th prop. is the same as Napier's second rule, at page 345.

27 *Prop.* Having given two adjacent numbers, of the 1000 natural numbers, with their logarithmic indices, or the measures of the proportions which those absolute or round numbers constitute with 1000, the greatest; the increments, or differences, of these logarithms, will be to the logarithm of the small element of the proportions, as the secants of the arcs whose cosines are the two absolute numbers, is to the greatest number, or the radius of the circle; so that, however, of the said two secants, the less will have to the radius a less proportion than the proposed difference has to the first of all,

but the greater will have a greater proportion, and so also will the mean proportional between the said secants have a greater proportion.

Thus if BC, CD be equal, also *bd* the logarithm of AB, and *cd* the logarithm of AC; then the proportion of *bc* to *cd* will be greater than the proportion of AG to AD, but less than that of AF to AD, and also less than that of the mean proportional between AF and AG to AD.



Corol. 1. The same obtains also when the two terms differ, not only by the unit of the small element, but by another unit, which may be ten fold, a hundred fold, or a thousand fold of that.

Corol. 2. Hence the differences will be obtained sufficiently exact, especially when the absolute numbers are pretty large, by taking the arithmetical mean between two small secants, or (if you will be at the labour) by taking the geometrical mean between two larger secants, and then by continually adding the differences, the logarithms will be produced.

Corol. 3. *Precept.* Divide the radius by each term of the assigned proportion, and the arithmetical mean (or still nearer the geometrical mean) between the quotients, will be the required increment; which being added to the logarithm of the greater term, will give the logarithm of the less term.

Example.

Let there be given the logarithm of 700, viz. 35667.4948, to find the log. to 699,

Here radius divided by 700 gives 1428571 &c.

and divided by 699 gives 1430672 &c.

the arithmetical mean is 142.962

which added to 35667.4948

gives the logarithm to 699   35810.4568

Corol. 4. *Precept* for the logarithms of sines.

The increment between the logarithms of two sines, is thus found: find the geometrical mean between the cosecants, and divide it by the difference of the sines, the quotient will be the difference of the logarithms.

Example.

0° 1′	sine 2909	cosec. 343774682
0 2	sine 5818	cosec. 171887319
	dif. 2909,	geom. mean 2428 nearly.

The quotient 80000 exceeds the required increment of the logarithms, because the secants are here so large.

Appendix. Nearly in the same manner it may be shown, that the second differences are in the duplicate proportion of the first, and the third in the duplicate of the second. Thus, for instance, in the beginning of the logarithms, the first difference is 100.00000, viz. equal to the difference of the numbers 100000.00000 and 99900.00000; the second, or difference of the differences, 10000; the third 20. Again, after arriving at the number of 50000.00000, the logarithms have for a difference 200.00000, which is to the first difference, as the number 100000.00000 to 50000.00000; but the second difference is 40000, in which 10000 is contained 4 times; and the third 328, in which 20 is contained 16 times. But since in treating of new matters we labour under the want of proper words, therefore lest we should become too obscure, the demonstration is omitted untried.

28 *Prop.* No number expresses exactly the measure of the proportion, between two of the 1000 numbers, constituted by the foregoing method.

29 *Prop.* If the measures of all proportions be expressed by numbers or logarithms; all proportions will not have assigned to them their due portion of measure, to the utmost accuracy.

30 *Prop.* If to the number 1000, the greatest of the chiliad, be referred others that are greater than it, and the logarithm of 1000 be made 0, the logarithms belonging to those greater numbers will be negative.

This concludes the first or scientific part of the work, the principles of which Kepler applies, in the second part, to the actual construction of the first 1000 logarithms, which construction is pretty minutely described. This part is intitled

" A very compendious method of constructing the Chiliad of Logarithms;" and it is not improperly so called, the method being very concise and easy. The fundamental principles are briefly these: That at the beginning of the logarithms, their increments or differences are equal to those of the natural numbers: that the natural numbers may be considered as the decreasing cosines of increasing arcs: and that the secants of those arcs at the beginning have the same differences as the cosines, and therefore the same differences as the logarithms. Then, since the secants are the reciprocals of the cosines, by these principles and the third corollary to the 27th proposition, he establishes the following method of constituting the 100 first or smallest logarithms to the 100 largest numbers, 1000, 999, 998, 997, &c, to 900. viz. Divide the radius 1000, increased with seven ciphers, by each of these numbers separately, disposing the quotients in a table, and they will be the secants of those arcs which have the divisors for their cosines; continuing the division to the 8th figure, as it is in that place only that the arithmetical and geometrical means differ. Then by adding successively the arithmetical means between every two successive secants, the sums will be the series of logarithms. Or, by adding continually every two secants, the successive sums will be the series of the double logarithms.

Besides the 100 logarithms, thus constructed, the author constitutes two others by continual bisection, or extractions of the square root, after the manner described in the second postulate. And first he finds the logarithm which measures the proportion between 100000.00 and 97656.25, which latter term is the third proportional to 1024 and 1000, each with two ciphers; and this is effected by means of twenty-four continual extractions of the square root, determining the greatest term of each of twenty-four classes of mean proportionals; then the difference between the greatest of these means and the first or whole number 1000, with ciphers, being as often doubled, there arises 2371.6526 for the logarithm sought, which made negative is the logarithm of 1024. Secondly, the like process is repeated for the proportion between the numbers 1000 and 500, from which arises 69314.7193 for the logarithm of 500; which he also calls the logarithm of duplication, being the measure of the proportion of 2 to 1.

Then from the foregoing he derives all the other logarithms in the chiliad, beginning with those of the prime numbers 1, 2, 3, 5, 7, &c, in the first 100. And first, since 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, are all in the continued proportion of 1000 to 500, therefore the proportion of 1024 to 1 is decuple of the proportion of 1000 to 500, and consequently the logarithm of 1 would be decuple of the logarithm of 500, if 0 were taken as the logarithm of 1024; but since the logarithm of 1024 is applied negatively, the logarithm of 1 must be diminished by as much: diminishing therefore 10 times the log. of 500, which is 693147.1928, by 2371.6526, the remainder 690775.5422 is the logarithm of 1, or of 100.00, which is set down in the table.

Nos.	Logarithms.
100	230258.5141
10	460517.0282
1	690775.5422
.1	921034.0563
.01	1151292.5703
.001	1381551.0844
·0001	1611809.5985

And because 1, 10, 100, 1000, are continued proportionals, therefore the proportion of 1000 to 1 is triple of the proportion of 1000 to 100, and consequently $\frac{1}{3}$ of the logarithm of 1 is to be set for the logarithm of 100, viz. 230258.5141, and this is also the logarithm of decuplication, or of the proportion of 10 to 1. And hence, multiplying this logarithm of 100 successively by 2, 3, 4, 5, 6, and 7, there arise the logarithms to the numbers in the decuple proportion, as in the margin.

Log. of 1	690775.5422
of 2 to 1	69314.7193
log. of 2	621460.8229
log. of 10	460517.0281
of 5 to 1	160943.7948
log. of 5	529831.7474

Also if the logarithm of duplication, or of the proportion of 2 to 1, be taken from the logarithm of 1, there will remain the logarithm of 2; and from the logarithm of 2 taking the logarithm of 10, there remains the logarithm of the proportion of 5 to 1; which taken from the logarithm of 1, there remains the logarithm of 5. See the margin.

For the logarithms of other prime numbers he has recourse to those of some of the first or greatest century of numbers, before found, viz. of 999, 998, 997, &c. And first, taking 960, whose logarithm is 4082.2001 ; then by adding to this logarithm the logarithm of duplication, there will arise the several logarithms of all these numbers, which are in duplicate proportion continued from 960, namely 480, 240, 120, 60, 30, 15. Hence the logarithm of 30 taken from the logarithm of 10, leaves the logarithm of the proportion of 3 to 1 ; which taken from the logarithm of 1, leaves the logarithm of 3, viz. 580914.3106. And the double of this diminished by the logarithm of 1, gives 471053.0790 for the logarithm of 9.

Next, from the logarithm of 990, or 9 × 10 × 11, which is 1005.0331, he finds the logarithm of 11, namely, subtracting the sum of the logarithms of 9 and 10 from the sum of the logarithm of 990 and double the logarithm of 1, there remains 450986.0106 the logarithm of 11.

Again, from the logarithm of 980, or 2 × 10 × 7 × 7, which is 2020.2711, he finds 496184.5228 for the logarithm of 7.

And from 5129.3303 the logarithm of 950, or 5 × 10 × 19, he finds 396331.6392 for the logarithm of 19.

In like manner the logarithm

to 998 or 4 × 13 × 19, gives the logarithm of 13 ;
to 969 or 3 × 17 × 19, gives the logarithm of 17 ;
to 986 or 2 × 17 × 29, gives the logarithm of 29 ;
to 966 or 6 × 7 × 23, gives the logarithm of 23 ;
to 930 or 3 × 10 × 31, gives the logarithm of 31.

And so on for all the primes below 100, and for many of the primes in the other centuries up to 900. After which, he directs to find the logarithms of all numbers composed of these, by the proper addition and subtraction of their logarithms, namely, in finding the logarithm of the product of two numbers, from the sum of the logarithms of the two factors take the logarithm of 1, the remainder is the logarithm of the product. In this way he shows that the logarithms of all numbers under 500 may be derived, except those of the following 36 numbers, namely, 127, 149, 167, 173, 179, 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449. Also, besides the composite numbers between 500 and 900, made up of the products of some numbers whose logarithms have been before determined, there will be 59 primes not composed of them; which, with the 36 above mentioned, make 95 numbers in all not composed of the products of any before them, and the logarithms of which he directs to be derived in this manner; namely, by considering the differences of the logarithms of the numbers interspersed among them; then by that method by which were constituted the differences of the logarithms of the smallest 100 numbers in a continued series, we are to proceed here in the discontinued series, that is, by prop. 27, corol. 3, and especially by the appendix to it, if it be rightly used, whence those differences will be very easily supplied.

This closes the second part, or the actual construction of the logarithms; after which follows the table itself, which has been before described, pa. 323. Before dismissing Kepler's work however, it may not be improper in this place to take notice of an erroneous property laid down by him in the appendix to the 27th prop. just now referred to; both because it is an error in principle, tending to vitiate the practice, and because it serves to show that Kepler was not acquainted with the true nature of the orders of differences of the logarithms, notwithstanding what he says above with respect to the construction of them by means of their several orders of differences, and that consequently he has no legal claim to any share in the discovery of the differential method, known at that time to Briggs, and it would seem to him alone, it being published in his logarithms in the same year, 1624, as Kepler's book, together with the true nature of the logarithmic orders of differences, as we shall presently see in the following account of his works. Now this error of Kepler's, here alluded to, is in that expression where he says the third differences are in the *duplicate* ratio of the second differences, like as the second differences are in the duplicate ratio of the first; or, in other words, that the third differences are as the *squares* of the second differences, as well as the second differences as the squares of the first; or that the third differences are as the *fourth powers* of the first differences: Whereas in truth the third differences are only as the *cubes* of the first differences. Kepler seems to have been led into this error by a mistake in his numbers, viz. when he says in that appendix, that "*the third difference is* 328, *in which* 20 *is contained* 16 *times*;" for when the numbers are accurately computed, the third difference comes out only 161, in which therefore 20 is contained only 8 times, which is the cube of 2, the number of times the one first difference contains the other. It would hence seem that Kepler had hastily drawn the above erroneous principle from this one numerical example, or little more, false as it is: for had he made the trial in many instances, though erroneously computed, they could not easily have been so uniformly so, as to afford the same false conclusion in all cases. And therefore from hence, and what he says at the conclusion of that appendix, it may be inferred, that he either never attempted the demonstration of the property in question, or else that finding himself embarrassed with it, and unable to accomplish it, he therefore dispatched it in the ambiguous manner in which it appears.

But it may easily be shown, not only that the third differences of the logarithms at different places, are as the cubes of the first differences; but, in general, that the numbers in any one and the same order of differences, at different places, are as that power of the numbers in the first differences, whose index is the same as that of the order: or that the second, third, fourth, &c differences, are as the second, third, fourth, &c powers of the first differences. For the several orders of differences, when the absolute numbers differ by indefinitely small parts, are as the several orders of fluxions of the logarithms; but if x be any number, then $\frac{m\dot{x}}{x}$ is the fluxion of the logarithm of x, to the modulus m, and the second fluxion, or the fluxion of this fluxion, is $-\frac{m\dot{x}^2}{x^2}$, since $\dot{x}$ is constant; and the third, fourth, &c fluxions, are $\frac{2m\dot{x}^3}{x^3}$, $-\frac{2.3m\dot{x}^4}{x^4}$, &c; that is, the first, second, third, fourth, fifth, sixth, &c orders of fluxions, are equal to the modulus m multiplied into each of these terms,

$$\frac{\dot{x}}{x},\ -\frac{1\dot{x}^2}{x^2},\ \frac{1.2\dot{x}^3}{x^3},\ -\frac{1.2.3\dot{x}^4}{x^4},\ \frac{1.2.3.4\dot{x}^5}{x^5},\ -\frac{1.2.3.4.5\dot{x}^6}{x^6},\ \&c;$$

where it is evident, that the fluxion of any order is as that power of the first fluxion, whose index is the same as the number of the order. And these quantities would actually be the several terms of the differences themselves, if the differences of the numbers were indefinitely small. But they vary the more from them, as the differences of the absolute numbers differ from $\dot{x}$, or as the said constant numerical difference 1 approaches towards the value of x the number itself. However, on the whole, the several orders vary proportionably, so as still sensibly to preserve the same analogy, namely, that two nth differences are in proportion as the nth powers of their respective first differences.

Of Briggs's Construction of his Logarithms.

Nearly according to the methods described in p. 349, 350, Mr. Briggs constructed the logarithms of the prime numbers, as appears from his relation of this business in the "Arithmetica Logarithmica," printed in 1624, where he details, in an ample manner, the whole construction and use of his logarithms. The work is divided into 32 chapters or sections. In the first of these, logarithms in a general sense are defined, and some properties of them illustrated. In the second chapter he remarks, that it is most convenient to make 0 the logarithm of 1; and on that supposition he exemplifies these following properties, namely, that the logarithms of all numbers are either the indices of powers, or proportional to them; that the sum of the logarithms of two or more factors, is the logarithm of their product; and that the difference of the logarithms of two numbers, is the logarithm of their quotient. In the third section he states the other assumption, which is necessary to limit his system of logarithms, namely, making 1 the logarithm of 10, as that which produces the most convenient form of logarithms: He hence also takes occasion to show that the powers of 10, namely 100, 1000, &c, are the only numbers which can have rational logarithms. The fourth section treats of the characteristic; by which name he distinguishes the integral, or first part, of a logarithm towards the left hand, which expresses one less than the number of integer places or figures, in the number belonging to that logarithm, or how far the first figure of this number is removed from the place of units; namely, that 0 is the characteristic of the logarithms of all numbers from 1 to 10; and 1 the characteristic of all those from 10 to 100; and 2 that of those from 100 to 1000; and so on.

He begins the fifth chapter with remarking, that his logarithms may chiefly be constructed by the two methods which were mentioned by Napier, as above related, and for the sake of which, he here premises several *lemmata*, concerning the powers of numbers and their indices, and how many places of figures are in the products of numbers, observing that the product of two numbers will consist of as many figures as there are in both factors, unless perhaps the product of the first figures in each factor be expressed by one figure only, which often happens, and then commonly there will be one figure in the product less than in the two factors; as also that, of any two of the terms, in a series of geometricals, the results will be equal by raising each term to the power denoted by the index of the other; or any number raised to the power denoted by the logarithm of the other, will be equal to this latter number raised to the power denoted by the logarithm of the former; and consequently if the one number be 10, whose logarithm is 1 with any number of ciphers, then any number raised to the power whose index is 1000 &c, or the logarithm of 10, will be equal to 10 raised to the power whose index is the logarithm of that number; that is, the logarithm of any number in this scale, where 1 is the logarithm of 10, is the index of that power of 10 which is equal to the given number. But the index of any integral power of 10, is one less than the number of places in that power; consequently the logarithm of any other number, which is no integral power of 10, is not quite one less than the number of places in that power of the given number whose index is 1000 &c, or the logarithm of 10.

Find therefore the 10th, or 100th, or 1000th &c, power of any number, as suppose 2, with the number of figures in such power; then shall that number of figures always exceed the logarithm of 2, though the excess will be constantly less than 1.

An example of this process is here given in the margin; where the 1st column contains the several powers of 2, the 2d their corresponding indices, and the 3d contains the number of places in the powers in the first column; and of these numbers in the third column, such as are on the lines of those indices that consist of 1 with ciphers, are continual approximations to the logarithm of 2, being always too great by less than 1 in the last figure, that logarithm being 30102999566398 &c.

And here, since the exact powers of 2 are not required, but only the number of figures they consist of, as shown by the third column, only a few of the first figures of the powers in the first column are retained, those being sufficient to determine the num-

Powers of 2	Indices.	No. of Places or logs.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10	4 log. of 2
10486	20	7 log. of 4
10995	40	13 log. of 16
12089	80	25 log. of 256
12676	100	31 log. of 2
16069	200	61 log. of 4
25823	400	121 log. 16
66680	800	241 log. 256
10715	1000	302 log. 2
11481	2000	603 log. 4
13182	4000	1205 log. 16
17377	8000	2409 log. 256
19950	10000	3011 log. 2
39803	20000	6021 log. 4
15843	40000	12042 log. 16
25099	80000	24083 log. 256
99900	100000	30103 log. 2
99801	200000	60206 log. 4
99601	400000	120412 &c.
99204	800000	240824
99006	1000000	301030
98023	2000000	602060
96085	4000000	1204120
92323	8000000	2408240
90498	10000000	3010300
81899	20000000	6020600
67075	40000000	12041200
44990	80000000	24082400
36846	100000000	30103000
13577	200000000	60206000
18433	400000000	120411999
33977	800000000	240823997
46129	1000000000	301029996

ber of places in them; and the multiplications in raising these powers are performed in a contracted way, so as to have the fifth or last figure in them true to the nearest unit. Indeed these multiplications might be performed in the same manner, retaining only the first three figures, and those to the nearest unit in the third place; which would make this a very easy way indeed of finding the logarithms of a few prime numbers.

It may also be remarked, that those several powers, whose indices are 1 with ciphers, are raised by thrice squaring from the former powers, and multiplying the first by the third of these squares; making also the corresponding doublings and additions of their indices: thus, the square of 2 is 4, and the square of 4 is 16, the square of 16 is 256, and 256 multiplied by 4 is 1024; in like manner, the double of 1 is 2, the double of 2 is 4, the double of 4 is 8, and 8 added to 2 makes 10. And the same for all the following powers and indices. The numbers in the third column, which show how many places are in the corresponding powers in the first column, are produced in the very same way as those in the second column, namely, by three duplications and one addition; only observing to subtract 1 when the product of the first figures are expressed by one figure; or when the first figures exceed those of the number or power next above them. It may further be observed, that, like as the first number in each quaternion, or space of four lines or numbers, in the third column, approximates to the logarithm of 2, the first number in the first quaternion of the first column; so the second, third, and fourth terms of each quaternion in the third column, approximate to the logarithm of 4, 16, and 256, the second, third, and fourth numbers in the first quaternion in the first column. And further, by cutting off one, two, three, &c, figures, as the index or integral part, from the said logarithms of 2, 4, 16, and 256, the first, second, third, and fourth numbers in the first quaternion of the first column, the remaining figures will be the decimal part of the logarithms of the corresponding first, second, third, and fourth numbers in the following second, third, fourth, &c,

quaternions: the reason of which is, that any number of any quaternion in the first column, is the tenth power of the corresponding term in the next preceding quaternion. So that the third column contains the logarithms of all the numbers in the first column: a property which, if Dr. Newton had been aware of, he could not easily have committed such gross mistakes as are found in a table of his, similar to that above given, in which most of the numbers in the latter quaternions are totally erroneous; and his confused and imperfect account of this method would induce one to believe that he did not well understand it.

In the sixth chapter our illustrious author begins to treat of the other general method of finding the logarithms of prime numbers, which he thinks is an easier way than the former, at least when the logarithm is required to a great many places of figures. This method consists in taking a great number of continued geometrical means between 1 and the given number whose logarithm is required; that is, first extracting the square root of the given number, then the root of the first root, the root of the second root, the root of the third root, and so on till the last root shall exceed 1 by a very small decimal, greater or less according to the intended number of places to be in the logarithm sought: then finding the logarithm of this small number, by methods described below, he doubles it as often as he made extractions of the square root, or, which is the same thing, he multiplies it by such power of 2 as is denoted by the said number of extractions, and the result is the required logarithm of the given number; as is evident from the nature of logarithms. The rule to know how far to continue this extraction of roots is, that the number of decimal places in the last root, be double the number of true places required to be found in the logarithm, and that the first half of them be ciphers; the integer being 1: the reason of which is, that then the significant figures in the decimal, after the ciphers, are directly proportional to those in the corresponding logarithms; such figures in the natural number being the half of those in the next preceding number, like as the logarithm of the last number is the half of the preceding logarithm. Therefore, any one such small number, with its logarithm, being once found, by the continual extractions of square roots out of a given number, as 10, and corresponding bisections of its given logarithm 1; the logarithm for any other such small number, derived by like continual extractions from another given number, whose logarithm is sought, will be found by one single proportion: which logarithm is then to be doubled according to the number of extractions, or multiplied at once by the like power of 2, for the logarithm of the number proposed. To find the first small number and its logarithm, our author begins with the number 10 and its logarithm 1, and extracts continually the root of the last number, and bisects its logarithm, as here registered in the margin, but to far more places of figures, till he arrives at the 53d and 54th roots, with their annexed logarithms, as here below:

	10, given n°.	1, its log.
1	3·162277 &c	0·5
2	1·778279	0·25
3	1·333521	0·125
4	1·154781	0·0625
5	1·074607	0·03125
	&c.	&c.

	Numbers.	Logarithms.
35	1·00000,00000,00000,25563,82986,40064,70	0·00000,00000,00000,11102,23024,62515,65404
54	1·00000,00000,00000,12781,91493,20032,35	0·00000,00000,00000,05551,11512,31257,82702

where the decimals in the natural numbers are to each other in the ratio of the logarithms, namely in the ratio of 2 to 1: and therefore any other such small number being found, by continual extraction or otherwise, it will then be as 12781&c, is to 5551&c, so is that other small decimal, to the corresponding significant figures of its logarithm. But as every repetition of this proportion requires both a very long multiplication and division, he reduces this constant ratio to another equivalent ratio whose antecedent is 1, by which all the divisions are saved: thus,

as 12781&c : 5551&c : : 1000&c : 43429448190325180 4,
that is, the logarithm of 1·00000,00000,00000,1
is 0·00000,00000,00000,04342,94481,90325,1804;

and therefore this last number being multiplied by any such small decimal, found as above by continual extraction, the product will be the corresponding logarithm of such last root.

But as the extraction of so many roots is a very troublesome operation, our author devises some ingenious contrivances to abridge that labour. And first, in the 7th chapter, by the following device, to have fewer and easier extractions to perform: namely, raising the powers from any given prime number, whose logarithm is sought, till a power of it be found such that its first figure on the left hand is 1, and the next to it either one or more ciphers; then, having divided this power by 1 with as many ciphers as it has figures after the first, or supposing all after the first to be decimals, the continual roots from this power are extracted till the decimal become sufficiently small, as when the first fifteen places are ciphers; and then by multiplying the decimal by 43429&c, he has the logarithm of this last root; which logarithm multiplied by the like power of the number 2, gives the logarithm of the first number, from which the extraction was begun: to this logarithm prefixing a 1, or 2, or 3, &c, according as this number was found by dividing the power of the given prime number by 10, or 100, or 1000, &c; and lastly, dividing the result by the index of that power, the quotient will be the required logarithm of the given prime number. Thus, to find the logarithm of 2: it is first raised to the 10th power, as in the margin, before the first figures come to be 10; then, dividing by 1000, or cutting off for decimals all the figures after the first or 1, the root is continually extracted out of the quotient 1,024, till the 47th extraction, which gives 1.00000,00000,00000,16851,60570,53949,77; the decimal part of which multi. by 43429 &c, gives 0.00000,00000,00000,07318,55936,90623,9368 for its logarithm: and this being continually doubled for 47 times, gives the logarithms of all the roots up to the first number: or being at once

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10

multiplied by the 47th power of 2, viz. 140737488355328, which is raised as in the margin, it gives 0·01029,99566,39811,95265,27744 for the logarithm of the number 1.024, true to 17 or 18 decimals: to this prefix 3, so shall 3.0102 &c be the logarithm of 1024: and lastly, because 2 is the tenth root of 1024, divide by 10, so shall 0.30102,99956,63981,1952 be the logarithm required to the given number 2.

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
1048576	20
1073741824	30
1099511627776	40
140737488355328	47

The logarithms of 1, 2, and 10 being now known; it is remarked that the logarithm of 5 becomes known; for since 10 ÷ 2 is = 5, therefore log. 10 − log. 2 = log. 5, which is 0.69897,00043,36018,8058; and that from the multiplications and divisions of these three 2, 5, 10, with the corresponding additions and subtractions of their logarithms, a multitude of other numbers and their logarithms are produced; so, from the powers of 2, are obtained 4, 8, 16, 32, 64, &c; from the powers of 5, these, 25, 125, 625, 3125, &c; also the powers of 5 by those of 10 give 250, 1250, 6250, &c; and the powers of 2 by those of 10, give 20, 200, 2000, &c; 40, 400, 80, 800, &c; likewise by division are obtained $2\frac{1}{2}$, $1\frac{1}{4}$, $12\frac{1}{2}$, $6\frac{1}{4}$, $1\frac{3}{5}$, $3\frac{1}{5}$, $6\frac{2}{5}$, &c.

Briggs then observes, that the logarithm of 3, the next prime number, will be best derived from that of 6, in this manner: 6 raised to the 9th power becomes 10077696, which divided by 10000000, gives 1.0077696, and the root from this continually extracted till the 46th,
is 1,00000,00000,00000,10998,59345,88155,71866;
the decimal part of which multiplied by 43429&c, gives 0.00000,00000,00000,04776,62844,78608,0304 for its logarithm; and this 46 times doubled, or multiplied by the 46th power of 2, gives 0.00336,12534,52792,69 for the logarithm

of 1.0077696; to which adding 7, the logarithm of the divisor 10000000, and dividing by 9, the index of the power of 6, there results 0.77815,12503,83643,63 for the logarithm of 6; from which subtracting the logarithm of 2, there remains 0.47712,12547,19662,44 for the logarithm of 3.

In the eighth chapter our ingenious author decribes an original and easy method of constructing, by means of differences, the continual mean proportionals which were before found by the extraction of roots. And this, with the other methods of generating logarithms by differences, in this book as well as in his "Trigonometria Britannica," are I believe the first instances that are to be found of making such use of differences, and show that he was the inventor of what may be called the "Differential Method." He seems to have discovered this method in the following manner: having observed that these continual means between 1 and any number proposed, found by the continual extraction of roots, approach always nearer and nearer to the halves of each preceding root, as is visible when they are placed together under each other; and indeed it is found that as many of the significant figures of each decimal part, as there are ciphers between them and the integer 1, agree with the half of those above them; I say, having observed this evident approximation, he subtracted each of these decimal parts, which he called A, or the first differences, from half the next preceding one, and by comparing together the remainders or second differences, called B, he found that the succeeding were always nearly equal to $\frac{1}{4}$ of the next preceding ones; then taking the difference between each second difference and $\frac{1}{4}$ of the preceding one, he found that these third differences, called C, were nearly in the continual ratio of 8 to 1; again taking the difference between each C and $\frac{1}{8}$ of the next preceding, he found that these fourth differences, called D, were nearly in the continual ratio of 16 to 1; and so on, the 5th E, 6th F, &c, differences, being nearly in the continual ratio of 32 to 1, of 64 to 1, &c.

These plain observations being made, they very naturally and clearly suggested to him the notion and method of constructing all the remaining numbers, from the differences of a few of the first, found by extracting the roots in the usual way. This will evidently appear from the annexed specimen of a few of the first numbers in the last example, for finding the logarithm of 6; where, after the 9th number, the rest are supposed to be constructed from the preceding differences of each, as here shown in the 10th and 11th. And it is evident that, in proceeding, the trouble will become always less and less, the differences gradually vanishing, till at last only the first differences remain; and that generally each less difference is shorter than the next greater, by as many

		1,00776,96	
1		1,00387,72833,36962,45663,84655,1	
2		1,00193,67661,36946,61675,87022,9	
3		1,00096,79146,39099,01728,89072,0	
4		1,00048,38402,68846,62985,49253,5	A
5		1,00024,18908,78824,68563,80872,7	A
		24,19201,34423,31492,74626,7	$\frac{1}{2}$A
		292,55598,62928,93754,0	B
6		1,00012,09581,26397,13459,43919,4	A
		12,09454,39412,34281,90436,3	$\frac{1}{2}$A
		73,13015,20822,46516,9	B
		73,13899,65732.23438,5	$\frac{1}{4}$B
		884,44909,76921,5	C
7		1,00006,04672,35055,30268,01600,5	A
		6,04690.63198,56779,71959,7	$\frac{1}{2}$A
		18,28143,25761,76359,2	B
		18,28253,80205,61629,2	$\frac{1}{4}$B
		110,54443,91270,0	C
		110,55613,72115,2	$\frac{1}{8}$C
		1169 80845,2	D
8		1,00003.02331,60565,65775,96479,4	A
		3,02336,17527,65484,00800,2	$\frac{1}{2}$A
		4,57021,99708,04320,8	B
		4,57035,81440,42589,8	$\frac{1}{4}$B
		13,81732,38269,0	C
		13,81805,48908,7	$\frac{1}{8}$C
		73,10639,7	D
		73,11302,8	$\frac{1}{16}$D
		663,1	E
9		1,00001,51164,65999,05672,95048,8	A
		1,51165,80252,82887,98239,7	$\frac{1}{2}$A
		1,14253,77215,03190,9	B
	Hitherto the	1,14255,49927,01080,2	$\frac{1}{4}$B
	smaller differences	1.72711,97889,3	C
	are found by sub-	1,72716,54783,6	$\frac{1}{8}$C
	tracting the larger from	4,56894,3	D
	the parts of the like pre-	4,56915.0	$\frac{1}{16}$D
	ceding ones.	20,7	E
		20,7	$\frac{1}{32}$E
	Here the greater differences	65	$\frac{1}{32}$E
	remain after subtracting	28555,89	$\frac{1}{16}$D
	the smaller from the parts	28555,24	D
	of the difference of	21588,99736,16	$\frac{1}{8}$C
	the next preceding	21588,71180,92	C
	number.	28563,44303,75797,72	$\frac{1}{4}$B
		28563.22715.04616,80	B
		75582,32999.52856,47524,40	$\frac{1}{2}$A
10		1,00000,75582,04436,30121,42907,60	A
		2	$\frac{1}{32}$E
		1784,70	$\frac{1}{16}$D
		1784,68	D
		2698.58897,62	$\frac{1}{8}$C
		2698,57112,94	C
		7140,80678,76154,20	$\frac{1}{4}$B
		7140,77980,19041,26	B
		37791 02218,15060.71453,80	$\frac{1}{2}$A
11		1,00000,37790,95077.37080,52412,54	A

places as there are ciphers at the beginning of the decimal in the number to be generated from the differences.

