MATHEMATICAL TRACTS.

No. I.

DETERMINANTS.

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TRACTS

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 1.

DETERMINANTS.

BY

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"That vast theory, transcendental in point of difficulty, elementary in regard to its being the basis of researches in the higher arithmetic, and in analytical geometry." —(M. HERMITE, quoted by Prof. SYLVESTER in *Phil. Mag.* 1852.)

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INTRODUCTION TO THE SERIES.

During some intervals of foreign travel, and consequent interruption of formal ministerial labor, I resolved to begin the preparation of a Series of Elementary Tracts upon the following subjects in the Modern Higher Mathematics; viz—

Trilinear Coordinates.
Invariants.
Theory of Surfaces.
Elliptic Integrals.
Quaternions.

Upon further reflection, I have concluded to introduce the Series by a treatise upon Determinants, brief and very elementary, but sufficiently inclusive and rigorous to support and explain the references to this theory which are involved in the ordinary exposition of the first three subjects of the proposed list.

In undertaking this labor, I hope to turn the attention of the students of my country, especially those who are desirous of becoming Mathematicians, to these studies, which at present lie considerably beyond the usual "Scientific Course," even in our best colleges, but which the demands of Physics and higher Engineering must soon bring within it. I purpose, therefore, to give a strictly elementary view of the principal developments of the Pure Mathematics since the year 1841. I mark this year, not only because it is the proper initial point at which to begin the proposed survey, but because the year itself was remarkably rich in mathematical productions.

Jacobi, in that year, exhibited the versatility of his genius, whose power twelve years before had been proved by his "*Nova Fundamenta,*" in giving to the world his celebrated Memoirs "*De formatione et proprietatibus Determinantium*" and *De Determinantibus Functionalibus,*" which have been the bases of all subsequent labors in the Theory of Determinants.

In the same year, Dr. Geo. Boole laid down the principles out of which has grown the Modern Higher Algebra.

In the year 1841, also, was published, in the seventh volume of "*Mémoires des Savans étrangers,*" the full text of the general theory of the Abelian Functions, although what was known as Abel's Theorem had appeared much earlier.

Before the close of that year, last and perhaps least, but confessedly of immense influence on the British Universities, was published Gregory's "Processes and Examples of the Differential and Integral Calculus."

At this period Modern Geometry was unknown. Indeed, till the appearance of Townsend's volumes, in 1863, it is believed that the only work in the English language on this subject was that of Dr. Mulcahy, and in any language that of Chasles, "*Traité de Géométrie Supérieure,*" which had then been published but little more than a decade.

Mathematicians have not only introduced a new language, which, taken in connexion with the new processes, makes modern mathematics absolutely unintelligible to one who has for a few years laid aside such studies, but also new functions, whose theory is regarded as a high subject of research. It would be simple pedantry to attempt an illustration of this in terms of the language itself; but we may select a function which is well known, and, ascending briefly the steps by which it has reached its present development, observe something of the spirit of modern mathematical analysis.

Take the theory of Elliptic Functions. Before the middle of the last century, mathematicians began to investigate the solutions of problems depending on the rectification of elliptic arcs.

Undoubtedly the first definite progress in the right direction was the discovery of Euler, which is recorded in sec 7. of *Novi Comm. Petrop.* for 1758-59, and which gives the integral of the differential equation

$$\frac{mdx}{(a+bx+cx^2+dx^3+ex^4)^{\frac{1}{2}}}=\frac{ndy}{(a+by+cy^2+dy^3+ey^4)^{\frac{1}{2}}}.$$

The next step was taken by Lagrange, who published, in the fourth volume (p. 98) of "*Mélanges de Philosophie et de Mathématique de Turin,*" an *à priori* solution of the same general equation which Euler had solved tentatively for special cases.

In 1775, John Landen published in the "Philosophical Transactions" his theorem, showing that any arc of an hyperbola is equal to the difference of two elliptic arcs. The extension of this theorem relating to the general theory of transformation is still the subject of research among mathematicians, among whom especially may be mentioned Richelot (see "*Die Landensche Transformation,*" Königsburg, 1868, also in several volumes of *Crelle's Journal.*)

In 1786, Legendre's first paper upon Elliptic Integrals was presented to the French Academy; and from that time onward, for a space of nearly fifty years, till his death, this subject chiefly engaged his attention; and when, in 1825, he presented to the Académie des Sciences the first volume of his "*Traité des Fonctions Elliptiques,*" it was supposed that the resources of the Integral Calculus in this direction were exhausted.

About this time, however, the young Norwegian Abel appeared upon the field; and, by bringing into his analysis the general Theory of Equations, was enabled to show that what had been done was but a small part of what might be expected; and immediately extended the boundaries of knowledge by proving his theorem for the comparison of all Transcendental Functions whatever, whose differentials are irrational from involving the second root of a rational function of the variable x. This is not the place to describe Abel's theorem; but the great research bestowed by modern mathematicians upon the Abelian Functions serves to show the spirit and line of a particular analysis, and the interest which attaches to a subject, which, under continual expansion for more than a century by minds of the highest mathematical power, still suggests for itself a much greater amplitude.

In the complete works of Abel, by Holmboe, we see the ease and power of that remarkable genius, for whom the principal mathematicians of his age, Poisson, Cauchy, and Legendre, foresaw the wreath of an enduring fame. Of the labors of Jacobi in this direction, whose work, "*Nova Fundamenta,*" appeared in the same year of Abel's death, 1829, it is not my intention to speak. Had Abel reached the patriarchal age of Legendre, he would still be living to write theorems for *future* generations. Abel died before he had completed his twenty-seventh year.

In *cahier* 23 of the "*Journal de l'Ecole Polytechnique,*" and in the 9th volume of *Liouville,* and in the 18th and 19th of *Comptes Rendus,* we find Abel's work proved and elucidated by Hermite and Liouville. In these journals, and in *Comptes Rendus* since 1843, the contributions of MM. Serret and Chasles would need especial study. So also "*Theorie der Abelschen Functionen,*" by Clebsch and Gordon, Professors in the University of Giessen (1866), and "*Théorie des Fonctions doublement périodiques et des Fonctions elliptiques,*" by Briot and Bouquet (1859).

It is not necessary to mention the greater number of distinguished Continental writers upon Abelian functions. Neumann of Halle, and Riemann of Tübingen (1863-4), Ivory, Bronwin, and Cayley, of Cambridge, are some of the well known writers upon these functions.

The student, however, should not fail to study the papers of Konigsberger (*Crelle,* Vol. 64) and Weirstrass on the solution of Hyperelliptic Functions (*Crelle,* Vol. 47); nor should a paper by Rosenhain, in "*Mémoires de l'Institut par divers Savans,*" be omitted, as also a report by Russell on Elliptic and Hyperelliptic Integrals before the British Association, from 1870 and now in progress.

But what is the use of such studies? If the array of illustrious names herein given do not sufficiently guarantee their importance, let me say that it is by such abstract and difficult labors men become mathematicians. What then? Well, suppose that it is shown that the secular inequalities resulting from the action of one planet on another are the same as if the mass of the disturbing planet were diffused along its orbit in the form of an elliptic ring of variable but indefinitely small thickness, and that it is inquired, what is the attraction exerted by such a ring upon an external point? The problem involves eventually two elliptic integrals, as Gauss shows, of the first and second kinds.

The final application, then, of the higher analysis must be the sufficient answer to all *cui bono* inquirers. Take, for instance, the original Bessel's functions in $L^m_{(z)}$, $Y^m_{(z)}$, and

$$Jn\,(z) = \frac{1}{\pi}\int_0^\pi \cos\,(z \sin \omega - n\omega)\, d\omega,$$

hitherto mostly in the hands of German mathematicians, and successfully applied to the solution of physical problems in heat, electricity, and the investigation of aerial vibrations in cylindrical spaces. A good example and illustration of this function may be seen in "*Studien über Bessell'schen Funktionen*," by Dr. Eugen Lommel, a paper in *Crelle*, Vol. 56, and one of high value by Strutt of Cambridge.

If the utility, then, of advanced modern mathematical study is not to be doubted, what provision can be made for its wider diffusion?

Now, the work of reducing the higher mathematics to the comprehension of ordinary readers, while confessedly a difficult and generally a thankless undertaking, has in some cases been attended with unlooked-for success.

Bowditch's notes upon "*Mécanique Céleste*," side by side with his translation; Mrs. Somerville's paraphrase of the same original work, and the excessive elementary labors of the Jesuit Fathers upon the *Principia*, were and are rewarded with the strongest expressions of appreciation. And there can be no doubt that similar labors will, in some circles, always be regarded with favor. Students must early know the goal, else their ambition may come too late. The equipment of a mathematican is now a very different thing from what it was thirty, or even ten, years ago. There should be some way by which, in very early years, the broad field of modern mathematics could be entered. Determinants should be taught constantly with common Algebra; Quaternions with Geometry; Trilinear Coordinates with the Cartesian; and Invariants, Co-variants, and Contravariants with the general Theory of Equations.

One grand principle should never be forgotten: the educational value of a subject is greatly modified by the the hands which administer it.

This is conspicuously true in mathematical teaching, whether by books or lectures. Let this be suggested. Every high subject has its easy elementary side, and there it may be pierced. The works of Cremona, Helmholtz, Tait, Sylvester, Clifford, and Cayley, may, in some of their elementary forms, be commingled with ordinary mathematical studies; and thus the ancient tasks of the student will be expanded and enlivened by fresh contributions from the great teachers of the world. Inspiration is needed for study, and study must deepen the inspiration.

The fundamental equations of Quaternions in i, j, k are easily exhibited to a class in Geometry in such manner as to become a source of real pleasure to them; and thus they may be incited to learn the power of an instrument which bids fair to stand unrivalled in the field of mathematical physics.

The rich stores of research and discovery entombed in the volumes of the learned societies of Europe, and in the mathematical journals, are something enormous; and my object is to bring, in a more elementary form, some of the more important subjects into a wider notice.

In regard to this tract on Determinants, it is very elementary, and intended to be more suggestive than exhaustive.

The works consulted in its preparation embrace the entire literature of the subject; viz.—The theory and practice of Determinants, by Baltzer, Brioschi, Spottiswoode, Salmon, Trudi, Dodgson (the two latter hardly worth consulting); the numberless papers in *Crelle* and *Liouville*, and in the Proceedings of the Royal Societies, from 1841; also a short account of Functional Determinants in the Analytical Mechanics of Prof. Peirce, of Harvard; and the chapter devoted to the subject by Todhunter, in his Theory of Equations, and those of Boole, Ferrers, and Whitworth.

W. J. W.

15, Regent Square,
London, W.C.; 1875.

ELEMENTARY DETERMINANTS.

CHAPTER I.

PRINCIPLES.

1. *Definitions.*—The common and general expression for a determinant of the nth order consists of the arrangement of n^2 quantities in n rows and n columns, as follows :—

$$a_{21} \qquad ? \qquad \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{12} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}$$

or, more briefly, $\Sigma\,(\pm\, a_{11}\; a_{22}\; \ldots\ldots\; a_{nn})$,

which is to be understood as expressing the sum of the $1 \,.\, 2 \,.\, 3 \ldots n$ products obtained by fully permuting the n suffixes, so that each product shall include all the suffixes, and the several products differ from each other by at least one variation of these suffixes.

Every variation in the suffixes introduces a change of sign.

The letters of the expression

$$\Sigma\,(\pm\, a_{11}\; a_{22}\; \ldots\ldots\; a_{nn}),$$

taken from the diagonal of the square, are called the *leading* letters; and these, together with all the others of the determinant, are called the *constituents.*

The products themselves, when formed, are called the *elements.**

* Called by Laplace resultants (*Hist. de l'Acad.* 1772).

Constituents are called conjugate to each other, when, considered in reference to their respective rows and columns, they hold the same positions.

Reserving other definitions till we have made some development of the subject, let us seek some simple illustrations in the formation of determinants.

2. Let us assume a determinant of two places, or the second order, as
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Writing together the leading letters $a_1 b_2$, we have the first element, or product, and permuting the suffixes we obtain the second $a_2 b_1$, since, by definition, a variation of the suffixes gives a change of sign.* These two products taken together form the *expansion* of the determinant. Hence we write
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

3. Connecting now these constituents with variables, let us find the conditions of co-existence of the homogeneous equations of the first degree
$$a_1 x + b_1 y = a_2 x + b_2 y,$$
and suppose c to be their common value, then
$$a_1 x + b_1 y = c,$$
$$a_2 x + b_2 y = c.$$

Eliminating y and x, we have
$$\left.\begin{aligned} (a_1 b_2 - a_2 b_1)\, x &= b_2 c - b_1 c \\ (a_1 b_2 - a_2 b_1)\, y &= a_1 c - a_2 c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$
and

Observe (1) that the coefficient $a_1 b_2 - a_2 b_1$ is common to both

* Laplace has not only stated the rule for the change of signs by *disarrangement*, which he refers to M. Cramer, but proves the more simple rule of Bezout by permutation of the suffixes. (*Hist. de l'Acad.* 1772, p. 295.)

variables ; (2) that this coefficient is identical with the value of the determinant
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$
as given above ; it may therefore be written in that form. The same remark will evidently apply to the second members b_2c-b_1c and a_1c-a_2c, and we may write (1) as

$$\left.\begin{aligned} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x &= \begin{vmatrix} c & b_1 \\ c & b_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y &= \begin{vmatrix} c & a_2 \\ c & a_1 \end{vmatrix} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2).$$

If now we regard $c = 0$, the second members of (1) and (2) vanish, and we have simply
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$
which must be interpreted as the condition of the co-existence of the two equations, when their second members vanish.

4. It will be sufficiently evident, from what has preceded, how a determinant of the second order, as last written, is to be expanded ; viz., by multiplying together the letters of each diagonal, beginning with the upper left-hand corner, and connecting the products with the negative sign. Observing this rule for forming the products, it plainly can make no difference with the result, if we write the rows as columns ; as,
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

5. It is also evident that the sign of the determinant will change when the rows, or columns, are interchanged ; as,
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}.*$$

* Laplace "*Sur le calcul intégral.*"

6. Let us now examine a determinant of the third order,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Each of these three determinants is formed by omitting in succession the row and column which contain a_1, b_1, c_1; *i.e.*, by writing in the above order the remainders of the second and third columns, the third and first, and first and second. The further expansion may be written out by the rule already given for the determinant of the second degree, that is, by diagonal multiplication. Otherwise, we may write down the leading letters a_1, b_2, c_3, and fully permute the suffixes. The sum of all the products thus obtained, with their appropriate signs, will express the true expansion. The rule for the signs being, as has been stated, that every variation of the suffixes yields a change of sign, or that an even number of permutations gives a *plus*, and an odd number, a *minus* sign. The permutation of the suffixes of the diagonal letters ($a_1\ b_2\ c_3$) gives six products, which result corresponds with products obtained from reducing each of the three equivalent determinants as written above; as,

$$a_1b_2c_3 - a_1b_3c_2 + b_1c_2a_3 - b_1c_3a_2 + c_1a_2b_3 - c_1a_3b_2.$$

7. It is easily shown that, if two rows, or two columns, become identical, the determinant vanishes; as,

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

which is perhaps sufficiently obvious without multiplying out at length.

8. We shall now proceed to show how this determinant arises. Having shown how a determinant of the third order may be reduced, the determinant itself being given, let us

seek, inversely, to construct a function, or its equivalent functions, by an actual process of elimination, such that its several products shall be identical with those of the given determinant.*

Let us seek, for example, the condition of the co-existence of the equations

$$a_1x + b_1y + c_1z = a_2x + b_2y + c_2z = a_3x + b_3y + c_3z.$$

If the common value be zero, then

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (3).$$

As to the manner of solving these equations so as to exhibit the required condition, two methods, at least, are open to us.†

(*a.*) We may multiply the second of the equations by l, the third by m, and add; then, whatever the value of the variables, l and m may be so taken as to cause two of their coefficients with which they are multiplied to disappear—that is, two of the coefficients of the second and third equations; and, since the equations are simultaneous, that of the third must vanish also. The equations will now contain only two unknowns, l and m, whence these may be determined from the second and third, and their values substituted will give the desired condition. So far as this relates to elimination, it is similar to the method employed by Laplace, referred to in a note below.

(*b.*) Otherwise, by eliminating alternately y and z from (3),

$$(a_2b_3 - a_3b_2)x + (c_2b_3 - b_2c_3)z = 0,$$

$$(a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = 0,$$

* This was also exhibited by Laplace, and its application to the resolution of linear equations. It might be of interest to compare the method of Lagrange, in his Memoir on the "Movement of the nodes and inclination of the orbits of planets," with the theorem of Malmsten for finding particular integrals by determinants.

† See Ferrers, Salmon, and Tait, on Determinants.

which, remembering to change signs in transposing, may evidently be written

$$\frac{x}{b_2c_3-c_2b_3}=\frac{y}{a_3c_2-a_2c_3}=\frac{z}{a_2b_3-a_3b_2}.$$

Dividing now the terms of the first of (3) respectively by these equals, we have

$$a_1(b_2c_3-c_2b_3)+b_1(a_3c_2-a_2c_3)+c_1(a_2b_3-a_3b_2)=0 \text{ ...... (4).}$$

Had we divided in the same manner the terms of the second and third equations of (3), we should have found identical relations; and, since the equations are simultaneous, we have therefore found the required condition. If now we perform in (4) the multiplications indicated, we shall have six products identical with those obtained above.

We see also that (4) may be written (Art. 5)

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}+b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}=0,$$

and hence also $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

9. If the products are written out, by permuting the suffixes, it is only necessary to observe the cyclic order; thus, if we have $a_1(b_2c_3)$ the next function of the order or line a must be $a_2(b_3c_1)$, and the third $a_3(b_1c_2)$.

Also that, while $(a_1b_2c_3)$ is of course identical with itself, it indicates the determinant equally in each of two positions; *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The one $=(a_1b_2c_3)=a_1(b_2c_3)+b_1(c_2a_3)+c_1(a_2b_3)$,

the other $=(a_1b_2c_3)=a_1(b_2c_3)+a_2(b_3c_1)+a_3(b_1c_2)$.

Taken together the products are equal, but the corresponding terms, after the first, are dissimilar. The reason of this remark will be obvious, when it is considered that the products may be derived otherwise than by permuting the suffixes.

10. We are in a position now to illustrate one or two important uses of determinants as a system of notation. The equation, for instance, of the straight line passing through two given points, may be written as a determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & y_2 \\ x & x_1 & x_2 \end{vmatrix} = 0.$$

In this form it is easily remembered; while, for practical use, greater brevity and clearness will be ensured. Suppose one of the points, as $(x_2 y_2)$, is changed to the origin; then, since its coordinates must vanish, the determinant becomes

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & 0 \\ x & x_1 & 0 \end{vmatrix} = 0.$$

Now, we know the equation of a line passing through the origin and a given point to be $y = \frac{y_1}{x_1} x$, which must be the value of the determinant if our notation holds true. Hence we write

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & 0 \\ x & x_1 & 0 \end{vmatrix} = \begin{vmatrix} y & y_1 \\ x & x_1 \end{vmatrix} = 0;$$

and we may thus here state what will be found true generally, *that if all the constituents but one of a column or row become zero, both the column and row which contain that constituent may be erased from the determinant.*

A second illustration may be found in the expression for the area of a triangle in terms of the coordinates of its vertices, the axes being rectangular.* This may be written as the

* Salmon's Conics, p. 30.

foregoing, with an additional suffix, as,

$$\tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0.$$ = area ?

If, now, two of the points, as $(x_1\ y_1)$, $(x_2\ y_2)$, be connected with the origin, the coordinates evidently of the other vertex become zero, and we have, therefore, simply

$$\tfrac{1}{2}\begin{vmatrix} y_1 & y_2 \\ x_1 & x_2 \end{vmatrix} = 0.$$

Other illustrations will occur to the reader.

11. *If any row or column be multiplied by any quantity, the determinant is multiplied by that quantity.*

$$x\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} xa & b & c \\ xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} ax & bx & cx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad \ldots\ldots(1).$$

A negative sign placed before the determinant is equivalent to interchanging one row or column with another parallel to it.

? (-)

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a & b \end{vmatrix} = \begin{vmatrix} b & a \\ b_1 & a_1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots (2).$$

It follows, from (1), that

$$\begin{vmatrix} x^2 & xy & xz \\ x & y_1 & z_1 \\ x & y_2 & z_2 \end{vmatrix} = x^2\begin{vmatrix} 1 & y & z \\ 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \end{vmatrix};$$

and, from (1) and (2), that ?(-)

$$\begin{vmatrix} a & b & c \\ a & b & c_1 \\ a & b & c_2 \end{vmatrix} = ab\begin{vmatrix} 1 & c & 1 \\ 1 & c_1 & 1 \\ 1 & c_2 & 1 \end{vmatrix}.$$

All these results are so nearly self-evident, or so easily verified, that it is only necessary to write them down.

12. *Minors.* The determinant

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + b \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} + c \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \ldots (1)$$

may be written more briefly as

$$\Delta = aA + bB + cC,$$

where Δ represents the primitive determinant, and A, B, and C the several minors formed, as is evident, by omitting in turn the column and row which contain a, b, c. The determinant may also be written $\Delta = aA + a_1B_1 + a_2C_1$, where A, B_1, and C_1 represent the minors when the rows of the determinant are written as columns—

$$\begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} = a \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + a_1 \begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} + a_2 \begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} \quad \ldots (2).$$

In comparing (1) and (2), it will be seen that the first minor in each of the two sets of minors is the same, but the others are unlike. It is important to observe this difference. The practical use will be seen in the solution of the following equations:

$$\begin{aligned} ax + by + cz &= e, \\ a_1x + b_1y + c_1z &= e_1, \\ a_2x + b_2y + c_2z &= e_2. \end{aligned}$$

Multiply the first by A, the second by B_1, and the third by C_1, and add, and we have $\Delta x = Ae + B_1e_1 + C_1e_2$, since y and z vanish. In like manner the values of y and z may be found.

It might be necessary to some readers to see the entire process written out; thus,

$$\begin{aligned} aAx + bAy + cAz &= e\,A, \\ a_1B_1x + b_1B_1y + c_1B_1z &= e_1B_1, \\ a_2C_1x + b_2C_1y + c_2C_1z &= e_2C_1; \end{aligned}$$

$$\text{or}\quad a\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} x + b\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} y + c\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} z = e\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} x + b_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} y + c_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} z = e_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix}$$

$$a_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} x + b_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} y + c_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} z = e_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix}$$

Adding and combining, we have

$$\begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} x + \begin{vmatrix} b & b_1 & b_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} y + \begin{vmatrix} c & c_1 & c_2 \\ c & c_1 & c_2 \\ b & b_1 & b_2 \end{vmatrix} z = \begin{vmatrix} e & e_1 & e_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix}.$$

The coefficients of y and z having two parallel lines identical vanish, and we have, changing the rows to columns for the final expression,

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} x = \begin{vmatrix} e & b & c \\ e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \end{vmatrix}$$

13. *If the constituents of any determinant be resolvable into the sum of n other constituents, the determinant is resolvable into the sum of n other determinants.*

Let $\Delta = aA + bB + cC$, where A, B, C have the meaning of Art 12. Increase a, b, c by x, y, z, respectively,*

$$\Delta_1 = (a+x)A + (b+y)B + (c+z)\,C$$

$$= (a+x)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + (b+y)\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} + (c+z)\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\text{Hence}\qquad \Delta_1 = \begin{vmatrix} a+x & a_1 & a_2 \\ b+y & b_1 & b_2 \\ c+z & c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 \\ y & b_1 & b_2 \\ z & c_1 & c_2 \end{vmatrix}$$

If we had used the sign of multiplication, or division,

* Salmon's Deter. p. 10.

between the quantities a and x, b and y, &c., we should have reached results already pointed out in Art. 11.

14. Since identical parallel lines in a determinant cause it to vanish, we might infer the same result if two given lines differ only by a constant factor; as,

$$\begin{vmatrix} ax & a \\ ax & a \end{vmatrix} = x \begin{vmatrix} a & a \\ a & a \end{vmatrix} = 0.$$

So also, having in mind the proof of the last Art., we might show that, when the sum of several lines differs from the given lines only by constant factors, the same result will follow; as,

$$\begin{vmatrix} la+ma_1 & a & a_1 \\ lb+mb_1 & b & b_1 \\ lc+mc_1 & c & c_1 \end{vmatrix} = l \begin{vmatrix} a & a & a_1 \\ b & b & b_1 \\ c & c & c_1 \end{vmatrix} + m \begin{vmatrix} a_1 & a & a_1 \\ b_1 & b & b_1 \\ c_1 & c & c_1 \end{vmatrix}$$

In the same manner, if to any line we add the sum of the other lines separately, or increased by constant factors, the determinant will vanish;* thus,

$$\begin{vmatrix} 1+na+mb & a & b \\ 1+na_1+mb_1 & a_1 & b_1 \\ 1+na_2+mb_2 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & a & b \\ 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \end{vmatrix} + \begin{vmatrix} na+mb & a & b \\ na_1+mb_1 & a_1 & b_1 \\ na_2+mb_2 & a_2 & b_2 \end{vmatrix}$$

And, as the last determinant vanishes, the remaining one is evidently the original. Hence we may easily verify the following:

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a-(b+c) & b & c \\ a_1-(b_1+c_1) & b_1 & c_1 \\ a_2-(b_2+c_2) & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a-a_2 & b-b_2 & c-c_2 \\ a_1-a_2 & b_1-b_2 & c_1-c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

&c. &c. &c.

15. *Determinants of the fourth order.*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

expresses the condition of the co-existence of four homogeneous

* Spottiswoode's Deter. p. 18.

equations of the first degree when the second members vanish (Art. 8). We may regard it as the sum of four determinants of the third order, each of which gives three other partial determinants, and each of these in turn gives two products. The whole number of products of a determinant of this order will therefore be

$$1 \,.\, 2 \,.\, 3 \,.\, 4,$$

a result identical with the number obtained by permuting the suffixes, as $a_1b_2c_3d_4$. This, as a determinant, may be expressed as four partial determinants,

$$a_1\,(b_2\,c_3\,d_4),\quad a_2\,(b_3\,c_4\,d_1),\quad a_3\,(b_4\,c_1\,d_2),\quad a_4\,(b_1\,c_2\,d_3).$$

This result may be obtained by the actual solution of four equations with four variables, or the law of formation as seen in the case of three unknowns would enable us to write

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & . & . \\ . & c_3 & . \\ . & . & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_3 & . & . \\ . & c_4 & . \\ . & . & d_1 \end{vmatrix} + a_3 \begin{vmatrix} b_4 & . & . \\ . & c_1 & . \\ . & . & d_2 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & . & . \\ . & c_2 & . \\ . & . & d_3 \end{vmatrix}$$

with this exception, we could not tell what signs to write before them.

These must be determined by considering the number of permutations which arise between $a_1(b_2\,c_3\,d_4)$ and $a_2(b_3\,c_4\,d_1)$. First, considering $a_1\,(b_2\,c_3\,d_4)$, which we assume to be *plus*, we exchange suffixes with a and b, which gives the required suffix for a. Then, let b and d exchange, which gives d_1, finally b and c, when the required element is reached in all three permutations; and therefore the sign must be negative, by definition, since the number is odd. The third element is obtained from $a_1\,(b_2\,c_3\,d_4)$ by permuting the suffixes of a and c, and b and d, an even number; and therefore the sign to be prefixed is *plus*.

The fourth element with three permutations is found, in the same manner, to be negative; and thus the whole number of products of this, and any other determinant, of any order, assuming the law of formation to be general, may be written out at once.

16. Before proceeding further, it may be interesting to work one or two examples for the sake of illustrating the reduction of determinants of the third order as exhibited under Art. 14, and show how the same principles may be applied to those of four places.

Ex. 1.—Let it be required to find the equation of a circle through three points, say (2, 3), (4, 5), (6, 1).

We shall evidently obtain three equations by substituting successively these coordinates of the three points in the general equation $x^2+y^2+2ax+2by+c=0$, viz.,

$$4a+\ \ 6b+c=-13,$$
$$8a+10b+c=-41,$$
$$12a+\ \ 2b+c=-37.$$

To obtain a, b, c we have

$$\begin{vmatrix} 4 & 6 & 1 \\ 8 & 10 & 1 \\ 12 & 2 & 1 \end{vmatrix} a = \begin{vmatrix} -13 & 6 & 1 \\ -41 & 10 & 1 \\ -37 & 2 & 1 \end{vmatrix}$$

$$8\begin{vmatrix} -2 & 2 & 0 \\ -1 & 4 & 0 \\ 3 & 1 & 1 \end{vmatrix} a = 2\begin{vmatrix} -13 & 3 & 1 \\ -28 & 2 & 0 \\ -24 & -2 & 0 \end{vmatrix}$$

$$8\begin{vmatrix} -2 & 2 \\ -1 & 4 \end{vmatrix} a = 8\begin{vmatrix} -7 & 2 \\ -6 & -2 \end{vmatrix}$$

$$a=-\frac{13}{3}.$$

Explanation.—The Δ, coefficient of a, has the two factors 2 and 4. The bottom row subtracted from each of the other rows gives zero for a constituent in two places, which, by Art. 8, causes Δ to reduce to the 2nd or lowest order.

The absolute term of the equation is first factored, and then the upper row is taken from each of the others, when it, like the other, reduces to a Δ of 2nd degree.

For the other values of the unknowns, we write for the determinant, which has been found to be -48, successively,

$$\Delta b = \begin{vmatrix} -13 & 1 & 4 \\ -41 & 1 & 8 \\ -37 & 1 & 12 \end{vmatrix} \quad \text{and} \quad \Delta c = \begin{vmatrix} -13 & 4 & 6 \\ -41 & 8 & 10 \\ -37 & 12 & 2 \end{vmatrix}$$

$$b = -\frac{8}{3}, \qquad c = \frac{61}{3}.$$

From these values the required equation can be formed. The reductions of the right members of these equations are effected in the same manner as for the value of a, almost by simple inspection.

Ex. 2.—Take the quadric (given by Dr. Salmon, p. 168 of his *Solid Geometry*)

$$7x^2 + 6y^2 + 5z^2 - 4yz - 4xy + 10x + 4y + 6z + 4 = 0,$$

differentiate it with respect to its variables, and we shall have

$$7x - 2y + 5 = 0,$$
$$-2x + 6y - 2z + 2 = 0,$$
$$-2y + 5z + 3 = 0,$$
$$5x + 2y + 3z + 4 = 0.$$

The determinant will then be, when written at length,

$$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 0 & 5 \\ -4 & 6 & -2 & 2 \\ -3 & -2 & 5 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -6 & -6 & -3 \\ 0 & 14 & 10 & 18 \\ 0 & 4 & 14 & 15 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

Explanation.—Obtained by taking the last column from the first; next, twice the bottom row from the first, and adding four and three times the same to the second and third rows; then thrice the last column from the first and second.

It is to be especially observed that the sign changes in the determinant when the factor 12 appears. The reason is obvious, since the whole determinant is

$$a_1(b_2c_3d_4) - a_2(b_3c_4d_1) + a_3(b_4c_1d_2) - a_4(b_1c_2d_3),$$

and $\quad a_1 = a_2 = a_3 = 0,$

while $a_4 = 1$, and therefore in this case the determinant reduces to

$$-(b_1c_2d_3).$$

$$= 12 \begin{vmatrix} 3 & 3 & 1 \\ 7 & 5 & 6 \\ 2 & 7 & 5 \end{vmatrix} = 12 \begin{vmatrix} 0 & 0 & 1 \\ -11 & -13 & 6 \\ -13 & -8 & 5 \end{vmatrix}$$

$$= 12 \begin{vmatrix} -11 & -13 \\ -13 & -8 \end{vmatrix} = -972.$$

Ex. 3.—To find an expression for three points in involution.

Substitute, in the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0,$$

$$x_1 = a_1 + a_2, \quad x_2 = b_1 + b_2, \quad x_3 = c_1 + c_2.$$

$$y_1 = a_1 a_2, \quad y_2 = b_1 b_2, \quad y_3 = c_1 c_2.$$

and we have

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ a_1 a_2 & b_1 b_2 & c_1 c_2 \end{vmatrix} = (c_1 - a_2)(b_1 - c_2)(a_1 - b_2) + (c_2 - a_1)(b_2 - c_1)(a_2 - b_1).$$

Ex. 4.—The following solution is given of the determinant proposed by Dr. Salmon (p. 12, *Deter.*)

$$\begin{vmatrix} 25 & -15 & 23 & -5 \\ -15 & -10 & 19 & 5 \\ 23 & 19 & -15 & 9 \\ -5 & 5 & 9 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 30 & 104 & -50 \\ 0 & -20 & 1 & 15 \\ 0 & 14 & -24 & 14 \\ -5 & 5 & 9 & -5 \end{vmatrix}$$

Explanation.—The sum of 2nd and 4th columns is taken from the 1st, 9 times last row is added to the first, once and twice the last are taken from 2nd and 3rd, the sum of 2nd and 3rd columns is taken from the first, the last row is added to the 1st and 2nd, and finally twice the 2nd row is added to the last. Sign changes twice — 1st, when $-a_4(b_1 c_2 d_3)$ alone remains of the determinant; and, 2nd, when by a still further reduction the determinant becomes $-a_2(a_3 b_1)$.

$$= +5 \begin{vmatrix} 0 & 80 & -36 \\ -12 & -23 & 29 \\ 0 & -70 & 72 \end{vmatrix} = 36 \times -600 \begin{vmatrix} 8 & -1 \\ -7 & 2 \end{vmatrix}$$

$$= -194400.$$

Ex. 5.—*Notation.*

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

This determinant expresses what will be at once recognised as expressing the elimination of x, y, z from

$$lx + my + nz = 0,$$
$$l_1 x + m_1 y + n_1 z = 0,$$
$$l_2 x + m_2 y + n_2 z = 0,$$

or the condition that three straight lines may be parallel to one plane.

Ex. 6.—Let S_1, S_2, S_3 be three circles of the form

$$S_1 = (x - a_1)^2 + (y - b_1)^2 - c_1^2 = 0;$$

find the circle orthotomic to these three.

Let S be the required circle; and, since S and S_1 are to be orthotomic, we must have

$$(a_1 - a)^2 + (b_1 - b)^2 = c_1^2 + c^2;$$

and, by eliminating a, b, and $(a^2 + b^2 - c^2)$ from the four resulting equations, we have the determinant

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ a_1^2 + b_1^2 - c_1^2 & a_1 & b_1 & 1 \\ a_2^2 + b_2^2 - c_2^2 & a_2 & b_2 & 1 \\ a_3^2 + b_3^2 - c_3^2 & a_3 & b_3 & 1 \end{vmatrix} = 0.$$

17. *Multiplication of Determinants.* — We have now to determine the product of one determinant by another. This we may accomplish by the method of *transformation.**

* Salmon, Spottiswoode, and Tait.

Let us take two systems of linear equations,

$$\begin{aligned} ax + by + cz &= v, \\ a_1 x + b_1 y + c_1 z &= v_1, \\ a_2 x + b_2 y + c_2 z &= v_2, \end{aligned}$$

and

$$\begin{aligned} dv + ev_1 + fv_2 &= 0, \\ d_1 v + e_1 v_1 + f_1 v_2 &= 0, \\ d_2 v + e_2 v_1 + f_2 v_2 &= 0. \end{aligned}$$

Substitute the values of v, v_1, &c. in the second, and, collecting terms, we shall have

$$\begin{aligned} (ad + ea_1 + fa_2)\, x + \&c. &= 0, \\ (ad_1 + e_1 a_1 + f_1 a_2)\, x + \&c. &= 0, \\ (ad_2 + e_2 a_1 + f_2 a_2)\, x + \&c. &= 0. \end{aligned}$$

The condition of coexistence of these equations (Art. 8) will be the determinant

$$\begin{vmatrix} ad + a_1 e + a_2 f & bd + b_1 e + b_2 f & cd + c_1 e + c_2 f \\ ad_1 + a_1 e_1 + a_2 f_1 & bd_1 + b_1 e_1 + b_2 f_1 & cd_1 + c_1 e_1 + c_2 f_1 \\ ad_2 + a_1 e_2 + a_2 f_2 & bd_2 + b_1 e_2 + b_2 f_2 & cd_2 + c_1 e_2 + c_2 f_2 \end{vmatrix} = 0 \ldots (1).$$

But it is evident that these two systems of equations may be treated as one, since the variables they contain are common to both; and we may inquire the condition of coexistence of these six equations

$$\begin{array}{lllllll} ax + by + cz & -v & & & & & = 0, \\ a_1 x + b_1 y + c_1 z & & -v_1 & & & & = 0, \\ a_2 x + b_2 y + c_2 z & & & -v_2 & & & = 0, \\ & dv & +ev_1 & +fv_2 & & & = 0, \\ & d_1 v & +e_1 v_1 & +f_1 v_2 & & & = 0, \\ & d_2 v & +e_2 v_1 & +f_2 v_2 & & & = 0. \end{array}$$

Here we may say, as before, the condition of co-existence of these equations is expressed by the determinant

$$\begin{vmatrix} a & b & c & -1 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & -1 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & -1 \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & d_1 & e_1 & f_1 \\ 0 & 0 & 0 & d_2 & e_2 & f_2 \end{vmatrix} = 0.$$

This determinant may evidently be written

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = A \times D,$$

or
$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \times \begin{vmatrix} d & e & f \\ d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

but this is no other than the condition expressed by (1) ; and therefore we say that (1) and (2) must be equal.

That is, the product of one determinant by another *is a determinant whose constituents consist of the sums of the products obtained by multiplying each column of the one determinant by the rows of the other.*

Ex. 1. $$\begin{vmatrix} \cos a & \cos b & \cos c \\ \cos a_1 & \cos b_1 & \cos c_1 \\ \cos a_2 & \cos b_2 & \cos c_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Ex. 2.—The equations of four planes intersecting in a point

are
$$\begin{aligned} lx + my + nz + d &= 0, \\ l_1 x + m_1 y + n_1 z + d_1 &= 0, \\ l_2 x + m_2 y + n_2 z + d_2 &= 0, \\ l_3 x + m_3 y + n_3 z + d_3 &= 0, \end{aligned}$$

and the determinant formed is evident; but if two of the planes pass through the axis of z, we shall have

$$\begin{vmatrix} l & m & n & d \\ l_1 & m_1 & n_1 & d_1 \\ l_2 & m_2 & 0 & 0 \\ l_3 & m_3 & 0 & 0 \end{vmatrix} = 0,$$

which is simply the product of the two determinants

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \quad \begin{vmatrix} n & d \\ n_1 & d_1 \end{vmatrix} = 0,$$

which may be multiplied by the rule.

Suppose, however, we wished to interpret the latter equation geometrically, in which case we see that either

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} n & d \\ n_1 & d_1 \end{vmatrix} = 0.$$

The first supposition marks the coincidence of the third and fourth planes; the second that the four planes intersect somewhere in the axis of z.

CHAPTER II.

FORMS OF INVERSE AND SKEW DETERMINANTS.

18. *Minors as constituents and as differential coefficients.*

We have already seen that a determinant may be written briefly by the aid of its minors as

$$\Delta = aA + bB + cC.$$

But since in any determinant we can interchange parallel lines and obtain the same result with a change of sign, when the number of such interchanges is odd, we can write a determinant of the third order, as above,

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1,$$

as the result of interchanging the first and second rows, and

$$\Delta = a_2 A_2 + b_2 B_2 + c_2 C_2$$

for the like process between the first and third, or evidently, in general,

$$\Delta = a_{1c} A_{1c} + a_{2c} A_{2c} \;\ldots\ldots\; a_{nc} A_{nc} \;\ldots\ldots\ldots\ldots\ldots (1),$$

where c is 1 ...... n.

If now we write

$$\begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}$$

we have what is called the *inverse* or reciprocal of a determinant of three places, that is, a determinant consisting of the *minors* corresponding to the *constituents* of the given determinant.

19. If, now, we differentiate (1), of the last Art., in respect to to a_{1c}, we must have

$$\frac{d\Delta}{da_{1c}} = A_{1c}, \quad \frac{d\Delta}{da_{2c}} = A_{2c}, \text{ \&c.}$$

That is, if we differentiate a determinant in respect to any constituent, the corresponding minor will be the differential coefficient.*

Hence, for a determinant of the nth order, we may write

$$\Delta = a_{1c}\frac{d\Delta}{da_{1c}} + a_{2c}\frac{d\Delta}{da_{2c}} \ldots\ldots a_n \frac{d\Delta}{da_{nc}} \ldots\ldots\ldots\ldots (1).$$

While this is a more cumbrous notation than that which it replaces, it has its advantages, which will become more apparent; for example, it enables us to distinguish, at once, between those determinants which do and do not, identically, vanish.

Since a determinant is the same in the sum of the products (Art. 14), whether we expand in the order of the rows or columns, we may write

$$\Delta = a_{k1}\frac{d\Delta}{da_{k1}} + a_{\ 2}\frac{d\Delta}{da_{k2}} \ldots\ldots a_{kn}\frac{d\Delta}{da_{kn}}.$$

It is equally evident, from what has preceded, that

$$a_{1k}\frac{d\Delta}{da_{1c}} + a_{2k}\frac{d\Delta}{da_{2c}} \ldots\ldots a_{nk}\frac{d\Delta}{da_{nc}} = 0,$$

since in these products we have *in fact* introduced into the given determinant a line parallel and identical with some other line, and therefore the determinant in such form vanishes identically.

This may be explained briefly thus: from what has preceded, it is manifest that, when we take the sum of the products of any line—that is, the sum of the products of all the constituents of that line by their corresponding minors—the determinant subsists; but if the minors do not correspond with their constituents, the determinant vanishes identically; hence, in general,

$$a_{1s}\frac{d\Delta}{da_{1c}} + a_{2s}\frac{d\Delta}{da_{2c}} \ldots\ldots a_{ns}\frac{d\Delta}{a_{nc}} = 0.$$

* The notation followed here is the same as that of Jacobi, Baltzer, Spottiswoode, and Brioschi.

20. We shall fail, perhaps, of our object unless we descend to special cases.

Let us take a determinant of four places

$$\begin{vmatrix} a_{11} & . & . & a_{14} \\ a_{21} & a_{22} & . & a_{24} \\ . & . & . & . \\ a_{41} & . & . & a_{44} \end{vmatrix}$$

The first minor is obtained by erasing one row and one column, the second minor by erasing two rows and two columns. Let A_{11} = the first minor, and A_{22} the second;

then $$\frac{d\Delta}{da_{11}} = A_{11}, \ \frac{d^2\Delta}{da_{11}\, da_{22}} = A_{22};$$

but $$\frac{d^2\Delta}{da_{11}^2} = 0.$$

So also, when we take the second differential in respect to either the first row or column, the result must be the same, since A_{11} does not contain any one of these constituents.

Hence, in general, we may write

$$\frac{d^2\Delta}{da_{11}\, da_{n1}} = 0 \text{ or } \frac{d^2\Delta}{da_{11}\, da_{1n}} = 0.$$

21. Since an interchange of two lines effects a change of sign, we must indicate a corresponding change in the ensuing differential coefficient.