He then concludes this chapter with an ingenious, but not obvious, method of finding the differences B, C, D, E, &c, belonging to any number, as suppose the 9th, from that number itself, independent of any of the preceding 8th, 7th, 6th, 5th, &c; and it is this: raise the decimal A to the 2d, 3d, 4th, 5th, &c powers; then will the 2d (B), 3d (C), 4th (D), &c differences, be as here below, viz.

$$\begin{array}{llllll}
B = \frac{1}{2}A^2, & & & & & \\
C = \frac{1}{2}A^3 + \frac{1}{8}A^4, & & & & & \\
D = \frac{7}{8}A^4 + & \frac{7}{8}A^5 + & \frac{7}{16}A^6 + & \frac{1}{8}A^7 + & \frac{1}{64}A^8, & \\
E = \ . & 2\frac{5}{8}A^5 + & 7A^6 + & 10\frac{15}{16}A^7 + & 12\frac{69}{128}A^8 + & 11\frac{11}{64}A^9 \&c. \\
F = \ . & . & 13\frac{9}{16}A^6 + & 81\frac{3}{8}A^7 + & 296\frac{87}{128}A^8 + & 834\frac{43}{128}A^9 \&c. \\
G = \ . & . & . & 122\frac{1}{16}A^7 + & 1510\frac{67}{128}A^8 + & 11475\frac{73}{128}A^9 \&c. \\
H = \ . & . & . & . & 1937\frac{95}{128}A^8 + & 47151\frac{93}{128}A^9 \&c. \\
I = \ . & . & . & . & . & 54902\frac{89}{128}A^9 \&c.
\end{array}$$

Thus in the 9th number of the foregoing example, omitting the ciphers at the beginning of the decimals, we have

$$\begin{array}{ll}
A = & 1.51164,65999,05672,95048,8 \\
A^2 = & - \ 2,28507,54430,06381,6726 \\
A^3 = & - \ - \ 3,45422,65239,48546,2 \\
A^4 = & - \ - \ - \ 5,22156,97802,288 \\
A^5 = & - \ - \ - \ - \ 7,89316,8205 \\
A^6 = & - \ - \ - \ - \ - \ 11,93168,1
\end{array}$$

Consequently,

$$\begin{array}{lrl}
\frac{1}{2}A^2 = & 1.14253,77215,03190,8363 & = B \\
\frac{1}{2}A^3 & 1,72711,32619,74273 & \\
\frac{1}{8}A^4 & - \ 65269,62225 & \\
\hline
\frac{1}{2}A^3 + \frac{1}{8}A^4 & 1,72711,97889,36498 & = C \\
\frac{7}{8}A^4 & 4,56887,35577 & \\
\frac{7}{8}A^5 & - \ 6,90652 & \\
\frac{7}{16}A^6 & - \ - \ 5 & \\
\hline
\frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6 & 4,56894,26234 & = D \\
2\frac{5}{8}A^5 & - \ 20,71957 & \\
7A^6 & - \ - \ 83 & \\
\hline
2\frac{5}{8}A^5 + 7A^6 & - \ - \ 20,72040 & = E
\end{array}$$

which agree with the like differences in the foregoing specimen.

In the 9th chapter, after observing that from the logarithms of 1, 2, 3, 5, and 10, before found, are to be determined, by addition and subtraction, the logarithms of all other numbers which can be produced from these by multiplication and division; for finding the logarithms of other prime numbers, instead of that in the 7th chapter, our author then shows another ingenious method of obtaining numbers beginning with 1 and ciphers, and such as to bear a certain relation to some prime number by means of which its logarithm may be found. The method is this: Find three products having the common difference 1, and such that two of them are produced from factors having given logarithms, and the third produced from the prime number, whose logarithm is required, either multiplied by itself, or by some other number whose logarithm is given: then the greatest and least of these three products being multiplied together, and the mean by itself, there arise two other products also differing by 1, of which the greater, divided by the less, gives for a quotient 1 with a small decimal, having several ciphers at the beginning. Then the logarithm of this quotient being found as before, from it will be deduced the required logarithm of the given prime number. Thus, if it be proposed to find the logarithm of the prime number 7; here $6 \times 8 = 48$, $7 \times 7 = 49$, and $5 \times 10 = 50$, will be the three products, of which the logarithms of 48 and 50, the 1st and 3d, will be given from those of their factors 6, 8, 5, 10: also $48 \times 50 = 2400$, and $49 \times 49 = 2401$ are the two new products, and $2401 \div 2400 = 1.00041\frac{2}{3}$ their quotient: then the least of 44 means between 1 and this quotient is 1.00000,00000,00000,02367,98249,04333,6405, which multiplied by 43429 &c, produces 0.00000,00000,00000,01028,40172,38387,29715 for its logarithm; which being 44 times doubled, or multiplied by 17592186044416, produces 0.00018,09183,45421,30 for the logarithm of the quotient $1.00041\frac{2}{3}$; which being added to the logarithm of the divisor 2400, gives the logarithm of the dividend 2401 ; then the half of this logarithm is the logarithm of 49 the root of 2401, and the half of this again gives 0.84509,80400,14256,82 for the logarithm of 7, which is the root of 49.—The author adds another example to illustrate this method; and then sets down the requisite factors, products, and quotients for finding the logarithms of all other prime numbers up to 100.

The 10th chapter is employed in teaching how to find the logarithms of fractions, namely by subtracting the logarithm of the denominator from that of the numerator, then the logarithm of the fraction is the remainder; which therefore is either abundant or defective, that is positive or negative, as the fraction is greater or less than 1.

In the 11th chapter is shown an ingenious contrivance for very accurately finding intermediate numbers to given logarithms, by the proportional parts. On this occasion, it is remarked, that while the absolute numbers increase uniformly, the logarithms increase unequally, with a decreasing increment; for which reason it happens, that either logarithms or numbers corrected by means of the proportional parts, will not be quite accurate, the logarithms so found being always too small, and the absolute numbers so found too great; but yet so however as that they approach much nearer to accuracy towards the end of the table, where the increments or differences become much nearer to equality, than in the former parts of the table. And from this property our author, ever fruitful in happy expedients to obviate natural difficulties, contrives a device to throw the proportional part, to be found from the numbers and logarithms, always near the end of the table, in whatever part they may happen naturally to fall. And it is this: Rejecting the characteristic of any given logarithm, whose number is proposed to be found, take the arithmetical complement of the decimal part, by subtracting it from 1.000&c, the logarithm of 10; then find in the table the logarithm next less than this arithmetical complement, together with its absolute number; to this tabular logarithm add the logarithm that was given, and the sum will be a logarithm necessarily falling among those near the end of the table; find then its absolute number, corrected by means of the proportional part, which will not be very inaccurate, as falling near the end of the table; this being divided by the absolute number, before found for the logarithm next less than the arithmetical complement, the quotient will be the required number answering to the given logarithm; which will be much more correct than if it had been found from the proportional part of the difference where it naturally happened to fall: and the reason of this operation is evident from the nature of logarithms. But as this divisor, when taken as the number answering to the logarithm next less than the arithmetical complement, may happen to be a large prime number; it is further remarked, that instead of this number and its logarithm, we may use the next less composite number, which has small factors, and *its* logarithms; because the division by those small factors, instead of by the number itself, will be performed by the short and easy way of division in one line. And for the more easy finding proper composite numbers and their factors, our author here subjoins an abacus, or list of all such numbers, with their logarithms and component factors, from 1000 to 10000; from which the proper logarithms and factors are immediately obtained by inspection. Thus, for example, to find the root of 10800, or the mean proportional between 1 and 10800: The logarithm of 10800 is 4.03342,37554,8695, the half of which is 2.01671,18777,4347 the logarithm of the number sought, the arithmetical complement of which log. is 0.98328,81222,5653; now the nearest log. to this in the abacus is 0.98227,12330,3957, and its annexed number is 9600, the factors of which are 2, 6, 8; to this last log. adding the log. of the number sought, the sum is 0.99898,31107,8304, whose absolute number, corrected by the proportional part, is 99766,12651,6521, which being divided continually by 2, 6, 8, the factors of 96, the last quotient is 103 92304845471; which is pretty correct, the true number being 103.92304845413 $= \sqrt{10800}$.

We now arrive at the 12th and 13th chapters, in which our ingenious author first of all teaches the rules of the Differential Method, in constructing logarithms by interpolation from differences. This is the same method which has since been more largely treated of by later authors, and particularly by the learned Mr. Cotes, in his "Canonotechnia." How Mr. Briggs came by it does not well appear, as he only delivers the rules, without laying down the principles or investigation of them. He divides the method into two cases, namely, when the second differences are equal or nearly equal, and when the differences run out to any length whatever. The former of these is treated in the 12th chapter; and he particularly adapts it to the interpolating 9 equidistant means between two given terms, evidently for this reason, that then the powers of 10 become the principal multipliers or divisors, and so the operations performed mentally. The substance of his process is this: Having given two absolute numbers with their logarithms, to find the logarithms of 9 arithmetical means between the given numbers: Between the given logarithms take the 1st difference, as well as between each of them and their next or equidistant greater and less logarithms; and likewise the second differences, or the two differences of these three first differences; then if these second differences be equal, multiply one of them severally by the numbers 45, 35, &c, in the annexed tablet, dividing each product by 1000, that is cutting off three figures from each; lastly, to $\frac{1}{10}$ of the 1st difference of the given logarithms, add severally the first five quotients, and subtract the other five, so shall the ten results be the respective first differences, to be continually added, to compose the required series of logarithms.

1	45	Additive products.
2	35	
3	25	
4	15	
5	5	
6	5	Subductive products.
7	15	
8	25	
9	35	
10	45	

Now this amounts to the same thing as what is at this day taught in the like case: we know that if A be any term of an equidistant series of terms, and a, b, c, &c, the first of the 1st, 2d, 3d, &c, order of differences; then the term z, whose distance from A is expressed by x, will be thus, $z = A + xa + x \cdot \frac{x-1}{2}b + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3}c$ + &c. And if now, with our author, we make the 2d differences equal, then c, d, e, &c, will all vanish, or be equal to 0, and z will become barely $= A + xa + x \cdot \frac{x-1}{2}b$.

Series of Terms.	The Differences.
A	
$A + \frac{1}{10}a + \frac{9}{200}b$	$\frac{1}{10}a + \frac{9}{200}b = \frac{1}{10}a + \frac{45}{1000}b$
$A + \frac{2}{10}a + \frac{16}{200}b$	$\frac{1}{10}a + \frac{7}{200}b = \frac{1}{10}a + \frac{35}{1000}b$
$A + \frac{3}{10}a + \frac{21}{200}b$	$\frac{1}{10}a + \frac{5}{200}b = \frac{1}{10}a + \frac{25}{1000}b$
$A + \frac{4}{10}a + \frac{24}{200}b$	$\frac{1}{10}a + \frac{3}{200}b = \frac{1}{10}a + \frac{15}{1000}b$
$A + \frac{5}{10}a + \frac{25}{200}b$	$\frac{1}{10}a + \frac{1}{200}b = \frac{1}{10}a + \frac{5}{1000}b$
$A + \frac{6}{10}a + \frac{24}{200}b$	$\frac{1}{10}a - \frac{1}{200}b = \frac{1}{10}a - \frac{5}{1000}b$
$A + \frac{7}{10}a + \frac{21}{200}b$	$\frac{1}{10}a - \frac{3}{200}b = \frac{1}{10}a - \frac{15}{1000}b$
$A + \frac{8}{10}a + \frac{16}{200}b$	$\frac{1}{10}a - \frac{5}{200}b = \frac{1}{10}a - \frac{25}{1000}b$
$A + \frac{9}{10}a + \frac{9}{200}b$	$\frac{1}{10}a - \frac{7}{200}b = \frac{1}{10}a - \frac{35}{1000}b$
$A + a$	$\frac{1}{10}a - \frac{9}{200}b = \frac{1}{10}a - \frac{45}{1000}b$

Therefore if we take x successively equally to $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$, &c, we shall have the annexed series of terms with their differences. Where it is to be observed, that our author had reduced the differences from the 1st to the 2d form, as he thought it easier to multiply by 5 than to divide by 2. Also all the last terms ($x \cdot \frac{x-1}{2}b$) are set down positive, because in the logarithms b is negative.—If the two 2d differences be only nearly equal, take an arithmetical mean between them, and proceed with it the same as above with one of the equal 2d differences.—He also shows how to find any one single term, independent of the rest; and concludes the chaper with pointing out a method of finding the proportional part more accurately than before.

In the 13th chapter our author remarks, that the best way of filling up the intermediate chiliads of his table, namely from 20000 to 90000, is by quinquisection, or interposing four equidistant means between two given terms; the method of performing this he thus particularly describes. Of the given terms, or logarithms, and two or three others on each side of them, take the 1st, 2d, 3d, &c, differences, till the last differences come out equal, which suppose to be the 5th differences: divide the first differences by 5, the 2d by 25, the 3d by 125, the 4th by 625, and the 5th by 3125, and call the respective quotients the 1st, 2d, 3d, 4th, 5th *mean* differences; or, instead of dividing by these powers of 5, multiply by their reciprocals $\frac{2}{10}$, $\frac{4}{100}$, $\frac{8}{1000}$, $\frac{16}{10000}$, $\frac{32}{100000}$; that is, multiply by 2, 4, 8, 16, 32, cutting off respectively one, two, three, four, five figures, from the end of the products, for the several mean differences: then the 4th and 5th of these mean differences are sufficiently accurate; but the 1st, 2d, and 3d are to be corrected in this manner; from the mean third differences subtract 3 times the 5th difference, and the remainders are the *correct* 3d differences; from the mean 2d differences subtract double the 4th differences, and the remainders are the correct 2d differences; lastly, from the mean 1st differences take the correct 3d differences, and $\frac{1}{5}$ of the 5th difference, and the remainders will be the correct first differences. Such are the corrections when the differences extend as far as the 5th. However, in completing those chiliads in this way, there will be only 3 orders of differences, as neither the 4th nor 5th will enter the calculation, but will vanish through their smallness: therefore the mean 2d and 3d differences will need no correction, and the mean first differences will be corrected by barely subtracting the 3d from them. These preparatory numbers being thus found, all the 2d differences of the logarithms required, will be generated by adding continually, from the less to the greater, the constant 3d difference; and the series of 1st differences will be found by adding the several 2d differences; and lastly, by adding continually these 1st differences to the 1st given logarithm &c, the required logarithmic terms will be generated.

These easy rules being laid down, Mr. Briggs next teaches how, by them, the remaining chiliads may best be completed: namely, having here the logarithm for all numbers up to 20000, find the logarithm to every 5 beyond this, or of 20005, 20010, 20015, &c, in this manner; to the logarithms of the 5th part of each of these, namely 4001, 4002, 4003, &c, add the constant logarithm of 5, and the sums will be the logarithms of all the terms of the series 20005, 20010, 20015, &c: and these logarithms will have the very same differences as those of the series 4001, 4002, 4003, &c; by means of which therefore interpose 4 equidistant terms by the rules above; and thus the whole canon will be easily completed.

Briggs here extends the rules for correcting the mean differences in quinquisection, as far as the 20th difference; he also lays down similar rules for trisection, and speaks of general rules for any other section, but omitted as being less easy. So that he appears to have been possessed of all that Cotes afterwards delivered in his "Canonotechnia sive Constructio Tabularum per Differentias," drawn from the Differential Method, as their general rules exactly agree, Briggs's mean and correct differences being by Cotes called round and quadrat differences, because he expresses them by the numbers 1, 2, 3, &c, written respectively within a small circle and square.

Briggs also observes, that the same rules equally apply to the construction of equidistant terms of any other kind, such as sines, tangents, secants, the powers of numbers, &c: and further remarks, that, of the sines of three equidifferent arcs, all the remote differences may be found by the rule of proportion, because the sines and their 2d, 4th, 6th, 8th, &c differences, are continued proportionals, as are also the 1st, 3d, 5th, 7th, &c differences, among themselves; and, like as the 2d, 4th, 6th, &c differences are proportional to the sines of the mean arcs, so also are the 1st, 3d, 5th, &c differences proportional to the cosines of the same arcs. Moreover, with regard to the powers of numbers, he remarks the following curious properties; 1st, that they will each have as many orders of differences as are denoted by the index of the power, the squares having two orders of differences, the cubes three, the 4th powers four, &c; 2d, that the last differences will be all equal, and each equal to the common difference of the sides or roots raised to the given power, and multiplied by $1 \times 2 \times 3 \times 4$ &c, continued to as many terms as there are units in the index: so, if the roots differ by 1, the second difference of the squares will be each 1×2 or 2, the 3d differences of the cubes each $1 \times 2 \times 3$ or 6, the 4th differences of the 4th powers each $1 \times 2 \times 3 \times 4$ or 24, and so on; and if the common difference of the roots be any other number n, then the last differences of the squares, cubes, 4th powers, 5th powers, &c, will be respectively $2n^2$, $6n^3$, $24n^4$, $120n^5$, &c.

Besides what was shown in the 11th chapter, concerning the taking out the logarithms of large numbers by means of proportional parts, Briggs employs the next or 14th chapter in teaching how, from the first ten chiliads only, and a small table of one page, here given, to find the number answering to any logarithm, and the logarithm to any number, consisting of fourteen places of figures*.

Having thus fully shown the construction and chief properties of his logarithms, our ingenious author, in the remaining eighteen chapters, exemplifies their uses in many curious and important subjects; such as The Rule-of-Three, or Rule of Proportion; finding the roots of given numbers; finding any number of mean proportionals between two given terms; with other arithmetical rules: also various geometrical subjects, as 1st, Having given the sides of any plane triangle, to find the area, the perpendicular, the angles, and the diameters of the inscribed and circumscribed circles; 2d, In a right-angled triangle, having given any two of these, to find the rest, viz. one leg and the hypotenuse, one leg and the sum or difference of the hypotenuse and the other leg, the two legs, one leg and the area, the area and the sum or difference of the legs, the hypotenuse and sum or difference of the legs, the hypotenuse and area, and the perimeter and area; 3d, Upon a given base, to describe a triangle, equal and isoperimetrical

* It is no more than a large exemplification of this method of Briggs's that has been printed so late as 1771, in a 4to tract, by Mr. Robert Flower, under the title of "The Radix, A New Way of making Logarithms." Though Briggs's work might not be known to this writer.

to another triangle given; 4th, To describe the circumference of a circle so, that the three distances from any point in it, to the three angles of a given plane triangle, shall be to one another in a given ratio; 5th, Having given the base, the area, and the ratio of the two sides, of a plane triangle, to find the sides; 6th, Given the base, difference of the sides, and area of a triangle, to find the sides; 7th, To find a triangle whose area and perimeter shall be expressed by the same number; 8th, Of four given lines, of which the sum of any three is greater than the fourth, to form a quadrilateral figure about which a circle may be described; 9th, Of the diameter, circumference, and area of a circle, and the surface and solidity of the sphere generated by it, having any one given, to find any one of the rest; 10th, Concerning the ellipse, spheroid, and gauging; 11th, To cut a line or a number in extreme and mean ratio; 12th, Given the diameter of a circle, to find the sides and areas of the inscribed and circumscribed regular figures of 3, 4, 5, 6, 8, 10, 12, and 16 sides; 13th, Concerning the regular figures of 7, 9, 15, 24, and 30 sides; 14th, Of isoperimetrical regular figures; 15th, Of equal regular figures; and 16th, Of the sphere and the 5 regular bodies; which closes this introduction. Such of these problems as can admit of it, are determined by elegant geometrical constructions, and they are all illustrated by accurate arithmetical calculations, performed by logarithms; for the exemplification of which they are purposely given.

At the end he remarks, that the chief and most necessary use of logarithms, is in the doctrine of spherical trigonometry, which he here promises to give in a future work, and which was accomplished in his Trigonometria Britannica, to the description of which we now proceed.

Of Briggs's Trigonometria Britannica.

At the close of the account of writings on the natural sines, tangents, and secants, we omitted the description of this work of our learned author, though it is perhaps the greatest of this kind, all things considered, that ever was executed by one person; purposely reserving the account of it to this place, not only as it is connected with the invention and construction of logarithms, but thinking it deserved more peculiar and distinguished notice, on account of the importance and originality of its contents. In the first place, we observe that the division of the quadrant, and the mode of construction, are both new; also the numbers are far more accurate, and are extended to more places, than they had ever been before. The circular arcs had always been divided in a sexagesimal proportion; but here the quadrant is divided into degrees and decimals, as this is a much easier mode of computation than by 60ths; the division being completed only to 100ths of degrees, though his design was to have extended it to 1000ths of degrees. And, besides his own private opinion, he was induced to adopt this mode of decimal divisions, partly at the request of other persons, and partly perhaps from the authority of Vieta, pa. 29 "Calendarii Gregoriani." And it is probable that computations by this decimal division would have come into general use, had it not been for the publication of Vlacq's tables, which came out in the interval, and were extended to every 10 seconds, or 6th parts of minutes. But besides this method, by a decimal division of the degrees, of which the whole circle contains 360, or the quadrant 90, in the 14th chapter he remarks that some other persons were inclined rather to adopt a complete decimal division of the whole circle, first into 100 parts, and each of these into 1000 parts; and for *their* sakes he subjoins a small table of the sines of every 40th part of the quadrant, and remarks, that from these few the whole may be made out, by continual quinquisections; namely, 5 times these 40 make 200, then 5 times these give 1000, thirdly 5 times these give 5000, and lastly, 5 times these give 25000 for the whole quadrant, or 100000 for the whole circumference.

But to return. Our author's large table consists of natural sines to 15 places, natural tangents and secants each to 10 places, logarithmic sines to 14 places, and logarithmic tangents to 10 places each, beside the characteristic. A most stupendous performance! The table is preceded by an introduction, divided into two books, the one containing an account of the truly ingenious construction of the table, by the author himself; and the other, its uses in trigonometry, &c, by Henry Gellibrand, professor of astronomy in Gresham College, who remarks in the preface, that the work was composed by the author about the year 1600; though it was only published by the direction of Gellibrand in 1633, it having been printed at Gouda under the care of Vlacq, and by the printer of his Trigonometria Artificialis, which came out the same year.

After briefly mentioning the common methods of dividing the quadrant, and constructing the tables of sines, &c, from the ancients down to his own time, he hastens to the description of his own peculiar and truly ingenious method, which is briefly this: having first divided the quadrant into a small number of parts, as 72, he finds the sine of one of those parts; then from it, the sines of the double, triple, quadruple, &c, up to the quadrant or 72 parts. He next quinquisects each of these parts, by interposing four equidistant means, by differences; he then quinquisects each of these; and finally each of these again; which completes the division as far as degrees and centesms. The rules for performing all these things he investigates, and illustrates, in a very ample manner. In treating of multiple and submultiple arcs, he gives general algebraical expressions for the sine or chord of any multiple whatever of a given arc, which he deduced from a geometrical figure, by finding the law for the series of successive multiple chords or sines, after the manner of Vieta; who was the first person that I know of, who laid down general rules for the chords of multiples and submultiples of arcs or angles: and the same was afterwards improved by Sir I. Newton, to such form, that radius, and double the cosine of the first given angle, are the first and second terms of all the proportions for finding the sines and cosines of the multiple angles. For assigning the coefficients of the terms in the multiple expressions, our author here delivers the construction of figurate or polygonal numbers, inserts a large table of them, and teaches their several uses; one of which is, that every other number, taken in the diagonal lines, furnishes the coefficients of the terms of the general equation, by which the sines and chords of multiple arcs are expressed, which he amply illustrates; and another, that the same diagonal numbers constitute the coefficients of the terms of any power of a binomial; which property was also mentioned by Vieta in his Angulares Sectiones, theor. 6, 7; and, before him, pretty fully treated of by Stifelius, in his Arithmetica Integra, fol. 44 et seq.; where he inserts and makes the like use of such a table of figurate numbers, in extracting the roots of all powers whatever. But it was perhaps known much earlier, as appears by the treatise on figurate numbers by Nicomachus, (see Malcolm's History, p. xviii). Though indeed Cardan seems to ascribe this discovery to Stifelius. See his Opus Novum de Proportionibus Numerorum, where he quotes it, and extracts the table and its use from Stifel's book. Cardan, in p. 135 &c, of the same work, makes use of a like table to find the number of variations, or conjugations, as he calls them. Stevinus too makes use of the same coefficients and method of roots as Stifelius. See his Arith. page 25. And even Lucas de Burgo extracts the cube root by the same coefficients, about the year 1470: but he does not go to any higher roots. And this is the first mention I have seen made of this law of the coefficients of the powers of a binomial, commonly called Sir I. Newton's binomial theorem, though it is very evident that Sir Isaac was not the first inventor of it: the part of it properly belonging to him seems to be, only the extending it to fractional indices, which was indeed an immediate effect of the general method of denoting all roots like powers, with fractional exponents, the theorem being not at all altered. However, it appears that our author Briggs was the first who taught the rule for generating the coefficients of the terms, successively one from another, of any power of a binomial, independent of those of any other power. For having shown, in his "Abacus Παγχρηστος" (which he so calls on account of its frequent and excellent use, and of which a small specimen is here annexed), that the numbers

ABACUS ΠΑΓΧΡΗΣΤΟΣ.							
H	G	F	E	D	C	B	A
−(8)	−(7)	+(6)	+( )	−(4)	−( )	+( )	(1)
1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2
	36	28	21	15	10	6	3
		84	56	35	20	10	4
			126	70	35	15	5
				126	56	21	6
					84	28	7
						36	8
							9

in the diagonal directions, ascending from right to left, are the coefficients of the powers of binomials, the indices being the figures in the first perpendicular column A, which are also the coefficients of the 2d terms of each power (those of the first terms, being 1, are here omitted); and that any one of these diagonal numbers is in proportion to the next higher in the diagonal, as the vertical of the former is to the marginal of the latter, that is, as the uppermost number in the column of the former is to the first or right-hand number in the line of the latter; having shown these things, I say, he thereby teaches the generation of the coefficients of any power, independently of all other powers, by the very same law or rule which we now use in the binomial theorem. Thus, for the 9th power; 9 being the coefficient of the 2d term, and 1 always that of the first, to find the 3d coefficient, we have 2 : 8 : : 9 : 36; for the 4th term, 3 : 7 : : 36 : 84; for the 5th term, 4 : 6 : : 84 : 126; and so on for the rest. That is to say, the coefficients of the terms in any power m, are inversely as the vertical numbers or first line 1, 2, 3, 4, . . m, and directly as the ascending numbers m, $m-1$, $m-2$, $m-3$, . . . 1, in the first column A; and that consequently those coefficients are found by the continual multiplication of these fractions $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, ... $\frac{1}{m}$, which is the very theorem as it stands at this day, and as applied by Newton to roots or fractional exponents, as it had before been used for integral powers. This theorem then being thus plainly taught by Briggs about the year 1600, it is surprizing how a man of such general reading as Dr. Wallis was, could be quite ignorant of it, as he plainly appears to be by the 85th chapter of his algebra, where he fully ascribes the invention to Newton, and adds, that he himself had formerly sought for such a rule, but without success: Or how Mr. John Bernoulli, in the 18th century, could himself first dispute the invention of this theorem with Newton, and then give the discovery of it to Pascal, who was not born till long after it had been taught by Briggs. See Bernoulli's Works, vol. 4, page 173. But it is not to be wondered that Briggs's remark was unknown to Newton, who owed almost every thing to genius, and deep meditation, but very little to reading: and there can be no doubt that he made the discovery himself, without any light from Briggs; and that he thought it was new for all powers in general, as it was indeed for roots and quantities with fractional and irrational exponents.

When the above table of the sums of figurate numbers is used by our author, in determining the coefficients of the terms of the equation, whose root is the chord of any submultiple of an arc, as when the section is expressed by any uneven number, he remarks, that the powers of that chord or root will be the 1st, 3d, 5th, 7th, &c, in the alternate uneven columns, A, C, E, G, &c, with their signs + or − as marked to the powers, continued till the highest power be equal to the index of the section; and that the coefficients of those powers are the sums of two continuous numbers in the same column with the powers, beginning with 1 at the highest power, and gradually descending one line obliquely to the right at each lower power: so, for a trisection, the numbers are 1 in C, and $1 + 2 = 3$ in A; and therefore the terms are $-1(3) + 3(1)$: for a quinquisection, the numbers are 1 in E,

$1 + 4 = 5$ in C, $2 + 3 = 5$ in A; so that the terms are $1(5) - 5(3) + 5(1)$: for a septisection, the numbers are 1 in G, $1 + 6 = 7$ in E, $4 + 10 = 14$ in C, and $3 + 4 = 7$ in A; and hence the terms are $- 1\,(7) + 7\,(5) - 14\,(3) + 7\,(1)$: and so on; the sum of all these terms being always equal to the chord of the whole or multiple arc. But when the section is denominated by an even number, the squares of the chords enter the equation, instead of the first powers as before, and the dimensions of all the powers are doubled, the coefficients being found as before, and therefore the powers and numbers will be those in the 2d, 4th, 6th, &c, columns: and the uneven sections may also be expressed the same way: hence, for a bisection the terms will be $- 1\,(4) + 4\,(2)$; for a trisection $1\,(6) - 6\,(4) + 9\,(2)$; for the quadrisection $- 1\,(8) + 8\,(6) - 20\,(4) + 16\,(2)$; for the quinquisection $1\,(10) - 10\,(8) + 35\,(6) - 50\,(4) + 25\,(2)$; and so on.

Our author subjoins another table, a small specimen of which is here annexed, in which the first column consists of the uneven numbers 1, 3, 5, &c, the rest being found by addition as before, and the alternate diagonal numbers themselves are the coefficients.

F	E	D	C	B	A
+ (6)	+ (5)	− (4)	− (3)	+ (2)	(1)
1	1	1	1	1	1
	7	6	5	4	3
		20	14	9	5
			30	16	7
				25	9
					11

The method is quite different from that of Vieta, who gives another table for the like purpose, a small part of which is here annexed, which is formed by adding, from the number 2, downwards obliquely towards the right; and the coefficients of the terms stand upon the horizontal line.

1st	Vieta's Table.				
2					
3	2d				
4	2				
5	5	3d			
6	9	2			
7	14	7	4th		
8	20	16	2		
9	27	30	9	5th	
10	35	50	25	2	6th

These angular sections were afterwards further discussed by Oughtred and Wallis. And the same theorems of Vieta and Briggs have been since given in a different form, by Herman and the Bernoullis, in the Leipsic Acts, and the Memoirs of the Royal Academy of Sciences. These theorems they expressed by the alternate terms of the power of a binomial, whose exponent is that of the multiple angle or section. And De Lagny, in the same Memoirs, first showed, that the tangents and secants of multiple angles are also expressed by the terms of a binomial, in the form of a fraction, of which some of those terms form the numerator, and others the denominator. Thus, if r express the radius, s the sine, c the cosine, t the tangent, and s the secant, of the angle A; then the sine, cosine, tangent, and secant of n times the angle, are expressed thus, viz.

$$\text{Sin. } n\text{A} = \frac{1}{r^{n-1}} \times \left(\frac{n}{1} c^{n-1} s - \frac{n.n-1.n-2}{1.2.3} c^{n-3} s^3 + \frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5} c^{n-5} s^5 \ \&c\right)$$

$$\text{Cosine } n\text{A} = \frac{1}{r^{n-1}} \times \left(c^n - \frac{n.n-1}{1.2} c^{n-2} s^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4} c^{n-4} s^4 \ \&c.\right)$$

$$\text{Tang. } n\text{A} = r \times \frac{\frac{n}{1} r^{n-1} t - \frac{n.n-1.n-2}{1.2.3} r^{n-3} t^3 + \frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5} r^{n-5} t^5 \ \&c.}{r^n - \frac{n.n-1}{1.2} r^{n-2} t^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4} r^{n-4} t^4 \ \&c.}$$

$$\text{Sec. } n\text{A} = r \times \frac{\text{s}^2 \text{ or } r^2 + t^2}{r^n - \frac{n.n-1}{1.2} r^{n-2} t^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4} r^{n-4} t^4 \ \&c.} :$$

where it is evident, that the series in the sine of n A, consists of the even terms of the power of the binomial $(c + s)^n$, and the series in the cosine of the uneven terms of the same power; also the series in the numerator of the tangent, consists of the even terms of the power $(r + t)^n$, and the denominator, both of the tangent and secant, consists of the uneven terms of the same power $(r + t)^n$. And if the diameter, chord, and chord of the supplement, be substituted for the radius, sine and cosine, in the expressions for the multiple sine and cosine, the result will give the chord, and chord of the supplement, of n times the arc or angle A. These, and various other expressions, for multiple and submultiple arcs, with other improvements in trigonometry, have also been given by Euler, and other eminent writers on the same subject.

The before mentioned De Lagny offered a project for substituting, instead of the common logarithms, a binary arithmetic, which he called the *natural logarithms*, and which he and Leibnitz seem to have both invented about the same time, independently of each other: but the project came to nothing. De Lagny also published, in several Memoirs of the Royal Academy, a new method of determining the angles of figures, which he called *Goniometry*. It consists in measuring, with a pair of compasses, the arc which subtends the angle in question: however, this arc is not measured in the usual way, by applying its extent to any preconstructed scale; but by examining what part it is of half the circumference of the same circle, in this manner: from the proposed angular point as a centre, with a sufficiently large radius, a semicircle being described, a part of which is the arc intercepted by the sides of the proposed angle, the extent of this arc is taken with a pair of fine compasses, and applied continually upon the arc of the semicircle, by which he finds how often it is contained in the semicircle, with usually a small arc remaining; in the same manner he measures how often this remaining arc is contained in the first arc; and what remains again is applied continually to the first remainder; and so the 3d remainder to the 2d, the 4th to the 3d, and so on till there be no remainder, or else till it become insensibly small. By this process he obtains a series of quotients, or fractional parts, one of another, which being properly reduced into one fraction, give the ratio of the first arc to the semicircumference, or of the proposed angle, to two right angles or 180 degrees, and consequently that angle in degrees, minutes, &c, if required, and that commonly, he says, to a degree of accuracy far exceeding the calculation of the same by means of any tables of sines, tangents or secants, notwithstanding the apparent paradox in this expression at first sight. Thus, if the 1st arc be 4 times contained in the semicircle, the remainder once contained in the first arc, the next 5 times in the second, and finally the fourth 2 times in the third: Here the quotients are 4, 1, 5, 2; consequently the fourth or last arc was $\frac{1}{2}$ the 3d; therefore the 3d was $\frac{1}{5\frac{1}{2}}$ or $\frac{2}{11}$ of the 2d, and the 2d was $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the 1st, and the first, or arc sought, was $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the semicircle; and consequently it contains $37\frac{1}{7}$ degrees, or $37° \, 8' \, 34''\frac{2}{7}$. Hence it is evident, that this method is in fact nothing more than an example of continued fractions, the first instance of which was given by lord Brouncker.