Thus, while $$\frac{d^2\Delta}{da_{11}\, da_{22}} = A_{22},$$

an exchange of a_{22} with a_{12} or a_{21} gives

$$\frac{d^2\Delta}{da_{21}\, da_{12}} = \frac{d^2\Delta}{da_{12}\, da_{21}} = -\frac{d^2\Delta}{da_{11}\, da_{22}} = A_{22},$$

since in either exchange the second minor is not affected, or in general

$$\frac{d^2\Delta}{da_{cc}\, da_{kk}} = -\frac{d^2\Delta}{da_{ck}\, da_{kc}}.$$

Evidently no *à priori* proof is needed here; a simple induction, as above, is sufficient: or, in other words, the theorem demands only a clear statement, when its truth is at once obvious.

22. In the case of a symmetrical determinant (Art. 1, def.), when $a_{21} = a_{12}$, we shall find, on differentiating the determinant in reference to any conjugate constituent, that the differential coefficient will be doubled, since the constituent function is supposed to enter twice; as, if $a_{12} = a_{21}$ and $a_{13} = a_{31}$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

and we have $\dfrac{d\Delta}{da_{12}} = 2A_{12}$, and, in general,

$$\frac{da_{ck}}{da_{kc}} = 1 \text{ and } \frac{d\Delta}{da_{ck}} = 2\,A_{ck}.$$

23. In the case, then, of a skew,* as the following, when the terms of the leading diagonal are zero, and the conjugates are of opposite signs, as

$$\begin{vmatrix} 0 & a_{12} & . & . & . & a_{1n} \\ a_{21} & 0 & . & . & . & . \\ a_{31} & . & 0 & . & . & . \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & . & . & . & 0 \end{vmatrix}$$

$$a_{12} = -a_{21} \text{ and } a_{1n} = -a_{n1};$$

in which case $\dfrac{da_{12}}{da_{21}} = -1$;

consequently $\dfrac{d\Delta}{da_{1n}} = A_{1n} - A_{n1} = 0$,

when the determinant is of the third and every odd order.

When the determinant is skew and of an even order, we shall have

$$A_{12} = -A_{21} \text{ and } \frac{d\Delta}{da_c} = 2A_{ck}.\dagger$$

* Salmon, p. 30; Crelle, Vol. 51, p. 264.
† Baltzer, p. 13.

That is, *when the skew symmetric determinant is of an odd degree, it vanishes; but if of an even degree, its differential coefficient in respect to any constituent function is equal to twice its corresponding minor.*

24. Referring again to equation (1), Art. 19, we see that, since $\frac{d\Delta}{da_{1\,c}} = A_{1\,c}$ is a determinant of the $n-1$ order, it may, as such, have an expansion similar to that equation. If, for example, the original determinant were of the fourth order, $\frac{d\Delta}{da_{1\,c}}$ would express a determinant whose outer row and column had been erased, in other words, a determinant of the third order.

Let us take, then, $\frac{d\Delta}{da_{k\,1}}$ to represent generally a determinant of the $n-1$ order, and suppose

$$\Delta = a_{c\,1}\frac{d\Delta}{da_{c\,1}} + a_{c\,2}\frac{d\Delta}{da_{c\,2}} \;\ldots\ldots\; a_{c\,n}\frac{d\Delta}{da_{c\,n}},$$

to represent a determinant of the n^n order; then, if we differentiate this equation in respect to $a_{k\,1}$, the left member will be identical with the proposed expression for the determinant of the $n-1$ order; that is,

$$\frac{d\Delta}{da_{k\,1}} = a_{c\,1}\frac{d^2\Delta}{da_{c\,1}\,da_{k\,1}} + a_{c\,2}\frac{d^2\Delta}{da_{c\,2}\,da_{k\,1}} \;\ldots\ldots\; a_{c\,n}\frac{d^2\Delta}{da_{c\,n}\,da_{k\,1}}.$$

The same equation, differentiated with respect to $a_{k\,2}$ and $a_{k\,n}$, will yield similar expressions for determinants of the $n-1$ order,

$$\frac{d\Delta}{da_{k\,2}} = a_{c\,1}\frac{d^2\Delta}{da_{c\,1}\,da_{k\,2}} + a_{c\,2}\frac{d^2\Delta}{da_{c\,2}\,da_{k\,2}} \;\ldots\; a_{c\,n}\frac{d^2\Delta}{da_{c\,n}\,da_{k\,2}},$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\frac{d\Delta}{da_{k\,n}} = a_{c\,1}\frac{d^2\Delta}{da_{c\,1}\,da_{k\,n}} + \;\ldots\ldots\ldots\ldots\ldots\ldots\; a_{c\,n}\frac{d^2\Delta}{da_{c\,n}\,da_{k\,n}}.$$

25. Remembering that the determinant subsists when the constituent function, and the function of the differential coefficient, as factors, are identical (Art. 18), we may write

$$\Delta = a_{1\,c}\frac{d\Delta}{da_{1\,c}} + a_{2\,c}\frac{d\Delta}{da_{2\,c}} \cdots\cdots a_{n\,c}\frac{d\Delta}{da_{n\,c}};$$

but
$$0 = a_{1\,1}\frac{d\Delta}{da_{1\,c}} + a_{2\,1}\frac{d\Delta}{da_{2\,c}} \cdots\cdots a_{n\,1}\frac{d\Delta}{da_{n\,c}},$$

when c and 1 are different.

We shall continue to use the differential notation, and apply it to the minors of the reciprocal of a determinant, as

$$\begin{vmatrix} \dfrac{d\Delta}{da_{1\,1}} & \dfrac{d\Delta}{da_{1\,2}} & \cdots\cdots & \dfrac{d\Delta}{da_{1\,n}} \\ \dfrac{d\Delta}{da_{2\,1}} & \dfrac{d\Delta}{da_{2\,2}} & \cdots\cdots & \dfrac{d\Delta}{da_{2\,n}} \\ \cdots & \cdots & & \cdots \\ \dfrac{d\Delta}{da_{n\,1}} & \dfrac{d\Delta}{da_{n\,2}} & \cdots\cdots & \dfrac{d\Delta}{da_{n\,n}} \end{vmatrix}.$$

We might use a different notation, as

$$\begin{vmatrix} A_{1\,1} & A_{1\,2} & \cdots\cdots & A_{1\,n} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ A_{n\,1} & A_{n\,2} & \cdots\cdots & A_{n\,n} \end{vmatrix}$$

but we prefer to familiarize the reader with the one we have adopted.

26. We now propose the following theorem:—*Any determinant other than skew, multiplied by its second differential coefficient, is equal to the difference of the products of the differential coefficients* $a_{1\,1}$, $a_{1\,2}$, $a_{2\,1}$, $a_{2\,2}$, *taken as conjugates.*

Confining, for the present, the demonstration to a particular case, let us write

$$0 = a_{1\,1}\frac{d\Delta}{da_{1\,2}} + a_{2\,1}\frac{d\Delta}{da_{2\,2}} \cdots\cdots a_{4\,1}\frac{d\Delta}{da_{4\,2}},$$

$$\Delta = a_{12}\frac{d\Delta}{da_{12}} + a_{22}\frac{d\Delta}{da_{22}} \;\cdots\cdots\; a_{42}\frac{d\Delta}{da_{42}},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$0 = a_{14}\frac{d\Delta}{da_{12}} + a_{24}\frac{d\Delta}{da_{22}} \;\cdots\cdots\; a_{44}\frac{d\Delta}{da_{42}}.$$

Multiply these equations by

$$\frac{d^2\Delta}{da_{11}\,da_{21}}, \quad \frac{d^2\Delta}{da_{11}\,da_{22}} \;\cdots\cdots\; \frac{d^2\Delta}{da_{11}\,da_{24}},$$

respectively; and, adding the results,

$$\Delta\frac{d^2\Delta}{da_{11}\,da_{22}} = \left\{\begin{array}{l} \left(a_{11}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\frac{d^2\Delta}{da_{11}\,da_{22}} \;\cdots\cdots\; a_{14}\frac{d^2\Delta}{da_{11}\,da_{24}}\right)\frac{d\Delta}{da_{12}} \\ \left(a_{21}\frac{d\Delta}{da_{11}\,da_{21}} + a_{22}\frac{d^2\Delta}{da_{11}\,da_{22}} \;\cdots\cdots\; a_{24}\frac{d^2\Delta}{da_{11}\,da_{24}}\right)\frac{d\Delta}{da_{22}} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \left(a_{41}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{42}\frac{d^2\Delta}{da_{11}\,da_{22}} \;\cdots\cdots\; a_{42}\frac{d^2\Delta}{da_{11}\,da_{24}}\right)\frac{d\Delta}{da_{42}} \end{array}\right.$$

The right member may be reduced as follows:—The first parenthesis becomes, by making one interchange of suffixes,

$$\left(a_{11}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\frac{d^2\Delta}{da_{21}\,da_{12}} \;\cdots\cdots\; a_{14}\frac{d^2\Delta}{da_{14}\,da_{21}}\right).$$

But $$\Delta = a_{11}\frac{d\Delta}{da_{11}} + a_{12}\frac{d\Delta}{da_{12}} \;\cdots\cdots\; a_{14}\frac{d\Delta}{da_{14}},$$

and $$-\frac{d\Delta}{da_{21}} = a_{11}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\frac{d^2\Delta}{da_{12}\,da_{21}} \;\cdots\cdots\; a_{14}\frac{d^2\Delta}{da_{14}\,da_{21}},$$

therefore $-\frac{d\Delta}{da_{21}}$ is the value of the parenthesis.

In the same manner, the second is found equal to $\frac{d\Delta}{da_{11}}$; and so also the third and fourth, *without change of sign.* That is, the values of the third and fourth parentheses appear to have the same sign. The *essential* sign must be determined from the rule of signs.

In this case we remember that

$$a_1(b_2 c_3 d_4) - a_2(b_3 c_4 d_1) + a_3(b_4 c_1 d_2) - a_4(b_1 c_2 d_3).$$

The parentheses after the second therefore destroy each other. The multipliers used interpose to change this order in the first and second, and hence we write as the result

$$\Delta \frac{d^2\Delta}{da_{11}\,da_{22}} = \frac{d\Delta}{da_{11}} \cdot \frac{d\Delta}{da_{22}} - \frac{d\Delta}{da_{21}} \cdot \frac{d\Delta}{da_{12}}.$$

This proof might, it is evident, have been made general. It is now, however, in a form to be readily verified.

27. Theorem second.—*A determinant formed from the first differential coefficients of the given determinant may be expressed in terms of the given determinant, and is equal to that determinant involved to a degree one less than its number of places.*

Let
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}$$

be the given determinant.

Its first differential coefficients, taken in order and arranged in square form, will then be

$$\Delta_1 = \begin{vmatrix} \frac{d\Delta}{da_{11}} & \frac{d\Delta}{da_{12}} & \ldots\ldots & \frac{d\Delta}{da_{1n}} \\ \frac{d\Delta}{da_{21}} & \frac{d\Delta}{da_{22}} & \ldots\ldots & \frac{d\Delta}{da_{2n}} \\ \cdots & \cdots & & \cdots \\ \frac{d\Delta}{da_{n1}} & \frac{d\Delta}{da_{n2}} & \ldots\ldots & \frac{d\Delta}{da_{nn}} \end{vmatrix}$$

Let now Δ_1 be multiplied by Δ, and we shall have

$$\Delta\Delta_1 =$$

$$\left| \begin{array}{ccccc} a_{11}\dfrac{d\Delta}{da_{11}} + \dots a_{1n}\dfrac{d\Delta}{da_{1n}} & a_{21}\dfrac{d\Delta}{da_{11}} + \dots a_{2n}\dfrac{d\Delta}{da_{1n}} & \dots\dots & a_{n1}\dfrac{d\Delta}{da_{11}} + \dots & \&c. \\ a_{11}\dfrac{d\Delta}{da_{21}} + \dots a_{1n}\dfrac{d\Delta}{da_{2n}} & a_{21}\dfrac{d\Delta}{da_{21}} + \dots a_{2n}\dfrac{d\Delta}{da_{2n}} & \dots\dots & \&c. & \&c. \\ \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots & \dots & \dots & \dots \\ a_{11}\dfrac{d\Delta}{da_{n1}} + \dots a_{1n}\dfrac{d\Delta}{da_{nn}} & \&c. \qquad \&c. & & a_{n1}\dfrac{d\Delta}{da_{n1}} + \dots & \&c. \end{array} \right.$$

Observing these products, it will be seen that all except those of the leading diagonal vanish identically; and hence we have

$$\Delta\Delta_1 = \begin{vmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta \end{vmatrix} = \Delta^n,$$

or
$$\Delta_1 = \Delta^{n-1},$$

which was to be proved.

28. We shall now begin to introduce, as we proceed with the general theory, some of the geometrical uses of determinants.

Mr. Spottiswoode, in Vol. 51, p. 262, and Prof. Cayley, in 32nd Vol. of *Crelle*, have discussed the subject of orthogonal substitutions in connection with skew determinants.*

We have already given a definition of a skew determinant; we will now show how to effect an orthogonal transformation of the third order, and express the values of the nine direction-cosines in terms of three independent variables, or in general how to connect n^2 quantities by $\frac{1}{2}n\,(n+1)$ relations,

* On the number of linear substitutions, see *Journal de l'Ecole Polytechnique*, Tom. 22, 38 cahier.

$\frac{1}{2}n(n-1)$ of them only being independent. Let us, for example, write the following linear equations:

$$x = a_{11}u + a_{12}v + a_{13}w,$$
$$y = a_{21}u + a_{22}v + a_{23}w,$$
$$z = a_{31}u + a_{32}v + a_{33}w,$$

and a derived system

$$X = a_{11}u + a_{21}v + a_{31}w,$$
$$Y = a_{12}u + a_{22}v + a_{32}w,$$
$$Z = a_{13}u + a_{23}v + a_{33}w,$$

where we will suppose $a_{ik} = -a_{ki}$ and $a_{ii} = 1$;

therefore, by addition, we have at once

$$x + X = 2u, \quad y + Y = 2v, \quad z + Z = 2w.$$

If, in the first system, we find the values of u, v, w, which we do by multiplying the equations respectively by

$$\frac{d\Delta}{da_{11}}, \ \frac{d\Delta}{da_{21}}, \ \frac{d\Delta}{da_{31}},$$

and adding, when

$$\Delta u = \frac{d\Delta}{da_{11}}x + \frac{d\Delta}{da_{21}}y + \frac{d\Delta}{da_{31}}z,$$

and, by a similar process, we obtain

$$\Delta v = \frac{d\Delta}{da_{12}}x + \frac{d\Delta}{da_{22}}y + \frac{d\Delta}{da_{32}}z,$$

$$\Delta w = \frac{d\Delta}{da_{13}}x, \ \&c.;$$

whence, by substituting for the values of u, v, w, $u = \frac{x+X}{2}$, &c., we obtain

$$\Delta X = \left(2\frac{d\Delta}{da_{11}} - \Delta\right)x + 2\frac{d\Delta}{da_{21}}y + 2\frac{d\Delta}{da_{31}}z,$$

$$\Delta Y = 2\frac{d\Delta}{da_{12}}x + \left(2\frac{d\Delta}{da_{22}} - \Delta\right)y + 2\frac{d\Delta}{da_{32}}z,$$

$$\Delta Z = 2\frac{d\Delta}{da_{13}}x + 2\frac{d\Delta}{da_{23}}y + \left(2\frac{d\Delta}{da_{33}} - \Delta\right)z.$$

Treating the second system in the same manner, we find

$$\Delta u = \frac{d\Delta}{da_{11}}X + \&c.,$$

$$\Delta v = \frac{d\Delta}{da_{21}}Y + \&c.,$$

$$\Delta w = \frac{d\Delta}{da_{31}}Z + \&c.;$$

and also, by substitution, taking value of x and instead of X, we find

$$\Delta x = \left(2\frac{d\Delta}{da_{11}} - \Delta\right)X + 2\frac{d\Delta}{da_{12}}Y + 2\frac{d\Delta}{da_{13}}Z,$$

$$\Delta y = 2\frac{d\Delta}{da_{21}}X + \&c.,$$

$$\Delta z = 2\frac{d\Delta}{da_{31}}X + \&c.,$$

or, more symmetrically,

$$\left.\begin{aligned} x &= c_{11}X + c_{12}Y + c_{13}Z \\ y &= c_{21}X + c_{22}Y + e_{23}Z \\ z &= c_{31}X + c_{32}Y + c_{33}Z \end{aligned}\right\} \cdots\cdots\cdots\cdots (1),$$

and

$$\left.\begin{aligned} X &= c_{11}x + c_{21}y + c_{31}z \\ Y &= c_{12}x + c_{22}y + c_{32}z \\ Z &= c_{13}x + c_{23}y + c_{33}z \end{aligned}\right\} \cdots\cdots\cdots\cdots (2),$$

where

$$\frac{2\frac{d\Delta}{da_{11}} - \Delta}{\Delta} = c_{11},\ \frac{2\frac{d\Delta}{da_{22}} - \Delta}{\Delta} = c_{22},\ \text{and}\ \frac{2\frac{d\Delta}{da_{12}}}{\Delta} = c_{12},\ \&c.$$

Now, if (1) and (2) are connected by an orthogonal substitution, we must have, by Solid Geom.,

$$c_{11}^2 + c_{12}^2 + c_{13}^2 = 1,$$
$$c_{11}c_{21} + c_{12}c_{22} + c_{13}c_{23} = 0, \text{ \&c. \&c.}$$

That is, the sums of the squares of the direction-cosines $= 1$, and the sums of their products taken two and two $= 0$, when the axes are rectangular. But these results immediately follow, if we substitute (2) in (1).

Proceeding now to give to c values corresponding to any given case, we see that the determinant must be analogous to the following *

$$\Delta = \begin{vmatrix} 1 & n & -m \\ -n & 1 & l \\ m & -l & 1 \end{vmatrix} = 1 + l^2 + m^2 + n^2,$$

and, forming the minors,

$$\begin{vmatrix} 1+l^2 & lm+n & nl-m \\ lm-n & 1+m^2 & mn+l \\ nl+m & mn-l & 1+n^2 \end{vmatrix}$$

and
$$\frac{2\dfrac{d\Delta}{da_{11}} - \Delta}{\Delta} = c_{11} = \frac{1+l^2-m^2-n^2}{1+l^2+m^2+n^2} = c_{22} = c_{33},$$

$$\frac{2\dfrac{d\Delta}{da_{12}}}{\Delta} = c_{12} = \frac{2\,(lm+n)}{1+l^2+m^2+n^2}, \text{ \&c. \&c.*}$$

where c_{11}, c_{12}, &c. represent the values of the nine direction-cosines in the given transformation.

29. Ex. 1. We may find an illustration of what has gone before in the following well-known geometrical relations.

* The values of l, m, n are $a \tan \frac{1}{2}\theta$, $b \tan \frac{1}{2}\theta$, $c \tan \frac{1}{2}\theta$, where the system is revolved through an angle θ, the direction-cosines of the old axes being a, b, c. (*Crelle*, vol. 51, p. 263.)

Suppose lmn, $l_1m_1n_1$, $l_2m_2n_2$ the direction-cosines of three right lines in reference to their three rectangular axes; a_1, a_2, a_3 the angles included between them:

$$\begin{aligned} l^2+m^2+n^2 &= 1, \quad & ll_1+mm_1+nn_1 &= \cos a_1, \\ l_1^2+m_1^2+n_1^2 &= 1, & ll_2+mm_2+nn_2 &= \cos a_2, \\ l_2^2+m_2^2+n_2^2 &= 1, & l_1l_2+m_1m_2+n_1n_2 &= \cos a_3. \end{aligned}$$

Now we are enabled to write

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & \cos a_3 & \cos a_2 \\ \cos a_3 & 1 & \cos a_1 \\ \cos a_2 & \cos a_1 & 1 \end{vmatrix}$$

For the above equations are true for every value of a_1, a_2, a_3, and therefore true when a_1 &c. $= 0$, as

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

which conforms to the condition, and is true when the lines are at right angles to each other, giving a determinant which has already been noticed (Art. 16).

Ex. 2. Another illustration is afforded by a determinant which is related to equations of a higher character than we had purposed to introduce at this stage of our progress, but we will just notice it.

Suppose a function of l is expressed in the following determinant,

$$\begin{vmatrix} a-l & d & e \\ d & b-l & f \\ e & f & c-l \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and suppose this function be multiplied by a function of $-l$; we may then write as the result

$$f(-l)\cdot f(l) = \begin{vmatrix} A-l^2 & D & E \\ D & B-l^2 & F \\ E & F & C-l^2 \end{vmatrix} \quad \ldots\ldots\ldots\ldots (2).$$

Determinant (1) expresses an equation of frequent occurrence in mathematical physics, as an instance of which the reader may examine Laplace's equation in g on the secular inequalities of the planets (*Mécanique Céleste*, Bk. II. sec. 56.)

Are the roots of such an equation *real?* Special cases had, of course, been resolved by the older mathematicians, as Cauchy and others; but the method by Sylvester (*Philosophical Mag.* 1852), depending upon the rule for the multiplication of determinants, is more simple and elegant. The method is shown above in (2), when $f(l) \cdot f(-l)$ is given, in which we find by expansion

$$\begin{aligned} A &= a^2+d^2+e^2, & D &= ef+d\,(a+b),\\ B &= b^2+f^2+d^2, & E &= fd+e\,(a+c),\\ C &= c^2+f^2+e^2, & F &= ed+f\,(b+c). \end{aligned}$$

With these values (2) becomes

$$l^6-Ll^4+Ml^2-N \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where, if L, M, and N are essentially positive, then, according to Des Cartes' rule of signs, we must have an equation for l^2, and therefore for $f(l)$, whose roots cannot be of the form of $(l-p)^2=-y^2$, and therefore negative, but must be essentially *real.* The only question to be considered is, what is the essential sign of L, M, and N? In the expansion of (2), we shall find that the L of (3) is equal to

$$a^2+b^2+c^2+2f^2+2e^2+2d^2,$$

$$M=(ab-d^2)^2+(ac-e^2)^2+(bc-f^2)^2 \\ +2\,(af-ed)^2+2\,(be-f)^2+(cd-f)^2,$$

and $$N=\begin{vmatrix} a & d & e \\ d & b & f \\ e & f & c \end{vmatrix}$$

where L, M, N are, it is evident, essentially positive.

Ex. 3. It might be well to mention one peculiar case in the multiplication of determinants, as exhibiting or suggesting an easy treatment of a large number of theorems. It may be found in *Crelle*, Vols. 39 and 51; it is also given by Salmon and Brioschi.

Suppose $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

the equation to a conic, a, b the semi-axes.

If, now, we take any three points on the curve and form a triangle, its area could be expressed at once by the determinant given in Art. 8, in terms of the co-ordinates of its vertices; and similarly, in this case, the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{y}{b} & \frac{y_1}{b} & \frac{y_2}{b} \\ \frac{x}{a} & \frac{x_1}{a} & \frac{x_2}{a} \end{vmatrix}$$

immediately suggests itself as expressing twice the area of the given triangle, $= \pm 2\,\frac{S}{ab}$, S being $=$ to the area of the triangle whose points are given (xy), (x_1y_1), (x_2y_2). If now we square this determinant, or multiply it by

$$\begin{vmatrix} \frac{x}{a} & \frac{y}{b} & -1 \\ \frac{x_1}{a} & \frac{y_1}{b} & -1 \\ \frac{x_2}{a} & \frac{y_2}{b} & -1 \end{vmatrix} = \mp 2\,\frac{S}{2ab},$$

we shall obtain a symmetrical determinant, as

$$\begin{vmatrix} a_1 & h & g \\ h & b_1 & f \\ g & f & c_1 \end{vmatrix} = -\frac{4S^2}{a^2 b^2},$$

where $$S=\tfrac{1}{2}ab\,(-a_1b_1c_1+a_1f^2+b_1g^2+c_1h^2-2fgh)^{\frac{1}{2}};$$

and since, if the points are on the curve, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0,\ \text{and}\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1=0,$$

$$a_1 = b_1 = c_1 = 0, \text{ and likewise } S = \tfrac{1}{2}ab\,(-2fgh)^{\frac{1}{2}},$$

which is the value of the determinant

$$\begin{vmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{vmatrix}$$

where $h = \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1,\ g = \dfrac{xx_2}{a^2} + \dfrac{yy_2}{b^2} - 1, f = \dfrac{x_1x_2}{a^2} + \dfrac{y_1y_2}{b^2} - 1,$

which can be reduced as follows:

Let c, d, e represent the sides of the triangle, and C, D, E the parallel semi-diameters respectively.

Then, from the nature of the ellipse, we have

$$\begin{aligned} \frac{c^2}{C^2} &= \frac{(x_1-x_2)^2}{a^2} + \frac{(y_1-y_2)^2}{b^2}, \\ \frac{d^2}{D^2} &= \frac{(x_2-x)^2}{a^2} + \frac{(y_2-y)^2}{b^2}, \\ \frac{e^2}{E^2} &= \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}, \end{aligned} \qquad \left\{ \begin{aligned} &\text{but } \frac{(x_1-x_2)^2}{a^2} + \frac{(y_1-y_2)^2}{b^2} \\ &\qquad = 2\left(1 - \frac{x_1x_2}{a_2} - \frac{y_1y_2}{b^2}\right), \end{aligned} \right.$$

and corresponding values for $\dfrac{d^2}{D^2}$ and $\dfrac{e^2}{E^2}$, which differ from the other values of h, g, and f by only the factor 2 and the negative sign.

Therefore, by substituting, we have

$$\frac{4S^2}{a^2b^2} = -\begin{vmatrix} 0 & \dfrac{c^2}{2C^2} & \dfrac{d^2}{2D^2} \\ \dfrac{c^2}{2C^2} & 0 & \dfrac{e^2}{2E^2} \\ \dfrac{d^2}{2D^2} & \dfrac{e^2}{2E^2} & 0 \end{vmatrix} = \frac{c^2d^2e^2}{4C^2D^2E^2},$$

therefore $$S = \tfrac{1}{4}ab\,\frac{cde}{CDE}.$$

30. It must be borne in mind that the examples here given are simply for illustration, and to satisfy the reader that the

principles employed are capable of wide application in all Co-ordinate Geometry.

Two theorems will now be added, which the reader will be able to prove in a manner more or less general.

1. *The square of a determinant of an even order can be expressed by a skew symmetric of an even order.*

2. *While a symmetric skew of an even order does not vanish, its inverse is a symmetric skew determinant.**

31. Let us now consider briefly *determinants arising from the roots of equations.*

It is well known that, by Sturm's theorem, we find the *number and places* of real roots—that the imaginary roots enter by pairs, and are equal in number to the variations of signs of the leading powers of x in all the functions.

Let
$$\Delta = \pm \begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ c_1 & c_2 & \ldots\ldots & c_n \\ c_1^2 & c_2^2 & \ldots\ldots & c_n^2 \\ \ldots & & \ldots\ldots & \ldots \\ c_1^{n-1} & c_2^{n-1} & \ldots\ldots & c_n^{n-1} \end{vmatrix}$$

be the determinant formed from the roots of the equation

$$x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \ldots\ldots C_0 = 0.$$

Substituting c for x, we write

$$\begin{aligned} c_1^n + C_{n-1}c_1^{n-1} + C_{n-2}c_1^{n-2} \ldots C_0 &= 0, \\ c_2^n + C_{n-1}c_2^{n-1} + \&c. \qquad &= 0, \\ \&c. \qquad \&c. & \\ c_n^n + C_{n-1}c_n^{n-1} + C_{n-2}c_n^{n-2} + \ldots &= 0. \end{aligned}$$

* These theorems have, in fact, already been exhibited, but their applications to linear equations *generally* will be seen in *Crelle*, Vols. 51 and 52, and, for earlier investigations of the theory of Substitutions, see *Euler*, Vols. 15 and 20 of *Novi Commentarii Acad. Petrop.* Compare also the formulæ given by Rodigues in *Liouville*, tom. 5, with those of Euler here cited in *N. C. A. P.* under *De motu corporum rigidorum.*

Let these equations be multiplied by any indeterminants, as $\kappa_1, \kappa_2 \ldots \kappa_n$, and assume

$$\kappa_1 c_1^s + \kappa_2 c_2^s + \kappa_3 c_3^s + \&c. = v \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

also
$$\kappa_1 + \kappa_2 + \ldots\ldots \kappa_n = 0,$$
$$\kappa_1 c_1 + \kappa_2 c_2 + \ldots\ldots \kappa_n c_n = 0,$$
$$\kappa_1 c_1^{n-1} + \kappa_2 c_2^{n-2} + \ldots\ldots \kappa_n c_n^{n-1} = 0;$$

whence, by a short algebraic process, we shall find

$$C_s = -\frac{1}{v}(\kappa_1 c_1^n + \kappa_2 c_2^n \ldots\ldots \kappa_n c_n^n) \ldots\ldots\ldots\ldots\ldots (2).$$

By differentiating the given determinant and employing the value of v, we have, from the determinant,

$$\frac{d\Delta}{dc_1^s} c_1^s + \frac{d\Delta}{dc_2^s} c_2^s + \ldots\ldots \frac{d\Delta}{dc_n^s} c_n^s = \Delta,$$

and, from (1),

$$\kappa_1 = \frac{d\Delta}{dc_1^s} \cdot \frac{v}{\Delta}, \quad \kappa_2 = \frac{d\Delta}{dc_2^s} \cdot \frac{v}{\Delta}, \quad \ldots\ldots \quad \kappa_n = \frac{d\Delta}{dc_n^s} \cdot \frac{v}{\Delta},$$

which values, substituted in (2), give

$$C_s = -\frac{1}{\Delta}\left(c_1^n \frac{d\Delta}{dc_1^s} + c_2^n \frac{d\Delta}{dc_2^s} \ldots + c_n^n \frac{d\Delta}{dc_n^s}\right),$$

which evidently represents the sums of the combinations of the roots taken $n-s$ and $n-s$.

Let us now seek for Δ in terms of the involved roots by their differences.

Let $$s = n-1,$$

and $$\phi(x) = (x-c_1)(x-c_2) \ldots (x-c_n);$$

and since κ_1, κ_2, &c. are any values

$$\kappa_1 = \frac{v}{\phi_1(c_1)}, \quad \kappa_2 = \frac{v}{\phi_1(c_2)}, \quad \ldots\ldots \quad \kappa_n = \frac{v}{\phi_1(c_n)},$$

therefore $$\frac{1}{\phi_1(c_1)} = \frac{1}{\Delta} \cdot \frac{d\Delta}{dc_1^{n-1}}, \quad \ldots\ldots \quad \frac{1}{\phi_1(c_n)} = \frac{1}{\Delta} \cdot \frac{d\Delta}{dc_n^{n-1}};$$

but
$$\frac{d\Delta}{dc_1^{n-1}} = \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ c_2 & c_3 & \cdots\cdots & c_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_2^{n-2} & c_3^{n-2} & \cdots\cdots & c_n^{n-2} \end{vmatrix} = \Delta_1.$$

Let $\phi'(x)$ designate
$$(x-c_2)(x-c_3)\ \cdots\cdots\ (x-c_n),$$
therefore
$$\frac{1}{\phi_1'(c_2)} = \frac{1}{\Delta_1}\cdot\frac{d\Delta_1}{dc_2^{n-2}}.$$

Similarly, if we put
$$\frac{d\Delta_1}{dc_2^{n-2}} = \Delta_2,\quad \frac{d\Delta_2}{dc_3^{n-3}} = \Delta_3,\quad \cdots\cdots\quad \frac{d\Delta_{n-2}}{dc_{n-1}} = \Delta_{n-1} = 1,$$
and $\phi_2(x) = (x-c_3)\ \ldots\ (x-c_n)$, $\phi_3(x) = (x-c_4)\ \ldots\ (x-c_n)$,
$$\phi_{n-2}(x) = (x-c_{n-1})(x-c_n);$$
there will result
$$\frac{1}{\phi_2'(c_3)} = \frac{1}{\Delta_2}\cdot\frac{d\Delta_2}{dc_3^{n-3}},\quad \cdots\cdots\quad \frac{1}{\phi_{n-2}'(c_{n-1})} = \frac{1}{\Delta_{n-1}};$$
whence, by multiplication, member by member,
$$\Delta = \phi'(c_1)\,\phi_1'(c_2)\,\phi_2'(c_3)\ \cdots\ \phi_{n-2}'(c_{n-1})$$
$$= (c_1-c_2)(c_1-c_3)\ldots(c_1-c_n)\ldots(c_2-c_n)\ldots(c_{n-1}-c_n)\ \cdots\cdots(3),$$
which is the product of the differences of n roots expressed as a determinant.

All this is easily generalized as follows:—

If, in
$$\Delta = \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ c_1 & c_2 & \cdots\cdots & c_n \\ c_1^2 & c_2^2 & \cdots\cdots & c_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ c_1^{n-1} & c_2^{n-1} & \cdots\cdots & c_n^{n-1} \end{vmatrix} \cdots\cdots\cdots\cdots\cdots\cdots (4),$$
we consider that this determinant would vanish if $c_1 = c_2$, and that therefore $c_1 - c_2$ must be a factor, and what is true of these

two roots is true of all the others considered two and two; hence we are enabled to write (3) at once.

Or we might prove true generally the method which is here exhibited as a special case,

$$\begin{vmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ c_1-c_3 & c_2-c_3 & c_3 \\ c_1^2-c_3^2 & c_2^2-c_3^2 & c_3^2 \end{vmatrix}$$

$$= (c_1-c_3)(c_2-c_3)\begin{vmatrix} 1 & 1 \\ c_1+c_3 & c_2+c_3 \end{vmatrix} = (c_1-c_3)(c_2-c_3)(c_2-c_1).$$

Ex.: Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ \cdots & \cdots & \cdots & \cdots \\ c_1^4 & c_2^4 & c_3^4 & c_4^4 \end{vmatrix} = (c_1-c_4)(c_2-c_4)(c_3-c_4)(c_2-c_3)(c_1-c_2)(c_1+c_2+c_3+c_4).$$

32. If now we proceed to form the square of (4) of the last Art., we may write the result

$$\begin{vmatrix} S_0 & S_1 & \cdots\cdots & S_{n-2} \\ S_1 & S_2 & \cdots\cdots & S_n \\ \cdots & \cdots & \cdots & \cdots \\ S_{n-1} & S_n & \cdots\cdots & S_{n-2} \end{vmatrix}$$

where S_0, S_2, ... S_n, &c. express the sum of the first, second, and nth powers of c_1, c_2, &c.

Thus, for example,

$$\begin{vmatrix} 1 & 1 \\ c_1 & c_2 \end{vmatrix}^2 = \begin{vmatrix} S_0 & S_1 \\ S_1 & S_2 \end{vmatrix} = (c_1-c_2)^2.$$

In the same manner we shall find

$$\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = \Sigma\,(c_1-c_2)^2(c_2-c_3)^2(c_3-c_1)^2,$$

&c. &c.

where $\Sigma =$ the sum of the products.

These determinants are of great practical value in the theory of equations, inasmuch as, with their aid, as with the functions of Sturm, we determine the number of variations of signs, and, as stated at the beginning of the preceding Art., this determines the number of pairs of imaginary roots.

But if these determinants are all positive, there will be no variations, and consequently all the roots of the equation will be real.

To those acquainted with the general theory of equations these hints will be sufficient to show the bearing of determinants upon this subject; the real object in this and the preceding Art. being to prepare the way for the solution of linear differential equations by the use of the determinant notation.

33. *When $n-1$ particular integrals are given, to find the n^{th}.* *

Let us take the general linear differential equation, coefficients being constant

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots R\frac{dy}{dx} + Ty = 0 \ldots\ldots\ldots\ldots (1).$$

If we separate the signs of operation from those of quantity, the part involving only signs of operation and constants may be considered as an operation performed on y, as

$$f\left(\frac{d}{dx}\right)y = 0.$$

From which we get y at once explicitly, if we are able to perform the inverse operation

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}.$$

This we cannot easily do in its general form, but we can conceive the operation $f\left(\frac{d}{dx}\right)$ to be made up of certain binomial operations, and then perform the inverse operation for each of these. We will, however, in this case proceed in a different manner.

Let us first assume the n particular integrals, that is, values

* See Malmsten, in *Crelle*, vol. 39.

that will satisfy (1), as y_1, $y_2 \dots y_n$; coefficients now being variable.

Proceeding as in Art 31, and placing

$$\begin{array}{l}
\kappa_1 y_1 + \kappa_2 y_2 \dots\dots \kappa_n y_n = 0 \\
\kappa_1 y_1^1 + \kappa_2 y_2^1 \dots\dots \kappa_n y_n^1 = 0 \\
\kappa_1 y_1^r + \kappa_2 y_2^r \dots\dots \kappa_n y_n^r = v \\
\dots \quad \dots \quad \dots \quad \dots \\
\kappa_1 y_1^{n-1} + \kappa_2 y_2^{n-1} \dots\dots \kappa_n y_n^{n-1} = 0
\end{array}$$

we obtain

$$A_r = -\frac{1}{v}(\kappa_1 y_1^n + \kappa_2 y_2^n + \dots \kappa_n y_n^n)^* \dots\dots\dots\dots (2).$$

Solving, as before, for the values of the indeterminates κ_1, $\kappa_2 \dots \kappa_n$, and substituting in (2), we find, since

$$\Delta = \begin{vmatrix} y_1 & y_2 & \dots\dots & y_n \\ y_1^1 & y_2^1 & \dots\dots & y_n^1 \\ \dots & \dots & & \dots \\ y_1^{n-1} & y_2^{n-1} & \dots\dots & y_n^{n-1} \end{vmatrix}$$

that $$A_r = -\frac{1}{\Delta}\left(y_1^n \frac{d\Delta}{dy_1^r} + y_2^n \frac{d\Delta}{dy_2^r} + \dots\dots y_n^n \frac{d\Delta}{dy_n^r}\right).$$

In differentiating this determinant, we get Δ', or

$$\Delta' = y_1^n \frac{d\Delta}{dy_1^{n-1}} + y_2^n \frac{d\Delta}{dy_2^{n-1}} \dots\dots y_n^n \frac{d\Delta}{dy_n^{n-1}};$$

therefore $$A_r = -\frac{\Delta'}{\Delta} \dots\dots\dots\dots\dots\dots\dots\dots\dots (4).$$

Resuming now equation (1):

Let us suppose the $n-1$ particular integrals y_1, y_2, $\dots y_{n-1}$ are known.†

Let
$$\left.\begin{array}{l}
y = y_1\kappa_1 + y_2\kappa_2 + \dots\dots y_{n-1}\kappa_{n-1} \\
0 = y_1\kappa_1' + y_2\kappa_2' + \dots\dots y_{n-1}\kappa_{n-1}' \\
0 = y_1^1\kappa_1' + y_2^1\kappa_2' + \dots\dots y_{n-1}^1\kappa_{n-1}' \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
0 = y_1^{n-3}\kappa_1' + y_2^{n-3}\kappa_2' + \dots\dots y_{n-1}^{n-3}\kappa_{n-1}'
\end{array}\right\} \dots\dots\dots (5).$$

* $r, n, n-1$ do not, of course, indicate powers.

† *Crelle*, vol. 39, p. 94.

Solving, we find

$$\kappa_1' : \kappa_2' : \ldots\ldots \kappa_{n-1}' :: \pm \begin{vmatrix} y_2 & y_3 & \ldots\ldots & y_{n-1} \\ y_2^1 & y_3^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_2^{n-3} & y_3^{n-3} & \ldots\ldots & y_{n-1}^{n-3} \end{vmatrix}$$

$$: \pm \begin{vmatrix} y_3 & y_4 & \ldots\ldots & y_1 \\ y_3^1 & y_4^1 & \ldots\ldots & y_1^1 \\ \ldots & \ldots & & \ldots \\ y_3^{n-3} & y_4^{n-3} & \ldots\ldots & y_1^{n-3} \end{vmatrix} : \ldots : \pm \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-2} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-2}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-3} & y_2^{n-3} & \ldots\ldots & y_{n-2}^{n-3} \end{vmatrix}$$

$$:: \Delta_1 : \Delta_2 : \ldots\ldots \Delta_{n-1}.$$

If now we differentiate successively equation (5), remembering the assumed relations between the $n-1$ functions, we shall have

$$\frac{dy}{dx} = y_1'\kappa_1 + y_2'\kappa_2 + \ldots\ldots y_{n-1}'\kappa_{n-1},$$

$$\frac{d^2y}{dx^2} = y_1''\kappa_1 + y_2''\kappa_2 + \ldots\ldots y_{n-1}''\kappa_{n-1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{d^{n-1}y}{dx^{n-1}} = y_1^{n-1}\kappa_1 + y_2^{n-1}\kappa_2 + \ldots\ldots y_{n-1}^{n-1}\kappa_{n-1}$$

$$+ y_1^{n-2}\kappa_1' + y_2^{n-2}\kappa_2' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}',$$

$$\frac{d^n y}{dx^n} = y_1^n\kappa_1 + y_2^n\kappa_2 + \ldots\ldots y_{n-1}^n\kappa_{n-1}$$

$$+ 2\left(y_1^{n-1}\kappa_1' + y_2^{n-1}\kappa_2' + \ldots\ldots y_{n-1}^{n-1}\kappa_{n-1}'\right)$$

$$+ y_1^{n-2}\kappa_1'' + y_2^{n-2}\kappa_2'' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}'';$$

which, substituted in (1), give

$$y_1^{n-2}\kappa_1'' + y_2^{n-2}\kappa_2'' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}'' + [A(y_1^{n-2}) + 2y_1^{n-1}]\kappa_1'$$

$$+ [A(y_2^{n-2}) + 2y_2^{n-1}]\kappa_2' + \ldots\ldots [A(y_{n-1}^{n-2}) + 2y_{n-1}^{n-1}]\kappa_{n-1}' = 0.$$

Let $\qquad \kappa_1' = U\Delta_1, \quad \kappa_2' = U\Delta_2 \ \ldots\ldots \ \kappa_n' = U\Delta_n;$

whence

$$\kappa_1'' = U'\Delta_1 + U\Delta_1', \quad \kappa_2'' = U'\Delta_2 + U\Delta_2' \;\ldots\ldots\; \kappa_n'' = U'\Delta_n + U\Delta_n'.$$

But
$$y_1^{n-2}\Delta_1 + y_2^{n-2}\Delta_2 + \ldots\ldots y_{n-1}^{n-2}\Delta_{n-1}$$

furnishes the determinant

$$\begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-1} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-2} & y_2^{n-2} & \ldots\ldots & y_{n-1}^{n-2} \end{vmatrix} = \Delta,$$

and
$$\frac{d\Delta}{dx} = \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-1} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots\ldots & y_{n-1}^{n-1} \end{vmatrix}$$

therefore
$$y_1^{n-2}\Delta_1' + y_2^{n-2}\Delta_2' + \ldots\ldots y_{n-1}^{n-2}\Delta_{n-1}' = 0,$$

and
$$y_1^{n-1}\Delta_1 - y_2^{n-1}\Delta_2 + \ldots\ldots y_{n-1}^{n-1}\Delta_{n-1} = \Delta';$$

therefore
$$U'\Delta + UA\Delta + 2U\Delta' = 0,$$

$$\frac{U'}{U} + \frac{2\Delta'}{\Delta} + A = 0.$$

By integrating this equation, we have

$$U = \frac{e^{-\int A\,dx}}{\Delta^2};$$

or, substituting for U the values of κ_1 and Δ_1, and integrating again,
$$\kappa_s = \int \frac{\Delta_1}{\Delta^s} e^{-\int A\,dx}\,dx.$$

If we write
$$\Delta = \frac{1}{\Delta} = \Sigma \pm y_1 \cdot y_2^1 \ldots\ldots y_{n-1}^{n-2},$$

we have
$$\kappa_s = (-1)^{n-1} \int \frac{d\Delta}{dy_s^{n-2}} e^{-\int A\,dx}\,dx.$$

Returning now to equation (4), we see that it can be written

$$A_r = \frac{\Delta'}{Ce^{-\int A\,dx}}, \text{ where } \Delta = Ce^{-\int A\,dx}.$$

A single instance is thus given in full, that the reader may judge for himself of the practical benefits of the determinant notation in conducting intricate analytical operations.