But to return from this long digression; Mr. Briggs next treats of interpolation by differences, and chiefly of quinquisection, after the manner used in the 13th chapter of his construction of logarithms, before described. He here proves that curious property of the sines and their several orders of differences, before mentioned, namely, that, of equidifferent arcs, the sines, with the 2d, 4th, 6th, &c differences, are continued proportionals; as also the cosines of the means between those arcs, and the 1st, 3d, 5th, &c differences. And to this treatise on interpolation by differences, he adds a marginal note, complaining that this 13th chapter of his "Arithmetica Logarithmica" had been omitted by Vlacq in his edition of it; as if he were afraid of an intention to deprive him of the honour of the invention of interpolation by successive differences. The note is this: "Modus correctionis à me traditus est Arithmeticæ Logarithmicæ capite 13, in editione Londinensis: Istud autem caput unà cum sequenti in editione Batava me inconsulto et inscio omissum fuit: nec in omnibus, editionis illius author, vir alioqui industrius et non indoctus, meam mentem videtur assequutus: Ideoque, ne quicquam desit cuiquam, qui integrum canonem conficere cupiat, quædam maxime necessaria illinc huc transferenda censui."

A large specimen of quinquisection by differences is then given, and he shows how it is to be applied to the construction of the whole canon of sines, both for 100th and 1000th parts of degrees; namely, for centesms, divide the quadrant first into 72 equal parts, and find their sines by the primary methods; then these quinquisected give 360 parts, a second quinquisection gives 1800 parts, and a third gives 9000 parts, or centesms of degrees: but for millesms, divide the quadrant into 144 equal parts; then one quinquisection gives 720, a second gives 3600, a third 18000, and a fourth gives 90000 parts, or millesms.

He next proceeds to the natural tangents and secants, which he directs to be raised in the same manner, by interpolations from a few primary ones, constructed from the known proportions between sines, tangents, and secants; excepting that half the tangents and secants are to be formed by addition and subtraction only, by means of some such theorems as these, namely, 1st, the secant of an arc is equal to the sum of the tangent of the same arc, and the tangent of half its complement, which will find every other secant; 2d, double the tangent of an arc added to the tangent of half its complement, is equal to the tangent of the sum of that arc and the said half complement, by which rule half the tangents will be found; &c.

In the two remaining chapters of this book are treated the construction of the logarithmic sines, tangents, and secants. This is preceded by some remarks on the origin and invention of them. Our author here observes, that logarithms may be of various kinds; that others had followed the plan of Baron Napier the first inventor, among whom Benjamin Ursinus is especially commended, who applied Napier's logarithms to every ten seconds of the quadrant; but that he himself, encouraged by the noble inventor, devised other logarithms that were much easier and more excellent*. He says he put 10, with ciphers, for the logarithm of radius; 9 for the logarithm sine of 5° 44′, whose natural sine is one 10th of the radius; 8 for that of 34′, whose natural sine is one 100th of the radius, and so on; thereby making 1 the loga-

* His words are: "Ego vero ipsius inventoris primi cohortatione adjutus, alios logarithmos applicandos censui, qui multo faciliorem usum habent, præstantiorem. Logarithmus radii circularis vel sinus totius, a me ponitur 10 &c."

rithm of the ratio of 10 to 1, which is the characteristic of his species of logarithms.

To construct the logarithmic sines, he directs first to divide the quadrant into 72 equal parts as before, and to find the logarithms of their natural sines as in the 14th chapter of his Arithmetica Logarithmica; after which, this number will be increased by quinquisection, first to 360, then to 1800, and lastly to 9000, or centesms of degrees. But if millesms of degrees be required, divide the quadrant first into 144 equal parts, and then by four quinquisections these will be extended to the following parts, 720, 3600, 18000, and 90000, or millesms of degrees. He remarks however, that the logarithmic sines of only half the quadrant need be found in this manner, as the other half may be found by mere addition, or subtraction, by means of this theorem, as the sine of half an arc is to half radius, so is the sine of the whole arc to the cosine of the said half arc. This theorem he illustrates with examples, and then adds a table of the logarithmic sines of the primary 72 parts of the quadrant, from which the rest are to be made out by quinquisection.

In the next chapter our author shows the construction of the natural tangents and secants more fully than he had done before, demonstrating and illustrating several curious theorems for the easy finding of them. He then concludes this chapter, and the book, with pointing out the very easy construction of the logarithmic tangents and secants by means of these three theorems:

1st, As cosine : sine : : radius : tangent,
2d, As tangent : radius : : radius : cotangent,
3d, As cosine : radius : : radius : secant.

So that in logarithms, the tangents are found by subtracting the cosines from the sines, adding always 10 or the radius; the cotangents are found by subtracting always the tangents from 20 or double the radius; and the secants are found by subtracting the cosines from 20 the double radius.—The 2d book, by Gellibrand, contains the use of the canon in plane and spherical trigonometry.

Besides Briggs's methods of constructing logarithms, above described, no others were given about that time. For as to the calculations made by Vlacq, his numbers being carried to comparatively but few places of figures, they were performed by the easiest of Briggs's methods, and in the manner which this ingenious man had pointed out in his two volumes. Thus, the 70 chiliads of logarithms, from 20000 to 90000, computed by Vlacq, and published in 1628, being extended only to 10 places, yield no more than two orders of mean differences, which are also the correct differences, in quinquisection, and therefore will be made out thus, namely, one-fifth of them by the mere addition of the constant logarithm of 5; and the other four-fifths of them by two easy additions of very small numbers, namely, of the 1st and 2d differences, according to the directions given in Briggs's Arith. Log. c. 13, p. 31. And as to Vlacq's logarithmic sines and tangents to every 10 seconds, they were easily computed thus; the sines for half the quadrant were found by taking the logarithms to the natural sines in Rheticus's canon; and then from these the logarithmic sines to the other half quadrant were found by mere addition and subtraction; and from these all the tangents by one single subtraction. So that all these operations might easily be performed by one person, as quickly as a printer could set up the types; and thus the computation and printing might both be carried on together. And hence it appears that there is no reason for admiration at the expedition with which these tables were said to have been brought out.

Of certain curves related to Logarithms.

About this time the mathematicians of Europe began to consider some curves which have properties analogous to logarithms. Edmund Gunter, it has been said, first gave the idea of a curve, whose abscisses are in arithmetical progression, while the corresponding ordinates are in geometrical progression, or whose abscisses are the logarithms of their ordinates; but I cannot find it noticed in any part of his writings. The same curve was afterwards considered by others, and named the *Logarithmic* or *Logistic* curve by Huygens, in his "Dissertatio de Causa Gravitatis," where he enumerates all the principal properties of this curve, showing its analogy to logarithms. Many other learned men have also treated of its properties; particularly Le Seur and Jacquier, in their commentary on Newton's Principia; by Dr. John Keill, in the elegant little tract on logarithms, subjoined to his edition of Euclid's Elements; and by Francis Maseres, Esq. cursitor baron of the exchequer, in his ingenious treatise on Trigonometry; in which books the doctrine of logarithms is copiously and learnedly treated, and their analogy to the logarithmic curve &c fully displayed.—It is indeed rather extraordinary that this curve was not sooner announced to the public; since it results immediately from baron Napier's manner of conceiving the generation of logarithms, by only supposing the lines which represent the natural numbers to be placed at right angles to that upon which the logarithms are taken. This curve greatly facilitates the conception of logarithms to the imagination, and affords an almost intuitive proof of the very important property of their fluxions, or very small increments, to wit, that the fluxion of the number is to the fluxion of the logarithm, as the number is to the subtangent; as also of this property, that, if three numbers be taken very nearly equal, so that their ratios to each other may differ but a little from a ratio of equality, as for exam. the three numbers 10000000, 10000001, 10000002, their differences will be very nearly proportional to the logarithms of the ratios of those numbers to each other: all which follows from the logarithmic arcs being very little different from their chords, when they are taken very small. And the constant subtangent of this curve is what was afterwards by Cotes called the *Modulus* of the system of logarithms: and since, by the former of the two properties above-mentioned, this subtangent is a 4th proportional to the fluxion of the number, the fluxion of the logarithm, and the number itself; this property afforded occasion to Mr. Baron Maseres to give the following definition of the modulus, which is the same in effect as Cotes's, but more clearly expressed, namely, that it is the limit of the magnitude of a 4th proportional to these three quantities, to wit, the difference of any two natural numbers that are nearly equal to each other, either of the said numbers, and the logarithm or measure of the ratio they have to each other. Or we may define the modulus to be the natural number at that part of the system of logarithms, where the fluxion of the number is equal to the fluxion of the logarithm, or where the numbers and logarithms have equal differences. And hence it follows, that the logarithms of equal numbers, or of equal ratios, in different systems, are to one another as the *moduli* of those systems. Further, the ratio whose measure or logarithm is equal to the modulus, and thence by Cotes called the *ratio modularis*, is by calculation found to be the ratio of 2·718281828459 &c to 1, or of 1 to ·367879441171 &c; the calculation of which number may be seen at full length in Mr. Baron Maseres's treatise on the Principles of Life Annuities, pa. 274 and 275.

The hyperbolic curve also afforded another source for developing and illustrating the properties and construction of logarithms. For the hyperbolic areas lying between the curve and one asymptote, when they are bounded by ordinates parallel to the other asymptote, are analogous to the logarithms of their abscisses, or parts of the asymptote. And so also are the hyperbolic sectors; any sector bounded by an arc of the hyperbola and two radii, being equal to the quadrilateral space bounded by the same arc, the two ordinates to either asymptote from the extremities of the arc, and the part of the asymptote intercepted between them. And though Napier's logarithms are commonly said to be the same as hyperbolic logarithms, it is not to be understood that hyperbolas exhibit Napier's logarithms only, but indeed all other possible systems of logarithms whatever. For, like as the right-angled hyperbola, the side of whose square inscribed at the vertex is 1, gives Napier's logarithms; so any other system of logarithms is expressed by the hyperbola whose asymptotes form a certain oblique angle, the side of the rhombus inscribed at the vertex of the hyperbola in this case also being still 1, the same as the side of the square in the right-angled hyperbola. But the areas of the square and rhombus, and consequently the logarithms of any one and the same number or ratio, differing according to the sine of the angle of the asymptotes. And the area of the square or rhombus, or any inscribed parallelogram, is also the same thing as what was by Cotes called the modulus of the system of logarithms; which modulus will therefore be expressed by the numerical measure of the sine of the angle formed by the asymptotes, to the radius 1; as that is the same with the number expressing the area of the said square or rhombus, the side being 1: which is another definition of the modulus to be added to those we remarked above, in treating of the logarithmic curve. And the evident reason of this is, that in the beginning of the generation of these areas, from the vertex of the hyperbola, the nascent increment of the abscisse drawn into the altitude 1, is to the increment of the area, as radius is to the sine of the angle of the ordinate and abscisse, or of the asymptotes; and at the beginning of the logarithms, the nascent increment of the natural numbers is to the increment of the logarithms, as 1 is to the modulus of the system. Hence we easily discover that the angle formed by the asymptotes of the hyperbola exhibiting Briggs's system of logarithms, will be 25 deg. 44 minutes, $25\frac{1}{2}$ seconds, this being the angle whose sine is 0·4342944819 &c, the modulus of this system.

Or indeed any one hyperbola will express all possible systems of logarithms whatever, namely, if the square or rhombus inscribed at the vertex, or, which is the same thing, any parallelogram inscribed between the asymptotes and the curve at any other point, be expounded by the modulus of the system; or, which is the same, by expounding the area, intercepted between two ordinates which are to each other in the ratio of 10 to 1, by the logarithm of that ratio in the proposed system.

As to the first remarks on the analogy between logarithms and the hyperbolic spaces; it having been shown by Gregory

St. Vincent, in his Quadratura Circuli et Sectionum Coni, published at Antwerp in 1647, that if one asymptote be divided into parts, in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and curve into equal portions; from hence it was shown by Mersenne, that, by taking the continual sums of those parts, there would be obtained areas in arithmetical progression, adapted to abscisses in geometrical progression, and which therefore were analogous to a system of logarithms. And the same analogy was remarked and illustrated soon after, by Huygens and many others, who showed how to square the hyperbolic spaces by means of the logarithms.

There are also innumerable other geometrical figures having properties analogous to logarithms; such as the equiangular spiral, the figures of the tangents and secants, &c; which it is not to our purpose to distinguish more particularly.

Of Gregory's Computation of Logarithms.

On the other hand, Mr. James Gregory, in his Vera Circuli et Hyperbolæ Quadratura, first printed at Patavi, or Padua, in the year 1667, having approximated to the hyperbolic asymptotic spaces by means of a series of inscribed and circumscribed polygons, thence shows how to compute the logarithms, which are analogous to those areas: and thus the quadrature of the hyperbolic spaces became the same thing as the computation of the logarithms. He here also lays down various methods to abridge the computation, with the assistance of some properties of numbers themselves, by which we are enabled to compose the logarithms of all prime numbers under 1000, each by one multiplication, two divisions, and the extraction of the square root. And the same subject is further pursued in his Exercitationes Geometricæ, to be described hereafter.

Mr. Gregory was born at Aberdeen in Scotland 1638, where he was educated. He was professor of mathematics in the college of St. Andrews, and afterwards in that of Edinburgh.

He died of a fever in December 1675, being only 36 years of age.

Of Mercator's Logarithmotechnia.

Nicholas Mercator, a learned mathematician, and an ingenious member of the Royal Society, was a native of Holstein in Germany, but spent most of his time in England, where he died in the year 1690, at about 50 years of age. He was the author of many works in geometry, geography, astronomy, astrology, &c.

In 1668, Mercator published his Logarithmotechnia, sive methodus construendi Logarithmos nova, accurata, et facilis; in which he delivers a new and ingenious method of computing the logarithms, on principles purely arithmetical; which, being curious and very accurately performed, I shall here give a rather full and particular account of that little tract, as well as of the small specimen of the quadrature of curves by infinite series, subjoined to it; and the more especially as this work gave occasion to the public communication of some of Sir Isaac Newton's earliest pieces, to evince that he had not borrowed them from this publication. So that it appears these two ingenious men had, independent of each other, in some instances fallen upon the same things.

Mercator begins this work by remarking that the word logarithm is composed of the words ratio and number, being as much as to say the number of ratios; which he observes is quite agreeable to the nature of them, for that a logarithm is nothing else but the number of *ratiunculæ* contained in the ratio which any number bears to unity. He then makes a learned and critical dissertation on the nature of ratios, their magnitude and measure, conveying a clearer idea of the nature of logarithms than had been given by either Napier or Briggs, or any other writer except Kepler, in his work before described; though those other writers seem indeed to have had in their own minds the same ideas on the subject as Kepler and Mercator, but without having expressed them so clearly. Our author indeed pretty closely follows Kepler in his modes of thinking and expression, and after him in plain and express terms calls logarithms the measures of ratios; and, in order to the right understanding that definition of them, he explains what he means by the magnitude of a ratio. This he does pretty fully, but not too fully, considering the nicety and subtlety of the subject of ratios, and their magnitude, with their addition to, and subtraction from, each other, which have been misconceived by very learned mathematicians, who have thence been led into considerable mistakes. Witness the oversight of Gregory St. Vincent, which Huygens animadverted on in the Εξετασις Cyclometriæ Gregorii a Sancto Vincentio, and which arose from not understanding, or not adverting to, the nature of ratios, and their proportions to one another. And many other similar mistakes might here be adduced of other eminent writers. From all which we must commend the propriety of our author's attention, in so judiciously discriminating between the magnitude of a ratio, as of a to b, and the fraction $\frac{a}{b}$, or quotient arising from the division of one term of the ratio by the other; which latter method of consideration is always attended with danger of errors and confusion on the subject; though in the 5th definition of the 6th book of Euclid this quotient is accounted the quantity of the ratio; but this definition is probably not genuine, and therefore very properly omitted by professor Simson in his edition of the Elements. And in those ideas on the subject of logarithms, Kepler and Mercator have been followed by Halley, Cotes, and most of the other eminent writers since that time.

Purely from the above idea of logarithms, namely, as being the measures of ratios, and as expressing the number of *ratiunculæ* contained in any ratio, or into which it may be divided, the number of the like equal *ratiunculæ* contained in some one ratio, as of 10 to 1, being supposed given, our author shows how the logarithm or measure of any other ratio may be found. But this however only by-the-bye, as not being the principal method he intends to teach, as his last and best, and which we arrive not at till near the end of the book, as we shall see below. Having shown then, that these logarithms, or numbers of small ratios, or measures of ratios, may be all properly represented by numbers, and that of 1, or the ratio of equality, the logarithm or measure being always 0, the logarithm of 10, or the measure of the ratio 10 to 1, is most conveniently represented by 1 with any number of ciphers; he then proceeds to show how the measures of all other ratios may be found from this last supposition. And he explains the principles by the two following examples.

First, to find the logarithm of 100·5*, or to find how many *ratiunculæ* are contained in the ratio of 100·5 to 1, the number of *ratiunculæ* in the decuple ratio, or ratio of 10 to 1, being 1.0000000.

The given ratio 100·5 to 1, he first divides into its parts, namely, 100·5 to 100, 100 to 10, and 10 to 1; the last two of which being decuples, it follows that the characteristic will be 2, and it only remains to find how many parts of the next decuple belong to the first ratio of 100·5 to 100. Now if each term of this ratio be multiplied by itself, the products will be in the duplicate ratio of the first terms, or this last ratio will contain a double number of parts; and if these be multiplied by the first terms again, the ratio of the last products will contain three times the number of parts; and so on, the number of times of the first parts contained in the ratio of any like powers of the first terms, being always denoted by the exponent of the power. If therefore the first terms, 100·5 and 100, be continually multiplied till the same powers of them have to each other a ratio whose measure is known, as suppose the decuple ratio 10 to 1, whose measure is 1.0000000; then the exponent of that power shows what mul. this measure 1.0000000, of the decuple ratio, is of the required measure of the first ratio 100·5 to 100; and consequently dividing 1.0000000 by that exponent, the quotient is the measure of the ratio 100·5 to 100 sought. The operation for finding this, he sets down as here follows; where the several multiplications are all performed in

* Mercator distinguishes his decimals from integers thus 100[5, or 100,5.

the contracted way, by inverting the figures of the multiple and retaining only the first number of decimals in each product.

		power
100·5000	. .	1
5001	. .	1
1005000		
5025		
1010025	. .	2
5200101	. .	2
1010025		
10100		
20		
5		
1020150	. .	4
0510201	. .	4
1020150		
20403		
102		
51		
1040706	. .	8
6070401	. .	8
1083068	. .	16
8603801	. .	16
1173035	. .	32
5303711	. .	32
1376011	. .	64
1106731	. .	64
1893406	.	128
6043981	.	128
3584985	.	256
5894853	.	256
12852116	.	512

This power being greater than the decuple of the like power of 100, which must always be 1 with ciphers, resume therefore the 256th power, and multiply it, not by itself, but by the next before, viz. by the 128th, thus

3584985	.	256
6043981	.	128
6787831	.	384
1106731	.	64
9340130	.	448
5303711	.	32
10956299	.	480

This power again exceeding the same power of 100 more than 10 times, therefore draw the same 448th, not into the 32d, but the next preceding, thus

9340130	.	448
8603801	.	16
10115994	.	464

This being again too much, instead of the 16th, draw it into the 8th, or next preceding, thus

9340130	.	448
6070401	.	8
9720329	.	456
0520201	.	4
9916193	.	460
5200101	.	2
10015603	.	462

Which power again exceeds the limit; theref. draw the 460th into the 1st, thus

9916193	.	460
5001	.	1
9965774	.	461

Since therefore the 462d power of 100·5 is greater, and the 461st power is less, than the decuple of the same power of 100, the ratio of 100·5 to 100 is contained in the decuple more than 461 times, but less than 462 times. Again,

Since the $\left\{\begin{array}{c}460\\461\\462\end{array}\right\}$ power is $\left\{\begin{array}{r}9916193\\9965774\\10015603\end{array}\right\}$ and the differences $\begin{array}{l}49581\\49829\end{array}\Big\}$ nearly equal;

therefore the proportional part which the exact power, or 10000000, exceeds the next less 9965774, will be easily and accurately found by the Golden Rule, thus:

The just power	. .	10000000
and the next less	. .	9965774
the difference	.	34226; then

As 49829 the dif. between the next less and greater,
: To 34226 the dif. between the next less and just,
: : So is 10000 : to 6868, the decimal parts; and therefore the ratio of 100·5 to 100, is 461·6868 times contained in the decuple or ratio of 10 to 1. Dividing now 1.0000000, the measure of the decuple ratio, by 461·6868, the quotient 00216597 is the measure of the ratio of 100·5 to 100; which being added to 2 the measure of 100 to 1, the sum 2.00216597 is the measure of the ratio of 100·5 to 1, that is, the log. of 100·5 is 2.00216597.—In the same manner he next investigates the log. of 99·5, and finds it to be 1.99782307.

A few observations are then added, calculated to generalize the consideration of ratios, their magnitude and affections. It is here remarked, that he considers the magnitude of the ratio between two quantities as the same, whether the antecedent be the greater or the less of the two terms: so, the magnitude of the ratio of 8 to 5, is the same as of 5 to 8; that is, by the magnitude of the ratio of either to the other, is meant the number of *ratiunculæ* between them, which will evidently be the same, whether the greater or less term be the antecedent. And he further remarks that, of different ratios, when we divide the greater term of each ratio by the less, that ratio is of the greater mass or magnitude, which produces the greater quotient, *et vice versa;* though those quotients are not proportional to the masses or magnitudes of the ratios. But when he considers the ratio of a greater term to a less, or of a less to a greater, that is to say, the ratio of greater or less inequality, as abstracted from the magnitude of the ratio, he distinguishes it by the word *affection*, as much as to say, greater or less affection, something in the manner of positive and negative quantities, or such as are affected with the signs

+ and −. The remainder of this work he delivers in several propositions, as follows.

Prop. 1. In subtracting from each other, two quantities of the same affection, to wit, both positive, or both negative; if the remainder be of the same affection with the two given, then is the quantity subtracted the less of the two, or expressed by the less number; but if the contrary, it is the greater.

Prop. 2. In any continued ratios, as $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, &c, (by which is meant the ratios of a to $a+b$, $a+b$ to $a+2b$, $a+2b$ to $a+3b$, &c,) of equidifferent terms, the antecedent of each ratio being equal to the consequent of the next preceding one, and proceeding from less terms to greater; the measure of each ratio will be expressed by a greater quantity than that of the next following; and the same through all their orders of differences, namely, the 1st, 2d, 3d, &c, differences; but the contrary, when the terms of the ratios decrease from greater to less.

Prop. 3. In any continued ratios of equidifferent terms, if the 1st or least be a, the difference between the 1st and 2d b, and c, d, e, &c, the respective first term of their 2d, 3d, 4th, &c, differences: then shall the several quantities themselves be as in the annexed scheme; where each term is composed of the first term, together with as many of the differences as it is distant from the first term, and to those differences joining, for coefficients, the numbers in the sloping or oblique lines contained in the annexed table of figurate numbers, in the same

$$\begin{array}{ll}
\text{1st term} & a \\
\text{2d term} & a + b \\
\text{3d term} & a + 2b + c \\
\text{4th term} & a + 3b + 3c + d \\
\text{5th term} & a + 4b + 6c + 4d + e \\
\quad\&\text{c.} & \qquad\qquad \&\text{c.}
\end{array}$$

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84		
1	5	15	35	70	126			
1	6	21	56	126				
1	7	28	84					
1	8	36						
1	9							

manner, he observes, as the same figurate numbers complete the powers raised from a binomial root, as had long before been taught by others. He also remarks, that this rule not only gives any one term, but also the sum of any number of successive terms from the beginning, making the 2d coefficient the first, the 3d the 2d, and so on; thus, the sum of the first 5 terms is $5a + 10b + 10c + 5d + e$.

1st term	a
2d term	$a - b$
3d term	$a - 2b + c$
4th term	$a - 3b + 3c - d$
5th term	$a - 4b + 6c - 4d + e$
&c.	&c.

In the 4th *prop.* it is shown, that if the terms decrease, proceeding from the greater to the less, the same theorems hold good, by only changing the sign of every other term, as in the margin.

Prop. 5 shows how to find any multiple nearly of a given ratio. To do this, take the difference of the terms of the ratio, which multiply by the index of the multiple, from the product subtract the same difference; add half the remainder to the greater term of the ratio, and subtract the same half from the less term, which give two terms expressing the required multiple a little less than the truth.—Thus, to quadruple the ratio $\frac{25}{28}$: the difference of the terms 3 multiplied by 4 makes 12, from which 3 deducted leaves 9, its half $4\frac{1}{2}$ added to the greater term 28 makes $32\frac{1}{2}$, and taken from the less term 25, leave $20\frac{1}{2}$: then $20\frac{1}{2}$ and $32\frac{1}{2}$ are the terms nearly of the quadruple sought, or reduced to whole numbers gives $\frac{41}{65}$, a little less than the truth.

Prop. 6 and 7 treat of the approximate multiplication and division of ratios, or, which is the same thing, the finding nearly any powers or any roots of a given fraction, in an easy manner. The theorem for raising any power, when reduced to a simpler form, is this, the m power of $\frac{a}{b}$, or $(\frac{a}{b})^m$, is $= \frac{s \mp md}{s \pm md}$ nearly, where s is $= a + b$, and $d = a \backsim b$, the sum and difference of the two numbers, and the upper or under signs take place according as $\frac{a}{b}$ is a proper or an improper fraction, that is, according as a is less or greater than b. And the th. for extracting the mth root of $\frac{a}{b}$, is $\sqrt[m]{\frac{a}{b}}$ or $(\frac{a}{b})^{\frac{1}{m}} = \frac{ms \mp d}{ms \pm d}$ nearly; which latter rule is also the same as the former, as will be evident by substituting $\frac{1}{m}$ instead of m in the first theorem. So that universally $(\frac{a}{b})^{\frac{m}{n}}$, is $= \frac{ns \mp md}{ns \pm md}$ nearly. These theorems however are nearly true only in some certain cases, namely when $\frac{a}{b}$ and $\frac{m}{n}$ do not differ greatly from unity. And in the 7th *prop.* the author shows how to find nearly the error of the theorems.

In the 8th *prop.* it is shown, that the measures of ratios of equidifferent terms, are nearly reciprocally as the arithmetical means between the terms of each ratio. So, of the ratios $\frac{16}{18}$, $\frac{33}{35}$, $\frac{50}{52}$, the mean between the terms of the first ratio is 17, of the 2d 34, of the 3d 51, and the measure of the ratios are nearly as $\frac{1}{17}$, $\frac{1}{34}$, $\frac{1}{51}$.

From this property he proceeds, in the 9th *prop.* to find the measure of any ratio less than $\frac{99.5}{100.5}$, which has an equal difference, 1, of terms. In the two examples, mentioned near the beginning, our author found the logarithm, or measure of the ratio, of $\frac{99.5}{100}$, to be $21769\frac{3}{10}$, and that of $\frac{100}{100.5}$ to be $21659\frac{7}{10}$; therefore the sum 43429 is the logarithm of $\frac{99.5}{100.5}$, or $\frac{99.5}{100} \times \frac{100}{100.5}$; or the logarithm of $\frac{99.5}{100.5}$ is nearer 43430, as found by other more accurate computations. Now to find the logarithm of $\frac{100}{101}$, having the same difference of terms, 1, with the former; it will be, by prop. 8, as 100·5 (the mean between 101 and 100) : 100 (the mean between 99·5 and 100·5) : : 43430 : 43213 the logarithm of $\frac{100}{101}$, or the difference between the logarithms of 100 and 101. But the log. of 100 is 2; therefore the logarithm of 101 is 2.0043213.—Again, to find the logarithm of 102, we must first find the logarithm of $\frac{101}{102}$; the mean between its terms being 101·5, therefore as 101·5 : 100 : : 43430 : 42788 the logarithm of $\frac{101}{102}$, or the difference of the logarithms of 101 and 102. But the logarithm of 101 was found above to be 2.0043213; therefore the log. of 102 is 2.0086001.—So that, dividing continually 868596 (the double of 434298 the logarithm of $\frac{99.5}{100.5}$ or $\frac{199}{201}$) by each number of the series 201, 203, 205, 207, &c, then add 2 to the 1st quotient, to the sum add the 2d quotient, and so on, adding always the next quotient to the last sum, the several sums will be the respective logarithms of the numbers in this series 101, 102, 103, 104, &c.

The next, or *prop.* 10, shows that, of two pair of continued ratios, whose terms have equal differences, the difference of the measures of the first two ratios, is to the difference of the measures of the other two, as the square of the common term in the two latter, is to that in the former, nearly. Thus, in the four ratios $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$; as the measure of $\frac{aa+2ab}{(a+b)^2}$ (the difference of the first two, or the quotient of the two fractions): is to the measure of $\frac{aa+8ab+15bb}{(a+4b)^2}$ :: so $(a+4b)^2$: is to $(a+b)^2$, nearly.

In *prop.* 11 the author shows that similar properties take place among two sets of ratios consisting each of 3 or 4 &c continued numbers.

Prop. 12 shows that, of the powers of numbers in arithmetical progression, the orders of differences which become equal, are the 2d differences in the squares, the 3d differences in the cubes, the 4th differences in the 4th powers, &c. And hence it is shown, how to construct all those powers by the continual addition of their differences; as had been long before more fully explained by Briggs.

In the next, or 13th *prop.* our author explains his compendious method of raising the tables of logarithms; showing how to construct the logarithms by addition only, from the properties contained in the 8th, 9th, and 12th props. For this purpose, he makes use of the quantity $\frac{a}{b-c}$, which by division he resolves into this infinite series $\frac{a}{b}+\frac{ac}{bb}+\frac{ac^2}{b^3}+\frac{ac^3}{b^4}$ &c (*in infin.*). Putting then $a=100$, the arithmetical mean between the terms of the ratio $\frac{99\cdot5}{100\cdot5}$, $b = 100000$, and c successively equal to 0·5, 1·5, 2·5, &c, that so $b - c$ may be respectively equal to 99999·5, 99998·5, 99997·5, &c, the corresponding means between the terms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, it is evident that $\frac{a}{b-c}$ will be the quotient of the 2d term divided by the 1st, in the proportions mentioned in the 8th and 9th propositions; and when all of these quotients are found, it remains then only to multiply them by the constant 3d term 43429, or rather 43429·8, of the proportion, to produce the logarithms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, till $\frac{10000}{10001}$; then adding these continually to 4, the logarithm of 10000, the least number, or subtracting them from 5, the logarithm of the highest term 100000, there will result the logarithms of all the absolute numbers from 10000 to 100000. Now when $c = 0\cdot5$, then

$$\frac{a}{b} = \cdot001, \frac{ac}{bb} = \cdot000000005, \frac{ac^2}{b^3} = \cdot000000000000025, \frac{ac^3}{b^4} = \cdot000000000000000000125$$

&c; therefore $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3}$ &c, is $= \cdot001000005000025000125$,

In like manner, if $c = 1\cdot5$, then $\frac{a}{b-c}$ will be $= \cdot001000015000225003375$,

and if $c = 2\cdot5$, then $\frac{a}{b-c}$ will be $= \cdot001000025000625015625$;

&c. But instead of constructing all the values of $\frac{a}{b-c}$ in the usual way of raising the powers, he directs them to be found by addition only, as in the last proposition. Having thus found all the values of $\frac{a}{b-c}$, the author then shows, that they may be drawn into the constant logarithm 43429 by addition only, by the help of the annexed table of its first 9 products.

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

The author then distinguishes which of the logarithms it may be proper to find in this way, and which from their component parts. Of these, the logarithms of all even numbers need not be thus computed, being composed from the number 2; which cuts off one-half of the numbers: neither are those numbers to be computed which end in 5, because 5 is one of their factors;

these last are $\frac{1}{10}$ of the numbers; and the two together $\frac{1}{2}$ + $\frac{1}{10}$ make $\frac{3}{5}$ of the whole: and of the other $\frac{2}{5}$, the $\frac{1}{3}$ of them, or $\frac{2}{15}$ of the whole, are composed of 3; and hence $\frac{3}{5} + \frac{2}{15}$, or $\frac{11}{15}$ of the numbers, are made up of such as are composed of 2, 3, and 5. As to the other numbers which may be composed of 7, of 11, &c; he recommends to find *their* logarithms in the general way, the same as if they were incomposites, as it is not worth while to separate them in so easy a mode of calculation. So that of the 90 chiliads of numbers, from 10000 to 100000, only 24 chiliads are to be computed.— Neither indeed are all of these to be calculated from the foregoing series for $\frac{a}{b-c}$, but only a few of them in that way, and the rest by the proportion in the 8th proposition. Thus, having computed the logarithms of 10003 and 10013, omitting 10023, as being divisible by 3, estimate the logarithms of 10033 and 10043, which are the 30th numbers from 10003 and 10013; and again omitting 10053, a multiple of 3, find the logarithms of 10063 and 10073. Then by prop. 8,

As 10048, the arithmetical mean between 10033 and 10063,
to 10018, the arithmetical mean between 10003 and 10033,
so 13006, the dif. between the logs. of 10003 and 10033,
to 12967, the dif. between the logs. of 10033 and 10063.

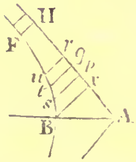
$$\text{That is, 1st, As} \left\{\begin{matrix} 10048 \\ 10078 \\ 10108 \end{matrix}\right\} : 10018 :: 13006 : \left\{\begin{matrix} 12967 \\ \&c. \\ \end{matrix}\right.$$

$$\text{Again, As} \left\{\begin{matrix} 10058 \\ 10088 \\ 10118 \end{matrix}\right\} : 10028 :: 12992 : \left\{\begin{matrix} 12953 \\ \&c. \\ \end{matrix}\right.$$

$$\text{And 3dly, As} \left\{\begin{matrix} 10068 \\ 10098 \\ \&c. \end{matrix}\right\} : 10038 :: 12979 : \left\{\begin{matrix} 12940 \\ \&c. \\ \end{matrix}\right.$$

And with this our author concludes his compendium for constructing the tables of logarithms.

He afterwards shows some applications and relations of the doctrine of logarithms to geometrical figures: in order to which, in *prop.* 14, he proves algebraically that, in the right-angled hyperbola, if from the vertex, and from any other point, there be drawn BI, FH perpendicular to the asymptote AH, or parallel to the other asymptote; then will AH : AI : : BI : FH. And,



In *prop.* 15, if AI = BI = 1, and HI $=a$; then will FH $= \frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5$ &c, *in infinitum*, by a continued algebraic division, the process of which he describes, step by step, as a thing that was new or uncommon. But that method of division had been taught before, by Dr. Wallis in his Opus Arithmeticum.

Prop. 16 is this: Any given number being supposed to be divided into innumerable small equal parts, it is required to assign the sum of any powers of the continual sums of those innumerable parts. For which purpose he lays down this rule; if the next higher power of the given number, above that power whose sum is sought, be divided by its exponent, the quotient will be the sum of the powers sought. That is, if N be the given number, and a one of its innumerable equal parts, then will

$a^n + (2a)^n + (3a)^n + (4a)^n$ &c . . . . N^n be $= \frac{N^{n+1}}{n+1}$: which theorem he demonstrates by a method of induction. And this, it is evident, is the finding the sum of any powers of an infinite number of arithmeticals, of which the greatest term is a given quantity, and the least indefinitely small. It is also remarkable, that the above expression is similar to the rule for finding the fluent to the given fluxion of a power, as afterwards taught by Sir I. Newton.

Mercator then applies this rule, in *prop.* 17, to the quadrature of the hyperbola. Thus, putting AI $= 1$, conceive the asymptote to be divided from I into innumerable equal parts, namely I$p = pq = qr = a$; then, by the 14th and 15th

$$\left.\begin{aligned} ps &= 1 - a + a^2 - a^3 \text{ \&c} \\ qt &= 1 - 2a + 4a^2 - 8a^3 \text{ \&c} \\ ru &= 1 - 3a + 9a^2 - 27a^3 \text{ \&c} \end{aligned}\right\}$$ But the area BIru is = the sum $ps + qt + ru$, which is =

$3 - 6a + 14a^2 - 36a^3$ &c, that is, equal to the number of terms contained in the line Ir, minus the sum of those terms, plus the sum of the squares of the same, minus the sum of their cubes, plus the sum of the 4th powers, &c. Putting now IA = 1, as before, and $Ip = 0{\cdot}1$ the number of terms, to find the area BI*ps*; by prop. 16 the sum of the terms will be $\frac{0{\cdot}1^2}{2} = {\cdot}005$, the sum of their squares = ·000333333, the sum of their cubes ·000025, the sum of the 4th powers ·000002, the sum of the 5th powers ·000000166, the sum of the 6th powers ·000000014, &c. Therefore the area BI*ps* is = ·1 − ·005 + ·000333333 − ·000025 + ·000002 − ·000000166 + ·000000014 &c = ·10033547 − ·005025166 = ·095310181 &c.

Again, putting I*q* = ·21 the number of terms, he finds in like manner the area BI*qt* = ·21 − ·02205 + ·003087 − ·000486202 + ·000081682 − ·000014294 + ·000002572 − ·000000472 + ·000000088&c = ·213171345 − ·022550984 = ·190620361 &c.

He then adds, hence it appears that, as the ratio of AI to A*p*, or 1 to 1·1, is half or subduplicate of the ratio of AI to A*q*, or 1 to 1·21, so the area BI*ps* is here found to be half of the area BI*qt*. These areas he computes to 44 places of figures, and finds them still in the ratio of 2 to 1.

The foregoing doctrine amounts to this, that if the rectangle BI × I*r*, which in this case is expressed by I*r* only, be put = A, AI being = 1, as before; then the area BI*ru*, or the hyperbolic logarithm of 1 + A, or of the ratio of 1 to 1 + A, will be equal to the infinite series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5$ &c; and which therefore may be considered as Mercator's quadrature of the hyperbola, or his general expression of an hyperbolic logarithm in an infinite series. And this method was further improved by Dr. Wallis in the Philos. Trans. for the year 1668.

In *prop.* 18 Mercator compares the hyperbolic *areolæ* with the *ratiunculæ* of equidifferent numbers, and observes that, the areola BI*ps* is the measure of the ratiuncula of AI to A*p*, the areola *spqt* is the measure of the ratiuncula of A*p* to A*q*, the areola *tqru* is the measure of the ratiun. of A*q* to A*r*, &c.

Finally, in the 19th *prop.* he shows how the sums of logarithms may be taken, after the manner of the sums of the *areolæ*. And hence infers, as a corollary, how the continual product of any given numbers in arithmetical progression may be obtained: for the sum of the logarithms is the logarithm of the continual product. He then remarks, that from the premises it appears, in what manner Mersennus's problem may be resolved, if not geometrically, at least in figures to any number of places. And thus closes this ingenious tract.

In the Philos. Trans. for 1668 are also given some further illustrations of this work, by the author himself. And in various places also in a similar manner are logarithms and hyperbolic areas treated of by Lord Brouncker, Dr. Wallis, Sir I. Newton, and many other learned persons.

Of Gregory's Exercitationes Geometricæ.

In the same year 1668 came out Mr. James Gregory's Exercitationes Geometricæ, in which are contained the following pieces:

1, Appendicula ad veram circuli et hyperbolæ quadraturam:

2, N. Mercatoris quadratura hyperbolæ geometricè demonstrata:

3, Analogia inter lineam meridianam planisphærii nautici et tangentes artificiales geometricè demonstrata; seu quod secantium naturalium additio efficiat tangentes artificiales:

4, Item, quot tangentium naturalium additio efficiat secantes artificiales:

5, Quadratura conchoidis:

6, Quadratura cissoidis: et

7, Methodus facilis et accurata componendi secantes et tangentes artificiales.

The first of these pieces, or the Appendicula, contains some further extension and illustration of his Vera circuli et hyperbolæ quadratura, occasioned by the animadversions made on that work by the celebrated mathematician and philosopher Huygens.

In the 2d is demonstrated geometrically, the quadrature of the hyperbola; by which he finds a series similar to Mercator's for the logarithm, or the hyperbolic space beyond the first ordinate (BI, fig. pa. 416). In like manner he finds another series for the space at an equal distance within that ordinate. These two series having all their terms alike, but all the signs of the one plus, and those of the other alternately plus and minus, by adding the two together, every other term is cancelled, and the double of the rest denotes the sum of both spaces. Gregory then applies these properties to the logarithms; the conclusion from all which may be thus briefly expressed:

$$\text{since } A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 \;\&c = \text{the log. of } \frac{1+A}{1},$$

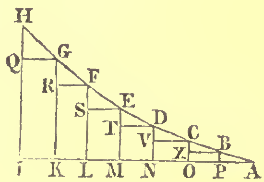
$$\text{and } A + \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 + \tfrac{1}{4}A^4 \;\&c = \text{the log. of } \frac{1}{1-A},$$

$$\text{theref. } 2A + \tfrac{2}{3}A^3 + \tfrac{2}{5}A^5 + \tfrac{2}{7}A^7 \;\&c = \text{the log. of } \frac{1+A}{1-A},$$

or of the ratio of $1 - A$ to $1 + A$. Which may be accounted Gregory's method of making logarithms.

The remainder of this little volume is chiefly employed about the nautical meridian, and the logarithmic tangents and secants. It does not appear by whom, nor by what accident, was discovered the analogy between a scale of logarithmic tangents and Wright's protraction of the nautical meridian line, which consisted of the sums of the secants. It appears however to have been first published, and introduced into the practice of navigation, by Henry Bond, who mentions this property in an edition of Norwood's Epitome of Navigation, printed about 1645; and he again treats of it more fully in an edition of Gunter's works, printed in 1653, where he teaches, from this property, to resolve all the cases of Mercator's sailing by the logarithmic tangents, independent of the table of meridional parts. This analogy had only been found to be nearly true by trials, but not demonstrated to be a mathematical property. Such demonstration seems to have been first discovered by Nicholas Mercator, who, desirous of making the most advantage of this and another concealed invention of his in navigation, by a paper in the Philos. Trans.

for June 4, 1666, invites the public to enter into a wager with him, on his ability to prove the truth or falsehood of the supposed analogy. This mercenary proposal however seems not to have been taken up by any one, and Mercator reserved his demonstration. The proposal however excited the attention of mathematicians to the subject itself, and a demonstration was not long wanting. The first was published about two years after by Gregory, in the tract now under consideration, and from thence and other similar properties, here demonstrated, he shows, in the last article, how the tables of logarithmic tangents and secants may easily be computed, from the natural tangents and secants. The substance of which is as follows:



Let AI be the arc of a quadrant, extended in a right line, and let the figure AHI be composed of the natural tangents of every arc from the point A, erected perpendicular to AI at their respective points: let AP, PO, ON, NM, &c, be the very small equal parts into which the quadrant is divided, namely, each $\frac{1}{60}$, or $\frac{1}{100}$ of a degree; draw PB, OC, ND, ME, &c, perpendicular to AI. Then it is manifest, from what had been demonstrated, that the figures ABP, ACO, &c, are the artificial secants of the arcs AP, AO, &c, putting 0 for the artificial radius. It is also manifest, that the rectangles BO, CN, DM, &c, will be found from the multiplication of the small part AP of the quadrant by each natural tangent. But, he proceeds, there is a little more difficulty in measuring the figures ABP, BCX, CDV, &c; for if the first differences of the tangents be equal, AB, BC, CD, &c, will not differ from right lines, and then the figures ABP, BCX, CDV, &c, will be right-angled triangles, and therefore any one, as HQG, will be = $\frac{1}{2}$QH × QG: but if the second differences be equal, the said figures will be portions of trilineal quadratices; for example, HQG will be a portion of a trilineal quadratix, whose axis is parallel to QH; and each of the last differences being z, it will be QHG $= \frac{1}{2}$QH × QG $- \frac{1}{12}$Z × QG; and if the 3d differences be equal, the said figures will be portions of trilineal cubices, and then shall QHG be equal $\frac{1}{2}$QH × QG $- \surd(\frac{1}{72}$QH × Z × QG$^2 - \frac{1}{1728}$Z^2 × QG$^2)$: when the 4th differences are equal, the said figures are portions of trilineal quadrato-quadratices, and the 4th differences are equal to 24 times the 4th power of QG, divided by the cube of the latus rectum; also when the 5th differences are equal, the said figures are portions of trilineal sursolids, and the 5th differences are equal to 120 times the sursolid of QG, divided by the 4th power of the latus rectum; and so on *in infinitum*. What has been here said of the composition of artificial secants from the natural tangents, it is remarked, may in like manner be understood of the composition of artificial tangents, from the natural secants, according to what was before demonstrated. It is also observed, that the artificial tangents and secants are computed, as above, on the supposition that 0 is the log. of 1, and 1000000000000 the radius, and 23025850929940456240178 70 the log. of 10; but that they may be more easily computed, namely by addition only, by putting $\frac{1}{60}$ of a degree $=$QG$=$AP$=1$, and the logarithm of 10 $=$ 7915704467897819; for by this means $\frac{1}{2}$QH × QG is $= \frac{1}{2}$QH $=$ QHG, and $\frac{1}{2}$QH × QG $- \frac{1}{12}$Z × QG $= \frac{1}{2}$QH $- \frac{1}{12}$Z $=$ QHG, also $\frac{1}{2}$QH × QG $- \surd(\frac{1}{72}$QH × Z × QG$^2 - \frac{1}{1728}$Z^2 × QG$^2)$ $= \frac{1}{2}$QH $- \surd(\frac{1}{72}$QH × Z $- \frac{1}{1728}$Z$^2)$ $=$ QHG: And finally, by one division only are found the artificial tangents and secants to 1000000000000000, the logarithm of 10, putting still 1 for radius, which are the differences of the artificial tangents and secants, in the table, from that artificial radius; and to make the operations easier in multiplying by the number 7915704467897819, or logarithm of 10, a table is set down of its products by the first 9 figures. But if AP or QG be $= \frac{1}{100}$ of a degree, the artificial tangents and secants will answer to 13192840779829703 as the logarithm of 10, the first 9 multiples of which are also placed in the table. But to represent the numbers by the artificial radius, rather than by the logarithm of 10, the author directs to add ciphers, &c.—And so much for Gregory's Exercitationes Geometricæ.

The same analogy between the logarithmic tangents and the meridian line, as also other similar properties, were afterwards more elegantly demonstrated by Dr. Halley in the Philos. Trans. for Feb. 1696, and various methods given for computing the same, by examining the nature of the spirals into which the rhumbs are transformed in the stereographical projection of the sphere, on the plane of the equator: the doctrine of which was rendered still more easy and elegant by the ingenious Mr. Cotes, in his Logometria, first printed in the Philos. Trans. for 1714, and afterwards in the collection of his works published in 1732, by his cousin Dr. Robert Smith, who succeeded him in the Plumian professorship of philosophy in the University of Cambridge.

The learned Dr. Isaac Barrow also, in his Lectiones Geometricæ, lect. xi. Append. first published in 1672, delivers a similar property, namely, that the sum of all the secants of any arc is analogous to the logarithm of the ratio of $r + s$ to $r - s$, or radius plus sine to radius minus sine; or, which is the same thing, that the meridional parts answering to any degree of latitude, are as the logarithms of the ratios of the versed sines of the distances from the two poles.

Mr. Gregory's method for making logarithms was further exemplified in numbers, in a small tract on this subject, printed in 1688, by one Euclid Speidell, a simple and illiterate person, and son of John Speidell, before mentioned among the first writers on logarithms.

Gregory also invented many other infinite series, and among them these here following, viz. a being an arc, t its tangent, and s the secant, to the radius r; then is

$$a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \text{ \&c.}$$

$$t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \text{ \&c.}$$

$$s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \text{ \&c.}$$

And if τ and σ denote the artificial or logarithmic tangent and secant of the same arc a, the whole quadrant being q, and $e = 2a - q$; then is

$$e = \tau - \frac{\tau^3}{6r^2} + \frac{\tau^5}{24r^4} - \frac{61\tau^7}{5040r^6} + \frac{277\tau^9}{72576r^8} \; \&c.$$

$$\tau = e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \; \&c.$$

$$\sigma = \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \; \&c.$$

And if s denote the artificial secant of 45°, and s + l the artificial secant of any arc a, the artificial radius being 0; then is

$$a = \tfrac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5} \; \&c.$$

The investigation of all which series may be seen at pa. 298 et seq. vol. 1, Dr. Horsley's commentary on Sir I. Newton's works, as they were given in the Commercium Epistolicum, no. xx, without demonstration, and where the number 2 is also wanting in the denominator of the first term of the series expressing the value of σ.

Such then were the ways in which Mercator and Gregory applied these their very simple series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4$ &c, and $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4$&c, for the purpose of computing logarithms. But they might, as I apprehend, have applied them to this purpose in a shorter and more direct manner, by computing, by their means, only a few logarithms of small ratios, in which the terms of the series would have decreased by the powers of 10, or some greater number, the numerators of all the terms being unity, and their denominators the powers of 10 or some greater number, and then employing these few logarithms, so computed, to the finding the logarithms of other and greater ratios, by the easy operations of mere addition and subtraction. This might have been done for the logarithms of the ratios of the first ten numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, to 1, in the following manner, communicated by Mr. Baron Maseres.

In the first place, the logarithm of the ratio of 10 to 9, or of 1 to $\frac{9}{10}$, or of 1 to $1 - \frac{1}{10}$, is equal to the series

$$\frac{1}{10} + \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} + \frac{1}{4 \times 10000} + \frac{1}{5 \times 100000} \&c.$$

In like manner are easily found the logarithms of the ratios of 11 to 10; and then, by the same series, those of 121 to 120, and of 81 to 80, and of 2401 to 2400; in all which cases the series would converge still faster than in the two first cases. We may then proceed by mere addition and subtraction of logarithms, as follows;

$$\begin{array}{l|l}
\text{Log.}\tfrac{11}{9} = \text{L.}\tfrac{11}{10} + \text{L.}\tfrac{10}{9}, & \text{L.}\tfrac{10}{4} = \text{L.}\tfrac{10}{9} + \text{L.}\tfrac{9}{4}, \\
\text{L.}\tfrac{121}{81} = 2\text{L.}\tfrac{11}{9}, & \text{L.}\tfrac{81}{16} = 2\text{L.}\tfrac{9}{4}, \\
\text{L.}\tfrac{121}{80} = \text{L.}\tfrac{121}{81} + \text{L.}\tfrac{81}{80}, & \text{L.}\tfrac{80}{16} = \text{L.}\tfrac{81}{16} - \text{L.}\tfrac{81}{80}, \\
\text{L.}\tfrac{120}{80} = \text{L.}\tfrac{121}{80} - \text{L.}\tfrac{121}{120}, & \text{L.}\tfrac{5}{1} = \text{L.}\tfrac{80}{16}, \\
\text{L.}\tfrac{120}{80} = \text{L.}\tfrac{3}{2}, & \text{L.}\tfrac{5}{2} = \text{L.}\tfrac{10}{4}, \\
\text{L.}\tfrac{9}{4} = 2\text{L.}\tfrac{3}{2}, & \text{L.}\tfrac{2}{1} = \text{L.}\tfrac{5}{1} - \text{L.}\tfrac{5}{2}.
\end{array}$$

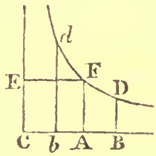
Having thus got the logarithm of the ratio of 2 to 1, or, in common language, the logarithm of 2, the logarithms of all sorts of even numbers may be derived from those of the odd numbers, which are their coefficients, with 2 or its powers. We may then proceed as follows:

$$\begin{array}{l|l}
\text{L. } 4 = 2\text{L. } 2, & \text{L. } 24 = \text{L. } 8 + \text{L. } 3, \\
\text{L. } 10 = \text{L.}\tfrac{10}{4} + \text{L. } 4, & \text{L.}2400 = \text{L. } 100 + \text{L. } 24, \\
\text{L. } 9 = \text{L.}\tfrac{9}{4} + \text{L. } 4, & \text{L.}2401 = \text{L.}\tfrac{2401}{2400} + \text{L. } 2400, \\
\text{L. } 3 = \tfrac{1}{2}\text{L. } 9, & \text{L. } 7 = \tfrac{1}{4}\text{L. } 2401, \\
\text{L. } 100 = 2\text{L. } 10, & \text{L. } 11 = \text{L.}\tfrac{11}{9} + \text{L. } 9, \\
\text{L. } 8 = 3\text{L. } 2, & \text{L. } 6 = \text{L. } 2 + \text{L. } 3.
\end{array}$$

Thus we have got the logarithms of 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. And this is, upon the whole, perhaps the best method of computing logarithms that can be taken. There have been indeed some methods discovered by Dr. Halley, and other mathematicians, for computing the logarithms of the ratios of prime numbers, to the next adjacent even numbers, which are still shorter than the application of the foregoing series. But those methods are less simple and easy to understand, and apply, than these series; and the computation of logarithms by these series, when their terms decrease by the powers of 10, or of some greater number, is so very short and easy (as we have seen in the foregoing computations of the logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, &c,) that it is not worth while to seek for any shorter methods of computing them. And this method of computing logarithms is very nearly the same with that of Sir I. Newton, in his second letter to Mr. Oldenburg, dated October 1676, as will be seen in the following article.

Of Sir Isaac Newton's Methods.

The excellent Sir I. Newton greatly improved the quadrature of the hyperbolical-asymptotic spaces by infinite series, derived from the general quadrature of curves by his method of fluxions; or rather indeed he invented that method himself, and the construction of logarithms derived from it, in the year 1665 or 1666, before the publication of either Mercator's or Gregory's books, as appears by his letter to Mr. Oldenburg dated October 24, 1676, printed in p. 634 *et seq.* vol. 3, of Wallis's works, and elsewhere. The quadrature of the hyperbola, thence translated, is to this effect. Let *d*FD be an hyperbola, whose centre is C, vertex F, and interposed square CAFE = 1. In CA take AB and A*b* on each side $= \frac{1}{10}$ or 0·1: And, erecting the perpendiculars BD, *bd*; half the sum of the spaces AD and A*d* will be $= 0{\cdot}1 + \frac{0{\cdot}001}{3} + \frac{0{\cdot}00001}{5} + \frac{0{\cdot}0000001}{7}$ &c. and the half diff. $= \frac{0{\cdot}01}{2} + \frac{0{\cdot}0001}{4} + \frac{0{\cdot}000001}{6} + \frac{0{\cdot}00000001}{8}$ &c. Which reduced will stand thus,



1·000000000000,	0·005000000000	The sum of these 0·1053605156577 is A*d*,
3333333333	250000000	and the differ. 0·0953101798043 is AD.
20000000	1666666	In like manner, putting AB and A*b*
142857	12500	each = 0·2, there is obtained
1111	100	A*d* = 0·2231435513142, and
9	1	AD = 0·1823215567939.
0·100335347731 0,	0·0050251679267	

Having thus the hyperbolic logarithms of the four decimal numbers 0·8, 0·9, 1·1, and 1·2; and since $\frac{1{\cdot}2}{0{\cdot}8} \times \frac{1{\cdot}2}{0{\cdot}9} = 2$, and 0·8 and 0·9 are less than unity; adding their logarithms to double the logarithm of 1·2, we have 0·6931471805597, the hyperbolic logarithm of 2. To the triple of this adding the log. of 0·8, because $\frac{2 \times 2 \times 2}{0{\cdot}8} = 10$, we have 2·3025850929933, the logarithm of 10. Hence by one addition are found the logarithms of 9 and 11: And thus the logarithms of all these prime numbers, 2, 3, 5, 11 are prepared. Further, by only depressing the numbers, above computed, lower in the decimal places, and adding, are obtained the logarithms of the decimals 0·98, 0·99, 1·01, 1·02; as also of these 0·998, 0·999, 1·001, 1·002. And hence, by addition and subtraction, will arise the logarithms of the primes 7, 13, 17, 37, &c. All which logarithms being divided by the above logarithm of 10, give the common logarithms to be inserted in the table.

And again, a few pages further on, in the same letter, he resumes the construction of logarithms, thus: Having found, as above, the hyperbolic logarithms of 10, 0·98, 0·99, 1·01, 1·02, which may be effected in an hour or two, dividing the last four logarithms by the logarithm of 10, and adding the index 2, we have the tabular logarithms of 98, 99, 100, 101, 102. Then, by interpolating nine means between each of these, will be obtained the logarithms of all numbers between 980 and 1020; and again interpolating 9 means between every two numbers from 980 to 1000, the table will be so far constructed. Then from these will be collected the logarithms of all the primes under 100, together with those of their multiples: all which will require only addition and subtraction; for

$$\sqrt[10]{\frac{9984 \times 1020}{9945}} = 2;\ \frac{10}{2} = 5;\ \sqrt{\frac{98}{2}} = 7;\ \frac{99}{9} = 11;\ \frac{1001}{7 \times 11} = 13;\ \frac{102}{6} = 17;$$

$$\frac{988}{4 \times 13} = 19;\ \frac{9936}{16 \times 27} = 23;\ \frac{986}{2 \times 17} = 29;\ \frac{992}{32} = 31;\ \frac{999}{27} = 37;\ \frac{984}{24} = 41;$$

$$\frac{989}{23} = 43;\ \frac{987}{27} = 47;\ \frac{9911}{11 \times 17} = 53;\ \frac{9971}{13 \times 13} = 59;\ \frac{9882}{2 \times 81} = 61;\ \frac{9849}{3 \times 49} = 67;$$

$$\frac{994}{14} = 71;\ \frac{9928}{8 \times 17} = 73;\ \frac{9954}{7 \times 18} = 79;\ \frac{996}{12} = 83;\ \frac{9968}{7 \times 16} = 39:\ \frac{9894}{6 \times 17} = 97.$$

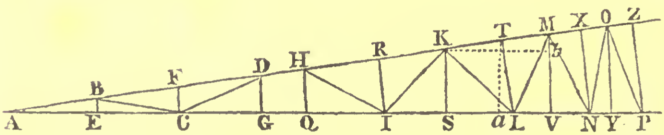
This quadrature of the hyperbola, and its application to the construction of logarithms, are still further explained by our celebrated author, in his treatise on Fluxions, published by Mr. Colson in 1736, where he gives all the three series for the areas AD, Ad, Bd, in general terms, the former the same as that published by Mercator, and the latter by Gregory; and he explains the manner of deriving the latter series from the former, namely by uniting together the two series for the spaces on each side of an ordinate, bounded by other ordinates at equal distances, every 2d term of each series is cancelled, and the result is a series converging much quicker than either of the former. And, in this treatise on fluxions, as well as in the letter before quoted, he recommends this as the most convenient way of raising a canon of logs. computing by the series the hyperbolic spaces answering to the prime numbers 2, 3, 5, 7, 11, &c, and dividing them by 2·302585092994045, which is the area corresponding to the number 10, or else multiplying them by its reciprocal 0·4342944819032518, for the common logarithms. "Then the logarithms of all the numbers in the canon which are made by the multiplication of these, are to be found by the addition of their logarithms, as is usual. And the void places are to be interpolated afterwards by the help of this theorem: Let n be a number to which a logarithm is to be adapted, x the difference between that and the two nearest numbers equally distant on each side, whose logarithms are already found, and let d be half the difference of the logarithms; then the required logarithm of the number n will be obtained by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c to the logarithm of the less number." This theorem he demonstrates by the hyperbolic areas, and then proceeds thus; "The two first terms $d + \frac{dx}{2n}$ of this series I think to be accurate enough for the construction of a canon of logarithms, even though they were to be produced to 14 or 15 figures; provided the number whose logarithm is to be found be not less than 1000. And this can give little trouble in the calculation, because x is generally an unit, or the number 2. Yet it is not necessary to interpolate all the places by the help of this rule. For the logarithms of numbers which are produced by the multiplication or division of the number last found, may be obtained by the numbers whose logarithms were had before, by the addition or subtraction of their logarithms.— Moreover, by the differences of the logarithms, and by their 2d and 3d differences, if there be occasion, the void places may be more expeditiously supplied; the foregoing rule being to be applied only when the continuation of some full places is wanted, in order to obtain those differences, &c." So that Sir I. Newton of himself discovered all the series for the above quadrature, which were found out, and afterwards published, partly by Mercator and partly by Gregory; and these we may here exhibit in one view all together, and that in a general manner for any hyperbola, namely putting CA $= a$, AF $= b$, and AB $=$ Ab $= x$; then will BD $= \frac{ab}{a+x}$, and $bd = \frac{ab}{a-x}$; whence the areas are as below, viz.

$$\text{AD} = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{A}d = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{B}d = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6} + \frac{2bx^9}{9a^8} \text{ \&c.}$$

In the same letter also, above quoted, to Mr. Oldenburg, our illustrious author teaches a method of constructing the trigonometrical canon of sines, by an easier method of multiple angles than that before delivered by Briggs, for the same purpose, because that in Sir Isaac's way radius or 1 is the first term, and double the sine or cosine of the first given angle is the 2d term, of all the proportions, by which the several successive multiple sines or cosines are found. The substance of the method is thus: The best foundation for the construction of the table of sines, is the continual addition of a given angle to itself, or to another given angle. As, if the angle A be to be added;



inscribe HI, IK, KL, LM, MN, NO, OP, &c, each equal to the radius AB; and to the opposite sides draw the perpendiculars BE, HQ, IR, KS, LT, MV, NX, OY, &c; so shall the angle A be the common difference of the angles HIQ, IKH, KLI, LMK, &c; their sines HQ, IR, KS, &c; and their cosines IQ, KR, LS, &c. Now let any one of them LMK, be given, then the rest will be thus found: Draw Ta and Kb perpendicular to SV and MV;

now because of the equiangular triangles ABE, TLa, KMb, ALT, AMV, &c, it will be AB : AE :: KT : Sa ($=\frac{1}{2}$LV $+\frac{1}{2}$LS) :: LT : Ta ($=\frac{1}{2}$MV $+\frac{1}{2}$KS,) and AB : BE :: LT : La ($=\frac{1}{2}$LS $-\frac{1}{2}$LV) :: KT ($=\frac{1}{2}$KM) : $\frac{1}{2}$Mb ($=\frac{1}{2}$MV $-\frac{1}{2}$KS.) Hence are given the sines and cosines KS, MV, LS, LV. And the method of continuing the progressions is evident. Namely,

$$\text{as AB : 2AE ::} \begin{cases} \text{LV : MT + MX :: MX : NV + NY \&c,} \\ \text{MV : NX + LT :: NX : OY + MV \&c,} \end{cases}$$

$$\text{or AB : 2BE ::} \begin{cases} \text{LV : NX − LT :: MX : OY − MV \&c,} \\ \text{MV : MT − MX :: NX : NV − NY \&c.} \end{cases}$$

And, on the other hand, AB : 2AE :: LS : KT + KR &c. Therefore put AB $= 1$, and make BE $\times$ LT $=$ La, AE $\times$ KT $=$ Sa, Sa $-$ La $=$ LV, 2AE $\times$ LV $-$ TM $=$ MX, &c.

The sense of these general theorems is this, that if p be any one among a series of angles in arithmetical progression, the angle d being their common difference, then as radius or

$$1 : 2\cos. d :: \begin{cases} \cos. p : \cos. p + d + \cos. p - d, \\ \sin. p : \sin. p + d + \sin. p - d, \end{cases}$$

$$1 : 2\sin. d :: \begin{cases} \cos. p : \sin. p + d - \sin. p - d, \\ \sin. p : \cos. p + d - \cos. p - d; \end{cases}$$

where the 4th terms of these proportions are the sums or differences of the sines or cosines of the two angles next less and greater than any angle p in the series; and therefore, subtracting the less extreme from the sum, or adding it to the difference, the result will be the greater extreme, or the next sine or cosine beyond that of the term p. And in the same manner are all the rest to be found. This method, it is evident, is equally applicable, whether the common difference d, or angle A, be equal to one term of the series or not: when it *is* one of the terms, then the whole series of sines and cosines becomes thus, viz, as $1 : 2 \cos. d$ ::

$\sin. d : \sin. 2d$ :: $\sin. 2d : \sin. d + \sin. 3d$ :: $\sin. 3d : \sin. 2d + \sin. 4d$ &c.

$\cos. d : 1 + \cos. 2d$ :: $\cos. 2d : \cos. d + \cos. 3d$ :: $\cos. 3d : \cos. 2d + \cos. 4d$ &c.

which is the very method contained in the directions given by Abraham Sharp, for constructing the canon of sines.

Sir I. Newton remarks, that it only remains to find the sine and cosine of a first angle A, by some other method; and for this purpose, he directs to make use of some of his own infinite series: thus, by them will be found 1·57079&c for the quadrantal arc, the square of which is 2·4694&c; divide this square by the square of the number expressing the ratio of 90 degrees to the angle A, calling the quotient z; then 3 or 4 terms of this series $1 - \frac{z}{2} + \frac{z^2}{24} - \frac{z^3}{720} + \frac{z^4}{40320}$ &c, will give the cosine of that angle A. Thus we may first find an angle of 5 degrees, and thence the table be computed to the series of every 5 degrees; then these interpolated to degrees or half degrees by the same method, and these interpolated again; and so on as far as necessary. But two-thirds of the table being computed in this manner, the remaining third will be found by addition or subtraction only, as is well known.