34. In the solution of simultaneous differential equations—that is, a system of equations with but one dependent variable, in which some form of this variable, as a function of the independent variables, must be found to satisfy all the equations—there is no reason why determinants may not be employed to effect the *elimination* (if this method be preferred to those of D'Alembert or Lagrange) as in the case of ordinary linear equations.

If, for example, we have three simultaneous differential equations of the form,

Ex. 1:

$$\frac{d}{dt}x + by + cz = 0,$$

$$ax + \frac{d}{dt}y + c'z = 0,$$

$$a'x + b'y + \frac{d}{dt}z = 0.$$

The condition of co-existence is the determinant

$$\begin{vmatrix} \frac{d}{dt} & b & c \\ a & \frac{d}{dt} & c' \\ a' & b' & \frac{d}{dt} \end{vmatrix} = \left[\frac{d}{dt}\left(\frac{d^2}{dt^2} - b'c'\right) + a\left(b'c - b\frac{d}{dt}\right) + a'\left(bc' - c\frac{d}{dt}\right)\right]x = 0;$$

i. e., $$\left[\frac{d^3}{dt^3} - (ab + a'c + b'c')\frac{d}{dt} + ab'c + a'bc'\right]x = 0,$$

and we can proceed to integrate at once this equation, giving rise to only three arbitrary constants.

Ex. 2.—Let us take four simultaneous equations.

The equations of Airy, for determining the secular variations of the eccentricity and longitude of the perihelion, will serve as an illustration:

$$\frac{d}{dt}u + a_1 v - a_2 v' = 0,$$

$$a_1 u - \frac{d}{dt}v - a_2 u' = 0,$$

$$\frac{d}{dt}u' + b_1 v' - b_2 v = 0,$$

$$b_1 u' - \frac{d}{dt}v' - b_2 u = 0.$$

The determinant for eliminating the variables u, v, u', v', is therefore

$$\begin{vmatrix} \frac{d}{dt} & a_1 & -a_2 & 0 \\ a_1 & -\frac{d}{dt} & 0 & -a_2 \\ 0 & -b_2 & b_1 & \frac{d}{dt} \\ -b_2 & 0 & -\frac{d}{dt} & b_1 \end{vmatrix}$$

$$= \frac{d^4}{dt^4} + (a_1^2 + b_1^2 + 2a_2b_2)\frac{d^2}{dt^2} + (a_1b_1 - a_2b_2)^2 = 0\,;$$

which can readily be integrated, and is, of course, symmetrical for either of the variables. This may be regarded as the equation in u.

35. One other example upon this point, and then we shall proceed to another subject.

Suppose we have a pair of linear partial differential equations, as

$$\frac{dV}{dx}dx + \frac{dV}{dy}dy = 0,$$

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy = 0,$$

where U and V are functions of x and y; then the condition of the dependence of these functions is expressed by the determinant

$$\begin{vmatrix} \frac{dV}{dx} & \frac{dV}{dy} \\ \frac{dU}{dx} & \frac{dU}{dy} \end{vmatrix} = 0.$$

This leads us to the consideration of what are called *functional determinants;* and the general proposition is that, *when a functional determinant of a system of functions vanishes, it expresses the condition of dependence of the functions;* that is, we may test the dependence of functions in a manner analogous to that which we have employed to determine the co-existence of linear equations.

CHAPTER III.

FUNCTIONAL DETERMINANTS.

36. As this subject is supposed to present some difficulties, and is of the highest interest in connection with geometrical researches, we shall seek in the first place to exhibit some of its principles in a very elementary form, and then proceed to show the field of application.

Suppose we have a series of functions $v_1, v_2 \dots v_n$ of as many variables $x_1, x_2 \dots x_n$, and by virtue of the relationship of these functions we are enabled to find

$$f(v_1 v_2 \dots v_n) = 0,$$

in other words, that they are connected by an equation which vanishes identically. This relationship is expressible as a determinant* (Art. 35)

$$\begin{vmatrix} \frac{dv_1}{dx_1} & \frac{dv_1}{dx_2} & \cdots\cdots & \frac{dv_1}{dx_n} \\ \frac{dv_2}{dx_1} & \frac{dv_2}{dx_2} & \cdots\cdots & \frac{dv_2}{dx_n} \\ \dots & \dots & & \dots \\ \frac{dv_n}{dx_1} & \frac{dv_n}{dx_2} & \cdots\cdots & \frac{dv_n}{dx_n} \end{vmatrix} = 0.$$

Then we say $v_1, v_2 \dots v_n$ are *dependent.*

To fix our thoughts by an illustration, suppose

$$v_1 = x + 2y + z,$$
$$v_2 = x - 2y + 3z,$$
$$v_3 = 2xy - xz + 4yz - 2z^2,$$

* On this subject, see Jacobi, *Crelle*, Vol. 22. Spottiswoode, *Crelle*, Vol. 51. Pierce's *Analytical Mechanics.*

then
$$\begin{vmatrix} \frac{dv_1}{dx} & \frac{dv_1}{dy} & \frac{dv_1}{dz} \\ \frac{dv_2}{dx} & \frac{dv_2}{dy} & \frac{dv_2}{dz} \\ \frac{dv_3}{dx} & \frac{dv_3}{dy} & \frac{dv_3}{dz} \end{vmatrix} = 0$$

becomes
$$xy \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix} = 0;$$

reducing, we find

$$-4(-x+4y-4z)+8(2y-z)-2(2x+4z)=0,$$

which, vanishing identically, shows the functions v_1, v_2, v_3 to be dependent.

37. Let us suppose the connecting equation to be

$$F(v_1 v_2 \dots v_n) = 0.$$

If now we differentiate this equation in respect to any one of the functions $v_1, v_2 \dots v_n$ under consideration, regarded as functions of the variables $x_1, x_2 \dots x_n$, we must have, in general,*

$$\frac{dF}{dv_r} = \frac{dF}{dx_1} \cdot \frac{dx_1}{dv_r} + \frac{dF}{dx_2} \cdot \frac{dx_2}{dv_r} \dots\dots \frac{dF}{dx_n} \cdot \frac{dx_n}{dv_r};$$

when $F = v_r$ the left member of this equation $= 1$. And, in general, if we replace F by v_s, we shall have, when $s = r$,

$$1 = \frac{dv_s}{dx_1} \cdot \frac{dx_1}{dv_s} + \frac{dv_s}{dx_2} \cdot \frac{dx_2}{dv_s} \dots\dots \frac{dv_s}{dx_n} \cdot \frac{dx_n}{dv_s} \dots\dots\dots (1).$$

If, however, s is not equal to r, we must have

$$0 = \frac{dv_s}{dx_1} \cdot \frac{dx_1}{dv_r} + \frac{dv_s}{dx_2} \cdot \frac{dx_2}{dv_r} \dots\dots \frac{dv_s}{dx_n} \cdot \frac{dx_n}{dv_r} \dots\dots\dots (2).$$

* Jacobi in *Crelle*, Vol. 22.

In the same manner,

$$\frac{dx_r}{dv_1}\cdot\frac{dv_1}{dx_r}+\ldots\ldots\frac{dx_r}{dv_n}\cdot\frac{dv_n}{dx_r}=1 \ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{dx_r}{dv_1}\cdot\frac{dv_1}{dx_s}+\ldots\ldots\frac{dx_r}{dv_n}\cdot\frac{dv_n}{dx_s}=0 \ldots\ldots\ldots\ldots\ldots (4).$$

By means of (1), (2), (3), (4), we are enabled to solve a system of equations analogous to the following:—

$$\left.\begin{array}{l} y\dfrac{dv}{dx}+y_1\dfrac{dv}{dx_1}\ \ldots\ldots\ y_n\dfrac{dv}{dx_n}=u \\ y\dfrac{dv_1}{dx}+y_1\dfrac{dv_1}{dx_1}\ \ldots\ldots\ y_n\dfrac{dv_1}{dx_n}=u_1 \\ \ldots\quad\ldots\quad\ldots\quad\ldots \\ y\dfrac{dv_n}{dx}+y_1\dfrac{dv_n}{dx_1}\ \ldots\ldots\ y_n\dfrac{dv_n}{dx_n}=u_n \end{array}\right\}\ldots\ldots\ldots\ldots (5).$$

If we multiply these equations by

$$\frac{dx}{dv},\ \frac{dx}{dv_1}\ \ldots\ldots\ \frac{dx}{dv_n},\ \text{\&c. \&c.},$$

and add, we shall have, by virtue of (3) and (4),

$$\left.\begin{array}{l} y\ =u\dfrac{dx}{dv}+u_1\dfrac{dx}{dv_1}\ \ldots\ldots\ u_n\dfrac{dx}{dv_n} \\ y_1=u\dfrac{dx_1}{dv}+u_1\dfrac{dx_1}{dv_1}\ \ldots\ldots\ u_n\dfrac{dx_1}{dv_n} \\ \ldots\quad\ldots\quad\ldots\quad\ldots \\ y_n=u\dfrac{dx_n}{dv}+u_1\dfrac{dx_n}{dv_1}\ \ldots\ldots\ u_n\dfrac{dx_n}{dv_n} \end{array}\right\}\ldots\ldots\ldots\ldots (6).$$

It is evident that, if the given functions $v, \ldots v_n$ are independent, and $y=y_1=y_n=0$, then $u=u_1=u_n=0$; or, in other words, if the given functions are independent, then (5) and (6) reduce in turn to 0.

38. The question now arises, how shall we express the solution of such systems as the above in the ordinary language

of determinants? If we examine the solution of system (5) of the preceding Art., we shall see that what might be denominated the modulus of transformation is the determinant

$$\Delta = \begin{vmatrix} \frac{dv}{dx} \cdot \frac{dv}{dx_1} & \cdots\cdots & \frac{dv}{dx_n} \\ \frac{dv_1}{dx} \cdot \frac{dv_1}{dx_1} & \cdots\cdots & \frac{dv_1}{dx_n} \\ \cdots \quad \cdots & & \cdots \\ \frac{dv_n}{dx} \cdot \frac{dv_n}{dx_1} & \cdots\cdots & \frac{dv_n}{dx_n} \end{vmatrix}$$

or

$$\begin{vmatrix} \Delta\frac{dx}{dv} & \Delta\frac{dx_1}{dv} & \cdots\cdots & \Delta\frac{dx_n}{dv} \\ \Delta\frac{dx}{dv_1} & \Delta\frac{dx_1}{dv_1} & \cdots\cdots & \Delta\frac{dx_n}{dv_1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta\frac{dx}{dv_n} & \Delta\frac{dx_1}{dv_n} & \cdots\cdots & \Delta\frac{dx_n}{dv_n} \end{vmatrix} = \Delta^n,$$

since, manifestly, writing

$$\begin{vmatrix} \frac{dx}{dv} \cdot \frac{dx}{dv_1} & \cdots\cdots & \frac{dx}{dv_n} \\ \cdots \quad \cdots & & \cdots \\ \cdots \quad \cdots & & \cdots \\ \frac{dx_n}{dv} \cdot \frac{dx_n}{dv_1} & \cdots\cdots & \frac{dx_n}{dv_n} \end{vmatrix} = \Delta',$$

we must have $\qquad \Delta \times \Delta' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$

Hence the notation to be adopted, which is all that is required, is sufficiently evident. If now we differentiate Δ' in respect to any one of its constituents, as $\frac{dx_i}{dv_k}$, we shall have $\frac{d\Delta'}{d\frac{dx_i}{dv_k}} = A$;

but, in consequence of (1), we are enabled to write $A = \Delta' \frac{dv_k}{dx_i}$, where $A =$ the corresponding minor, and therefore

$$\frac{d\Delta'}{d\dfrac{dx_i}{dv_k}} = \Delta' \frac{dv_k}{dx_i}.$$

In the same manner, in general,

$$\frac{d\Delta}{d\dfrac{dv_i}{dx_k}} = \Delta \frac{dx_k}{dv_i}, \quad \text{therefore} \quad \frac{d\Delta'}{d\dfrac{dx_i}{dv_k}} \cdot \frac{d\Delta}{d\dfrac{dv_i}{dx_k}} = \frac{dv_k}{dx_i} \cdot \frac{dx_k}{dv_i}.$$

The same course of reasoning may be applied to the connecting equations. That is, if

$$f_1 = 0 \;\ldots\ldots\; f_n = 0$$

connect the variables $x_1, x_2 \ldots x_n$ with $v_1, v_2 \ldots v_n$; then, inversely, if we find from $f_1 = 0$, &c., the values of $v_1, v_2 \ldots v_n$, and substitute these in the same equations, these will vanish identically; or, since we may eliminate $n-1$ of the variables from these equations, each may be treated as the function of a single variable and the given functions; therefore

$$\frac{df_k}{dv_1} \cdot \frac{dv_1}{dx_k} + \frac{df_k}{dv_2} \cdot \frac{dv_2}{dx_k} \;\ldots\ldots\; \frac{df_k}{dv_n} \cdot \frac{dv_n}{dx_k} = \frac{df_k}{dx_k}.$$

If $k = 1, 2 \ldots n$, we shall obtain n equations, from which, eliminating the differentials, a linear partial differential equation will arise, which shall be satisfied by the primitive equation under consideration, as $f_k = 0$.

Proceeding in a manner similar to that for obtaining (1), we write, finally,

$$\Delta \Sigma \left(\pm \frac{df_1}{dv_1} \frac{df_2}{dv_2} \;\ldots\ldots\; \frac{df_n}{dv_n} \right) = (-1)^n \Sigma \left(\pm \frac{df_1}{dx_1} \frac{df_2}{dx_2} \;\ldots\ldots\; \frac{df_n}{dx_n} \right).$$

The general application of these principles to the transformation of multiple integrals, as

$$\int^n V\,dv_1\,dv_2 \ldots\ldots dv_n,$$

where the functions $v_1 \ldots v_n$ are connected with the same number of other variables $x_1 \ldots x_n$ by equations similar to those assumed above, will not be considered.

It may, however, be remarked that, in transforming from one set of variables to another, the formula of transformation

$$\int^n V\,dv_1 \ldots\ldots dv_n = \int^n V\frac{df_1}{dx_1} \ldots\ldots \frac{df_n}{dx_n}\,.\,dx_1 \ldots\ldots dx_n$$

reduces at once to

$$\int^n V\,dv_1 \ldots\ldots dv_n = \int^n V.\,\Delta\,dx_1 \ldots\ldots dx_n,$$

and

$$\int^n V\,dx_1 \ldots\ldots dx_n = \int^n V.\,\Delta'\,dv_1 \ldots\ldots dv_n.$$

On this subject, see Baltzer, p. 64.

39. *The Jacobian.*—The determinant already considered,

$$\begin{vmatrix} \dfrac{dU_1}{dx_1} & \cdots\cdots & \dfrac{dU_1}{dx_n} \\ \cdots & \cdots & \cdots \\ \dfrac{dU_n}{dx_1} & \cdots\cdots & \dfrac{dU_n}{dx_n} \end{vmatrix}$$

which, after Jacobi, is called the Jacobian, and generally denoted by J, has received considerable attention in the theory of elimination. The principal proposition is that, if a system of homogeneous equations be satisfied by a set of values, these values will satisfy both the Jacobian and its differential in regard to all the variables.

Let us take a system of three equations

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0;$$

J will then be written

$$\begin{vmatrix} \frac{du_1}{dx} & \frac{du_1}{dy} & \frac{du_1}{dz} \\ \cdots & \cdots & \cdots \\ \frac{du_3}{dx} & \frac{du_3}{dy} & \frac{du_3}{dz} \end{vmatrix} = \Delta = J;$$

and let us assume, what is not difficult to prove, that

$$x\frac{du_1}{dx} + y\frac{du_1}{dy} + z\frac{du_1}{dz} = au_1,$$

$$x\frac{du_2}{dx} + y\frac{du_2}{dy} + z\frac{du_2}{dz} = au_2,$$

$$x\frac{du_3}{dx} + y\frac{du_3}{dy} + z\frac{du_3}{dz} = au_3.$$

Solving for x, we have, Art. 12,

$$(1) \ldots\ldots \Delta x = U_1 au_1 + U_2 au_2 + U_3 au_3,$$

where U_1, U_2, &c. = the minors. We see here that, if u_1, u_2, u_3 vanish, the determinant vanishes.

Differentiating (1) in respect to x and y,

$$(2) \ldots\ldots \Delta + x\frac{d\Delta}{dx} = au_1\frac{dU_1}{dx} + au_2\frac{dU_2}{dx} + au_3\frac{dU_3}{dx}$$
$$+ a\left(\frac{du_1}{dx}U_1 + \frac{du_2}{dx}U_2 + \frac{du_3}{dx}U_3\right),$$

$$(3) \ldots\ldots x\frac{d\Delta}{dy} = au_1\frac{dU_1}{dy} + au_2\frac{dU_2}{dy} + au_3\frac{dU_3}{dy}$$
$$+ a\left(\frac{du_1}{dy}U_1 + \frac{du_2}{dy}U_2 + \frac{du_3}{dy}U_3\right).$$

But the first parenthesis = Δ, Art. 12, and the second parenthesis = 0.

Again, introducing the supposition $u_1 = u_2 = u_3 = 0$, we see that $\frac{d\Delta}{dy}$ and $\frac{d\Delta}{dx}$ must vanish, since (2) and (3) in this case, in consequence of (1), reduce to 0.

The application of this principle is obvious; for if we have three equations homogeneous in the second degree, their J will be of the third, and each of its differentials of the second, and these three new equations will be satisfied by the values common to the given equations. We have then

$$u_1 = 0, \quad \frac{d\Delta}{x} = 0,$$

$$u_2 = 0, \quad \frac{d\Delta}{y} = 0,$$

$$u_3 = 0, \quad \frac{d\Delta}{z} = 0,$$

to eliminate x^2, y^2, z^2, xy, zy, xz; and therefore the *eliminant*, that is, the eliminating determinant, can be formed.

When the given equations are of the third degree homogeneous, J is of the sixth, its differentials of the fifth; and by using Sylvester's dialytic process, we can eliminate the twenty-one quantities of an equation of the fifth degree.

40. *The Hessian.*—We will now show how to form this important determinant. Let V be any homogeneous function of n variables, analogous to $(a, b, c, d \between x, y)^3$, and, taking its second differential coefficients in respect to each of the variables, we write, for the special case,

$$H = \begin{vmatrix} ax+by & bx+cy \\ bx+cy & cx+dy \end{vmatrix}$$

This is called the Hessian, after the late Dr. Otto Hesse, of Munich.

The degree of the determinant will be $n(p-2)$, where $p =$ the degree of the function, and n the number of variables.

If we connect the variables x and y with two others u and z by the equations

$$x = eu + fz,$$

$$y = e_1 u + f_1 z,$$

calling the transformed function V', and taking its second differentials, and indicating the Hessian thus formed by H' we may write

$$H' = H \times \Delta^2,$$

where

$$\Delta = \begin{vmatrix} e & f \\ e_1 & f_1 \end{vmatrix}$$

In orthogonal substitutions, $\Delta^2 = 1$, and $H = H'$.

Hesse has shown, in the use of this theorem, that if $V = 0$ be an equation to a plane curve of the nth order, the vanishing of the Hessian indicates the condition by which the curve reduces to a pencil of n right lines; and in like manner, if $V = 0$ be an equation to a surface, this surface reduces to a cone when H vanishes.*

* *Crelle*, vol. 42, p. 123.

CHAPTER IV.

SOME APPLICATIONS.

41. In proceeding to the common applications of what has been explained, it will be necessary to introduce some of the terms of higher algebra; thus,

$$(a,\ b,\ c\between x,\ y)^2$$

is called a binary quadratic, which, written fully, is simply

$$ax^2+2bxy+cy^2,$$

and since it is a homogeneous function it is called also a *quantic*. If any quantic is to be considered apart from numerical coefficients, it is written

$$(a,\ b,\ c\between x,\ y)^n.$$

Let us now take the first expression, and linearly transform it, substituting

$$x = lx + my,$$
$$y = l'x + m'y,$$

and we shall have $Ax^2+2Bxy+Cy^2$

as the transformed function. If, now, we compare the Hessian of the given and the transformed expression, we shall find the relation given in the last Art. to be true, viz.,

$$H' = H\,.\,\Delta^2,$$

or, in full, $$AC-B^2 = (ac-b^2)\,(lm'-l'm)^2.$$

Now H and H' are called respectively the *discriminants* of the given and transformed quadratic.

42. When a quantic has been transformed as above, any function of its coefficients is called an *invariant*. Hence $ac-b^2$ is also, by definition, an invariant; and, in general, a quantic of the quadratic class, irrespective of its variables, has no other invariant than its own discriminant, and in such cases the two terms indicate identical functions. Now, when we take the Hessian of any quantic, or what is sometimes called the second emanant, we obtain the *covariant* of the quantic, that is, a function of the coefficients involving the variables of the given quantic.

43. *Study first.*—Let us write the quadric surface

$$ax^2+by^2+cz^2+2exy+2fxz+2hyz+2gx+2iy+2kz+d=0;$$

the discriminant will then be

$$\begin{vmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & k \\ g & i & k & d \end{vmatrix} = 0 \text{ .... (1)},$$

which may be formed in the manner already described, or we may transform to any parallel axes drawn through $x'y'z'$ by writing $x+x'$ for x, &c., and we shall find certain relations connecting the coefficients a, b, &c. with a', b', &c.; in other words, that there are functions of the given coefficients equal to the same functions of the transformed coefficients.

By taking the differentials in respect to each of the variables, we shall find the new coefficient of x to be

$$2(ax'+ey'+fz'+g),$$

and the condition that this general equation shall represent a cone will be the determinant of the following equations,

$$\left.\begin{aligned} ax'+ey'+fz'+g&=0 \\ ex'+by'+hz'+i&=0 \\ fx'+hy'+cz'+k&=0 \\ gx'+iy'+kz'+d&=0 \end{aligned}\right\} \text{ .... (2).}$$

The determinant of which is the same as (1); the coordinates of the new vertex satisfying each of the above equations.

Forming now the first minors of this determinant, we have

$$a\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} - e\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} + f\begin{vmatrix} i & e & b \\ k & f & h \\ d & g & i \end{vmatrix} - g\begin{vmatrix} e & b & h \\ f & h & c \\ g & i & k \end{vmatrix}$$

and the second minors

$$ab\begin{vmatrix} c & k \\ k & d \end{vmatrix} + ah\begin{vmatrix} k & h \\ d & i \end{vmatrix} + ai\begin{vmatrix} h & c \\ i & k \end{vmatrix} \text{ \&c.}$$

Considering the first and second minors, we see that

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} \times b = \begin{vmatrix} b & i \\ i & d \end{vmatrix} \times \begin{vmatrix} b & h \\ h & c \end{vmatrix} - \begin{vmatrix} b & h \\ i & k \end{vmatrix}^2,$$

since

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} \times b + h^2 i^2 - h^2 i^2 = (bc-h^2)(bd-i^2) - (bk-hi)^2;$$

and we shall find, in general, that any first minor, multiplied by a constituent, is expressible in terms of the second minors, formed from this first minor.

Thus we shall find E; *i.e.*, second first minor, or

$$\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} \times c + fhk^2 - fhk^2$$
$$= \begin{vmatrix} e & f \\ h & c \end{vmatrix} . \begin{vmatrix} c & k \\ k & d \end{vmatrix} - \begin{vmatrix} h & c \\ i & k \end{vmatrix} . \begin{vmatrix} f & c \\ g & k \end{vmatrix}$$

Also A, or

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} . d = \begin{vmatrix} c & k \\ k & d \end{vmatrix} . \begin{vmatrix} b & i \\ i & d \end{vmatrix} - \begin{vmatrix} h & k \\ i & d \end{vmatrix}^2 \text{ ...... (3),}$$

and

$$\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} . d + * - * = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} c & k \\ k & d \end{vmatrix} - \begin{vmatrix} h & k \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} \quad \text{.........(4)}.$$

If, now, A and $E = 0$, (4) becomes

$$\begin{vmatrix} h & k \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} c & k \\ k & d \end{vmatrix}$$

and (3) reduces to $\begin{vmatrix} c & k \\ k & d \end{vmatrix} . \begin{vmatrix} b & i \\ i & d \end{vmatrix} = \begin{vmatrix} h & k \\ i & d \end{vmatrix}^2$

And these, multiplied together, give

$$\begin{vmatrix} b & i \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} h & k \\ i & d \end{vmatrix}$$

but minor F, *i. e.*,

$$\begin{vmatrix} e & b & i \\ f & h & k \\ g & i & d \end{vmatrix} . d + * - * = \begin{vmatrix} b & i \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} - \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} h & k \\ i & d \end{vmatrix}$$

hence, on the supposition that $A = E = 0$, we have $F = 0$.

Continuing our analysis, we find that, when $A = 0 = E$, or $E = F = 0$, we have $\Delta = 0.$*

In general, we write the following, analogous to (2) and (3),

$$c\frac{d\Delta}{da} = \frac{d^2\Delta}{da\,db} \cdot \frac{d^2\Delta}{da\,dd} - \left(\frac{d^2\Delta}{da\,di}\right)^2,$$

$$c\frac{d\Delta}{de} = \frac{d^2\Delta}{dd\,de} \cdot \frac{d^2\Delta}{da\,db} - \frac{d^2\Delta}{da\,di} \cdot \frac{d^2\Delta}{db\,dg},$$

$$b\frac{d\Delta}{df} = \frac{d^2\Delta}{dd\,df} \cdot \frac{d^2\Delta}{da\,dc} - \frac{d^2\Delta}{da\,dk} \cdot \frac{d^2\Delta}{dc\,dg}.$$

* For an extension of the geometrical applications, herein considered, to tangential coordinates, and the determination of circular sections, see *Philosophical Mag.*, vol. xiv., 4th series, p. 393.

Assume
$$\left.\begin{aligned} eE-fF+gG&=0 \\ hE-cF+kG&=0 \\ iE-kF+dG&=0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (5),$$

which gives the determinant for the elimination of G and F,

$$\begin{vmatrix} e & -f & g \\ h & -c & k \\ i & -k & d \end{vmatrix} E = E \, . \, E = 0\,; \ i.e., \ E=0\,;$$

therefore $$F=0 \text{ and } G=0.$$

By equations similar to (5), as

$$aA+fF-gG=0,$$
$$\&c. \qquad \&c.,$$

we may show that $AB=0$, from which it follows that, if $B=0$, $H=I=0$; or, when $A=0$, $F=G=0$.

It can be shown that equations (5) are true when $\Delta=0$ and $A=0$ in all cases.

Let us now, in view of these suppositions and results, consider the nature of the surface given at the head of this Article.

Suppose $\dfrac{d^2\Delta}{da\,dd}=0$, $\dfrac{d^2\Delta}{db\,dd}=0$, and $-\dfrac{d^2\Delta}{dc\,dd}=0$,

and consequently $\dfrac{d^2\Delta}{dd\,dh}=0$; and suppose also a to be negative; then, multiplying the surface by a, subtract $(ey+fz+g)^2$, and finally let $ax'=ax+ey+fz+g$, and we shall have

$$a^2x'^2+2\,\frac{d^2\Delta}{dc\,di}\,y+2\,\frac{d^2\Delta}{db\,dk}\,z+\frac{d^2\Delta}{db\,dc}=0,$$

an equation to a *parabolic cylinder.*

The three latter suppositions applied to the surface reduce it to

$$a^2x'^2+\frac{d^2\Delta}{dc\,dd}\,y^2+2\left(\frac{d^2\Delta}{dd\,dh}\,z+\frac{d^2\Delta}{dc\,di}\right)y+\frac{d^2\Delta}{db\,dd}\,z^2$$
$$+\,2\,\frac{d^2\Delta}{db\,dk}\,z+\frac{d^2\Delta}{db\,dc}=0.$$

This equation, multiplied by $\frac{d^2\Delta}{dc\,dd}$, adding and subtracting $\left(\frac{d^2\Delta}{dd\,dh}z + \frac{d^2\Delta}{dc\,di}\right)^2$, and finally making

$$\frac{d^2\Delta}{dc\,dd}y' = \frac{d^2\Delta}{dc\,dd}y + \frac{d^2\Delta}{dd\,dh}z + \frac{d^2\Delta}{dc\,di},$$

we obtain

$$a^2\frac{d^2\Delta}{dc\,dd}x'^2 + \left(\frac{d^2\Delta}{dc\,dd}\right)^2 y'^2 + a\left(\frac{d\Delta}{dd}z^2 + 2\frac{d\Delta}{dk}z + \frac{d\Delta}{dc}\right) = 0.$$

If, now, we multiply this by $\frac{d\Delta}{dd}$, add and subtract $a\left(\frac{d\Delta}{dk}\right)^2$, and put $\frac{d\Delta}{dd}z$ for $\frac{d\Delta}{dd}z + \frac{d\Delta}{dk}$, we have

$$a^2\frac{d^2\Delta}{dc\,dd}\frac{d\Delta}{dd}x'^2 + \left(\frac{d^2\Delta}{dc\,dd}\right)^2\frac{d\Delta}{dd}y'^2 + a\left(\frac{d\Delta}{dd}\right)^2 z'^2 + a\frac{d^2\Delta}{dc\,dd}\Delta = 0,$$

which is the equation to an *ellipsoid* when Δ is negative, and $\frac{d\Delta}{dd}$, $\frac{d^2\Delta}{dc\,dd}$ both positive.

When Δ is positive, and $\frac{d\Delta}{dd}$, $\frac{d^2\Delta}{dc\,dd}$ either one or both negative, this equation represents a *hyperboloid of one sheet.* If Δ be negative it represents a *hyperboloid of two sheets*, if it vanishes a *cone.* Also, if $\frac{d\Delta}{dd} = 0$, it is the equation to an *elliptic* or *hyperbolic paraboloid*, according as $\frac{d^2\Delta}{dc\,dd}$ is positive or negative. In the same manner it represents an *elliptic* or *hyperbolic cylinder* when $\frac{d\Delta}{dd} = \frac{d\Delta}{dk} = 0$, and $\frac{d^2\Delta}{dc\,dd}$ is positive or negative.

To find the plane perpendicular to the chord to which it is conjugate; *i.e.*, the diametral plane.

Let l, m, n be the direction cosines of the chord, the plane in question will be, from equations (1),

$$l(ax+ey+fz+g)+m(ex+by+hz+i)+n(fx+hy+cz+k)=0,$$

provided l, m, n are proportional to the coefficients of the variables x, y, z; and we shall have, in that case,

$$\left.\begin{array}{l} la+me+nf=pl \\ le+mb+nh=pm \\ lc+mh+nc=pn \end{array}\right\} \dots\dots\dots\dots\dots\dots (6);$$

therefore, to find p, we have the determinant

$$\begin{vmatrix} a-p & e & f \\ e & b-p & h \\ c & h & c-p \end{vmatrix}$$

The value of p being found and substituted in equations (6), we shall obtain the values of l, m, n, and thus be able to find *the three diametral planes of the surface.*

We conclude this study with the simple remark that we are not here concerned with teaching Modern Geometry, but with an exercise for the practice of Determinants, and to indicate their use in the investigation of *loci.*

44. The Jacobian, which has already been described, deserves, on account of its importance, a special consideration.

Study second.—(a) Let V and V_1 be two functions, homogeneous in the second degree, J is then

$$\begin{vmatrix} \dfrac{dV}{dx} & \dfrac{dV}{dy} \\ \dfrac{dV_1}{dx} & \dfrac{dV_1}{dy} \end{vmatrix} = 0,$$

which, under the conditions mentioned, determines the foci of involution of two pairs of points.

(*b*) Let $S_1=0$, $S_2=0$, $S_3=0$ be three circles, and $u=0$ the equation to the circle orthotomic. The polar of any point on u (xyz), with regard to each of the given circles, will pass through a single point. Let u_1, u_2, u_3, &c. represent the differentials $\frac{du}{dx}$, &c., then

$$lu_1+mu_2+nu_3=0,$$

$$lv_1+mv_2+nv_3=0,$$

$$lw_1+mw_2+nw_3=0.$$

The determinant of which is a Jacobian and $=u$, the equation of the circle orthotomic required.

(*c*) If we proceed to the conicoids, as V, V_1, V_2, the equations of the three polars will be

$$V'x+V''y+V'''z=0,$$

$$V_1'x+V_1''y+V_1'''z=0,$$

$$V_2'x+V_2''y+V_2'''z=0;$$

therefore
$$\begin{vmatrix} \frac{dV'}{dx} & \frac{dV''}{dy} & \frac{dV'''}{dz} \\ \frac{dV_1'}{dx} & \frac{dV_1''}{dy} & \frac{dV_1'''}{dz} \\ \frac{dV_2'}{dx} & \frac{dV_2''}{dy} & \frac{dV_2'''}{dz} \end{vmatrix} = J,$$

and is a curve of the third order, in other words, the locus of a point whose polars, in regard to V, V_1, V_2, meet in a point.

(*d*) It is easily shown that two conics always intersect in four points.

Let V and V_1 intersect, and through these points draw V_2. Then the J of the three conics is the equation to the curve which cuts V_2 *in six points.*

(*e*) If we form the Hessian of $lV+mV_1+nV_2$; then, if we examine the coefficients of l, m, n, we shall find them invariants of V, V_1, V_2, one of which vanishes whenever $lV+mV_1+nV_2$ represents two planes; the other vanishes (as shown by Prof. Cayley) when any two of the eight points of intersection coincide, and their J is a curve of the sixth order, when V, V_1, V_2 represent quadrics, and this curve is the locus of a point whose polar planes meet in a line.

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MATHEMATICAL TRACTS.

No. II

TRILINEAR COORDINATES.

[illegible] Davidson
June 14/77

TRACTS

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 2.

TRILINEAR COORDINATES.

BY

REV. W. J. WRIGHT, PH.D.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

"Ἐθέλω σοι εἰπεῖν ὥσπερ οἱ γεωμέτραι."
PLATO: GORGIAS.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1877.

My acknowledgments are due to R. Tucker, Esq., M.A., Honorary Secretary of the London Mathematical Society, for valuable assistance rendered in passing these sheets through the press.—W. J. W.

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PREFACE TO TRACT NO. II.

MINISTERIAL and other duties have prevented the earlier appearance of this Tract. The delay has afforded an opportunity to those persons who have become acquainted with the proposed plan of this Series of expressing their opinion upon the merits of such an undertaking.

A considerable number of Professors and Amateurs have been pleased to signify their approval of this effort, and to give me more than deserved commendations. I have no object in referring to this, except so far as to certify that the purpose in view is a good one, and that the means adopted, while novel, are likely to prove in a fair measure successful. I take this opportunity of again urging upon those to whom these Tracts may come the great importance of the study of the Modern Mathematics, not only in their various subjects as educative instruments, but also as the best media of investigation. The extent and value of the new methods, together with the duties of those capable of teaching them, are happily expressed in a letter to me from M. Hermite, dated Paris, October 28, 1876, who will probably pardon the liberty I take with his communication, on the ground that the following extract is of public importance:—

"Les vues exposées par vous, Monsieur, dans la préface de cet ouvrage [Tract No. I.] sur les obligations qu'imposent à l'enseignement les grands progrès de la science de notre temps, je les adopte pleinement, et, autant qu'il m'a été possible, j'ai essayé de m'y conformer dans mon Cours d'Analyse de l'Ecole Polytechnique. Une grande transformation s'est déjà faite et continue encore de se faire dans le domaine de l'Analyse; des voies nouvelles plus fécondes et je crois aussi plus faciles ont été ouvertes, et c'est l'œuvre de ceux qui veulent servir la science et leur pays de discerner ce que les éléments peuvent recevoir de l'immense élaboration qui s'est accomplie depuis Gauss jusqu'à Riemann."

I am also indebted to Prof. Benj. Peirce, of Harvard, for a communication in reference to the form of Laplace's equation for secular perturbations, referred to on p. 41 of Tract No. I.

Without detracting from the value of the Ancient Geometry, it is believed that a considerable portion might be omitted, if necessary, to give place to the Modern, and that our regular college curriculum would be greatly enriched by such substitution.

In any event, I hold it to be the duty of every teacher of Geometry, whether in the form of analysis or synthesis, to incorporate in his instructions large masses of the New Geometry, unless, indeed, there happens to be a Chair devoted to this especial science.

In presenting Trilinear Coordinates, it is not proposed to supersede the Cartesian, nor even to regard them as inseparable from them; but to show (as Dr. Salmon has shown) the peculiar province and power of each. In this Tract it has not been thought necessary to advance far in this comparison. The student will quickly see where he can most advantageously employ the one or the other, or, leaving both, press into his service the Triangular or Tangential Coordinates.

All that could be attempted in a work of this size is to give a syllabus of the more common equational forms, and to exhibit, in as simple a manner as possible, their genesis.

There are other systems of Coordinates which space did not allow me to exhibit; the *quadrilinear,* which involves four straight lines as lines of reference, is one of some importance.

Another form of Coordinates I will just mention, the description of which has been communicated to me by Rev. Thos. Hill, D.D., LL.D., late President of Harvard. These Coordinates consist in defining a curve by expressing the length of a perpendicular let fall from the origin upon a normal as the function of its direction. Thus, if θ represent the angle contained by the perpendicular and the axis of X, then $p = f(\theta)$. These are known in this form as Watson's Coordinates. Dr. Hill has modified this system, and succeeded in achieving some very interesting results. (See Proceedings of the American Association for the Advancement of Science, 1873—75.)

For my first interest in the subject of this Tract I am indebted to a paper read before the Royal Society of Edinburgh in 1865, and published in the *Messenger of Mathematics* of the year following, the author of which, Rev. Hugh Martin, D.D., has exhibited in that paper much of the power and originality which characterise his well-known treatise upon "The Atonement."

It may be said, however, that works upon Modern Geometry do in general suggest the treatment of their subjects by the method of Trilinear Coordinates. They do, indeed, suggest far more than has been attempted here. In the works of Mulcahy, Townsend, Salmon, Ferrers, Whitworth, the recent volumes of Dr. Booth, Carnot, Steiner, Serret, Rouché and Comberousse, Bobillier, Cremona, Briot and Bouquet, Chasles, together with the journals *Annali di Matematica pura ed applicata* (of which Cremona is co-editor), *Comptes Rendus des Séances,* that of Crelle and Borchardt, *Nouvelles Annales de Mathématiques,* may be found much that leads to, and much more that leads beyond, that which now follows.

Books, at best, are but poor substitutes for the living teacher. Under familiar, oral teaching the difficulties which otherwise too frequently envelope the student rapidly disappear. Hence I would again emphasize the importance of admitting these subjects to our colleges as parts of the regular course.

Since the publication of Tract No. I., the heads of two of our leading Universities have made haste to inform me that some parts of the Modern Mathematics I am endeavouring to enforce and popularise are taught in their colleges. I profoundly wish that these exceptions were made the rule.

W. J. W.

CHAMBERSBURG, PA.; 1877.

TRILINEAR COORDINATES.

CHAPTER I.

FUNDAMENTAL EQUATIONS.

1. The fundamental equation of the straight line in Trilinear Coordinates is

$$l\alpha + m\beta + n\gamma = 0.$$

2. The apparatus for expressing this conception consists of a triangle of reference, whose sides are called the three lines of reference.

3. The angular points of this triangle are indicated by A at the vertex, B at the left, and C at the right; the lengths of the sides opposite these angles by a, b, c; and the perpendicular distances of any point from BC, CA, AB by α, β, γ.

The distance α may be described as reckoned downward or upward from the given point, β to the right, and γ to the left.

4. We may say, in general, that the position of a point in a plane is known implicitly when its perpendicular distances from any two sides of the proposed triangle are given. Its perpendicular distance from the third side is then given by these data, for manifestly

$$\frac{2\Delta - (\beta b + c\gamma)}{a} = \alpha,$$

where $\Delta =$ area of given triangle.

5. By attention to the figure, which scarcely need be drawn, we are clearly presented with the equation

$$a\alpha + b\beta + c\gamma = 2\Delta \text{ ....(1)},$$

which is found by taking the sum of the areas of the three triangles APC, APB, BPC, $\frac{b\beta}{2}$, $\frac{c\gamma}{2}$, $\frac{a\alpha}{2}$ respectively.

Observing that $\frac{a}{2} = r \sin A$, $\frac{b}{2} = r \sin B$, $\frac{c}{2} = r \sin C$, (1) may be written

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \frac{\Delta}{r} = \mathrm{V},$$

where r = radius of the circumscribing circle. These equations hold, whether the point is situated below BC, within the triangle, or above the vertex.

In the first case, by convention, $a\alpha$ is regarded as negative; in the second, each term as positive; while in the last $a\alpha$ is alone positive.

6. It will be observed also that the point is equally determined if the ratios of the three perpendiculars are given, for we see at once that each ratio determines a locus, which is a line drawn through the angle upon which the point is situated. The point sought is at the intersection of these lines.

7. Before proceeding further, it may be well to exhibit in full the process for deriving the equation of the straight line (Art. 1).

Let P_1, P_2 be the given points; $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ their coordinates; and P_1P_2 the straight line whose equation is to be determined. Take any point P on this line, and let its coordinates be α, β, γ; then, by similar triangles,

$$PP_1 : PP_2 :: \alpha_1 - \alpha : \alpha - \alpha_2 : \beta_1 - \beta : \beta - \beta_2 : \gamma_1 - \gamma : \gamma - \gamma_2;$$

or, taking the last two ratios, we are immediately presented with the two determinants (D. 2; *i.e.*, Tract No. I., Art. 2)

$$\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} = \begin{vmatrix} \gamma & \beta \\ \gamma_1-\gamma_2 & \beta_1-\beta_2 \end{vmatrix};$$

likewise

$$\begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix} = \begin{vmatrix} \alpha & \gamma \\ \alpha_1-\alpha_2 & \gamma_1-\gamma_2 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} \beta & \alpha \\ \beta_1-\beta_2 & \alpha_1-\alpha_2 \end{vmatrix}.$$

If now we multiply these equations respectively by α, β, γ, and add, we shall have at once

$$\alpha \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} + \beta \begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix} + \gamma \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 0 \;\ldots\ldots\; (1).$$

Let now these determinants in this last result be represented by l, m, n respectively, and we have

$$l\alpha+m\beta+n\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

And since l, m, n represent constants, and since also α, β, γ are the coordinates of any point of the line, this equation expresses, as before stated in (Art. 1), the conception of the general equation of the straight line in trilinear coordinates.

Cor. 1.—This is also plainly the equation of a straight line through two given points.

Cor. 2.—The ratios represented by l, m, n are manifestly constant whatever the position of P on the locus, which involves also the deduction that this locus must be a straight line.