Various other improvements in logarithms and trigonometry are owing to the same excellent personage; such as, the series for expressing the relation between circular arcs and their sines, cosines, versed-sines, tangents, &c; namely, the arc being a, the sine s, the versed-sine v, cosine c, tangent t, radius 1, then is

$$a = s + \tfrac{1}{6}s^3 + \tfrac{3}{40}s^5 + \tfrac{5}{112}s^7 + \tfrac{35}{1152}s^9 + \text{\&c.}$$
$$a = v^{\frac{1}{2}} + \tfrac{1}{6}v^{\frac{3}{2}} + \tfrac{3}{40}v^{\frac{5}{2}} + \tfrac{5}{112}v^{\frac{7}{2}} + \tfrac{35}{1152}v^{\frac{9}{2}} + \text{\&c.}$$
$$a = t - \tfrac{1}{3}t^3 + \tfrac{2}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \text{\&c.}$$
$$s = a - \tfrac{1}{6}a^3 + \tfrac{1}{120}a^5 - \tfrac{1}{5040}a^7 + \tfrac{1}{362880}a^9 - \text{\&c.}$$
$$c = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \tfrac{1}{720}a^6 + \tfrac{1}{40320}a^8 - \text{\&c.}$$
$$v = \tfrac{1}{2}a^2 - \tfrac{1}{24}a^4 + \tfrac{1}{720}a^6 - \tfrac{1}{40320}a^8 + \tfrac{1}{3628800}a^{10} - \text{\&c.}$$
$$t = a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 + \tfrac{17}{315}a^7 + \tfrac{62}{2835}a^9 + \text{\&c.}$$

Of Dr. Halley's Method.

Many other improvements in the construction of logarithms are also derived from the same doctrine of fluxions, as we shall show hereafter. In the mean time proceed we to the ingenious method of the learned Dr. Edmund Halley, secretary to the Royal Society, and the second astronomer royal, having succeeded Mr. Flamsteed in that honourable office in the year 1719, at the Royal Observatory at Greenwich, where he died the 14th January 1742, in the 86th year of his age. His method was first printed in the Philosophical Transactions for the year 1695, and it is entitled "A most compendious and facile method for constructing the logarithms, exemplified and demonstrated from the nature of numbers, without any regard to the hyperbola, with a speedy method for finding the number from the given logarithm."

Instead of the more ordinary definition of logarithms, as *numerorum proportionalium æquidifferentes comites*, in this tract our learned author adopts this other, *numeri rationem exponentes*, as being better adapted to the principle on which logarithms are here constructed, where those quantities are not considered as the logarithms of the numbers, for example, of 2, or of 3, or of 10, but as the logarithms of the ratios of 1 to 2, or 1 to 3, or 1 to 10. In this consideration he first pursues the idea of Kepler and Mercator, remarking that any such ratio is proportional to, and is measured by, the number of equal ratiunculæ contained in each; which ratiunculæ are to be understood as in a continued scale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals, is to that infinite number of the like and equal ratiunculæ between any other two terms, as the logarithm of the one ratio, is to the logarithm of the other: thus, if there be supposed between 1 and 10 an infinite scale of mean proportionals, whose number is 100000 &c in infinitum; then between 1 and 2 there will be 30102 &c of such proportionals; and between 1 and 3 there will be 47712 &c of them; which numbers therefore are the logarithms of the ratios of 1 to 10, 1 to 2, and 1 to 3. But for the sake of *his* mode of constructing logarithms, he changes this idea of *equal* ratiunculæ, for that of other ratiunculæ, so constituted, as that the *same* infinite number of them shall be contained in the ratio of 1 to every other number whatever; and that therefore these latter ratiunculæ will be of *unequal* or different magnitudes in all the different ratios, and in such sort, that in any one ratio, the *magnitude* of each of the ratiunculæ in this latter case, will be as the *number* of them in the former. And therefore, if between 1 and any number proposed, there be taken any infinity of mean proportionals, the infinitely small augment or decrement of the first of those means from the first term 1, will be a ratiuncula of the ratio of 1 to the said number; and as the number of all the ratiunculæ in these continued proportionals is the same, their sum, or the whole ratio, will be directly proportional to the magnitude of one of the said ratiunculæ in each ratio. But it is also evident that the first of any number of means, between 1 and any number, is always equal to such root of that number, whose index is expressed by the number of those proportionals from 1: so, if m denote the number of proportionals from 1, then the first term after 1 will be the mth root of that number. Hence, the indefinite root of any number being extracted, the differentiola of the said root from unity, shall be as the logarithm of that number. So if there be required the log. of the ratio of 1 to $1+q$; the first term after 1 will be $(1+q)^{\frac{1}{m}}$, and theref. the required log. will be as $(1+q)^{\frac{1}{m}}-1$. But, $(1+q)^{\frac{1}{m}}$ is $= 1 + \frac{1}{m}q + \frac{1}{m}\cdot\frac{1-m}{2m}q^2 + \frac{1}{m}\cdot\frac{1-m}{2m}\cdot\frac{1-2m}{3m}q^3$ &c; or by omitting the 1 in the compound numerators, as infinitely small in respect of the infinite number m, the same series will become $1+\frac{1}{m}q+\frac{1}{m}\cdot\frac{-m}{2m}q^2+\frac{1}{m}\cdot\frac{-m}{2m}\cdot\frac{-2m}{3m}q^3$ &c, or by abbreviation it is $1+\frac{1}{m}q-\frac{1}{2m}q^2+\frac{1}{3m}q^3-\frac{1}{4m}q^4$ &c; and hence, finding the differentiola by subtracting 1, the logarithm of the ratio of 1 to $1+q$ is as $\frac{1}{m}\times(q-\frac{1}{2}q^2+\frac{1}{3}q^3-\frac{1}{4}q^4+\frac{1}{5}q^5-\frac{1}{6}q^6$ &c.) Now the index m may be taken equal to any infinite number, and thus all the varieties of scales of logarithms may be produced: so, if m be taken 1000000&c, the theorem will give Napier's logarithms; but if m be taken equal to 230258&c, there will arise Briggs's logarithms.

This theorem being for the increasing ratio of 1 to $1+q$: if that for the decreasing ratio of 1 to $1-q$ be also sought, it will be obtained by a proper change of the signs, by which the decrement of the first of the infinite number of proportionals, will be found to be $\frac{1}{m}$ into $q+\frac{1}{2}q^2+\frac{1}{3}q^3+\frac{1}{4}q^4$ &c, which therefore is as the logarithm of the ratio of 1 to $1-q$.

Hence the terms of any ratio being a and b, q becomes $\frac{b-a}{a}$, or the difference divided by the less term, when it is an increasing ratio; or $q = \frac{b-a}{b}$ when the ratio is decreasing, or as b to a. Therefore the logarithm of the same ratio may be doubly expressed; for, putting x for the difference $b - a$ of the terms, it will be

$$\text{either } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{\&c.}$$

$$\text{or } \frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \text{\&c.}$$

But if the ratio of a to b be supposed divided into two parts, namely, into the ratio of a to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$, and the ratio of $\frac{1}{2}z$ to b, then will the sum of the logarithms of those two ratios be the logarithm of the ratio of a to b. Now by substituting in the foregoing series, the logarithms of those two ratios will

$$\text{be } \frac{1}{m} \text{ into } \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c.}$$

$$\text{and } \frac{1}{m} \text{ into } \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c; and hence the sum,}$$

$$\text{or } \frac{1}{m} \text{ into } \frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \frac{2x^9}{9z^9} + \text{\&c,}$$

will be the logarithm of the ratio of a to b.

Further, if from the logarithm of the ratio of a to $\frac{1}{2}z$, be taken that of $\frac{1}{2}z$ to b, we shall have the logarithm of the ratio of ab to $\frac{1}{4}z^2$; and the half of this gives that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical mean. And consequently the logarithm of this ratio will be equal to half the difference of that of the above two ratios, and will therefore be $\frac{1}{m}$ into $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \text{\&c.}$

The above series are similar to some that were before given by Newton and Gregory, for the same purpose, deduced from the consideration of the hyperbola. But the rule which is properly our author's own, is that which follows, and is derived from the series above given for the logarithm of the sum of two ratios. For the ratio of ab to $\frac{1}{4}z^2$ or $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$, having the difference of its terms $\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$ or $(\frac{1}{2}b - \frac{1}{2}a)^2$ or $\frac{1}{4}x^2$, which in the case of finding the logs. of prime numbers is always 1, if we call the sum of the terms $\frac{1}{4}z^2 + ab = y^2$, the log. of the ra. of $\sqrt{ab}$ to $\frac{1}{2}a+\frac{1}{2}b$ or $\frac{1}{2}z$ will be found to be

$$\frac{1}{m} \text{ into } \frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \&c.$$

And these rules our learned author exemplifies by some cases in numbers, to show the easiest mode of application in practice.

Again, by means of the same binomial theorem he resolves, with equal facility, the reverse of the problem, namely, from the log. given, to find its number or ratio: For, as the log. of the ratio of 1 to $1+q$ was proved to be $(1+q)^{\frac{1}{m}}-1$, and that of the ratio of 1 to $1-q$ to be $1-(1-q)^{\frac{1}{m}}$; hence, calling the given logarithm L, in the former

case it will be $(1+q)^{\frac{1}{m}} = 1 + \text{L}$,

and in the latter $(1-q)^{\frac{1}{m}} = 1 - \text{L}$;

and therefore $\left.\begin{array}{l} 1+q = (1+\text{L})^m \\ \text{and } 1-q = (1-\text{L})^m \end{array}\right\}$ that is, by the binomial theorem,

$$1+q = 1+m\text{L}+\tfrac{1}{2}m^2\text{L}^2+\tfrac{1}{6}m^3\text{L}^3+\tfrac{1}{24}m^4\text{L}^4+\tfrac{1}{120}m^5\text{L}^5 \ \&c,$$

and $1-q = 1-m\text{L}+\tfrac{1}{2}m^2\text{L}^2-\tfrac{1}{6}m^3\text{L}^3+\tfrac{1}{24}m^4\text{L}^4-\tfrac{1}{120}m^5\text{L}^5$ &c,

m being any infinite index whatever, differing according to the scale of logarithms, being 1000&c in Napier's or the hyperbolic logarithms, and 2302585&c in Briggs's.

If one term of the ratio, of which L is the logarithm, be given, the other term will be easily obtained by the same rule: For if L be Napier's logarithm, of the ratio of a the less term, to b the greater, then, according as a or b is given, we shall have,

$$b = a \text{ into } 1 + \text{L} + \tfrac{1}{2}\text{L}^2 + \tfrac{1}{6}\text{L}^3 + \tfrac{1}{24}\text{L}^4 + \&c,$$

$$a = b \text{ into } 1 - \text{L} + \tfrac{1}{2}\text{L}^2 - \tfrac{1}{6}\text{L}^3 + \tfrac{1}{24}\text{L}^4 - \&c.$$

Hence, by help of the logarithms contained in the tables, may easily be found the number to any given log. to a great extent. For if the small difference between the given log. L and the nearest tabular logarithm, either greater or less, be called l, and the number answering to the tabular logarithm a, when it is less than the given logarithm, but b when greater; it will follow, that the number answering to the log. L, will be

$$\text{either } a \text{ into } 1 + l + \tfrac{1}{2}l^2 + \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 + \tfrac{1}{120}l^5 + \&c,$$

$$\text{or } b \text{ into } 1 - l + \tfrac{1}{2}l^2 - \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 - \tfrac{1}{120}l^5 + \&c,$$

which series converge so quickly, l being always very small, that the first two terms $1 \pm l$ are generally sufficient to find the number to 10 places of figures.

Dr. Halley subjoins also an easy approximation for these series; by which it appears, that the number answering to the log. is nearly $\frac{1+\frac{1}{2}l}{1-\frac{1}{2}l} \times a$ or $\frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} \times b$ in Napier's logs.; and $\frac{n+\frac{1}{2}l}{n-\frac{1}{2}l} \times a$ or $\frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times b$ in Briggs's logarithms; where n is $=$ $434294481903\&c = \frac{1}{m}$.

Of Mr. Sharp's Methods.

The labours of Mr. Abraham Sharp, of Little-Horton, near Bradford in Yorkshire, in this branch of mathematics, were very great and meritorious. His merit however consisted rather in the improvement and illustration of the methods of former writers, than in the invention of any new ones of his own. In this way he greatly extended and improved Dr. Halley's method, above described, as also those of Mercator and Wallis; illustrating these improvements by extensive calculations, and by them computing table 5 of my collection of Mathematical Tables, consisting of the logarithms of all numbers to 100, and of all prime numbers to 1100, each to 61 places. He also composed a neat compendium of the best methods for computing the natural sines, tangents, and secants, chiefly from the rules before given by Newton; and by Newton's or Gregory's series $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$ &c, for the arc in terms of the tangent, he computed the circumference of the circle to 72 places, namely from the arc of 30 degrees, whose tangent t is $= \sqrt{\frac{1}{3}}$ to the radius 1. Other surprizing instances of his industry and labour appear in his Geometry Improv'd, printed in 1717, and signed A. S. Philomath, from which the 5th table of logarithms above-mentioned was extracted. This ingenious man was sometime assistant at the Royal Observatory to Mr. Flamsteed the first astronomer royal; and, being one of the most accurate and indefatigable computers that ever existed, he was for many years the common resource for Mr. Flamsteed, Sir Jonas Moore, Dr. Halley, &c, in all intricate and troublesome calculations. He afterwards retired to his native place at Little-Horton, where, after a life spent in intense study and calculations, he died the 18th July 1742, in the 91st year of his age.

Of the Construction of Logarithms by Fluxions.

It appears by the very definition and description given by Napier of his logarithms, as stated in page 341 of this vol. that the fluxion of his, or the hyperbolic logarithm, of any number, is a fourth proportional to that number, its logarithm, and unity; or, which is the same thing, that it is equal to the fluxion of the number divided by the number: For the description shows, that $z1 : za$ or $1 :: \dot{z1}$ the fluxion of $za : \dot{za}$, which therefore is $= \frac{\dot{z1}}{z1}$; but $\dot{za}$ is also equal to the fluxion of the logarithm A&c, by the description; therefore the fluxion of the logarithm is equal to $\frac{\dot{z1}}{z1}$, the fluxion of the quantity divided by the quantity itself. The same thing appears again at art. 2 of that little piece, in the appendix to his Constructio Logarithmorum, entitled Habitudines Logarithmorum et suorum naturalium numerorum invicem, where he observes that, as any greater quantity is to a less, so is the velocity of the increment or decrement of the logarithms at the place of the less quantity, to that at the greater. Now this velocity of the increment or decrement of the logarithms being the same thing as their fluxions, that proportion is this, $x : a ::$ flux. log. a : flux. log. x; hence if a be $= 1$, as at the beginning of the table of numbers, where the fluxion of the logs. is the index or characteristic c, which is also 1 in Napier's or the hyperbolic logarithms, and ·43429&c in Briggs's, the same proportion becomes $x : 1 :: c :$ flux. log. x; but the constant fluxion of the numbers is also $\dot{1}$, and therefore that proportion is also this, $x : \dot{x} :: c : \frac{c\dot{x}}{x} =$ the fluxion of the log. of x; and in the hyperbolic logs. where c is $= 1$, it becomes $\frac{\dot{x}}{x} =$ the fluxion of Napier's or the hyperbolic logarithm of

x. This same property has also been noticed by many other authors since Napier's time. And the same, or a similar property, is evidently true in all systems of logarithms whatever, namely, that the modulus of the system is to any number, as the fluxion of its logarithm is to the fluxion of the number.

Now from this property, by means of the doctrine of fluxions, are derived other ways for making logarithms, which have been illustrated by many writers on this branch, as Craig, John Bernoulli, and almost all the writers on fluxions. And this method chiefly consists in expanding the reciprocal of the given quantity in an infinite series, then multiplying each term by the fluxion of the said quantity, and lastly taking the fluents of the terms; by which there arises an infinite series of terms for the logarithm sought. So, to find the logarithm of any number N; put any compound quantity for N, as suppose $\frac{n+x}{n}$;

then the flux. of the log. or $\frac{\dot{N}}{N}$ being $\frac{\dot{x}}{n+x} = \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4}$ &c,

the fluents give log. of N or log. of $\frac{n+x}{n} = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

And writing $-x$ for x gives log. $\frac{n-x}{n} = -\frac{x}{n} - \frac{x^2}{2n^2} - \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

Also, because $\frac{n}{n \pm x} = 1 \div \frac{n \pm x}{n}$, or log. $\frac{n}{n \pm x} = 0 - \log. \frac{n \pm x}{n}$

$$\text{theref. log. } \frac{n}{n+x} = -\frac{x}{n} + \frac{x^2}{2n^2} - \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \text{ \&c,}$$

$$\text{and log. } \frac{n}{n-x} = +\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \text{ \&c.}$$

And by adding and subtracting any of these series, to or from one another, and multiplying or dividing their corresponding numbers, various other series for logarithms may be found, converging much quicker than these do.

In like manner, by assuming quantities otherwise compounded, for the value of N, various other forms of logarithmic series may be found by the same means.

Of Mr. Cotes's Logometria.

Mr. Roger Cotes was elected the first Plumian professor of astronomy and experimental philosophy in the university of

Cambridge, January 1706, which appointment he filled with the greatest credit, till he died the 5th of June 1716, in the prime of life, having not quite completed the 34th year of his age. His early death was a great loss to the mathematical world, as his genius and abilities were of the brightest order, as is manifest by the specimens of his performance given to the public. Among these is, his Logometria, first printed in number 338 of the Philosophical Transactions, and afterwards in his Harmonia Mensurarum, published in 1722, with his other works, by his relation and successor, in the Plumian professorship, Dr. Robert Smith. In this piece he first treats, in a general way, of measures of ratios, which measures, he observes, are quantities of any kind, whose magnitudes are analogous to the magnitudes of the ratios, these magnitudes mutually increasing and decreasing together in the same proportion. He remarks, that the ratio of equality has no magnitude, because it produces no change by adding and subtracting; that the ratios of greater and less inequality, are of different affections; and therefore if the measure of the one of these be considered as positive, that of the other will be negative; and the measure of the ratio of equality nothing: That there are endless systems of these, which have all their measures of the same ratios proportional to certain given quantities, called *moduli*, which he defines afterwards, and the ratio of which they are the measures, each in its peculiar system, is called the modular ratio, *ratio modularis*, which ratio is the same in all systems. He then adverts to logarithms, which he considers as the numerical measures of ratios, and he describes the method of arranging them in tables, with their uses in multiplication and division, raising of powers and extracting of roots, by means of the corresponding operations of addition and subtraction, multiplication and division.

After this introduction, which is only a slight abridgment of the doctrine long before very amply treated of by others, and particularly by Kepler and Mercator, we arrive at the first proposition, which has justly been censured as obscure and imperfect, seemingly through an affectation of brevity, intricacy, and originality, without sufficient room for a display of this quality. The reasoning in this proposition, such as it is, seems to be something between that of Kepler and the principles of fluxions, to which the quantities and expressions are nearly allied. However, as it is my duty rather to narrate than explain, I shall here exhibit it exactly as it stands. This proposition is, to determine the measure of any ratio, as for instance that of AC to AB, and which is effected in this manner: Conceive the difference BC to be divided into innumerable very small particles, as PQ, and the ratio between AC and AB into as many such very small ratios, as between AQ and AP: then if the magnitude of the ratio between AQ and AP be given, by dividing, there will also be given that of PQ to AP; and therefore, this being given, the magnitude of the ratio between AQ and AP may be expounded by the given quantity $\frac{PQ}{AP}$; for, AP remaining constant, conceive the particle PQ to be augmented or diminished in any proportion, and in the same proportion will the magnitude of the ratio between AQ and AP be augmented or diminished: Also, taking any determinate quantity M, the same may be expounded by $M \times \frac{PQ}{AP}$; and therefore the quantity $M \times \frac{PQ}{AP}$ will be the measure of the ratio between AQ and AP. And this measure will have divers magnitudes, and be accommodated to divers systems, according to the divers magnitudes of the assumed quantity M, which therefore is called the *modulus* of the system. Now, like as the sum of all the ratios AQ to AP is equal to the proposed ratio AC to AB, so the sum of all the measures $M \times \frac{PQ}{AP}$, found by the known methods, will be equal to the required measure of the said proposed ratio.

The general solution being thus dispatched, from the general expression, Cotes next deduces other forms of the measure, in several corollaries and scholia: as 1st, the terms AP, AQ, approach the nearer to equality as the small difference PQ is less; so that either M $\times \frac{PQ}{AP}$ or M $\times \frac{PQ}{AQ}$ will be the measure of the ratio between AQ and AP, to the modulus M. 2d, That hence the modulus M, is to the measure of the ratio between AQ and AP, as either AP or AQ is to their difference PQ. 3d, The ratio between AC and AB being given, the sum of all the $\frac{PQ}{AP}$ will be given; and the sum of all the M $\times \frac{PQ}{AP}$ is as M: therefore the measure of any given ratio, is as the modulus of the system from which it is taken. 4th, Therefore, in every system of measures, the modulus will always be equal to the measure of a certain determinate and immutable ratio; which therefore he calls the modular ratio. 5th, To illustrate the solution by an example: let z be any determinate and permanent quantity, x a variable or indeterminate quantity, and $\dot{x}$ its fluxion; then, to find the measure of the ratio between $z + x$ and $z - x$, put this ratio equal to the ratio between y and 1, expounding the number y by AP, its fluxion $\dot{y}$ by PQ, and 1 by AB: then the fluxion of the required measure of the ratio between y and 1 is M $\times \frac{\dot{y}}{y}$. Now, for y, restore its val. $\frac{z+x}{z-x}$, and for $\dot{y}$ the flux. of that val. $\frac{2z\dot{x}}{(z-x)^2}$, so shall the flux. of the measure become 2M $\times \frac{z\dot{x}}{zz-xx}$, or 2M into $\frac{\dot{x}}{z}+\frac{\dot{x}x^2}{z^3}+\frac{\dot{x}x^4}{z^5}+$ &c; and therefore that measure will be 2M into $\frac{x}{z}+\frac{x^3}{3z^3}+\frac{x^5}{5z^5}+$ &c. In like manner the measure of the ratio between $1 + v$ and 1, will be found to be - - - - M into $v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 +$ &c. And hence, to find the number from the logarithm given, he reverts the series in this manner: If the last measure be called m, we

$$\text{shall have } \frac{m}{M} \text{ or } Q = v - \tfrac{1}{2}v + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 \text{ \&c,}$$
$$\text{therefore } Q^2 = \quad v^2 - v^3 + \tfrac{11}{12}v^4 - \tfrac{5}{6}v^5 \text{ \&c,}$$
$$\text{and } Q^3 = \quad v^3 - \tfrac{3}{2}v^4 + \tfrac{7}{4}v^5 \text{ \&c,}$$
$$\text{and } Q^4 = \quad v^4 - 2v^5 \text{ \&c,}$$
$$\text{and } Q^5 = \quad v^5 \text{ \&c;}$$

then, by adding continually, we shall have,

$$Q + \tfrac{1}{2}Q^2 = v - \tfrac{1}{6}v^3 + \tfrac{5}{24}v^4 - \tfrac{13}{60}v^5 \ \&c,$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 = v - \tfrac{1}{24}v^4 + \tfrac{3}{40}v^5 \ \&c,$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 = v - \tfrac{1}{120}v^5 \ \&c,$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 + \tfrac{1}{120}Q^5 = v \ \&c,$$

that is $v = Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 + \frac{1}{120}Q^5$ &c. And therefore the required ratio of $1 + v$ to 1, is equal to the ratio of $1 + Q + \frac{1}{2}Q^2$ &c to 1. Now put $m = M$, or $Q = 1$, and the above will become the ratio of $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$ &c to 1, for the constant modular ratio. In like manner, if the ratio between 1 and $1 - v$ be proposed, the measure of this ratio will come out M into $v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4$ &c; which being called m, and $\frac{m}{M} = Q$, that ratio will be the ratio of 1 to $1 - Q + \frac{1}{2}Q^2 - \frac{1}{6}Q^3 + \frac{1}{24}Q^4$ &c. And hence, taking $m = M$, or $Q = 1$, the said modular ratio will also be the ratio of 1 to $1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}$ &c. And the former of these expressions, for the modular ratio, comes out the ratio of 2.718281828459 &c to 1, and the latter the ratio of 1 to 0.367879441171 &c, which number is the reciprocal of the former.

In the 2d prop. the learned author gives directions for constructing Briggs's canon of logarithms, namely, first by the general series 2M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} +$ &c, finding the logarithms of a few such ratios as that of 126 to 125, 225 to 224, 2401 to 2400, 4375 to 4374, &c, from which the logarithm of 10 will be found to be 2.302585092994 &c, when M is 1; but since Briggs's log. of 10 is 1, therefore as 2.302585 &c is to the mod. 1, so is 1 (Briggs's log. of 10) to 0.434294481903&c, which therefore is the modulus of Briggs's logarithms. Hence he deduces the logarithms of 7, 5, 3, and 2. In like manner are the logarithms of other prime numbers to be found, and from them the logarithms of composite numbers by addition and subtraction only.

Cotes then remarks, that the first term of the general series 2M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} +$ &c, will be sufficient for the logarithms of intermediate numbers between those in the table, or even for numbers beyond the limits of the table. Thus, to find the logarithm answering to any intermediate number; let a and e be two numbers, the one the given number, and the other the nearest tabular number, a being the greater, and e the less of them; put $z = a + e$ their sum, $x = a - e$ their difference, $\lambda =$ the logarithm of the ratio of a to e, that is the excess of the logarithm of a above that of e: so shall the said difference of their logarithms be $\lambda = 2\text{M} \times \frac{x}{z}$ very nearly. And, if there be required the number answering to any given intermediate logarithm, because λ is $= \frac{2\text{M}x}{z} = \frac{2\text{M}x}{2a - x}$ or $\frac{2\text{M}x}{2e + x}$, theref. $x = \frac{\lambda a}{\text{M} + \frac{1}{2}\lambda}$ or $\frac{\lambda e}{\text{M} - \frac{1}{2}\lambda}$ very nearly.

In the 3d prop. the ingenious author teaches how to convert the canon of logarithms into logarithms of any other system, by means of their *moduli*. And, in several more propositions, he exemplifies the canon of logarithms in the solution of various important problems in geometry and physics; such as the quadrature of the hyperbola, the description of the logistica, the equiangular spiral, the nautical meridian, &c, the descent of bodies in resisting mediums, the density of the atmosphere at any altitude, &c, &c.

Of Dr. Taylor's Construction of Logarithms.

Dr. Brook Taylor, a very learned mathematician, and secretary to the Royal Society, who died at Somerset-house, Nov. 1731, gave the following method of constructing logarithms, in number 352 of the Philosophical Transactions. His method is founded on these three considerations: 1st, that the sum of the logarithms of any two numbers, is the logarithm of the product of those numbers; 2d, that the logarithm of 1 is nothing, and consequently that the nearer any number is to 1, the nearer will its logarithm be to 0; 3d, that the product of two numbers or factors, of which the one is greater and the other less than 1, is nearer to 1 than that factor is which is on the same side of 1 with itself; so of the two numbers $\frac{2}{3}$ and $\frac{4}{3}$, the product $\frac{8}{9}$ is less than 1, but yet nearer to it than $\frac{2}{3}$ is, which is also less than 1. On these principles he founds the present approximation, which he explains by the following example.

To find the relation between the logs. of 2 and 10 : In order to this, he assumes two fractions, as $\frac{128}{100}$ and $\frac{8}{10}$, or $\frac{2^7}{10^2}$, and $\frac{2^3}{10}$, whose numerators are powers of 2, and their denominators powers of 10, the one fraction being greater, and the other less than unity or 1. Having set these two down, in the form of decimal fractions, below each other in the first column of the following table, and in the second column A and B for their logarithms, expressing by an equation how they are

1,280000000000	A = . . =	$7l2 -$	$2l10$	$l2 > 0,28,$
0,800000000000	B = . . =	$3l2 -$	$l10$	$< 0,33$
1,024000000000	C = A + B =	$10l2 -$	$3l10$	$> 0,300$
0,990352031429	D = B + 9C =	$93l2 -$	$28l10$	$< 0,30107$
1,004336277664	E = C + 2D =	$169l2 -$	$59l10$	$> 0,301020$
0,998959536107	F = D + 2E =	$485l2 -$	$146l10$	$< 0,3010309$
1,000162894165	G = E + 4F =	$2136l2 -$	$643l10$	$> 0,30102996$
0,999936281874	H = F + 6G =	$13301l2 -$	$4004l10$	$< 0,301029997$
1,000035441215	I = G + 2H =	$28738l2 -$	$8651l10$	$> 0,3010299951$
0,999971720830	K = H + I =	$42039l2 -$	$12655l10$	$< 0,3010299959$
1,000007161046	L = I + K =	$70777l2 -$	$21306l10$	$> 0,30102999562$
0,999993203514	M = K + 3L =	$254370l2 -$	$76573l10$	$< 0,30102999567$
1,000000364511	N = L + M =	$325147l2 -$	$97879l10$	$> 0,301029995635$
0,999999764687	O = M + 18N =	$6107016l2 -$	$1838335l10$	$< 0,301029995640$
comp.ar.235313				
0 = 364511O + 235313N =		$2302585925187l2 -$	$693147400972l10$	$> 0,30102999566398$

composed of the logarithms of 2 and 10, the numbers in question, those logarithms being denoted thus, $l2$ and $l10$. Then multiplying the two numbers in the first column together, there is produced a third number 1,024, against which is written C, for its logarithm, expressing likewise by an equation in what manner C is formed of the foregoing logarithms A and B. And in the same manner the calculation is continued throughout; only observing this compendium, that before multiplying the two last numbers already entered in the table, to consider what power of one of them must be used to bring the product the nearest that can be to unity. Now after having continued the table a little way, this is found by only dividing the differences of the numbers from unity one by the other, and taking the nearest quotient for the index of the power sought. Thus, the second and third numbers in the table being 0,8 and 1,024, their differences from unity are 0,200 and 0,024; hence 0,200 ÷ 0,024 gives 9 for the index; and therefore multiplying the 9th power of 1,024 by 0,8, produces the next number 0,990352031429, whose logarithm is $D = B + 9C$.

When the calculation is continued in this manner till the numbers become small enough, or near enough to 1, the last logarithm is supposed equal to nothing, which gives an equation expressing the relation of the logarithms, and thence the required logarithm is determined. Thus, supposing $G = 0$, we have $2136l2 - 643l10 = 0$, and hence, because the logarithm of 10 is 1, we obtain $l2 = \frac{643}{2136} = 0,30102996$, too small in the last figure only; which so happens, because the number corresponding to G is greater than 1. And in this manner are all the numbers in the third or last column obtained, which are continual approximations to the logarithm of 2.

There is another expedient, which renders this calculation still shorter, and it is founded on this consideration: that when x is small, $(1+x)^n$ is nearly $= 1 + nx$. Hence if $1+x$ and $1 - z$ be the two last numbers already found in the first column of the table, the product of their powers $(1+x)^m \times (1-z)^n$ will be nearly $= 1$; and hence the relation of m and n may be thus found, $(1+x)^m \times (1-z)^n$ is nearly $= (1+mx) \times (1-nz) = 1 + mx - nz - mnxz = 1 + mx - nz$ nearly, which being also $= 1$ nearly, therefore $m : n :: z : x :: l.(1-z) : l.(1+x)$; whence $xl.(1-z) + zl.(1+x) = 0$. For example, let 1,024 and 0,990352 be the last numbers in the table, their logs. being C and D: here we have $1,024 = 1 + x$, and $0,990352 = 1 - z$; conseq. $x = 0,024$, and $z = 0,009648$, and hence the ratio $\frac{z}{x}$ in small numbers is $\frac{201}{500}$. So that, for finding the logarithms proposed, we may take $500D + 201C = 48510l2 - 14603l10 = 0$; which gives $l2 = 0,3010307$. And in this manner are found the numbers in the last line of the table.

Of Mr. Long's Method.

In number 339 of the Philosophical Transactions, are given a brief table and method for finding the logarithm to any number, and the number to any logarithm, by Mr. John Long, B. D. Fellow of C. C. C. Oxon. This table and method are similar to those described in chap. 14, of Briggs's Arith. Log. differing only in this, that in this table, by Mr. Long, the logarithms, in each class, are in arithmetical progression, the common difference being 1; but in Briggs's little table, the column of natural numbers has the like common difference. The table consists of eight classes of logarithms, and their corresponding numbers, as follow:

L.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.
,9	7,943282347	,009	1,020939484	,00009	1,000207254	,0000009	1,000002072
,8	6,309573445	8	1,018591388	8	1,000184224	8	1,000001842
,7	5,011872336	7	1,016248694	7	1,000161194	7	1,000001611
,6	3,981071706	6	1,013911586	6	1,000138165	6	1,000001381
,5	3,162277660	5	1,011579454	5	1,000115136	5	1,000001151
,4	2,511886432	4	1,009252886	4	1,000092106	4	1,000000921
,3	1,995262315	3	1,006931669	3	1,000069086	3	1,000000690
,2	1,584893193	2	1,004615794	2	1,000046053	2	1,000000460
,1	1,258925412	1	1,002305238	1	1,000023026	1	1,000000230
,09	1,230268771	,0009	1,002074475	,000009	1,000020724	,00000009	1,000000207
8	1,202264435	8	1,001843766	8	1,000018421	8	1,000000184
7	1,174897555	7	1,001613109	7	1,000016118	7	1,000000161
6	1,148153621	6	1,001382506	6	1,000013816	6	1,000000138
5	1,122018454	5	1,001151956	5	1,000011513	5	1,000000115
4	1,096478196	4	1,000921459	4	1,000009210	4	1,000000092
3	1,071519305	3	1,000691015	3	1,000006908	3	1,000000069
2	1,047128548	2	1,000460623	2	1,000004605	2	1,000000046
1	1,023292992	1	1,000230285	1	1,000002302	1	1,000000023

where, because the logarithms in each class are the continual multiples 1, 2, 3, &c, of the lowest, it is evident that the natural numbers are so many scales of geometrical proportionals, the lowest being the common ratio, or the ascending numbers are the 1, 2, 3, &c, powers of the lowest, as expressed by the figures 1, 2, 3, &c, of their corresponding logarithms. Also the last number in the first, second, third, &c class, is the 10th, 100th, 1000th, &c root of 10; and any number in any class, is the 10th power of the corresponding number in the next following class.

To find the logarithm of any number, as suppose of 2000, by this table, Look in the first class for the number next less than the first figure 2, and it is 1,995262315, against which is 3 for the first figure of the logarithm sought. Again, dividing 2, the number proposed, by 1,995262315, the number found in the table, the quotient is 1,002374467; which being looked for in the second class of the table, and finding neither its equal nor a less, 0 is therefore to be taken for the second figure of the logarithm; and the same quotient 1,002374467 being looked for in the third class, the next less is there found to be 1,002305238, against which is 1 for the third figure of the logarithm; and dividing the quotient 1,002374467 by the said next less number 1,002305238, the new quotient is 1,000069070; which being sought in the fourth class, gives 0, but sought in the fifth class gives 2, which are the fourth and fifth figures of the logarithm sought: again, dividing the last quotient by 1,000046053, the next less number in the table, the quotient is 1,000023015, which gives 9 in the 6th class for the 6th figure of the logarithm sought: and again dividing the last quotient by 1,000020724, the next less number, the quotient is 1,000002291, the next less than which, in the 7th class, gives 9 for the 7th figure of the logarithm: and dividing the last quotient by 1,000002072, the quotient is 1,000000219, which gives 9 in the 8th class for the 8th figure of the log.: and again the last quotient 1,000000219 being divided by 1,000000207, the next less, the quotient 1,000000012 gives 5 in the same 8th class, when one figure is cut off, for the 9th figure of the logarithm sought. All which figures collected together give 3,301029995 for Briggs's log. of 2000, the index 3 being supplied; which logarithm is true in the last figure.

To find the number answering to any given logarithm, as suppose to 3,3010300: omitting the characteristic, against the other figures 3, 0, 1, 0, 3, 0, 0, as in the first column in the margin, are the several numbers as in the 2d column, found from their respective 1st, 2d, 3d, &c classes; the effective numbers of which multiplied continually together, the last product is 2,000000019966, which, because the characteristic is 3, gives 2000,000019966, or 2000 only, for the required number, answering to the given logarithm.

3	1,995262315
0	0
1	1,002305238
0	0
3	1,000069080
0	0
0	0

Of Mr. Jones's Method.

In the 61st volume of the Philosophical Transactions, is a small paper on logarithms, which had been drawn up, and left unpublished, by the learned and ingenious William Jones, Esq. The method contained in this memoir, depends on an application of the doctrine of fluxions, to some properties drawn from the nature of the exponents of powers. Here all numbers are considered as some certain powers of a constant determinate root: so, any number x may be considered as the z power of any root r, or that $x = r^z$ is a general expression for all numbers, in terms of the constant root r, and a variable exponent z. Now the index z being the logarithm of the number x, therefore, to find this logarithm, is the same thing, as to find what power of the radical r is equal to the number x.

From this principle, the relation between the fluxions of any number x, and its logarithm z, is thus determined: Put $r = 1 + n$; then is $x = r^z = (1+n)^z$, and $x + \dot{x} = (1+n)^{z+\dot{z}} = (1+n)^z \times (1+n)^{\dot{z}} = x \times (1+n)^{\dot{z}}$, which by expanding $(1+n)^{\dot{z}}$, omitting the 2d, 3d, &c powers of $\dot{z}$, and writing q for $\frac{n}{1+n}$, becomes $x + x\dot{z} \times (q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \&c)$; therefore $\dot{x} = ax\dot{z}$, putting a for the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, or $f\dot{x} = x\dot{z}$, putting $f = \frac{1}{a}$.

Now when $r = 1 + n = 10$, as in the common logarithms of Briggs's form; then $n = 9$, $q = ,9$, and the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, gives $a = 2,302585$&c, and theref. its recip. $f = ,434294$&c. But if $a = 1 = f$, the form will be that of Napier's logarithms.

From the above form $x\dot{z}=f\dot{x}$, or $\dot{z} = \frac{f\dot{x}}{x}$, are then deduced many curious and general properties of logarithms, with the several series heretofore given by Gregory, Mercator, Wallis, Newton, and Halley. But of all these series, that one which our author selects for constructing the logarithms, is this, putting $N = \frac{r-p}{r+p}$, the logarithm of $\frac{r}{p}$ is $= 2f \times : N + \frac{1}{3}N^3 + \frac{1}{5}N^5 + \frac{1}{7}N^7 +$ &c, in the case in which $r - p$ is $= 1$, and consequently in that case $N = \frac{1}{2r-1}$ or $\frac{1}{2p+1}$; which series will then converge very fast.

Hence, having given any numbers, p, q, r, &c, and as many ratios a, b, c, &c, composed of them, the difference between the two terms of each ratio being 1; as also the logarithms A, B, C, &c, of those ratios given: to find the logarithms P, Q, R, &c, of those numbers; supposing $f = 1$. For instance, if $p = 2$, $q = 3$, $r = 5$; and $a = \frac{9}{8} = \frac{3^2}{2^3}$, $b = \frac{16}{15} = \frac{2^4}{3 \cdot 5}$, $c = \frac{25}{24} = \frac{5^2}{3 \cdot 2^3}$. Now the logarithms A, B, C, of these ratios a, b, c, being found by the above series, from the nature of powers we have these three equations,

$$\left.\begin{aligned} A &= 2Q - 3P \\ B &= 4P - Q - R \\ C &= 2R - Q - 3P \end{aligned}\right\} \text{which equations reduced give}$$

$$\begin{aligned} P &= 3A + 4B + 2C = \text{log. of } 2. \\ Q &= 5A + 6B + 3C = \text{log. of } 3. \\ R &= 7A + 9B + 5C = \text{log. of } 5. \end{aligned}$$

And hence $P + R = 10A + 13B + 7C$ is $=$ the logarithm of 2×5 or 10.

An elegant tract on logarithms, as a comment on Dr. Halley's method, was also given by Mr. Jones, in his Synopsis Palmariorum Matheseos, published in the year 1706. And, in the Philosophical Transactions, he communicated various improvements in goniometrical properties, and the series relating to the circle and to trigonometry.

The memoir above described was delivered to the Royal Society by their then librarian, Mr. John Robertson, a worthy, ingenious, and industrious man, who also communicated to the Society several little tracts of his own relating to logarithmical subjects; he was also the author of an excellent treatise on the Elements of Navigation in two volumes; and he was successively mathematical master to Christ's hospital in London; head master to the royal naval academy at Portsmouth; and librarian, clerk, and housekeeper, to the Royal Society; at whose house, in Crane Court, Fleet-street, he died in 1776, aged 64 years.

And among the papers of Mr. Robertson, I have, since his death, found one containing the following particulars relating to Mr. Jones, which I here insert, as I know of no other account of his life, &c, and as any true anecdotes of such extraordinary men must always be acceptable to the learned.—This paper is not in Mr. Robertson's hand writing, but in a kind of running law-hand, and is signed R. M. 12 Sept. 1771.

" William Jones, Esquire, F. R. S. was born at the foot of Bodavon mountain [Mynydd Bodafon], in the parish of Llanfihangel tre'r Bardd, in the isle of Anglesey, North Wales, in the year 1675. His father John George* was a farmer, of a good family, being descended from Hwfa ap Cynddelw, one of the 15 tribes of North Wales. He gave his two sons the common school education of the country, reading, writing, and accounts, in English, and the latin grammar. Harry his second soon took to the farming business; but William the eldest, having an extraordinary turn for mathematical studies, determined to try his fortune abroad from a place where the same was but of little service to him; he accordingly came to London, accompanied by a young man, Rowland Williams, afterwards an eminent perfumer in Wych-street. The report in the country is, that Mr. Jones soon got into a merchant's counting-house, and so gained the esteem of his master, that he gave him the command of a ship for a West-India voyage; and that upon his return he set up a mathematical school,

* " It is the custom in several parts of Wales for the name of the father to become the surname of his children. John George the father was commonly called Sion Siors of Llambado, to which parish he moved, and where his children were brought up."

and published his book of navigation*; and that upon the death of the merchant he married his widow: that Lord Macclesfield's son being his pupil, he was made secretary to the chancellor, and one of the D. tellers of the exchequer—and they have a story of an Italian wedding which caused great disturbance in Lord Macclesfield's family, but compromised by Mr. Jones; which gave rise to a saying, that Macclesfield was the making of Jones, and Jones the making of Macclesfield." Mr. Jones died July 3, 1749, being vice-president of the Royal Society; and left one daughter, and a young son, who was the late Sir William Jones, one of the judges in India, and highly esteemed for his great abilities, extensive learning, and eminent patriotism.

Of Mr. Andrew Reid and Others.

Andrew Reid, Esq. published in 1767 a quarto tract, under the title of An Essay on Logarithms, in which he also shows the computation of logarithms, from principles depending on the binomial theorem and the nature of the exponents of powers, the logarithms of numbers being here considered as the exponents of the powers of 10. He hence brings out the usual series for logarithms, and largely exemplifies Dr. Halley's most simple construction.

Besides the authors whose methods have been here particularly described, many others have treated on the subject of logarithms, and of the sines, tangents, secants, &c; among the principal of whom are Leibnitz, Euler, Maclaurin, Wolfius, and professor Simson, in an elegant geometrical tract on logarithms, contained in his posthumous works, printed in 4to at Glasgow, in the year 1776, at the expense of the very learned Earl Stanhope, and by his Lordship disposed of in

* This tract on navigation, intitled, "A New Compendium of the whole Art of Practical Navigation," was published in 1702, and dedicated "to the reverend and learned Mr. John Harris, M. A. and F. R. S." the author, I apprehend, of the "Universal Dictionary of Arts and Sciences," under whose roof Mr. Jones says he composed the said treatise on Navigation.

presents among gentlemen most eminent for mathematical learning.

Of Mr. Dodson's Anti-logarithmic Canon.

The only remaining considerable work of this kind published, that I know of, is the Anti-logarithmic Canon of Mr. James Dodson, an ingenious mathematician, which work he published in folio in the year 1742; a very great performance, containing all the logs. under 100000, and their corresponding natural numbers to 11 places of figures, with all their differences and the proportional parts; the whole arranged in the order contrary to that used in the common tables of numbers and logarithms, the exact logarithms being here placed first, and increasing continually by 1, from 1 to 100000, with their corresponding nearest numbers in the columns opposite to them; and, by means of the differences and proportional parts, the logarithm to any number, or the number to any logarithm, each to 11 places of figures, is readily found. This work contains also, besides the construction of the natural numbers to the given logarithms, "precepts and examples, showing some of the uses of logarithms, in facilitating the most difficult operations in common arithmetic, cases of interest, annuities, mensuration, &c; to which is prefixed an introduction, containing a short account of logarithms, and of the most considerable improvements made, since their invention, in the manner of constructing them."

The manner in which these numbers were constructed, consists chiefly in imitations of some of the methods before described by Briggs, and is nothing more than generating a scale of 100000 geometrical proportionals, from 1 the least term, to 10 the greatest, each continued to 11 places of figures; and the means of effecting this, are such as easily flow from the nature of a series of proportionals, and are briefly as follow. First, between 1 and 10 are interposed 9 mean proportionals; then between each of these 11 terms there are interposed 9 other means, making in all 101 terms; then between each of these a 3d set of 9 means, making in all 1001 terms; again between each of these a 4th set of 9 means, making in all 10001 terms; and lastly, between each two of these terms, a 5th set of 9 means, making in all 100001 terms, including both the 1 and the 10. The first four of these 5 sets of means, are found each by one extraction of the 10th root of the greater of the two given terms, which root is the least mean, and then multiplying it continually by itself, according to the number of terms in the section or set; and the 5th or last section is made by interposing each of the 9 means by help of the method of differences before taught. Namely, putting 10, the greatest term, $= A$, $A^{\frac{1}{10}} = B$, $B^{\frac{1}{10}} = C$, $C^{\frac{1}{10}} = D$, $D^{\frac{1}{10}} = E$, and $E^{\frac{1}{10}} = F$; now extracting the 10th root of A or 10, it gives $1,2589254118 = B = A^{\frac{1}{10}}$, for the least of the 1st set of means; and then multiplying it continually by itself, we have B, B^2, B^3, B^4, &c, to $B^{10} = A$, for all the 10 terms: 2dly, the 10th root of 1,2589254118 gives $1,0232929923 = C = B^{\frac{1}{10}} = A^{\frac{1}{100}}$, for the least of the 2d class of means; which being continually multiplied gives C, C^2, C^3, &c, to $C^{100} = B^{10} = A$, for all the 2d class of 100 terms: 3dly, the 10th root of 1,0232929923 gives $1,0023052381 = D = C^{\frac{1}{10}} = B^{\frac{1}{100}} = A^{\frac{1}{1000}}$, for the least of the 3d class of means; which being continually multiplied, gives D, D^2, D^3, &c, to $D^{1000} = C^{100} = B^{10} = A$, for the 3d class of 1000 terms: 4thly, the 10th root of 1,0023052381 gives $1,0002302850 = B = D^{\frac{1}{10}} = C^{\frac{1}{100}} = B^{\frac{1}{1000}} = A^{\frac{1}{10000}}$, for the least of the 4th class of means, which being continually multiplied, gives E, E^2, E^3, &c, to $E^{10000} = D^{1000} = C^{100} = B^{10} = A$, for the 4th class of 10000 terms. Now these 4 classes of terms, thus produced, require no less than 11110 multiplications of the least means by themselves; which however are much facilitated by making a small table of the first 10, or even 100 products, of the constant multiplier, and from it only taking out the proper lines, and adding them together: and these 4 classes of numbers always prove themselves at every 10th term, which must always agree with the corresponding successive terms of the preceding class. The remaining 5th class is constructed by means of differences, being much easier than the method of continual multiplication, the 1st and 2d differences only being used, as the 3d difference is too small to enter the computation of the sets of 9 means, between each two terms of the 4th class. And the several 2d differences, for each of these sets of 9 means, are found from the properties of a set of proportionals, 1, r, r^2, r^3, &c, as disposed in the 1st column of the annexed table, and their several orders of differences as in the other columns of the table; where it is evident that

Terms.	1st dif.	2d dif.	3d dif.	&c.
$1\times$	$(r-1)\times$	$(r-1)^2\times$	$(r-1)^3\times$	
1	1	1	1	&c.
r	r	r	r	
r^2	r^2	r^2	r^2	
r^3	r^3	r^3	r^3	
&c.	&c.	&c.	&c.	

each column, both that of the given terms of the progression, and those of their orders of differences, forms a scale of proportionals, having the same common ratio r; and that each horizontal line, or row, forms a geometrical progression, having all the same common ratio $r-1$, which is also the 1st difference of each set of means: so, $(r-1)^2$ is the 1st of the 2d differences, and which is constantly the same, as the 3d differences become too small in the required terms of our progression to be regarded, at least near the beginning of the table: hence, like as 1, $r-1$, and $(r-1)^2$ are the 1st term, with its 1st and 2d differences; so r^n, $r^n.(r-1)$, and $r^n.(r-1)^2$, are any other term with its 1st and 2d differences. And by this rule the 1st and 2d differences are to be found, for every set of 9 means, viz, multiplying the 1st term of any class (which will be the several terms of the series E, E^2, E^3, &c, or every 10th term of the series F, F^2, F^3, &c) by $r-1$, or $F-1$, for the 1st difference, and this multiplied by $F-1$

again for the true 2d difference, at the beginning of that class. Thus, the 10th root of 1,0002302850, or E, gives 1,000023026116 for F, or the 1st mean of the lowest class, therefore $F - 1 = r - 1 =$,000023026116, is its 1st difference, and the square of it is $(r-1)^2 =$,0000000005302 its 2d diff.; then is ,000023026116F^{10n} or ,000023026116E^n, the 1st difference, and ,0000000005302F^{20n} or ,0000000005302E^{2n} is the 2d difference, at the beginning of the nth class of decades. And this 2d difference is used as the constant 2d difference through all the 10 terms, except towards the end of the table, where the differences increase fast enough to require a small correction of the 2d difference, which Mr. Dodson effects by taking a mean 2d difference among all the 2d differences, in this manner; having found the series of 1st differences $(F-1).E^n$, $(F-1).E^{n+1}$, $(F-1).E^{n+2}$, &c, he takes the differences of these, and $\frac{1}{10}$ of them gives the mean 2d differences to be used, namely, $\frac{F-1}{10}(E^{n+1} - E^n)$, $\frac{F-1}{10}(E^{n+2} - E^{n+1})$, &c, are the mean 2d differences. And this is not only the more exact, but also the easier way. The common 2d difference, and the successive 1st differences, are then continually added, through the whole decade, to give the successive terms of the required progression.

TRACT XXII.

SOME PROPERTIES OF THE POWERS OF NUMBERS.

1. Of any two square numbers, at any distance from each other in the natural series of the squares 1^2, 2^2, 3^2, 4^2, &c, the mean proportional between the two squares, is equal to the less square plus its root multiplied by the difference of the roots, that is, by the distance in the series between the two square numbers, or by 1 more than the number of squares between them. The same mean proportional, is also equal to the greater of the two squares, minus its root the same

number of times taken. That is, $mn = mm + dm = nn - dn$; where d is $= n - m$, the distance between the two squares m^2, n^2. For, since $n = m + d$; multiply by m, then $mn = mm + md$, which is the first part of the proposition. Again, $m = n - d$; multiply this by n, then $mn = nn - nd$, which is the latter part.

2. An arithmetical mean between the two squares mm and nn, exceeds their geometrical mean, by half the square of the difference of their roots, or of their distance in the series. For, by the first section, $mn = mm + dm$, and also $mn = nn - dn$; add these two together, and the sums are $2mn = mm + nn - d(n-m) = mm + nn - dd$; divide by 2, then $mn = \frac{1}{2}mm + \frac{1}{2}nn - \frac{1}{2}dd$.

3. Of three adjacent squares in the series, the geometrical mean between the extremes, is less by 1 than the middle square. For, let the three squares be m^2, $(m+1)^2$, $(m+2)^2$; then the mean between the extremes, $m(m+2) = mm + 2m$ is $= (m+1)^2 - 1$.

In like manner, the mean between the extremes, of any three squares, whose common distance or difference of their roots is d, is less than the middle square by the square of the distance dd.

4. The difference between the two adjacent squares mm, nn, or $nn - mm$, is $(m+1)^2 - m^2 = 2m + 1$. In like manner, the difference between n^2 and the next following square p^2, or $p^2 - n^2$, is $2n + 1$; and so on. Hence, the difference of these differences, or the 2d difference of the squares, is $2(n - m) = 2$, which is constant, because $n - m = 1$. And thus, the 2d differences being constantly the number 2, all the first differences will be found by the continual addition of this number 2; and then the whole series of squares themselves will be found by the continual addition of the first differences. Thus, the

2d difs.	2,	2,	2,	2,	2,	2,	2,	2,	2,	2,	&c.
1st difs.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	&c.
squares,	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	&c.

5. Again, if m^3, n^3, p^3, be three adjacent cubes; then $\left.\begin{array}{l} n^3 - m^3 = 3m^2 + 3m + 1 \\ p^3 - n^3 = 3n^2 + 3n + 1 \end{array}\right\}$; and the differences of these first differences is $3(n^2 - m^2) + 3(n - m) = 6(m + 1)$, the 2d difference. In like manner, the next 2d difference will be $6(n + 1)$. Then the dif. of these 2d differences is $6(n - m) = 6$ the 3d difference, which therefore is constant. Now, supposing the series of cubes to begin from 0, the first of each of the several orders of differences will be found by making $m = 0$, in the general expression for each order: thus, $6(m + 1)$ becomes 6 for the first of the 2d differences; and $3m^2 + 3m + 1$ becomes 1 for the first of the 1st differences. And hence is found all the others, as in this table.

3d difs.	6,	6,	6,	6,	6,	6,	6,	6,	6,	&c.
2d difs.	6,	12,	18,	24,	30,	36,	42,	48,	54,	&c.
1st difs.	1,	7,	19,	37,	61,	91,	127,	169,	217,	&c.
cubes	0,	1,	8,	27,	64,	125,	216,	343,	512,	&c.

And thus may all the powers of the series of natural numbers 1, 2, 3, 4, 5, &c, be found, by addition only, adding continually the numbers throughout the several orders of differences. And here it is remarkable, that the number of the orders of differences, will be the same as the index of the powers to be formed; that is, in the series of squares, there are two orders of differences; in the cubes, three; in the 4th powers, four, &c: or, which is the same thing, of the squares, the 2d differences are equal to each other; of the cubes, the 3d differences are equal; of the 4th power, the 4th diffs. are equal; &c. Further, the 2d diffs. in the squares are $1.2 = 2$; the 3d diffs. in the cubes $1.2.3 = 6$; the 4th diffs. in the 4th powers $1.2.3.4 = 24$; and so on. And from these properties were found, by continual additions only, all the series of squares and cubes in the table at the end of this volume, and in my large Table of the Products and Powers of Numbers, published in 1781, by the Board of Longitude.

TRACT XXIII.

A NEW AND EASY METHOD FOR THE SQUARE ROOTS OF NUMBERS.—FROM MY MATHEMATICAL MISCEL. P. 323.

Problem.—Having given any nonquadrate number N; it is required to find a simple vulgar fraction $\frac{n}{d}$, the value of which shall be within any degree of nearness to $\sqrt{N}$, the surd root of N.

Investigation.—Since $\sqrt{N}$ is $=\frac{n}{d}$ nearly, or $d^2N=n^2$ nearly; let d^2N be $= n^2-D$. Then, since n, d, and N, are all integers by the supposition, D must also be an integer; and the smaller that integer is, the nearer will the value of $\frac{n}{d}$ be to $\sqrt{N}$, as is evident: therefore let $D = 1$ the smallest integer; then is $d^2N = n^2-1$, or $n^2 = d^2N + 1$: suppose this to be $= (dx-1)^2 = d^2x^2 - 2dx + 1$, where x is evidently some near value of $\sqrt{N}$; from this equation we have $d=\frac{2x}{x^2-N}$, and consequently $n=\sqrt{(d^2N-1)}=\frac{x^2+N}{x^2-N}$; hence theref. $\sqrt{N}=\frac{n}{d}$ is $=\frac{x^2+N}{2x}$ nearly.

Thus then the function $\frac{x^2+N}{2x}$ is an approximate value of $\sqrt{N}$, where x is to be assumed of any value whatever; but the nearer it is taken to $\sqrt{N}$, the nearer will the value of the fraction be to $\sqrt{N}$ required. And since $\frac{x^2+N}{2x}$ is always nearer to $\sqrt{N}$ than what x is, therefore assume any integer, or rational fraction, for x, but the nearer to $\sqrt{N}$ the more convenient, and write that assumed value of it in this expression, instead of it, so shall we have a nearer approximate rational value of $\sqrt{N}$; then use this last found value of $\sqrt{N}$ instead of x, in the same expression, and there will result a still nearer rational value of $\sqrt{N}$; and thus, by always substituting the last found value for x, in the fraction $\frac{x^2+N}{2x}$ or $\frac{1}{2}x + \frac{N}{2x}$, the result will be a still nearer value. And thus we may proceed to any degree of proximity required.

But a theorem somewhat easier for this continual substitution, may be thus raised: $\frac{n}{d}$ being any one approximate value of $\sqrt{N}$, write it instead of x, in the general function $\frac{x^2+N}{2x}$, then we have $\frac{n^2+Nd^2}{2dn}$ for the general approximation. That is, having assumed or found any one approximation $\frac{n}{d}$, the numerator of the next nearer approximation will be equal to the sum of the square of the numerator n and N times the square of the denominator of this one, and the denominator of the new one will be double the product of the numerator and denominator of this.

Or, a still easier continual approximation is $\frac{2n^2-1}{2dn} = \frac{n}{d} - \frac{1}{2dn}$, which is equal to the former, because n^2 is $= d^2N + 1$.

Example 1.—To find near rational values of the square root of the number 2.—Here N $= 2$. Take $1\frac{1}{2}$ or $\frac{3}{2}$ for the first value of x, as being nearly equal to $\sqrt{2}$. Then $n = 3$, and $d = 2$; therefore $\frac{2n^2-1}{2dn} = \frac{18-1}{12} = \frac{17}{12} = 1{\cdot}416$&c, for the next nearer value of $\sqrt{2}$. Again, take $\frac{17}{12} = \frac{n}{d}$; then $\frac{2n^2-1}{2dn} = \frac{2 \times 17^2 - 1}{2 \times 17 \times 12} = \frac{577}{408} = 1{\cdot}414215$, true for $\sqrt{2}$ to the last figure. And writing again $\frac{577}{408}$ for $\frac{n}{d}$, we obtain $\frac{665857}{470832} = 1{\cdot}414213562376$ for the value of $\sqrt{2}$, true to the last figure, which should be a 3, instead of a 6.

This small number is but an unfavourable example of the method, notwithstanding the ease and expedition with which the root has been so quickly obtained. For, the larger the given number N is, the quicker will the theorem approximate. Thus, taking for

Example 2.—To find the root of the number 920. Here N $= 920$, and $x = 30$ nearly. Now we must first use the rule $\frac{x^2+N}{2x}$, because x is taken $= 30$, below the true value. Hence then $\frac{x^2+N}{2x} = \frac{900+920}{60} = \frac{1820}{60} = \frac{91}{3} = 30\frac{1}{3}$ the second value of $\sqrt{920}$. Next make $\frac{91}{3} = \frac{n}{d}$; then $\frac{2n^2-1}{2dn} = \frac{2\times 91^2-1}{2\times 91\ \times 3} = \frac{16561}{546} =$ 30·33150183, differing from the truth but by 6 in the tenth place of figures, the true number being 30·33150177.

And in this way may the square roots, in the table at the end of this volume, be easily found.

TRACT XXIV.

TO CONSTRUCT THE SQUARE AND CUBE ROOTS AND THE RECIPROCALS OF THE SERIES OF THE NATURAL NUMBERS.

1. *For the Square Roots.*

Since the square root of $a^2 + n$ is $a + \frac{n}{2a} - \frac{n^2}{8a^3} + \frac{n^3}{16a^5} - $ &c: therefore the series of the square roots of a^2, $a^2 + 1$, $a^2 + 2$, $a^2 + 3$, &c, and their 1st, 2d, 3d, 4th, &c differences, will be as below:

Nos.	Square Roots.	1st Diffs.	2d Diffs.	3d Diffs.
a^2	a			
a^2+1	$a + \frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{4a^3} - \frac{3}{8a^5}$	$\frac{3}{8a^5}$ &c.
a^2+2	$a + \frac{2}{\cdot} - \frac{4}{\cdot} + \frac{8}{\cdot}$	$\frac{1}{\cdot} - \frac{3}{\cdot} + \frac{7}{\cdot}$	$\frac{1}{\cdot} - \frac{6}{\cdot}$	$\frac{3}{\cdot}$
a^2+3	$a + \frac{3}{\cdot} - \frac{9}{\cdot} + \frac{27}{\cdot}$	$\frac{1}{\cdot} - \frac{5}{\cdot} + \frac{19}{\cdot}$	$\frac{1}{\cdot} - \frac{9}{\cdot}$	
a^2+4	$a + \frac{4}{\cdot} - \frac{16}{\cdot} + \frac{64}{\cdot}$	$\frac{1}{\cdot} - \frac{7}{\cdot} + \frac{37}{\cdot}$		

Where, the columns of fractions having in each of them the same denominator, after the first line, in each class, a dot is written in the place of the denominators, to save the too frequent repetition of the same quantities. Now it is evident that, in every class, both of roots and of every set of differences, the first terms are all alike; and therefore, by the subtractions, it happens that every class of differences contains one term fewer than the one immediately preceding it. These differences are to be employed in constructing tables of square roots; and the extent to which the orders of differences are to be continued, must be regulated by the number of decimal figures to which the roots in the table are to be carried. In the above specimen the differences are continued as far as the 3d order, where the common first term is $\frac{5}{8a^5}$, which may be sufficiently small for constructing all the preceding orders of differences, and then the series of roots themselves, as far as to 7 places of decimals in each, when we commence with the number 1024, for the first square a^2, the root of which is 32. After this, the squares 1025, 1026, 1027, &c, continually increasing, their roots 32+, &c, proceed increasing also; but the series of numbers, in every order of differences, are all in a decreasing progression; so that the following orders are all found by taking each latter difference from the one immediately above it. Then, to construct the table of roots, having found the first term of each order of differences, as far as necessary, suppose to the 3d order; subtract that continually from the first of the 2d differences, which will complete the series of this order of differences. Then these being taken each from the first difference, the successive remainders will form the whole series of first differences.—Lastly, these first differences added continually with the first square root, will form the whole series of roots, from the first rational root, suppose 32, the root of the square number 1024, to be continued to the next rational root 33, or root of the next square number 1089. Then begin again, from this last square number, in like manner, with a new series of roots and differences, which are to be continued to the third square number 1156, the root of which is the next rational root 34. Then the like process is to be repeated again, and continued from the 3d to the 4th square number. And so on, continuing from each successive square number, to the next following one, as far as necessary; the last of each series of roots and differences always verifying the whole series from square to square.

The computation may begin at 1024, for the series of squares 1024, 1089, 1156, &c, their differences being 65, 67, 69, &c, and their roots 32, 33, 34, &c, as in the margin; in order to find the intermediate or irrational roots, to any proposed extent in decimals. The roots will be obtained true to different numbers of figures, according to the number of the orders of differences employed.

Roots.	Squares.	Diffs.
32	1024	
		65
33	1089	
		67
34	1156	
		69
35	1225	
		71
36	1296	

The first differences only will give the roots true to 5 places of figures, in commencing with the square 1024; the 2d differences will give the roots true to 9 places; the 3d differences to 12 places; and so on, as here below.

First, To find the Diffs.

$\frac{1}{2a} = 0{\cdot}015625$

$\frac{-1}{8a^3} = \ldots -38147$

$\frac{+1}{16a^5} = \ldots\ldots +18\frac{1}{2}$

1st dif. 0·015621187

$\frac{1}{4a^3} =$ 0·000007629

$\frac{-3}{8a^5} = \ldots\ldots -11$

2d dif. 0·000007618

$\frac{3}{8a^5} =$ 3d dif. . . 11

Then for the Roots.

3d Dif.	2d Difs.	1st Difs.	Roots.
·0⁷1 +	·00000762	·01562119	32·00000000
	761	·01561357	32·01562119
	760	01560596	32·03123476
	758	·01559836	32 04684072
	757	·01559078	32·06243908
	756	·01558321	32·07802986
	756	·01557565	32·09361307
	754	·01556809	32·10918872
	753	·01556055	32·12475681
	752	·01555302	32·14031736
	750	·01554550	32·15587038
	750	·01553800	32·17141588
		·01553050	32·18695388

2. *For the Cube Roots.*

In the series and contrivances for constructing a table of cube roots of numbers, the process is exactly similar to that for the square roots, just above explained, in every respect, differing only in the terms of the general series by which the root of the binomial is expressed, viz, the series for $\sqrt[3]{(a^3+n)}$, instead of the series for $\sqrt{(a^2+n)}$. So that, all the explanation, and forms of process, being the same here, as in the former case, for the square roots, the repetition of these may here be dispensed with, and we shall only need to set down the series of roots and differences, with the calculation from them.