8. *The condition of concurrence.*—Let the straight lines be

$$l_1\alpha+m_1\beta+n_1\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$
$$l_2\alpha+m_2\beta+n_2\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

These equations, regarded as simultaneous, must have α, β, γ as the coordinates of a common point. To obtain the ratios, we are presented with the determinants (D., Arts. 10, 12)

$$\begin{vmatrix} 1 & 1 & 1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix};$$

otherwise

$$\alpha : \beta : \gamma :: \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}.$$

Hence the trilinear ratios of the point of intersection are determined.

Cor.—The general equation of a straight line passing through their point of intersection may be written

$$l\alpha + m\beta + n\gamma = k(l_1\alpha + m_1\beta + n_1\gamma) \quad \text{.........}(3),$$

where k is any constant; for the locus of (3) must pass through every point common to the loci of (1) and (2).

9. Three straight lines, as

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$
$$l_3\alpha + m_3\beta + n_3\gamma = 0,$$

present the determinant (D. 6)

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

as the condition that three straight lines shall have a point in common.

10. *The condition of parallelism.*—Let the two straight lines be

$$l\alpha + m\beta + n\gamma = 0,$$
$$l_1\alpha + m_1\beta + n_1\gamma = 0.$$

Let us find the condition of parallelism.

Suppose $\alpha, \beta, \gamma, f, g, h$ the coordinates of two points in the former; $\alpha_1, \beta_1, \gamma_1, f_1, g_1, h_1$ the coordinates of any two points in the latter.

If these lines are parallel, the geometry of the figure requires

$$\alpha - f : \beta - g : \gamma - h :: \alpha_1 - f_1 : \beta_1 - g_1 : \gamma_1 - h_1.$$

Let us seek an expression for this in terms of the constants of the given equations and the triangle of reference.

Remembering (Art. 5) that

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

and consequently $\quad af + bg + ch = 2\Delta,$

we have $\quad a(\alpha - f) + b(\beta - g) + c(\gamma - h) = 0 \quad \text{..........(1).}$

Also, since $\quad l\alpha + m\beta + n\gamma = 0,$

and $\quad lf + mg + nh = 0,$

we obtain $\quad l(\alpha - f) + m(\beta - g) + n(\gamma - h) \quad \text{..............(2).}$

Equations (1) and (2) give the eliminant (D. 39)

$$\begin{vmatrix} 1 & 1 & 1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0,$$

from which we derive the ratios

$$\alpha - f : \beta - g : \gamma - h :: \begin{vmatrix} m & n \\ b & c \end{vmatrix} : \begin{vmatrix} n & l \\ c & a \end{vmatrix} : \begin{vmatrix} l & m \\ a & b \end{vmatrix},$$

likewise

$$\alpha_1 - f_1 : \beta_1 - g_1 : \gamma_1 - h_1 :: \begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix};$$

hence

$$\begin{vmatrix} m & n \\ b & c \end{vmatrix} : \begin{vmatrix} n & l \\ c & a \end{vmatrix} : \begin{vmatrix} l & m \\ a & b \end{vmatrix} :: \begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix}.$$

Multiplying each of these ratios by l_1, m_1, n_1, and remembering how they were derived, we have, by restoring,

$$l_1 \begin{vmatrix} m & n \\ b & c \end{vmatrix} + m_1 \begin{vmatrix} n & l \\ c & a \end{vmatrix} + n_1 \begin{vmatrix} l & m \\ a & b \end{vmatrix}$$

$$:: l_1 \begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} + m_1 \begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} + n_1 \begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix}.$$

These are (D. 6) the expanded determinants for

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} :: \begin{vmatrix} l_1 & m_1 & n_1 \\ l_1 & m_1 & n_1 \\ a & b & c \end{vmatrix}.$$

By (D. 7) the right-hand determinant vanishes, and hence

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0 \text{ *}$$

is the condition of parallelism; or, by reverting this determinant, it can be written (D. 12)

$$aA + bB + cC = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and in this form is easily remembered.

11. *Excursus on the straight line.*—We have obtained (Art. 5) the equation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \frac{\Delta}{r} = \text{a constant};$$

and therefore we may write

$$l\alpha + m\beta + n\gamma + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0$$

as the parallel of the line

$$l\alpha + m\beta + n\gamma.$$

This follows from the analogy of the Cartesian coordinates, where, it will be remembered, two lines differing by only a constant are parallel. Also, if two equations are so connected that their difference is ever a constant, their sum represents their parallel and is situated half-way between them.

In the last Art., equation (3) is the result, in fact, of elimination between three equations, one of which is the impossible equation $$a\alpha + b\beta + c\gamma = 0;$$

impossible at least in any finite conception, since we have

* That this determinant may rigorously be equated to 0 is evident from the consideration of the ratios, when it will be seen we have been, in fact, concerned with only *one* equation.

proved it equal, in every position of the origin, to the area of the triangle of reference. Here again, after the analogy of the Cartesian, of which trilinear coordinates may be regarded as a particular case,* we may interpret

$$a\alpha+b\beta+c\gamma=0$$

as a line situated at an infinite distance from the origin, or we may say that every straight line may be regarded as parallel to the straight line at infinity.

Thus, analytically:

The ratios (Art. 8) $\alpha:\beta:\gamma$ express the relations of the coordinates of the point of intersection of two straight lines. The *actual* values are evidently given by the three equations,

$$a\alpha+b\beta+\ c\gamma=2\Delta,$$

$$l_1\alpha+m_1\beta+n_1\gamma=0,$$

$$l_2\alpha+m_2\beta+n_2\gamma=0,$$

where $$\frac{\alpha}{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}}=\frac{2\Delta}{\begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}$$

Writing A for the minor in the one case, and A_1 for the determinant in the latter, we have

$$\alpha=\frac{2\Delta A}{A_1}.$$

When α becomes infinite, A_1 becomes zero. But this expresses the condition of the straight line at infinity; that is, the point of intersection lies at an infinite distance.

But this is also the condition of parallelism of two straight lines.

* Salmon's Conics, p. 64.

The determinant, therefore, to represent parallel straight lines, may be written

$$A_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

which identically vanishes, and where it will be seen the ratios $l : m : n$ are merged in, and have become identical with, $a : b : c$.

12. *The condition of collinearity.*—Let the three points $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ be determined in the same straight line. We see it is only necessary to accent the α, β, γ of equation (1), (Art. 7), change α_1 to α_2, β_1 to β_2, &c., and we can write the condition at once

$$\alpha_1 \begin{vmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{vmatrix} + \beta_1 \begin{vmatrix} \gamma_2 & \alpha_2 \\ \gamma_3 & \alpha_3 \end{vmatrix} + \gamma_1 \begin{vmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{vmatrix} = 0.$$

By (D. 6),

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0,$$

which is the condition determining three points in a straight line.

The following well-known theorem will illustrate this :—

Let P be a point within the triangle of reference. Through this point let straight lines be drawn from A, B, C to meet the opposite sides respectively in A_1, B_1, C_1; these are the angular points of a triangle whose sides, when produced, will meet the corresponding sides of the first triangle in three points which lie in a straight line.

Suppose f, g, h the coordinates of the point P; α, β, γ those of any point, as A_1. Then $\alpha = 0$; and, by similar triangles, g and h will be the ratios.

A_1 will therefore be represented by $0, g, h$.

For B_1, g of course is 0, f and h its ratios, since $\beta : \gamma :: g : h$. Hence, in the same manner, B_1 is represented by

$$f, \ 0, \ h.$$

Hence the line joining A_1, B_1 is (Art. 7, Cor. 1)

$$\alpha \begin{vmatrix} g & h \\ 0 & h \end{vmatrix} + \beta \begin{vmatrix} h & 0 \\ h & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & g \\ f & 0 \end{vmatrix} = 0.$$

Otherwise $$\alpha gh + \beta hf - \gamma fg = 0 \quad \text{.....................} \quad (1),$$

which may be written, the line

$$gh, \quad hf, \quad -fg,$$

using the coefficients only to represent the line.

Recurring again to equation (1), (Art. 7), we see that, if γ_1, γ_2 are each $= 0$, we must have, for (2) of the same Article,

$$0\alpha + 0\beta + n\gamma = 0 \quad \text{.......................} (2).$$

But this condition attaches to the line AB, which is therefore represented by $0, \; 0, \; 1.$

The intersection of A_1B_1 and AB is therefore the concurrence of (1) and (2), which (Art. 8) is the point

$$fh, \quad -gh, \quad 0;$$

or, by ratios, $$f, \quad -g, \quad 0.$$

BC and B_1C_1 will intersect, similarly, in

$$0, \quad g, \quad -h;$$

and AC, A_1C_1 in $$-f, \quad 0, \quad h.$$

Hence, since $$\begin{vmatrix} f & -g & 0 \\ -f & 0 & h \\ 0 & g & -h \end{vmatrix} = 0,$$

the lines (AB, A_1B_1), (BC, B_1C_1), (AC, A_1C_1) meet in points which are collinear.

13. Another illustration of the use of these coordinates is found in the proof that the straight line joining the middle points of two sides of the triangle of reference is parallel to the third side. If the points be taken on BC and AC, then

equations (1) and (2) of the last Article will represent the lines to be considered, remembering only to accent two of the coordinates, when (1) becomes

$$gh_1, \quad hf_1, \quad -f_1 g\,;$$

and (2) $$0 \quad 0 \quad 1.$$

Substituting these in the determinant of parallelism (Art. 10), we find the required expression

$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & 0 & 1 \end{vmatrix} = am - bl = ahf_1 - bgh_1,$$

by giving m and l their values; and since, if the given lines are parallel, $h = h_1$, we may write

$$af_1 = bg,$$

which accords with the geometry of the figure.

14. *The condition of perpendicularity.*—The more common method of determining this condition is by establishing, in the first place, the angular relation of a given straight line to two of the sides of the triangle of reference. For this purpose the internal bisector of one of the angles may be used as an axis. A line drawn through the vertex A, for instance, may be regarded as known when its inclination to the bisector of this angle is determined. The equation of such a line evidently is concerned with but the two coordinates β, γ.

Two lines thus drawn may be represented by

$$t\beta = s\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

and $$t_1\beta = s_1\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

and their angular relations to the internal bisector of the angle A by θ and θ_1.

If now these lines be conceived as drawn parallel respectively to the given lines,

$$l\alpha + m\beta + n\gamma = 0,$$
$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$

whose condition of perpendicularity is sought, we may write, regarding only the ratios of β and γ,

$$(ma - lb)\,\beta + (na - lc)\,\gamma = 0,$$

$$(m_1 a - l_1 b)\,\beta + (n_1 a - l_1 c)\,\gamma = 0,$$

which are of the form of (1) and (2).

Equation (1) may be treated as follows:—

$$\sin\left(\frac{A}{2}+\theta\right) : \sin\left(\frac{A}{2}-\theta\right) :: t : s.$$

This, by composition, division, alternation, and reducing, becomes

$$\tan\theta : \tan\frac{A}{2} :: t-s : t+s.$$

Similarly,

$$\tan\theta_1 : \tan\frac{A}{2} :: t_1-s_1 : t_1+s_1.$$

But the condition of perpendicularity in general is

$$\tan\theta\tan\theta_1+1 = 0;$$

therefore, by reduction and supplying values, we get

$$mm_1a^2+nn_1a^2+ll_1(b^2+c^2-2bc\cos A)$$
$$-(nl_1+n_1l)(ac-ab\cos A) - (lm_1+l_1m)(ab-ac\cos A)$$
$$-(mn_1+m_1n)(a^2\cos A) = 0,$$

which, remembering that

$$b^2+c^2-2bc\cos A = a^2, \qquad c-b\cos A = a\cos B,$$
$$b-c\cos A = a\cos C,$$

becomes

$$ll_1-(mn_1+m_1n)\cos A + mm_1-(nl_1+n_1l)\cos B$$
$$+ nn_1-(lm_1+l_1m)\cos C = 0,$$

the condition necessary.

General Exercises.

1. To prove whether perpendiculars upon the opposite sides meet.

We perceive that the perpendicular divides any angle of the triangle into parts which are the complements of the remaining two angles.

Therefore the equation of AD is

$$\cos B \,.\, \beta = \cos C \,.\, \gamma,$$

or, more fully,

$$\beta : \gamma :: \sin CAD : \sin BAD :: \cos C : \cos B.$$

Similarly, $$\cos A \,.\, \alpha = \cos B \,.\, \beta,$$

and $$\cos C \,.\, \gamma = \cos A \,.\, \alpha.$$

If we write the equations of these perpendiculars in order, we see that α does not appear in the first or AD, β in the second or BE, and γ is wanting in the last or CF; and remembering that these are the coefficients of a linear equation,

as, $$0\alpha + \cos B \,.\, \beta - \cos C \,.\, \gamma = 0,$$
$$\&c. \qquad \&c.,$$

and remembering also that, by Art. 8, the problem is simply elimination between these three equations, the condition of concurrence, as we have already seen, is presented by the determinant

$$\begin{vmatrix} 0 & \cos B & -\cos C \\ -\cos A & 0 & \cos C \\ \cos A & -\cos B & 0 \end{vmatrix} = 0.$$

2. On the sides of the triangle of reference, as bases, are constructed three triangles, similar and so placed that the adjacent base angles are equal, and each base angle respectively equal to the vertex most remote; thus:

$$A_1BC = AB_1C = ABC_1, \qquad B_1CA = BC_1A = BCA_1,$$

and $$C_1AB = CA_1B = CAB_1;$$

then will AA_1, BB_1, CC_1 cointersect.

Since the point A_1 falls without the triangle of reference, but within the angle A, the ordinate α must be negative. The same applies to β at the point B_1, &c.

We first seek the perpendiculars on a, b, c from A_1, which are, in order,

$$S \,.\, \sin C_1, \qquad S \,.\, \sin (C + C_1), \qquad S_1 \,.\, \sin (B + B_1),$$

where $$S = \frac{a \sin B_1}{\sin A_1}, \quad \text{and} \quad S_1 = \frac{a \sin C_1}{\sin A_1}.$$

Dividing these by the first to obtain the ratios, we have for the coordinates

of A_1,	-1,	h,	g,
of B_1,	h,	-1,	f,
of C_1,	g,	f,	-1,

where f represents $\dfrac{\sin(A+A_1)}{\sin A_1}$,

" g " $\dfrac{\sin(B+B_1)}{\sin B_1}$,

" h " $\dfrac{\sin(C+C_1)}{\sin C_1}$.

The ratios of A are	1,	0,	0,
" B "	0,	1,	0,
" C "	0,	0,	1.

Hence the equation of the line joining the two points A and A_1 is (Art. 7)

$$0 \,.\, \alpha + g \,.\, \beta - h \,.\, \gamma = 0;$$

for BB_1, $$-f \,.\, \alpha + 0 \,.\, \beta + h \,.\, \gamma = 0;$$

for CC_1, $$f \,.\, \alpha - g \,.\, \beta + 0 \,.\, \gamma = 0.$$

By (Art. 9) the determinant of concurrence is formed from these three lines; that is,

$$\begin{vmatrix} 0 & g & -h \\ -f & 0 & h \\ f & -g & 0 \end{vmatrix} = 0.$$

3. In the same manner, from the same figure, prove that (BC, B_1C_1), (CA, C_1A_1), (AB, A_1B_1) respectively meet in points which are collinear.

15. *A straight line through a given point and parallel to a given straight line.*

Let (l, m, n) be the given straight line, (f, g, h) the given point, (l_1, m_1, n_1) the required straight line.

The condition of parallelism of

$$l_1\alpha+m_1\beta+n_1\gamma=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$l\,\alpha+m\,\beta+n\,\gamma=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

by (Art. 10), is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0;$$

that is,

$$l_1\begin{vmatrix} m & n \\ b & c \end{vmatrix} + m_1\begin{vmatrix} n & l \\ c & a \end{vmatrix} + n_1\begin{vmatrix} l & m \\ a & b \end{vmatrix} = 0 \ldots\ldots\ldots (3).$$

If the locus passes through f, g, h, we must have

$$l_1 f+m_1 g+n_1 h=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

We are now furnished with three equations to eliminate $l_1 m_1 n_1$; viz., (1), (3), and (4).

Hence

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f & g & h \\ A & B & C \end{vmatrix} = 0$$

is the equation sought, where A, B, C stand for the minors of (3).

16. To show that $\begin{vmatrix} b & c \\ R & S \end{vmatrix} = 2\Delta(\alpha-\alpha_1)$,

where

$$R = \begin{vmatrix} \gamma & \alpha \\ \gamma_1 & \alpha_1 \end{vmatrix}, \quad S = \begin{vmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{vmatrix}$$

are the second and first determinants formed from the coordinates of two points (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$,

$$\begin{vmatrix} b & c \\ R & S \end{vmatrix} = \begin{vmatrix} 0 & -c & b \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 0 \\ \alpha & \beta & 2\Delta \\ \alpha_1 & \beta_1 & 2\Delta \end{vmatrix} = 2\Delta(\alpha-\alpha_1).$$

Similarly, if $$Q = \begin{vmatrix} \beta & \gamma \\ \beta_1 & \gamma_1 \end{vmatrix},$$

$$\begin{vmatrix} c & a \\ S & Q \end{vmatrix} = 2\Delta(\beta-\beta_1), \quad \text{and} \quad \begin{vmatrix} a & b \\ Q & R \end{vmatrix} = 2\Delta(\gamma-\gamma_1).$$

17. *Deduced coordinates of the triangle of reference.*

1st. *Of the angular points.*

At A, $$\beta = 0, \quad \gamma = 0.$$

Hence $$a\alpha = 2\Delta, \quad \alpha = \frac{2\Delta}{a}.$$

At B, similarly, $$0, \quad \frac{2\Delta}{b}, \quad 0.$$

At C, $$0, \quad 0, \quad \frac{2\Delta}{c}.$$

2nd. *Of the middle point of BC.*

Evidently $b\beta =$ area of triangle $= c\gamma$,

and $$\alpha = 0,$$

Hence $0, \ \dfrac{\Delta}{b}, \ \dfrac{\Delta}{c}$ are the coordinates.

3rd. *Of the foot of the perpendicular from A upon BC.*

The perpendicular $= \dfrac{2\Delta}{a}$, by 1st case.

Hence $$\frac{2\Delta}{a}\cos C = \beta, \quad \frac{2\Delta}{a}\cos B = \gamma;$$

and $$0, \quad \frac{2\Delta}{a}\cos C, \quad \frac{2\Delta}{a}\cos B$$

are the required coordinates.

4th. *Of the centre of the inscribed circle.*

The point being equally distant from the three lines of reference, we must have

$$\alpha = \beta = \gamma = \frac{2\Delta}{a+b+c}.$$

Ex.—Prove that

$$\alpha = r\cos A,\quad \beta = r\cos B,\quad \gamma = r\cos C,$$

r being radius of circumscribed circle.

18. *Distance between two points.*

Various expressions may be deduced. One only is here given; others will be given hereafter.

Let B_1C_1, B_1A_1, drawn parallel to the sides of the triangle of reference BC, BA respectively, be two sides of a quadrilateral $B_1C_1PA_1$ inscribed in a circle whose diameter is B_1P; (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ the coordinates of B_1, P; $r =$ the distance between them. Through C_1 draw a diameter C_1D. Join A_1C_1, DA_1.

$$\begin{aligned} A_1C_1^2 &= PC_1^2 + PA_1^2 - 2PC_1 \,.\, PA_1 \cos C_1PA_1 \\ &= PC_1^2 + PA_1^2 + 2PC_1 \,.\, PA_1 \cos B \;\;\ldots\ldots\ldots\ldots (1). \end{aligned}$$

The angle at $D =$ the angle $C_1B_1A_1 = B$,

$$A_1C_1 = C_1D \sin B = PB_1 \sin B = r \sin B,$$

$$PC_1 = \alpha - \alpha_1,\quad PA_1 = \gamma - \gamma_1.$$

Substituting these values in (1),

$$r^2 \sin^2 B = (\alpha-\alpha_1)^2 + (\gamma-\gamma_1)^2 + 2(\alpha-\alpha_1)(\gamma-\gamma_1)\cos B,$$

which is the required equation.

This, however, may be made symmetrical with the determinants formed from the coordinates of the points B_1, P. These determinants are represented in (Art. 16) by Q, R, S; also in the same Article it was shown that

$$\frac{\begin{vmatrix} b & c \\ R & S \end{vmatrix}}{2\Delta} = \alpha - \alpha_1, \quad\text{and}\quad \frac{\begin{vmatrix} a & b \\ Q & R \end{vmatrix}}{2\Delta} = \gamma - \gamma_1.$$

Hence $\qquad 4\Delta^2 r^2 \sin^2 B = X^2 + Y^2 + 2XY\cos B,$

where $\qquad X = \begin{vmatrix} b & c \\ R & S \end{vmatrix},\quad Y = \begin{vmatrix} a & b \\ Q & R \end{vmatrix}.$

Developing the values of X and Y, we find

$$4\Delta^2 r^2 \sin^2 B = b^2 (Q^2 + R^2 + S^2 - 2RS \cos A - 2SQ \cos B - 2QR \cos C).$$

By (Art. 5), $\qquad 2\Delta \sin B = Vb.$

Therefore

$$r = \frac{1}{V} \sqrt{(Q^2 + R^2 + S^2 - 2RS \cos A - 2SQ \cos B - 2QR \cos C)}.$$

19. *The area of a triangle from the trilinear coordinates of the angular points.*

The area of a triangle expressed as a determinant in Cartesian coordinates, the axes being rectangular, is (Art. 10, D.)

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

which, referred to oblique axes, becomes

$$\frac{1}{2} \operatorname{cosec} \omega \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

Let $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ take the places of x and y, and multiply and divide by V, and we shall have

$$\frac{\operatorname{cosec} C}{2V} \begin{vmatrix} V & V & V \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

Multiply the last row by $\sin A$ and the second by $\sin B$, and then take the sum of these new rows from the first.

Observing (Art. 5) that

$$\alpha \sin A + \beta \sin B + \gamma \sin C = V,$$

we are enabled to write

$$\text{Area} = \frac{\operatorname{cosec} C}{2V}\begin{vmatrix} \gamma_1 \sin C & \gamma_2 \sin C & \gamma_3 \sin C \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

$$= \frac{1}{2V}\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

or, more symmetrically,

$$= \frac{1}{2V}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

It will be observed that the angle ω, between the axes, is changed to one of the angles of the triangle of reference in passing from the Cartesian to the trilinear system.

20. *Perpendicular distance of a point from a line.*

Let the coordinates of the point be $(\alpha_2, \beta_2, \gamma_2)$; (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ the coordinates of two points in the line; r the distance between these last-named points; and p the perpendicular sought.

Then $pr =$ twice the area of the triangle of which the three points are the vertices.

By the preceding Article,

$$p = \frac{1}{r}\cdot\frac{1}{V}D.$$

But, by (Art. 18), $\quad r = \dfrac{1}{V}Z,$

where $Z =$ the radical part in the final value of r.

Hence $$p = \frac{D}{Z} = \frac{\begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}}{Z}$$

$$= \frac{Q\alpha_2 + R\beta_2 + S\gamma_2}{Z} \quad\text{..........................(1).}$$

We have already seen (Art. 7) that the equation to a line joining two points is

$$l\alpha+m\beta+n\gamma=0.$$

But the numerator of (1) is, in fact, the same expression under another form. As general equations of a straight line joining two points they must be identical.

Hence we may write

$$p=\frac{l\alpha+m\beta+n\gamma}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}}.$$

CHAPTER II.

THE EQUATION IN TERMS OF PERPENDICULARS—TANGENTIAL AND TRIANGULAR COORDINATES—IMAGINARIES.

21. It is necessary, as we proceed, to introduce the equation of the straight line under somewhat different forms. We have considered a point as determined by its perpendicular distances from the three sides of the triangle of reference. A line joining two of these points has thus far occupied our attention.

Let now the perpendicular distances of the three angular points A, B, C from a straight line be p, q, r, and let it be required to find the equation to this straight line in terms of these quantities.

We will assume two points on this straight line, one upon each side of the perpendicular p; $d =$ the distance between them.

$\frac{pd}{2} =$ area of the triangle formed by these two points and the point A.

For the coördinates of the point A we have (Art. 17)

$$\frac{2\Delta}{a},\quad 0,\quad 0.$$

Let (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ be the coordinates of the two given points.

Hence (Art. 19)

$$pd = \frac{1}{V}\begin{vmatrix} \frac{2\Delta}{a} & 0 & 0 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

also $$qd = \frac{1}{V}\begin{vmatrix} 0 & \frac{2\Delta}{b} & 0 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

and $$rd = \frac{1}{V}\begin{vmatrix} 0 & 0 & \frac{2\Delta}{c} \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

Multiplying these equations by $a\alpha$, $b\beta$, $c\gamma$ respectively, and adding, we have (D. 7)

$$(ap\alpha + bq\beta + cr\gamma)\, d = \frac{2\Delta}{V}\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} = 0,$$

that is, $$ap\alpha + bq\beta + cr\gamma = 0.$$

This is only another form, or a special case, of

$$l\alpha + m\beta + n\gamma = 0,$$

in which l, m, n are proportional to ap, bq, cr, or to the determinants of (Art. 7, eq. 1).

22. *The perpendicular distance of a point from the line*

$$ap\alpha + bq\beta + cr\gamma = 0 \text{ ....}(1).$$

Let the given point be (f, g, h), through which a parallel is drawn; d the perpendicular distance required.

Then the distances from A, B, C to this parallel will be represented by the perpendiculars $d \pm p$, $d \pm q$, $d \pm r$, which, substituted in (1), give

$$a(d \pm p)\,\alpha + b(d \pm q)\,\beta + c(d \pm r)\,\gamma = 0.$$

But, by hypothesis, this line passes through (f, g, h).

Hence $$a(d \pm p) f + b(d \pm q)\, g + c(d \pm r)\, h = 0,$$

$$(af + bg + ch)\, d = \mp (apf + bqg + crh),$$

which gives $$d = \pm \frac{apf + bqg + crh}{2\Delta},$$

an equation for the perpendicular distance.

23. The equation

$$d = \frac{apf + bqg + crh}{2\Delta}$$

evidently gives the altitude of a triangle whose vertex is f, g, h, and the equation of the base

$$apf + bqg + crh = 0.$$

The equations of the sides will differ only in the perpendicular; hence these may be written

$$ap_1f + bq_1g + cr_1h = 0,$$
$$ap_2f + bq_2g + cr_2h = 0.$$

With these two equations, and

$$af + bg + ch = 2\Delta,$$

the values of f, g, h may be determined; that is,

$$af : bg : ch : 2\Delta :: \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

$$\frac{af}{2\Delta} = \frac{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

Multiplying this equality by p, and the second and third equations formed from the above proportion by q and r respectively, and adding, we have

$$d = \frac{apf + bqg + crh}{2\Delta} = \frac{\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

24. We will now show the method of expressing the position of a right line by coordinates, and that of a point by an equation.*

Let p, q, r be the unknown, α, β, γ the known, coordinates; then, by the equation we have just considered, we are enabled to determine a relation between p, q, r which will be true for any right line drawn through the fixed point of which α, β, γ are the coordinates; that is,

$$a\alpha p + b\beta q + c\gamma r = 0,$$

which is called an equation, in tangential coordinates, of the point whose trilinear coordinates are α, β, γ.

25. *To find the tangential equation to the point of intersection of two right lines.*

Let (p_1, q_1, r_1), (p_2, q_2, r_2) be the tangential coordinates of the two lines; (α, β, γ) the trilinear coordinates of their point of intersection. Evidently, then, (α, β, γ) is a point on each of two lines whose perpendicular distances from A, B, C are p_1, q_1, r_1; p_2, q_2, r_2.

We first determine the ratios of the trilinear coordinates of the point.

We have (Art. 21)

$$ap_1\alpha + bq_1\beta + cr_1\gamma = 0,$$

and

$$ap_2\alpha + bq_2\beta + cr_2\gamma = 0;$$

hence

$$a\alpha : b\beta : c\gamma :: \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

Multiply each of these ratios by p, q, r respectively, and add; then each of these ratios

$$= \frac{a\alpha p + b\beta q + c\gamma r}{\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

* Salmon's Conics, p. 65.

The numerator expresses a relation between p, q, r by the preceding Article; but the denominator evidently expresses the same relation.

Hence
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0$$
is the equation required.

Otherwise, suppose
$$a\alpha p + b\beta q + c\gamma r = 0 \quad \text{.....................} \quad (1)$$
the equation of the point of intersection, which must be satisfied by the coordinates of any line drawn through that point; but (p_1, q_1, r_1), (p_2, q_2, r_2) by hypothesis are perpendiculars from the points of reference upon lines drawn through the point of intersection.

Hence
$$a\alpha p_1 + b\beta q_1 + c\gamma r_1 = 0 \quad \text{.....................} \quad (2),$$
$$a\alpha p_2 + b\beta q_2 + c\gamma r_2 = 0 \quad \text{.....................} \quad (3);$$
that is, (1), (2), (3) furnish the determinant
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0,$$
the same relation and equation as before.

26. *Tangential equation of a point at infinity.*

The point is clearly the intersection of two parallels.

Let (p_1, q_1, r_1) and (p_1+t, q_1+t, r_1+t) be the parallels. But the condition of parallelism (Art. 10) is
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_1+t & q_1+t & r_1+t \end{vmatrix} = 0,$$
which may be written
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ t & t & t \end{vmatrix} = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

The last determinant identically vanishes, as will be seen, if a common factor can be taken so as to make the first row unity; in other words, if $p=q=r$. That is, points at infinity are comprised upon the line

$$p=q=r;$$

and equation (1) of last Article reduces to

$$a\alpha+b\beta+c\gamma=0,$$

a relation which has already been interpreted (Art. 11).

27. Since we define the equation to a point in these coordinates as an equation satisfied by the coordinates of all right lines drawn through the point, it follows that, if

$$L=0,$$
$$V=0,$$

be two equations representing two points in tangential coordinates, then the equation

$$L+kV=0$$

being satisfied, as it evidently is, by the coordinates of L and V, must express a point on the line joining the given points.

28. Reserving for the present the further development and the application of tangential coordinates, we will just mention a system of coordinates known by the term *triangular.*

Instead of the trilinear equation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

we may write $$\frac{a\alpha}{2\Delta}+\frac{b\beta}{2\Delta}+\frac{c\gamma}{2\Delta}=1;$$

and, denoting the ratios of the left member by x, y, z, we have

$$x+y+z=1.$$

The ratios $\frac{a\alpha}{2\Delta}$, $\frac{b\beta}{2\Delta}$, $\frac{c\gamma}{2\Delta}$ evidently represent the ratios of the triangles BPC, APC, APB to the triangle of reference.

It is clear how the trilinear coordinates α, β, γ are related to x, y, z; for, if we divide x by $a\alpha$, we have $\frac{1}{2\Delta}$. In the same manner, y divided by $b\beta = \frac{1}{2\Delta}$; so that

$$\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma} = \frac{1}{2\Delta}.$$

29. The coordinates of the middle point of BC are, in triangular coordinates, $0, \ \frac{1}{2}, \ \frac{1}{2}$.

This appears, since $b\beta = \Delta$.

But $b\beta = 2\Delta y$;

hence $y = \frac{1}{2}$;

similarly, $z = \frac{1}{2}$;

while $x = 0$.

30. We have seen (Art. 17) that the coordinates of the foot of the perpendicular from A upon BC are

$$0, \quad \frac{2\Delta}{a} \cos C, \quad \frac{2\Delta}{a} \cos B.$$

These expressions, transformed as above, become

$$0, \quad \frac{b \cos C}{a}, \quad \frac{c \cos B}{a}.$$

Referring again to (Art. 17), we find the coordinates of the centre of the inscribed circle, which, transferred into the triangular system, become

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{1}{a+b+c}.$$

Transforming the area of a triangle (Art. 19), we have

$$\frac{1}{2V} \cdot \frac{8\Delta^3}{abc} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

= (Art. 5)

$$\frac{8\Delta^3}{abc}\cdot\frac{1}{2(\alpha\sin A+\beta\sin B+\gamma\sin C)}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \Delta\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The equation to a straight line joining two points is, in a similar manner, found to be (Art. 7)

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The condition of concurrence is the same for both systems.

The equation to the right line at infinity (Art. 11) becomes in triangular coordinates

$$x+y+z=0;$$

and consequently the condition of parallelism (Art. 10) is easily transformed to

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

The equation to the perpendicular (Art. 14, Ex. 1)

$$\beta\cos B=\gamma\cos C,$$

transformed from trilinear to triangular coordinates, is

$$y\cot B=z\cot C.$$

Thus a great number of similar transformations might be written out.

These, however, must suffice for the present, and these are probably sufficient to give the reader a correct idea of such changes when they become necessary. The useful applications of these several systems of coordinates must be learned chiefly from a study of lines of a higher order than the first.

31. The principle detailed by Dr. Salmon* is equally ap-

* Conics, p. 33.

plicable to trilinear or triangular coordinates, or any system in which a point is determined by coordinates; that is, if

$$u = l\alpha + m\beta + n\gamma = 0,$$

and $$v = l_1\alpha + m_1\beta + n_1\gamma = 0,$$

then will $$u + kv = 0 \quad \text{.........................(1)}$$

represent a line passing through the intersection of u and v, which line, it is evident, can be made to represent any particular line by giving particular values to the arbitrary constant k.

Let us try a simple application.

Suppose the triangle of reference circumscribed by lines whose equations are

$$u = 0, \quad v = 0, \quad w = 0;$$

that is, representing A_1B_1, B_1C_1, C_1A_1. Let A_1B_1 be produced to some point B_2, and from B_2 let B_2C_2 be drawn, and let B_2C_2 be the line whose equation is to be determined. Join B_2C_1. But C_1 is the point of intersection of $v = 0$ and $w = 0$. Hence, from what has just preceded, B_2C_1 will be represented by

$$v + kw = 0.$$

Also, since B_2C_1 and A_1B_1 (produced) meet in B_2, the line B_2C_2, which is drawn through their intersection, will, by the same considerations, be represented by

$$k_1u + v + kw = 0,$$

which, written symmetrically and in the usual form, becomes

$$\lambda u + \mu v + \nu w = 0.$$

It is manifest that this proof is not restricted to lines forming a triangle. It is equally plain that they should not be parallel.

32. In order that a point may be determined upon the line

$$l\alpha + m\beta + n\gamma = 0,$$

its coordinates must simultaneously satisfy the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

If such values prove to be irrational, they are, by convention, said to be the coordinates of an imaginary point. Since quadratics involve two roots—sometimes imaginary—there will be the same number of intersections, if the question is one of intersection, real or imaginary,—or, more exactly, real, coincident, or imaginary. But as this truth is so well known and so fully exhibited in Cartesian Geometry, we shall here consider only what is peculiar to our subject.

33. It is evident that the imaginary roots of a trilinear equation of the second degree must be of the form

$$\alpha+\alpha_1\sqrt{-1},\quad \beta+\beta_1\sqrt{-1},\quad \gamma+\gamma_1\sqrt{-1} \;\ldots\ldots\ldots (a).$$

Suppose these roots to be the coordinates of an imaginary point. Then, by the last Article, these must satisfy the relation

$$a\alpha+b\beta+c\gamma = 2\Delta.$$

Making the substitution, we have

$$(a\alpha+b\beta+c\gamma)+(a\alpha_1+b\beta_1+c\gamma_1)\sqrt{-1} = 2\Delta \;\ldots\ldots (1);$$

wherefore $$a\alpha_1+b\beta_1+c\gamma_1 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and $$a\alpha+b\beta+c\gamma = 2\Delta \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From which we see that (1) is made up of both real and imaginary parts; the imaginary parts satisfying (2) the equation to the line at infinity (Art. 11); while (3) is of course satisfied by its own coordinates. The reader will learn to distinguish between the coordinates of an imaginary point and those of an imaginary point at infinity; that is, if the coordinates (a) had been regarded as the coordinates of an imaginary point (or proportional to them) at infinity, both (2) and (3) must have been written $=0$.

34. The equation to an imaginary right line may be written

$$(l+l_1\sqrt{-1})f+(m+m_1\sqrt{-1})g+(n+n_1\sqrt{-1})h = 0.$$

35. Writers upon equations of the second degree representing right lines in Cartesian coördinates are accustomed to dispose of the contingency of two imaginary roots by referring both to two imaginary lines drawn through the origin, thus determining a *real* point. So now we say that every imaginary right line passes through one real point, and but one.

If we consider the equation of (Art. 34), we see that the real and imaginary parts are not coincident, and consequently the factor $\sqrt{-1}$ does not divide out; hence the equation may be expressed
$$u+v\sqrt{-1}=0 \text{ ....}(1),$$
in which u and v are functions of the coordinates of the given straight line. This equation is manifestly entirely similar to equation (1), (Art. 31).

It is also to be observed that u and v are of the first degree, and hence
$$u=0$$
and
$$v=0$$
are satisfied by real values, which values satisfy (1), which passes through the point of intersection of u and v, and therefore each straight line passes through a *real point*.

36. Suppose the equation to a straight line
$$lf+mg+nh=0$$
to pass through an imaginary point whose coordinates are given in [Art. 33 (a)]; then
$$l\alpha+m\beta+n\gamma+(l\alpha_1+m\beta_1+n\gamma_1)\sqrt{-1}=0;$$
and consequently
$$l\alpha+m\beta+n\gamma=0,$$
$$l\alpha_1+m\beta_1+n\gamma_1=0,$$
which equations determine the ratios of l, m, n; or we may determine them fully by the determinant
$$\begin{vmatrix} f & g & h \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}=0,$$

which is the equation to the straight line drawn through the imaginary point whose coordinates are

$$\alpha+\alpha_1\sqrt{-1},\quad \beta+\beta_1\sqrt{-1},\quad \gamma+\gamma_1\sqrt{-1};$$

and consequently

$$\alpha-\alpha_1\sqrt{-1},\quad \beta-\beta_1\sqrt{-1},\quad \gamma-\gamma_1\sqrt{-1}.$$

Therefore, since imaginary roots enter by pairs into an equation, the imaginary points of intersection of two lines (curves) *will be found upon real straight lines by twos.*

37. If we have an equation of the form

$$l\beta^2-m\beta\gamma+n\gamma^2=0 \quad\ldots\ldots\ldots\ldots(1),$$

we can evidently subject it to the same reasoning which is applied to the quadratic

$$x^2-pxy+qy^2=0.*$$

Each equation is reducible to the form

$$(\beta-s\gamma)(\beta-s_1\gamma)=0;$$

that is, the two straight lines

$$\beta-s\gamma=0 \quad\ldots\ldots\ldots\ldots(2),$$

$$\beta-s_1\gamma=0 \quad\ldots\ldots\ldots\ldots(3),$$

are real or imaginary according as we find, by the resolution of (1) for the ratio $\beta:\gamma$, that $4ln$ is less or greater than m^2.†

Examining (2) and (3) in the light of (Art. 31), we see that these lines intersect in the point A of the triangle of refernece.

38. *Excursus on imaginary right lines and points.*

It is evident, from what has immediately preceded, that this portion of the subject is capable of considerable expansion, and that this system of coordinates is eminently fitted to deal with the Infinite and Imaginary. From what has already been said in reference to the adaptation of the reasoning

* Salmon's Conics, p. 69.

† Algebra, Bourdon, p. 159.

employed in Cartesian methods to trilinear coordinates, the views of high authorities upon these results are interesting.

Poncelet* has discovered and illustrated geometrically the *rationale* of the principles which, upon purely analytical grounds, we are enabled to re-discover, apply, and extend; he has pointed out the correspondence of points, some real and some imaginary, and taught that theorems concerning imaginary points and lines may be extended to real points and lines, and hence shown how to indicate the properties of a figure when some of the lines and points are real and some imaginary. By the method of trilinear coordinates we are enabled quickly to generalize all those theorems which are concerned with the line at infinity. For example, if four points on a conic, or four tangents to a conic, are given, and it is required to find the locus of the centre of the conic, we proceed to find the locus of the pole of the line

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

which also gives us, the conditions being the same, the locus of the pole of any line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0.$$

In applying the method of projections, the analytic shows its superiority over the synthetic method, by proving the general theorem at once, rather than by inferring it by the projection from a more elementary state of the figure.

As to the results reached in our discussion of parallelism, and what we have said upon the theory and use of the line

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

nothing is affirmed beyond what has been received, almost without dissent, from the first, both upon geometrical and analytical considerations. See Chasles (Géom. Sup.), Townsend (Vol. I., p. 16, Art. 136), Salmon (Conics, pp. 64, 318), Poncelet (Proj. Persp., p. 53), Hamilton (Quaternions, p. 90).

* Traité des Propriétés Projectives des Figures.

38. *Tangent of angle between two lines.*

Let
$$l\,\alpha+m\,\beta+n\,\gamma=0,$$
$$l_1\alpha+m_1\beta+n_1\gamma=0,$$
be the given lines.

If θ, θ_1 be their respective inclinations to one of the lines of reference, then, by the reasoning in Art. 14, we must have as the tangent of the difference of the two angles, that is, the tangent of the required angle,

$$\tan(\theta-\theta_1)=\frac{\tan\theta-\tan\theta_1}{1+\tan\theta\tan\theta_1},$$

which becomes, by a laborious reduction,

$$\frac{\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \sin A & \sin B & \sin C \end{vmatrix}}{\mathrm{P}}$$

where P is the sinister member of the equation of perpendicularity given on page 17.

This is probably the simplest form possible in trilinear coordinates.

Examples under Chapters I. and II.

1. Express the parallelism of
$$l\alpha+m\beta+n\gamma=0,$$
with AC in the triangle of reference.

Ans.:
$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & 1 & 0 \end{vmatrix}=0.$$

2. The same line with BC; with AB.

3. What relation of two lines is expressed by the determinant

$$\begin{vmatrix} a & b & c \\ 1 & 0 & \cos B \\ 0 & 1 & \cos A \end{vmatrix} = 0;$$

and what are the lines?

4. What condition is expressed by

$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & -1 & 1 \end{vmatrix} = 0\,?$$

5. Find the angle between the lines

$$\alpha = \gamma \cos B,$$

and

$$\beta = \gamma \cos A.$$

6. If $u+v\sqrt{-1}=0$, $u'+v'\sqrt{-1}=0$ are imaginary straight lines having a real point of intersection, then the four real straight lines $u=0$, $v=0$, $u'=0$, $v'=0$ are concurrent.

7. What is the determinant expressing the equation of the right line drawn through the intersections of the pairs of lines

$$2au+bv+cw=0, \quad bv+cw=0;$$

$$2bu+av+cw=0, \quad av-cw=0\,?$$

CHAPTER III.

THE TRILINEAR METHOD APPLIED TO CONICS.

39. We will now call attention to the fact, which may not have escaped the notice of the reader, that trilinear equations are always homogeneous. If not so in form, they can be made so by a very simple process. Since

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

we may write $$\frac{a\alpha + b\beta + c\gamma}{2\Delta} = 1;$$

and therefore any term of an equation may be multiplied by this fraction without affecting the pre-existing relation of equality. Thus, if we have

$$\alpha^2 - 2\alpha\beta + \gamma = 2,$$

we may proceed to raise each non-homogeneous term to the second order, as

$$\alpha^2 - 2\alpha\beta + \gamma\left(\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right) = 2\left[\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right]^2.$$

40. Another consideration, which has been referred to, may be here emphasized; viz., that we are not concerned with the absolute values of the coordinates, but with their ratios; and this advantage we derive from the principle of homogeneity which belongs to every trilinear equation; thus,

$$\alpha^2 - 2\alpha\beta + \gamma^2 = 0$$

is, in fact, $$\left(\frac{\alpha}{\gamma}\right)^2 - 2\left(\frac{\alpha}{\gamma}\right)\cdot\frac{\beta}{\gamma} + 1 = 0,$$

in which only the ratios $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\gamma}$ appear. Beyond these ratios it is not necessary for us to inquire.

41. It may be desirable to find the equation to the same locus, but referred to another triangle of reference.

First Transformation,

when the equations of the sides of the new triangle are given.

These sides being represented by equations in terms of the perpendiculars from the angular points of the original triangle, we have (Art. 21)

$$\begin{array}{ll} \text{coordinates of } A, & (p, p_1, p_2)\,; \\ \qquad\text{,,} \qquad B, & (q, q_1, q_2)\,; \\ \qquad\text{,,} \qquad C, & (r, r_1, r_2)\,; \end{array}$$

that is,

$$apf + bqg + crh = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$ap_1f + bq_1g + cr_1h = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$ap_2f + bq_2g + cr_2h = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where (f, g, h) are the old coordinates of any point P.

To find the locus of the homogeneous equation

$$F(f, g, h) = 0.$$

When referred to (1), (2), (3), we observe that f represents the perpendicular from (f_1, g_1, h_1)—these being the new coordinates of P—on the line joining B and C. Therefore

$$(f_1, g_1, h_1), \quad (q, q_1, q_2), \quad (r, r_1, r_2)$$

indicate the angular points of a triangle whose area is found by Art. 19,

$$\text{double area} = af = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ q & q_1 & q_2 \\ r & r_1 & r_2 \end{vmatrix}.$$

Similarly, $$bg = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ r & r_1 & r_2 \\ p & p_1 & p_2 \end{vmatrix}, \quad ch = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ p & p_1 & p_2 \\ q & q_1 & q_2 \end{vmatrix},$$

from which the values of f, g, h are readily determined. Hence, representing these determinants by Q, R, S respectively, we may write

$$F\left(\frac{Q}{a}, \frac{R}{b}, \frac{S}{c}\right) = 0$$

as the equation with new lines of reference, the degree not being changed by transformation.

42. *Second Transformation*,

coordinates of the new points of reference being given.

A triangle drawn within or without the original triangle will sufficiently represent the construction.

Let the perpendiculars from A_1, B_1, C_1, the new points of reference, upon BC be denoted by p, p_1, p_2; on AC by q, q_1, q_2; on AB by r, r_1, r_2; a_1, b_1, c_1 the sides of the new triangle; f_1, g_1, h_1 the new coordinates, and f, g, h the old coordinates, of any point P. Then, by Art. 21, we find

$$a_1 p f_1 + b_1 p_1 g_1 + c_1 p_2 h_1 = 0,$$
$$a_1 q f_1 + b_1 q_1 g_1 + c_1 q_2 h_1 = 0,$$
$$a_1 r f_1 + b_1 r_1 g_1 + c_1 r_2 h_1 = 0.$$

Representing these equations by Q, R, S respectively, we have, by Art. 22, the distance of P from each of the sides of the original triangle expressed in a simple form; that is,

$$f = \frac{Q}{2\Delta_1}, \;\; g = \frac{R}{2\Delta_1}, \;\; h = \frac{S}{2\Delta_1},$$

the old coordinates expressed in terms of the new.

43. We shall now pass on to the consideration of curves of the second degree. An important property of these curves was conceived by the early geometers; viz., that every curve of this degree might be regarded as a conic section. What then can be easily shown may be stated here, that the section of a right circular cone by any plane can be expressed by a

homogeneous equation of the second degree in trilinear coordinates. This can be readily proved by selecting particular lines of reference; and since, by the preceding Articles, we may transform to any other lines without affecting the degree of the equation, we may regard this as a general truth irrespective of the lines of reference.

Let us write

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0$$

as the general equation of the second degree in trilinear coordinates.

This equation, it will be seen, contains six terms; but as the nature of the curve does not depend upon the independent magnitude of these coefficients, we may simply regard their mutual ratios, or, in other words, assign a particular value to one of the coefficients, varying the values of the others.

Here, then, as in the Cartesian coordinates, we can find the equation to the conic described through five points. There are, in other words, five constants to be determined whose values substituted in the general equation will give the equation of the conic through five points; that is,

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \\ \alpha_1^2 & \beta_1^2 & \gamma_1^2 & \beta_1\gamma_1 & \gamma_1\alpha_1 & \alpha_1\beta_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_5^2 & \beta_5^2 & \gamma_5^2 & \beta_5\gamma_5 & \gamma_5\alpha_5 & \alpha_5\beta_5 \end{vmatrix}=0.$$

44. *Concurrence of the straight line and conic.*

The well-known property that every right line meets a curve of the second degree in two real, coincident, or imaginary points* is readily exhibited.

Writing the general equations of the conic and straight line,

$$l\alpha+m\beta+n\gamma=0,$$

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0,$$

* Salmon's Conics, p. 132.

and the simultaneous relation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

we see from these three equations that we are enabled first to express β and γ as functions of α of the first degree, which substituted gives us a quadratic, and this in turn furnishes two roots, determining two points of intersection.

45. *Excursus upon the fundamental form of the equation to a conic section in trilinear coordinates.*

Conceive the vertex of a right circular cone placed at the origin O of x, y, z coordinates; XYZ the plane of section, and also the triangle of reference in trilinear coordinates; $\theta_1, \theta_2, \theta_3$ the angles which the perpendicular upon this plane makes with OX, OY, OZ; OX and OY supposed to be at right angles to the axis of the cone OZ and to one another; P a point on the curve and the origin of α, β, γ.

Let a, b, c be the perpendicular distances of P from the coordinate planes, and d the diagonal from O in the lower face of the parallelepipedon.

Then the perpendiculars from P on $XY = \alpha$, on $XZ = \beta$, on $YZ = \gamma$.

By the geometry of the figure, a (the perpendicular distance of P from the plane OYZ) $= \alpha \sin\theta_1$, $b = \beta \sin\theta_2$, and c (the distance of P from the plane OXY) $= \gamma \sin\theta_3$,

$$d^2 = a^2 + b^2,$$

$$c = d \tan\theta,$$

where $\theta =$ semi-vertical angle of the cone.

Hence $$c^2 = (a^2 + b^2)\tan^2\theta;$$

that is, $$\gamma^2 \sin^2\theta_3 = (\alpha^2 \sin^2\theta_1 + \beta^2 \sin^2\theta_2)\tan^2\theta,$$

or $$\alpha^2 \sin^2\theta_1 \tan^2\theta + \beta^2 \sin^2\theta_2 \tan^2\theta - \gamma^2 \sin^2\theta_3 = 0,$$

which also may be written

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

where it is understood that the signs are not all the same.

We have, therefore, derived an equation to a conic section homogeneous and of the second degree in trilinear coordinates; and in turn it may easily be shown that the general equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u_1\beta\gamma + 2v_1\gamma\alpha + 2w_1\alpha\beta = 0$$

may be made to take the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0;$$

and hence every equation of the second degree may be said to express some section of a right circular cone.

In the genesis of this equation it is evident how we might proceed to make some applications in tri-dimensional Geometry. For instance, let us take some function of x, y, z as an equation to a surface in three rectangular coordinates, as,

$$f(x, y, z) = 0;$$

and let $$x \cos\theta_1 + y \cos\theta_2 + z \cos\theta_3 = p$$

be the equation to any plane; also let the traces of the coordinate planes upon the plane of section be the lines of reference; then, if x, y, z be the coordinates of any point P upon the given surface, and if θ_1, θ_2, θ_3 be the angles which the proposed plane makes with the original plane, we must have

$$x = \alpha \sin\theta_1, \quad y = \beta \sin\theta_2, \quad z = \gamma \sin\theta_3;$$

and consequently, by substituting in the given surface, we obtain the trilinear equation to the section, that is,

$$f(\alpha \sin\theta_1, \ \beta \sin\theta_2, \ \gamma \sin\theta_3) = 0.$$

In the same manner, the equation to the section of the ellipsoid $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by the same plane would evidently become

$$\frac{\alpha^2 \sin^2\theta_1}{a^2} + \frac{\beta^2 \sin^2\theta_2}{b^2} + \frac{\gamma^2 \sin^2\theta_3}{c^2} = 1 \quad \text{.........} \ (1),$$

which is easily rendered homogeneous by first finding the identical relation among α, β, γ in the given plane; that is, by substituting the values of x, y, z as above when

$$\frac{\alpha \sin 2\theta_1 + \beta \sin 2\theta_2 + \gamma \sin 2\theta_3}{2p} = 1;$$

and consequently (1) becomes

$$\frac{\alpha^2 \sin^2 \theta_1}{a^2} + \frac{\beta^2 \sin^2 \theta_2}{b^2} + \frac{\gamma^2 \sin^2 \theta_3}{c^2} - \left[\frac{\alpha \sin 2\theta_1 + \beta \sin 2\theta_2 + \gamma \sin 2\theta_3}{2p}\right]^2 = 0.$$

46. By the last Article we are enabled at once to interpret such an equation as

$$\alpha\beta - k\gamma x = 0 \quad \text{.......................} \quad (1),$$

where, by ordinary abridged notation, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $x = 0$ are the equations to four straight lines, and k is any constant.

In considering the given equation, we see that it is of the second degree, and satisfied when $\alpha = 0$ and $\gamma = 0$ are at the same time satisfied; and hence we infer that the conic represented by the equation passes through the intersection of these lines. In the same manner, another point is determined by the intersection of $\beta = 0$ and $x = 0$, and so on for the four sets of lines determining four points through which the conic must pass; that is, (1) represents a conic circumscribing a quadrilateral whose sides are α, β, γ, and x.

From this we readily pass to the interpretation of the similar equation

$$\alpha\beta - k\gamma^2 = 0 \quad \text{.......................} \quad (2),$$

which indicates that two of the opposite sides, γ and x, are coincident. And as each of the lines $\alpha = 0$ and $\beta = 0$ can meet the conic in but two points, they must be conceived as drawn from a point without, and hence as tangents to the conic at the points respectively where the coinciding lines meet the conic.*

* Salmon's Conics, p. 223.

47. *The triangle of reference self-conjugate with regard to the conic.*

Returning to the equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

we see that it expresses no possible locus while l, m, n are regarded as all positive or all negative; but, as we saw in Art. 45, these are not all of the same sign.

Let l, m be positive, n negative, and for them write u^2, v^2, w^2 respectively; then

$$u^2\alpha^2 + v^2\beta^2 - w^2\gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$u^2\alpha^2 + (v\beta + w\gamma)(v\beta - \gamma) = 0.$$

After the analogy of equation (2) of Art. 46, the lines

$$v\beta + w\gamma = 0,$$

$$v\beta - w\gamma = 0,$$

must be tangents, and α their chord of contact; in other words, the line $\alpha = 0$, which is the equation of BC, a side of the triangle of reference, is the chord of contact of a pair of tangents from the vertex A.

It is equally admissible to write (1)

$$v^2\beta^2 + u^2\alpha^2 - w^2\gamma^2 = 0,$$

whence we see, as before, $\beta = 0$, which is the equation of AC, a side of the triangle of reference, is the chord of contact of the lines

$$u\alpha + w\gamma = 0,$$

and

$$u\alpha - w\gamma = 0,$$

which are tangents from the vertex B; or, still further, (1) may take the form

$$u^2\alpha^2 + v^2\beta^2 - w^2\gamma^2 = 0,$$

or

$$(u\alpha + v\beta\sqrt{-1})(u\alpha - v\beta\sqrt{-1}) - w^2\gamma^2 = 0;$$

whence

$$u\alpha + v\beta\sqrt{-1} = 0,$$

$$u\alpha - v\beta\sqrt{-1} = 0,$$

are the imaginary tangents from the vertex C, and $\gamma = 0$ their chord of contact, which is also the equation of AB.

Therefore, as we see, each side of the triangle of reference becomes in turn the chord of contact of tangents from the opposite angle, that is, the *polar* of that point with respect to the conic; and, conversely, each vertex is seen to be the *pole* of the opposite side, or the triangle may be described as self-conjugate with respect to the conic; which was to be shown.*

48. *Intercepts of a directed line upon the curve*

$$l\alpha^2+m\beta^2+n\gamma^2=0 \quad\text{.....................}\quad (1).$$

Let $(\alpha_1, \beta_1, \gamma_1)$ be the point from which the directed line h is drawn to meet the conic; s_1, s_2, s_3 sines of the given direction all measured in the same direction, the first from h to the parallel of BC, the second measured from the same point to the parallel of AC, and the third in the same direction round to the parallel of AB, of the sides, respectively, of the triangle of reference.

Then, evidently,
$$\alpha = \alpha_1+s_1h,$$
$$\beta = \beta_1+s_2h,$$
$$\gamma = \gamma_1+s_3h.$$

These values substituted in (1) give a quadratic in h; that is,

$$h^2(ls_1^2+ms_2^2+ns_3^2)+2h(ls_1\alpha_1+ms_2\beta_1+ns_3\gamma_1) + (l\alpha_1^2+m\beta_1^2+n\gamma_1^2)=0,$$

The two values of h obtained from this equation will be the lengths of the intercepts from the given point.

Suppose this point to be on the curve, we shall then have

$$l\alpha_1^2+m\beta_1^2+n\gamma_1^2=0,$$

and consequently but one value to h (one intercept becoming zero), which is manifestly the length of a chord in the given direction.

49. *Locus of middle points of parallel chords.*

Let the curve be the same as in the last Article; $(\alpha_1, \beta_1, \gamma_1)$

* Salmon's Conics, p. 227.

a point on the locus, and the chord from this point represented by h, whose direction is given by its sines, s_1, s_2, s_3.

Then, as before,

$$\alpha = \alpha_1 + s_1 h, \quad \beta = \beta_1 + s_2 h, \quad \gamma = \gamma_1 + s_3 h.$$

Hence the intercepts of the curve are given by

$$l(\alpha_1 + s_1 h)^2 + m(\beta_1 + s_2 h)^2 + n(\gamma_1 + s_3 h)^2 = 0,$$

which, by the supposition, are equal; that is, the two values of h will appear with opposite signs; and since they must be equal, their sum, or the coefficient of h, will be equal to 0, and consequently*

$$ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1 = 0,$$

a straight line giving the relation, in fact, of *any* point on the locus, and hence the equation required.

50. *Tangent to a point on the conic.*

Let now the point $(\alpha_1, \beta_1, \gamma_1)$ be on the conic. We have seen, by Art. 48, that when this point lies on the conic,

$$l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2 = 0;$$

and therefore the quadratic in h of that Article reduces to

$$(ls_1^2 + ms_2^2 + ns_3^2)\,h + 2(ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1) = 0,$$

which gives the length of the chord.

When, now, the direction becomes that of the tangent, the length of this chord, that is h, becomes zero, and we have

$$ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

But if (α, β, γ) be any point on this tangent, we must have

$$s_1 = \frac{\alpha - \alpha_1}{h}, \quad s_2 = \frac{\beta - \beta_1}{h}, \quad s_3 = \frac{\gamma - \gamma_1}{h},$$

which values substituted in (1) give

$$l\alpha_1\alpha + m\beta_1\beta + n\gamma_1\gamma = l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2 = 0.$$

* Bourdon's Algebra, p. 160.

Hence $$l\alpha_1\alpha+m\beta_1\beta+n\gamma_1\gamma=0$$

expresses the required relation, and is therefore the equation sought.

51. *Coordinates of centre.*

The reasoning is similar to that of Art. 49. The direction of a diameter being s_1, s_2, s_3, the lengths of intercepts by the curve in this direction, measured from the centre $(\alpha_1, \beta_1, \gamma_1)$, will be given by the same equation as in that Article; and since the quadratic must, by the premises, give equal roots with opposite signs, the coefficient of h will $=0$; that is,

$$ls_1\alpha_1+ms_2\beta_1+ns_3\gamma_1=0 \quad\text{.................}\quad (1).$$

For the actual determination of α_1, β_1, γ_1 we have

$$a\alpha_1+b\beta_1+c\gamma_1=2\Delta \quad\text{....................}\quad (2);$$

but since $$a\alpha+b\beta+c\gamma=2\Delta,$$

and $$\alpha=\alpha_1+s_1h,\quad \beta=\beta_1+s_2h,\quad \gamma=\gamma_1+s_3h,$$

we have $$as_1+bs_2+cs_3=0 \quad\text{.......................}\quad (3).$$

Comparing (1) and (3), we get

$$\frac{l\alpha_1}{a}=\frac{m\beta_1}{b}=\frac{n\gamma_1}{c}.$$

These ratios will enable us to find the values of α_1, β_1, γ_1; thus, by dividing (2) by $\frac{l\alpha_1}{a}$ or its equals, we have

$$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=\frac{2\Delta a}{l\alpha_1}=\frac{2\Delta b}{m\beta_1}=\frac{2\Delta c}{n\gamma_1};$$

and therefore the coordinates of the centre are determined.

It will be observed that these coordinates enable us to determine the condition that the conic may be a parabola; the centre of a parabola being infinitely distant, its coordinates must satisfy the relation

$$a\alpha_1+b\beta_1+c\gamma_1=0.$$

Making the substitution, we have the required condition,

that is, $$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0.$$

52. *Equation of circle with respect to which the triangle of reference is self-conjugate.*

It may be inferred from Art. 47 that when an equation of the second degree does not involve $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, the conic, in such case, is so related to the triangle of reference that each side is the polar, with respect to this conic, of the opposite vertex.

Let one side, as CA, cut this conic in two points $(f_1, 0, h_1)$, $(f_2, 0, h_2)$, and let this chord be bisected. Then the equation of the straight line from the vertex to the point of bisection

is, evidently, $$\frac{\alpha}{f_1+f_2}=\frac{\gamma}{h_1+h_2} \quad\text{.......................... (1)},$$

which line passes through the centre of the conic.

We have seen (Art. 47) that the conic may be written

$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2=0,$$

where, in this case, nothing is assumed as to which of the coefficients l, m, n should be attributed the negative sign.

Now f_1 and f_2 are identical with the values of α given by this equation.

We can eliminate β and γ by the relation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

remembering that $\beta=0$; and we have the quadratic

$$\alpha^2-\frac{4\Delta w^2 a}{u^2c^2+w^2a^2}\alpha+\frac{4\Delta^2}{u^2c^2+w^2a^2}=0;$$

and therefore $$f_1+f_2=\frac{4\Delta w^2 a}{u^2c^2+w^2a^2},$$

since this coefficient is the sum of the roots of α; by the same reasoning we have

$$h_1 + h_2 = \frac{4\Delta u^2 c}{w^2a^2 + u^2c^2},$$

and consequently (1) becomes

$$\frac{\alpha}{w^2 a} = \frac{\gamma}{u^2 c},$$

a line on which the centre lies, which may be written

$$\frac{u^2\alpha}{a} = \frac{w^2\gamma}{c}.$$

Similarly, $$\frac{u^2\alpha}{a} = \frac{v^2\beta}{b}.$$

Now it is a property of the circle that a line joining any point to the centre is perpendicular to the polar; therefore the line

$$\frac{u^2\alpha}{a} - \frac{w^2\gamma}{c} = 0,$$

which is drawn from the centre to the vertex B, is perpendicular to $\beta = 0$.

But, by the figure, we have

$$\frac{\alpha}{\cos C} = \frac{\gamma}{\cos A};$$

therefore $$\frac{u^2}{a \cos A} = \frac{w^2}{c \cos C};$$

similarly, $$\frac{u^2}{a \cos A} = \frac{v^2}{b \cos B};$$

and the equation of the circle becomes

$$a \cos A\, \alpha^2 + b \cos B\, \beta^2 + c \cos C\, \gamma^2 = 0,$$

or $$\sin 2A\, \alpha^2 + \sin 2B\, \beta^2 + \sin 2C\, \gamma^2 = 0.$$

The circle thus represented will be imaginary unless the triangle of reference have an obtuse angle.

53. *The inscribed triangle.*

Returning now to the general equation of the second degree,

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u_1\beta\gamma + 2v_1\gamma\alpha + 2w_1\alpha\beta = 0,$$

we see that if $\beta = 0,$

$\gamma = 0,$

the equation reduces to $u = 0.$

But this is the condition that the curve should pass through the vertex A. In the same manner, it may be shown that when $v = 0$ and $w = 0$ it will pass through B and C.

Under these conditions the equation reduces to

$$u_1\beta\gamma + v_1\gamma\alpha + w_1\alpha\beta = 0 \quad\text{.................}\quad (1),$$

which also may now be written without the subscripts. We may therefore write it

$$u\beta\gamma + \alpha\,(v\gamma + w\beta) = 0\,;$$

and since every straight line cuts the curve in two points, the line $v\gamma + w\beta = 0$ must pass through the point where $\alpha = 0$ and $\beta = 0$, since these values alone will satisfy the equation; but these points are coincident, and determine the vertex A. This line could not therefore be drawn within the curve, for it would then meet it in three points; it must be drawn without, and therefore is the tangent at A.

Equation (1) is an equation of the second degree, and represents evidently, from what has been said, a curve circumscribing the triangle of reference, satisfied when any two coordinates $= 0$, in which case each vertex lies upon the locus.

54. The conic $u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$

will give values for the intercepts by the curve upon a straight line from a given point; the equation to the tangent at any point; the locus of middle points of parallel chords, in precisely the same manner as has already been shown in preceding Articles.

Let us here seek the condition that any straight line should be a tangent to the conic.

Since $$u\beta\gamma+v\gamma\alpha+w\alpha\beta=0$$

represents a conic described about the triangle of reference, it passes through the point, as we have seen, where $\beta=0$ and $\gamma=0$.

Let $$f\alpha+g\beta+h\gamma=0$$

be the straight line. If by means of this equation we eliminate α from the equation to the given conic, we must have, evidently, coincident values for $\beta:\gamma$.

Now the quadratic which results,

$$\frac{\beta^2}{\gamma^2}+\frac{\beta}{\gamma}\left(\frac{gv-fu+hw}{gw}\right)+\frac{vh}{gw}=0,$$

will give equal values for $\frac{\beta}{\gamma}$ when the value of the radical is zero; that is, when

$$4hgvw-(gv+hw-fu)^2=0,$$

or $$u^2f^2+v^2g^2+w^2h^2-2vwgh-2uwhf-2uvgf=0.$$

This may also be written in the form

$$\pm\sqrt{uf}\pm\sqrt{vg}\pm\sqrt{wh}=0,$$

which can be verified by clearing of radicals; and this is the condition that the straight line

$$f\alpha+g\beta+h\gamma=0$$

may touch the curve

$$u\beta\gamma+v\gamma\alpha+w\alpha\beta=0.$$

Dr. Salmon has called this the tangential equation of the curve.

55. *Pascal's hexagon:* the opposite sides of a hexagon inscribed in a conic meet, if produced, in collinear points.

Let the triangle of reference be inscribed in the curve, and let Ax_1, Bx_2, Cx_3 be three of the sides of the inscribed hexagon.

Since, if (f_1, g_1, h_1), (f_2, g_2, h_2), (f_3, g_3, h_3) be the coordinates of x_1, x_2, x_3 respectively, we shall have, by the figure,

$$\beta : \gamma :: g_1 : h_1 ;$$

Ax_1 will therefore be represented by

$$0, \quad h_1, \ -g_1 ;$$

and x_2C by $\qquad g_2, \ -f_2, \quad 0.$

Hence (Art. 12) the point of intersection of these sides is

$$g_1f_2, \quad g_1g_2, \quad h_1g_2.$$

The side Bx_2 will be subject to the coordinates f_2 and h_2; the side x_3A to h_3 and g_3; hence these sides will intersect in the point

$$f_2h_3, \quad h_2g_3, \quad h_3h_2.$$

The sides Cx_3 and x_1B will, in like manner, intersect in the point $\qquad f_3f_1, \quad f_1g_3, \quad h_1f_3.$

Hence we have, by the determinant of collinearity, the condition

$$\begin{vmatrix} g_1f_2 & g_1g_2 & h_1g_2 \\ f_2h_3 & h_2g_3 & h_3h_2 \\ f_3f_1 & f_1g_3 & h_1f_3 \end{vmatrix} = 0 \quad \text{..............} \quad (1),$$

which, as is evident, is also the condition that the three points x_1, x_2, x_3 lie on one conic with the vertices of reference.

For let (f_1, g_1, h_1), (f_2, g_2, h_2), (f_3, g_3, h_3) be the three given points on the conic, and let the conic be represented by

$$ugh+vhf+wfg=0.$$

Then will the three vertices of reference lie on this conic; and if the curve pass through the given points we must have

$$ug_1h_1+vh_1f_1+wf_1g_1=0,$$
$$ug_2h_2+vh_2f_2+wf_2g_2=0,$$
$$ug_3h_3+vh_3f_3+wf_3g_3=0;$$

and the determinant by which u, v, and w are eliminated is

$$\begin{vmatrix} g_1h_1 & h_1f_1 & f_1g_1 \\ g_2h_2 & h_2f_2 & f_2g_2 \\ g_3h_3 & h_3f_3 & f_3g_3 \end{vmatrix}=0,$$

which is identical in result with the condition given in (1).

EXERCISES.

1. A triangle being inscribed in a conic, are the points collinear in which each side intersects the tangents at the opposite vertex?

2. Prove the theorem of Hermes, that if $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be two points on the conic

$$u\beta\gamma+v\gamma\alpha+w\alpha\beta=0,$$

then the equation to the straight line joining them is

$$\frac{u\alpha}{\alpha_1\alpha_2}+\frac{v\beta}{\beta_1\beta_2}+\frac{w\gamma}{\gamma_1\gamma_2}=0.$$

3. When does $u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$

represent an hyperbola?

4. What is the chord of contact of the tangents

$$u(\beta+\gamma) + (\sqrt{v} \pm \sqrt{w})^2 \alpha = 0\,?$$

5. What is the condition of concurrence of the normals at the vertices of the triangle of reference to the above conic?

CHAPTER IV.

POLE AND POLAR—RECIPROCATION.

56. *Inscribed Conic.*

Any conic inscribed in the triangle of reference may be represented by

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

which, cleared of radicals, is

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

as we have seen (Art. 54), where it expressed a particular condition.

If we examine this equation, we shall find that it may be written in each of the three following forms:

$$4mn\beta\gamma - (m\beta + n\gamma - l\alpha)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

$$4nl\,\alpha\gamma - (n\gamma + l\alpha - m\beta)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2),$$

$$4ml\,\alpha\beta - (l\alpha + m\beta - n\gamma)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

from which, as they differ only by a constant from the equation interpreted in Art. 54, we conclude from parallel reasoning that each represents a conic section in which the factors of the first terms equated separately to zero are tangents to the curve in whose equation they respectively appear, and the second terms are the squares of their respective chords of contact.

Hence the lines of reference are tangents, and the conic is an inscribed conic.

57. *Conversely, every conic whose lines of reference are the sides of a circumscribed triangle will have an equation of the form*

$$\pm\sqrt{l\alpha} \pm \sqrt{m\beta} \pm \sqrt{n\gamma} = 0,$$

since every conic may be represented by

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0.$$

If the triangle of reference be circumscribed, the side BC will be a tangent and be represented by $\alpha=0$. This value substituted in the general equation gives

$$v\beta^2+2u_1\beta\gamma+w\gamma^2=0,$$

which, from the nature of the case, must have equal roots, that is, the left-hand member of the equation must be a perfect square; hence $\qquad u_1^2=vw;$

that is, $\qquad u_1=\pm\sqrt{vw};$

and similarly $\qquad v_1=\pm\sqrt{wu},$

$$w_1=\pm\sqrt{uv},$$

are the necessary and sufficient conditions that the conic should touch the lines $\beta=0$ and $\gamma=0$.

Substituting these values in the general equation, and remembering to write l^2, m^2, n^2 for u, v, w, we have

$$\pm\sqrt{l\alpha}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0 \qquad (1),$$

which was to be proved.

58. Four conics may be inscribed in the triangle of reference so related that the points of contact shall lie on the lines represented by $\qquad \pm l\alpha\pm m\beta\pm n\gamma=0.$

For it is evident that (1) of the last Article may be written

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2\pm 2mn\beta\gamma\pm 2nl\gamma\alpha\pm 2lm\alpha\beta=0,$$

which, writing all the doubtful signs negative, or one negative only at a time, breaks up into the equations to four conics, and we are presented with four interpretations similar to (1), (2), (3) of Art. 56. If the double signs be taken otherwise, the locus will become simply two coincident straight lines.

These equations therefore, as representing conics, have twelve points of contact lying three and three on the above four straight lines. It may be observed that the actual sign of the quantities under the radicals in equation (1) of the last Article depends upon which sign is taken with the coefficients of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

The process for finding tangent, intercepts, centre of conic, &c. is similar to that already exhibited in the last Chapter, and need not be repeated.

59. *Brianchon's Hexagon: the three opposite diagonals of every hexagon described about a conic concur.*

The method of proof is quite similar to that already exhibited. Let three sides be produced for the triangle of reference; $ABCDEF$ the hexagon; AB, CD, EF the sides produced.

If
$$l_1\alpha + m_1\beta + n_1\gamma = 0$$
be the equation to AF,
$$l_2\alpha + m_2\beta + n_2\gamma = 0$$
to that of BC, and
$$l_3\alpha + m_3\beta + n_3\gamma = 0$$
to that of DE; then the diagonals AD and FC will be represented as follows:—

The point A, $\gamma = 0$, and $l_1\alpha + m_1\beta = 0$;

" D, $\alpha = 0$, and $m_3\beta + n_3\gamma = 0$;

(AD), $\quad l_1 m_3\alpha + m_1 m_3\beta + m_1 n_3\gamma = 0$;

(FC), $\quad l_1 n_2\alpha + n_1 m_2\beta + n_1 n_2\gamma = 0$;

(BE), $\quad l_2 l_3\alpha + l_3 m_2\beta + l_2 n_3\gamma = 0$.

Hence, (Art. 8),
$$\begin{vmatrix} l_1 m_3 & m_1 m_3 & m_1 n_3 \\ l_1 n_2 & n_1 m_2 & n_1 n_2 \\ l_2 l_3 & l_3 m_2 & l_2 n_3 \end{vmatrix} = 0.$$

The condition that the three lines AF, BC, and DE shall touch the conic

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

is found by first finding the condition of tangency of each of these lines, which is, for AF,

$$\frac{l}{l_1} + \frac{m}{m_1} + \frac{n}{n_1} = 0;$$

for BC, $$\frac{l}{l_2} + \frac{m}{m_2} + \frac{n}{n_2} = 0;$$

for DE, $$\frac{l}{l_3} + \frac{m}{m_3} + \frac{n}{n_3} = 0;$$

and therefore $$\begin{vmatrix} \frac{1}{l_1} & \frac{1}{m_1} & \frac{1}{n_1} \\ \frac{1}{l_2} & \frac{1}{m_2} & \frac{1}{n_2} \\ \frac{1}{l_3} & \frac{1}{m_3} & \frac{1}{n_3} \end{vmatrix} = 0,$$

which, it is seen, is the same condition as above.

From what follows on reciprocation it will be evident that, by reciprocating Pascal's Theorem, Brianchon's Theorem may be obtained.

60. It is proper here to notice a different form of notation which is frequently employed in this subject.

Suppose $f(\alpha, \beta, \gamma) = 0$ to represent the equation to the curve, and s_1, s_2, s_3 the direction-sines of the tangent at the point $(\alpha_1, \beta_1, \gamma_1)$; (α, β, γ) any point on the tangent; and h the distance between these two points. Then, following the reasoning of Art. 48, the intercepts on h will be given by substituting the new values,

$$\alpha = \alpha_1 + s_1 h, \quad \beta = \beta_1 + s_2 h, \quad \gamma = \gamma_1 + s_3 h,$$

in the above equation; that is, by

$$f(\alpha_1+s_1h,\ \beta_1+s_2h,\ \gamma_1+s_3h) = 0,$$

which, when expanded as we have already seen in the Article referred to, will consist of some function of $(\alpha_1, \beta_1, \gamma_1)$, a coefficient of h, and a coefficient of h^2 which is some function of the direction-sines, or

$$f(\alpha_1, \beta_1, \gamma_1) + h^2 f(s_1, s_2, s_3) + h\left(s_1\frac{df}{d\alpha_1} + s_2\frac{df}{d\beta_1} + s_3\frac{df}{d\gamma_1}\right) = 0.$$

If now $(\alpha_1, \beta_1, \gamma_1)$ lie on the curve, we must have

$$f(\alpha_1, \beta_1, \gamma_1) = 0;$$

also one of the intercepts becomes zero, and since the line is a tangent the length of the chord is zero, that is, the coefficient of h vanishes, and we have

$$s_1\frac{df}{d\alpha_1} + s_2\frac{df}{d\beta_1} + s_3\frac{df}{d\gamma_1} = 0.$$

By substituting for s_1, s_2, s_3 their values, we shall obtain twice the function in $(\alpha_1, \beta_1, \gamma_1)$ which $= 0$, that is,

$$\alpha_1\frac{df}{d\alpha_1} + \beta_1\frac{df}{d\beta_1} + \gamma_1\frac{df}{d\gamma_1} = 2f(\alpha_1\beta_1\gamma_1),$$

as may be shown by taking the differential coefficients of

$$u\alpha_1^2 + v\beta_1^2 + w\gamma_1^2 + 2u_1\beta_1\gamma_1 + 2v_1\gamma_1\alpha_1 + 2w_1\alpha_1\beta_1 = 0,$$

in respect to $\alpha_1, \beta_1, \gamma_1$ respectively, multiplying the differential coefficients by each of these coordinates and adding, when we shall find that $\quad 2f(\alpha_1, \beta_1, \gamma_1) = 0,$

since $(\alpha_1, \beta_1, \gamma_1)$ is supposed to be on the curve. There will remain, therefore,

$$\alpha\frac{df}{d\alpha_1} + \beta\frac{df}{d\beta_1} + \gamma\frac{df}{d\gamma_1} = 0,$$

the equation to the tangent at $(\alpha_1, \beta_1, \gamma_1)$, since it expresses a relation among the coordinates of any point in the line.

61. *Polar of a point in respect to the conic.*

Let the fixed point be $(\alpha_1, \beta_1, \gamma_1)$; $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ the coordinates of points of contact of tangents from the given point. Then we can show, by an extension of the reasoning of the last Article, that

$$\alpha_2 \frac{df}{d\alpha_1} + \beta_2 \frac{df}{d\alpha_1} + \gamma_2 \frac{df}{d\alpha_1} = 0$$

is the tangent from $(\alpha_1, \beta_1, \gamma_1)$ to the point of contact $(\alpha_2, \beta_2, \gamma_2)$;

and likewise $$\alpha_3 \frac{df}{d\alpha_1} + \beta_3 \frac{df}{d\beta_1} + \gamma_3 \frac{df}{d\gamma_1} = 0$$

is the equation to the tangent at $(\alpha_3, \beta_3, \gamma_3)$. Therefore these equations express the fact that the line joining these points of contact is a locus whose equation is

$$\alpha \frac{df}{d\alpha_1} + \beta \frac{df}{d\beta_1} + \gamma \frac{df}{d\gamma_1} = 0,$$

that is, the polar with respect to the conic

$$f(\alpha, \beta, \gamma) = 0;$$

or we may proceed otherwise. Defining the polar of a given point as the locus of the intersection of tangents drawn to the points of section by a straight line through the given point, we should have for the equation through the three points in the same straight line

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0 \;\ldots(1),$$

where (α, β, γ) is the given point, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_1, \beta_1, \gamma_1)$ the points of section in which any straight line cuts the conic.

The intersection of tangents,

$$\alpha_1 \frac{df}{d\alpha} + \beta_1 \frac{df}{d\beta} + \gamma_1 \frac{df}{d\gamma} = 0,$$

and

$$\alpha_2 \frac{df}{d\alpha} + \beta_2 \frac{df}{d\beta} + \gamma_2 \frac{df}{d\gamma} = 0,$$

will be

$$\frac{\dfrac{df}{d\alpha}}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\dfrac{df}{d\beta}}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\dfrac{df}{d\gamma}}{\alpha_1\beta_2 - \alpha_2\beta_1} \quad \text{.........} \ (2).$$

Equation (1) with (2) gives

$$\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} = 0.$$

This equation being independent of $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ is the relation at the intersection of the tangents; it is therefore the locus required, and, by definition, the polar of (α, β, γ).

62. *Coordinates of the pole of a straight line in respect to a conic.*

Let

$$f\alpha + g\beta + h\gamma = 0$$

be the equation to the straight line, and

$$\phi(\alpha, \beta, \gamma) = 0$$

to that of the conic. If (α, β, γ) be the coordinates of the required point, then its polar, by the last Article, is

$$\alpha \frac{d\phi}{d\alpha} + \beta \frac{d\phi}{d\beta} + \gamma \frac{d\phi}{d\gamma} = 0;$$

and since this is the same as the given line, we have

$$\frac{\dfrac{d\phi}{d\alpha}}{f} = \frac{\dfrac{d\phi}{d\beta}}{g} = \frac{\dfrac{d\phi}{d\gamma}}{h};$$

that is,

$$\frac{u\alpha + w_1\beta + v_1\gamma}{f} = \frac{v\beta + u_1\gamma + w_1\alpha}{g} = \frac{w\gamma + v_1\alpha + u_1\beta}{h}.$$

Putting each member $= -s$, we have

$$u\alpha + w_1\beta + v_1\gamma + sf = 0,$$

$$v\beta + u_1\gamma + w_1\alpha + sg = 0,$$

$$w\gamma + v_1\alpha + u_1\beta + sh = 0\,;$$

and consequently

$$\frac{\alpha}{\begin{vmatrix} f & g & h \\ v_1 & u_1 & w \\ w_1 & v & u_1 \end{vmatrix}} = \frac{\beta}{\begin{vmatrix} f & g & h \\ u & w_1 & v_1 \\ v_1 & u_1 & w \end{vmatrix}} = \frac{\gamma}{\begin{vmatrix} f & g & h \\ w_1 & v & u_1 \\ u & w_1 & v_1 \end{vmatrix}}$$

which, with $$a\alpha + b\beta + c\gamma = 2\Delta,$$

determine the coordinates required.

63. *Centre of conic.*

If the equation be $\phi(\alpha, \beta, \gamma) = 0$,

and $(\alpha_1, \beta_1, \gamma_1)$ be the centre; then the roots of

$$\phi(\alpha_1 + s_1 h, \beta_1 + s_2 h, \gamma_1 + s_3 h) = 0$$

will be equal and opposite in sign.

Hence, since the coefficient of h must vanish in the quadratic,

therefore $$s_1 \frac{d\phi}{d\alpha_1} + s_2 \frac{d\phi}{d\beta_1} + s_3 \frac{d\phi}{d\gamma_1} = 0.$$

Bearing in mind the proportionality of s_1, s_2, s_3, that is, the relation $$as_1 + bs_2 + cs_3 = 0,$$

we have $$\frac{\dfrac{d\phi}{d\alpha_1}}{a} = \frac{\dfrac{d\phi}{d\beta_1}}{b} = \frac{\dfrac{d\phi}{d\gamma_1}}{c},$$

which, fully written out, will give determinants similar in form to those of the last Article, with a, b, c in place of f, g, h, and $\alpha_1, \beta_1, \gamma_1$ in place of α, β, γ.

64. *The conic will break up into two right lines* when we have the condition

$$\begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} = 0.$$

For suppose the two lines into which

$$\phi(\alpha, \beta, \gamma) = 0$$

breaks up to be represented by

$$f\alpha + g\beta + h\gamma = 0,$$
$$f_1\alpha + g_1\beta + h_1\gamma = 0.$$

Then will

$$\phi(\alpha, \beta, \gamma) = (f\alpha + g\beta + h\gamma)(f_1\alpha + g_1\beta + h_1\gamma),$$

and $$\frac{d\phi}{d\alpha} = f(f_1\alpha + g_1\beta + h_1\gamma) + f_1(f\alpha + g\beta + h\gamma),$$

with corresponding values for $\frac{d\phi}{d\beta}$ and $\frac{d\phi}{d\gamma}$. Hence, reverting to a principle already explained (Art. 31),

$$\frac{d\phi}{d\alpha} = 0, \quad \frac{d\phi}{d\beta} = 0, \quad \frac{d\phi}{d\gamma} = 0,$$

are straight lines which pass through the intersection of the given lines; that is, the lines

$$u\,\alpha + w_1\beta + v_1\gamma = 0,$$
$$w_1\alpha + v\,\beta + u_1\gamma = 0,$$
$$v_1\alpha + u_1\beta + w\,\gamma = 0,$$

concur, and give the above determinant.

65. When some of the four points of intersection of two conics become coincident, some of the common chords will

coincide; others will touch at a common point, that is, become tangents. There will, in general, be three pairs of common chords; if two points coincide, the conics touch; if the two remaining points also coincide, the conics have double contact and a chord of contact.

66. *Equation to the asymptotes.*

From Art. 64 we can easily form the equation to any pair of common chords. Thus, if

$$\phi(\alpha, \beta, \gamma) = Q\alpha^2 + R\beta^2 + S\gamma^2 + 2Q_1\beta\gamma + 2R_1\gamma\alpha + 2S_1\alpha\beta = 0$$

and

$$\phi_1(\alpha, \beta, \gamma) = u\alpha^2 + v\beta^2 + w\gamma^2 + 2u_1\beta\gamma + 2v_1\gamma\alpha + 2w_1\alpha\beta = 0$$

represent the two conics, the locus in question will have the equation (Art. 31),

$$\phi(\alpha, \beta, \gamma) + k\phi_1(\alpha, \beta, \gamma) = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

which must be so conditioned in k as to represent two straight lines, hence (Art. 64) *

$$\begin{vmatrix} Q + ku & S_1 + kw_1 & R_1 + kv_1 \\ S_1 + kw_1 & R + kv & Q_1 + ku_1 \\ R_1 + kv_1 & Q_1 + ku_1 & S + kw \end{vmatrix} = 0.$$

If now $\phi(\alpha, \beta, \gamma) = 0$ breaks up into two coincident straight lines, as,

$$(f\alpha + g\beta + h\gamma)^2 = 0,$$

we shall find

$$k = \frac{\begin{vmatrix} u & w_1 & v_1 & f \\ w_1 & v & u_1 & g \\ v_1 & u_1 & w & h \\ f & g & h & 0 \end{vmatrix} (=U),}{\begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} (=W),}$$

* Ferrers, p. 85.

which, substituted in (1), gives

$$\phi(\alpha, \beta, \gamma)\, W + U\phi_1(\alpha, \beta, \gamma) = 0,$$

or $$(f\alpha + g\beta + h\gamma)^2\, W + U\phi_1(\alpha, \beta, \gamma) = 0 \quad \ldots\ldots\ldots (2).$$

This equation now represents, under the above condition, not a pair of common chords, but a pair of common tangents whose chord of contact is

$$f\alpha + g\beta + h\gamma = 0.$$

We have now only to introduce the condition that the chord of contact is at infinity; that is, that

$$a\alpha + b\beta + c\gamma = 0;$$

wherefore (2) becomes

$$(a\alpha + b\beta + c\gamma)^2 \begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} + \phi_1(\alpha, \beta, \gamma) \begin{vmatrix} u & w_1 & v_1 & a \\ w_1 & v & u_1 & b \\ v_1 & u_1 & w & c \\ a & b & c & 0 \end{vmatrix} = 0,$$

which is the equation of the asymptotes.