Now the general form of the series for $\sqrt[3]{(a^3+n)}$, or the cube root of a^3+n, is $a+\frac{n}{3a^2}-\frac{n^2}{9a^5}+\frac{5n^3}{81a^8}-\frac{10a^4}{243a^{11}}$ &c: therefore, expounding n by 1, 2, 3, &c, the series of the cube roots of a^3, a^3+1, a^3+2, a^3+3, &c, with their 1st, 2d, 3d, &c differences, will be as below:

Nos.	Cube Roots.	1st Diffs.	2d Diffs.	3d Diffs.
a^3	a			
a^3+1	$a+\frac{1}{3a^2}-\frac{1}{9a^5}+\frac{5}{81a^8}$	$\frac{1}{3a^2}-\frac{1}{9a^5}+\frac{5}{81a^8}$	$\frac{2}{9a^5}-\frac{10}{27a^8}$	$\frac{10}{27a^8}$
a^3+2	$a+\frac{2}{\cdot}-\frac{4}{\cdot}+\frac{40}{\cdot}$	$\frac{1}{\cdot}-\frac{3}{\cdot}+\frac{35}{\cdot}$	$\frac{2}{\cdot}-\frac{20}{\cdot}$	$\frac{10}{\cdot}$ &c.
a^3+3	$a+\frac{3}{\cdot}-\frac{9}{\cdot}+\frac{135}{\cdot}$	$\frac{1}{\cdot}-\frac{5}{\cdot}+\frac{95}{\cdot}$	$\frac{2}{\cdot}-\frac{30}{\cdot}$	
a^3+4	$a+\frac{4}{\cdot}-\frac{16}{\cdot}+\frac{320}{\cdot}$	$\frac{1}{\cdot}-\frac{7}{\cdot}+\frac{185}{\cdot}$		

Now here all the series converge faster than the like series for the square roots; because here the denominators, having higher powers, are larger than those in the former; consequently fewer terms will suffice in this case, than were requisite in the former, for an equal degree of accuracy, in all the differences and roots. The calculation for a few terms here follows.

First, To find the Diffs.

$\frac{1}{3a^2}$ = ·0033333333

$\frac{-1}{9a^5}$ = ... 11111

$\frac{+5}{81a^8}$ = ... 6

1st dif. ·0033322228

$\frac{2}{9a^5}$ = ·0000022222

$\frac{-10}{27a^8}$ = ... 37

2d dif ·0000022185

$\frac{10}{27a^8}$ = 3d dif. . . 37

Then for the Roots.

3d Dif.	2d Diffs.	1st Diffs.	Cube Roots.
$\cdot 0^8 37$	$\cdot 0^5 22185$	·0033322228	10·0000000000
	22148	33300043	0033322228
	22111	33277895	0066622271
	22074	33255784	0099900166
	22037	33233710	0133155950
	22000	33211673	0166389660
	21963	33189673	0199601333
	21926	33167710	0232791006
	21889	33145784	0265958716
	21852	33123895	0299104500
	21815	33102043	0332228395
	21778	33080228	0365330438
		33058450	0398410666
			0431469116

3. *For the Reciprocals of Numbers.*

The reciprocals of the natural numbers a, $a+1$, $a+2$, $a+3$, &c, are denoted by the fractions $\frac{1}{a}$, $\frac{1}{a+1}$, $\frac{1}{a+2}$, $\frac{1}{a+3}$, &c, where a is any integer number to commence with; which reciprocals, with their several orders of differences here follow.

Recips.	1st Diffs.	2d Diffs.	3d Diffs.
$\frac{1}{a}$			
$\frac{1}{a+1}$	$\frac{1}{a.a+1}$	$\frac{1.2}{a.a+1.a+2}$	$\frac{1.2.3}{a.a+1.a+2.a+3}$
$\frac{1}{a+2}$	$\frac{1}{a+1.a+2}$	$\frac{1.2}{a+1.a+2.a+3}$	$\frac{1.2.3}{a+1.a+2.a+3.a+4}$
$\frac{1}{a+3}$	$\frac{1}{a+2.a+3}$		

Here, if we would employ only the column of first differences, by actually multiplying the terms in their denominators, these, with their two orders of differences, will be as follow.

Denoms.	1st Dif.	2d D.
a^2+a		
	$2a+2$	
a^2+3a+2		2
	$2a+4$	
a^2+5a+6		2
	$2a+6$	
$a^2+7a+12$		

Where the first differences are in arithmetical progression, and the 2d differences equal, viz, the constant number 2. Hence the series of denominators will be very soon constructed, by two easy additions, the first of which is by the constant number 2. So, for instance, if a be $=1000$, then the first differences, and their denominators, will be thus: Where the column of first diffs. increases always by the number 2, and the column of denominators is constructed by adding the several first differences. These denominators are so large, that a very few figures in their quotients, will be sufficient to form, by one addition for each, the original column of reciprocals, to a great many places of figures. And these reciprocals will be verified and corrected at every 10th number; for any reciprocal whose denominator ends with a cipher, will have the same signifi-

1st Dif.	Denoms.
2002	1001000
2004	1003002
2006	1005006

cant figures as the reciprocal of its 10th part, which, it is supposed, has been before found.

The above first differences and denominators will be sufficient to construct the table of reciprocals, commencing with the number 1000, as far as 9 places of decimals, the constant 2d difference being 2 in the 9th place, for a considerable way. Thus, dividing 1 by the several denominators above set down, gives for their quotients the annexed column of first diffs, and thence their annexed reciprocals, &c.

1st Diffs.	Reciprocals.	Nos.
·000000999	·000999001	1001
·000000997	·000998004	1002
·000000995	·000997009	1003
·000000993	·000996016	1004

But if a table of reciprocals be desired to a greater number of decimals, we might take in, and employ, the column of 2d differences also; by which means we should obtain the series of reciprocals to 12 places of decimals. And so on, for still more figures.

From the last two or three Tracts, may be constructed, or may be easily continued further, such tables as here next follow, of the reciprocals, squares, cubes, and roots of the natural series of integer numbers; the use of which is evidently to shorten the trouble of arithmetical calculations. The structure of the table is evident: the first column contains the natural series of numbers, from 1 to 1000; the 2d the squares of the same; the 3d the cubes; the 4th the reciprocals; the 5th the square roots; and lastly the cube roots of the same. The decimals, in the columns of reciprocals and roots, are all set down to the nearest figure in the last decimal place; that is, when the next figure, beyond the last place set down in the table, came out a 5 or more, the last figure was increased by 1; otherwise not; except in the repetends, which occurred among the reciprocals, where the real last figure is always set down. Those reciprocals which in the table have less than seven places of figures, are such as terminate, and are complete within that number, having nothing remaining; such as ·5 the reciprocal of 2, ·25 the reciprocal of 4, &c. The manner and cases of applying these numbers are generally evident: but it may be remarked, that the column of reciprocals (which are no other than the decimal values of the quotients, resulting from the division of unity, or 1, by each of the several numbers, from 1 to 1000), is not only useful in showing, by inspection, the quotient when the dividend is unity or 1, but is also applied with much advantage in changing many divisions into multiplications, whatever the dividend or numerators may be, which are much easier performed, being done by only multiplying the reciprocal of the divisor, as found in the table, by the dividend, for the quotient. It will also apply to good purpose in summing the terms of many converging series, as in the 8th of these Tracts, in which a few of the first terms, to be found by division, are taken out of this table, and then added together.

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
1	1	1	1	1·0000000	1·000000
2	4	8	5	1·4142136	1·259921
3	9	27	3333333	1·7320508	1·442250
4	16	64	25	2·0000000	1·587401
5	25	125	2	2·2360680	1·709976
6	36	216	1666666	2·4494897	1·817121
7	49	343	1428571	2·6457513	1·912933
8	64	512	125	2·8284271	2·000000
9	81	729	1111111	3·0000000	2·080084
10	100	1000	1	3·1622777	2·154435
11	121	1331	0909090	3·3166248	2·223980
12	144	1728	0833333	3·4641016	2·289428
13	169	2197	0769230	3·6055513	2·351335
14	196	2744	0714285	3·7416574	2·410142
15	225	3375	0666666	3·8729833	2·466212
16	256	4096	0625	4·0000000	2·519842
17	289	4913	0588235	4·1231056	2·571282
18	324	5832	0555555	4·2426407	2·620741
19	361	6859	0526316	4·3588989	2·668402
20	400	8000	05	4·4721360	2·714418
21	441	9261	0476190	4·5825757	2·758923
22	484	10648	0454545	4·6904158	2·802039
23	529	12167	0434783	4·7958315	2·843867
24	576	13824	0416666	4·8989795	2·884499
25	625	15625	04	5·0000000	2·924018
26	676	17576	0384615	5·0990195	2·962496
27	729	19683	0370370	5·1961524	3·000000
28	784	21952	0357143	5·2915026	3·036589
29	841	24389	0344828	5·3851648	3·072317
30	900	27000	0333333	5·4772256	3·107232
31	961	29791	0322581	5·5677644	3·141381
32	1024	32768	03125	5·6568542	3·174802
33	1089	35937	0303030	5·7445626	3·207534
34	1156	39304	0294118	5·8309519	3·239612
35	1225	42875	0285714	5·9160798	3·271066
36	1296	46656	0277777	6·0000000	3·301927
37	1369	50653	0270270	6·0827625	3·332222
38	1444	54872	0263158	6·1644140	3·361975
39	1521	59319	0256410	6·2449980	3·391211
40	1600	64000	025	6·3245553	3·419952
41	1681	68921	0243902	6·4031242	3·448217
42	1764	74088	0238095	6·4807407	3·476027
43	1849	79507	0232558	6·5574385	3·503398
44	1936	85184	0227272	6·6332496	3·530348
45	2025	91125	0222222	6·7082039	3·556893
46	2116	97336	0217391	6·7823300	3·583048
47	2209	103823	0212766	6·8556546	3·608826
48	2304	110592	0208333	6·9282032	3·634241
49	2401	117649	0204082	7·0000000	3·659306
50	2500	125000	02	7·0710678	3·684031

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
51	2601	132651	0196078	7·1414284	3·708430
52	2704	140608	0192308	7·2111026	3·732511
53	2809	148877	0188679	7·2801099	3·756286
54	2916	157464	0185185	7·3484692	3·779763
55	3025	166375	0181818	7·4161985	3·802953
56	3136	175616	0178571	7·4833148	3·825862
57	3249	185193	0175439	7·5498344	3·848501
58	3364	195112	0172414	7·6157731	3·870877
59	3481	205379	0169492	7·6811457	3·892996
60	3600	216000	0166666	7·7459667	3·914867
61	3721	226981	0163934	7·8102497	3·936497
62	3844	238328	0161290	7·8740079	3·957892
63	3969	250047	0158730	7·9372539	3·979057
64	4096	262144	015625	8·0000000	4·000000
65	4225	274625	0153846	8·0622577	4·020726
66	4356	287496	0151515	8·1240384	4·041240
67	4489	300763	0149254	8·1853528	4·061548
68	4624	314432	0147059	8·2462113	4·081656
69	4761	328509	0144928	8·3066239	4·101566
70	4900	343000	0142857	8·3666003	4·121285
71	5041	357911	0140845	8·4261498	4·140818
72	5184	373248	0138888	8·4852814	4·160168
73	5329	389017	0136986	8·5440037	4·179339
74	5476	405224	0135135	8·6023253	4·198336
75	5625	421875	0133333	8·6602540	4·217163
76	5776	438976	0131579	8·7177979	4·235824
77	5929	456533	0129870	8·7749644	4·254321
78	6084	474552	0128205	8·8317609	4·272659
79	6241	493039	0126582	8·8881944	4·290841
80	6400	512000	0125	8·9442719	4·308870
81	6561	531441	0123457	9·0000000	4·326749
82	6724	551368	0121950	9·0553851	4·344481
83	6889	571787	0120482	9·1104336	4·362071
84	7056	592704	0119048	9·1651514	4·379519
85	7225	614125	0117647	9·2195445	4·396830
86	7396	636056	0116279	9·2736185	4·414005
87	7569	658503	0114943	9·3273791	4·431047
88	7744	681472	0113636	9·3808315	4·447960
89	7921	704969	0112360	9·4339811	4·464745
90	8100	729000	0111111	9·4868330	4·481405
91	8281	753571	0109890	9·5393920	4·497942
92	8464	778688	0108696	9·5916630	4·514357
93	8649	804357	0107527	9·6436508	4·530655
94	8836	830584	0106383	9·6953597	4·546836
95	9025	857375	0105263	9·7467943	4·562903
96	9216	884736	0104166	9·7979590	4·578857
97	9409	912673	0103093	9·8488578	4·594701
98	9604	941192	0102041	9·8994949	4·610436
99	9801	970299	0101010	9·9498744	4·626065
100	10000	1000000	01	10·0000000	4·641589

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
101	10201	1030301	0099009	10·0498756	4·657010
102	10404	1061208	0098039	10·0995049	4·672330
103	10609	1092727	0097087	10·1488916	4·687548
104	10816	1124864	0096154	10·1980390	4·702669
105	11025	1157625	0095238	10·2469508	4·717694
106	11236	1191016	0094340	10·2956301	4·732624
107	11449	1225043	0093458	10·3440804	4·747459
108	11664	1259712	0092592	10·3923048	4·762203
109	11881	1295029	0091743	10·4403065	4·776856
110	12100	1331000	0090909	10.4880885	4·791420
111	12321	1367631	0090090	10·5356538	4·805896
112	12544	1404928	0089286	10·5830052	4·820284
113	12769	1442897	0088496	10·6301458	4·834588
114	12996	1481544	0087719	10·6770783	4·848808
115	13225	1520875	0086957	10·7238053	4·862944
116	13456	1560896	0086207	10·7703296	4·876999
117	13689	1601613	0085470	10·8166538	4·890973
118	13924	1643032	0084746	10·8627805	4·904868
119	14161	1685159	0084034	10·9087121	4·918685
120	14400	1728000	0083333	10·9544512	4·932424
121	14641	1771561	0082645	11·0000000	4·946088
122	14884	1815848	0081967	11·0453610	4·959675
123	15129	1860867	0081300	11·0905365	4·973190
124	15376	1906624	0080645	11·1355287	4·986631
125	15625	1953125	008	11·1803399	5·000000
126	15876	2000376	0079365	11·2249722	5·013298
127	16129	2048383	0078740	11·2694277	5·026526
128	16384	2097152	0078125	11·3137085	5·039684
129	16641	2146689	0077519	11·3578167	5·052774
130	16900	2197000	0076923	11·4017543	5·065797
131	17161	2248091	0076336	11·4455231	5·078753
132	17424	2299968	0075757	11·4891253	5·091643
133	17689	2352637	0075188	11·5325626	5·104469
134	17956	2406104	0074627	11·5758369	5·117230
135	18225	2460375	0074074	11·6189500	5·129928
136	18496	2515456	0073529	11·6619038	5·142563
137	18769	2571353	0072993	11·7046999	5·155137
138	19044	2628072	0072464	11·7473444	5·167649
139	19321	2685619	0071942	11·7898261	5·180101
140	19600	2744000	0071429	11·8321596	5·192494
141	19881	2803221	0070922	11·8743421	5·204828
142	20164	2863288	0070423	11·9163753	5·217103
143	20449	2924207	0069930	11·9582607	5·229[illegible]21
144	20736	2985984	0069444	12·0000000	5·241482
145	21025	3048625	0068966	12·0415946	5·253588
146	21316	3112136	0068493	12·083[illegible]0	5·265637
147	21609	3176523	0068027	12·1243557	5·277632
148	21904	3241792	0067567	12·1655251	5·289572
149	22201	3307949	0067114	12·2065556	5·301459
150	22500	3375000	0066666	12·2474487	5·313293

Numb.	Square.	Cube.	Recipr.	Sq Root.	C. Root.
151	22801	3442951	0066225	12·2882057	5·325074
152	23104	3511808	0065789	12·3288280	5·336803
153	23409	3581577	0065359	12·3693169	5·348481
154	23716	3652264	0064935	12·4096736	5·360108
155	24025	3723875	0064516	12·4498996	5·371685
156	24336	3796416	0064103	12·4899960	5·383213
157	24649	3869893	0063694	12·5299641	5·394690
158	24964	3944312	0063291	12·5698051	5·406120
159	25281	4019679	0062893	12·6095202	5·417501
160	25600	4096000	00625	12·6491106	5·428835
161	25921	4173281	0062112	12·6885775	5·440122
162	26244	4251528	0061728	12·7279221	5·451362
163	26569	4330747	0061350	12·7671453	5·462556
164	26896	4410944	0060975	12·8062485	5·473703
165	27225	4492125	0060606	12·8452326	5·484806
166	27556	4574296	0060241	12·8840987	5·495865
167	27889	4657463	0059880	12·9228480	5·506879
168	28224	4741632	0059524	12·9614814	5·517848
169	28561	4826809	0059172	13·0000000	5·528775
170	28900	4913000	0058824	13·0384048	5·539658
171	29241	5000211	0058480	13·0766968	5·550499
172	29584	5088448	0058140	13·1148770	5·561298
173	29929	5177717	0057803	13·1529464	5·572054
174	30276	5268024	0057471	13·1909060	5·582770
175	30625	5359375	0057143	13·2287566	5·593445
176	30976	5451776	0056818	13·2664992	5·604079
177	31329	5545233	0056497	13·3041347	5·614673
178	31684	5639752	0056180	13·3416641	5·625226
179	32041	5735339	0055866	13·3790882	5·635741
180	32400	5832000	0055555	13·4164079	5·646216
181	32761	5929741	0055249	13·4536240	5·656652
182	33124	6028568	0054945	13·4907376	5·667051
183	33489	6128487	0054645	13·5277493	5·677411
184	33856	6229504	0054348	13·5646600	5·687734
185	34225	6331625	0054054	13·6014705	5·698019
186	34596	6434856	0053763	13·6381817	5·708267
187	34969	6539203	0053476	13·6747943	5·718479
188	35344	6644672	0053191	13·7113092	5·728654
189	35721	6751269	0052910	13·7477271	5·738794
190	36100	6859000	0052632	13·7840488	5·748897
191	36481	6967871	0052356	13·8202750	5·758965
192	36864	7077888	0052083	13·8564065	5·768998
193	37249	7189057	0051813	13·8924440	5·778996
194	37636	7301384	0051546	13·9283883	5·788960
195	38025	7414875	0051282	13·9642400	5·798890
196	38416	7529536	0051020	14·0000000	5·808786
197	38809	7645373	0050761	14·0356688	5·818648
198	39204	7762392	0050505	14·0712473	5·828476
199	39601	7880599	0050251	14·1067360	5·838272
200	40000	8000000	005	14·1421356	5·848035

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
201	40401	8120601	0049751	14·1774469	5·857765
202	40804	8242408	0049504	14·2126704	5·867464
203	41209	8365427	0049261	14·2478068	5·877130
204	41616	8489664	0049020	14·2828569	5·886765
205	42025	8615125	0048780	14·3178211	5·896368
206	42436	8741816	0048544	14·3527001	5·905941
207	42849	8869743	0048309	14·3874946	5·915481
208	43264	8998912	0048077	14·4222051	5·924991
209	43681	9123329	0047817	14·4568323	5·934473
210	44100	9261000	0047619	14·4913767	5·943911
211	44521	9393931	0047393	14·5258390	5·953341
212	44944	9528128	0047170	14·5602198	5·962731
213	45369	9663597	0046948	14·5945195	5·972091
214	45796	9800344	0046729	14·6287388	5·981426
215	46225	9938375	0046512	14·6628783	5·990727
216	46656	10077696	0046296	14·6969385	6·000000
217	47089	10218313	0046083	14·7309199	6·009244
218	47524	10360232	0045872	14·7648231	6·018463
219	47961	10503459	0045662	14·7986486	6·027650
220	48400	10648000	0045454	14·8323970	6·036811
221	48841	10793861	0045249	14·8660687	6·045943
222	49284	10941048	0045045	14·8996644	6·055048
223	49729	11089567	0044843	14·9331845	6·064126
224	50176	11239424	0044643	14·9666295	6·073177
225	50625	11390625	0044444	15·0000000	6·082201
226	51076	11543176	0044248	15·0332964	6·091199
227	51529	11697083	0044053	15·0665192	6·100170
228	51984	11852352	0043860	15·0996689	6·109115
229	52441	12008989	0043668	15·1327460	6·118032
230	52900	12167000	0043478	15·1657509	6·126925
231	53361	12326391	0043290	15·1986842	6·135792
232	53824	12487168	0043103	15·2315462	6·114634
233	54289	12649337	0042918	15·2643375	6·153449
234	54756	12812904	0042735	15·2970585	6·162239
235	55225	12977875	0042553	15·3297097	6·171005
236	55696	13144256	0042373	15·3622915	6·179747
237	56169	13312053	0042194	15·3948043	6·188403
238	56644	13481272	0042017	15·4272486	6·197154
239	57121	13651919	0041841	15·4596248	6·205821
240	57600	13824000	0041660	15·4919334	6·214404
241	58081	13997521	0041494	15·5241747	6·223083
242	58564	14172488	0041322	15·5563492	6·231678
243	59049	14348907	0041152	15·5884573	6·240251
244	59536	14526784	0040984	15·6204994	6·248800
245	60025	14706125	0040816	15·6524758	6·257324
246	60516	14886936	004065	15·6843871	6·265826
247	61009	15069223	0040486	15·7162336	6·274304
248	61504	15252992	0040323	15·7480157	6·282760
249	62001	15138219	0040161	15·7797338	6·291194
250	62500	15625000	004	15·8113883	6·299604

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
251	63001	15813251	0039841	15·8429795	6·307992
252	63504	16003008	0039683	15·8745079	6·316359
253	64009	16194277	0039526	15·9059737	6·324704
254	64516	16387064	0039370	15·9373775	6·333025
255	65025	16581375	0039216	15·9687194	6·341325
256	65536	16777216	0039063	16·0000000	6·349602
257	66049	16974593	0038911	16·0312195	6·357859
258	66564	17173512	0038760	16·0623784	6·366095
259	67081	17373979	0038610	16·0934769	6·374310
260	67600	17576000	0038462	16·1245155	6·382504
261	68121	17779581	0038314	16·1554944	6·390676
262	68644	17984728	0038168	16·1864141	6·398827
263	69169	18191447	0038023	16·2172747	6·406958
264	69696	18399744	0037878	16·2480768	6·415068
265	70225	18609625	0037736	16·2788206	6·423157
266	70756	18821096	0037594	16·3095064	6·431226
267	71289	19034163	0037453	16·3401346	6·439275
268	71824	19248832	0037313	16·3707055	6·447305
269	72361	19465109	0037175	16·4012195	6·455314
270	72900	19683000	0037037	16·4316767	6·463304
271	73441	19902511	0036900	16·4620776	6·471274
272	73984	20123648	0036765	16·4924225	6·479224
273	74529	20346417	0036630	16·5227116	6·487153
274	75076	20570824	0036496	16·5529454	6·495064
275	75625	20796875	0036363	16·5831240	6·502956
276	76176	21024576	0036232	16·6132477	6·510829
277	76729	21253933	0036101	16·6433170	6·518684
278	77284	21484952	0035971	16·6733320	6·526519
279	77841	21717639	0035842	16·7032931	6·534335
280	78400	21952000	0035714	16·7332005	6·542132
281	78961	22188041	0035587	16·7630546	6·549911
282	79524	22425768	0035461	16·7928556	6·557672
283	80089	22665187	0035336	16·8226038	6·565415
284	80656	22906304	0035211	16·8522995	6·573139
285	81225	23149125	0035088	16·8819430	6·580844
286	81796	23393656	0034965	16·9115345	6·588531
287	82369	23639903	0034843	16·9410743	6·596202
288	82944	23887872	0034722	16·9705627	6·603854
289	83521	24137569	0034602	17·0000000	6·611488
290	84100	24389000	0034483	17·0293864	6·619106
291	84681	24642171	0034364	17·0587221	6·626705
292	85264	24897088	0034246	17·0880075	6·634287
293	85849	25153757	0034130	17·1172428	6·641851
294	86436	25412184	0034014	17·1464282	6·649399
295	87025	25672375	0033898	17·1755640	6·656930
296	87616	25934336	0033783	17·2046505	6·664443
297	88209	26198073	0033670	17·2336879	6·671940
298	88804	26463592	0033557	17·2626765	6·679419
299	89401	26730899	0033445	17·2916165	6·686882
300	90000	27000000	0033333	17·3205081	6·694328

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
301	90601	27270901	0033223	17·3493516	6·701758
302	91204	27543608	0033113	17·3781472	6·709172
303	91809	27818127	0033003	17·4068952	6·716569
304	92416	28094464	0032895	17·4355958	6·723950
305	93025	28372625	0032787	17·4642492	6·731316
306	93636	28652616	0032680	17·4928557	6·738665
307	94249	28934443	0032573	17·5214155	6·745997
308	94864	29218112	0032468	17·5499288	6·753313
309	95481	29503629	0032362	17·5783958	6·760614
310	96100	29791000	0032258	17·6068169	6·767899
311	96721	30080231	0032154	17·6351921	6·775168
312	97344	30371328	0032051	17·6635217	6·782422
313	97969	30664297	0031949	17·6918060	6·789661
314	98596	30959144	0031847	17·7200451	6·796884
315	99225	31255875	0031746	17·7482393	6·804091
316	99856	31554496	0031646	17·7763888	6·811284
317	100489	31855013	0031546	17·8044938	6·818461
318	101124	32157432	0031447	17·8325545	6·825624
319	101761	32461759	0031348	17·8605711	6·832771
320	102400	32768000	003125	17·8885438	6·839903
321	103041	33076161	0031153	17·9164729	6·847021
322	103684	33386248	0031056	17·9443584	6·854124
323	104329	33698267	0030960	17·9722008	6·861211
324	104976	34012224	0030864	18·0000000	6·868284
325	105625	34328125	0030769	18·0277564	6·875343
326	106276	34645976	0030675	18·0554701	6·882388
327	106929	34965783	0030581	18·0831413	6·889419
328	107584	35287552	0030488	18·1107703	6·896435
329	108241	35611289	0030395	18·1383571	6·903436
330	108900	35937000	0030303	18·1659021	6·910423
331	109561	36264691	0030211	18·1934054	6·917396
332	110224	36594368	0030120	18·2208672	6·924355
333	110889	36926037	0030030	18·2482876	6·931300
334	111556	37259704	0029940	18·2756669	6·938232
335	112225	37595375	0029851	18·3030052	6·945149
336	112896	37933056	0029762	18·3303028	6·952053
337	113569	38272753	0029674	18·3575598	6·958943
338	114244	38614472	0029586	18·3847763	6·965819
339	114921	38958219	0029499	18·4119526	6·972682
340	115600	39304000	0029412	18·4390889	6·979532
341	116281	39651821	0029326	18·4661853	6·986369
342	116964	40001688	0029240	18·4932420	6·993191
343	117649	40353607	0029155	18·5202592	7·000000
344	118336	40707584	0029070	18·5472370	7·006796
345	119025	41063625	0028986	18·5741756	7·013579
346	119716	41421736	0028902	18·6010752	7·020349
347	120409	41781923	0028818	18·6279360	7·027106
348	121104	42144192	0028736	18·6547581	7·033850
349	121801	42508549	0028653	18·6815417	7·040581
350	122500	42875000	0028751	18·7082869	7·047208

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
351	123201	43243551	0028490	18·7349940	7·054003
352	123904	43614208	0028409	18·7616630	7·060696
353	124609	43986977	0028329	18·7882942	7·067376
354	125316	44361864	0028248	18·8148877	7·074043
355	126025	44738875	0028169	18·8414437	7·080698
356	126736	45118016	0028090	18·8679623	7·087341
357	127449	45499293	0028011	18·8944436	7·093970
358	128164	45882712	0027933	18·9208879	7·100588
359	128881	46268279	0027855	18·9472953	7·107193
360	129600	46656000	0027777	18·9736660	7·113786
361	130321	47045881	0027701	19·0000000	7·120367
362	131044	47437928	0027624	19·0262976	7·126935
363	131769	47832147	0027548	19·0525589	7·133492
364	132496	48228544	0027473	19·0787840	7·140037
365	133225	48627125	0027397	19·1049732	7·146569
366	133956	49027896	0027322	19·1311265	7·153090
367	134689	49430863	0027248	19·1572441	7·159599
368	135424	49836032	0027174	19·1833261	7·166095
369	136161	50243409	0027100	19·2093727	7·172580
370	136900	50653000	0027027	19·2353841	7·179054
371	137641	51064811	0026954	19·2613603	7·185516
372	138384	51478848	0026882	19·2873015	7 191966
373	139129	51895117	0026810	19·3132079	7·198405
374	139876	52313624	0026738	19·3390796	7·204832
375	140625	52734375	0026666	19·3649167	7·211247
376	141376	53157376	0026596	19·3907194	7·217652
377	142129	53582633	0026525	19·4164878	7·224045
378	142884	54010152	0026455	19·4422221	7·230427
379	143641	54439939	0026385	19·4679223	7·236797
380	144400	54872000	0026316	19·4935887	7·243156
381	145161	55306341	0026247	19·5192213	7·249504
382	145924	55742968	0026178	19·5448203	7·255841
383	146689	56181887	0026110	19·5703858	7·262167
384	147456	56623104	0026042	19·5959179	7·268482
385	148225	57066625	0025974	19·6214169	7·274786
386	148996	57512456	0025907	19·6468827	7·281079
387	149769	57960603	0025840	19·6723156	7·287362
388	150544	58411072	0025773	19·6977156	7·293633
389	151321	58863869	0025707	19·7230829	7·299893
390	152100	59319000	0025641	19·7484177	7·306143
391	152881	59776471	0025575	19·7737199	7·312383
392	153664	60236288	0025510	19·7989899	7·318611
393	154449	60698457	0025445	19·8242276	7·324829
394	155236	61162984	0025381	19·8494332	7·331037
395	156025	61629875	0025316	19·8746069	7·337234
396	156816	62099136	0025252	19·8997487	7·343420
397	157609	62570773	0025189	19·9248588	7·349597
398	158404	63044792	0025126	19·9499373	7·355762
399	159201	63521199	0025063	19·9749844	7·361917
400	160000	64000000	0025	20·0000000	7·368063

Numb.	Square.	Cube.	Recipr.	Sq Root.	C. Root.
401	160801	64481201	0024938	20·0249844	7·374198
402	161604	64964808	0024876	20·0499377	7·380322
403	162409	65450827	0024814	20·0748599	7·386437
404	163216	65939264	0024752	20·0997512	7·392542
405	164025	66430125	0024691	20·1246118	7·398636
406	164836	66923416	0024631	20·1494417	7·404720
407	165649	67419143	0024570	20·1742410	7·410794
408	166464	67917312	0024510	20·1990099	7·416859
409	167281	68417929	0024450	20·2237484	7·422914
410	168100	68921000	0024390	20·2484567	7·428958
411	168921	69426531	0024331	20·2731349	7·434993
412	169744	69934528	0024272	20·2977831	7·441018
413	170569	70444997	0024213	20·3224014	7·447033
414	171396	70957944	0024155	20·3469899	7·453039
415	172225	71473375	0024096	20·3715488	7·459036
416	173056	71991296	0024038	20·3960781	7·465022
417	173889	72511713	0023981	20·4205779	7·470999
418	174724	73034632	0023923	20·4450483	7·476966
419	175561	73560059	0023866	20·4694895	7·482924
420	176400	74088000	0023810	20·4939015	7·488872
421	177241	74618461	0023753	20·5182845	7·494810
422	178084	75151448	0023697	20·5426386	7·500740
423	178929	75686967	0023641	20·5669638	7·506660
424	179776	76225024	0023585	20·5912603	7·512571
425	180625	76765625	0023529	20·6155281	7·518473
426	181476	77308776	0023474	20·6397674	7·524365
427	182329	77854483	0023419	20·6639783	7·530248
428	183184	78402752	0023364	20·6881609	7·536121
429	184041	78953589	0023310	20·7123152	7·541986
430	184900	79507000	0023256	20·7364414	7·547841
431	185761	80062991	0023202	20·7605395	7·553688
432	186624	80621568	0023148	20·7846097	7·559525
433	187489	81182737	0023095	20·8086520	7·565353
434	188356	81746504	0023041	20·8326667	7·571173
435	189225	82312875	0022989	20·8566536	7·576984
436	190096	82881856	0022936	20·8806130	7·582786
437	190969	83453453	0022883	20·9045450	7·588579
438	191844	84027672	0022831	20·9284495	7·594363
439	192721	84604519	0022779	20·9523268	7·600138
440	193600	85184000	0022727	20·9761770	7·605905
441	194481	85766121	0022676	21·0000000	7·611662
442	195364	86350888	0022624	21·0237960	7·617411
443	196249	86938307	0022573	21·0475652	7·623151
444	197136	87528384	0022522	21·0713075	7·628883
445	198025	88121125	0022472	21·0950231	7·634606
446	198916	88716536	0022422	21·1187121	7·640321
447	199809	89314623	0022371	21·1423745	7·646027
448	200704	89915392	0022321	21·1660105	7·651725
449	201601	90518849	0022272	21·1896201	7·657414
450	202500	91125000	0022222	21·2132034	7·663094

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
451	203401	91733851	0022173	21·2367606	7·668766
452	204304	92345408	0022124	21·2602916	7·674430
453	205209	92959677	0022075	21·2837967	7·680085
454	206116	93576664	0022026	21·3072758	7·685732
455	207025	94196375	0021978	21·3307290	7·691371
456	207936	94818816	0021930	21·3541565	7·697002
457	208849	95443993	0021882	21·3775583	7·702624
458	209764	96071912	0021834	21·4009346	7·708238
459	210681	96702579	0021786	21·4242853	7·713844
460	211600	97336000	0021739	21·4476166	7·719442
461	212521	97972181	0021692	21·4709106	7·725032
462	213444	98611128	0021645	21·4941853	7·730614
463	214369	99252847	0021598	21·5174348	7·736187
464	215296	99897344	0021552	21·5406592	7·741753
465	216225	100544625	0021505	21·5638587	7·747310
466	217156	101194696	0021459	21·5870331	7·752860
467	218089	101847563	0021413	21·6101828	7·758402
468	219024	102503232	0021368	21·6333077	7·763936
469	219961	103161709	0021322	21·6564078	7·769462
470	220900	103823000	0021277	21·6794834	7·774980
471	221841	104487111	0021231	21·7025344	7·780490
472	222784	105154048	0021186	21·7255610	7·785992
473	223729	105823817	0021142	21·7485632	7·791487
474	224676	106496424	0021097	21·7715411	7·796974
475	225625	107171875	0021053	21·7944947	7·802453
476	226576	107850176	0021008	21·8174242	7·807925
477	227529	108531333	0020964	21·8403297	7·813389
478	228484	109215352	0020921	21·8632111	7·818845
479	229441	109902239	0020877	21·8860686	7·824294
480	230400	110592000	0020833	21·9089023	7·829735
481	231361	111284641	0020790	21·9317122	7·835168
482	232324	111980168	0020747	21·9544984	7·840594
483	233289	112678587	0020704	21·9772610	7·846013
484	234256	113379904	0020661	22·0000000	7·851424
485	235225	114084125	0020619	22·0227155	7·856828
486	236196	114791256	0020576	22·0454077	7·862224
487	237169	115501303	0020534	22·0680765	7·867613
488	238144	116214272	0020492	22·0907220	7·872994
489	239121	116930169	0020450	22·1133444	7·878368
490	240100	117649000	0020408	22·1359436	7·883734
491	241081	118370771	0020367	22·1585198	7·889094
492	242064	119095488	0020325	22·1810730	7·894446
493	243049	119823157	0020284	22·2036033	7·899791
494	244036	120553784	0020243	22·2261108	7·905129
495	245025	121287375	0020202	22·2485955	7·910460
496	246016	122023936	0020162	22·2710575	7·915784
497	247009	122763473	0020121	22·2934968	7·921100
498	248004	123505992	0020080	22·3159136	7·926408
499	249001	124251499	0020040	22·3383079	7·931710
500	250000	125000000	002	22·3606798	7·937005

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
501	251001	125751501	0019960	22·3830293	7·942293
502	252004	126506008	0019920	22·4053565	7·947573
503	253009	127263527	0019881	22·4276615	7·952847
504	254016	128024064	0019841	22·4499443	7·958114
505	255025	128787625	0019801	22·4722051	7·963374
506	256036	129554216	0019763	22·4944438	7·968627
507	257049	130323843	0019724	22·5166605	7·973873
508	258064	131096512	0019685	22·5388553	7·979112
509	259081	131872229	0019646	22·5610283	7·984344
510	260100	132651000	0019608	22·5831796	7·989569
511	261121	133432831	0019569	22·6053091	7·994788
512	262144	134217728	0019531	22·6274170	8·000000
513	263169	135005697	0019493	22·6495033	8·005205
514	264196	135796744	0019455	22·6715681	8·010403
515	265225	136590875	0019417	22·6936114	8·015595
516	266256	137388096	0019380	22·7156334	8·020779
517	267289	138188413	0019342	22·7376340	8·025957
518	268324	138991832	0019305	22·7596134	8·031129
519	269361	139798359	0019268	22·7815715	8·036293
520	270400	140608000	0019231	22·8035085	8·041451
521	271441	141420761	0019194	22·8254244	8·046603
522	272484	142236648	0019157	22·8473193	8·051748
523	273529	143055667	0019120	22·8691933	8·056886
524	274576	143877824	0019084	22·8910463	8·062018
525	275625	144703125	0019048	22·9128785	8·067143
526	276676	145531576	0019011	22·9346899	8·072262
527	277729	146363183	0018975	22·9564806	8·077374
528	278784	147197952	0018939	22·9782506	8·082480
529	279841	148035889	0018904	23·0000000	8·087579
530	280900	148877000	0018868	23·0217289	8·092672
531	281961	149721291	0018832	23·0434372	8·097758
532	283024	150568768	0018797	23·0651252	8·102838
533	284089	151419437	0018762	23·0867928	8·107912
534	285156	152273304	0018727	23·1084400	8·112980
535	286225	153130375	0018692	23·1300670	8·118041
536	287296	153990656	0018657	23·1516738	8·123096
537	288369	154854153	0018622	23·1732605	8·128144
538	289444	155720872	0018587	23·1948270	8·133186
539	290521	156590819	0018553	23·2163735	8·138223
540	291600	157464000	0018518	23·2379001	8·143253
541	292681	158340421	0018484	23·2594067	8·148276
542	293764	159220088	0018450	23·2808935	8·153293
543	294849	160103007	0018416	23·3023604	8·158304
544	295936	160989184	0018382	23·3238076	8·163309
545	297025	161878625	0018349	23·3452351	8·168308
546	298116	162771336	0018315	23·3666429	8·173302
547	299209	163667323	0018282	23·3880311	8·178289
548	300304	164566592	0018248	23·4093998	8·183269
549	301401	165469149	0018215	23·4307490	8·188244
550	302500	166375000	0018181	23·4520788	8·193212

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
551	303601	167284151	0018149	23·4733892	8·198175
552	304704	168196608	0018116	23·4946802	8·203131
553	305809	169112377	0018083	23·5159520	8·208082
554	306916	170031464	0018051	23·5372040	8·213027
555	308025	170953875	0018018	23·5584380	8·217965
556	309136	171879616	0017986	23·5796522	8·222898
557	310249	172808693	0017953	23·6008474	8·227825
558	311364	173741112	0017921	23·6220236	8·232746
559	312481	174676879	0017889	23·6431808	8·237661
560	313600	175616000	0017857	23·6643191	8·242570
561	314721	176558481	0017825	23·6854386	8·247474
562	315844	177504328	0017794	23·7065392	8·252371
563	316969	178453547	0017762	23·7276210	8·257263
564	318096	179406144	0017730	23·7486842	8·262149
565	319225	180362125	0017699	23·7697286	8·267029
566	320356	181321496	0017668	23·7907545	8·271903
567	321489	182284263	0017637	23·8117618	8·276772
568	322624	183250432	0017606	23·8327506	8·281635
569	323761	184220009	0017575	23·8537209	8·286493
570	324900	185193000	0017544	23·8746728	8·291344
571	326041	186169411	0017513	23·8956063	8·296190
572	327184	187149248	0017483	23·9165215	8·301030
573	328329	188132517	0017452	23·9374184	8·305865
574	329476	189119224	0017422	23·9582971	8·310694
575	330625	190109375	0017391	23·9791576	8·315517
576	331776	191102976	0017361	24·0000000	8·320335
577	332929	192100033	0017331	24·0208243	8·325147
578	334084	193100552	0017301	24·0416306	8·329954
579	335241	194104539	0017271	24·0624188	8·334755
580	336400	195112000	0017241	24·0831892	8·339551
581	337561	196122941	0017212	24·1039416	8·344341
582	338724	197137368	0017182	24·1246762	8·349125
583	339889	198155287	0017153	24·1453929	8·353904
584	341056	199176704	0017123	24·1660919	8·358678
585	342225	200201625	0017094	24·1867732	8·363446
586	343396	201230056	0017065	24·2074369	8·368209
587	344569	202262003	0017036	24·2280829	8·372966
588	345744	203297472	0017007	24·2487113	8·377718
589	346921	204336469	0016978	24·2693222	8·382465
590	348100	205379000	0016949	24·2899156	8·387206
591	349281	206425071	0016920	24·3104916	8·391942
592	350464	207474688	0016891	24·3310501	8·396673
593	351649	208527857	0016863	24·3515913	8·401398
594	352836	209584584	0016835	24·3721152	8·406118
595	354025	210644875	0016807	24·3926218	8·410832
596	355216	211708736	0016779	24·4131112	8·415541
597	356409	212776173	0016750	24·4335834	8·420245
598	357604	213847192	0016722	24·4540385	8·424944
599	358801	214921799	0016694	24·4744765	8·429638
600	360000	216000000	0016666	24·4948974	8·434327

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
601	361201	217081801	0016639	24·5153013	8·439009
602	362404	218167208	0016611	24·5356883	8·443687
603	363609	219256227	0016584	24·5560583	8·448360
604	364816	220348864	0016556	24·5764115	8·453027
605	366025	221445125	0016529	24·5967478	8·457689
606	367236	222545016	0016501	24·6170673	8·462347
607	368449	223648543	0016474	24·6373700	8·466999
608	369664	224755712	0016447	24·6576560	8·471647
609	370881	225866529	0016420	24·6779254	8·476289
610	372100	226981000	0016393	24·6981781	8·480926
611	373321	228099131	0016367	24·7184142	8·485557
612	374544	229220928	0016340	24·7386338	8·490184
613	375769	230346397	0016313	24·7588368	8·494806
614	376996	231475544	0016287	24·7790234	8·499423
615	378225	232608375	0016260	24·7991935	8·504034
616	379456	233744896	0016234	24·8193473	8·508641
617	380689	234885113	0016207	24·8394847	8·513243
618	381924	236029032	0016181	24·8596058	8·517840
619	383161	237176659	0016155	24·8797106	8·522432
620	384400	238328000	0016129	24·8997992	8·527018
621	385641	239483061	0016103	24·9198716	8·531600
622	386884	240641848	0016077	24·9399278	8·536177
623	388129	241804367	0016051	24·9599679	8·540749
624	389376	242970624	0016026	24·9799920	8·545317
625	390625	244140625	0016	25·0000000	8·549879
626	391876	245314376	0015974	25·0199920	8·554437
627	393129	246491883	0015949	25·0399681	8·558990
628	394384	247673152	0015924	25·0599282	8·563537
629	395641	248858189	0015898	25·0798724	8·568080
630	396900	250047000	0015873	25·0998008	8·572618
631	398161	251239591	0015848	25·1197134	8·577152
632	399424	252435968	0015823	25·1396102	8·581680
633	400689	253636137	0015798	25·1594913	8·586204
634	401956	254840104	0015773	25·1793566	8·590723
635	403225	256047875	0015748	25·1992063	8·595238
636	404496	257259456	0015723	25·2190404	8·599747
637	405769	258474853	0015699	25·2388589	8·604252
638	407044	259694072	0015674	25·2586619	8·608752
639	408321	260917119	0015649	25·2784493	8·613248
640	409600	262144000	0015625	25·2982213	8·617738
641	410881	263374721	0015601	25·3179778	8·622224
642	412164	264609288	0015576	25·3377189	8·626706
643	413449	265847707	0015552	25·3574447	8·631183
644	414736	267089984	0015528	25·3771551	8·635655
645	416025	268336125	0015504	25·3968502	8·640122
646	417316	269586136	0015480	25·4165301	8·644585
647	418609	270840023	0015456	25·4361947	8·649043
648	419904	272097792	0015432	25·4558441	8·653497
649	421201	273359449	0015408	25·4754784	8·657946
650	422500	274625000	0015385	25·4950976	8·662391

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
651	423801	275894451	0015361	25·5147010	8·666831
652	425104	277167808	0015337	25·5342907	8·671266
653	426409	278445077	0015314	25·5538647	8·675697
654	427716	279726264	0015291	25·5734237	8·680123
655	429025	281011375	0015267	25·5929678	8·684545
656	430336	282300416	0015244	25·6124969	8·688963
657	431649	283593393	0015221	25·6320112	8·693376
658	432964	284890312	0015198	25·6515107	8·697784
659	434281	286191179	0015175	25·6709953	8·702188
660	435600	287496000	0015151	25·6904652	8·706587
661	436921	288804781	0015129	25·7099203	8·710982
662	438244	290117528	0015106	25·7203607	8·715373
663	439569	291434247	0015083	25·7487864	8·719759
664	440896	292754944	0015060	25·7681975	8·724141
665	442225	294079625	0015038	25·7875939	8·728518
666	443556	295408296	0015015	25·8069758	8·732891
667	444889	296740963	0014993	25·8263431	8·737260
668	446224	298077632	0014970	25·8456960	8·741624
669	447561	299418309	0014948	25·8650343	8·745984
670	448900	300763000	0014925	25·8843582	8·750340
671	450241	302111711	0014903	25·9036677	8·754691
672	451584	303464448	0014881	25·9229628	8·759038
673	452929	304821217	0014859	25·9422435	8·763380
674	454276	306182024	0014837	25·9615100	8·767719
675	455625	307546875	0014814	25·9807621	8·772053
676	456976	308915776	0014793	26·0000000	8·776382
677	458329	310288733	0014771	26·0192237	8·780708
678	459684	311665752	0014749	26·0384331	8·785029
679	461041	313046839	0014728	26·0576284	8·789346
680	462400	314432000	0014706	26·0768096	8·793659
681	463761	315821241	0014684	26·0959767	8·797967
682	465124	317214568	0014663	26·1151297	8·802272
683	466489	318611987	0014641	26·1342687	8·806572
684	467856	320013504	0014620	26·1533937	8·810868
685	469225	321419125	0014599	26·1725047	8·815159
686	470596	322828856	0014577	26·1916017	8·819447
687	471969	324242703	0014556	26·2106848	8·823730
688	473344	325660672	0014535	26·2297541	8·828009
689	474721	327082769	0014514	26·2488095	8·832285
690	476100	328509000	0014493	26·2678511	8·836556
691	477481	329939371	0014472	26·2868789	8·840822
692	478864	331373888	0014451	26·3058929	8·845085
693	480249	332812557	0014430	26·3248932	8·849344
694	481636	334255384	0014409	26·3438797	8·853598
695	483025	335702375	0014388	26·3628527	8·857849
696	484416	337153536	0014368	26·3818119	8·862095
697	485809	338608873	0014347	26·4007576	8·866337
698	487204	340068392	0014327	26·4196896	8·870575
699	488601	341532099	0014306	26·4386081	8·874809
700	490000	343000000	0014286	26·4575131	8·879040

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
701	491401	344472101	0014265	26·4764046	8·883266
702	492804	345948008	0014245	26·4952826	8·887488
703	494209	347428927	0014225	26·5141472	8·891706
704	495616	348913664	0014205	26·5329983	8·895920
705	497025	350402625	0014184	26·5518361	8·900130
706	498436	351895816	0014164	26·5706605	8·904336
707	499849	353393243	0014144	26·5894716	8·908538
708	501264	354894912	0014124	26·6082694	8·912736
709	502681	356400829	0014104	26·6270539	8·916931
710	504100	357911000	0014085	26·6458252	8·921121
711	505521	359425431	0014065	26·6645833	8·925307
712	506944	360944128	0014045	26·6833281	8·929490
713	508369	362467097	0014025	26·7020598	8·933668
714	509796	363994344	0014006	26·7207784	8·937843
715	511225	365525875	0013986	26·7394839	8·942014
716	512656	367061696	0013966	26·7581763	8·946180
717	514089	368601813	0013947	26·7768557	8·950343
718	515524	370146232	0013928	26·7955220	8·954502
719	516961	371694959	0013908	26·8141754	8·958658
720	518400	373248000	0013888	26·8328157	8·962809
721	519841	374805361	0013870	26·8514432	8·966957
722	521284	376367048	0013850	26·8700577	8·971100
723	522729	377933067	0013831	26·8886593	8·975240
724	524176	379503424	0013812	26·9072481	8·979376
725	525625	381078125	0013793	26·9258240	8·983508
726	527076	382657176	0013774	26·9443872	8·987637
727	528529	384240583	0013755	26·9629375	8·991762
728	529984	385828352	0013736	26·9814751	8·995883
729	531441	387420489	0013717	27·0000000	9·000000
730	532900	389017000	0013699	27·0185122	9·004113
731	534361	390617891	0013680	27·0370117	9·008222
732	535824	392223168	0013661	27·0554985	9·012328
733	537289	393832837	0013643	27·0739727	9·016430
734	538756	395446904	0013624	27·0924344	9·020529
735	540225	397065375	0013605	27·1108834	9·024623
736	541696	398688256	0013587	27·1293199	9·028714
737	543169	400315553	0013569	27·1477439	9·032802
738	544644	401947272	0013550	27·1661554	9·036885
739	546121	403583419	0013532	27·1845544	9·040965
740	547600	405224000	0013513	27·2029410	9·045041
741	549081	406869021	0013495	27·2213152	9·049114
742	550564	408518488	0013477	27·2396769	9·053183
743	552049	410172407	0013459	27·2580263	9·057248
744	553536	411830784	0013441	27·2763634	9·061309
745	555025	413493625	0013423	27·2946881	9·065367
746	556516	415160936	0013405	27·3130006	9·069422
747	558009	416832723	0013387	27·3313007	9·073472
748	559504	418508992	0013369	27·3495887	9·077519
749	561001	420189749	0013351	27·3678644	9·081563
750	562500	421875000	0013333	27·3861279	9·085603

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
751	564001	423564751	0013316	27·4043792	9·089639
752	565504	425259008	0013298	27·4226184	9·093672
753	567009	426957777	0013280	27·4408455	9·097701
754	568516	428661064	0013263	27·4590604	9·101726
755	570025	430368875	0013245	27·4772633	9·105748
756	571536	432081216	0013228	27·4954542	9·109766
757	573049	433798093	0013210	27·5136330	9·113781
758	574564	435519512	0013193	27·5317998	9·117793
759	576081	437245479	0013175	27·5499546	9·121801
760	577600	438976000	0013158	27·5680975	9·125805
761	579121	440711081	0013141	27·5862284	9·129806
762	580644	442450728	0013123	27·6043475	9·133803
763	582169	444194947	0013106	27·6224546	9·137797
764	583696	445943744	0013089	27·6405499	9·141788
765	585225	447697125	0013072	27·6586334	9·145774
766	586756	449455096	0013055	27·6767050	9·149757
767	588289	451217663	0013038	27·6947648	9·153737
768	589824	452984832	0013021	27·7128129	9·157713
769	591361	454756609	0013004	27·7308492	9·161686
770	592900	456533000	0012987	27·7488739	9·165656
771	594441	458314011	0012970	27·7668868	9·169622
772	595984	460099648	0012953	27·7848880	9·173585
773	597529	461889917	0012937	27·8028775	9·177544
774	599076	463684824	0012920	27·8208555	9·181500
775	600625	465484375	0012903	27·8388218	9·185452
776	602176	467288576	0012887	27·8567766	9·189401
777	603729	469097433	0012870	27·8747197	9·193347
778	605284	470910952	0012853	27·8926514	9·197289
779	606841	472729139	0012837	27·9105715	9·201228
780	608400	474552000	0012821	27·9284801	9·205164
781	609961	476379541	0012804	27·9463772	9·209096
782	611524	478211768	0012788	27·9642629	9·213025
783	613089	480048687	0012771	27·9821372	9·216950
784	614656	481890304	0012755	28·0000000	9·220872
785	616225	483736025	0012739	28·0178515	9·224791
786	617796	485587656	0012723	28·0356915	9·228706
787	619369	487443403	0012706	28·0535203	9·232618
788	620944	489303872	0012690	28·0713377	9·237527
789	622521	491169069	0012674	28·0891438	9·240433
790	624100	493039000	0012658	28·1069386	9·244335
791	625681	494913671	0012642	28·1247222	9·248234
792	627264	496793088	0012626	28·1424946	9·252130
793	628849	498677257	0012610	28·1602557	9·256022
794	630436	500566184	0012594	28·1780056	9·259911
795	632025	502459875	0012579	28·1957444	9·263797
796	633616	504358336	0012563	28·2134720	9·267679
797	635209	506261573	0012547	28·2311884	9·271559
798	636804	508169592	0012531	28·2488938	9·275435
799	638401	510082399	0012516	28·2665881	9·279308
800	640000	512000000	00125	28·2842712	9·283177

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
801	641601	513922401	0012484	28·3019434	9·287044
802	643204	515849608	0012469	28·3196045	9·290907
803	644809	517781627	0012453	28·3372546	9·294767
804	646416	519718464	0012438	28·3548938	9·298623
805	648025	521660125	0012422	28·3725219	9·302477
806	649636	523606616	0012407	28·3901391	9·306327
807	651249	525557943	0012392	28·4077454	9·310175
808	652864	527514112	0012376	28·4253408	9·314019
809	654481	529475129	0012361	28·4429253	9·317859
810	656100	531441000	0012346	28·4604989	9·321697
811	657721	533411731	0012330	28·4780617	9·325532
812	659344	535387328	0012315	28·4956137	9·329363
813	660969	537366797	0012300	28·5131549	9·333191
814	662596	539353144	0012285	28·5306852	9·337016
815	664225	541343375	0012270	28·5482048	9·340838
816	665856	543338496	0012255	28·5657137	9·344657
817	667489	545338513	0012240	28·5832119	9·348473
818	669124	547343432	0012225	28·6006993	9·352285
819	670761	549353259	0012210	28·6181760	9·356095
820	672400	551368000	0012195	28·6356421	9·359901
821	674041	553387661	0012180	28·6530976	9·363704
822	675684	555412248	0012165	28·6705424	9·367505
823	677329	557441767	0012151	28·6879766	9·371302
824	678976	559476224	0012136	28·7054002	9·375096
825	680625	561515625	0012121	28·7228132	9·378887
826	682276	563559976	0012106	28·7402157	9·382675
827	683929	565609283	0012092	28·7576077	9·386460
828	685584	567663552	0012077	28·7749891	9·390241
829	687241	569722789	0012063	28·7923601	9·394020
830	688900	571787000	0012048	28·8097206	9·397796
831	690561	573856191	0012034	28·8270706	9·401569
832	692224	575930368	0012019	28·8444102	9·405338
833	693889	578009537	0012005	28·8617394	9·409105
834	695556	580093704	0011990	28·8790582	9·412869
835	697225	582182875	0011976	28·8963666	9·416630
836	698896	584277056	0011962	28·9136646	9·420387
837	700569	586376253	0011947	28·9309523	9·424141
838	702244	588480472	0011933	28·9482297	9·427893
839	703921	590589719	0011919	28·9654967	9·431642
840	705600	592704000	0011905	28·9827535	9·435388
841	707281	594823321	0011891	29·0000000	9·439130
842	708964	596947688	0011876	29·0172363	9·442870
843	710649	599077107	0011862	29·0344623	9·446607
844	712336	601211584	0011848	29·0516781	9·450341
845	714025	603351125	0011834	29·0688637	9·454071
846	715717	605495736	0011820	29·0860791	9·457799
847	717409	607645423	0011806	29·1032644	9·461524
848	719104	609800192	0011792	29·1204396	9·465247
849	720801	611960049	0011779	29·1376046	9·468966
850	722500	614125000	0011765	29·1547595	9·472682

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
851	724201	616295051	0011751	29·1719043	9·476395
852	725904	618470208	0011737	29·1890390	9·480106
853	727609	620650477	0011723	29·2061637	9·483813
854	729316	622835864	0011710	29·2232784	9·487518
855	731025	625026375	0011696	29·2403830	9·491219
856	732736	627222016	0011682	29·2574777	9·494918
857	734449	629422793	0011669	29·2745623	9·498614
858	736164	631628712	0011655	29·2916370	9·502307
859	737881	633839779	0011641	29·3087018	9·505998
860	739600	636056000	0011628	29·3257566	9·509685
861	741321	638277381	0011614	29·3428015	9·513369
862	743044	640503928	0011601	29·3598365	9·517051
863	744769	642735647	0011587	29·3768616	9·520730
864	746496	644972544	0011574	29·3938769	9·524406
865	748225	647214625	0011561	29·4108823	9·528079
866	749956	649461896	0011547	29·4278779	9·531749
867	751689	651714363	0011534	29·4448637	9·535417
868	753424	653972032	0011521	29·4618397	9·539081
869	755161	656234909	0011507	29·4788059	9·542743
870	756900	658503000	0011494	29·4957624	9·546402
871	758641	660776311	0011481	29·5127091	9·550058
872	760384	663054848	0011468	29·5296461	9·553712
873	762129	665338617	0011455	29·5465734	9·557363
874	763876	667627624	0011442	29·5634910	9·561010
875	765625	669921875	0011429	29·5803989	9·564655
876	767376	672221376	0011416	29·5972972	9·568297
877	769129	674526133	0011403	29·6141858	9·571937
878	770884	676836152	0011390	29·6310648	9·575574
879	772641	679151439	0011377	29·6479325	9·579208
880	774400	681472000	0011363	29·6647939	9·582839
881	776161	683797841	0011351	29·6816442	9·586468
882	777924	686128968	0011338	29·6984848	9·590093
883	779689	688465387	0011325	29·7153159	9·593716
884	781456	690807104	0011312	29·7321375	9·597337
885	783225	693154125	0011299	29·7489496	9·600954
886	784996	695506456	0011287	29·7657521	9·604569
887	786769	697864103	0011274	29·7825452	9·608181
888	788544	700227072	0011261	29·7993289	9·611791
889	790321	702595369	0011249	29·8161030	9·615397
890	792100	704969000	0011236	29·8328678	9·619001
891	793881	707347971	0011223	29·8496231	9·622603
892	795664	709732288	0011211	29·8663690	9·626201
893	797449	712121957	0011198	29·8831056	9·629797
894	799236	714516984	0011186	29·8998328	9·633390
895	801025	716917375	0011173	29·9165506	9·636981
896	802816	719323136	0011161	29·9332591	9·640569
897	804609	721734273	0011148	29·9499583	9·644154
898	806404	724150792	0011136	29·9666481	9·647736
899	808201	726572699	0011123	29·9833287	9·651316
900	810000	729000000	0011111	30·0000000	9·654893

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
901	811801	731432701	0011099	30·0166620	9·658468
902	813604	733870808	0011086	30·0333148	9·662040
903	815409	736314327	0011074	30·0499584	9·665609
904	817216	738763264	0011062	30·0665928	9·669176
905	819025	741217625	0011050	30·0832179	9·672740
906	820836	743677416	0011038	30·0998339	9·676301
907	822649	746142643	0011025	30·1164407	9·679860
908	824464	748613312	0011013	30·1330383	9·683416
909	826281	751089429	0011001	30·1496269	9·686970
910	828100	753571000	0010989	30·1662063	9·690521
911	829921	756058031	0010977	30·1827765	9·694069
912	831744	758550528	0010965	30·1993377	9·697615
913	833569	761048497	0010953	30·2158899	9·701158
914	835396	763551944	0010941	30·2324329	9·704698
915	837225	766060875	0010929	30·2489669	9·708236
916	839056	768575296	0010917	30·2654919	9·711772
917	840889	771095213	0010905	30·2820079	9·715305
918	842724	773620632	0010893	30·2985148	9·718835
919	844561	776151559	0010881	30·3150128	9·722363
920	846400	778688000	0010870	30·3315018	9·725888
921	848241	781229961	0010858	30·3479818	9·729410
922	850084	783777448	0010846	30·3644529	9·732930
923	851929	786330467	0010834	30·3809151	9·736448
924	853776	788889024	0010823	30·3973683	9·739963
925	855625	791453125	0010810	30·4138127	9·743475
926	857476	794022776	0010799	30·4302481	9·746985
927	859329	796597983	0010787	30·4466747	9·750493
928	861184	799178752	0010776	30·4630924	9·753998
929	863041	801765089	0010764	30·4795013	9·757500
930	864900	804357000	0010753	30·4959014	9·761000
931	866761	806954491	0010741	30·5122926	9·764497
932	868624	809557568	0010730	30·5286750	9·767992
933	870489	812166237	0010718	30·5450487	9·771484
934	872356	814780504	0010707	30·5614136	9·774974
935	874225	817400375	0010695	30·5777697	9·778461
936	876096	820025856	0010684	30·5941171	9·781946
937	877969	822656953	0010672	30·6104557	9·785428
938	879844	825293672	0010661	30·6267857	9·788908
939	881721	827936019	0010650	30·6431069	9·792386
940	883600	830584000	0010638	30·6594194	9·795861
941	885481	833237621	0010627	30·6757233	9·799333
942	887364	835896888	0010616	30·6920185	9·802803
943	889249	838561807	0010604	30·7083051	9·806271
944	891136	841232384	0010593	30·7245830	9·809736
945	893025	843908625	0010582	30·7408523	9·813198
946	894916	846590536	0010571	30·7571130	9·816659
947	896809	849278123	0010560	30·7733651	9·820117
948	898704	851971392	0010549	30·7896086	9·823572
949	900601	854670349	0010537	30·8058436	9·827025
950	902500	857375000	0010526	30·8220700	9·830475

Numb.	Square.	Cube.	Recipr.	Sq. Root.	C. Root.
951	904401	860085351	0010515	30·8382879	9·833923
952	906304	862801408	0010504	30·8544972	9·837369
953	908209	865523177	0010493	30·8706981	9·840812
954	910116	868250664	0010482	30·8868904	9·844253
955	912025	870983875	0010471	30·9030743	9·847692
956	913936	873722816	0010460	30·9192497	9·851128
957	915849	876467493	0010449	30·9354166	9·854561
958	917764	879217912	0010438	30·9515751	9·857992
959	919681	881974079	0010428	30·9677251	9·861421
960	921600	884736000	0010416	30·9838668	9·864848
961	923521	887503681	0010406	31·0000000	9·868272
962	925444	890277128	0010395	31·0161248	9·871694
963	927369	893056347	0010384	31·0322413	9·875113
964	929296	895841344	0010373	31·0483494	9·878530
965	931225	898632125	0010363	31·0644491	9·881945
966	933156	901428696	0010352	31·0805405	9·885357
967	935089	904231063	0010341	31·0966236	9·888767
968	937024	907039232	0010331	31·1126984	9·892174
969	938961	909853209	0010320	31·1287648	9·895580
970	940900	912673000	0010309	31·1448230	9·898983
971	942841	915498611	0010299	31·1608729	9·902383
972	944784	918330048	0010288	31·1769145	9·905781
973	946729	921167317	0010277	31·1929479	9·909177
974	948676	924010424	0010267	31·2089731	9·912571
975	950625	926859375	0010256	31·2249900	9·915962
976	952576	929714176	0010246	31·2409987	9·919351
977	954529	932574833	0010235	31·2569992	9·922738
978	956484	935441352	0010225	31·2729915	9·926122
979	958441	938313739	0010215	31·2889757	9·929504
980	960400	941192001	0010204	31·3049517	9·932883
981	962361	944076141	0010194	31·3209195	9·936261
982	964324	946966168	0010183	31·3368792	9·939636
983	966289	949862087	0010173	31·3528308	9·943009
984	968256	952763904	0010163	31·3687743	9·946379
985	970225	955671625	0010152	31·3847097	9·949747
986	972196	958585256	0010142	31·4006369	9·953113
987	974169	961504803	0010132	31·4165561	9·956477
988	976144	964430272	0010121	31·4324673	9·959839
989	978121	967361669	0010111	31·4483704	9·963198
990	980100	970299000	0010101	31·4642654	9·966554
991	982081	973242271	0010091	31·4801525	9·969909
992	984064	976191488	0010081	31·4960315	9·973262
993	986049	979146657	0010070	31·5119025	9·976612
994	988036	982107784	0010060	31·5277655	9·979959
995	990025	985074875	0010050	31·5436206	9·983304
996	992016	988047936	0010040	31·5594677	9·986648
997	994009	991026973	0010030	31·5753068	9·989990
998	996004	994011992	0010020	31·5911380	9·993328
999	998001	997002999	0010010	31·6069613	9·996665
1000	1000000	1000000000	001	31·6227767	10.000000

END OF VOL. I.

ERRATA.

Page 264, line 5 from the bottom, for 5^5, read $5r^5$.

Page 266, line 1, for a^2b^2, read $4a^2b^2$.

Page 430, line 22, for $\frac{2}{5}t^5$, read $\frac{1}{5}t^5$.

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