Cor. 1.—Since every parabola has one tangent altogether at an infinite distance, the vanishing of the second determinant in the above equation expresses the condition that the conic may be a parabola.*

Cor. 2.—The conic will be a rectangular hyperbola when the asymptotes are at right angles to one another; that is, when (Art. 14) the two straight lines into which the conic breaks are subject to the condition of perpendicularity,

$$ll_1 - (mn_1 + m_1n)\cos A + mm_1 - (nl_1 + n_1l)\cos B$$
$$+ nn_1 - (lm_1 + l_1m)\cos C.$$

In other words, if the conic be

$$\phi_1(\alpha, \beta, \gamma) = 0,$$

* Salmon's Conics, p. 224.

the required condition becomes

$$u+v+w-2u_1 \cos A-2v_1 \cos B-2w_1 \cos C=0.$$

Def. — W is called the discriminant of the function $\phi_1(\alpha, \beta, \gamma)$, and U the bordered discriminant of the same function (D. 43), where f, g, h, as the coefficients of α, β, γ, are $=\frac{a}{2\Delta}, \frac{b}{2\Delta}, \frac{c}{2\Delta}$ respectively. For the conic,

$$l\alpha^2+m\beta^2+n\gamma^2=0,$$

$$W=lmn,$$

$$U=-(a^2mn+b^2nl+c^2lm)\frac{1}{4\Delta^2}.$$

Numerous other functions may be determined.

67. Space does not permit extended illustration of the use of the abridged notation thus far exhibited. The reader can easily apply it. For instance, if the function be $\varphi_1(\alpha, \beta, \gamma)$, and we wish to express the equation of the straight line at infinity in terms of the derived functions, the required equation might be written

$$f\frac{d\phi_1}{d\alpha}+g\frac{d_1}{d\beta}+h\frac{d\phi_1}{d\gamma}=0;$$

and since $$a\alpha+b\beta+c\gamma=0 \qquad (1)$$

represents the straight line at infinity, we have

$$\frac{uf+w_1g+v_1h}{a}=\frac{w_1f+vg+u_1h}{b}=\frac{v_1f+u_1g+wh}{c}.$$

These equivalents represented by $-k$ give us

$$uf+w_1g+v_1h+ak=0 \qquad (2),$$

$$w_1f+vg+u_1h+bk=0 \qquad (3),$$

$$v_1f+u_1g+wh+ck=0 \qquad (4).$$

Eliminating now between (1), (2), (3), (4), we obtain the condition that the minors of the bordered discriminant in respect to its f, g, h are proportional to f, g, h of the given equation, which minors being represented by A, B, C, the equation becomes

$$A\frac{d\phi_1}{d\alpha}+B\frac{d\phi_1}{d\beta}+C\frac{d\phi_1}{d\gamma}=0$$

as the straight line at infinity.

68. *The equation of the nine-point circle.*

We first find the condition that the conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0$$

may represent a circle. If the conic be a circle, $f(s_1, s_2, s_3)$ is constant, that is, all diameters will be equal; and since, in the equation for finding the lengths of the intercepts,

$$f(\alpha,\beta,\gamma)+h\left(s_1\frac{df}{d\alpha}+s_2\frac{df}{d\beta}+s_3\frac{df}{d\gamma}\right)+h^2f(s_1, s_2, s_3)=0,$$

the coefficient of h vanishes, we have

$$f(\alpha,\beta,\gamma)+h^2f(s_1, s_2, s_3)=0,$$

which gives the radius in the given direction. To reduce this, we may express the condition that diameters in three directions (that is, directions of the lines of reference) are equal.

We have, therefore, to express this,

$$h^2=\frac{f(\alpha,\beta,\gamma)}{f(x)}=\frac{f(\alpha,\beta,\gamma)}{f(y)}=\frac{f(\alpha,\beta,\gamma)}{f(z)}.$$

Wherefore $\qquad f(x)=f(y)=f(z);$

or, for direction of BC,

$$s_1=0,\quad s_2=\sin C,\quad s_3=-\sin B.$$

Hence $\qquad f(x)=f(0, c, -b).$

Similarly, for CA and AB,

$$f(y) = f(-c, \quad 0, \quad a),$$
$$f(z) = f(\ b, \ -a, 0),$$

from the proportionality of $\sin A$, $\sin B$, $\sin C$.

Hence we have the two conditions,

$$vc^2 + wb^2 - 2u_1bc = wa^2 + uc^2 - 2v_1ca = ub^2 + va^2 - 2w_1ab.$$

In the second place, we see that, if the curve pass through the middle points of the sides of reference, α, β, γ must in succession be taken $= 0$; whence

$$\left.\begin{aligned} vc^2 + wb^2 + 2u_1bc &= 0 \\ wa^2 + uc^2 + 2v_1ca &= 0 \\ ub^2 + va^2 + 2w_1ab &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which follows from the condition involved, that

$$b\beta = c\gamma = a\alpha.$$

Comparing the two sets of equations, we find

$$w_1ab = v_1ca = u_1bc.$$

Hence, if equations (1) are true, they will hold whatever the value of u_1. Let $u_1 = -a$.

The resolution of these equations gives

$$u = 2a \cos A,$$
$$v = 2b \cos B,$$
$$w = 2c \cos C.$$

Hence the circle which passes through the middle points of the sides of reference (*the nine-point circle*) becomes

$$\alpha^2 \sin A \cos A + \beta^2 \sin B \cos B + \gamma^2 \sin C \cos C$$
$$- \beta\gamma \sin A - \gamma\alpha \sin B - \alpha\beta \sin C = 0,$$

or $\quad \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$
$$- 2\beta\gamma \sin A - 2\gamma\alpha \sin B - 2\alpha\beta \sin C = 0.$$

Cor. 1.—If now $\alpha = 0$,

$$\beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma \sin A = 0.$$

But since $$2 \sin A = 2 \sin (B+C),$$

$$\beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma (\sin B \cos C + \sin C \cos B) = 0.$$

This breaks up into the factors

$$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C) = 0.$$

The circle therefore meets BC in two points.

The one when, by the hypothesis,

$$\alpha = 0, \quad b\beta = c\gamma, \quad i.e., \ \beta \sin B = \gamma \sin C,$$

which determines the middle of BC.

The other is evidently when

$$\alpha = 0, \quad \beta \cos B = \gamma \cos C,$$

the foot of the perpendicular from A. Similarly for the other sides.

Cor. 2.—The last equation of this Article shows that the nine-point circle passes through the points of intersection of the circumscribed circle and the circle in respect to which the triangle of reference is self-conjugate.

Cor. 3.—The difference between the equations of the circumscribed circle and the circle through the middle points of the sides of the triangle is

$\alpha \cos A + \beta \cos B + \gamma \cos C = 0$ multiplied by a constant,

since $$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$= (\alpha \cos A + \beta \cos B + \gamma \cos C)(\alpha \sin A + \beta \sin B + \gamma \sin C).$$

But $\alpha \sin A + \beta \sin B + \gamma \sin C$ is a constant, and therefore $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$ is their radical axis, or the homological axis of the triangle of reference and that formed by joining the feet of the perpendiculars.

Cor. 4.—By similar reasoning we find that the same circle passes through the middle points of the sides of the triangles of which the point of intersection of perpendiculars is the vertex. Nine points are therefore determined.

POLAR RECIPROCALS.

69. Reciprocation—the principle of duality, or that analysis (or synthesis) which, while determining the distribution of points, coordinately fixes the position of lines—though of great interest, is altogether too large a subject for this Tract. Some theorems may be introduced. In general, we may say that to reciprocate involves interchanging "angular points" for "sides," "inscribing" for "circumscribing," "join" for "intersect," &c. &c.

For instance, if the well-known theorem, that "If two triangles be inscribed in a conic, their sides will be tangent to a conic," be reciprocated, we may write, "If two triangles circumscribe one conic, their vertices will lie on a conic."

This proof and its reciprocal may be exhibited by a common process in triangular and tangential coordinates (Arts. 24, 27). Let vertices of one triangle (sides of the same) be represented by (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3); let the other be the triangle of reference, and suppose

$$f(p, q, r) = 0$$

the tangential equation of the conic passing through the points of reference; or the equation may be represented in both systems by $$lqr + mpr + npq = 0.$$

Then the equations to the one triangle will be

$$\frac{lp}{p_2 p_3} + \frac{mq}{q_2 q_3} + \frac{nr}{r_2 r_3} = 0,$$

$$\frac{lp}{p_3 p_1} + \frac{mq}{q_3 q_1} + \frac{nr}{r_3 r_1} = 0,$$

$$\frac{lp}{p_1 p_2} + \frac{mq}{q_1 q_2} + \frac{nr}{r_1 r_2} = 0.$$

By comparing (Arts. 56, 57), we see that by this form of representation the inscribed conic (circumscribing) may be expressed by

$$\sqrt{Lp} + \sqrt{Mq} + \sqrt{Nr} = 0;$$

that is, this conic will be inscribed in (circumscribe) both triangles provided the conditions of tangency be satisfied,

$$\left.\begin{array}{l} \dfrac{Lp_2p_3}{l} + \dfrac{Mq_2q_3}{m} + \dfrac{Nr_2r_3}{n} = 0 \\ \dfrac{Lp_3p_1}{l} + \dfrac{Mq_3q_1}{m} + \dfrac{Nr_3r_1}{n} = 0 \\ \dfrac{Lp_1p_2}{l} + \dfrac{Mq_1q_2}{m} + \dfrac{Nr_1r_2}{n} = 0 \end{array}\right\} \text{............ (1).}$$

But since the given points (p_1, q_1, r_1, &c.), (vertices), lie by hypothesis on the conic, the condition must be expressed by

$$\begin{vmatrix} \dfrac{1}{p_1} & \dfrac{1}{q_1} & \dfrac{1}{r_1} \\ \dfrac{1}{p_2} & \dfrac{1}{q_2} & \dfrac{1}{r_2} \\ \dfrac{1}{p_3} & \dfrac{1}{q_3} & \dfrac{1}{r_3} \end{vmatrix} = 0,$$

which condition satisfies equations (1), and proves the theorem and its reciprocal. The determinant follows, it is evident, as the eliminant of the equations,

$$\frac{l}{p_1} + \frac{m}{q_1} + \frac{n}{r_1} = 0,$$

$$\frac{l}{p_2} + \frac{m}{q_2} + \frac{n}{r_2} = 0,$$

$$\frac{l}{p_3} + \frac{m}{q_3} + \frac{n}{r_3} = 0.$$

70. If m, n, p, q are the poles of the sides of a polygon $abcd$, then the points a, b, c, d are the poles of the sides of the polygon $mnpq$.

The conic with respect to which the poles and polars are taken is the auxiliary conic.

The reciprocal of a conic is a conic.

By Art. 60, the polar is given by

$$\alpha \frac{d\phi}{d\alpha} + \beta \frac{d\phi}{d\beta} + \gamma \frac{d\phi}{d\gamma} = 0.$$

If therefore (f, g, h) be any point on the reciprocal curve, its polar with respect to the auxiliary conic,

$$U\alpha^2 + V\beta^2 + W\gamma^2 = 0 \quad \text{..................... (1)},$$

will be given by the equation

$$Uf\alpha + Vg\beta + Wh\gamma = 0 \quad \text{..................... (2)}.$$

Let the conic to be reciprocated be

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad \text{..................... (3)}.$$

To find the condition that (2) may touch (3), we eliminate α between the equation of the conic and the line; and if the line be a tangent, the values of $\beta : \gamma$ must be equal (Art. 57), and we obtain

$$\frac{U^2f^2}{l} + \frac{V^2g^2}{m} + \frac{W^2h^2}{n} = 0.$$

This being of the second degree, giving two points of intersection of the straight line, is a conic, and is the reciprocal of (3) with respect to (1).

71. Two straight lines are conjugate when each passes through the pole of the other. Required to express this condition. Let

$$f_1\alpha + g_1\beta + h_1\gamma = 0,$$

$$f_2\alpha + g_2\beta + h_2\gamma = 0,$$

be the two lines. Then (Art. 62) we may express the condition by the equation

$$\begin{vmatrix} \dfrac{d^2\phi}{d\alpha^2} & \dfrac{d^2\phi}{d\alpha\, d\beta} & \dfrac{d^2\phi}{d\alpha\, d\gamma} & f_1 \\ \dfrac{d^2\phi}{d\beta\, d\alpha} & \dfrac{d^2\phi}{d\beta^2} & \dfrac{d^2\phi}{d\beta\, d\gamma} & g_1 \\ \dfrac{d^2\phi}{d\gamma\, d\alpha} & \dfrac{d^2\phi}{d\gamma\, d\beta} & \dfrac{d^2\phi}{d\gamma^2} & h_1 \\ f_2 & g_2 & h_2 & 0 \end{vmatrix} = 0,$$

where $\phi(\alpha, \beta, \gamma) = u\alpha^2 + v\beta^2 + w\gamma^2 + \&c.$

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MATHEMATICAL TRACTS.

No. III.

INVARIANTS.

Dec 23/[illegible]

Morris Plains, N. J.
Nov. 22d 1882

Prof. Geo. Davidson,

Dear Sir: I rep[illegible]
the delayed & devious
course of your lett[er]
rec'd to-day - Thom[as]
No. III goes to yo[ur]
address to-day -
Very Respect[fully]
W. J. Wrig[ht]

[illegible]
Dec 1882

TRACTS

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 3.

INVARIANTS.

BY

REV. W. J. WRIGHT, PH.D.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

"Τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν."
PLATO, Rep. VII., 527, *b*.

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

1879.

My acknowledgments are due to R. Tucker, Esq., M.A., Honorary Secretary of the London Mathematical Society, for valuable assistance rendered in passing these sheets through the press.—W. J. W.

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PREFACE TO TRACT NO. III.

This Tract takes up the general Theory of Invariants.

It is published in pursuance of a purpose, announced in the first number of this series, to give an account of the principal new methods, processes, and extensions which, since 1841, have been introduced into the study of Mathematics. The chief requisite to this undertaking, which undoubtedly is one of considerable magnitude, is evidently a sufficiently comprehensive reading upon these various subjects.

The English, German, French, and Italian Mathematicians have contributed to their journals and learned societies innumerable memoirs and treatises, whose value and bearing upon the matter in hand the reader cannot determine without some degree of careful examination. It also frequently happens that the time consumed in tracing a fugitive paper is in inverse ratio to its importance.

The reader who wishes to read fully upon this Theory may adopt one of two methods. He may begin at the beginning, reading in order of time the papers of its chief authors and expounders, commencing with the essay of the late Dr. Boole in the Cambridge Mathematical Journal for 1841, and follow this with the numerous papers of living authors—Sylvester, Cayley, Hermite, and Salmon—papers extending through the subsequent volumes of the Cambridge and the Cambridge and Dublin Mathematical Journals, and the Philosophical Magazine, together with the various contributions of Clebsch and Aronhold, and others, in *Crelle*, from Vol. 39 to Vol. 69. Or he may take a reverse course, beginning with the Lessons of Salmon, and those of Serret, on Modern Higher Algebra, which, as compends of this and connecting Theories, are in the main works of great excellence, though oftentimes not as clear and satisfactory as could be desired, or as full and explicit as may be found elsewhere; and then he may extend his reading to the Journals above mentioned, together with the proceedings of the contemporaneous societies — as the Philosophical Transactions, *Comptes Rendus*, &c. But, whatever course he may take, he will doubtless never be able clearly to determine to what authorship he is to ascribe some parts and illustrations of the Theory.

The best reading-room for this work, so far as I can judge, after an experience of nearly two years in European libraries, is that of the British Museum.

In view of the extensive literature upon this subject, it may be asked, what can be accomplished by a work of the size of this Tract? Its actual value, evidently, remains to be seen; but I believe that within these pages the reader will find such an account of the Theory as will enable him to gain a knowledge of its principal propositions, and also to judge, from the explained applications, of its real value in Geometry. The computations of Invariants, Chapter IV., will afford such a guide in the various applications that he will probably be at little loss in extending them at his pleasure. I have been desirous of making these calculations so fully, that no one with a fair geometrical knowledge need fail of understanding how each result was obtained. Nowhere else can such work be found in so elementary a form, and for this reason I hope it may prove acceptable to those persons whose time and opportunities of study are somewhat limited, and to those also who are unwilling to obtain and to read the larger works.

In reviewing the notes which I had taken of the principal contributions to this Theory, I found that I had frequently omitted the proper credits, either through sheer neglect, or want of sufficient knowledge; and hence, without attempting to supply these omissions, as could not well be done in the absence of the books and journals, it was concluded to omit them nearly altogether.

The number of persons who have obtained the preceding Tracts of this series, and who have expressed themselves in terms highly favorable to their publication, is deemed sufficient evidence that they are meeting a public want. One thing which was expected has certainly followed,—a goodly number of my countrymen have been awakened to look, for the first time, upon a vast untraversed domain of mathematical knowledge. To these persons, at least, there can be no doubt as to the direction of the goal. It is now clearly and definitely fixed that mathematical researches will, for a long time to come, be mainly conducted through the media of methods and processes, to whose exposition these Tracts are devoted. The time is not far distant, if it has not already arrived, when a knowledge of these subjects will be considered as necessary to the equipment of a mathematician as the Calculus. It is not meant by this, that it is the duty of every mathematician to make a specialty of algebraic forms, either with or without their geometrical interpretation. But it is meant that the modern treatment of the Higher Geometry should be studied as a part of the general preparation necessary to a student of Physical Science.

W. J. W.

Cape May Point, N.J.;
April, 1879.

INVARIANTS.

CHAPTER I.

PROLEGOMENA.

1. The Theory of Invariants, as will appear, is based upon a knowledge of the General Theory of Equations and several of its later important extensions. Some of these extensions must be stated, because, although perhaps familiar to the reader as commonly or formerly expressed, they may not be easily recognised in their modern dress or terminology; others, because they have no existence outside of their present form.

2. *Symmetric Functions.*—If the general equation be

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \&c. = 0,$$

the Newtonian formulæ give us

$$S_1 = -a_1, \quad S_2 = a_1^2 - 2a_2,$$
$$S_3 = -a_1^3 + 3a_1 a_2 - 3a_3, \ \&c.,$$

or, as they are written by Hirsch, Cayley, and others,

$$\begin{aligned}
\Sigma\alpha &= -a_1, \\
\Sigma\alpha^2 &= a_1^2 - 2a_2; \qquad \Sigma\alpha\beta = a_2, \\
\Sigma\alpha^3 &= -a_1^3 + 3a_1 a_2 - 3a_3, \\
\Sigma\alpha^2\beta &= -a_1 a_2 + 3a_3, \\
\Sigma\alpha\beta\gamma &= -a_3, \ \&c.;
\end{aligned}$$

in which we have expressed the sum of the roots and the sum of their products by twos, by threes, &c.

3. If we consider any one of these products, as $a_1 a_2$, we say that its weight is $1+2$, or, in general, that the weight of any term is the sum of the suffixes. Looking at these functions,

however far we may extend them, we see that they are symmetrical as to weight. The *order* is estimated by the number of factors in each term. Hence $a_1a_2a_3$ is of the third order, and its weight is $1+2+3$. This being stated, it is easy to see, by inspection of the several functions written above, that the weight of $\Sigma\alpha^t\beta^u$ is $t+u$, and the order the greater of t, u. The order of $\Sigma\alpha\beta\gamma$ (being the sum of the products in threes) can evidently be, so far as the coefficients of the given equation are concerned, only unity. If, therefore, we regard α as the leading root, appearing in every function, we might predicate the degree of the function upon the degree of α. In this case any symmetric function of the p^{th} order must contain more or less terms involving α^p. There will then be p factors each including α. Hence $\Sigma\alpha^3$, $\Sigma\alpha^3\beta\gamma$ are each of the third order in the coefficients of the given equation; that is, the highest order in any term is three. In general, then, the order of any symmetric function is determined by the highest degree in any one root, while the weight is estimated by the total degree of the roots as factors. The literal part, then, of any symmetric function can thus be at once written out. For the sake of clearness, it is necessary to notice that the functions of roots in this manner may be expressed in terms of the coefficients of the given equation, as will be seen by solving the linear equations just written for S_1, S_2, &c.; and, consequently, any function of the differences of roots can be expressed in the same terms.

3. *Symmetric functions of the differences of roots.*—These we shall see are *invariants*. For the present let us consider what relation such functions ought to satisfy. We begin by observing the effect upon the coefficients of the given equation of increasing or diminishing all the roots by the same quantity. There will plainly be no change in the resulting functions of the differences of roots. Let then $x+l$ be substituted for x, and we have

$$x^n+(a_1+nl)\,x^{n-1}$$
$$+\left[a_2+(n-1)\,la_1+\tfrac{1}{2}n\,(n-1)\,l^2\right]x^{n-2}+\&c.=0.$$

Next, observe the form of any function f of the coefficients a_1, a_2, a_3, &c., when a_1, a_2, &c. are changed into a_1+da_1, a_2+da_2, &c.

This form will be

$$f+\left[\frac{df}{da_1}\,da_1+\frac{df}{da_2}\,da_2+\&c.\right]+\frac{1}{1.2}\left[\frac{d^2f}{da_1^2}(da_1)^2+\&c.\right]\ldots(1).$$

But, by the substitution of $x+l$ for x, a_1 becomes a_1+nl, and a_2 becomes $a_2+(n-1)\,la_1+\frac{1}{2}n\,(n-1)\,l^2$.

Clearly, then, if this substitution were made in any function of the coefficients a_1, a_2, &c., and the result arranged with reference to l, we must have, by (1),

$$f+l\left[n\frac{df}{da_1}+(n-1)\,a_1\frac{df}{da_2}+(n-2)\,a_2\frac{df}{da_3}\right]+\&c.=0.$$

This is true whatever l may be.

Let $l=0$, and we have, as the condition which any function of the differences will satisfy,

$$n\frac{df}{da_1}+(n-1)\,a_1\frac{df}{da_2}+(n-2)\,a_2\frac{df}{da_3}+\&c.=0.$$

This relation is both necessary and sufficient in order that the given function of the coefficients should remain unchanged by the substitution of $x+l$ for x in the given equation.

We can now write, not only the literal part, but the coefficients of any symmetric function. For instance, if we are to form $\Sigma(\beta-\gamma)^2$, we see that its order is 2 and its weight 2. There can be no more than two factors in any term, while the weight for each term must be 2. It must be of the form $Aa_2+Ba_1^2$. By the above differential equation,

$$[A\,(n-1)+2nB]\,a_1=0.$$

This gives $B=-\dfrac{n-1}{2n}$, when $A=1$; or the function can differ by only a factor from $(n-1)\,a_1^2-2na_2$.

We may see that this factor is unity by supposing $\gamma = 1$ and the other roots 0; then $a_2 = 0$; and $a_1 = 1$, since $\alpha + \beta + \&c. = a_1$, and $\alpha(\beta + \gamma + \&c.) + \beta\gamma + \&c. = a_2$.

4. The homogeneous equation

$$(a_0, a_1 \ldots\ldots a_n)(x, y)^n$$

or $$a_0x^n + na_1x^{n-1}y + \tfrac{1}{2}n(n-1)a_2x^{n-2}y^2 + \ldots\ldots + a_ny^n = 0$$

reduces to the general equation of Art. 2 by dividing by a_0y^n. And it is plain that the differential equation of the last Article will undergo a corresponding change. Hence, for the substitution of $x+l$ for x, we must write

$$a_0\frac{df}{da_1} + 2a_1\frac{df}{da_2} + 3a_2\frac{df}{da_3} + \&c. = 0,$$

while for the substitution $y+l$ for y it must be written in a reverse order, that is,

$$na_1\frac{df}{da_0} + (n-1)a_2\frac{df}{da_1} + (n-2)a_3\frac{df}{da_2} + \&c. = 0.$$

5. *The symmetric function of the homogeneous equation in* $\frac{x}{y}$.—Suppose α one of the roots, then $\frac{x}{y} = \alpha$. That is, any system of values, as $\frac{x_1}{y_1} = \alpha$, in other words, any ratio which is $= \alpha$, will satisfy the homogeneous equation. Or, we may state it thus: any symmetric function expressed in terms of its roots, as x_1, x_2, x_3, &c., may be reduced to the corresponding function of a homogeneous equation of the same degree, by dividing each x_1, x_2, &c. by y_1, y_2, &c., and then multiplying this result by any power of y_1y_2, &c. that will clear it of fractions.

Hence we may write any function of the differences as the sum of products of determinants

$$\Sigma \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \&c. \times (y_1y_2\&c.)^n,$$

where $n =$ the variable power necessary to clear of fractions. Thus, to form for $(a, b, c, d)(x, y)^3$ the sum of the products of the squares of the differences of the roots, we have the ratios, or roots,

$$\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3},$$

that is, $$\Sigma \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^2 \times \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^2 \times \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}^2.$$

In this case the order is 4 and weight 6.

The form is therefore

$$Aa_3^2 a_0^2 + Ba_3 a_2 a_1 a_0 + Ca_3 a_1^3 + Da_2^3 a_0 + Ea_2^2 a_1^2 \ldots\ldots (1).$$

Operating with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3}$, we get

$$(B+6A)\, a_3 a_2 a_0^2 + (3C+2B)\, a_3 a_1^2 a_0$$
$$+ (2E+6D+3B)\, a_2^2 a_1 a_0 + (4E+3C)\, a_2 a_1^3 = 0,$$

which gives us, taking $A = 1$, and equating each term to 0,

$$B = -6, \quad C = 4, \text{ \&c.};$$

or, since $a_0 = a$ and $a_3 = d$, we have

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

The result would be the same had we required the product of the squares of the differences

$$(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2.$$

The order being the same, 4, and the weight also 6, the form would be

$$Aa_3^2 + Ba_3 a_2 a_1 + Ca_3 a_1^3 + Da_2^3 + Ea_2^2 a_1^2.$$

To render this homogeneous, as if derived from a homogeneous equation in x, y, each factor must be divided by a_0, and the whole multiplied by the highest power of a_0 in any denominator. It would then be identical with (1).

6. The eliminant* or resultant of a system of equations is that function of the coefficients whose vanishing expresses that the equations are simultaneous. If we have as many independent equations as we have variables, we can ordinarily, by direct elimination, arrive at such a function freed from any of the assumed variables. This function is generally indicated by Δ.

7. *Eliminant by symmetric functions.*—The product of the several roots of an equation is a symmetric function, as $\Sigma\alpha\beta\gamma$ or $\Sigma\alpha^2\beta$. If we have α, β, γ as the roots of the equation $f(x)=0$, and α, β_1, γ_1 as the roots of $f_1(x)=0$, then, since they have a common root α, the eliminant condition is involved.

If the first set of roots be substituted in the second equation, $f_1(x)$, the result for the value α will vanish; therefore the continued product

$$f_1(\alpha)\times f_1(\beta)\times f_1(\gamma)$$

will vanish; and consequently will conform to the definition of an eliminant, since it is plain, being a symmetric function of the roots of $f(x)$, it can be expressed in terms of the given coefficients of $f(x)=0$ and $f_1(x)=0$, however they may be written.

From this it is seen that the eliminant is a function of the differences of the roots of the two or more equations.

If the equations are homogeneous, $f(x, y)=0$, $f_1(x, y)=0$, they may be treated as non-homogeneous by dividing each equation by the coefficient of the highest power of x and the highest power of y. To illustrate this form of operation, let us find the eliminant of

$$ax^2+2bxy+cy^2=0\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$a_1x^2+2b_1xy+c_1y^2=0\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

---

* Thus $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}=0$, $\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}^2-\begin{vmatrix} b & c \\ b_1 & c_1 \end{vmatrix}\times\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}=0$

are determinant expressions for the eliminants of

$\begin{matrix} ax+b=0 \\ a_1x+b_1=0 \end{matrix}$ and $\begin{matrix} ax^2+bx+c=0 \\ a_1x+b_1x+c_1=0 \end{matrix}$ respectively.

or, written in the non-homogeneous form,

$$z^2+nz+n_1=0, \quad z^2+mz+m_1=0.$$

The symmetric function is then

$$(\alpha^2+m\alpha+m_1)(\beta^2+m\beta+m_1)=0,$$

or

$$\alpha^2\beta^2+m\alpha\beta(\alpha+\beta)+m_1(\alpha^2+\beta^2)+m^2\alpha\beta+mm_1(\alpha+\beta)+m_1^2=0,$$

But $\alpha^2+\beta^2=n^2-2n_1$ (Art. 2), $\alpha\beta=n_1$, $\alpha+\beta=-n$.

Hence $$(n_1-m_1)^2+(m-n)(n_1m-nm_1)=0.$$

Giving m, n, m_1, n_1 their values, we have

$$\left(\frac{c}{a}-\frac{c_1}{a_1}\right)^2+\left(\frac{2b_1}{a_1}-\frac{2b}{a}\right)\left(\frac{2b_1c}{aa_1}-\frac{2bc_1}{aa_1}\right)=0,$$

or $$(ca_1-c_1a)^2+4(b_1a-ba_1)(b_1c-bc_1)=0,$$

the eliminant. This method is useful in this place simply as an exercise in symmetric functions. In practice, it would be far easier to eliminate directly.

8. *The order.*—By inspecting this example and others, we are enabled to determine inductively the order of the eliminant in the coefficients. The symmetric function consists of as many factors as there are units in the degree of the first equation, but each of these factors involves the coefficients of the second in the first degree. On the other hand, the entire product consists of the several symmetric functions of the roots of the first equation, and the highest degree of these is the same as that of the second equation; hence it is evident that the orders of the coefficients in the eliminant are the same as those of the given equations, but taken in an inverse order; that is, the coefficients of the first equation have the order of the second, and the contrary.

If, for instance, there were three homogeneous equations in three variables of the 2nd, 3rd, and 4th orders, then the eliminant would be a homogeneous function of the 12th order in the coefficients of the first equation, of the 8th in those of the second, and of the 6th in those of the third.

9. *The weight.*—It is not so easy to determine the weight. But we may begin by considering that the eliminant is a symmetric function of the differences between the roots of the first and second equations expressed in terms of their cofficients, and then the number of these differences is equal to the product of the orders of the equations. If we multiply each root by any factor as k, we do, in effect, multiply each difference by k; and consequently, the eliminant, which is the product of these differences, is multiplied by k to a power equal to the product of the degrees of the equations. Now, each root in the equations

$$a_0x^n + na_1x^{n-1}y + \tfrac{1}{2}n(n-1)\,a_2x^{n-2}y^2 + \&c. = 0 \ldots\ldots (1),$$
$$b_0x^m + mb_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,b_2x^{m-2}y^2 + \&c. = 0 \ldots (2),$$

will, it is evident, be multiplied by k when we multiply a_1, b_1; a_2, b_2; by k, k^2, &c.; and therefore each term of the eliminant would involve k to the mn^{th} degree. In this manner we can readily determine the weight of each term, which we shall find to be constant, that is, mn for each term.

10. It is easy to see, from the definition of an eliminant and from the results of (Art. 3), that the eliminant must satisfy the differential equations there given; or, if referred to equations (1) and (2) of the last Article, must be of the form

$$a_0\frac{d\Delta}{da_1} + 2a_1\frac{d\Delta}{da_2} + 3a_2\frac{d\Delta}{da_3} + \&c. + b_0\frac{d\Delta}{db_1} + \&c. = 0,$$

where Δ represents the eliminant of equations (1) and (2).

11. *The eliminant of three equations in three variables.*—The eliminant vanishing, the equations are simultaneous. This can be brought under the system of two equations. For, solving between any two equations, and substituting these values in the third, the product of these substitutions must vanish, since, by hypothesis, there is a community of values between the

different sets, the number of which must equal the weight of the eliminant of those two equations, that is, the product of their degrees. These substituted successively in the remaining equation, and multiplied together, will furnish the requisite symmetric functions by which the coefficients of the solved equations may be expressed, which gives the eliminant whose weight is equal to the product of the degrees of the three equations. For four equations we proceed in the same manner, solving for three and substituting these values in the fourth.

12. In reviewing this method of elimination, it will be seen to be of the widest generality, and all its results susceptible of very satisfactory proof. It is not introduced for any use in actual elimination, but that the reader may here avail himself of important assistance in the study of the Theory of Invariants.

13. The reader interested in determinants will naturally seek some form for elimination by this method. That of Euler leading in this direction is of high theoretical value. Two equations, homogeneous or otherwise, are supposed to be satisfied by a common root of the first degree; then the first, multiplied by all the remaining factors of the second, is evidently equal to the second multiplied by all the remaining factors of the first; as, if we have

$$x^2-(a+b)\,x+ab=0,$$
$$x^2-(a+c)\,x+ac=0,$$

then $(x-c)\,\{x^2-(a+b)\,x+ab\}=(x-b)\,\{x^2-(a+c)\,x+ac\}$;

or, in general, if we multiply the homogeneous equation $f(x, y)=0$ by any arbitrary function of a degree one less than $f_1(x, y)=0$, and the latter by any arbitrary function with a degree one less than the former equation, and then equate term to term, we shall have a number of equations equal to the sum of the degrees of the two given equations, and the eliminant will of course appear in the form of a determinant.

To eliminate between

$$ax^2+2bxy+cy^2,$$
$$a_1x^3+3b_1x^2y+3c_1xy^2+dy^3:$$

$$(kx^2+klxy+ly^2)(ax^2+2bxy+cy^2)$$
$$=(k_1x+l_1y)(a_1x^3+3b_1x^2y+3c_1xy^2+dy^3);$$

equating like terms,

$$ka-k_1a_1=0,$$
$$2bk+kla-3k_1b_1-l_1a_1=0,$$
$$kc+la+2klb-3k_1c_1-3l_1b_1=0,$$
$$2lb+klc-k_1d-3l_1c_1=0,$$
$$lc-l_1d=0.$$

Eliminating k and l, we have

$$\begin{vmatrix} a & 0 & 0 & -a_1 & 0 \\ 2b & 0 & a & -3b_1 & -a_1 \\ c & a & 2b & -3c_1 & -3b_1 \\ 0 & 2b & c & -d & -3c_1 \\ 0 & c & 0 & 0 & -d \end{vmatrix}=0$$

as the eliminant.

14. The various other methods, such as Bezout's method,* Sylvester's dialytic process, the uses of the Jacobian in elimination, explained in (D. 39),† since they do not illustrate the Theory of Invariants, may be omitted.

We will now pass at once to a subject which is intimately connected with that theory.

15. *Discriminants.*—If an equation, or quantic, as it is called when it is not equated to 0, be differentiated with

* I must qualify this statement, so far as it relates to Bezout's method. It is well known by those acquainted with Dr. Sylvester's researches, that what he calls a Bezoutiant is the discriminant of a quadratic function in any number of variables, and is expressible as a symmetrical determinant which is written, as in (D. 22), with a double suffix. The eliminant of two equations of the n^{th} degree may be similarly expressed. The use of the Bezoutiant in the theory of equations is exhibited in a Memoir by Sylvester, *Phil. Trans.*, 1853, p. 513.

† Tract No. 1, Determinants.

respect to its variables, the eliminant of these several differentials is the discriminant. As the quantic is understood to be homogeneous, it is evident that the discriminant must be homogeneous also. The order of the discriminant is clearly the product of the degrees of the differentials of which it is the eliminant.

Observing the same order of the suffixes a_0, a_1, &c., the weight of the discriminant will depend upon the number of differentials and the order of the quantic. Thus, for a binary quadratic the weight must be 2; for a ternary cubic, that is, a quantic containing three variables, $3(3-1)^2$. This arises from a slight modification of the reasoning in Art. 9. The weight would be evidently $(n-1)$, taken as many times as a factor as there are variables, were it not for the consideration that all but one of these differentials begin with a coefficient whose relation to the leading variable is the same as in the original quantic; in other words, with a suffix one greater than the first differential which begins with a_0; hence the number of suffixes must be increased in this proportion. If $p =$ the number of differentials, $(n-1)^p$ must be increased by $(n-1)^{p-1}$, that is,

$$(n-1)^p + (n-1)^{p-1} = n(n-1)^{p-1},$$

which is the sum of the suffixes for each term of the discriminant.

16. If we divide the homogeneous equation by y^n, the result is reducible to a product of factors, as

$$\begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} \times \begin{vmatrix} x & y \\ x_2 & y_2 \end{vmatrix} \times \begin{vmatrix} x & y \\ x_3 & y_3 \end{vmatrix} \times \text{\&c.} = 0.$$

Comparing this product with

$$a_0 x^n + n a_1 x^{n-1} + \text{\&c.} = 0,$$

we see that $$y_1 y_2 y_3 \text{ \&c.} = a_0,$$

since the product of

$$(xy_1 - x_1 y)(xy_2 - x_2 y)(xy_3 - x_3 y) \text{ \&c.} \equiv Q \quad \ldots\ldots\ldots \quad (1)$$

gives $y_1 y_2$ &c. for the coefficient of x^n.

17. The discriminant is equal to the continued product of the squares of the differences of the roots of the given quantic taken two and two.

Suppose x_1y_1, x_2y_2, x_3y_3, &c. are the roots of (1) above,

$$\text{then}\quad \frac{dQ}{dx} = y_1\,(xy_2-yx_2)(xy_3-yx_3)\ \&\text{c.} + y_2\,(xy_1-x_1y)\ \&\text{c.}$$

Observing the effect of substituting x_1y_1 in $\frac{dQ}{dx}$, which is $y_1\,(x_1y_2-y_1x_2)$ &c.; substituting in the same manner x_2y_2, x_3y_3, &c. in the same equation, and taking the continued product, we must have

$$y_1y_2\ \&\text{c.}\ (x_1y_2-y_1x_2)^2\ (x_1y_3-y_1x_3)^2\ \&\text{c.} = 0 \ldots\ldots\ldots (1),$$

which, as we have seen, is the eliminant (Art. 7) of Q and $\frac{dQ}{dx}$. The same product, divided by y_1y_2 &c. $= a_0$, will give the discriminant.

This will more fully appear when we consider that

$$nQ = x\frac{dQ}{dx} + y\frac{dQ}{dy};$$

then, when we have substituted successively all the roots of $\frac{dQ}{dx} = 0$ in Q, we shall have for the continued product $y_1y_2y_3$ &c. multiplied by a similar result of substituting the same roots in $\frac{dQ}{dy}$. But this latter result is evidently the discriminant. Hence, if (1) be divided by a_0, that is, if the eliminant of the quantic and its first differential with reference to x be divided by the product of the y's, we shall obtain the same result as if we had found the eliminant of the first differentials with reference to x and y.

18. Enough preliminary matter has now been introduced to enable the reader to follow with profit all that will follow.

To those who wish to pursue the theory of discriminants further, and desire to study an interesting geometrical application, the theorem of Joachimsthal, taken as the basis of an investigation on the nature of cones circumscribing surfaces having multiple lines, by Dr. Salmon ("Cambridge and Dublin Math. Journal," 1847 and 1849) would probably prove as fruitful in this direction as any that could be mentioned.*

* The theorem above alluded to is included in the following statement. If we have the quantic $(a_0, a_1 \ldots a_{n-1}, a_n \between x, y)^n$, and a_1 contain a factor t, and if a_0 contain t^2 as a factor, the discriminant will be divisible by t^2; also, if a_2 contain t as a factor, and if a_1 and a_0 contain t^2 and t^3 respectively, then the discriminant will be divisible by t^6, and so on. The application by Dr. Salmon was that, if $a_0 + a_1x + a_2x^2 +$ &c. be the equation to a surface, and if xy be a double line, a_0 will contain y in the second, and a_1 in the first degree. The discriminant in respect to x is divisible by y^2, and the locus is a tangent cone.

CHAPTER II.

FORMATION OF INVARIANT FUNCTIONS.

19. The definition of an invariant and covariant of a single quantic has already been given (D. 42). In pursuance of this, we might proceed at once to show how in general such functions can be formed, and then give some explanation of the geometrical importance of the theory; but, for the sake of clearness, we will commence with one of the simplest examples of an invariant function.

20. The determinant of a system of linear equations is an invariant of that system, because, as it will be remembered, when the variables are all transformed by the same linear substitution, the determinant of the transformed equations is equal to the determinant of the given equations multiplied by the modulus of transformation (D. 42). In other words, the determinant (function of the coefficients) of the given equations, which remains unaltered by the transformation, is called an invariant. The equations of (D. 17) will exactly illustrate this.

When the linear equations

$$\left.\begin{aligned} dv + ev_1 + fv_2 &= 0 \\ d_1v + e_1v_1 + f_1v_2 &= 0 \\ d_2v + e_2v_1 + f_2v_2 &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (1)$$

are transformed by the substitutions

$$\left.\begin{aligned} ax + by + cz &= v \\ a_1x + b_1y + c_1z &= v_1 \\ a_2x + b_2y + c_2z &= v_2 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (2),$$

then the determinant of the transformed system will be equal to $(de_1f_2) \times (ab_1c_2)$, which may be written

$$f\,(\text{transformed}) = \Delta^n f\,(\text{given}),$$

where $\Delta =$ the modulus $=$ the determinant formed from the sinister members of (2).

The f, which is also a determinant expressing the coexistence of equations (1), is in this case called an *invariant*.

21. It is not difficult to see from the above that a somewhat complicated problem is now presented to us. We are to trace the effect of linear transformation upon the same functions of the coefficients of an equation, to determine the number of the functions which remain unaltered by such transformation, and to deduce convenient rules for their formation. It was seen, for instance (D. 41), that when the binary quadratic $(abc\between x, y)^2$ was linearly transformed by the substitution of

$$x = lx + my,$$
$$y = l_1x + m_1y,$$

we wrote $Ax^2 + 2Bxy + Cy^2$ as the transformed quadratic, in which

$$A = al^2 + 2bll_1 + cl_1^2,$$
$$C = am^2 + 2bmm_1 + cm_1^2,$$
$$B = alm + b\,(lm_1 + l_1m) + cl_1m_1,$$

and from which we obtained

$$AC - B^2 = (ac - b^2)(lm_1 - l_1m)^2.$$

The invariant in this case, $ac - b^2$, is no other than the discriminant of the given quadratic $ax^2 + 2bxy + cy^2$.

22. Proceeding now to the binary cubic $(a, b, c, d\between x, y)^3$, we obtain its discriminant; that is, we find its two differentials, and by direct elimination their eliminant, which is the discriminant, viz.,

$$4\,(bd - c^2)(b^2 - ac) + (ad - bc)^2,$$

which, in form, is the same as that obtained in Art. 7. And here, again, we say that the invariant in this case is no other than the discriminant.

23. Since $ac-b^2$ is an invariant of the quadratic

$$ax^2+2bxy+cy^2,$$

we can, by the introduction of a constant, derive not only two invariants after the analogy of $ac-b^2$, but one other whose constituents are derived from the coefficients of the transformed system.

Thus, if we have the two quadratics,

$$sx^2+2txy+uy^2,$$
$$s_1x^2+2t_1xy+u_1y^2,$$

we may multiply the second by k, an arbitrary constant, and obtain by addition

$$(s+ks_1)\,x^2+2\,(t+kt_1)\,xy+(u+ku_1)\,y^2,$$

which, by a transformation identical with that in Art. 21, becomes

$$(S+kS_1)\,X^2+2\,(T+kT_1)\,XY+(U+kU_1)\,Y^2;$$

and the invariant, consequently, by symmetry, is

$$(S+kS_1)(U+kU_1)-(T+kT_1)^2$$
$$=\Delta^2\left[(s+ks_1)(u+ku_1)-(t+kt_1)^2\right].$$

Since this equation is satisfied by any value of k, and therefore identical, the coefficients of the like powers of k must be equal, and therefore

$$SU-T^2=\Delta^2\,(su-t^2),$$
$$S_1U_1-T_1^2=\Delta^2\,(s_1u_1-t_1^2),$$
$$SU_1+S_1U-2TT_1=\Delta^2\,(su_1+s_1u-2tt_1).$$

The last of these is therefore an invariant of a system of two quantics.

And here it would be well to observe that, if we had operated upon the given invariant $su-t^2$ with $s_1\dfrac{d}{ds}+t_1\dfrac{d}{dt}+u_1\dfrac{d}{du}$, the result would have been the same as that actually obtained by the substitution of $s+ks_1$ for s, &c.; and thus, in general, if we have an invariant of any known quantic, we may find the invariant of a system of two or more simultaneous quantics of the same degree, either by substitution or by the use of the operator, as above.

24. As we have stated, the binary quadratic and cubic have no other invariant than their respective determinants; but, as we shall see, binaries of a higher degree may have two or more functions unaltered by transformation. For instance, if we take the binary quartic

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and operate upon it with the symbols* $\frac{d}{dy}$ and $-\frac{d}{dx}$, that is, substitute $\frac{d}{dy}$ for x and $-\frac{d}{dx}$ for y; we shall find that the result

$$ae - 4bd + 3c^2$$

will conform to the definition of an invariant. The same will be true if we expand the determinant formed from the fourth differentials of the quartic, viz.,

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace + 2bcd - ad^2 - eb^2 - c^3 \ldots\ldots\ldots (1).$$

That these two invariants may be derived from the binary quartic, may be shown by actually transforming the given

* The theory of these symbols must be reserved for another part of the subject; see Arts. 28, 33. The actual process is to introduce the symbols into the quantic, thus obtaining a differential symbol. Thus, by substituting $-\frac{d}{dx}$ for y, and $\frac{d}{dy}$ for x in the given quantic, we have

$$a\frac{d^4}{dy^4} - 4b\frac{d^4}{dy^3dx} + 6c\frac{d^4}{dy^2dx^2} - 4d\frac{d^4}{dy\,dx^3} + e\frac{d^4}{dx^4}.$$

Operating upon the quartic with this symbol, we get

$$48ae - 192bd + 144c^2.$$

Hence the relation is $\quad ae - 4bd + 3c^2.$

If the quantic had been of odd degree, the result would have vanished.

quartic, equating the values of A, B, C, &c., and we should find

$$AE-4BD+3C^2=\Delta^4(ae-4bd+3c^2),\ *$$

and also another, (1), which entering into the discriminant forms still a third. These latter, however, do not at present concern us beyond the assurance that they exist, the theory of their formation being reserved for future consideration.

25. The Theory of Covariants will be found to be immediately connected with the Theory of Invariants. This follows from the fact that the covariant is a function not only of the coefficients, but also of the variables of the given quantic; that is,

$$f(S,\ U,\ \&c.\ X,\ Y,\ \&c.)=\Delta^m f(s,\ u,\ \&c.\ x,\ y,\ \&c.)$$

To illustrate this, let us take the Hessian (D. 40) of the quartic

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4 \ \ldots\ldots\ldots\ldots\ (1),$$

and we have

$$\begin{vmatrix} ax^2+2bxy+cy^2, & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2, & cx^2+2dxy+ey^2 \end{vmatrix} \ \ldots\ldots\ldots\ (2),$$

which, expanded, gives the invariant form $mn-l^2$, and differs in form only from the invariant of the quadratic (Art. 23) $ac-b^2$ by the variables of the given quantic.

Looking at this example a little further, we see that (1) and (2) contain the same powers of the variables, and equally the same coefficients. Hence the invariant of the covariant in this case can be no other than the invariant of (1), and this conclusion may easily be seen to be general.

26. Covariants may be formed by substituting in the given quantic $x+kx_1$ and $y+ky_1$ for x and y. The coefficients of the several powers of k form covariants, and, taken in order, are called emanants of the quantic. Thus, if we take the binary cubic

$$ax^3+3bx^2y+3cxy^2+dy^3,$$

* To transform this quartic, ax^4 &c., the reader has only to repeat the process of Art. 21 on a larger scale.

and substitute as proposed, we shall find the several emanants to be $\left(x_1 \frac{d}{dx} + y_1 \frac{d}{dy}\right)^n$, in which form the coefficients of the ascending powers of k appear.

The emanants of any quantic can in general be expressed in the form

$$\left(x_1 \frac{du}{dx} + y_1 \frac{du}{dy}\right)^{1, 2 \dots n} \dots\dots\dots\dots\dots\dots\dots (1)$$

as first, second, and n^{th} emanants.

If we take the second power, to get the second emanant, we may write

$$x_1^2 m^2 + 2x_1 y_1 mn + y_1^2 n^2 \dots\dots\dots\dots\dots\dots\dots (2)$$

as the result, where m and n represent the differentials $\frac{du}{dx}$, $\frac{du}{dy}$, and x_1, y_1 are regarded as cogredient to,* or vary as, the given variables. Now it is easy to see that, if we regard (1) or (2) as a function of x_1, y_1, and the original variables as constants, and proceed to form the invariants, these invariants will in turn represent covariants of the given quantic, if we then conceive x, y as variables.

This will appear at once by reference to Art. 23, where we were enabled, after transforming the quadratic, to write

$$SU - T^2 = \Delta^2 (su - t^2).$$

Transform now (2), and write its invariant, and we shall have

$$\frac{d^2V}{dX^2} \cdot \frac{d^2V}{dY^2} - \left(\frac{d^2V}{dX\,dY}\right)^2 = \Delta^2 \left[\frac{d^2v}{dx^2} \cdot \frac{d^2v}{dy^2} - \left(\frac{d^2v}{dx\,dy}\right)^2\right] \dots (3),$$

where $V =$ the transformed, and $v =$ the original quantic.

* To exhibit this, let $(abc \between x, y)^2$ be the given quantic.

Making the substitutions, we have

$$k^2 (ax_1^2 + 2bx_1y_1 + cy_1^2) \text{ and } 2k \left\{ axx_1 + b(xy_1 + x_1y) + cyy_1 \right\}$$

as first and second emanants; but, by hypothesis, x_1, y_1 are cogredient to xy; hence each of the coefficients above resume the quadratic form, in other words, they become identical. Hence there are not, as we shall see, two covariants to the quadratic $(abc \between x, y)^2$, but one (as there is no function of the differences of the roots), and that one must be the quantic itself.

This invariant is now the covariant of the given quantic, as is evident algebraically if we compare (2) with the form for the second emanant,

$$\left(x_1 \frac{dv}{dx} + y_1 \frac{dv}{dy}\right)^2,$$

in which x, y are first treated as constants, by which supposition we form the invariant, and then as variables, so that the result conforms to the definition of an invariant. It is plain also that (3) is the expanded determinant formed from the quadratic emanant, and in this sense may be regarded as a discriminant of the quadratic function, and is therefore an invariant with the limitation that its variables are regarded as constants. The first emanants of a system of linear equations yield a determinant. Hence in general we may say that the Jacobian (D. 39) of the first emanants of a system of linear equations—that is, the first differentials of these equations regarded as functions of x_1, y_1, z_1—will form a determinant which is a covariant of the system.

27. *Inverse Substitution.*—In Trilinear Coordinates α, β, γ are used ordinarily to express the coordinates of the point. Let now

$$x\alpha + y\beta + z\gamma = 0$$

be the equation of a straight line in which α, β, γ are the tangential coordinates of the line, that is, its perpendicular distances from the three points of reference; and x, y, z the perpendicular distances of any point in the line from the three lines of reference, that is, its trilinear coordinates (T. 2, 25).*

By transforming this equation to new axes by linear substitution, it will be seen that, while the trilinear coordinates are transformed by direct substitution, the tangential coordinates are transformed at the same time by the inverse substitution.

Let

$$x = l_1 X + m_1 Y + n_1 Z,$$
$$y = l_2 X + \&c.,$$
$$z = l_3 X + \&c.$$

* Tract No. II., Art. 25.

Then the new equation of the line may be written

$$AX+BY+\Gamma Z = 0 \qquad (1),$$

where
$$\begin{aligned} A &= l_1\alpha + l_2\beta + l_3\gamma, \\ B &= m_1\alpha + m_2\beta + m_3\gamma, \\ \Gamma &= n_1\alpha + n_2\beta + n_3\gamma. \end{aligned}$$

The two sets of coordinates in this case are said to be contragredient to each other; and in general it may be stated that the tangential coordinates, whether of a line or a plane, will be transformed by a different—that is, inverse—substitution, from the coordinates representing different points. The latter are said to be cogredient, as x, x_1, y, y_1, &c., because transformed by the same substitution; while the former are said to be contragredient, because tranformed by an inverse substitution.

28. This may be stated in another form; for, since the equation which was a function of x, y, z has been transformed to a function of X, Y, Z, the total differential coefficients with respect to the latter are functions of those with respect to the former.

We have, from (1) of the last Article,

$$\alpha(l_1X+m_1Y+n_1Z) + \beta(l_2X+m_2Y+n_2Z) + \gamma(l_3X+m_3Y+n_3Z) = 0;$$

and therefore
$$\frac{d}{dX} = l_1\frac{d}{dx} + l_2\frac{d}{dy} + l_3\frac{d}{dz},$$

since
$$\frac{dx}{dX} = l_1, \text{ \&c.}$$

Comparing x, y, z with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, we see that the substitution which linearly transforms the one will linearly transform the other, but by a reciprocal relation as expressed by the determinants of the coefficients

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \qquad \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

If now $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ vanish, as will happen only when the discriminant of the quantic or system vanishes, then $\frac{d}{dX}$ will necessarily vanish also.

29. The consideration of inverse substitution leads directly to a function well known in geometry as the *contravariant.* For it will be seen at once that, if a quantic which is a function of two sets of variables x, y, z; α, β, γ, be linearly transformed, the function involving the coefficients and the variables, regarded as transformed by the inverse substitution, must be similar to the covariant, but which is called the contravariant; that is,

$$f(A_0, A_1, \text{\&c. } \mathrm{A}, \mathrm{B}, \Gamma) = \Delta^m f(a_0, a_1, \text{\&c. } \alpha, \beta, \gamma).$$

It is evident also, from what has preceded, that the contravariant may be deduced in a manner similar to that exhibited in Art. 23. If we take the binary quadratic,

$$ax^2 + 2bxy + cy^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1),$$

and combine it with $\qquad k(x\alpha + y\beta)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$

we shall have

$$(a + k\alpha^2)\,x^2 + 2\,(b + k\alpha\beta)\,xy + (c + k\beta^2)\,y^2 \;\ldots\ldots\ldots (3).$$

If now (1) becomes, by linear transformation,

$$AX^2 + 2BXY + CY^2,$$

and (2) becomes $\qquad k(X\mathrm{A} + Y\mathrm{B})^2,$

then (3) becomes

$$(A + k\mathrm{A}^2)\,X^2 + 2\,(B + k\mathrm{AB})\,XY + (C + k\mathrm{B}^2)\,Y^2,$$

and the invariant form gives

$$(A+k\text{A}^2)(C+k\text{B}^2)-(B+k\text{AB})^2$$
$$=\Delta^2[(a+k\alpha^2)(c+k\beta^2)-(b+k\alpha\beta)^2].$$

And since we may equate the coefficients of the like powers of k, we have

$$\text{AB}^2-2B\text{AB}+C\text{A}^2=\Delta^2(a\beta^2-2b\alpha\beta+c\alpha^2)\ \ldots\ldots\ldots(1),$$

that is, $a\beta^2-2b\alpha\beta+c\alpha^2$, differing only by a power of the modulus from the corresponding function of the transformed coefficients and variables α, β, is a contravariant.

In reviewing now three functions thus considered, it will be seen that they all equally possess the property of invariance.

30. In general, when α, β, γ are regarded as contragredient to x, y, z, the contravariant may be expressed, by application of the preceding Article,

$$A_0X^n+\&\text{c.}+k(X\text{A}+Y\text{B}+Z\Gamma)^n$$
$$=a_0x^n+\&\text{c.}+k(x\alpha+y\beta+z\gamma)^n,$$

and the invariant would be

$$f(A_0+k\text{A}^n,\ A_1+k\text{A}^{n-1}\text{B},\ \&\text{c.})=\Delta^m f(a_0+k\alpha^n,\ a_1+k\alpha^{n-1}\beta,\ \&\text{c.})$$

We have thus to develop the sum of two functions, which, by Taylor's Theorem, gives us for the coefficients of the constant k,

$$\left(\alpha^n\frac{d}{da_0}+\alpha^{n-1}\beta\frac{d}{da_1}+\alpha^{n-1}\gamma\frac{d}{db_1}+\ldots\ldots\right)^r P.*$$

If $r=1$, this formula gives us what has been called the first evectant.

* $P=$ the invariant of the quantic.

To apply this, we know that $ac-b^2$ is the invariant of the quadratic. We have then

$$\Delta^2\left(\alpha^2\frac{d}{da}+\alpha\beta\frac{d}{db}+\beta^2\frac{d}{dc}\right)(ac-b^2)$$
$$=\left(A^2\frac{d}{dA}+AB\frac{d}{dB}+B^2\frac{d}{dC}\right)(AC-B^2),$$

which is identical with (1) of the last Article.

If the quantic be a ternary quadratic, as

$$a_0x^2+a_1y^2+a_2z^2+2a_3yz+2a_4xz+2a_5xy,$$

we shall have for an invariant

$$\begin{vmatrix} a_0 & a_5 & a_4 \\ a_5 & a_1 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix} = a_0a_1a_2+2a_3a_4a_5-a_0a_3^2-a_1a_4^2-a_2a_5^2=P,$$

which is the discriminant of the quantic; whence

$$\left(\alpha^2\frac{d}{da_0}+\beta^2\frac{d}{da_1}+\gamma^2\frac{d}{da_2}+\beta\gamma\frac{d}{da_3}+\alpha\gamma\frac{d}{da_4}+\alpha\beta\frac{d}{da_5}\right)P$$
$$=(a_1a_2-a_3^2)\,\alpha^2+(a_0a_2-a_4^2)\,\beta^2+(a_0a_1-a_5^2)\,\gamma^2$$
$$+2\,(a_4a_5-a_0a_3)\,\beta\gamma+2\,(a_3a_5-a_1a_4)\,\alpha\gamma+2\,(a_3a_4-a_2a_5)\,\alpha\beta$$

is a contravariant, and in geometry expresses the condition that a given line represented by a trilinear equation shall touch the given conic, or, in other words, is the tangential equation of the conic.*

It is to be observed that the discriminant of the first evectant of the second degree can be written as a determinant:

$$\frac{dE}{d\alpha}=0,$$

$$\frac{dE}{d\beta}=0,$$

$$\frac{dE}{d\gamma}=0,$$

* Salmon's "Conics," p. 249.

and may be regarded as the invariant of a system, or of the given contravariant.

It is also to be observed that there are instances of functions involving both x, y, z and α, β, γ, and which do not change by transformation of the quantic, that is,

$$f(A_0, A_1 \ldots X, Y \ldots A, B \ldots) = \Delta^m f(a_0, a_1 \ldots x, y \ldots \alpha, \beta \ldots),$$

which have received the general name of *mixed concomitants.*

That such functions may easily be formed may be seen by examining the covariant (1) of Art. 26.

If we subject it to the operation of finding the coefficients by Taylor's Theorem, we shall have $\left(\alpha^2 \frac{d}{da} + \alpha\beta \frac{d}{db} + \beta^2 \frac{d}{dc}\right) C$, where $C =$ the covariant in question.

31. We may here perhaps interest the reader by introducing an illustration of the geometrical application of invariants. It is well known that when we transform from one rectangular system to another, that $a+b$ and $ab-h^2$ remain unaltered by the transformation. Suppose it were inquired as to the form these quantities take when the transformation is made from rectangular (or oblique) to oblique axes, where a, b, h are constants in the quadratic

$$ax^2 + 2hxy + by^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let the transformation be from axes inclined at the angle ω to axes of any other inclination, as Ω. Then, by making the proper substitutions, (1) becomes

$$AX^2 + 2HXY + BY^2 \text{ (Art. 21).}$$

By symmetry, $x^2 + 2xy \cos\omega + y^2$ would become

$$X^2 + 2XY \cos\Omega + Y^2,$$

as either expresses the square of any point from the origin. Adopting now a method with which we are familiar, we say

$$ax^2 + 2hxy + by^2 + k(x^2 + 2xy \cos\omega + y^2)$$
$$= AX^2 + 2HXY + BY^2 + k(X^2 + 2XY \cos\Omega + Y^2).$$

If we determine k so that the first side of the equation may become a perfect square, the second will become a perfect square also, that is, k must be one of the roots of

$$k^2 \sin^2 \omega + (a+b-2h\cos\omega)\,k + ab - h^2 = 0.$$

This value of k will make the left member a perfect square. A similar quadratic will be found in the right-hand member, which will make it also a perfect square. Both members become perfect squares for the same value of k, and are therefore equal.

Equating coefficients of corresponding terms, we have, what we already knew (Art. 5) in form,

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{A+B-2H\cos\Omega}{\sin^2\Omega},$$

$$\frac{ab-h^2}{\sin^2\omega} = \frac{AB-H^2}{\sin^2\Omega}.$$

(This elegant demonstration is due to the late Dr. Geo. Boole. See *Cambridge Math. Jour.*, N. S., VI. 87.)

32. From Art. 28, we learn that x, y, z; α, β, γ sustain a reciprocal relation to each other. The same is to be observed of x, y, z and $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$. The transformation of the former transforms the latter, but by an inverse method. In this way the contravariant is obtained, which, as has been remarked, possesses the property of invariance. If now in the contravariant we substitute $\frac{d}{dx}$, &c., we shall obtain a function which, containing signs of operation, and being itself unchanged by transformation, may be called an operating symbol—a type form which, if applied to the quantic or to its covariants, must give either an invariant or a covariant according as the variables disappear or remain after differentiation.

$c\alpha^2 - 2b\alpha\beta + a\beta^2$ being a contravariant of $ax^2 + 2bxy + cy^2$, we obtain, by applying to the quadratic the operating symbol $c\frac{d^2}{dx^2} - 2b\frac{d^2}{dx\,dy} + a\frac{d^2}{dy^2}$, the invariant $ac-b^2$.

33. Proceeding upon the principle now before us, we are enabled to generate, as will be seen, the three functions considered, by means of simple substitution. Since $f(\beta, -\alpha)$ becomes, by a linear transformation, $\frac{f(B, -A)}{\Delta}$, that is, a contravariant, we have then, in a binary quadratic, only to substitute β and $-\alpha$ for x and y to obtain the contravariant.

If, therefore, we write α with the negative sign, there is no reason why we should not say that $-\alpha$, β are transformed by the same rules as x, y. The symbols $\frac{d}{dx}$, &c., which we regarded as contragredient to x, y, may be with equal reason called cogredient to $-x$, y; and, conversely, $\frac{d}{dy}$, $-\frac{d}{dx}$ may be taken as cogredient to x, y. Hence, if we substitute these symbols in either the quantic or its covariants, we obtain a new set of functions of the same form. The exception is seen in the binary quartics, where, for instance in the quadratic, the substitution gives $4(ac-b^2)$, an invariant.

34. A fuller investigation of the quadratic, in the general theory, will lead to what is perhaps already sufficiently evident, that the quadratic $(a, b, c \between x, y)^2$ has no covariant but the quantic itself.* We have seen that its discriminant is the invariant $ac-b^2$, and its contravariant $c\alpha^2 - 2b\alpha\beta + a\beta^2$; and since $ac-b^2$ is an invariant, we learn, from Art. 26, that the second emanant is a quadratic in x_1, y_1, and its discriminant is a covariant, for a quantic higher than the second degree. We know (Art. 22) that the invariant of

$$(a, b, c, d \between x, y)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is

$$a^2d^2 + 4ac^3 - 6abcd + 4db^3 - 3b^2c^2 \ldots\ldots\ldots\ldots\ldots (2).$$

* An invariant being a function of the differences of roots, there can be no such function formed other than the given quantic. See Note, p. 24.

Hence for every quantic higher than the third we have the covariant

$$\left[\left(\frac{d^3}{dx^3}\frac{d^3}{dy}\right)^2 + 4\frac{d^3}{dx^3}\left(\frac{d^3}{dx\,dy}\right)^3 - 6\frac{d^3}{dx^3}\frac{d^3}{dx^2\,dy}\frac{d^3}{dx\,dy^2}\frac{d^3}{dy^3} + 4\frac{d^3}{dy^3}\frac{d^3}{dx^2\,dy} - 3\left(\frac{d^3}{dx^2\,dy}\frac{d^3}{dx\,dy^2}\right)^2\right]u.$$

The covariant of (1) may be found by forming the evectant (Art. 30)

$$\left(\alpha^3\frac{d}{da} + \alpha^2\beta\frac{d}{db} + \alpha\beta^2\frac{d}{dc} + \beta^3\frac{d}{dd}\right)P,$$

where $P = (2)$.

Then, by substituting x, y for α, β, we have

$$x^3(ad^2-3bcd+2c^3) + 3x^2y(-acd+2b^2d-bc^2) + 3xy^2(-abd+2ac^2-b^2c) + y^3(a^2d-3abc+2b^3).$$

And thus generally for binaries, when any invariant is known.

35. If we take any quantic, and observe the effect of any linear substitution, it is easy to see that its invariant will remain unchanged if for x we substitute y or lx, and y for x. It will be seen that the order or degree of the invariant is still constant, and also that the weight, which is estimated by taking the sum of the suffixes of the factors of the several terms, is constant for each invariant.

If s, s_1, s_2, &c. represent the suffixes before transformation, $n-s$, $n-s_1$, $n-s_2$ &c. will represent the suffixes of the same coefficients after transformation, and we shall have

$$s+s_1+s_2\ \&c. = n-s+n-s_1+n-s_2\ \&c.,$$

or

$$2w = nt,$$

where $w =$ the weight of the suffixes for each term of the coefficients, and $t =$ the degree or order of the invariant. In other words, the weight is $= \frac{1}{2}nt$.

In this way the invariant of any quantic may be written at once, the required degree being known.

If, for instance, an invariant of a binary quartic of the second degree in the coefficients is required, we have

$$w = \tfrac{1}{2}nt = 4.$$

There will be as many terms of the proposed invariant as the sum of two numbers 0 ... 4 inclusive can be written; hence

$$A_0 a_0 a_4 + A_1 a_2 a_2 + A_2 a_1 a_3 \text{ .... } (1)$$

is the required invariant. The values of A_0, &c. will depend upon other considerations. The first is, that an invariant must be a function of the differences of the roots; for it is to be unchanged when we effect the transformation by substituting $x+l$ for x; it must therefore satisfy a differential equation for the function of the differences of the roots, as

$$a_0 \frac{dP}{da_1} + 2a_1 \frac{dP}{da_2} + 3a_2 \frac{dP}{da_3} + 4a_3 \frac{dP}{da_4} + \text{\&c.} = 0 \text{ ...... } (2).$$

The second consideration is, that the coefficients thus obtained are clearly proportional.

Applying then (2) to (1), we have

$$(A_2 + 4A_0)\, a_0 a_3 + (4A_1 + 3A_2)\, a_1 a_2 = 0.$$

Taking $A_0 = 1$, we find the invariant to be

$$a_0 a_4 - 4a_1 a_3 + 3a_2^2;$$

or, using the coefficients of the quartic,

$$ae - 4bd + 3c^2.$$

Thus the differential equation furnishes the conditions to determine the values of A_0, &c.

If the number of conditions is greater than these coefficients, there is no additional invariant; if the same, one more, or one alone; if less, more than one. If we wished to obtain the discriminant of the quartic, which is also an invariant, by this method, or rather if we wished to obtain an invariant of the sixth order in the coefficients, we should find the number of ways in which 12 can be written as the sum of 6 numbers from $0 \dots 4$, and we should have as many conditions as $\frac{1}{2}nt-1$ or 11 can be written as the sum of 6 numbers from $0 \dots 4$.

36. We may arrive at the covariant in the same manner.

If n represent the degree of the quantic, n_1 the degree of the covariant, in x and y, and m the degree of x in any term, we have

$$m+s+s_1+s_2 \,\&c. = n_1-m+n-s+n-s_1+n-s_2 \,\&c.$$

Calling $m+s+s_1$ &c. the weight, the equation gives

$$w = \tfrac{1}{2}(nt+n_1).$$

If we wished to form, for instance, the quartic covariant to the quartic of the second degree in the coefficients, we could, instead of taking the Hessian of the quantic, which would give the required covariant, estimate the terms multiplying each variable, since $t=2$, $n=4$, $m=4$, and, if we are concerned with the coefficient of x^4, $n_1=4$.

The weight would then be 6, and hence

$$4+s+s_1=6,$$
$$s+s_1=2.$$

There are therefore two terms multiplying x^4 each of the second degree, that is, a_0a_2 and a_1a_1, or ac and b^2.

In the same manner we find, for the terms which multiply x^3,

$$3+s+s_1=6.$$

Hence the terms are a_0a_3 and a_1a_2, or ad and bc, &c. &c.

Now it will be perceived we do not know how by this process to connect ac and b^2, ad and bc.

To ascertain this relation, let us suppose that

$$A_0x^{n_1}+n_1A_1x^{n_1-1}y+\frac{n_1(n_1-1)}{2}A_2x^{n_1-2}y^2+\&c. \;\dots\dots\; (1)$$

represents the covariant.

Suppose also

$$a_0\frac{d}{da_1}+a_1\frac{d}{da_2}\text{ \&c. and } na_1\frac{d}{da_0}+(n-1)\,a_2\frac{d}{da_1}\text{ \&c.}$$

to be represented by α and β. If now, in (1), we suppose the same substitution as was made in the original quantic, and that

$$\frac{da_0}{d\alpha}=0,\quad \frac{da_1}{d\alpha}=a_0,\quad \frac{da_2}{d\alpha}=2a_1,\quad \frac{da_3}{d\alpha}=3a_2,$$

then
$$\frac{df}{d\alpha}=a_0\frac{df}{da_1}+2a_1\frac{df}{da_2}+3a_2\frac{df}{da_3}+\text{\&c.};$$

and we can write, on the supposition that these changes are identical,

$$\frac{dA_0}{d\alpha}=0,\text{ \&c. as above,}$$

and, for the same reason,

$$\frac{dA_0}{d\beta}=n_1A_1,\text{ \&c.}$$

Thus, when A_0 is a function of the differences, we can find all the other terms of the covariant; that is, we can, by successive differentiation, pass from one term to the other, and thus, by the use of these two operators, determine the exact form of the coefficients of the covariant. Thus, in the case of the quadratic covariant to the quartic, we found A_0 to be of the form $Aa_0a_2+Ba_1a_1$, which, operated upon by $\frac{dA_0}{d\alpha}$, becomes $(A+2B)a_0a_1=0$. If $A=1$, then $B=-1$, and $A_0=a_0a_2-a_1a_1$. Operate upon this latter with $\frac{dA_0}{d\beta}$, which in this case is

$$4a_1\frac{d}{da_0}+3a_2\frac{d}{da_1}+2a_3\frac{d}{da_2}+a_4\frac{d}{da_3},$$

and we get
$$2\,(a_0a_3-a_2a_1)=A_1.$$

Again, operating with $\frac{dA_1}{d\beta}$ upon $a_0a_3-a_2a_1$, and we have

$$4a_1a_3+a_4a_0-2a_3a_1-3a_2a_2=A_2=a_4a_0+2a_1a_3-3a_2a_2.$$

Operating then upon this latter with $\frac{dA_2}{d\beta}$, we obtain

$$2\,(a_1 a_4 - a_2 a_3) = A_3.$$

And finally we have

$$\frac{dA_3}{d\beta} = a_2 a_4 - a_3 a_3 = A_4.$$

The covariant then, written fully, is

$$(ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3y + (ae + 2bd - 3c^2)\,x^2y^2 \\ + 2\,(be - cd)\,xy^3 + (ce - d^2)\,y^4.$$

We see, therefore, that A_0 is the source of the covariant, and we can readily write

$$A_0\,x^{n_1} + \frac{dA_0}{d\beta}\,x^{n_1-1}y + \frac{d^2A_0}{d\beta^2}\,\frac{x^{n_1-2}\,y^2}{1\,.\,2} + \frac{d^3A_0}{d\beta^3}\,\frac{x^{n_1-3}\,y^3}{1\,.\,2\,.\,3} + \&c.$$

as the law of derivation.

That A_0 is appropriately called the source is evident from its repeated use, being, in fact, operated upon by each successive differential symbol, as is seen on p. 36.

CHAPTER III.

THEORY OF LEAST OR CANONICAL FORMS.

37. WHEN a quantic has been reduced to the least form in which it can be written, and yet retain its generality, it is said to be reduced to its *canonical form.* The theory has been presented by Dr. Sylvester (see *Philosophical Magazine*, Nov. 1851). The name canonical seems to have been first applied by Hermite. The number of constants remains in most cases implicitly the same.

Since $lx+my$ may be represented by X, and $l'x+m'y$ by Y, a cubic in two variables may be represented by X^3+Y^3. This is evident, as the entire number of constants is implied in this form.

The quadratic $(a, b, c \between x, y)^2$ can be reduced with four constants* to the form x^2+y^2, or to a similar form Az^2+Bt^2 containing the original number. But the binary quadratic in geometrical investigations is so completely manageable in its

* To reduce $2x^2+14x+29$ to the sum of two squares.

We have $$(lx+my)^2+(l'x+m'y)^2$$
as the transformed quadratic, or
$$(x+t)^2+(x+t')^2 = 2x^2+14x+29,$$
where $$t = \frac{m}{l}, \text{ and } t' = \frac{m'}{l'},$$
whence $$t^2+t'^2 = 29,$$
and $$t+t' = 7 \text{ or } t=2, \; t'=5,$$
while the coefficient of x is plainly 1, therefore $(x+2)^2+(x+5)^2$ is the expression.

original form that its reduction to a sum of squares is not a matter of much interest.

But the reduction of the cubic is of more practical importance, since, independent of geometrical considerations, the reduction to the sum of two cubes furnishes a method of solution of numerical equations. The cubic

$$ax^3+3bx^2y+3cxy^2+dy^3$$

becomes, we will suppose, by transformation

$$AX^3+3BX^2Y+3CXY^2+DY^3,$$

and, remembering that the Hessian

$$\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dx\,dy}\right)^2$$

gives a covariant which may be transformed in the same manner and into a function of the same constants as before, that is,

$$\Delta^2\left[\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dx\,dy}\right)^2\right]=\frac{d^2U}{dX^2}\cdot\frac{d^2U}{dY^2}-\left(\frac{d^2U}{dX\,dY}\right)^2 \quad \ldots\ldots(1),$$

we see that the transformed becomes $ADXY$ when B and C vanish.

Or, since we are simply seeking the factors into which the Hessian may break up when B and C vanish, we may omit the factor Δ^2 (being composed of the constants of transformation), and examine the left-hand member of (1) for the required factors X, Y.

With these conditions, the Hessian cannot differ by more than a factor from XY.

As an illustration, let us take

$$4x^3+30x^2+78x+70=0=u.$$

The Hessian is

$$\begin{vmatrix} 2x+5 & 5x+13 \\ 5x+13 & 13x+35 \end{vmatrix} = x^2+5x+6.$$

Taking the factors of this, $x+2$ and $x+3$, we have

$$A(x+2)^3+D(x+3)^3$$

for the determination of A and D by comparison with the given quantic

$$A+\quad D=4,$$
$$8A+27D=70,$$

or
$$A=2,\quad D=2.$$

Hence
$$2(x+2)^3+2(x+3)^3=u,$$

that is, $(x+2)^3+(x+3)^3$ differs by only a factor from u, and therefore

$$(x+2)+(x+3)=0$$

gives $x=-\frac{5}{2}$ as a root of the given cubic. The other roots of this cubic being imaginary, it is evident that not every cubic can be reduced to this form, since it must differ from one which contains three real factors, or one containing a square factor.

In the latter case, we could evidently express the canonical form of the given cubic by $(lx+my)^2(l'x+m'y)$ or $(x+t)^2(x+t')$ or x^2y.

To reduce $x^3+7x^2+16x+12$ to the form x^2y.

We have
$$x^3+(t'+2t)x^2+(2tt'+t^2)x+t^2t',$$

whence
$$t'+2t=7,$$
$$2tt'+t^2=16,$$
$$t^2t'=12,$$

$\therefore$
$$t=2,\quad t'=3.$$

38. *The canonizant.*—This is a name given by Dr. Sylvester to a determinant which is used in the extension of the method of the last Article. The theory assumes that a quantic of the

fifth degree can be reduced to a sum of three terms of the fifth degree, one of the seventh degree to the sum of four terms of the seventh degree, and thus for every odd degree; and then proceeds to make the assumption good in the following manner. The transformation is supposed to be effected, as before, by letting

$$s = lx + my, \quad t = l'x + m'y, \quad v = l''x + m''y.$$

The theorem then requires that

$$(a, b, c, d, e, f \between x, y)^5 = s^5 + t^5 + v^5.$$

Since the right-hand member of this equation contains implicitly as many constants as the given quantic, it must be capable of expressing that quantic when s, t, v have been properly determined.

Let $u =$ the left-hand member, and U the right-hand member of the above equation; then, by successive differentiation, we shall have

$$U \begin{vmatrix} \frac{d^4}{dx^4} & \frac{d^4}{dx^3\,dy} & \frac{d^4}{dx^2\,dy^2} \\ \frac{d^4}{dx^3\,dy} & \frac{d^4}{dx^2\,dy^2} & \frac{d^4}{dx\,dy^3} \\ \frac{d^4}{dx^2\,dy^2} & \frac{d^4}{dx\,dy^3} & \frac{d^4}{dy^4} \end{vmatrix} = U \begin{vmatrix} \frac{d^4}{dx^4} & \cdot & \cdot \\ \cdot & \frac{d^4}{dx^2\,dy^2} & \cdot \\ \cdot & \cdot & \frac{d^4}{dy^4} \end{vmatrix},$$

or the symmetrical determinants,

$$\begin{vmatrix} ax+by & bx+cy & cx+dy \\ bx+cy & cx+dy & dx+ey \\ cx+dy & dx+ey & ex+fy \end{vmatrix} = \begin{vmatrix} l^2s & l'^2t & l''^2v \\ lms & l'm't & l''m''v \\ m^2s & m'^2t & m''^2v \end{vmatrix}$$

$$\times \begin{vmatrix} l^2 & l'^2 & l''^2 \\ lm & l'm' & l''m'' \\ m^2 & m'^2 & m''^2 \end{vmatrix} = s\,.\,t\,.\,v \begin{vmatrix} l & l' \\ m & m' \end{vmatrix}^2 . \begin{vmatrix} l' & l'' \\ m' & m'' \end{vmatrix}^2 . \begin{vmatrix} l'' & l \\ m'' & m \end{vmatrix}^2 .$$

That is, if the expansion of

$$\begin{vmatrix} ax+by & . & . \\ . & cx+dy & . \\ . & . & ex+fy \end{vmatrix}$$

yields the factors s, t, v, then these factors will differ from the factors $(x+ty)$, $(x+t'y)$, $(x+t''y)$ by only numerical coefficients; and, consequently,

$$(a, b, c, d, e, f \between x, y)^5 = T(x+ty)^5 + T'(x+t'y)^5 + T''(x+t''y)^5.$$

39. In the same manner, to find the condition that a quantic of even degree can be reduced to the sum of $\frac{n}{2}$ n^{th} powers, where n is even.

The nature of this condition is seen from the last Article. The determinant formed from the n differentials will, on the supposition that the quantic can be reduced to the sum of $\frac{n}{2}$ n^{th} powers, vanish by the same process which proved that a quantic of odd degree, as for instance the fifth, could be reduced to a sum of three powers of the same degree. The determinant formed from the n differentials in the latter case being a covariant, gave the necessary factors s, t, v, while, in the case now under consideration, the proposed determinant, it will be seen, gives an invariant whose vanishing proves that the quantic can be reduced to a sum of powers each of the n^{th} degree.

To see if $2x^4+12x^3+30x^2+36x+17 = S$ can be reduced to a sum of two fourth powers, we take the fourth differentials as in the last Article and we obtain the determinant

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{vmatrix} = 0.$$

The vanishing of this determinant shows that in this case the reduction is possible. To obtain the binomials, we equate like powers of S and s, v, in $S = s^4 + v^4$, and we find

$$s = x+1, \quad v = x+2.$$

40. It is hardly necessary to carry this proof into the higher powers. But it may be said, in general, that if the quantic does not break up into sums of powers of binomials, it will be sufficient to add to these powers some multiple of their product or product of their powers, as

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4 = s^4 + t^4 + 6Ds^2t^2,$$

and $$(a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6 = s^6 + t^6 + u^6 + Estu.$$

That these are the least or canonical forms may be seen by extending the proof already given. The subject in such form as developed by Sylvester and others would not be necessary here.

41. *Combinants.*—We have seen (Art. 20) that the eliminant of a system of linear equations is an invariant. An invariant or eliminant of a system of equations or quantics of a uniform degree higher than the first is called a *combinant.* One peculiarity of the combinant is that it satisfies the equation

$$\frac{a_1 dC}{da} + \frac{b_1 dC}{db} + \&\text{c.} = 0,$$

where C is the combinant of

$$a\,x^n + nb\,x^{n-1} + \&\text{c.} = 0 \dots\dots\dots\dots\dots\dots\dots (1),$$

$$a_1 x^n + nb_1 x^{n-1} + \&\text{c.} = 0 \dots\dots\dots\dots\dots\dots\dots (2).$$

$$\&\text{c.} \qquad \&\text{c.}$$

42. Another peculiarity to be observed is, that if a pair of quantics have a common factor, their Jacobian will contain this factor in the second degree.

Take the equations as above, and form their Jacobian, and the truth of this will be evident; or, let a be a common factor in

$$u = ax^2 + 3ay^2,$$
$$v = 2ax^2 + ay^2,$$

then
$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = -20a^2xy.$$

If, in (1) and (2), (Art. 41), $n = 2$, we shall have, for J,

$$\begin{vmatrix} ax + by & bx + cy \\ a_1x + b_1y & b_1x + c_1y \end{vmatrix} = J,$$

whose discriminant we find to be

$$4(ab_1 - a_1b)(bc_1 - b_1c) - (ac_1 - a_1c)^2.$$

This, as we have seen, is the eliminant of u and v as quadratics; or, in other words, we find that in this case, at least, the J of u and v contains their eliminant as a factor: and this is a truth to be observed when $n = 3, 4,$ &c., in which cases the discriminant of J, as is evident, will be composed of the eliminant of the quantics and some other factor whose form may be determined.

43. If (1) and (2) above be represented by u and v, then $u + hv$ represents a locus common to u and v; and, by assigning varying values to h, we shall obtain a system of quantics some of which will contain square factors; and in the involution of points formed by these quantics there will be as many double points as there are quantics which contain square factors. The number of these is seen to be $2(n-1)$, or is the same as the order in the coefficients of the discriminant. The number of double points may then be determined by the Jacobian of u and v. If $u + hv$ has a factor $(x-a)^2$, then a will satisfy $\frac{du}{dx} + h\frac{dv_1}{dx} = 0$, and $\frac{du}{dy} + h\frac{dv_1}{dy} = 0$. It will evidently satisfy the

equation obtained by eliminating h, and therefore the Jacobian of (1) and (2). We thus have an easy method of determining the number of double points resulting from the involution of these quantics. In this form we see that h can be so determined that $u+hv$ shall contain the square factor $(x-a)^2$; and, by adding another condition, we may determine the value of constants so that the quantic shall contain $(x-a)^3$. The coefficient in (1) and (2) will then be of the degree $3(n-2)$. Conversely, if $(x-a)^3$ exists as a factor in $u+hv+mt$, this factor in the first degree will exist in the three second differential coefficients, and consequently in their eliminant with respect to h and m; that is

$$\begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2v}{dx^2} & \frac{d^2t}{dx^2} \\ \frac{d^2u}{dx\,dy} & \frac{d^2v}{dx\,dy} & \frac{d^2t}{dx\,dy} \\ \frac{d^2u}{dy^2} & \frac{d^2v}{dy^2} & \frac{d^2t}{dy^2} \end{vmatrix} = 0,$$

which gives the number of triple points in the above system $u+hv+mt$; and, if $y=1$, it expresses the number of these points on the axis of x, or $3(n-2)$, which fulfils the condition of a combinant.

44. *Tact-invariant.*—When we find the eliminant of (1) and (2), and equate it to zero, we express the condition that the two curves may be tangent to each other. If we express also the existence of a cubic factor in any quantic of the series $u+hv$, that is, a cuspidal curve, by $U=0$ and by $V=0$, one having two double points, or two square factors, and by $W=0$, what has been described above as the tact-invariant; then the discriminant of $u+hv$ with respect to h will contain U, V, W as factors.

If u and v are tangent to each other, then the discriminant of $u+hv$ will, as a function of h, have a square factor; in other words, when expressed geometrically, it is the condition that a curve has a double point.

45. The tact-invariant is of the order $3n(n-1)$ in the coefficients. Hence, if we have three surfaces L, P, Q (of l, m, n degrees), the condition that two of the lmn points of intersection will coincide is called, in this case, the tact-invariant, and the coefficients of L are in the degree $mn(2l+n+m-4)$, and so of P and Q.

The tact-invariant of two surfaces aL and P have the coefficients of L in the degree $m(l^2+2lm+3m^2-4l-8m+6)$.* These results are obtained from quantics of four variables.

The geometrical importance of these results will be further seen.

46. As to *the number of invariants of a binary quantic*, we have already seen that a quadratic has one, that a cubic has one, each of these being the discriminant of the given quantic.

If we take the next in order, the quantic

$$(a, b, c, d, e \between x, y)^4,$$

we can easily determine the number of ordinary invariants, omitting from our enumeration those which are expressible as rational and integral functions of the same or lower degrees. Remembering that the invariant must satisfy the differential equation

$$a_0 \frac{dI}{da_1} + 2a_1 \frac{dI}{da_2} + 3a_2 \frac{dI}{da_3} + \&c. = 0,$$

and that the last invariant must be of the order 2 in the coefficients, it must therefore be of the weight 4 in the coefficients, that is,

$$Aa_4a_0 + Ba_3a_1 + Ca_2a_2.$$

Operating upon this with the differential equation, we have

$$4Aa_3a_0 + 3Ba_2a_1 + Ba_0a_3 + 4Ca_1a_2,$$

* Terquem's *Annales*, Vol. XIX., and *Quarterly Journal*, Vol. I.

which, taking A as 1, gives for B, -4, and for C, 3; and we have by substitution

$$a_4 a_0 - 4a_3 a_1 + 3a_2 a_2,$$

or

$$ae - 4bd + 3c^2,$$

as the invariant function, which is the same as would have resulted by actual transformation. Had we followed the latter method, we should have found that the function of the new would be equal to the old when multiplied by the fourth power of the modulus, $(lm' - l'm)^4$ or Δ^4, or, written fully,

$$AE - 4BD + 3C^2 = \Delta^4 (ae - 4bd + 3c^2) \text{ .... } (1).$$

Proceeding now to the invariant of the third order in the coefficients, we see that the weight would be 6, and must be of the general form

$$Aa_0 a_4 a_2 + Ba_1 a_2 a_3 + Ca_0 a_3 a_3 + Da_1 a_1 a_4 + Ea_2 a_2 a_2,$$

which embraces all possible forms.

By applying the differential equations as before, we have

$$ace + 2bcd - ad^2 - eb^2 - c^3,$$

or

$$ACE + 2BCD - AD^2 - EB^2 - C^3$$
$$= \Delta^6 (ace + 2bcd - ad^2 - eb^2 - c^3) \text{ .... } (2).$$

If the Δ^6 does not follow clearly by symmetry, the actual transformation will make it evident. If we proceed to the fourth order in the coefficients of another invariant, we shall find only a function of those already found, which therefore is not to be counted in the enumeration.

47. *Absolute Invariants.*—If we eliminate Δ between (1) and (2) in the above, we shall obtain what has been called an absolute invariant, that is,

$$(ACE + 2BCD - AD^2 - EB^2 - C^3)^2 (ae - 4bd + 3c^2)^3$$
$$= (AE - 4BD + 3C^2)^3 (ace + 2bcd - ad^2 - eb^2 - c^3)^2.$$

And if I and T represent the invariants (1) and (2), their ratio $I^3 : T^2$, as is seen, is unchanged by transformation.

48. As to the discriminant of the quartic which is the eliminant of its two first differentials, we shall see that we can arrive at a method of derivation by means of (1) and (2). We have only to remember that the eliminant vanishes if the differentials have a common factor, and that this factor will exist if the binary quantic contains a square factor. We have only, to arrive at this condition, to suppose the first two coefficients to vanish; the quantic then has a square factor, since it is divisible by y^2. It is clear, also, that the invariant of such a quantic must vanish. The one contains the other as a factor when the two first coefficients a and b vanish. Or we may state it thus:—The invariant is a symmetric function of the differences of the roots, and the discriminant is the product of the squares of the differences between any two roots (Art. 17); that is, the invariant, on the above supposition that the roots are equal, as expressed in the terms of the roots, must contain the difference between the roots taken two and two. Now, since the ratio of $I^3 : T^2$ is unchanged by transformation, a new invariant may be constructed from them, and we see that I^3-27T^2 will vanish when a and b are each 0; that is, I becomes $3c^2$, and T, $-c^3$, on that supposition. And since we know (Art. 15) that this form gives us the required order in the coefficients, we conclude it to be the discriminant, that is,

$$(ae-4bd+3c^2)^3-27\,(ace+2bcd-ad^2-eb^2-c^3)^2,$$

which, being of the form of $I^3 \pm kT^2$, is not commonly reckoned as distinct from I and T; and thus generally when, as in this case, I and T are expressible as an invariant, a function both rational and integral of I and T, such function is not counted as a new invariant. We would infer also, in the same manner, that, if I and T are invariants of the same degree, then $I \pm kT$ need not be counted.

To sum up our number of invariants thus far, we have

$ac-b^2$ the invariant of the quadratic $(a, b, c\between x, y)^2$,

which is the discriminant.

Next, $$a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2$$

is the discriminant of the cubic $(a, b, c, d\between x, y)^3$ (Art. 5); that is, it is the eliminant of its two first differentials.

This is its only invariant (Art. 22).

And, lastly, the I and T of the quartic $(a, b, c, d, e\between x, y)^4$ just considered, which are two ordinary invariants.

49. *The Series of Covariants.*—It follows from the definitions of invariants and covariants, and may easily be verified, that every invariant of a covariant is an invariant of the original quantic, and the contrary; consequently the quadratic can have no other covariant than the quadratic itself; or we say that this fact follows immediately from the consideration that there are no differences of roots—there being in this case but one difference—and that there can be no function of the differences of the roots. But in the cubic, since a symmetric function of differences of roots, and differences between x and one or more of the roots, is a covariant, we can form a covariant distinct from the cubic. The form of this covariant,

$$(a^2d-3abc+2b^3,\ abd-2ac^2+b^2c,$$
$$-acd+2b^2d-bc^2,\ 3bcd-ad^2-2c^3\between x, y)^3,$$

has been investigated in Art. 34, and the process need not be repeated here. We have also the Hessian which Dr. Salmon writes

$$H=\begin{vmatrix} a & b & c \\ b & c & d \\ y^2 & -xy & x^2 \end{vmatrix} = (ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2.$$

These two covariants examined in connection with the quantic itself, which is also a covariant, show at once that the list for the cubic is complete. For we see that the coefficient of x^2 in each case is a, $ac-b^2$, $a^2d-3abc+2b^3$, which are called the leaders. Recurring now to the discussion (Art. 35), we find that whatever analytical relation exists between the leaders of covariants, that same or similar relation will hold with the covariants as a whole. This being the case, we need only operate upon these leaders in order to discover the successive covariants.

Thus H above is the Jacobian of the first covariant $(a, b, c, d \between x, y)^3$, or V, and the original quantic $(a, b, c \between x, y)^2$, say; so also the third covariant in the above series, whose leader is $a^2d-3abc+2b^3$, is the Jacobian of the above Hessian, and the original quantic, which in this case, the cubic, is V, and thus each succeeding covariant, is found by taking the Jacobian of the last covariant of the series and the original quantic, whatever that may be. For the cubic this last covariant is indicated by J.

50. The question whether any other covariants may be properly added to this list, as regards the cubic, may be examined as follows. We see that a, ac, a^2d, &c. are divisible by a. We find then what new functions, rational and integral, of these leaders may be formed which contain a. In this case, the leaders of H, J, $ac-b^2$, $a^2d-3abc+2b^3$, become, on the supposition that $a=0$, $4H'^3+J'^2=0$. It therefore contains some power of a. Performing the operation indicated by $4H^3+J^3=0$ and dividing by a^2, we obtain the discriminant of the cubic $\quad a^2d^2-6abcd-3b^2c^2+4ac^3+4db^3$.

Now it must be remembered that a covariant, as also an invariant, is by definition a function of differences of the roots, and that a covariant is known when its source or leading coefficient is known (Art. 36); hence these leaders, as well as resulting invariants, will satisfy concurrently the differential equation $V\left(a_0\dfrac{d}{da_1}+2a_1\dfrac{d}{da_2}+3a_2\dfrac{d}{da_3}+\text{\&c.}\right)=0$,

where V is any leader or invariant.

From this fact, and in conformity with the definition, we might, for the purposes of this classification, include the invariants with the covariant of a quantic. The above discriminant, then, may be classed with the coefficients of the covariants.

Regarded in this light, we shall find that a quantic of the n^{th} degree will have n covariants, including the quantic itself, so that each other covariant, multiplied by some power of the quantic, will be equal to a rational and integral function of the n covariants. Thus, at once, if we represent the discriminant (invariant) by Δ, we shall have

$$\Delta V^2 = J^2 + 4H^3,*$$

or, using the canonical forms,

$$a^2d^2\,(ax^3 + dy^3)^2 = a^2d^2\,(ax^3 - dy^3)^2 + 4\,(adxy)^3.$$

51. If in Δ we let $a = 0$, we have left a quantity containing coefficients which cannot be eliminated by combining with $-b^2$ or $2b^3$. In other words, no new functions of $ac - b^2$, $a^2d - 3abc + 2b^3$ can be formed divisible by a. Hence we may say for the cubic the list is complete.

52. The covariants of the quartic are first the Hessian,† and then the Jacobian of this Hessian and the quartic itself must be taken. We find H to be

$$\{ac - b^2,\ 2\,(ad - bc),\ ae + 2bd - 3c^2,\ 2\,(be - cd),\ ce - d^2 \between x,\ y\}^4.$$

The Jacobian has its first term, or leader, $a^2d - 3abc + 2b^3$ &c., which, by Prof. Cayley's symbolical representation (where the Hessian of every binary quantic is written $\overline{12}^2$, and the Jacobian of H and the quantic $\overline{12}^2$, $\overline{13}$), is easily distinguished, and indicates a basis of calculation.

* Prof. Cayley, "Phil. Trans.," 1854.

† Known in geometry as the Harmonic Conic.

53. We might state here more fully the principle of this symbolic representation.

In Arts. 30 and 34, it was shown that $\frac{d}{dx}, \frac{d}{dy}$, &c., regarded as operating symbols contragredient to x, y, &c., while transformed by a direct substitution x, y, &c., will be transformed by an inverse substitution, and the contrary; and that, representing $\frac{d}{dx}, \frac{d}{dy}$, &c., by α, β, &c., operating symbols could be formed which, substituted in the quartic, a covariant or invariant could be formed according as the variables were or were not removed by differentiation. We can thus form an operative symbol for a system of quartics by a system of determinants formed of α, β, &c. Thus $\alpha_1\beta_2-\alpha_2\beta_1$, represented by $\overline{12}$, is an invariant symbol of operation. If we operate on two quantics S and V, the result of the operation upon their product SV by $\overline{12}$ is the Jacobian.

If these are quadratics,

$$(a, b, c \between x, y)^2, \quad (a_1, b_1, c_1 \between x_2, y_2)^2,$$

then the result of the operative symbol $\overline{12}^2$, or

$$\alpha_1^2\beta_2^2-2\alpha_1\beta_1\alpha_2\beta_2+\alpha_2^2\beta_1^2,$$

on SV will be an invariant, *i.e.*, $ac_1+ca_1-2bb_1$.

In the same manner, $\overline{12}^2\,\overline{13}$ expresses the operative symbol (or its effect upon a binary quantic)

$$(\alpha_1\beta_2-\alpha_2\beta_1)^2(\alpha_1\beta_3-\beta_3\alpha_1).$$

54. We have then, as the effect of $\overline{12}$ on SV, the Jacobian

$$\frac{dS}{dx}\frac{dV}{dy}-\frac{dS}{dy}\frac{dV}{dx},$$

and the application to any two quantics may be expressed by

$$\left(\frac{dS}{dx}\frac{dV}{dy}-\frac{dS}{dy}\frac{dV}{dx}\right)^n,$$

or $\overline{12}^n$.

In the former case, the exponent of the power does not apply to S and V, but only to the symbols of differentiation. The result is, of course, the same in both cases—an invariant if $n =$ the degree of the quantic, since all the variables are removed by differentiation, or a covariant if n is less than the degree of the quantic. From this it will immediately appear that, if by this process, we wish to form the covariant of a single quantic, we have only to make $S = V$. Thus, if we desired to form the covariant of a single quantic with the symbol $\overline{12}^2$, or

$$\left(\frac{dS}{dx}\frac{dV}{dy} - \frac{dS}{dy}\frac{dV}{dx}\right)^2,$$

we have only to make $S = V$, and the latter symbol becomes

$$2\left[\frac{d^2S}{dx^2}\frac{d^2S}{dy^2} - \left(\frac{d^2S}{dx\,dy}\right)^2\right],$$

which, applied to two quadratics S and V, would in this case give $2\,(ac - b^2)$. Hence, in general, the quantic to be operated upon may be conceived to be the product of two or more quantics S, V, T, &c., whose variables are distinguished by subscripts, as x_1, y_1, x_2, y_2, &c., and when the differentiation is complete the variables are written solely x, y. Since $\overline{32}$ and $\overline{23}$ are clearly the same with opposite signs, as also $\overline{12}$ and $\overline{21}$, it will appear that either of these symbols with odd powers will, when applied to any single function as SV, cause it to vanish. Following this analogy, we can easily write the symbol for a system of ternary quadratics. If, for x_1, y_2, &c., we write $\frac{d}{dx_1}$, $\frac{d}{dy_2}$, &c. (in which the cogredient variables can be written as a determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \overline{123}),$$

we shall have, when the symbol $\overline{123}^2$ is applied to the ternary quadratics

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$a_1x^2+b_1y^2+\&c.$$
$$a_2x^2+b_2y^2+\&c.,$$

$$6\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 6\,(abc+2fgh-af^2-bg^2-ch^2),$$

i.e., six times the discriminant of the ternary quadratic when $a=a_1=a_2$ &c.

CHAPTER IV.

COMPUTATION AND GEOMETRICAL APPLICATION OF INVARIANTS.

55. The attentive reader of the preceding pages will have now no great difficulty in making a variety of important applications of the Invariant Theory.

It is shown in works on the Conic Sections, that if V and V_1 represent two conics, there are three values of k for which $kV \pm V_1$ represents a pair of right lines.

We take

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

as the general homogeneous equation of the second degree in three variables; and this is intimately connected with

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots (1);$$

the latter being derived from the former by making $z = 1$.

This latter *may* represent two right lines, and does in general, when its coefficients fulfil the relation

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

which is obtained by the resolution of (1) as a quadratic; the above determinant being the condition necessary to make the quantity under the radical a perfect square. If we call

$$V_1 = a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy,$$

then $\qquad \Delta_1 = a_1b_1c_1 + 2f_1g_1h_1 - a_1f_1^2 - b_1g_1^2 - c_1h_1^2.$

It is not difficult to see that the three values of k, for which $kV \pm V_1$ represents a pair of right lines, is obtained by substituting $ka+a_1$, $kb+b_1$, &c., for a, b, c, &c., in Δ. Writing this result in full, we shall find that k^3 will have Δ for its coefficient; k^2 and k will have functions for their coefficients, which may be represented by θ and θ_1; and lastly, that Δ_1 appears as the absolute term; that is,

$$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0.$$

The value of

$$\theta = (bc-f^2)\,a_1 + (ca-g^2)\,b_1 + (ab-h^2)\,c_1$$
$$+2\,(gh-af)\,f_1 + 2\,(hf-bg)\,g_1 + 2\,(fg-ch)\,h_1 \;\ldots\; (2),$$

and $$\theta_1 = (b_1c_1 - f_1^2)\,a + \&c.,$$

the same as θ, the accents being interchanged.

Now between $$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0 \;\ldots\ldots\ldots\ldots\; (3),$$

and $$kV + V_1 = 0,$$

we may eliminate k, which gives

$$\Delta V_1^3 - \theta V_1^2 V + \theta_1 V_1 V^2 - V^3 \Delta_1 = 0,$$

denoting the three pairs of lines which join the four points of intersection of V and V_1.

56. Since any two conics have a common self-conjugate triangle, and since they may be written

$$V = ax^2 + by^2 + cz^2 = 0,$$

$$V_1 = a_1x^2 + b_1y^2 + c_1z^2 = 0,$$

(see T., Arts. 45, 47, 56,) or

$$V_1 = x^2 + y^2 + z^2 = 0,$$

where x is written for $x\sqrt{a_1}$, &c., we obtain, by Invariants, the three values for which $kV_1 + V$ represents right lines.

Then Δ reduces to abc,

$$\theta = ab + bc + ac, \quad \theta_1 = a + b + c, \quad \Delta_1 = 1;$$

or, were we to substitute $ka + a_1$, &c., in abc, we must have, for the required condition,

$$k^3 + k^2(a+b+c) + k(ab+ac+bc) + abc = 0,$$

which is satisfied by $-a$, $-b$, $-c$.

For another example, let us take the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 = V,$$

and the circle $\quad (x-x_1)^2 + (y-y_1)^2 - r^2 = 0 = V_1.$

In forming Δ from V, we must remember to affect the result by the negative sign, since c or the coefficient of z^2, as well as z^2 itself, is reduced to unity with the minus sign. Hence

$$\Delta = -\frac{1}{a^2b^2}.$$

To obtain θ we must recur to the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

From which we find, by comparing the values of the coefficients with those in the preceding Article, that

$$\begin{array}{ll} a_1 = 1, & \\ b_1 = 1, & a = \dfrac{1}{a^2}, \\ g_1 = x_1, & \\ f_1 = y_1, & b = \dfrac{1}{b^2}, \\ c_1 = x_1^2 + y_1^2 - r^2. & c = -1. \end{array}$$

From these values we find θ to be

$$-\frac{1}{b^2} - \frac{1}{a^2} + \frac{x_1^2 + y_1^2 - r^2}{a^2b^2},$$

* Students who are familiar with Salmon's "Conic Sections," will at once recognize these examples. It is believed that the treatment here given them will completely remove the difficulties which hitherto have been experienced by many in their solution.

or $$\frac{1}{a^2b^2}(x_1^2 + y_1^2 - r^2 - a^2 - b^2).$$

In the same manner, by interchanging the accents, we find

$$\theta_1 = (x_1^2 + y_1^2 - r^2 - y_1^2)\frac{1}{a^2} + (x_1^2 + y_1^2 - r^2 - x_1^2)\frac{1}{b^2} + (1-0)(-1)$$
$$= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

and $$\Delta_1 = x_1^2 + y_1^2 - r^2 - y_1^2 - x_1^2 = -r^2;$$

from which the equation in k is formed.

If, instead of the ellipse, we had taken the circle

$$x^2 + y^2 - r^2 = 0,$$

V_1 remaining as before, accenting the r, we should have had

$$\Delta = -r^2,$$

since $$a = 1, \quad b = 1, \quad c = -r^2;$$

$$\theta = (-r^2-0)\,1 + (-r^2-0)\,1 + (1-0)(x_1^2 + y_1^2 - r_1^2)$$
$$= x_1^2 + y_1^2 - 2r^2 - r_1^2, \text{ by (2) of Art. 55;}$$

$$\theta_1 = (x_1^2 + y_1^2 - r_1^2 - y_1^2)1 + (x_1^2 + y_1^2 - r_1^2 - x_1^2)1 + (1-0)(-r^2)$$
$$= x_1^2 + y_1^2 - 2r_1^2 - r^2,$$
$$\Delta_1 = -r_1^2,$$

as in the previous case.

57. Since Δ, Δ_1, θ, θ_1 are invariants of the system of conics under consideration, their computation should be carefully studied, because in solid, as we shall see, as well as in plane geometry, these functions are fundamental.

Take the parabola $y^2 = 2px$, and V_1 as before, the circle

$$(x-x_1)^2 + (y-y_1)^2 = r_1^2.$$

Here $b=1$, while p corresponds to g in the more general equation, as is evident from (1), Art. 55, the other coefficients reducing to zero.

We have then $\qquad \Delta = -bg^2 = -p^2,$

$$\theta = (0-p^2)\,1 + (0+2p)\,(-x_1) = -p\,(2x_1+p),$$

$$\theta_1 = (x_1^2 + y_1^2 - r_1^2 - x_1^2)\,1 + (0+x_1)(-2p) = y_1^2 - 2px_1 - r_1^2,$$

$$\Delta = -r_1^2, \text{ as before.}$$

If $\qquad x^2+y^2 = r^2$ and $(x-x_1)^2+(y-y_1)^2 = r_1^2$

represent two circles, and d the distance between their centres, we have, as before,

$$\Delta = -r^2, \quad \theta = d^2-2r^2-r_1^2, \quad \theta_1 = d^2-r^2-2r_1^2, \quad \Delta_1 = -r_1^2.$$

58. If we turn to equation (3), Art. 55, and observe its degree, and remember that two conics always intersect in four points, and that four points may be connected by six lines, viz., 12, 13, 14, 23, 24, 34, we may conclude that this equation is that of the three pairs of chords of intersection of the two conics.

An easy application of this equation is found in the problem, to find the locus of the intersection of normals to a conic from the ends of a chord which passes through a given point.

The equation of the normal to an ellipse is

$$a^2xy_1 - b^2x_1y = c^2x_1y_1.$$

If we interchange the accents, the right line becomes a curve,

in fact, an hyperbola $\quad a^2x_1y - b^2xy_1 = c^2xy,$

expressing that the point on the normal is known, and that the point on the curve is sought; consequently, we see that the intersections of the given ellipse and the equation last written are points whose normals will pass through the given point; that is, $x_1\,y_1$.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0=V,$$

$$2\,(a^2xy_1-b^2x_1y-c^2x_1y_1)=V_1.$$

This latter equation, it is evident, should be, as has been done, multiplied by 2 in order to sustain the fixed numerical relation expressed in the corresponding coefficients of the general equation. The equation of the six chords joining the feet of normals through xy, the locus required when satisfying the given point, is readily formed by substituting the requisite invariants in equation (3), referred to above.

We have then

$$\Delta=-\frac{1}{a^2b^2},\qquad \theta=0,$$

since $$h_1=c^2,$$

$$g_1=b^2y',\text{ and } a_1=b_1=c_1=0,$$

$$f_1=-a^2x';$$

and therefore $$\theta_1=-(a^2x_1^2-c^4+b^2y_1^2),$$

$$\Delta_1=-2a^2b^2c^2x_1y_1.$$

Hence, if $\alpha\beta$ represent the given point, we have

$$\frac{8}{a^2\,b^2}\,(a^2\beta x-b^2\alpha y-c^2\alpha\beta)^3+\&c.=0,$$

an equation of the third degree, reducing to a conic when the axis is a part of the locus.

59. In the cubic for k, its values, for which $kV\pm V_1$ represents right lines, remain the same without reference to the coordinates in which V and V_1 are taken. In other words, the relation between the coefficients Δ, θ, &c., remains unaltered by a change of coordinates, and these coefficients for the new system are equal to those of the old, multiplied by the square of the modulus of transformation, or in general by some power of that modulus. (Art. 20.)

60. If 1 and 2 of the four points of intersection of two conics coincide, then 13 and 23 will coincide with 14 and 24. In this case the cubic in k will have two equal roots. Let us take the differential coefficient of this equation, and proceed as if to find their greatest common divisor. This condition may be expressed as

$$(\theta\theta_1-9\Delta\Delta_1)^2-4(\theta^2-3\Delta\theta_1)(\theta_1^2-3\Delta_1\theta)=0.*$$

In this case the conics are said to touch each other, though it must not be supposed that there are not also two other real or imaginary points in which the conics meet. A great variety of examples will at once occur to the reader which will illustrate the foregoing. We might exhibit an application of the last example, Art. 58. Expressing that the two curves touch, we must have, since $\theta=0$,

$$27\Delta\Delta_1^2+4\theta_1^3=0.$$

Now that this equation will apply to the finding of the evolute of the given curve—that is, the ellipse—we have only to remember that the coordinates of the centre of the osculatory circle and those of the evolute coincide, that two of the normals coincide which can be drawn through each point of the evolute; and we have

$$27a^2b^2c^4x^2y^2+(a^2x^2+b^2y^2-c^4)^3=0$$

as the required equation.

61. Before passing to other applications, we may discuss the conditions under which Δ_1, θ, and θ_1 vanish.

If $\Delta_1=0$, how shall we interpret θ and θ_1? Since V_1 breaks up into two right lines when $\Delta_1=0$, we may represent these lines by α and β, and then instead of $V+kV_1$ we may write $V+2k\alpha\beta$, whose discriminant may be found by substituting $h+k$ for h in Δ, from which we obtain

$$\Delta+2k(fg-ch)-ck^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(1).$$

* This condition may be found by equating the discriminant of the given cubic in k (Article 55) to zero.

But when the coefficient of k vanishes, that is, when $fg = ch$, we have the condition that the pole of the axis of x in the general equation should lie on the axis of y; in other words, in this case, that the lines α and β are conjugate with respect to V.

Now the vanishing of the discriminant indicates, as we know, a double point in the curve, and hence the vanishing of (1) shows us that the point $\alpha\beta$ lies on the curve V; that is, the coefficient of k^2 vanishes when, in this case, $c = 0$; and consequently that, when $\theta_1 = 0$, the intersection of the two lines is on V.

More generally, the geometrical interpretation of $\theta = 0$ may be shown if we take the trilinear equation (T. 47) of the general form
$$ax^2 + by^2 + cz^2 = 0,$$
in which the triangle of reference is self-conjugate in respect to V_1. We have then
$$\theta = (bc - 0)\, a_1 + (ca - 0)\, b_1 + (ab - 0)\, c_1.$$
Again, from (T. 53), we see that
$$f_1 yz + g_1 zx + h_1 xy = 0. \quad \text{.....................(2)}$$
represents a curve circumscribing the triangle of reference. Hence we say that, if V_1 has the form of (2), θ will vanish, since, in that case, $a_1 = b_1 = c_1 = 0$; that is, θ will vanish when the triangle of reference inscribed in V_1 is self-conjugate in respect to V. If we reverse this relation, taking the triangle of reference as self-conjugate in respect to V_1,
$$\theta = (bc - f^2)\, a_1 + (ca - g^2)\, b_1 + (ab - h^2)\, c_1 \quad \text{........(3)},$$
since in this case $\quad f_1 = g_1 = h_1 = 0.$

We see that (3) will vanish if we impose the condition of equal roots in the general equation; that is, if $bc = f^2$, &c., which is the condition of coincident tangents, or that x as a line should touch V; that is, that the triangle should circumscribe V while self-conjugate in respect to V_1, in which case $\theta = 0$.

62. Since $V = k\alpha^2$ represents a conic having double contact with V, α being the chord of contact, if now V represent the general equation in x, y, z, and $lx + my + nz$ the equation of a line in trilinear coordinates, the equation of any conic having double contact with V on the points of intersection with the given conic, can be written

$$kV + (lx + my + nz)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and suppose it were required to so determine k that this equation may represent two right lines. In this case Δ remains unaffected, but θ evidently becomes

$$(bc - f^2)\, l^2 + (ca - g^2)\, m^2 + (ab - h^2)\, n^2 + 2\,(gh - af)\, mn$$
$$+ 2\,(hf - bg)\, nl + 2\,(fg - ch)\, lm = 0.$$

But since, by hypothesis, V_1 breaks up into two right lines, $\Delta_1 = 0$, and θ also vanishes, since there is double contact, or the intersection of the two lines is on V; hence the cubic in k

reduces to $$\Delta k^3 + \theta k^2 = 0.$$

In other words, there are two roots $= 0$, and we have

$$k\Delta + \theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

to determine the other. When there are two equal roots, the conics touch each other (Art. 59). Hence, finding the value of k in (2), and substituting it in (1), we have

$$\theta V = \Delta\,(lx + my + nz)^2,$$

which is the equation of the pair of tangents at the points where the conic is cut by the given line. Where $\theta = 0$, representing its new value as above, we have the condition that the line touches the conic, and the tangents coincide with the line.

63. It may be well here to remind the beginner, that by a tangent is understood, analytically, in general, a line meeting the curve in two coincident points, and that when

the curve breaks up, as we have supposed, into two right lines, the only tangent which can meet such a locus must be on the intersection of these right lines; and since a curve of the second degree may always have two tangents, both tangents must coincide with the line at the point of intersection.

We know that, when V and V_1 represent conics, $V+kV_1=0$ represents a conic passing through their points of intersection. If now the condition were sought that the line $lx+my+nz=0$ should pass through one of these points, we may equate z in V to 0 and in the equation of line $=1$; and then, substituting the value of y found from the equation of the line in $V=0$, we have a quadratic in x whose condition of equal roots we wrote in the last Article, viz.,

$$\theta = (bc-f^2)\,l^2+(ca-g^2)\,m^2+(ab-h^2)\,n^2+2\,(gh-af)\,mn \\ +2\,(hf-by)\,nl+2\,(fg-ch)\,lm.$$

Let this right member now be represented by Σ, the condition that the given line touches V. If in this expression we write $a+ka_1$ for a, $b+kb_1$ for b and c, we shall manifestly have the same condition for $V+kV_1$, or any conic of the system, which we had for V. Hence, multiplying out, we have, for the coefficient of k,

$$(bc_1+b_1c-2ff_1)\,l^2+(ca_1+c_1a-2gg_1)\,m^2+(ab_1+a_1b-2hh_1)\,n^2 \\ +2\,(gh_1+g_1h-af_1-a_1f)\,mn+2\,(hf_1+h_1f-bg_1-b_1g)\,nl \\ +2\,(fg_1+f_1g-ch_1-c_1h)\,lm.$$

Representing this by Φ,* and the coefficient of k^2 by Σ_1, we have

$$k^2\Sigma_1+k\,\Phi+\Sigma=0.$$

The condition that this equation should have equal roots is $\Phi^2=4\Sigma\Sigma_1$; or is the condition that the given line should pass through one of the four points; and as the envelope of this

* When $\Phi=0$ we have the condition that the given line shall be cut harmonically by V and V_1. It is also to be observed that this condition is a contravariant of the system of conics V and V_1.

system is clearly only these four points, the equation last written may be regarded as the envelope of the system. It is to be remembered that we are here really discussing functions which remain unaltered by change of axis, because, if V and V_1 by transformation to a new set of co-ordinates become $\overline{V}$ and $\overline{V}_1$, then $V+kV_1$ becomes $\overline{V}+k\overline{V}_1$, k still remaining constant.

Now $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ is the determinant whose vanishing is the condition that the general equation may represent right lines. Differentiating this function with reference to each of its letters, we have the coefficients of Σ above. Also both Φ and Σ_1 are functions of Δ, in such manner as to possess the character of invariance.

64. If we seek the condition that

$$l\alpha+m\beta+n\gamma=0$$

shall touch the conic represented by the general trilinear equation (T. 43)

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0. \quad \text{.........}(1),$$

we shall have the condition represented by Σ, as above. For the coefficients there given, $bc-f^2$, &c., we may write A, B, &c., or

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \quad \text{......}(2),$$

which is sometimes called the tangential equation of the conic. If between this equation and the equation of the line we eliminate n, we shall have a quadratic in $\frac{l^2}{m^2}$, and the condition of two equal roots, or that it breaks up into straight lines; or, which in this case is the same thing, the envelope of the line is

$$(BC-F^2)\,\alpha^2+(CA-G^2)\,\beta^2+\&\text{c.}=0 \quad \text{.........}(3),$$

an equation symmetrical with Σ,—the latter in l, m, n and its coefficients in small letters, the former in α, β, γ and its coefficients in large letters. We may see, then, that the envelope of a line whose coefficients fulfil the condition Σ is the conic (1), for we have only to substitute for A, B, &c., their values $bc-f^2$, &c., and (2) becomes $\Delta V = 0$ when $V = (1)$. Consequently, if we write the trilinear equation corresponding to

$$\Sigma + k\Sigma_1 = 0,$$

we have
$$\Delta V + kD + k^2\Delta_1 V_1 = 0 \qquad\qquad (4),$$

in which D is symmetrical with Φ; that is,

$$D = (BC_1 + B_1C - 2FF_1)\,\alpha^2 + \&c.,$$

an equation in x, y, z when V and V_1 have the meaning we have heretofore assigned them.

The envelope of the system (3) is

$$D^2 = 4\Delta\Delta_1 VV_1;$$

but the envelope in this case is the four common tangents. Hence this is the equation of the four common tangents to the two conics.

To illustrate this, take the two conics

$$x^2 + 3y^2 + 5z^2 = 0,$$

$$2x^2 + 4y^2 + 6z^2 = 0.$$

$$\Delta = 15,\ \Delta_1 = 48,\ A = 15,\ B = 5,\ C = 3,$$

$$A_1 = 24,\ B_1 = 12,\ C_1 = 8;$$

$$D = 2\,(18+20)\,x^2 + 12\,(10+6)\,y^2 + 30\,(4+6)\,z^2.$$

Hence

$$(76x^2 + 192y^2 + 300z^2)^2 = 2880\,(x^2 + 3y^2 + 5z^2)\,(2x^2 + 4y^2 + 6z^2)$$

is the equation of the four common tangents to the two conics.

If
$$3y^2 - 2x^2 - 4xy = 0 = V,$$

and
$$\frac{x^2}{4} + \frac{y^2}{3} - 1 = 0 = V_1,$$

what is D?

65. As has been before intimated, an invariant is a function whose vanishing indicates some property of the curve independent of the axis to which it is referred. In the same manner, as we know, covariants are particular loci whose relation to the equations whence they were derived is independent of the axes of these given equations. In other words, the two functions agree so far as axes are concerned.

Turning our attention now to covariants, which, as we have seen, contain the given variables, we may find our illustration in the system of conics we have been considering, V and V_1, which we will again refer (Article 56) to their self-conjugate triangle, that is,

$$V = ax^2 + by^2 + cz^2,$$

$$V_1 = x^2 + y^2 + z^2.$$

If we proceed now as in the last Article, we find

$$A = bc, \quad B = ca, \quad C = ab,$$

$$A_1 = B_1 = C_1 = 1;$$

consequently

$$D = (ab + bc)x^2 + (bc + ba)y^2 + (ac + cb)z^2 \quad \ldots\ldots(1);$$

equation (2) of the preceding Article becomes

$$Al^2 + Bm^2 + Cn^2 = 0,$$

or the condition that a line should touch V. Hence the locus of the poles with regard to V_1 of the tangents to V is

$$Ax^2 + By^2 + Cz^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Adding (1) and (2), we have

$$(A + B + C)(x^2 + y^2 + z^2) = D.$$

Or, since (Art. 56) $\quad \theta = A + B + C,$

we have $\theta V_1 = D$ as the equation of the polar conic of V with respect to V_1 in terms of the conics of the system and the conic D. The locus $\theta V_1 = D$ is therefore a covariant of V and V_1, and this relation will not be altered when V and V_1 are transformed to othes axes. Similarly, $\theta_1 V = D$ is a locus, a covariant, the polar conic of V_1 in regard to V.

Returning to $\Phi = 0$ (see note, Art. 63), we see that it becomes, retaining the same expressions for V and V_1,

$$(b+c)\,l^2 + (c+a)\,m^2 + (a+b)\,n^2 = 0,$$

which may be called the tangential equation of the conic enveloped by a line cut harmonically by V and V_1. Now the trilinear equation, as found from this, is of the form of equation (3) of the last Article, that is,

$$BCx^2 + CAy^2 + ABz^2 = 0,$$

or

$$(c+a)(a+b)\,x^2 + (a+b)(b+c)\,y^2 + (c+a)(b+c)\,z^2 = 0 \ldots (1),$$

since in this case $A = (b+c)$, &c.

Adding the value of D to (1), and reducing, we have

$$\theta V_1 + \theta_1 V - D = 0$$

as the equation, a locus, covariant with V and V_1, expressing in terms of these conics a conic enveloped by a line cut harmonically by the conics in question. If D breaks up into two right lines, we have simply $\Delta = 0$ in equation (1),

or $$(ab+ac)(bc+ba)(ac+ab) = 0.$$

66. It would be a profitable exercise for the reader, at this stage, to reduce a few conics to the forms

$$x^2 + y^2 + z^2 = 0,$$

$$ax^2 + by^2 + cz^2 = 0.$$

This can be done with the help of

$$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

That is, the roots of this equation will give us the new a, b, c; then we shall have

$$x^2 + y^2 + z^2 = V, \quad ax^2 + by^2 + cz^2 = V_1,$$

when V and V_1 are the given conics.

We shall still need one more equation, and for this we can conveniently use equation (1) of the preceding Article,

$$(ab+ac)x^2+(bc+ab)y^2+(ac+cb)z^2 = D,$$

with this caution, that, as the discriminant of V is 1, D must be divided by Δ to put the three equations upon the same relation.

Thus, if V and V_1 are

$$x^2-2xy+2y^2-4x+6y=0,$$

$$3x^2-6xy+5y^2-2x-1=0,$$

we see these are of the general form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The Δ of the first is -5 (Art. 55),

$$\theta=-14,\ \theta_1=-9,\ \Delta_1=-11;$$

and since (Art. 56) the roots of (1), when the conics are referred to their self-conjugate triangle, are $-a$, $-b$, $-c$, the actual form of (1) for numerical use must be

$$\Delta k^3-\theta k^2+\theta_1 k-\Delta_1=0,$$

or in this case

$$-5k^3+14k^2-9k+11=0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In order to calculate the covariant D, we must first know A, B, C, A_1, B_1, C_1, &c.

These may be computed by equation (2), Art. 64.

V gives us	V_1 gives
$a=1,$	$a_1=3,$
$h=-1,$	$h_1=-3,$
$b=2,$	$b_1=5,$
$g=-2,$	$g_1=-1,$
$f=3,$	$f_1=0,$
$c=0;$	$c_1=-1.$

As given in Art. 63,

$$A=bc-f^2,\quad B=ca-g^2,\quad C=ab-h^2,$$

$$F=gh-af,\quad G=hf-bg,\quad H=fg-ch.$$

In the same manner $A_1 = b_1c_1 - f_1^2$, &c.

The value of D must be computed from the general equation, which we now write in full,

$$(BC_1+B_1C-2FF_1)x^2+(CA_1+C_1A-2GG_1)y^2$$
$$+(AB_1+A_1B-2HH_1)z^2+2(GH_1+G_1H-AF_1-A_1F)yz$$
$$+2(HF_1+H_1F-BG_1-B_1G)xz$$
$$+2(FG_1+F_1G-CH_1-C_1H) = D.$$

Now suppose the roots of (1) to be represented by a, b, c (the new a, b, c), and we shall have

$$X^2+Y^2+Z^2 = V,$$
$$aX^2+bY^2+cZ^2 = V_1,$$
$$(ab+ac)X^2+(bc+ab)Y^2+(ac+ab)Z^2 = \frac{D}{\Delta}.$$

From which we can obtain the values of X, Y, Z, which were required. The reader can complete this example. These computations are important on account of their frequent occurrence in geometrical investigations, as will be seen in a succeeding Tract.

67. Another of a large class of examples will show how invariants determine the situation of a conic, as for example a fixed locus.

Let us take V, a curve circumscribing the triangle of reference (T., Art. 53),

that is, $$2(u\beta\gamma+v\gamma\alpha+w\alpha\beta) = 0.$$

Let V_1 be touched by two sides of the triangle. This can be represented by the tangential equation, in this case (T., Art. 54), by $$\alpha^2+\beta^2+\gamma^2-2\beta\gamma-2\gamma\alpha-2\alpha\beta(1+wk),$$

since $\alpha=0$, $\beta=0$, in each case, satisfies the equation, giving perfect squares. Then will $kV+V_1$, a conic passing through their intersections, be touched by the third side of the triangle. Computing the invariants as before, we have

$$\Delta = 2uvw,$$

$$\theta = -u^2-v^2-w^2-2vw-2uw-2uv-2uvwk$$
$$= -(u+v+w)^2-2uvwk,$$

$$\theta_1 = 2(u+v+w)(2+wk), \quad \Delta_1 = -(2+hk)^2.$$

From which we obtain

$$\theta_1^2 = 4\Delta\Delta_1 k + 4\theta\Delta,$$

and, eliminating the parameter k between this last equation and $kV+V_1$, we have the envelope of the third side of the triangle of reference; or—which, in this case, is the same thing—by substituting the value of k, derived from that equation, in the latter, we obtain, plainly, a fixed conic touched by the third side, that is,

$$(\theta_1^2-4\theta\Delta)\,V = 4\Delta\Delta_1 V_1.$$

When $\theta_1^2 = 4\theta\Delta$, $k=0$, and is simply the condition that the three sides of the triangle are touched by V_1.

68. If l and m are any lines at right angles to each other through a focus, we can construct an equation, a particular form of

$$u^2\alpha^2+v^2\beta^2 = w^2\gamma^2, \qquad \text{(T., Art. 47)}$$

that is,

$$l^2+m^2 = e^2\gamma^2,$$

where γ, the polar of the focus, is the directrix. If $e=0$, as in the circle, we have the equation which determines the direction of the points at infinity on any circle; or, in other words,

$$l^2+m^2=0$$

is the tangential equation of these points, or the condition that the line

$$lx+my+n=0$$

should pass through one of them.

Now the necessary relation between these constants, in order that this line may touch the curve represented by the general equation, sometimes called the tangential equation of the curve, is given Art. 63, equation (2). Distinguishing this by S, let us proceed to examine the discriminant formed from

$$S+k(l^2+m^2),$$

which is $$\Delta^2+k\Delta(a+b)+k^2(ab-h^2).$$

Form also the discriminant of

$$S+kS_1,$$

which is $$\Delta^2+k\Delta\theta_1+k^2\Delta_1\theta+k^3\Delta_1^2,$$

and we see that $a+b$ corresponds to θ_1 and $ab-h^2$ to θ. Hence we say that, the invariants of any conic and a pair of points at infinity being formed, we can express the condition, by placing $\theta_1=0$, that the curve is an equilateral hyperbola, and by $\theta=0$, that it is a parabola. This result follows from the theory of invariants,—viz., that whatever homogeneous relation is seen to exist in the one case will also exist in the other, irrespective of the coordinates in which the curves are expressed or the axes to which they are referred.

We now seek for the corresponding expression in Trilinear Coordinates. The length of the perpendicular on one of these four imaginary common tangents from any point must be infinite. Hence the denominator of p (T., 20) must be put $=0$, that is,

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0$$

must be the general tangential equation of the points in question in trilinear coordinates. Combining this with S, as before, we find that θ_1 corresponds to

$$a+b+c-2f\cos A-2g\cos B-2h\cos C,$$

which, equated to 0, is the condition that the conic $S+kS_1$ shall represent an equilateral hyperbola.

In this computation the coefficient of k only, it is evident, need be formed, which divided by Δ must give the condition sought. To find the condition that the curve shall represent a parabola, it will be necessary to form the coefficient of k^3 and then divide this result by Δ_1.

68. By the theory of foci, the four tangents drawn through the two imaginary points at infinity on any circle form a quadrilateral, in which two of these vertices are real and the foci of the conic. Now, since $S+kS_1$ touches the four tangents common to S and S_1, it will represent these two vertices or foci in question, when k has been so determined that the conic $(S+kS_1)$ reduces to a pair of points, with the condition that S_1 represents the two points at infinity.

To find these foci, we proceed to find the value of k in

$$(ab-h^2)\,k^2+\Delta\,(a+b)\,k+\Delta^2=0,$$

which, substituted in $S+k\,(l^2+m^2)$, gives two factors, viz.,

$$\left(l\frac{x_1}{z_1}+m\frac{y_1}{z_1}+n\right)\left(l\frac{x_2}{z_2}+m\frac{y_2}{z_2}+n\right),$$

in which $\frac{x_1}{z_1}$, $\frac{y_1}{z_1}$ and $\frac{x_2}{z_2}$, $\frac{y_2}{z_2}$ are the coordinates of the foci, one value of k giving the real and the other the imaginary foci.

As a simple illustration, let us seek the coordinates of the focus of

$$x^2+2xy+y^2-2x-2y+2=0.$$

Here $ab-h^2=0$, and consequently

$$\Delta^2+k\Delta\,(a+b)+k^2\,(ab-h^2)$$

reduces to

$$2k\Delta+\Delta^2=0.$$

But

$$\Delta=2+2-1-1-2=0.$$

Hence S, or

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm+k\,(l^2+m^2),$$

reduces to $\quad l^2+m^2-2lm \quad$ or $\quad (l-m)\,(l-m).$

Therefore the cordinates of the focus are 1, 1, if we regard z_1 as the linear unit in the equation of the line

$$lx_1 + my_1 + nz_1.$$

But if these variables are conceived of as functions of one another, or the line as a function of the variables, then, as $z_1 = 0$ and the coordinates are represented by $\frac{x_1}{z_1}, \frac{y_1}{z_1}$, these become infinite, which result is still consistent with the geometrical conception of the foci of the parabola, where one focus is regarded as at infinity.

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